Continuous Random Variables

Introductory course on Statistics and Probability

Nicholas A. Pearson

Università degli Studi di Trieste

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Random Variables

A **random variable** (r.v.) X is a variable whose value is a numerical outcome of a random phenomenon, that is, it is a well defined but unknown number.

There are two main types of random variables: **discrete** (if it has a finite list of possible outcomes), and **continuous** (if it can take any value in an interval).

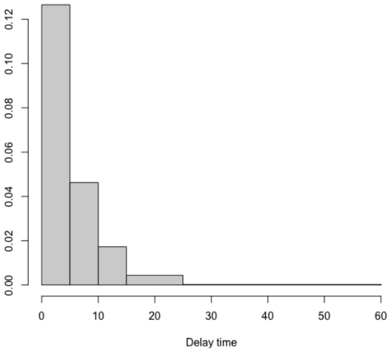
- C the delay time of the airplane
- C the weight of a newborn
- C the duration of a phone call with your mother

For **continuous random variables** we can assign probabilities only to a range of values, using a mathematical function, we express the probability distribution in the continuous space.

Delay time of a flight

About the delay time of a flight, we might observe the following data and plot them using a histogram.

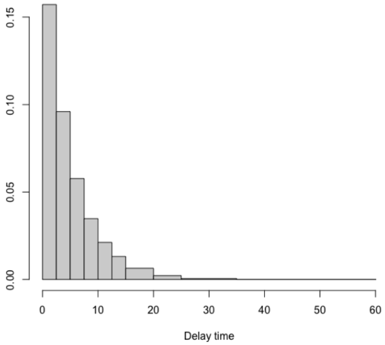
Delay	P
(0,5]	0.633
(5,10]	0.231
(10,15]	0.086
(15,25]	0.043
(25,Inf]	0.006



Delay time of a flight

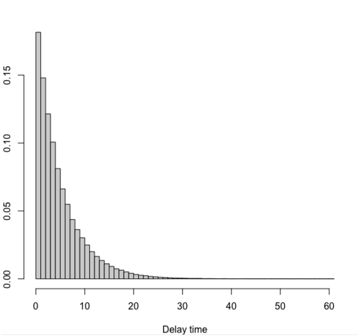
If we increase the number of intervals and make each interval "smaller":

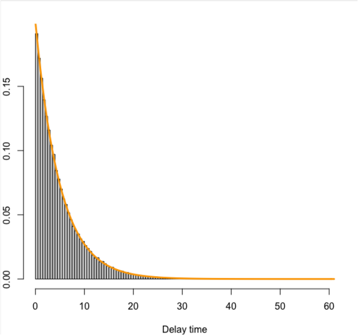
Delay	P
(0,2.5]	0.393
(2.5,5]	0.240
(5,7.5]	0.144
(7.5,10]	0.087
(10,12.5]	0.053
(12.5,15]	0.033
(15,20]	0.032
(20,25]	0.011
(25,35]	0.005
(35,Inf]	0.001



Delay time of a flight

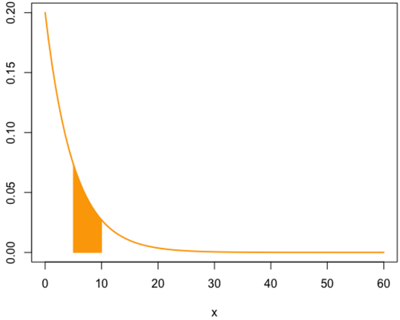
Assuming that we have infinite observations we can increase the number of classes and approximate the histogram with a curve: the **probability density function**.





Probability density function

The interpretation of the probability density function is analogous to the histogram: the areas represent probabilities rather than frequencies.



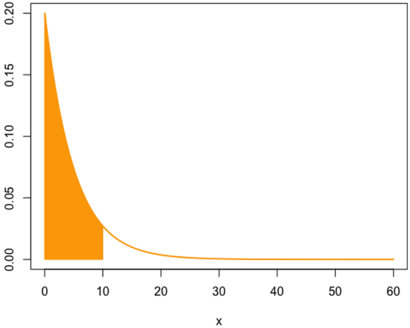
E.g. the probability of observing a value between 5 and 10 is:

$$P(5 \leq X \leq 10) = 0.234$$

Cumulative distribution function

A relevant quantity is:

$$P(X \leq x)$$

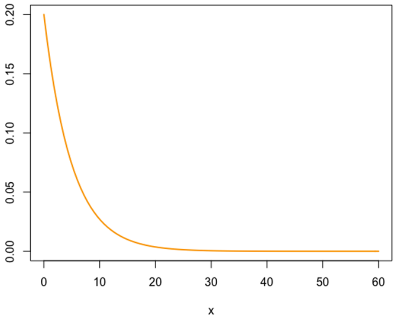


Example: the probability of observing a value smaller than 10 is:

$$P(X \leq 10) = 0.865$$

Probability density function

The probability density function is a real-valued function assuming non-negative values.



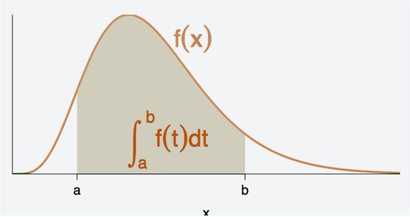
The function in the plot is:

$$f(x) = \frac{1}{5} e^{-x/5}, \qquad x \geq 0$$

Probability density function

The **probability density function** of a continuous random variable X is a non-negative function $f(x)$. The probability that X lies within a given interval is obtained as the area under the curve of $f(x)$ over that interval.

$$\int_a^b f(t)dt = P(a \leq X \leq b)$$



Probability density function

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$$\int_a^b f(t)dt = P(a \leq X \leq b)$$

The probability density function satisfies the following properties:

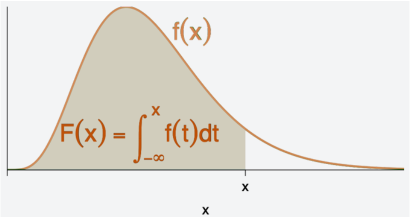
(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(t)dt = 1$

Cumulative distribution function

The **cumulative distribution function** of a random variable X is the function:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$



Cumulative distribution function

The **cumulative distribution function** of a random variable X is the function:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

and satisfies the following properties:

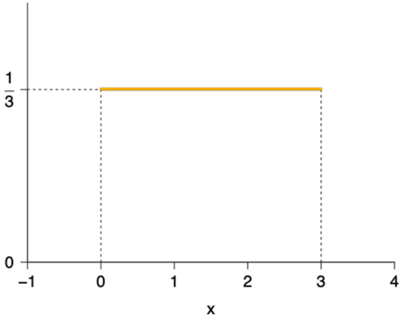
i. $F(x) \geq 0, \forall x \in \mathbb{R}$;
ii. $F(x)$ is not decreasing;
iii. $\lim_{x \to -\infty} F(x) = 0$;
iv. $\lim_{x \to +\infty} F(x) = 1$.

and note that:

$$P(a \leq X \leq b) = F(b) - F(a)$$

Example

There are several density functions, any non-negative function that integrates to 1 is a density function.



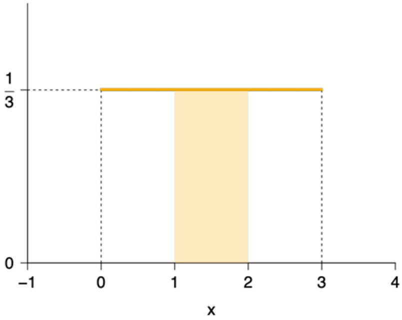
Uniform between 0 and 3

$$f(x) = \begin{cases} 1/3 & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(1 \leq X \leq 2)$

Example

There are several density functions, any non-negative function that integrates to 1 is a density function.



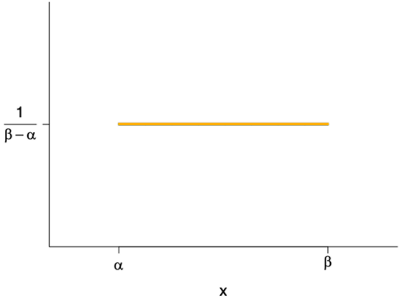
Uniform between 0 and 3

$$f(x) = \begin{cases} 1/3 & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P(1 \leq X \leq 2) = (2-1) \times \frac{1}{3} = \frac{1}{3}$$

Probability density functions and parameters

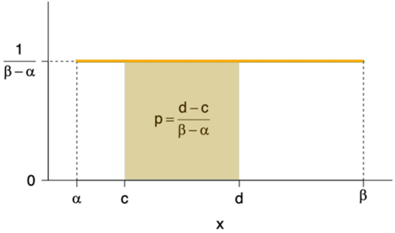
Often it is useful to define a probability density function up to one or more parameters, which is the equivalent of defining a set of density functions.



$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

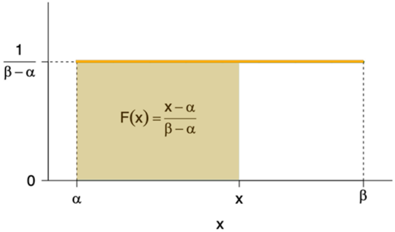
$f(x)$ is a probability density function for every α and β, with $\beta > \alpha$.

Uniform distribution on $[\alpha, \beta]$



$$P(c \leq X \leq d) = \frac{d-c}{\beta-\alpha}$$

with $\alpha < c < d < \beta$



$$F(x) = \begin{cases} 0 & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

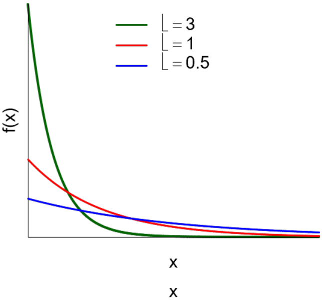
Exponential distribution

The **exponential distribution** is a continuous probability distribution that describes the time between events in a Poisson process, where events occur continuously and independently at a constant average rate.

It has rate parameter λ and is defined as:

$$f(x) = \lambda e^{-\lambda x}$$

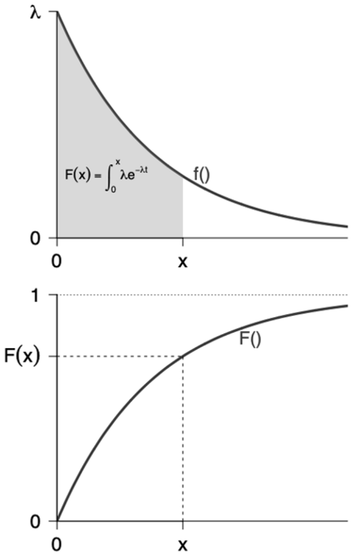
for $x \geq 0$ and $\lambda > 0$.



Exponential distribution

The cumulative distribution function is:

$$F(x) = \int_0^x \lambda e^{-\lambda t} dt$$
$$= [-e^{-\lambda t}]_0^x$$
$$= 1 - e^{-\lambda x}$$



Expected value and variance

Let $f(x)$ be the probability density function of a continuous random variable X.

$$E(X) = \int xf(x)dx$$

$$E(h(X)) = \int h(x)f(x)dx$$

$$V(X) = \int (x - E(X))^2 f(x)dx$$

Moreover, the following properties hold (using the properties of the integrals)

- $E(aX + b) = aE(X) + b$
- $V(X) = E(X^2) - [E(X)]^2$
- $V(aX + b) = a^2 V(X)$

Uniform dist.: expected value and variance

Let X be a random variable following the Uniform distribution between α and β, thus

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

then

$$E(X) = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha} dx = \frac{1}{\beta-\alpha} \left[\frac{x^2}{2}\right]_{\alpha}^{\beta} = \frac{1}{\beta-\alpha} \frac{\beta^2-\alpha^2}{2} = \frac{\alpha+\beta}{2}$$

$$E(X^2) = \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha} dx = \frac{1}{\beta-\alpha} \left[\frac{x^3}{3}\right]_{\alpha}^{\beta} = \frac{1}{\beta-\alpha} \frac{\beta^3-\alpha^3}{3} = \frac{\beta^2+\alpha\beta+\alpha}{3}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{(\beta-\alpha)^2}{12}$$

Exponential distr: expected value and variance

Let X be a random variable following the Exponential distribution with rate parameter $\lambda > 0$, thus

$$f(x) = \lambda e^{-\lambda x}$$

for $x \geq 0$, then, by integration by parts,

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x} dx = [-xe^{-\lambda x}]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = [-\frac{e^{-\lambda x}}{\lambda}]_0^{\infty} = \frac{1}{\lambda}$$

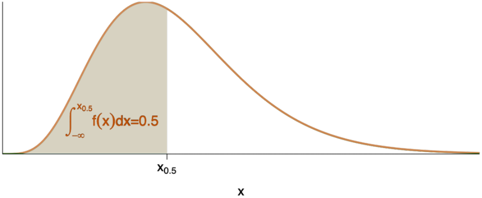
$$E(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x} dx = [-x^2 e^{-\lambda x}]_0^{\infty} + \int_0^{\infty} 2xe^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Median

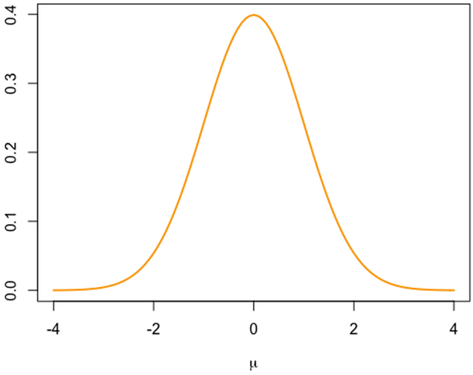
We define the **median** of a continuous random variable X with pdf $f(x)$ the value $Me(X)$ or $x_{0.5}$ such that:

$$F(Me(X)) = \int_{-\infty}^{Me(X)} f(x)dx = 0.5$$



Normal distribution

The most popular continuous distribution is the Normal (or Gaussian) distribution and its density has the following shape.



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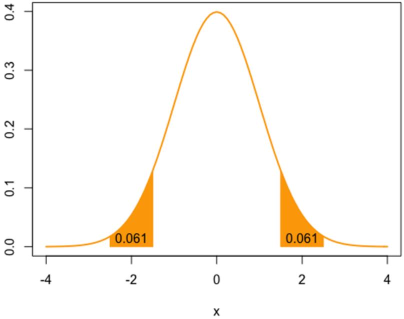
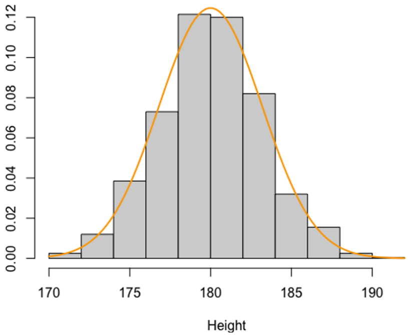


- The most likely values to occur are the ones around the center (the mean)
- It is symmetric, then symmetric deviations on the right and left have the same probability

Example

The distribution of the male students' height.



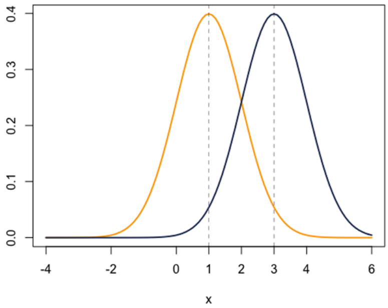
Normal distribution: parameters

The probability density function of the **normal distribution** is:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Thus if a r.v. X follows a normal distribution we write:

$$X \sim N(\mu, \sigma^2)$$



- μ is the mean of the distribution
- For different μ values, the distribution shifts on the x axis

Normal distribution: parameters

The probability density function of the **normal distribution** is:

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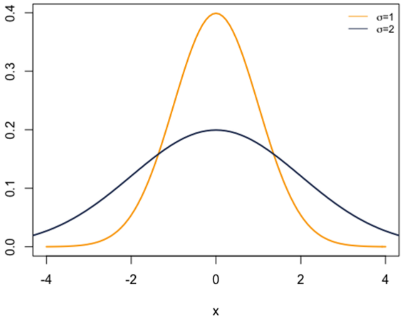

- σ^2 is the variance of the distribution
- For different σ values, the distribution remains centered on the same value but becomes wider: larger values (in absolute values) of the r.v. are more likely

Normal distribution: parameters

The probability density function of the **normal distribution** is:

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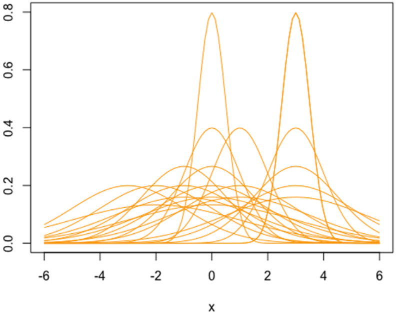


Varying μ and σ, we can obtain an infinite number of distributions.

Normal distribution: probabilities

Assume that $X \sim N(\mu, \sigma^2)$, then its pdf is:

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

we can use it to compute probabilities of a Normal rv

$$p = P(a \leq X \leq b)$$

where

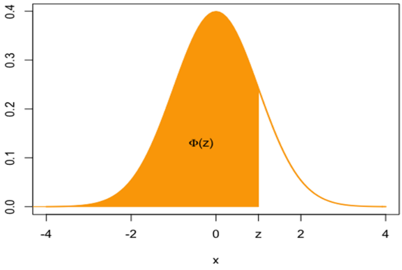
$$p = \int_a^b f(t)dt = F(b) - F(a)$$

Unfortunately, *we do not have a closed form to compute this probability*, but we can refer to a particular normal distribution...

Standard Normal distribution

The standard normal distribution, that is $\mu = 0$ and $\sigma = 1$, plays an important role.

$$Z \sim N(0,1) \rightarrow f(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



We define:

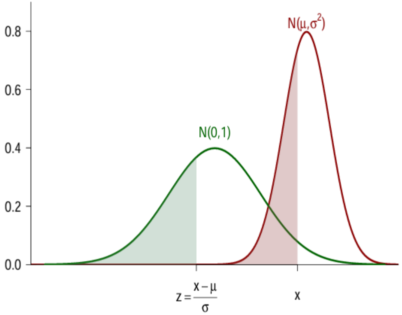
$$\Phi(z) = P(Z \leq z)$$

the area under the curve between $-\infty$ and z. Note that with Φ we call the cumulative distribution function of the standard normal distribution

Standard Normal vs Normal(μ, σ)

If $X \sim N(\mu, \sigma^2)$, then

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$



That is, the red area (equal to $P(X \leq x)$) is equal to the green one. Knowing $\Phi(z)$, we can compute any probability associated with a generic normal distribution $N(\mu, \sigma^2)$.

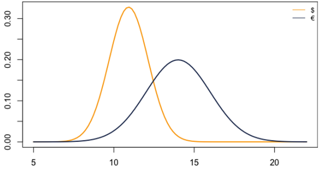
Normal distribution: transformations

Let X be distributed as a normal distribution $N(\mu, \sigma^2)$, and let a, and b be two real numbers, then:

$$Y = aX + b$$

follows a normal distribution:

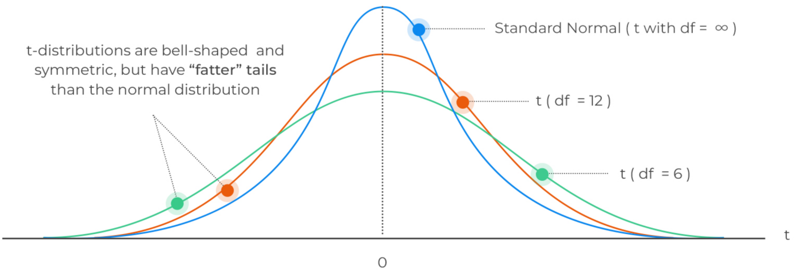
$$Y \sim N(a\mu + b, a^2\sigma^2)$$



As an example, if the stock price on a given day expressed in euros follows a Normal distribution $N(14, \sigma^2 = 2)$, then the value in dollars ($1\$ = 0.78$ euros) is still distributed as a Normal with mean 0.78×14 and variance $2 \times (0.78)^2$.

Normal vs Student's t

Another relevant continuous distribution is the **Student't** distribution t_ν, governed by the parameter ν (the degrees of freedom).



It is closely related to the Standard Normal (when $\nu \to \infty, \ t_\nu \overset{approx}{\sim} \mathcal{N}(0,1)$ but, in general, it has **heavier tails**. It is widely used in statistical inference.