Random Signals & Noise

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1 Important Facts

1.1 Probability Measure

If X and Y are statistically-independent random variables, then

$$P(X = x, Y = y) = P(X = x)P(Y = x)$$

Or more generally, if A and B are independent events then

$$P(A \cap B) = P(A)P(B)$$

1.2 CDF

The cumulative distribution function (CDF), F_X for a random variable X is defined as

$$F_X(x) = P(X \leq x)$$

Several corollaries of this definition and the definition of a probability measure are

1. $(\forall x)\ 0 \leq F_X(x) \leq 1$
2. $\lim_{x\to\infty} F_X(x) = 1$
3. $\lim_{x\to-\infty} F_X(x) = 0$
4. $F_X(x)$ is non-strictly monotonically increasing in x.
5. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

1.2.1 Joint CDF

The joint CDF is defined as

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = F_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

The joint PDF is defined as

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \frac{\partial^n F_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

1.3 PDF

The probability density function (PDF), f_X for a random variable X is defined as

$$f_X(x) := \frac{dF_X(x)}{dx}$$

Some corollaries to this are

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$
4. $\int_{x_1}^{x_2} f_X(t)\,dt = P(x_1 < X \leq x_2)$

1.3.1 Marginal PDFS

For two R.V.s X and Y and joint PDF $f_{XY}(x, y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \tag{$\star$}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \tag{$\star$}$$

These are called the marginal PDFs of X and Y. They are often denoted $f(x)$ and $f(y)$, respectively. In general cannot determine the joint PDF from the marginal PDFs. Statistically independent R.V.s are an exception.

1.3.2 Independence

Two random variables are statistically independent iff

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

1.4 Moments

The nth moment of a random variable X is defined as

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx \tag{$\star$}$$

Suppose also that we've defined a transformation between random variables X and Y as

$$Y = g(X)$$

then the nth moment of Y is

$$E[Y^n] = \int_{-\infty}^{\infty} y^n f_Y(y)\, dy = \int_{-\infty}^{\infty} [g(x)]^n f_X(x)\, dx \tag{$\star$}$$

1.5 Transformations

Define a mapping g as

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) \end{aligned}$$

Then $Y = g(X)$ defines a new random variable, Y. We can determine Y from X using a few methods.
The first relies on the set I_y define as

$$I_y := \{x \mid g(x) \leq y\}$$

The definition imples

$$F_Y(y) = P(Y \leq y) = P(X \in I_y) \tag{$\star$}$$

The second assumes that the function g is invertible. Then we can determine f_Y directly as

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} = \frac{f_X(x)}{|g'(x)|}\Big|_{x=g^{-1}(y)}$$

1.5.1 Transformations of Multiple Random Variables

Let $Z = g(X, Y)$ Then

$$E[Z] = E_{XY}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)\,dxdy$$

More specifically, we can find the expectation of the sum of the R.V.s as

$$E[Z] = E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f_{XY}(x,y)\,dxdy$$

which by properties of integration reduces to

$$E[X+Y] = E[X] + E[Y] \tag{$\star$}$$

1.5.2 Correlation

First define the correlation as

$$E[XY] = \int_{-\infty}^{\infty} (xy)f_{XY}(x,y)\,dxdy$$

and we say

$$X \text{ and } Y \text{ are uncorrelated} \iff E[XY] = E[X]E[Y] \tag{$\star$}$$

note that

$$X \text{ and } Y \text{ are statistically independent} \implies X \text{ and } Y \text{ are uncorrelated} \tag{$\star$}$$

1.5.3 Covariance

Let X and Y be two R.V.s Then the variance of the sum is

$$Var(X+Y) = Var(X) + Var(Y) + 2\cdot\mathrm{cov(X,Y)}$$

Where the covariance is defined as

$$\mathrm{cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = E[XY] - E[X]E[Y]} \tag{$\star$}$$

Note, that if X and Y are uncorrelated then

$$\mathrm{cov(X,Y)} = 0 \tag{$\star$}$$

An important corollary of this is

$$X \text{ and } Y \text{ are uncorrelated} \implies \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

Note that if X and Y are statistically independent then $E[XY] = E[X]E[Y]$ this can be shown by splitting the joint PDF and integrating each part separately. This implies the important concept

$$X \text{ and } Y \text{ independent} \implies X \text{ and } Y \text{ uncorrelated} \tag{$\star$}$$

1.6 Characteristic Function

Consider the transformation $Y = g(X) = e^{j\omega X}$. Then

$$\Phi_X(\omega) := E[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx \tag{$\star$}$$

we can think of this as the Fourier transform of the PDF

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)e^{-j\omega x}\,d\omega \tag{$\star$}$$

Consider X and Y R.V.s. Then we can determine the distribution of their sum via the convolution. To see this consider $Z = X + Y$. Consider the CDF of Z

$$P(Z \leq z) = P(X + Y \leq z)$$

so we want to consider all $y \leq z - x$, can find all such (x, y) pairs by integrating the joint PDF

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y)\, dxdy$$

by change of variables

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z} f_{XY}(x, v - x)\, dxdv$$

Fubini's theorem

$$\int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{XY}(x, v - x)\, dvdx$$

and by differentiating with respect to z

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx \tag{$\star$}$$

if the variables are independent then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = (f_X * f_Y)(z) \tag{$\star$}$$

A corollary of this (from induction), for $Z = X_1 + X_2 + \cdots + X_n$ all X_i independent of each other.

$$\Phi_Z = \Phi_{X_1} \Phi_{X_2} \ldots \Phi_{X_n} \tag{$\star$}$$

Remember that the FT of a Gaussian is a Gaussian. As a corollary to the previous we immediately have that, for $Z = X + Y$, X, Y are Gaussians that Z is a Gaussian (with mean $E[X] = E[Y]$ and variance $Var(X) + Var(Y)$)

1.6.1 Obtaining Raw Moments

consider

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

notice that when you take the nth derivate of Φ_X

$$\frac{\partial^n \Phi_X(\omega)}{\partial \omega^n} = \int_{-\infty}^{\infty} \frac{\partial^n e^{j\omega x}}{\partial \omega^n} f_X(x)\, dx = j^n \int_{-\infty}^{\infty} x^n e^{j\omega x} f_X(x)\, dx$$

If we evaluate this at $\omega = 0$

$$\frac{\partial^n \Phi_X(0)}{\partial \omega^n} = j^n E[x^n] \iff j^{-n} \frac{\partial^n \Phi_X(0)}{\partial \omega^n} = E[x^n]$$

1.7 Conditional PDFs and CDFs

Remember that

$$P(A \mid M) = \frac{P(A \cap M)}{P(M)} \quad P(M) > 0$$

We define the conditional CDF this way

$$F_X(x \mid M) = P(X \leq x \mid M) = \frac{P(X \leq x, M)}{P(M)} \tag{$\star$}$$

Interesting, conditional CDF is a valid CDF. We can differentiate to obtain the conditional PDF

$$f_X(x \mid M) = \frac{dF_X(x \mid M)}{dx} \tag{$\star$}$$

We'll note that the conditional PDF is also a valid PDF.

1.7.1 Relationship between joint and the conditional PDFs

This is analagous to Bayes' rule, except for PDFs

$$f_Y(y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} \tag{$\star$}$$

This notation is bad but is used in engineering, more precisely

$$f_Y(y \mid X = k) = \frac{f_{XY}(x, y)}{f_X(x)}\Big|_{x=k}$$

Notice above, that the denominator is a single value, the value of the marginal at $x = k$ and the numerator is a function of y. This can be expressed more concisely as

$$f(y \mid x) = \frac{f(x, y)}{f(x)}$$

If we are dealing with a mixed random variable, that is entirely discrete (composed of solely deltas) We can state the former as

$$f(y \mid x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

Rather

$$f(y \mid X = k) = \frac{P(X = k, Y = y)}{P(X = k)}$$

For some $k \in \mathbb{R}$.

1.7.2 Conditional Expectation

consider $X = g(Y)$. Then

$$E[g(Y) \mid M] = \int_{-\infty}^{\infty} g(y) f_Y(y \mid M)\, dy$$

We might regard $E[y \mid x]$ as a function of x in the sense

$$E[y \mid x] = \int_{-\infty}^{\infty} y f_Y(y \mid x)\, dx$$

1.8 Specific Continuous Distributions

X is
Uniform

- $E[X] = \dfrac{a + b}{2}$
- $Var(X) = \dfrac{(b - a)^2}{12}$

Gaussian

- $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\dfrac{-(x - \mu_x)^2}{2\sigma_x^2}\right)$
- $E[x] = \mu_x$
- $Var(x) = \sigma_x^2$

Exponential

- $f_X(x) = U(x) \cdot [ae^{-a}]$

- $E[x] = 1/a$
- $Var(x) = 1/a^2$

Rayleigh

- $\alpha > 0, \quad f_X(x) = U(x)\left[\frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}}\right]$
- $E[x] = \alpha\sqrt{\frac{\pi}{2}}$
- $Var(x) = \left(2 - \frac{\pi}{2}\right)\alpha^2$