Probability

Introductory course on Statistics and Probability

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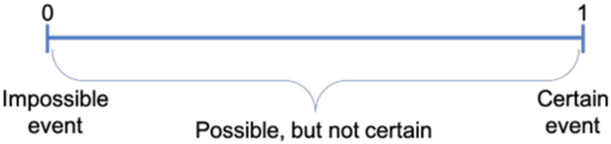
Introduction to Probability

Probability helps us recognize randomness, preventing it from being mistaken for deterministic patterns.



Probability focuses on uncertain events, such as:

- A roulette ball landing on black
- Tomorrow's weather in Trieste being sunny
- Juventus winning the Italian league Football League
- Whether a defendant is guilty

Subjective Interpretation

Bruno de Finetti in "Theory of probability" (1970) introduced **Subjective probabilities** - that is, **the degree of belief in the occurrence of an event attributed by a given person at a given instant and with a given set of information**.

There are two main approaches to interpreting probability:

- **Subjective or Bayesian**: probability as degree of belief - similar to what de Finetti proposed
- **Frequentist or Objective**: probability as long-run frequency in repeatable experiments. It is based on observation rather than theorizing.

In this introductory course we will focus on the objective approach.

Introduction to Probability

- De Finetti's idea emphasizes that probability is a measure of an individual's degree of belief about an event occurring.
- Psychologists have shown experimentally that heuristics, or mental shortcuts, often lead to errors when assessing uncertainty.
- Individuals may rely on these mental shortcuts, which can be influenced by biases and cognitive errors, when assigning probabilities subjectively.
- I toss a coin eight times and record the outcome of Heads or Tails. Which result is more likely?

  HHHHTTTT or **HTTHHTHH**

  Both sequences have an equal probability of occurring in a fair coin toss!

Introduction to Probability

- By saying that probability is the 'degree of belief' we have given a good definition, but this is not of help when it comes to numerically determining this degree of belief.
- What is the probability that it will rain tomorrow?
  - Personal guess $\rightarrow$?
  - Based on average monthly rainfall $\rightarrow$ 28% (P = 0.28)
  - Based on OSMER forecast $\rightarrow$ 5% (P = 0.05)

Introduction to Probability

- *All evaluations are valid*: they are opinions, each is legitimate, but perhaps some are more reasonable or informed than others.
- For example, OSMER, for a relatively trivial event like 'rain tomorrow,' uses complex meteorological models and combines lots of information.

For now, we will look at some cases where the evaluation is simple and intuitive. In such situations, it will be easy to understand some rules for combining probabilities using four operations and a bit of logic.

Frequentist Interpretation

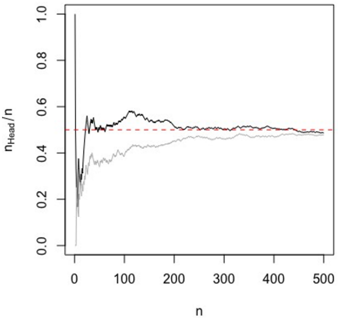
The **Frequentist interpretation** is based on:

- Probability is not something we can directly observe.
- In order to estimate it, we link it to something observable.
- **Example**: an event $E =$ 'getting Heads (H) in a coin toss'.
- This event is repeatable, meaning we can toss a coin many times.

Let's do it, or imagine doing it, and at each toss, calculate the percentage of heads observed up to that point.

Frequentist Interpretation

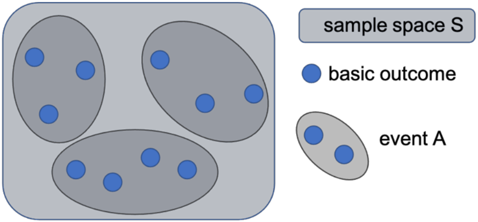
Here are the outcomes from 500 tosses: chance behavior is unpredictable in the short run but has a regular and predictable pattern in the long run.



The **probability** of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions.

Random Experiment

- A **random experiment/phenomenon** is a process leading to two or more possible **outcomes**, without knowing exactly which one will occur.
- The **sample space** S of a random experiment is the set of all the possible **basic outcomes**, that is, outcomes that cannot occur together.
- An **event** A is an outcome or a set of possible outcomes.



Classical Probability

It is the proportion of times that an event will occur, assuming that all outcomes in a sample space are equally likely to occur. It is based on **reasoning and symmetry**.

$$P(A) = \frac{N_A}{N}$$

where N_A is the number of outcomes that satisfy A, and N is the total number of outcomes in the sample space.

Example: tossing a fair dice - $P(4) = 1/6$.

Relative Frequency Probability

It is the the limit of the proportion of times that an event A occurs in a large number of trials, n. Probability is defined as the **long-run frequency of an event**.

$$P(A) = \frac{n_A}{n}$$

where n_A is the number of A outcomes and n is the total number of trials or outcomes.

Example: tossing a fair dice 1000 times. If 4 shows up 167 times out of 1000 tosses - $P(4) \approx 167/1000 = 0.167$.

Random Experiment: Examples

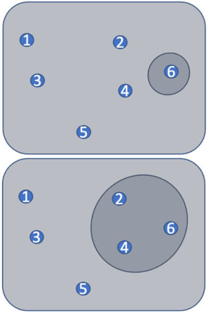
Rolling a dice

- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
- Event: A = get a 6

$$P(A) = N_A/N = 1/6$$

- Event: B = get an even number = $\{2, 4, 6\}$

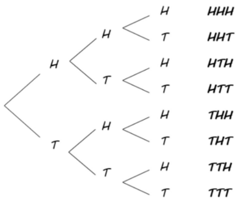
$$P(B) = N_B/N = 3/6 = 1/2$$



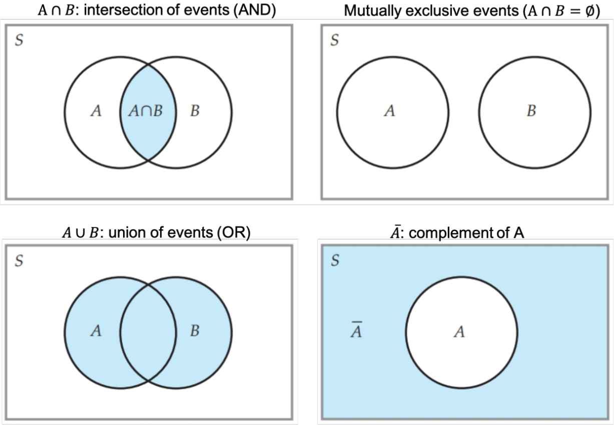
Random Experiment: Examples

Tossing a coin three times

- S = {HHH, HHT, HTT, TTT, TTH, THH, THT, HTH}
- A = two heads = {HHT, THH, HTH}. $P(A) = N_A/N = 3/8$
- B = more than one head = {HHT, THH, HTH, HHH}. $P(B) = N_B/N = 4/8 = 1/2$



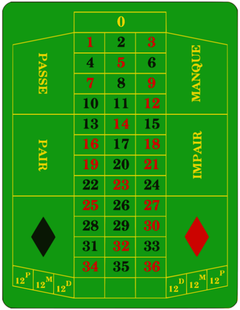
Elementary events can be combined to define more complex events, called **compound events**. There are rules for combining the probabilities of elementary events to calculate the probability of compound events.



Collectively Exhaustive Events: given K events $E_1, E_2, \ldots, E_K$ in the sample space S, if $E_1 \cup E_2 \cup \cdots \cup E_K = S$, the K events are said to be collectively exhaustive.

Roulette Example

- S= $\{0, 1, \ldots, 36\}$
- A= first column= $\{1, 4, 7, \ldots, 34\}$
- B= red = $\{1, 3, 5, 7, 8, 12, 14, 16, 18, \ldots\}$
- C= $\{0\}$
- $A \cap B$ = first column AND red = $\{1, 7, 16, 19, 25, 34\}$
- $A \cup B$ = first column OR red = $\{1, 3, 4, 5, 7, 9, 10, 12, \ldots, \} = A + B - (A \cap B)$
- $A \cap C$ = first column AND $\{0\} = \emptyset$
- $A \cup C$ = first column OR $\{0\} = \{0, 1, 4, 7, \ldots, 34\} = A + C$
- $\bar{C} = \{1, 2, 3, \ldots, 36\} = S - C$



Postulates of Probability

The set of all basic outcomes contained in a sample space is **mutually exclusive** and **collectively exhaustive**.

- $0 \leq P(A) \leq 1$: any probability of an event is a number between 0 and 1.
- $P(A) = \sum_A P(\text{basic outcome}_i) = P(O_1) + P(O_2) + \dots$ basic outcomes are mutually exclusive.
- $P(S) = 1$: all possible basic outcomes together must have probability 1, as they are collectively exhaustive.
- **Complement rule**: $P(\bar{A}) = 1 - P(A)$, since A and $\bar{A}$ are mutually exclusive and collectively exhaustive.
- **Addition rule of probability**:
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Complement rule

We want to prove that: $P(\bar{A}) = 1 - P(A)$

Note that A and $\bar{A}$ are collectively exhaustive

$$A \cup \bar{A} = S$$

and that A and $\bar{A}$ are mutually exclusive:

$$P(A \cup \bar{A}) = P(A) + P(\bar{A})$$

then:

$$P(A) + P(\bar{A}) = P(A \cup \bar{A}) = P(S) = 1$$

. Also follows that $P(0) = \emptyset$

Addition rule of probability

We want to prove that: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note that $A \cup B = A \cup (\bar{A} \cap B)$, A and $\bar{A} \cap B$ are mutually exclusive, then:

$$P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

Moreover, we can write $B = (A \cap B) \cup (\bar{A} \cap B) =$ with $(A \cap B)$ and $(\bar{A} \cap B)$ mutually exclusive

$$P(B) = P(A \cap B) + P(\bar{A} \cap B) \rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

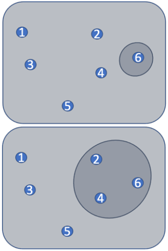
Then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

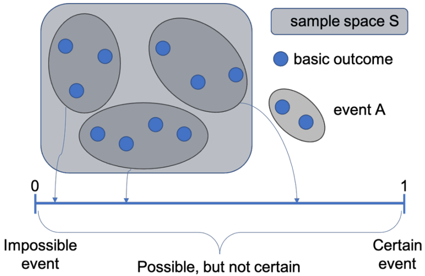
Rolling a dice

- Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
- Event: A = get a 6
- Event: B = get an even number = $\{2, 4, 6\}$
- $P(A) = 1/6$
- P(B) = P(get a 2) +P(get a 4) + P(get a 6) = 3/6 = 1/2
- P(S)=P(get a 1) + P(get a 2) + ... + P(get a 6) = 1
- $P(\bar{A}) = 1 - 1/6 = 5/6$
- $P(A \cap B)$ =P(get a 6) = 1/6
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/6 + 1/2 - 1/6 = 1/2$

Note that A implies B ($A \subset B$), then $P(A) \leq P(B)$



Probability Model



A **probability model** is a mathematical description of a random experiment consisting of a sample space and a way of assigning probabilities to events.

It is a convenient way to describe the distribution of an experiment's outcomes and involves listing all possible outcomes and their associated probabilities.

Exercise A

Canada has two official languages, English and French. Choose a Canadian randomly and ask "What is your mother tongue?". Here is the distribution of responses, combining many separate languages from the broad Asian/Pacific region:

Language	English	French	Asian/Pacific	Other
Probability	0.59	0.23	0.07	?

a) Complete the table

b) What is the probability that a Canadian's mother tongue is not English?

$$S = \{\text{English, French, Asian/Pacific, Other}\}$$

Exercise B

The sales manager wants to estimate the probability that a car will be returned for service during the warranty period. The following table shows a manager's probability assesssment for the number of returns.

Number of returns	0	1	2	3	4
Probability	0.28	0.36	0.23	0.09	0.04

Let A be the event "the number of returns will be more than two", and let B be the event "the number of returns will be less than four".

a) Find the probability of event A
b) Find the probability of event B
c) Find the probability of the complement of event A
d) Find the probability of the intersection of events A and B
e) Find the probability of the union of events A and B

Exercise B

Number of returns	0	1	2	3	4
Probability	0.28	0.36	0.23	0.09	0.04

Let A be the event "the number of returns will be more than two", and let B be the event "the number of returns will be less than four".

a)

$$\begin{aligned} P(A) &= P(\text{return more than } 2) = P(\text{returns } > 2) \\ &= P(\text{returns} = 3) + P(\text{returns} = 4) \end{aligned}$$

b)

$$\begin{aligned} P(B) &= P(\text{return less than } 4) = P(\text{returns } < 4) \\ &= P(\text{returns} = 0) + P(\text{returns} = 1) \\ &+ P(\text{returns} = 2) + P(\text{returns} = 3) \\ &= 1 - P(\text{returns} = 4) \end{aligned}$$

Exercise B

Number of returns	0	1	2	3	4
Probability	0.28	0.36	0.23	0.09	0.04

Let A be the event "the number of returns will be more than two", and let B be the event "the number of returns will be less than four".

c)

$$P(\bar{A}) = 1 - P(A) = 1 - P(\text{returns} > 2)$$

d)

$$P(A \cap B) = P(\text{returns} > 2 \text{ AND returns } < 4) = P(\text{returns} = 3)$$

e)

$$P(A \cup B) = P(\text{returns} > 2 \text{ OR returns} < 4) = 1$$

Exercise C

A cell phone company found that 75% of costumers want text messaging on their phone, 80% want photo capability and 65% want both.
What is the probability that a customer will want at least one of the two?
Define the events:

$$
\begin{aligned}
\text{A} &= \text{costumer wants text messaging} \\
\text{B} &= \text{costumer wants photo capability} \\
A \cap B &= \text{customer wants text messaging AND photo capability}
\end{aligned}
$$

We know that $P(A) = 0.75$, $P(B) = 0.80$ and $P(A \cap B) = 0.65$. We need $P(A \cup B)$= probability that a costumer wants text messaging OR photo capability.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.75 + 0.80 - 0.65$$

Exercise D

Motor vehicles sold in the US are classified as either cars or light trucks and as either domestic or imported. Last year, 80% of the new vehicles sold to individuals were domestic, 54% were light trucks and 47% were domestic light trucks.

Choose a vehicle sale at random and compute P(domestic ∪ light truck).

$$P(\text{domestic}) = 0.80$$

$$P(\text{truck}) = 0.54$$

$$P(\text{domestic} \cap \text{light truck}) = 0.47$$

Then:

$$\begin{aligned} P(\text{domestic} \cup \text{light truck}) &= P(\text{domestic}) + P(\text{light truck}) \\ &- P(\text{domestic} \cap \text{light truck}) \\ &= 0.8 + 0.54 - 0.47 \end{aligned}$$

Exercise D

$$P(\text{domestic}) = 0.80$$

$$P(\text{truck}) = 0.54$$

$$P(\text{domestic} \cap \text{light truck}) = 0.47$$

$$P(\text{domestic} \cup \text{light truck}) = 0.87$$

	Domestic	Imported	
Light truck			
Car			

	Domestic	Imported	
Light truck	0.47		0.54
Car			
	0.80		

The total row/column can be obtained from the joint probabilities by the addition rule

$$P(\text{truck}) = P(\text{domestic} \cap \text{light truck}) + P(\text{domestic} \cup \text{light truck})$$

Exercise D

	Domestic	Imported	
Light truck	0.47	0.07	0.54
Car	0.33	0.13	0.46
	0.80	0.20	

The probability that a vehicle is truck is:

$$P(\text{truck}) = P(\text{domestic}\cap\text{light truck}) + P(\text{domestic}\cup\text{light truck}) = 0.47 + 0.07$$

Does knowing that the chosen vehicle is imported, change the probability that it is a truck?

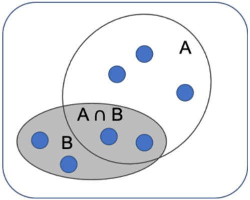
$$P(\text{truck}|\text{imported}) = \frac{P(\text{truck} \cap \text{imported})}{P(\text{imported})} = \frac{0.07}{0.20} = 0.35$$

The probability of an even can change if we known that some other event has occurred.

Conditional Probability

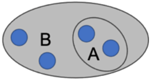
Let A and B be two events, the **conditional probability** of event A, **given that** event B has occurred, is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided that } P(B) > 0$$



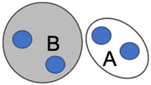
Similarly,

$$P(B|A) = \frac{P(B \cap A)}{P(A)}, \text{ provided that } P(A) > 0$$



Note that:

- If $A \subset B$, then $P(A|B) = 1$.
- If A and B are mutually exclusive, then $P(A|B) = 0$.



Multiplication Rule

From the definition of conditional probability follows the **multiplication rule**. Given two events A and B, the probability of their intersection is:

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

Namely, for both events to occur, the first one must occur, i.e., $P(A)$, and then, given that the first occurred, the second must occur, i.e. $P(B|A)$.

Statistical Independence

Two events A and B are said to be **statistically independent** if and only if $P(A \cap B) = P(A)P(B)$.

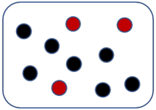
From the multiplication rule, we can also derive $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

That is, the information about the occurrence of B is of no value in determining P(A) (and vice versa).

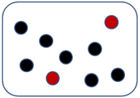
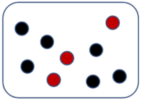
Examples: Jar of Balls

A jar contains 3 red balls and 7 black balls.

- A = draw a red ball. $P(A) = 3/10 = 0.3$.
- B = the second ball drawn is red (replacing the first ball).
  - $P(B|A) = P(B|\bar{A}) = 3/10$.
  - A and B are independent.



- B = the second ball drawn is red (not replacing the first ball).
  - $P(B|A) = 2/9$ and $P(B|\bar{A}) = 3/9$.
  - A and B are not independent.

We have to prove that $P(B|A) = P(B)$ in the first scenario and $P(B|A) \neq P(B)$ in the second one.

Law of Total Probability

Given the events A and B with $0 < P(B) < 1$, then:

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B})$$

- B and $\bar{B}$ are mutually exclusive and collectively exhaustive, so $P(B \cup \bar{B}) = 1$.
- $P(B)$ and $P(\bar{B})$ act as weights in considering the conditional probabilities.

In general, given the events A and $B_1, B_2, \ldots$ with $B_i \cap B_j = \emptyset, i \neq j$ and collectively exhaustive, then:

$$P(A) = \sum_j P(A \cap B_j) = P(B_j)P(A|B_j)$$

Examples: Jar of Balls

A jar contains 3 red balls and 7 black balls.

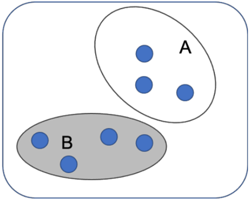
- A = draw a red ball. $P(A) = 3/10 = 0.3$.
- B = the second ball drawn is red (replacing the first ball).
  - $P(B|A) = P(B|\bar{A}) = 3/10$.
  - A and B are independent.
  - $P(B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A}) = 0.3(3/10) + 0.7(3/10) = 3/10$
- B = the second ball drawn is red (not replacing the first ball).
  - $P(B|A) = 2/9$ and $P(B|\bar{A}) = 3/9$.
  - A and B are not independent.
  - $P(B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A}) = 0.3(2/9) + 0.7(3/9) = 3/10$.

Rules

- **Multiplication rule**
  If A and B are independent then $P(A \cap B) = P(A)P(B)$, not otherwise
- **Addition rule**
  If A and B are mutually exclusive then
  $P(A \cup B) = P(A) + P(B)$, not otherwise

Mutually exclusive events $\neq$ independent events.

Independence cannot be depicted in the Venn diagram because it involves the probabilities of the events rather than the outcomes.
$P(A|B) = 0$
A and B are not independent.



Summary of Rules

- **Complement rule** $P(\bar{A}) = 1 - P(A)$
- **Addition rule** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ if the events are mutually exclusive: $P(A \cup B) = P(A) + P(B)$.
- **Conditional probability** $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and $P(B|A) = \frac{P(B \cap A)}{P(A)}$
- **Multiplication rule** $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$ if the events are independent: $P(A|B) = P(A)$ and $P(A \cap B) = P(B)P(A) = P(A)P(B)$.
- **Law of total probability** $P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B})$