PS4 q1

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a. Solve for the normalized output, $y^* = [Y^*]/Y_T$, as a function of input [L] at steady state. You should express the output in terms of the following dimensionless variables and parameters:

$x^* = [X^*]/X_T$; $\kappa_1 = K_1/X_T$; $\kappa_2 = K_2/X_2$; $\kappa_3 = K_3/X_3$; $\kappa_4 = K_4/X_4$; $\kappa_D = k_{off}/(k_{on}[L])$;

$\theta_B = [R^*]/R_T$; $V_1/V_2 = (\gamma_1 \theta_B/V_2)^*R_T$; $V_3/V_4 = (\gamma_3 x^*/V_4)^*X_T$;

$$\kappa_3 = \frac{K_3}{Y_T} \qquad \kappa_4 = \frac{K_4}{Y_T}$$

$$\frac{d[R^*]}{dt} = k_{on}[R][L] - k_{off}[R^*]$$

$$\frac{d[X^*]}{dt} = -\frac{d[X]}{dt} = \frac{V_1[X]}{K_1+[X]} - \frac{V_2[X^*]}{K_2+[X^*]}$$

where $V_1 = \gamma_1[R^*]$

$$\frac{d[Y^*]}{dt} = -\frac{d[Y]}{dt} = \frac{V_3[Y]}{K_3+[Y]} - \frac{V_4[Y^*]}{K_4+[Y^*]}$$

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where $V_3 = \gamma_3[X^*]$

$R_T = [R] + [R^*]$

$X_T = [X] + [X^*]$

$Y_T = [Y] + [Y^*]$

At steady state, $\frac{d[Y^*]}{dt} = 0$, $\frac{V_3[Y]}{K_3+[Y]} = \frac{V_4[Y^*]}{K_4+[Y^*]} \longrightarrow \frac{V_3}{V_4} = \frac{(K_3+[Y])[Y^*]}{[Y](K_4+[Y^*])}$

$$\xrightarrow{\text{apply } Y_T = [Y]+[Y^*]} \frac{V_3}{V_4} = \frac{(K_3+Y_T-[Y^*])[Y^*]}{(Y_T-[Y^*])(K_4+[Y^*])} \quad \frac{1/Y_T}{1/Y_T} \longrightarrow \frac{V_3}{V_4} = \frac{\left(\frac{K_3}{Y_T}+1-y^*\right)y^*}{(1-y^*)\left(\frac{K_4}{Y_T}+y^*\right)}$$

$$\xrightarrow{\text{substitute}} \frac{V_3}{V_4} = \frac{(\kappa_3+1-y^*)y^*}{(1-y^*)(\kappa_4+y^*)} \longrightarrow \kappa_3 y^* + y^* - y^{*2} = \frac{V_3}{V_4}\left(\kappa_4 + y^* - y^*\kappa_4 - y^{*2}\right)$$

$$\longrightarrow y^{*2}\left(1+\frac{V_3}{V_4}\right) + y^*\left(\kappa_3+1-\frac{V_3}{V_4}+\kappa_4\frac{V_3}{V_4}\right) - \frac{V_3}{V_4}\kappa_4 = 0$$

$$\xrightarrow{\text{solve quadratic}} y^* = -\left(\kappa_3+1-\frac{V_3}{V_4}+\kappa_4\frac{V_3}{V_4}\right) + \sqrt{\left(\kappa_3+1-\frac{V_3}{V_4}+\kappa_4\frac{V_3}{V_4}\right)^2 + 4\left(-1+\frac{V_3}{V_4}\right)\left(\frac{V_3}{V_4}\kappa_4\right)}\left(\frac{1}{2\left(-1+\frac{V_3}{V_4}\right)}\right)$$

x^* can be solved for in the same way:

$$x^* = -\left(\kappa_1+1-\frac{V_1}{V_2}+\kappa_2\frac{V_1}{V_2}\right) + \sqrt{\left(\kappa_1+1-\frac{V_1}{V_2}+\kappa_2\frac{V_1}{V_2}\right)^2+4\left(-1+\frac{V_1}{V_2}\right)\left(\frac{V_1}{V_2}\kappa_2\right)}\left(\frac{1}{2\left(-1+\frac{V_1}{V_2}\right)}\right)$$

solve for θ_B:

$$k_{on}[R][L] = k_{off}[R^*] \longrightarrow k_{on}(R_T-[R^*])[L] = k_{off}[R^*] \longrightarrow \frac{k_{off}}{k_{on}[L]} = \frac{(R_T-[R^*])}{[R^*]} \quad \frac{1/R_T}{1/R_T}$$

$$\longrightarrow \kappa_D = \frac{1-\theta_B}{\theta_B} \longrightarrow \kappa_D\theta_B = 1-\theta_B \longrightarrow \theta_B(\kappa_D+1) = 1 \longrightarrow \theta_B = \frac{1}{\kappa_D+1} = \frac{1/\kappa_D}{1+1/\kappa_D}$$

solve for x^* in terms κ_D

$$-\left(\kappa_1+1-\frac{V_1}{V_2}+\kappa_2\frac{V_1}{V_2}\right) + \sqrt{\left(\kappa_1+1-\frac{V_1}{V_2}+\kappa_2\frac{V_1}{V_2}\right)^2+4\left(-1+\frac{V_1}{V_2}\right)\left(\frac{V_1}{V_2}\kappa_2\right)}\left(\frac{1}{2\left(-1+\frac{V_1}{V_2}\right)}\right)$$

$$V_1 = \gamma_1[R^*] = \gamma_1\theta_B R_T = \gamma_1 R_T\left(\frac{1}{\kappa_D+1}\right)$$

$$\longrightarrow x^* = -\left(\kappa_1+1-\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{\kappa_D+1}\right)+\kappa_2\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{\kappa_D+1}\right)\right) + \sqrt{\left(\left(\kappa_1+1-\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{\kappa_D+1}\right)+\kappa_2\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{\kappa_D+1}\right)\right)\right)^2+4\left(-1+\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{\kappa_D+1}\right)\right)\left(\kappa \ldots\right.}$$

solve for y^* in terms x^*

$$\boxed{y^* = -\left(\kappa_3+1-\frac{\gamma_3 x^*[X]_T}{V_4}+\kappa_4\frac{\gamma_3 x^*[X]_T}{V_4}\right) + \sqrt{\left(\kappa_3+1-\frac{\gamma_3 x^*[X]_T}{V_4}+\kappa_4\frac{\gamma_3 x^*[X]_T}{V_4}\right)^2 + 4\left(-1+\frac{\gamma_3 x^*[X]_T}{V_4}\right)\left(\frac{\gamma_3 x^*[X]_T}{V_4}\kappa_4\right)}\left(\frac{1}{2\left(-1+\frac{\gamma_3 x^*[X]_T}{V_4}\right)}\right)}$$

$$\overline{\quad 2\,\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{K_D+1}\right)\Big)} \left(\frac{1}{2\left(1+\frac{\gamma_1 R_T}{V_2}\left(\frac{1}{K_D+1}\right)\right)} \right)$$