ON THE HALPERN METHOD WITH ADAPTIVE ANCHORING PARAMETERS

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Abstract. We establish the convergence of a speed-up version of the Halpern iteration with adaptive anchoring parameters in the general geodesic setting of Hadamard spaces, generalizing a recent result by He, Xu, Dong and Mei from a linear to a nonlinear setting. In particular, our results extend the fast rates of asymptotic regularity obtained by these authors for the first time to a nonlinear setting. Our approach relies on a quantitative study of these previous results in the linear setting, combined with certain optimizations and an elimination of the weak compactness arguments employed crucially in the linear setting, which not only allows for the lift of the result to a nonlinear setting but also streamlines the previous convergence analysis considerably. This work is set in the context of recent developments in proof mining, and as a byproduct of our approach, we further obtain quantitative information in the form of highly uniform rates of metastability of low complexity, which are new already in the context of Hilbert spaces.

Keywords: Halpern iteration; adaptive anchoring parameters; Hadamard spaces; rates of asymptotic regularity; rates of metastability
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1. Introduction

Let H be a Hilbert space and $T : H \to H$ be a nonexpansive map, i.e.

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in H$. We write $\mathrm{Fix}(T) := \{x \in H \mid Tx = x\}$ for the set of fixed points of T.

It is a fundamental problem of nonlinear analysis to study methods for finding fixed points of nonexpansive maps, not at least since a very wide variety of computational problems can be captured by associated fixed point equations of nonexpansive operators. This task is made highly nontrivial by the fact that nonexpansive maps may fail to have fixed points at all, and even if they do, usual methods such as the Picard iteration or the Krasnoselskii-Mann iteration may even fail to converge or only converge weakly in the infinite-dimensional setting.

The so-called Halpern iteration alleviates the problem of weak convergence by utilizing a so-called anchor point $u \in H$ together with a sequence of parameters $(\lambda_n) \subseteq [0, 1]$ to construct the sequence

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n$$

from a given arbitrary starting point $x_0 \in X$. This iteration, originally studied by Halpern [22] in the special case of $u = 0$, can be shown to strongly converge to the projection of the anchor

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u onto the set of fixed points $\mathrm{Fix}(T)$ under suitable conditions on the anchoring parameters. Besides the two conditions

$$\lim_{k\to\infty} \lambda_k = 0 \text{ and } \sum_{k=0}^{\infty} \lambda_k = \infty,$$

already isolated as necessary conditions by Halpern [22], work in the years following the introduction of this method has focused on various, successively improving, conditions on these parameters sufficient for guaranteeing strong convergence, notably as in the works of Halpern [22], Lions [38], Wittmann [54], Reich [44] and Xu [56], to only name a few. Beyond these technicalities, Halpern's iteration is also very important from a conceptual standpoint, as in the case where T is a linear operator and $\lambda_n = 1/(n+1)$ (a choice which was for the first time possible through the conditions introduced in Wittmann's work [54]), the Halpern iteration reduces to the usual ergodic averages, and the convergence result consequently presents itself as a nonlinear variant of the von Neumann mean ergodic theorem.

Quantitatively, Halpern's iteration is known to quickly produce approximate fixed points for suitable anchoring parameters as expressed through the following rate of asymptotic regularity

$$\|x_n - Tx_n\| \leq \frac{2}{n+1}\|x_0 - p\|$$

in the case $\lambda_n = 1/(n+2)$ (and assuming $u = x_0$) where $p \in \mathrm{Fix}(T)$ is an arbitrary fixed point of T, as recently established by Lieder [36] as well as Sabach and Shtern [46]. Moreover, this rate is known to be optimal in general.

To further speed up the asymptotic regularity of Halpern's iteration, in the recent work [23], He, Xu, Dong and Mei proposed an iteration where each anchoring parameter λ_n is chosen adaptively (in a rather novel way compared to previous adaptive parameter selections, see the discussion in [23]) along the iteration to increase the speed of asymptotic regularity in concrete applied circumstances and still guarantee the strong convergence of Halpern's iteration. This adaptive iteration concretely takes the following form: given a point $x_0 \in H$, define a sequence (x_n) recursively via the clauses

$$x_{n+1} := \begin{cases} x_n & \text{if } Tx_n = x_n, \\ \left(\frac{1}{\varphi_n+1}\right) x_0 + \left(\frac{\varphi_n}{\varphi_n+1}\right) Tx_n & \text{if } Tx_n \neq x_n, \end{cases}$$

where

$$\varphi_n := \frac{2\langle x_n - Tx_n, x_0 - x_n\rangle}{\|x_n - Tx_n\|^2} + 1$$

in the case where $Tx_n \neq x_n$. As shown in [23], this method in particular allows for the following estimate on the asymptotic regularity

$$\|x_n - Tx_n\| \leq \frac{2}{\varphi_{n-1}+1}\|x_0 - p\| \leq \frac{2}{n+1}\|x_0 - p\|$$

for all $n \geq 1$, where $p \in \mathrm{Fix}(T)$ is again an arbitrary fixed point of T. The latter inequality follows from the fact that $\varphi_n \geq n+1$ and so one obtains an analogous result on the convergence speed of Halpern's iteration as given in [36]. Beyond matching the speed of the optimal convergence result for Halpern's iteration due to [36], as illustrated in [23], certain examples of application can be constructed where the above asymptotic regularity result yields substantially better theoretical estimates on the asymptotic regularity than the previous ones. Even further, experimental results presented in [23] illustrate that the use of adaptive anchoring parameters leads to a considerable speedup in very general practical circumstances.

It is now the aim of this paper to study the above adaptive Halpern iteration in the nonlinear context of Hadamard spaces (i.e. complete CAT(0) spaces; see the next section for precise definitions), one of the most important classes of nonlinear hyperbolic spaces and which present a suitable nonlinear analogue to that of Hilbert spaces. Indeed, the benefit of this general geometric framework of Hadamard spaces is that it encompasses important classes of spaces beyond Hilbert spaces, such as $\mathbb{R}$-trees and complete simply connected Riemannian manifolds of nonpositive sectional curvature, among many others. These spaces naturally occur in many advanced optimization and fixed point scenarios, but it is often difficult to efficiently extend many of the methods usually employed in the linear setting. The benefit of Halpern-type iterations is in that way twofold. For one, based on their geodesic structure, Halpern-type iterations naturally lend themselves to generalizations into this hyperbolic context of Hadamard spaces, as first observed in [47]. Second, similarly already highlighted in [47], also the benefit of strong convergence of Halpern-type iterations carries over to the nonlinear setting. Concretely, in the context of Hadamard spaces, there is a natural extension of the notion of weak convergence from linear spaces, which is commonly called Δ-convergence. This notion goes back to the work of Kirk and Panyanak [27], who provided an extension of Lim's notion of Δ-convergence [37] to CAT(0) spaces. In fact, as shown by Espínola and Fernández-León [17] (see also the discussion in [2]), an equivalent version of that notion was already studied by Jost [24], developed there under the name of weak convergence. Like in the linear context, one can actually in general only guarantee Δ-convergence for many, if not most, of the commonly studied fixed point iterations and optimization methods in the context of Hadamard spaces. However, for Halpern-type iterations, one retains strong convergence in the way that the iteration converges in the usual sense w.r.t. the metric in that setting, a fact which is also practically of high importance in these nonlinear contexts as this type of convergence is often the only one with a clear geometric intuition.

We now turn to a more precise description of the contributions of the present paper. Our first contribution is a lift of the above adaptive Halpern iteration to these nonlinear spaces, which we achieve by considering the following analogous iteration in a Hadamard space X: given an arbitrary starting point $x_0 \in X$ simultaneously serving as an anchor, define a sequence (x_n) via

$$x_{n+1} := \begin{cases} x_n & \text{if } Tx_n = x_n, \\ \left(\frac{1}{\varphi_n+1}\right) x_0 \oplus \left(\frac{\varphi_n}{\varphi_n+1}\right) Tx_n & \text{if } Tx_n \neq x_n, \end{cases} \tag{$\mathrm{H_{adp}}$}$$

where $T : X \to X$ is a nonexpansive map, i.e.

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$ similar to before, and

$$\varphi_n := \frac{2\left\langle \overrightarrow{Tx_n x_n}, \overrightarrow{x_n x_0} \right\rangle}{d^2(x_n, Tx_n)} + 1$$

for $Tx_n \neq x_n$. Here, we write $\lambda x \oplus (1-\lambda)y$ for the convex combination of two points $x, y \in X$, an operation which is naturally defined through the use of geodesics in Hadamard spaces, and given $x, y, z, w \in X$, $\langle \overrightarrow{xy}, \overrightarrow{zw} \rangle$ denotes the so-called quasi-linearization function of the space X, a natural nonlinear generalization of the inner product in Hilbert spaces which operates on two pairs of points, denoted by $\overrightarrow{xy}$ and $\overrightarrow{zw}$ (again, we refer to the next section for a precise introduction to these notions).

In particular, as a second contribution, we show that this iteration satisfies an analogous linear asymptotic regularity result in this metric context by establishing the inequality

$$d(x_n, Tx_n) \leq \frac{8}{\varphi_{n-1}+1} d(x_0, p) \leq \frac{8}{n+1} d(x_0, p)$$

for all $n \geq 1$, where $p \in \mathrm{Fix}(T) := \{x \in X \mid Tx = x\}$ is a fixed point of T. Again, the latter inequality follows from the fact that $\varphi_n \geq n+1$. The proof of this result, although similar in nature to the respective result in Hilbert spaces, is rather nontrivial to establish in this general setting of Hadamard spaces and relies on a modified argument compared to that given in [23], which crucially relies on the linearity of the underlying space. Quantitatively, this worsens the inequality given in [23] for Hilbert spaces by a factor of 4, where an argument based on the linear structure of the Hilbert space (akin to [36]) can be given to remove that factor and obtain the previous rates. However, the constant improves the best known for Halpern's iteration in the context of Hadamard spaces recently obtained in [8] (see also [10]). It remains an open question whether the associated inequality for Halpern's iteration in Hadamard spaces given in [8] is optimal and analogously we do not know if the above inequality is optimal for the adapted Halpern iteration in Hadamard spaces.

Beyond this, as a third contribution, we in particular also show the metric convergence of the above iteration in Hadamard spaces towards a fixed point of T. In particular, it should be emphasized that, similar to the ordinary Halpern iteration, the convergence is in terms of the metric and hence is "strong" and does not rely on any notion of weak convergence in Hadamard spaces, like for example Δ-convergence discussed before. As every Hilbert space is a Hadamard space, this result for $(\mathrm{H}_{\mathrm{adp}})$ in particular also contains the strong convergence result from [23] as a special case but, as our results here show, it is certainly not limited to its original linear context. This also in particular answers an open problem phrased in [10] which asked for extensions of the method given in [23] to more general, nonlinear settings and to provide similar asymptotic regularity results as in [23].

The approach we took toward obtaining these results relies on the logical methodology of proof mining, a program in mathematical logic which aims at the extraction of quantitative and effective information from prima facie non-effective proofs (we refer to the seminal monograph [29] for a comprehensive overview of both theoretical as well as applied aspects of this program up to 2008 and to the survey [32] for comments on more recent applications to nonlinear analysis). In particular, proof mining has previously been highly successful in providing quantitative versions of the convergence result for Halpern's iteration and its many variants as considered in the literature, with notable instances ranging from the first analysis of Wittmann's proof of the strong convergence of Halpern's iteration given in the seminal paper of Kohlenbach [31] and the first rates of asymptotic regularity for Halpern's iteration given by Leuştean [34] and the subsequent extension of these results to Hadamard spaces in [33] by an analysis of a corresponding convergence proof by Saejung [47] to analyses of involved extensions of the Halpern iteration like in [48] for the modified Mann iteration introduced in [26] (and extended to nonlinear spaces in [11]) as well as Tikhonov-Mann-type methods and their extensions as introduced in [57] and [5] and analyzed in [8, 9, 10, 14] (with [8] notable as therein, linear rates of asymptotic regularity were obtained for the first time in the context of applications of proof mining) or the recently introduced Halpern-Mann method [15, 35].

Now, we concretely arrived at the results presented here by first deriving a finitary quantitative variant of the above result from [23] set in linear spaces, which by virtue of this analysis was accompanied with an elementary proof which was stripped of any infinitary arguments and in particular made no use of sequential weak compactness.[1] This elementary proof then allowed for a more straightforward generalization to the nonlinear setting of Hadamard spaces, yielding a corresponding finitary quantitative convergence result therein which then can be brought in the form of a usual convergence result by "forgetting" about these finitary quantitative aspects.[2] However, we want to emphasize that while this logical perspective was crucial in obtaining the present results, the paper does not rely on any notions from logic at all.

Therefore, as a sort of byproduct of our approach, our fourth contribution is that we also obtain a quantitative version of our convergence result in the form of a rate of metastability for the sequence (x_n), i.e. in the form of a function $\rho(k, f)$ that bounds the quantifier $\exists n \in \mathbb{N}$ in the expression

$$\forall k \in \mathbb{N}\, \forall f : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right)$$

in terms of k and f, where we write $[a; b] = [a, b] \cap \mathbb{N}$. This so-called metastability property is, albeit noneffectively, equivalent to the usual Cauchy property for (x_n) and presents a highly fruitful phrasing of that property in the context of quantitative considerations on convergence, a fact that is not only put forward by proof mining (we refer to the discussion in [29]) but was also crucially highlighted in the work of Tao on finitary analysis [52, 53] (where the term metastability was actually coined). In particular, rates of metastability are in general the best one can hope for if one aims at computable information on convergence statements as computable rates of convergence are in general ruled out for wide classes of iterations (including Halpern's iteration), as can be shown by adapting results from recursion theory [50]. Further, the rate of metastability that we give is highly uniform, depending only on a few number-theoretic data bounding crucial parameters of the iteration, and they in particular do not depend on the space or the points of the iteration. Importantly, since our convergence result in particular contains the one presented in [23] as a special case, also our quantitative results apply to the context of [23] where they are also already novel.

Our fifth and final contribution are numerical experiments that complement the experiments performed in [23]. Concretely, we consider a special Hadamard space based on $\mathbb{R}^2$ in which a certain slightly shifted variant of the Rosenbrock function is convex. In that context, the ordinary Halpern iteration and, respectively, the adapted Halpern iteration can be used to approximate a minimizer of said function through the use of its proximal map, which is a special nonexpansive map in that context. These experiments confirm the empirical results from [23] and show that, as in Hilbert spaces, the adapted Halpern iteration strongly outperforms the ordinary Halpern iteration (with anchoring parameters chosen in the usual open loop manner) also in this general setting of Hadamard spaces.

[1] This crucial absence of the use of weak compactness in proofs originating from analyses as provided by proof mining was already a main feature of the first application of proof mining to Halpern's iteration given in [31] and can be in particular explained as an a priori feature of such proofs by underlying logical methods, as highlighted in [18].

[2] This type of strategy for generalizing results in nonlinear analysis by generalizing quantitative finitary variants as provided by proof mining methods was frequently used in some recent works, and we in particular refer to the works [40, 42, 43] which were explicitly obtained in that manner and we refer to [41] for further discussions on this type of strategy.

The rest of the paper is now organized as follows: In Section 2, we give the necessary background on the theory of Hadamard spaces together with some basic lemmas needed for the main results, later presented in Section 3. Section 4 presents our numerical experiments.

2. Preliminaries and basic lemmas

A triple (X, d, W) is called a hyperbolic space [28] if (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a function satisfying for all $x, y, z, w \in X$ and $\lambda, \lambda' \in [0,1]$:

(W1) $d(W(x, y, \lambda), z) \leq (1-\lambda)d(x, z) + \lambda d(y, z)$,
(W2) $d(W(x, y, \lambda), W(x, y, \lambda')) = |\lambda - \lambda'|d(x, y)$,
(W3) $W(x, y, \lambda) = W(y, x, 1-\lambda)$,
(W4) $d(W(x, y, \lambda), W(z, w, \lambda)) \leq (1-\lambda)d(x, z) + \lambda d(y, w)$.

Hyperbolic spaces allow for discussions in generalized settings where the central arguments rely on the notion of convex combinations, and the reader may easily convince themselves that the function W has all the natural properties one would expect from a convex combination. Several similar notions exist in the literature. The convexity function W was first considered by Takahashi in [51] where a triple (X, d, W) satisfying (W1) is called a convex metric space.

The notion used here was introduced by Kohlenbach in [28] motivated by proof-theoretical considerations, and it is frequently considered the nonlinear generalization of convexity in normed spaces. We note that this notion is more general than that of hyperbolic spaces in the sense of Reich and Shafrir [45], and slightly more restrictive than the setting due to Goebel and Kirk [19] of spaces of hyperbolic type. In particular, we note that the class of hyperbolic spaces includes the normed spaces and their convex subsets (with W being the usual linear convex combination), and the Hilbert ball [20], among many more examples (we refer in particular to the seminal monograph [6]). We refer to [28] for further motivating considerations on these spaces.

To ease the notation, we write the more intuitive expression $\lambda x \oplus (1-\lambda)y$ for the point $W(x, y, 1-\lambda)$. One easily sees that

$$d(x, \lambda x \oplus (1-\lambda)y) = (1-\lambda)d(x, y) \quad \text{and} \quad d(y, \lambda x \oplus (1-\lambda)y) = \lambda d(x, y).$$

A subset $C \subseteq X$ is said to be convex if for all $\lambda \in [0,1]$, $\lambda x \oplus (1-\lambda)y \in C$, whenever $x, y \in C$. Further, over a general metric space X, we write

$$\overline{B}_r(x) := \{y \in X \mid d(x, y) \leq r\}$$

for the closed ball around a point $x \in X$ with radius $r > 0$.

An important subclass of hyperbolic spaces is that of CAT(0) spaces. These spaces, introduced by Aleksandrov [1] and named as such by Gromov [21], are characterized as the hyperbolic spaces that satisfy the CN^- property (which, in the presence of (W1)–(W4), is equivalent to the Bruhat-Tits CN-inequality [7]): for all $x, y, z \in X$ and $\lambda \in [0,1]$,

$$d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y). \tag{CN^-}$$

This relation extends beyond the midpoint (see e.g. [13, Lemma 2.5]) and we have

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y). \tag{CN^+}$$

It follows from the work of Berg and Nikolaev [4] that CAT(0) spaces can be equivalently characterized as the hyperbolic spaces that satisfy the inequality

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leq d(x,y)d(u,v) \tag{CS}$$

for all $x, y, u, v \in X$, where the expression on the left-hand side is the so-called quasi-linearization function, defined for all $x, y, u, v \in X$ by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right)$$

using $\overrightarrow{xy}$ to denote pairs (x,y) as before. In any metric space, this function is the unique function $X^2 \times X^2 \to \mathbb{R}$ satisfying the following properties for all $x, y, u, v \in X$ (see [4, Proposition 14]):

(1) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x,y)$,
(2) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$,
(3) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$,
(4) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle + \langle \overrightarrow{xy}, \overrightarrow{vw} \rangle = \langle \overrightarrow{xy}, \overrightarrow{uw} \rangle$.

Therefore, this function enjoys properties similar to those of an inner product, and the condition (CS) can be regarded as a metric version of the Cauchy-Schwarz inequality. Indeed, CAT(0) spaces are often considered the canonical nonlinear counterpart of inner product spaces (in which case $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle x-y, u-v \rangle$). A complete CAT(0) space is called a Hadamard space and is the canonical nonlinear generalization of a Hilbert space (we refer to the seminal monograph [6] for a comprehensive overview of CAT(0) and Hadamard spaces and also refer to [3] for a shorter treatment focused on aspects of convex analysis and optimization).

We have the following essential results regarding the quasi-linearization function.

Lemma 2.1. *For any metric space X and $x, y, z \in X$:*

$$d^2(x,y) = d^2(x,z) + d^2(z,y) + 2\langle \overrightarrow{xz}, \overrightarrow{zy} \rangle.$$

Proof. Immediate from the definition of the quasi-linearization function. □

Lemma 2.2. *Let X be a* CAT(0) *space, $x, y, z \in X$ and $\lambda \in [0,1]$. Then*

$$\lambda \langle \overrightarrow{zy}, \overrightarrow{xy} \rangle \leq \langle \overrightarrow{zy}, \overrightarrow{wy} \rangle,$$

where $w = \lambda x \oplus (1-\lambda)y$.

Proof. Using (CN$^+$), we have the following:

$$\begin{aligned}
2\lambda \langle \overrightarrow{zy}, \overrightarrow{xy} \rangle - 2\langle \overrightarrow{zy}, \overrightarrow{wy} \rangle &= \lambda\left(d^2(z,y) + d^2(x,y) - d^2(z,x)\right) \\
&\quad - \left(d^2(z,y) + d^2(w,y) - d^2(z,w)\right) \\
&= \lambda\left(d^2(z,y) + d^2(x,y) - d^2(z,x)\right) \\
&\quad - d^2(z,y) - \lambda^2 d^2(x,y) + d^2(z,w) \\
&\leq \lambda d^2(z,y) + \lambda d^2(x,y) - \lambda d^2(z,x) \\
&\quad - d^2(z,y) - \lambda^2 d^2(x,y) + \lambda d^2(x,z) \\
&\quad + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y) \\
&= d^2(y,z)\left(\lambda - 1 + (1-\lambda)\right) \\
&\quad + \lambda d^2(x,y)\left(1 - \lambda - (1-\lambda)\right) \\
&\quad + \lambda d^2(x,z) - \lambda d^2(z,x) \\
&= 0.
\end{aligned}$$

□

Lemma 2.3. *Let X be a* CAT(0) *space, let $x, y, z \in X$ and let $\lambda \in [0,1]$. Then*
$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda^2 d^2(x,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz}\rangle + (1-\lambda)^2 d^2(y,z).$$

Proof. Using (CN$^+$) and the definition of the quasi-linearization function, we have
$$\begin{aligned} d^2(\lambda x \oplus (1-\lambda)y, z) &\leq \lambda d^2(x,z) - \lambda(1-\lambda)d^2(x,y) + (1-\lambda)d^2(y,z) \\ &= \lambda^2 d^2(x,z) + \lambda(1-\lambda)(d^2(x,z) + d^2(y,z) - d^2(x,y)) + (1-\lambda)^2 d^2(y,z) \\ &= \lambda^2 d^2(x,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz}\rangle + (1-\lambda)^2 d^2(y,z). \end{aligned}$$
□

The next lemma will be useful in the sequel.

Lemma 2.4. *Let X be a metric space. For any $r > 0$ and points $x, y, z \in X$ such that $d(x,y), d(x,z) \leq r$, we have $d^2(x,y) \leq d^2(x,z) + 2r \cdot d(y,z)$.*

Proof. By triangle inequality, $d(x,y) - d(x,z) \leq d(y,z)$, and so
$$d^2(x,y) - d^2(x,z) = (d(x,y) - d(x,z))(d(x,y) + d(x,z)) \leq 2rd(y,z),$$
entailing the result. □

We recall a result regarding sequences of real numbers due to Xu [55], which is frequently used in the study of Halpern-type iterations.

Lemma 2.5. *Consider sequences of nonnegative real numbers (a_n), (b_n), $(c_n) \subseteq [0,\infty)$, and $(\lambda_n) \subseteq (0,1)$ such that $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n b_n + c_n$, for all $n \in \mathbb{N}$. If*
$$(i)\ \sum \lambda_n = \infty, \quad (ii)\ \limsup b_n \leq 0, \quad (iii)\ \sum c_n < \infty,$$
then $\lim a_n = 0$.

The first quantitative formulations of the previous lemma featured in [31, 34], and here we will require a variant of [33, Lemmas 5.2 and 5.3] (see also [39, Lemmas 14 and 16]). As we use a slight modification of said results, we include the proof for completeness.

Lemma 2.6. *Let $(a_n) \subseteq [0,\infty)$ be a bounded sequence and $D \in \mathbb{N} \setminus \{0\}$ be an upper bound on (a_n). Consider sequences $(\lambda_n) \subseteq [0,1]$ and $(b_n) \subseteq \mathbb{R}$. For any $k, A, B \in \mathbb{N}$ and any $\Gamma : \mathbb{N} \to \mathbb{N}$, if*
$$\forall n \in [A;B] \left(a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n b_n + \frac{1}{3(B+1)(k+1)} \text{ and } b_n \leq \frac{1}{3(k+1)} \right),$$
and $\sum_{k=0}^{\Gamma(L)} \lambda_k > L$ for all $L \leq A + \lceil \ln(3D(k+1)) \rceil$, then
$$\forall n \in [\Theta;B] \left(a_n \leq \frac{1}{k+1} \right),$$
where $\Theta := \Gamma(A + \lceil \ln(3D(k+1)) \rceil) + 1$.

Proof. Assume that $\Theta \leq B$, otherwise the conclusion is trivially true. Inductively, we have
$$a_{A+m+1} \leq \left(\prod_{n=A}^{A+m} (1-\lambda_n) \right) a_A + \left(1 - \prod_{n=A}^{A+m} (1-\lambda_n) \right) \frac{1}{3(k+1)} + \frac{m+1}{3(B+1)(k+1)}$$
for all $m \leq B - A$. Hence we get
$$a_{A+m+1} \leq D \cdot \left(\prod_{n=A}^{A+m} (1-\lambda_n) \right) + \frac{2}{3(k+1)} \tag{+}$$

for $m \leq B - A$. Take $M := \Gamma\left(A + \lceil \ln\left(3D(k+1)\right)\rceil\right) - A$. Note that $M \in \mathbb{N}$, otherwise since $(\lambda_n) \subseteq [0,1]$ and $\lceil \ln(3D(k+1))\rceil \geq 2$, we would get the contradiction

$$A + 2 \leq A + \lceil \ln(3D(k+1))\rceil < \sum_{n=0}^{\Gamma(A+\lceil \ln(3D(k+1))\rceil)} \lambda_n \leq \sum_{n=0}^{A} \lambda_n \leq A + 1.$$

For $m \geq M$, we now have

$$\sum_{n=0}^{A+m} \lambda_n \geq \sum_{n=0}^{\Gamma(A+\lceil \ln(3D(k+1))\rceil)} \lambda_n > A + \ln\left(3D(k+1)\right) \geq \sum_{n=0}^{A-1} \lambda_n + \ln\left(3D(k+1)\right),$$

and so $\sum_{n=A}^{A+m} \lambda_n \geq \ln\left(3D(k+1)\right)$ for such m. Since for $x \geq 0$ we have $1 - x \leq \exp(-x)$, we get

$$\prod_{n=A}^{A+m} (1 - \lambda_n) \leq \exp\left(-\sum_{n=A}^{A+m} \lambda_n\right) \leq \frac{1}{3D(k+1)}$$

for $m \geq M$. Therefore, together with (+), we can conclude that

$$\forall m \in [M; B - A] \left(a_{A+m+1} \leq \frac{1}{k+1}\right),$$

entailing the result. $\square$

For S a nonempty convex closed subset of a Hadamard space X, and for any $u \in X$, let $P_S(u)$ denote the metric projection of u onto S. Then, by [12], $P_S(u)$ is characterized as the unique point $x \in S$ satisfying

$$\forall y \in S\left(\langle \overrightarrow{ux}, \overrightarrow{yx}\rangle \leq 0\right).$$

For the case at hand, the set S will be the set of fixed points of a nonexpansive map T, which is easily seen to be closed, convex, and will be assumed to be nonempty.

We shall require a quantitative version regarding the characterization of the projection. The first proof mining studies on the metric projection are due to Kohlenbach in [30] and [31]. The formulation that we use here (essentially) featured in [15], and is a simple (nonlinear) variation of the corresponding result in [18].

Proposition 2.7 (essentially Proposition 4.4 in [15])**.** *Let X be a* CAT(0) *space and let $T : X \to X$ be a nonexpansive map. Let $x_0 \in X$ and let $b \in \mathbb{N}$ be such that $b \geq d(x_0, p)$ for some $p \in \mathrm{Fix}(T)$. For any $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, there exist $n \leq 24b(\widetilde{h}_f^{(R)}(0) + 1)^2$ and $x \in \overline{B}_b(p)$ such that*

$$d(x, Tx) \leq \frac{1}{f(n) + 1}$$

$$\text{and} \quad \forall y \in \overline{B}_b(p)\left(d(y, Ty) \leq \frac{1}{n+1} \to \langle \overrightarrow{x_0x}, \overrightarrow{yx}\rangle \leq \frac{1}{k+1}\right),$$

with $R := 4b^4(k+1)^2$ and $h_f(m) := \max\{f(24b(m+1)^2), 24b(m+1)^2\}$, where $\widetilde{h}_f^{(R)}$ is the R-fold composition of $\widetilde{h}_f$ with $\widetilde{h}_f(m) := \max\{h_f(m') \mid m' \leq m\}$.

3. Main results

For the remainder of this paper, unless said otherwise, we let (X,d,W) be a CAT(0) space. Further, let (x_n) be the iteration defined via $(\mathrm{H}_{\mathrm{adp}})$ for a given nonexpansive map $T : X \to X$ where we assume $\mathrm{Fix}(T) \neq \emptyset$. As we care for the asymptotic behavior of the sequence generated by $(\mathrm{H}_{\mathrm{adp}})$, we assume throughout that $x_n \neq Tx_n$ for all $n \in \mathbb{N}$.

We begin with the essential properties of the adaptive parameters. Crucially, as established in [23], in the normed context these satisfy

$$\|x_{n+1} - Tx_{n+1}\|^2 \leq \frac{2}{\varphi_n}\langle x_{n+1} - Tx_{n+1}, x_0 - x_{n+1}\rangle$$

as well as $\varphi_{n+1} > \varphi_n \geq n+1$ which is then used to derive the corresponding asymptotic regularity and also the main convergence results. In the context of CAT(0) spaces, we get the following extended result:

Lemma 3.1. *For all $n \in \mathbb{N}$:*

(1) $d^2(x_{n+1}, Tx_{n+1}) \leq \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle$,
(2) $\varphi_{n+1} > \varphi_n \geq n+1$.

Proof. For item (1), note that by (CN^+) and since $x_{n+1} = \left(\frac{1}{\varphi_n+1}\right)x_0 \oplus \left(\frac{\varphi_n}{\varphi_n+1}\right)Tx_n$, we get

$$d^2(x_{n+1}, Tx_{n+1}) \leq \frac{1}{\varphi_n+1}d^2(x_0, Tx_{n+1}) + \frac{\varphi_n}{\varphi_n+1}d^2(Tx_n, Tx_{n+1}) - \frac{\varphi_n}{(\varphi_n+1)^2}d^2(x_0, Tx_n).$$

Thus, we get

$$d^2(x_0, Tx_{n+1}) \geq (\varphi_n+1)d^2(x_{n+1}, Tx_{n+1}) - \varphi_n d^2(Tx_n, Tx_{n+1}) + \frac{\varphi_n}{\varphi_n+1}d^2(x_0, Tx_n).$$

Using Lemma 2.1, we get

$$d^2(x_0, Tx_{n+1}) = d^2(x_{n+1}, Tx_{n+1}) + d^2(x_0, x_{n+1}) + 2\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle$$

and combined with the previous result, this yields

$$\begin{aligned}
&d^2(x_{n+1}, Tx_{n+1}) + d^2(x_0, x_{n+1}) + 2\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle \\
&\qquad \geq (\varphi_n+1)d^2(x_{n+1}, Tx_{n+1}) - \varphi_n d^2(Tx_n, Tx_{n+1}) + \frac{\varphi_n}{\varphi_n+1}d^2(x_0, Tx_n)
\end{aligned}$$

which, using the nonexpansivity of T, in particular implies

$$\begin{aligned}
&d^2(x_{n+1}, Tx_{n+1}) \\
&\quad \leq \frac{1}{\varphi_n}d^2(x_0, x_{n+1}) + \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle + d^2(Tx_n, Tx_{n+1}) - \frac{1}{\varphi_n+1}d^2(x_0, Tx_n) \\
(*) \qquad &\quad \leq \frac{1}{\varphi_n}d^2(x_0, x_{n+1}) + \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle + d^2(x_n, x_{n+1}) - \frac{1}{\varphi_n+1}d^2(x_0, Tx_n).
\end{aligned}$$

Using Lemma 2.1 again, we get

$$\begin{aligned}
d^2(x_n, x_{n+1}) &= d^2(x_n, Tx_n) + d^2(Tx_n, x_{n+1}) - 2\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_{n+1}Tx_n}\right\rangle \\
(+) \qquad &= d^2(x_n, Tx_n) + \frac{1}{(\varphi_n+1)^2}d^2(Tx_n, x_0) - 2\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_{n+1}Tx_n}\right\rangle.
\end{aligned}$$

From the definition of φ_n, we now have

$$\begin{aligned}
\varphi_n &= \frac{2\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_0x_n}\right\rangle}{d^2(x_n,Tx_n)} + 1 \\
&= \frac{2\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_0x_n}\right\rangle + d^2(x_n,Tx_n)}{d^2(x_n,Tx_n)} \\
&= \frac{2\left(\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_0x_n}\right\rangle + \left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_nTx_n}\right\rangle\right) - \left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_nTx_n}\right\rangle}{d^2(x_n,Tx_n)} \\
&= \frac{2\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_0Tx_n}\right\rangle}{d^2(x_n,Tx_n)} - 1
\end{aligned}$$

and hence

$$d^2(x_n,Tx_n) = \frac{2}{\varphi_n+1}\left\langle \overrightarrow{x_nTx_n}, \overrightarrow{x_0Tx_n}\right\rangle. \tag{$\circ$}$$

Using Lemma 2.2 combined with the previous (+) and ($\circ$) then yields

$$d^2(x_n,x_{n+1}) \leq \frac{1}{(\varphi_n+1)^2} d^2(Tx_n,x_0). \tag{$\diamond$}$$

Noting that

$$\frac{1}{\varphi_n} d^2(x_0,x_{n+1}) = \frac{\varphi_n}{(\varphi_n+1)^2} d^2(x_0,Tx_n),$$

we therefore derive using ($*$) and ($\diamond$) that

$$\begin{aligned}
&d^2(x_{n+1},Tx_{n+1}) \\
&\quad\leq \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle \\
&\qquad + \frac{1}{\varphi_n} d^2(x_0,x_{n+1}) + d^2(x_n,x_{n+1}) - \frac{1}{\varphi_n+1} d^2(x_0,Tx_n) \\
&\quad\leq \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle \\
&\qquad + \frac{\varphi_n}{(\varphi_n+1)^2} d^2(x_0,Tx_n) + \frac{1}{(\varphi_n+1)^2} d^2(Tx_n,x_0) - \frac{1}{\varphi_n+1} d^2(x_0,Tx_n) \\
&\quad= \frac{2}{\varphi_n}\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle
\end{aligned}$$

which is the claim.

We now prove item (2) by induction on n. For $n = 0$, we clearly have $\varphi_0 = 1$. For the induction step, assume $\varphi_n \geq n+1$. Then by item (1), we get

$$\varphi_{n+1} = \frac{2\left\langle \overrightarrow{Tx_{n+1}x_{n+1}}, \overrightarrow{x_{n+1}x_0}\right\rangle}{d^2(x_{n+1},Tx_{n+1})} + 1 \geq \varphi_n + 1 \geq n+2.$$

This inequality also entails $\varphi_{n+1} > \varphi_n$. $\square$

Lemma 3.2. *The sequence (x_n) is bounded and*

$$d(Tx_n, p) \leq d(x_n, p) \leq d(x_0, p)$$

holds for any $p \in \mathrm{Fix}(T)$.

Proof. As T is nonexpansive and $p \in \mathrm{Fix}(T)$, we have $d(Tx_n, p) \leq d(x_n, p)$. We can then show $d(x_n, p) \leq d(x_0, p)$ by induction on n. Immediately, this holds for $n = 0$ and if $d(x_n, p) \leq d(x_0, p)$ holds, then

$$d(x_{n+1}, p) \leq \frac{1}{\varphi_n + 1} d(x_0, p) + \frac{\varphi_n}{\varphi_n + 1} d(Tx_n, p) \leq \frac{1}{\varphi_n + 1} d(x_0, p) + \frac{\varphi_n}{\varphi_n + 1} d(x_n, p) \leq d(x_0, p).$$

□

Akin to [23], the above can then be immediately employed to give the following asymptotic regularity result, in particular yielding a linear rate of asymptotic regularity, which is the first main result of our paper.

Theorem 3.3. *The iteration (x_n) generated by the scheme $(\mathrm{H}_{\mathrm{adp}})$ satisfies*

$$d(x_n, Tx_n) \leq \frac{4}{\varphi_{n-1}} d(x_0, p) \leq \frac{8}{\varphi_{n-1} + 1} d(x_0, p) \leq \frac{8}{n + 1} d(x_0, p)$$

for any $n \geq 1$ and $p \in \mathrm{Fix}(T)$. In particular, the sequence (x_n) is asymptotically regular relative to T with a rate $\alpha(k) := 4b(k + 1)$, i.e.

$$\forall k \in \mathbb{N}\ \forall n \geq \alpha(k) \left(d(x_n, Tx_n) \leq \frac{1}{k + 1} \right),$$

where $b \in \mathbb{N}$ is such that $b \geq d(x_0, p)$ for some $p \in \mathrm{Fix}(T)$.

Proof. From Lemma 3.1 and Lemma 3.2, we get

$$d(x_{n+1}, Tx_{n+1}) \leq \frac{2}{\varphi_n} d(x_0, x_{n+1}) \leq \frac{4}{\varphi_n} d(x_0, p) \leq \frac{4}{n + 1} d(x_0, p)$$

for any $n \in \mathbb{N}$. The rate α is immediate from these inequalities. Further, since $\varphi_n \geq 1$, we have $1/\varphi_n \leq 2/(\varphi_n + 1)$ and so in particular

$$\frac{4}{\varphi_n} \leq \frac{8}{\varphi_n + 1}$$

which, with $\varphi_n \geq n + 1$, yields the other inequalities. □

We now move on to the main convergence result. For that, we fix a $p \in \mathrm{Fix}(T) \neq \emptyset$ and we fix a $b \in \mathbb{N} \setminus \{0\}$ such that $b \geq d(x_0, p)$.

As discussed in the introduction, we establish our convergence result by means of an initial quantitative result which gives a rate of metastability for the sequence. For that, we in following now give the necessary quantitative lemmas required for that construction. We begin with the following approximate projection result:

Lemma 3.4. *For any natural number $k \in \mathbb{N}$ and function $f : \mathbb{N} \to \mathbb{N}$, there exist $n \leq \beta(k, f)$ and $x \in \overline{B}_b(p)$ such that $d(x, Tx) \leq 1/(f(n) + 1)$ and*

$$\forall m \geq n \left(\langle \overrightarrow{x_0 x}, \overrightarrow{x_m x} \rangle \leq \frac{1}{k + 1} \right),$$

where $\beta(k, f) := 96b^2(\widetilde{h}^{(R)}_{f_\alpha}(0) + 1)^2 + 4b$, with $\widetilde{h}_{(\cdot)}$ and R as defined in Proposition 2.7 and with the function $f_\alpha(n) := f(\alpha(n)) = f(4b(n + 1))$.

Proof. Given k and f, apply Proposition 2.7, to get an $n_0 \leq 24b(\widetilde{h}^{(R)}_{f_\alpha}(0)+1)^2$ and $x \in \overline{B}_b(p)$ such that $d(x,Tx) \leq 1/(f_\alpha(n_0)+1)$ and

$$\forall y \in \overline{B}_b(p) \left(d(y,Ty) \leq \frac{1}{n_0+1} \to \langle \overrightarrow{x_0 x}, \overrightarrow{yx} \rangle \leq \frac{1}{k+1} \right).$$

From Lemma 3.2, we have $(x_n) \subseteq \overline{B}_b(p)$, and by Theorem 3.3,

$$\forall m \geq 4b(n_0+1) \left(d(x_m, Tx_m) \leq \frac{1}{n_0+1} \right).$$

Therefore,

$$\forall m \geq 4b(n_0+1) \left(\langle \overrightarrow{x_0 x}, \overrightarrow{x_m x} \rangle \leq \frac{1}{k+1} \right)$$

and clearly the result holds with $n = 4b(n_0+1) \leq \beta(k,f)$. □

The proof of the main convergence result given in [23] now proceeds with a case distinction of whether the adaptive parameters $\lambda_n := 1/(\varphi_n+1)$ satisfy

$$\sum_{n=0}^{\infty} \lambda_n = \infty \text{ or } \sum_{n=0}^{\infty} \lambda_n < \infty,$$

deducing the convergence of the sequence in each of the cases. Quantitatively, we will resolve this case distinction by first analyzing the two cases individually, giving a rate of metastability in dependence on a quantitative reformulation of the associated condition on the series over the parameters. In a final step, these results will then be joined together. We now begin with the case of $\sum_{n=0}^{\infty} \lambda_n = \infty$:

Lemma 3.5. *For any $k \in \mathbb{N}$ and functions $f, \Gamma : \mathbb{N} \to \mathbb{N}$, if*

$$\forall L \leq \omega_1(k,f,\Gamma) \left(\sum_{n=0}^{\Gamma(L)} \lambda_n > L \right),$$

then

$$\exists n \leq \omega_2(k,f,\Gamma)\ \exists x \in \overline{B}_b(p)\ \forall i \in [n; n+f(n)] \left(d^2(x_i, x) \leq \frac{1}{k+1} \right),$$

with

$$\begin{aligned} A(n) &:= \max\left\{n, 144b^2(k+1)\right\}, \\ A_0 &:= A(\beta(18k+17,F)), \\ \Theta(n) &:= \Gamma\left(A(n) + \lceil \ln\left(12b^2(k+1)\right) \rceil\right) + 1, \\ F(n) &:= 48b^3\left(\Theta(n) + f(\Theta(n)) + 1\right)(k+1), \\ \omega_1(k,f,\Gamma) &:= A_0 + \lceil \ln\left(12b^2(k+1)\right) \rceil, \\ \omega_2(k,f,\Gamma) &:= \max\left\{\Theta(n) \mid n \leq \beta(18k+17,F)\right\}, \end{aligned}$$

and where β is as in Lemma 3.4.

Proof. Let k, f and Γ be given. Applying the Lemma 3.4 (with $k = 18k+17$ and $f = F$), we conclude the existence of an $n_0 \leq \beta(18k+17,F)$ and $x \in \overline{B}_b(p)$ such that $d(x,Tx) \leq \frac{1}{F(n_0)+1}$ and

$$\forall m \geq n_0 \left(\langle \overrightarrow{x_0 x}, \overrightarrow{x_m x} \rangle \leq \frac{1}{18(k+1)} \right).$$

Note that for all $m \geq 144b^2(k+1)$, we have

$$d(x_m, Tx_m) \leq \frac{1}{36b(k+1)},$$

by Theorem 3.3. Using Lemma 3.1, a fortiori we also have

$$\lambda_m = \frac{1}{\varphi_m + 1} \leq \frac{1}{m+2} \leq \frac{1}{36b^2(k+1)}$$

for such m. For all $m \in \mathbb{N}$, using Lemmas 2.3 and 2.4, we thus get

$$\begin{aligned} d^2(x_{m+1}, x) &\leq (1-\lambda_m)^2 d^2(Tx_m, x) + \lambda_m^2 d^2(x_0, x) + 2\lambda_m(1-\lambda_m)\left\langle \overrightarrow{x_0 x}, \overrightarrow{Tx_m x} \right\rangle \\ &\leq (1-\lambda_m) d^2(x_m, x) + 4bd(x, Tx) \\ &\qquad + \lambda_m^2 d^2(x_0, x) + 2\lambda_m(1-\lambda_m)\langle \overrightarrow{x_0 x}, \overrightarrow{x_m x} \rangle + 2\lambda_m d(x, x_0) d(x_m, Tx_m) \\ &\leq (1-\lambda_m) d^2(x_m, x) + \lambda_m b_m + \frac{4b}{F(n_0)+1}, \end{aligned}$$

with

$$b_m := 4b^2\lambda_m + 2(1-\lambda_m)\langle \overrightarrow{x_0 x}, \overrightarrow{x_m x} \rangle + 4bd(x_m, Tx_m).$$

Writing $a_n := d^2(x_n, x)$, for which we have $(a_n) \subseteq [0, 4b^2]$, we derive for all $m \in \mathbb{N}$:

$$\begin{aligned} a_{m+1} &\leq (1-\lambda_m)a_m + \lambda_m b_m + \frac{4b}{F(n_0)+1} \\ &\leq (1-\lambda_m)a_m + \lambda_m b_m + \frac{1}{3(4b^2)(\Theta(n_0) + f(\Theta(n_0)) + 1)(k+1)}. \end{aligned}$$

Moreover, for $m \geq A(n_0)$, we get

$$b_m \leq \frac{4b^2}{36b^2(k+1)} + \frac{2}{18(k+1)} + \frac{4b}{36b(k+1)} \leq \frac{1}{3(k+1)}.$$

Hence, under this assumption, by Lemma 2.6, we conclude that

$$\forall i \in [\Theta(n_0); \Theta(n_0) + f(\Theta(n_0))] \left(d^2(x_i, x) \leq \frac{1}{k+1} \right)$$

Therefore, the result clearly holds with $n := \Theta(n_0)$. □

Lemma 3.6. *For any $k \in \mathbb{N}$ and functions $f, \Gamma : \mathbb{N} \to \mathbb{N}$, if*

$$\forall L \leq \Omega_1(k, f, \Gamma) \left(\sum_{n=0}^{\Gamma(L)} \lambda_n > L \right),$$

then

$$\exists n \leq \Omega_2(k, f, \Gamma)\ \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

with $\Omega_i(k, f, \Gamma) := \omega_i(4(k+1)^2 - 1, f, \Gamma)$, for $i \in \{1, 2\}$ and where the ω_i are as in Lemma 3.5.

Proof. Given k, f and Γ, apply Lemma 3.5 with $4(k+1)^2 - 1$ to conclude

$$\exists n \leq \Omega_2(k, f, \Gamma)\ \exists x \in \overline{B}_b(p)\ \forall i \in [n; n + f(n)] \left(d^2(x_i, x) \leq \frac{1}{4(k+1)^2} \right)$$

under the assumption that $\sum_{n=0}^{\Gamma(L)} \lambda_n > L$ for all $L \leq \Omega_1(k, f, \Gamma)$. Hence, for $i \in [n; n + f(n)]$, we get $d(x_i, x) \leq \frac{1}{2(k+1)}$ and the result follows by triangle inequality. □

We now consider the case of $\sum_{n=0}^{\infty} \lambda_n < \infty$. For that, we crucially rely on the following finitary quantitative variant of the monotone convergence theorem and an associated result for series:

Lemma 3.7 (folklore, see e.g. [29] and [53]). *For all $k, b \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ and for any nondecreasing finite sequence*

$$(a_n)_{n=0}^{\phi_b(k,f)} \subseteq [0, b],$$

there exists an $n \in \mathbb{N}$ such that $n + f(n) \leq \phi_b(k, f)$ and

$$\forall i, j \in [n; n + f(n)] \left(|a_i - a_j| \leq \frac{1}{k+1} \right)$$

with $\phi_b(k, f) := f_+^{(b(k+1))}(0)$ where $f_+^{(b(k+1))}$ is the $b(k+1)$-fold composition of $f_+(n) := n+f(n)$.

Proof. Suppose that there are k, b and f as well as a nondecreasing finite sequence $(a_n)_{n=0}^{\phi_b(k,f)} \subseteq [0, b]$ such that

$$\forall n \in \mathbb{N} \left(n + f(n) \leq \phi_b(k, f) \to \left(a_{n+f(n)} - a_n > \frac{1}{k+1} \right) \right).$$

Considering the sequence $f_+^{(i)}(0)$ for $i \leq b(k + 1)$, we thus get

$$a_{f_+^{(i+1)}(0)} > \frac{1}{k+1} + a_{f_+^{(i)}(0)}$$

for $i < b(k + 1)$ so that

$$a_{\phi_b(k,f)} > \frac{b(k+1)}{k+1} = b,$$

which is a contradiction. In particular, let now n be such that $n + f(n) \leq \phi_b(k, f)$ and $a_{n+f(n)} - a_n \leq 1/(k + 1)$. For any $i < j \in [n; n + f(n)]$, we then have

$$|a_i - a_j| = a_j - a_i \leq a_{n+f(n)} - a_n \leq \frac{1}{k+1}. \qquad \square$$

Lemma 3.8. *Let $(\alpha_l) \subseteq [0, \infty)$ be a given sequence. For any $k, L \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, if*

$$\sum_{l=0}^{\phi_L(k,\hat{f})} \alpha_l \leq L,$$

then

$$\exists n \leq \phi_L(k, \hat{f}) + 1 \left(\sum_{l=n}^{n+f(n)} \alpha_l \leq \frac{1}{k+1} \right),$$

where $\hat{f}(n) := f(n + 1) + 1$ and where $\phi_{(\cdot)}$ is as in Lemma 3.7.

Proof. The sequence $(\sum_{l=0}^{n} \alpha_l)$ for $n = 0, \dots, \phi_L(k, \hat{f})$ is a monotone nondecreasing finite sequence in $[0, L]$. Thus, it follows from Lemma 3.7 that there is a $\hat{n} \leq \phi_L(k, \hat{f})$ such that

$$\forall i, j \in [\hat{n}; \hat{n} + \hat{f}(\hat{n})] \left(\left| \sum_{l=0}^{i} \alpha_l - \sum_{l=0}^{j} \alpha_l \right| \leq \frac{1}{k+1} \right).$$

In particular, for $n = \hat{n} + 1$ and $i = n + f(n) = \hat{n} + \hat{f}(\hat{n})$ and $j = \hat{n}$, we have

$$\sum_{l=n}^{n+f(n)} \alpha_l = \sum_{l=0}^{n+f(n)} \alpha_l - \sum_{l=0}^{\hat{n}} \alpha_l \leq \frac{1}{k+1}$$

and so this $n \leq \phi_L(k, \hat{f}) + 1$ yields the claim. ☐

This now allows for the following quantitative version of the case of $\sum_{n=0}^{\infty} \lambda_n < \infty$:

Lemma 3.9. *For any $k, L \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, if*

$$\sum_{n=0}^{\Phi_1(k,f,L)} \lambda_n \leq L,$$

then

$$\exists n \leq \Phi_2(k, f, L)\ \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Phi_1(k, f, L) := \phi_{L_b}(k, \hat{f}),$$
$$\Phi_2(k, f, L) := \phi_{L_b}(k, \hat{f}) + 1,$$

with $L_b := (10L + 2)b$ and where $\phi_{(\cdot)}$ is as in Lemma 3.7 and $\hat{f}$ is as in Lemma 3.8.

Proof. Using Lemma 3.1, we have

$$d(x_{n+1}, Tx_{n+1}) \leq \frac{2}{\varphi_n} d(x_0, x_{n+1})$$

and using Lemma 3.2, we have $d(x_0, x_{n+1}) \leq 2b$. Rewriting this with $\lambda_n = 1/(\varphi_n + 1)$, we get that

$$d(x_{n+1}, Tx_{n+1}) \leq \frac{4b}{\varphi_n} \leq \frac{4b\lambda_n}{1 - \lambda_n} \leq 8b\lambda_n$$

as $\lambda_n \leq 1/(n + 2) \leq 1/2$.

But then, for $n \geq 1$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{1}{\varphi_n + 1} d(x_0, x_n) + \frac{\varphi_n}{\varphi_n + 1} d(Tx_n, x_n) \\ &\leq 2b\lambda_n + 8b\lambda_{n-1} \end{aligned}$$

so that, since we have $\sum_{n=0}^{\Phi_1(k,f,L)} \lambda_n \leq L$, we get

$$\sum_{n=1}^{\Phi_1(k,f,L)} d(x_{n+1}, x_n) \leq 10bL$$

and in particular

$$\sum_{n=0}^{\Phi_1(k,f,L)} d(x_{n+1}, x_n) \leq (10L + 2)b.$$

Using Lemma 3.8, we get

$$\exists n \leq \Phi_2(k, f, L) \ \left(\sum_{l=n}^{n+f(n)} d(x_{l+1}, x_l) \leq \frac{1}{k+1} \right)$$

and for $i, j \in [n; n + f(n)]$ (say with $i \leq j$), we have

$$d(x_i, x_j) \leq \sum_{l=i}^{j} d(x_l, x_{l+1}) \leq \sum_{l=n}^{n+f(n)} d(x_l, x_{l+1}) \leq \frac{1}{k+1}.$$

☐

The following abstract result then allows to "join" the rates of metastability produced by the previous lemmas which dealt with each case individually. To phrase the result concisely, we use $\mathbb{N}^\mathbb{N}$ to denote the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

Lemma 3.10. *Let $\mathcal{S}(k, f, n)$ be some statement on $k, n \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, and (λ_n) be a sequence of nonnegative real numbers. Assume the existence of functions $\Omega_1, \Omega_2 : \mathbb{N}\times\mathbb{N}^\mathbb{N}\times\mathbb{N}^\mathbb{N} \to \mathbb{N}$ and $\Phi_1, \Phi_2 : \mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that*

(1) $\forall k \in \mathbb{N}\ \forall f, \Gamma : \mathbb{N} \to \mathbb{N}$

$$\left(\forall L \leq \Omega_1(k, f, \Gamma) \left(\sum_{n=0}^{\Gamma(L)} \lambda_n > L\right) \to \exists n \leq \Omega_2(k, f, \Gamma) \left(\mathcal{S}(k, f, n)\right)\right),$$

(2) $\forall k, L \in \mathbb{N}\ \forall f : \mathbb{N} \to \mathbb{N}$

$$\left(\sum_{n=0}^{\Phi_1(k,f,L)} \lambda_n \leq L \to \exists n \leq \Phi_2(k, f, L) \left(\mathcal{S}(k, f, n)\right)\right).$$

Then for any $k \in \mathbb{N}$ and any $f : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \rho(k, f) \left(\mathcal{S}(k, f, n)\right),$$

where

$$\rho(k, f) := \max \left\{\Omega_2(k, f, \Gamma_0), \max \left\{\Phi_2(k, f, L) \mid L \leq \Omega_1(k, f, \Gamma_0)\right\}\right\},$$

with $\Gamma_0(L) := \Phi_1(k, f, L)$.

Proof. Fix $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. If $\sum_{n=0}^{\Gamma_0(L)} \lambda_n > L$ for all $L \leq \Omega_1(k, f, \Gamma_0)$, then by (1) with $\Gamma = \Gamma_0$ we have

$$\exists n \leq \Omega_2(k, f, \Gamma_0) \leq \rho(k, f) \left(\mathcal{S}(k, f, n)\right).$$

If on the contrary there exists some $L_0 \leq \Omega_1(k, f, \Gamma_0)$ such that

$$\sum_{n=0}^{\Gamma_0(L_0)} \lambda_n = \sum_{n=0}^{\Phi_1(k,f,L_0)} \lambda_n \leq L_0,$$

then by (2) with $L = L_0$, we have

$$\exists n \leq \Phi_2(k, f, L_0) \leq \rho(k, f) \left(\mathcal{S}(k, f, n)\right),$$

which concludes the proof. □

Combining the two previous cases contained in Lemmas 3.6 and 3.9 using the above Lemma 3.10, we obtain the following finitary quantitative convergence result for the sequence (x_n):

Theorem 3.11. *For any $k \in \mathbb{N}$ and any $f : \mathbb{N} \to \mathbb{N}$:*

$$\exists n \leq \rho(k, f)\ \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k + 1}\right),$$

where

$$\rho(k, f) := \max \left\{\Omega_2(k, f, \Gamma_0), \max \left\{\Phi_2(k, f, L) \mid L \leq \Omega_1(k, f, \Gamma_0)\right\}\right\},$$

with $\Gamma_0(L) := \Phi_1(k, f, L)$ and where the Ω_i are as in Lemma 3.6 and the Φ_i are as in Lemma 3.9.

Proof. The result immediately follows from the Lemmas 3.6 and 3.9 together with Lemma 3.10, taking

$$\mathcal{S}(k, f, n) :\equiv \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right)$$

in the latter. □

Now, as said before, we can then immediately deduce the following "usual" convergence result for the sequence (x_n), which is our second main result of this paper:

Theorem 3.12. *Let X be a Hadamard space. Let (x_n) be generated via* $(\mathrm{H}_{\mathrm{adp}})$ *for a given nonexpansive map T with* $\mathrm{Fix}(T) \neq \emptyset$*. Then the sequence (x_n) converges to a fixed point of T.*

Proof. Either $x_n = Tx_n$ for some $n \in \mathbb{N}$, where the claim is then immediate. Or $x_n \neq Tx_n$ for all $n \in \mathbb{N}$, where by Theorem 3.11 we in particular get that the sequence (x_n) is metastable, i.e. that

$$\forall k \in \mathbb{N}\, \forall f : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in [n; n + f(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right).$$

This immediately implies that the sequence is Cauchy, i.e. that

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in \mathbb{N} \left(d(x_i, x_j) \leq \frac{1}{k+1} \right).$$

For suppose not. Then there is a $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there are $i, j \geq n$ such that $d(x_i, x_j) > 1/(k+1)$. Define a function $f : \mathbb{N} \to \mathbb{N}$ by setting $f(n) := \max\{i - n, j - n\}$ for such a pair of indices i, j. Then we have a k and f such that for any $n \in \mathbb{N}$:

$$\exists i, j \in [n; n + f(n)] \left(d(x_i, x_j) > \frac{1}{k+1} \right)$$

which is a contradiction to the above metastability property. As the sequence (x_n) is thus Cauchy, it converges to some limit since the space X is complete. We denote the limit by x. By Theorem 3.3, we get $d(x, Tx) = \lim d(x_n, Tx_n) = 0$ and so $x \in \mathrm{Fix}(T)$. □

Remark 3.13. *As already highlighted in the introduction, the convergence here is "strong" in the sense that it is convergence in regards to the metric and not in any weak sense like Δ-convergence. This result is therefore similar to the "strong" convergence result of Saejung* [47] *for the ordinary Halpern iteration and in particular contains the strong convergence result proved in* [23] *for the adapted Halpern iteration in Hilbert spaces. As also mentioned in the introduction, the quantitative results from Theorem 3.11 on that strong convergence are already novel in that linear context.*

4. Numerical experiments

In this section, we illustrate the practical effectiveness of the adaptive Halpern iteration $(\mathrm{H}_{\mathrm{adp}})$ in Hadamard spaces through an experimental evaluation. As highlighted in the introduction, numerical experiments are already presented in the work [23] which in particular show the effective speedup achieved by the adaptive anchoring parameters. Concretely, the iteration $(\mathrm{H}_{\mathrm{adp}})$ already outperforms the ordinary Halpern iteration (with anchoring parameters chosen in the usual open loop manner) in the various practical test cases over Euclidean spaces presented in [23], including the so-called LASSO problem and the de-blurring of the well-known cameraman test image.

The purpose of the present section in that way is to augment these results with an experimental case study in a genuine Hadamard space to illustrate that the speedup perceived in linear contexts persists in this extended hyperbolic context.

As a test case, we utilize a common benchmark problem from the literature, which to our knowledge first appeared in [16]: For the underlying space, we consider $\mathbb{R}^2$ with a non-Euclidean metric d defined by

$$d(x,y) := \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is then straightforward (albeit a bit tedious) to verify that $\mathbb{R}^2$ with this metric d forms a Hadamard space with the respective convex combination

$$W(x,y,\lambda) := ((1-\lambda)x_1 + \lambda y_1, ((1-\lambda)x_1 + \lambda y_1)^2 - (1-\lambda)(x_1^2 - x_2) - \lambda(y_1^2 - y_2)).$$

This space offers a more elementary structure, making it more accessible than, for example, the Poincaré upper half-plane model (see e.g. [6]) which is perhaps the most common example of a Hadamard space that is not a Hilbert space and is also sometimes chosen as the underlying space for numerical evaluations (see e.g. [49]).

Over this space, we now want to minimize a perturbed variant of the Rosenbrock function $f : \mathbb{R}^2 \to \mathbb{R}$, that is

$$f(x) := 100((x_2+1) - (x_1+1)^2)^2 + x_1^2$$

for $x = (x_1, x_2)$, originating as a test case in this nonlinear context, to our knowledge, in [16]. In similarity to the usual Rosenbrock function on $\mathbb{R}^2$, while this function has a unique minimum at $(0,0)$, it lies in a very narrow valley which makes it hard to find for most iterative methods. We can use the (adaptive or ordinary) Halpern method to approximate a minimizer of said function by rephrasing the corresponding minimization problem as a fixed point problem. Concretely, while f is not convex on $\mathbb{R}^2$ in the usual sense, it is convex on $(\mathbb{R}^2, d, W)$ (see e.g. [16]). Hence, we can utilize its corresponding proximal map

$$\mathrm{Prox}_f(x) := \mathrm{argmin}_{y\in\mathbb{R}^2}\left\{ f(y) + \frac{1}{2}d^2(x,y)\right\},$$

which, in this general context of Hadamard spaces with no linear structure, goes back to the work [25]. As f is convex and lower semi-continuous over $(\mathbb{R}^2, d, W)$, it follows (see e.g. [25]) that Prox_f is well-defined, nonexpansive and that

$$\mathrm{Fix}(\mathrm{Prox}_f) = \mathrm{argmin} f.$$

Hence, we can apply both the ordinary Halpern iteration (with anchoring parameters chosen in the usual open loop manner), that is the iteration (y_n) defined recursively, given a start point y_0, by

$$\text{(H)} \qquad y_{n+1} := \left(\frac{1}{n+1}\right) y_0 \oplus \left(\frac{n}{n+1}\right) T y_n,$$

as well as the adapted Halpern iteration ($\mathrm{H_{adp}}$), generating a sequence (x_n) through a start point x_0 as before, with the map $T = \mathrm{Prox}_f$ to approximate the minimizer of f. With (H), we effectively consider here a Halpern-type variant of the seminal proximal point algorithm, which is also well-studied in its ordinary form in Hadamard spaces (we refer to [2] and the references listed therein), while ($\mathrm{H_{adp}}$) presents a novel type of adapted Halpern-type proximal point method (both with constant proximal parameters).

To simulate these iterations, we wrote a MATLAB script, all components of which utilize standard MATLAB functionalities. The only thing of note is the evaluation of the proximal map $T = \mathrm{Prox}_f$, which we approximated in MATLAB via the built-in function fminsearch with start value $[0, 0]$ during each iteration. With initial/anchor point $x_0 = y_0 = (1, -1)$, the script then provided the following numerical evaluation of these methods: The first graph, given in Figure 1, shows the value of $d(x_n, Tx_n)$ for (x_n) generated via ($\mathrm{H}_{\mathrm{adp}}$), represented by the solid line, and $d(y_n, Ty_n)$ for (y_n) generated via (H), represented by the dotted-dashed line.

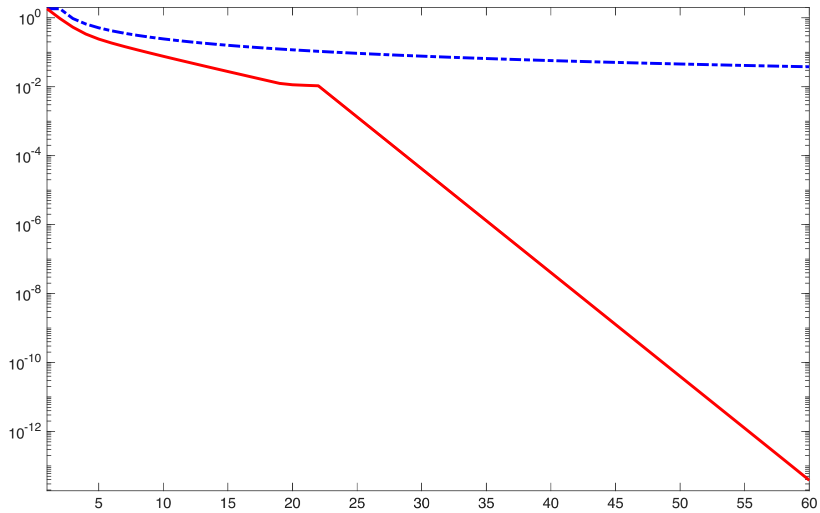

FIGURE 1. Values of $d(x_n, Tx_n)$ (solid line) and $d(y_n, Ty_n)$ (dotted-dashed line) for (x_n) generated via ($\mathrm{H}_{\mathrm{adp}}$) and (y_n) via (H).

The complexity of ($\mathrm{H}_{\mathrm{adp}}$) is, as the graph shows, negligible against that of (H). Indeed, we had to restrict the iteration number to $n \leq 60$ to avoid overflow errors caused by the high accuracy of the adaptive Halpern iteration achieved after that. To make this quite striking difference between these methods more apparent, Table 1 lists the precise iteration complexities for errors up to 10^{-6}.

ε	($\mathrm{H}_{\mathrm{adp}}$)	(H)
10^{-1}	9	24
10^{-2}	23	225
10^{-3}	26	2238
10^{-4}	29	22362
10^{-5}	33	223608
10^{-6}	36	2236070

TABLE 1. Number of iterations required by (x_n) generated via ($\mathrm{H}_{\mathrm{adp}}$) and (y_n) via (H) to achieve $d(x_n, Tx_n) < \varepsilon$ and $d(y_n, Ty_n) < \varepsilon$.

The second graph, given in Figure 2, shows the adaptive values φ_n against n. As before, here we had to restrict $n \leq 60$ as the φ_n grow so fast that already φ_n for n slightly above 60 causes an overflow.

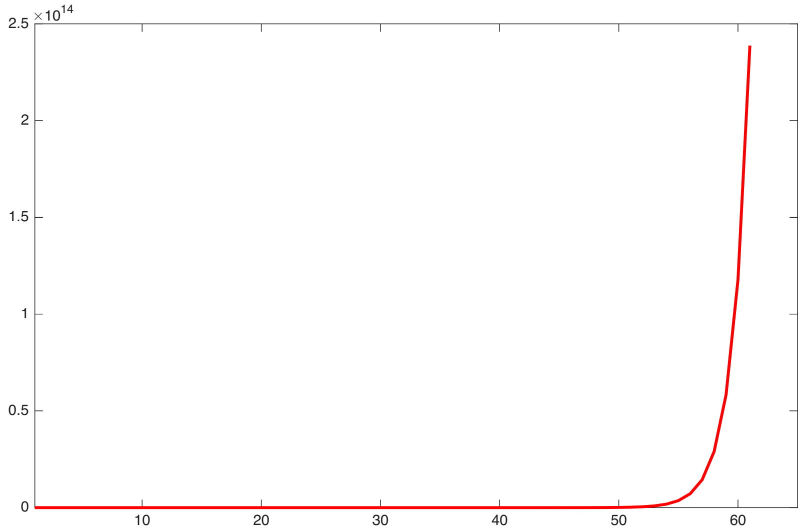

FIGURE 2. Value of φ_n for $n \leq 60$ calculated during the iteration (x_n) generated via $(\mathrm{H}_{\mathrm{adp}})$.

As is immediately apparent, the numerical results indeed show a huge difference between (H) and $(\mathrm{H}_{\mathrm{adp}})$ and hence confirm the empirical results from [23] that the adapted Halpern iteration strongly outperforms the ordinary Halpern iteration (with anchoring parameters chosen in the usual open loop manner) also in this general setting of Hadamard spaces.

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