QUANTITATIVE RESULTS FOR A TSENG-TYPE PRIMAL-DUAL METHOD FOR COMPOSITE MONOTONE INCLUSIONS

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Abstract. We provide quantitative results on a seminal Tseng-type primal-dual splitting algorithm for solving monotone inclusions due to Combettes and Pesquet which involves a mixture of sums, linear compositions and parallel sums of set-valued and Lipschitzian operators. For that, we first give quantitative results on a version of Tseng's forward-backward-forward splitting algorithm including error terms and variable parameters, partially extending previous work of Treusch and Kohlenbach, to which the method of Combettes and Pesquet is then reduced.

Keywords: Splitting algorithms, Tseng's algorithm, monotone inclusions, rates of convergence, rates of metastability, proof mining

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1. Introduction

This paper provides a quantitative analysis of a seminal Tseng-type splitting algorithm introduced by Combettes and Pesquet [4] for simultaneously solving a primal together with a dual inclusion problem, both being formulated using very general composite operators involving a mixture of sums, linear compositions and parallel sums of monotone set-valued and Lipschitzian operators.

Concretely, we show that if the individual sums of the operators involved are uniformly monotone, then one has a simple effective simultaneous full rate for the strong convergence for the individual components of the sequence produced by the algorithm, which respectively correspond to the primal and the dual inclusion problem, which is moreover highly uniform in the way that it only depends (in addition to a given error of precision) on certain norm bounds on the starting parameters, a rate of convergence for the sum of the error terms involved in the method and moduli witnessing the uniform monotonicity (in the sense of [9]) for the operators (cf. Theorem 4.8).

Without any uniform monotonicity assumption the algorithm converges weakly (as shown in [4]) but even in the finite dimensional case, there is in general no computable rate of convergence as one can show using results from computability theory due to Specker [16] (see also the discussions in [11, 14]). In such a situation, the next best thing is to construct an effective so-called rate of metastability of a sequences (x_n) in question, namely a function $\Delta(\varepsilon, g)$ bounding the n in the expression[1]

$$(-) \qquad \forall \varepsilon > 0\ \forall g \in \mathbb{N}^{\mathbb{N}}\ \exists n \in \mathbb{N}\ \forall i, j \in [n; n+g(n)]\ (\|x_i - x_j\| < \varepsilon)$$

in terms of ε and g. This metastability property $(-)$, considered already in [6] and with the name later coined through the work of Tao [17, 18] on finitary aspects of analysis, is a

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[1] Here, and in the following, we write $[n; m] := [n, m] \cap \mathbb{N}$.

(noneffectively) equivalent phrasing of the usual Cauchy property for the sequence (x_n). In that way, we then construct a simultaneous rate of metastability for all the components of the algorithm in the finite dimensional case which does not use any uniform monotonicity but which depends on a modulus witnessing the total boundedness of bounded balls in the space(s) at hand, which can be in particular computed from the dimension(s) (cf. Theorem 4.2).

Such a rate of metastability is also obtainable (for arbitrary Hilbert spaces) for the components of the sequence produced by the algorithm individually if the sums of the corresponding operators related to the individual component are uniformly monotone while no such requirement is placed on the other operators involved in the problem. (cf. Theorem 4.7).

The central paradigm pioneered in the work of Combettes and Pesquet [4] for approaching such complex composite primal-dual inclusion problems is to transfer the problem to a higher-dimensional composite space where all individual components are operated on simultaneously and the primal-dual inclusion is rephrased as the zero problem of a sum of a suitably defined monotone and a Lipschitzian operator. This perspective allows one to reduce the algorithm developed in [4] to a special instance corresponding to these two defined operators of an inexact version of the usual seminal splitting method of Tseng [20] augmented with error terms and variable scalars as introduced in [3]. Hence, in this paper we actually first analyze that latter algorithm in its full abstract generality, similarly providing rates of convergence under a uniform monotonicity assumption for the sum of the two operators (cf. Theorem 3.19) and a rate of metastability under a relative compactness assumption (cf. Theorem 3.13). For the extraction of the rate of convergence of that algorithm under the uniform monotonicity assumption, we in particular built upon the work [19] where such a rate was recently established in the situation without error terms and constant scalars. Here it turns out that the quantitative nature of the extended version of Tseng's algorithm can actually be understood through to general macros for quasi-Fejér monotone algorithms for both constructing rates of convergence under a metric regularity assumption as developed in [12] (see also [15]) and for constructing rates of metastability in the locally compact case as developed in [11]. As this reductive paradigm of approaching composite primal-dual monotone inclusions has, since the work of Combettes and Pesquet, become very influential in the literature over the last years (spawning in particular further seminal works like [2, 21] due to Boţ and Hendrich as well as Vũ where this paradigm is utilized to also provide Douglas-Rachford-type and forward-backward-type methods for solving related primal-dual monotone inclusion problems), we believe that a quantitative perspective on such reductive approaches as first given in this paper will be beneficial in providing similar results also for these methods (in particular in combination with previous works on the quantitative nature of the underlying Douglas-Rachford and forward-backward splitting methods as given in [19]).

Lastly, we want to remark that the aforementioned macros for quantitative results on quasi-Fejér monotone sequences, as well as the other results utilized and developed in this paper, have been established using the logic-based methodology of proof mining (see [7] for a comprehensive book treatment until 2008 and [8] for a survey of further more recent applications to, in particular, nonlinear analysis). However, as common in proof mining, all the results and proofs given in this work are formulated in a way which avoids any reference to mathematical logic.

The paper is now organized as follows: In Section 2, we briefly discuss the minimal necessary background from convex analysis and monotone operator theory required for this paper. In Section 3, we then provide the previously discussed quantitative results for the inexact variant of Tseng's algorithm from [3]. These are finally used in Section 4 to obtain corresponding results for the primal-dual splitting algorithm from [4].

2. Preliminaries

In this brief section, we shortly survey the necessary notions and notations for this paper. We begin with the (very minimal) background in set-valued monotone operator theory over Hilbert spaces.

Over a (real) Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, a set-valued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is called monotone if

$$\langle x - y, u - v \rangle \geq 0 \text{ for all } (x, u), (y, v) \in A$$

and maximally monotone if any monotone operator $B \supseteq A$ satisfies $A = B$. Here, and in the rest of the paper, we write $(x, u) \in A$ to state that (x, u) is included in the graph of A, that is $u \in Ax$. The resolvent of an operator A is the mapping $J_A := (\mathrm{Id} + A)^{-1}$, which, if A is monotone, is single-valued and firmly nonexpansive (see e.g. [1]) and, by the well-known theorem of Minty, the resolvent is additionally total if, and only if, A is further maximally monotone. The operator A is called uniformly monotone at a point x if there exists an increasing function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at 0 such that

$$\langle x - y, u - v \rangle \geq \phi(\|x - y\|) \text{ for all } (x, u), (y, v) \in A.$$

Given two set-valued operators $A, B : \mathcal{H} \to 2^{\mathcal{H}}$, their parallel sum is the operator $A \Box B := (A^{-1} + B^{-1})^{-1}$. Further, if $L : \mathcal{H} \to \mathcal{G}$ is a bounded linear operator for another (real) Hilbert space $\mathcal{G}$, we denote by $L^* : \mathcal{G} \to \mathcal{H}$ its adjoint which is uniquely defined by $\langle Lx, y \rangle = \langle x, L^* y \rangle$. Also, for a finite collection of Hilbert spaces $\mathcal{G}_i$, $i = 1, \dots, m$, we denote by $\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ their direct sum, that is their product space equipped with the inner product

$$\langle (x_i)_{1 \leq i \leq m}, (y_i)_{1 \leq i \leq m} \rangle := \sum_{i=1}^{m} \langle x_i, y_i \rangle$$

as well as the induced norm. Note that for a tuple $(x_i)_{1 \leq i \leq m} \in \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$, we have

$$\|x_j\| = \sqrt{\|x_j\|^2} \leq \sqrt{\sum_{i=1}^{m} \|x_i\|^2} = \|(x_i)_{1 \leq i \leq m}\|$$

for any $j = 1, \dots, m$. Lastly, given a point $x \in \mathcal{H}$ and a radius $r > 0$, we write $\overline{B}_r(x)$ for the closed ball around x with radius r.

Further, for quantitative aspects of the convergence of series, let $(a_n) \subseteq [0, \infty)$ be such that $\sum_{n=0}^{\infty} a_n < \infty$. We call $\alpha : (0, \infty) \to \mathbb{N}$ a Cauchy modulus for $\sum_{n=0}^{\infty} a_n < \infty$ if

$$\forall \varepsilon > 0 \left(\sum_{n=\alpha(\varepsilon)}^{\infty} a_n < \varepsilon \right).$$

We write $\lceil \cdot \rceil$ for the usual ceiling function, that is $\lceil x \rceil$ is the least integer z satisfying $z \geq x$.

3. Tseng's splitting algorithm with error terms and variables parameters

We begin with the fundamental splitting method of Tseng [20], also often called the forward-backward-forward method. Concretely, let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to \mathcal{H}$ be two mappings such that A is maximally monotone and B is monotone and $\frac{1}{\beta}$-Lipschitzian for some $\beta \in (0, \infty)$. In these data, Tseng's splitting algorithm proceeds via the following iterative scheme for a given starting point $x_0 \in \mathcal{H}$ and $\gamma \in (0,\infty)$:

$$\begin{cases} y_n := x_n - \gamma B x_n, \\ z_n := J_{\gamma A} y_n, \\ r_n := z_n - \gamma B z_n, \\ x_{n+1} := x_n - y_n + r_n. \end{cases}$$

As (essentially) shown in [20] (see also e.g. [1]), under the assumption that $\mathrm{zer}(A+B) \neq \emptyset$, it follows that the iteration (x_n) weakly converges to a solution $x \in \mathrm{zer}(A+B)$ and if A or B is assumed to be uniformly monotone, then (x_n) and (z_n) strongly converge to the unique solution $x \in \mathrm{zer}(A+B)$.

To model errors in the evaluation of the operators B and J_A, and simultaneously allow for regularization parameters in the resolvent of A, a slight modification of Tseng's method was proposed in the work of Briceño-Arias and Combettes [3]. Concretely, for operators A and B as well as a starting point $x_0 \in \mathcal{H}$ as above, the iteration proposed in [3] takes the following form:

$$(*) \qquad \begin{cases} y_n := x_n - \gamma_n (B x_n + a_n), \\ z_n := J_{\gamma_n A} y_n + b_n, \\ r_n := z_n - \gamma_n (B z_n + c_n), \\ x_{n+1} := x_n - y_n + r_n, \end{cases}$$

where now $(a_n), (b_n), (c_n) \subseteq \mathcal{H}$ are additional error sequences which are absolutely summable, i.e. $\sum_{n=0}^{\infty} \|a_n\|, \sum_{n=0}^{\infty} \|b_n\|, \sum_{n=0}^{\infty} \|c_n\| < \infty$, and $(\gamma_n) \subseteq (0,\infty)$ is an additional sequence of parameters.

For that method, the following convergence result was obtained:

Theorem 3.1 ([3])**.** *Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to \mathcal{H}$ be two mappings such that A is maximally monotone and B is monotone and $\frac{1}{\beta}$-Lipschitzian for some $\beta \in (0,\infty)$. Suppose that $\mathrm{zer}(A+B) \neq \emptyset$. Let $(x_n), (z_n)$ be defined as in $(*)$ for sequences $(a_n), (b_n), (c_n) \subseteq \mathcal{H}$ with*

$$\sum_{n=0}^{\infty} \|a_n\|, \sum_{n=0}^{\infty} \|b_n\|, \sum_{n=0}^{\infty} \|c_n\| < \infty,$$

and a parameter sequence $(\gamma_n) \subseteq [\frac{1}{k}, (1-\frac{1}{k})\beta]$ for some $k \geq 2$. Then:

(1) The sequences (x_n) and (z_n) weakly converge to a point $x \in \mathrm{zer}(A+B)$.
(2) If A or B is uniformly monotone at a point $x \in \mathrm{zer}(A+B)$, then (x_n) and (z_n) strongly converge to x.

In this section, we provide a quantitative study of this inexact version of Tseng's splitting algorithm. Concretely, we give both a rate of metastability of this method in finite dimensional Hilbert spaces as well as a rate of convergence of the method even in infinite dimensional Hilbert spaces, however in the context of a uniform monotonicity assumption. This latter result extends the previous quantitative analysis of Tseng's method under uniform monotonicity assumptions carried out in [19] to this inexact and parametrized version (where it however should be noted that the assumption of the totality of the operator B, i.e. that $\mathrm{dom}(B) = \mathcal{H}$, is relaxed in [19], as is common with exact variants of Tseng's method).

To begin, as in [3], we define the auxiliary sequences

$$\begin{cases} \tilde{y}_n := x_n - \gamma_n B x_n, \\ \tilde{z}_n := J_{\gamma_n A} \tilde{y}_n, \\ \tilde{r}_n := \tilde{z}_n - \gamma_n B \tilde{z}_n, \end{cases} \quad \text{and} \quad \begin{cases} e_n := y_n - r_n - \tilde{y}_n + \tilde{r}_n, \\ u_n := \gamma_n^{-1}(x_n - \tilde{z}_n) + B\tilde{z}_n - Bx_n. \end{cases}$$

Throughout this section, if not stated otherwise, we assume that

$$\sum_{n=0}^{\infty} \|a_n\|, \sum_{n=0}^{\infty} \|b_n\|, \sum_{n=0}^{\infty} \|c_n\| \leq R.$$

Further, we take $b \geq \beta$ for some $b \in \mathbb{N} \setminus \{0\}$ and, defining $d_n := 3b\|a_n\| + 2\|b_n\| + b\|c_n\|$, we take $\alpha : (0,\infty) \to \mathbb{N}$ to be a Cauchy modulus for $\sum_{n=0}^{\infty} d_n < \infty$. Also, we let $M_0, N_0 \in \mathbb{N}$ be such that[2] $\|x_0 - \tilde{z}_0\|, \|\tilde{y}_0 - \tilde{z}_0\| \leq M_0$ and $\|Bx_0\| \leq N_0$.

The next part of this section now lists necessary basic results for the two quantitative convergence results for Tseng's method that we present afterwards.

3.1. Basic lemmas.

Lemma 3.2. *It holds that $\sum_{n=0}^{\infty} \|e_n\| \leq (4b+2)R$ and α is a Cauchy modulus for $\sum_{n=0}^{\infty} \|e_n\| < \infty$.*

Proof. As in the proof of Theorem 2.5 from [3] one shows that

$$\|e_n\| \leq 3\beta\|a_n\| + 2\|b_n\| + \beta\|c_n\| \leq 3b\|a_n\| + 2\|b_n\| + b\|c_n\| = d_n$$

using $b \geq \beta$. From this, the claims follow immediately using that R is a bound for $\sum_{n=0}^{\infty} \|a_n\|$, $\sum_{n=0}^{\infty} \|b_n\|$, $\sum_{n=0}^{\infty} \|c_n\|$ and using that α is a Cauchy modulus for $\sum_{n=0}^{\infty} d_n < \infty$. □

Lemma 3.3. *Let $x \in \mathrm{zer}(A+B)$. Then for any $n \in \mathbb{N}$, it holds that*

$$\|x_{n+1} - x\| \leq \|x_n - \tilde{y}_n + \tilde{r}_n - x\| + \|e_n\| \leq \|x_n - x\| + \|e_n\|.$$

Hence, $\|x_n - x\|$ and $\|x_n - \tilde{y}_n + \tilde{r}_n - x\|$ are bounded and in particular

$$\|x_n - x\|, \|x_n - \tilde{y}_n + \tilde{r}_n - x\| \leq L := d + (4b+2)R$$

for all $n \in \mathbb{N}$ where $\|x_0 - x\| \leq d$.

Proof. The first two inequalities are established as (2.13) in [3]. From that, we get

$$\|x_n - x\|, \|x_n - \tilde{y}_n + \tilde{r}_n - x\| \leq \|x_0 - x\| + \sum_{n=0}^{\infty} \|e_n\|$$

whereby the bounds then follow using Lemma 3.2. □

Lemma 3.4. *Let $x \in \mathrm{zer}(A+B)$. Then for any $n \in \mathbb{N}$, it holds that*

$$\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \frac{1}{k^2}\|x_n - \tilde{z}_n\|^2 + \varepsilon_n$$

where $\varepsilon_n := 2\mu\|e_n\| + \|e_n\|^2$, provided that $\mu \geq \sup\{\|x_n - \tilde{y}_n + \tilde{r}_n - x\| \mid n \in \mathbb{N}\}$. Furthermore

$$\sum_{n=0}^{\infty} \varepsilon_n \leq 2\mu(4b+2)R + (4b+2)^2R^2$$

and for $\|x_0 - x\| \leq d$, μ can be taken as $\mu = L = d + (4b+2)R$.

[2] To have a bound on $\|\tilde{y}_0 - \tilde{z}_0\|$ is convenient but actually redundant as we have $\|\tilde{y}_0 - \tilde{z}_0\| \leq \|\tilde{y}_0 - x_0\| + \|x_0 - \tilde{z}_0\| \leq \gamma_0\|Bx_0\| + M_0 \leq bN_0 + M_0$, using in particular that $\gamma_0 \leq (1 - \frac{1}{k})\beta \leq \beta \leq b$.

Proof. The first inequality is shown as (2.14) in [3]. The bound on μ follows from Lemma 3.3 and the bound on $\sum_{n=0}^{\infty} \varepsilon_n$ follows using Lemma 3.2. □

Lemma 3.5. *For any $n \in \mathbb{N}$, it holds that $\|z_n - \tilde{z}_n\| \leq d_n$. Furthermore, α is a rate of convergence for $\|z_n - \tilde{z}_n\| \to 0$, i.e.*

$$\forall \varepsilon > 0\ \forall n \geq \alpha(\varepsilon)\left(\|z_n - \tilde{z}_n\| < \varepsilon\right).$$

Proof. By definition of $z_n, \tilde{z}_n$ as well as $y_n, \tilde{y}_n$ together with the nonexpansivity of the resolvents, we get

$$\begin{aligned}
\|z_n - \tilde{z}_n\| &= \|J_{\gamma_n A} y_n + b_n - J_{\gamma_n A}\tilde{y}_n\| \\
&\leq \|y_n - \tilde{y}_n\| + \|b_n\| \\
&= \|x_n - \gamma_n(Bx_n + a_n) - (x_n - \gamma_n Bx_n)\| + \|b_n\| \\
&= \gamma_n \|a_n\| + \|b_n\| \\
&\leq b\|a_n\| + \|b_n\| \leq d_n.
\end{aligned}$$

As α is a Cauchy modulus for $\sum_{n=0}^{\infty} d_n < \infty$, we have

$$\|z_n - \tilde{z}_n\| \leq d_n \leq \sum_{k=n}^{\infty} d_n \leq \sum_{k=\alpha(\varepsilon)}^{\infty} d_n < \varepsilon$$

for any $n \geq \alpha(\varepsilon)$ and any $\varepsilon > 0$. □

Lemma 3.6. *For any $n \in \mathbb{N}$, it holds that $\|J_{\gamma_n A}\tilde{y}_0 - J_{\gamma_0 A}\tilde{y}_0\| \leq (1 + bk)M_0$.*

Proof. Using the so-called resolvent identity (see e.g. Proposition 23.31 in [1]), we get

$$J_{\gamma_0 A}\tilde{y}_0 = J_{\gamma_n A}\left(\frac{\gamma_n}{\gamma_0}\tilde{y}_0 + \left(1 - \frac{\gamma_n}{\gamma_0}\right) J_{\gamma_0 A}\tilde{y}_0\right).$$

Together with the nonexpansivity of the resolvent, we get

$$\begin{aligned}
\|J_{\gamma_n A}\tilde{y}_0 - J_{\gamma_0 A}\tilde{y}_0\| &= \left\|J_{\gamma_n A}\tilde{y}_0 - J_{\gamma_n A}\left(\frac{\gamma_n}{\gamma_0}\tilde{y}_0 + \left(1 - \frac{\gamma_n}{\gamma_0}\right) J_{\gamma_0 A}\tilde{y}_0\right)\right\| \\
&\leq \left\|\tilde{y}_0 - \left(\frac{\gamma_n}{\gamma_0}\tilde{y}_0 + \left(1 - \frac{\gamma_n}{\gamma_0}\right) J_{\gamma_0 A}\tilde{y}_0\right)\right\| \\
&\leq \left|1 - \frac{\gamma_n}{\gamma_0}\right| \|\tilde{y}_0 - J_{\gamma_0 A}\tilde{y}_0\| \\
&\leq (1 + bk)M_0.
\end{aligned}$$

□

Lemma 3.7. *Let $x \in \mathrm{zer}(A + B)$ and $\|x_0 - x\| \leq d$. Then for any $n \in \mathbb{N}$, it holds that*

$$\|\tilde{z}_n - x\| \leq H := 2L + bN_0 + (2 + bk)M_0 + 3d$$

where $L := d + (4b + 2)R$ as before.

Proof. For any $n \in \mathbb{N}$, we have (using Lemma 3.6)

$$\begin{aligned}
\|\tilde{z}_n - x\| &= \|J_{\gamma_n A}\tilde{y}_n - x\| \\
&= \|J_{\gamma_n A}\tilde{y}_n - J_{\gamma_n A}\tilde{y}_0 + J_{\gamma_n A}\tilde{y}_0 - x_0 + x_0 - x\| \\
&\leq \|J_{\gamma_n A}\tilde{y}_n - J_{\gamma_n A}\tilde{y}_0\| + \underbrace{\|J_{\gamma_0 A}\tilde{y}_0 - x_0\|}_{=\|\tilde{z}_0 - x_0\|} + \|J_{\gamma_0 A}\tilde{y}_0 - J_{\gamma_n A}\tilde{y}_0\| + \|x_0 - x\| \\
&\leq \|\tilde{y}_n - \tilde{y}_0\| + (2 + bk)M_0 + d \\
&= \|x_n - \gamma_n B x_n - x_0 + \gamma_0 B x_0\| + (2 + bk)M_0 + d.
\end{aligned}$$

Now, using that B is $\frac{1}{\beta}$-Lipschitzian, we get for all $n \in \mathbb{N}$ that

$$\begin{aligned}
\|x_n - \gamma_n B x_n - x_0 + \gamma_0 B x_0\| &\leq \|x_n - x_0\| + \gamma_n \|B x_n - B x_0\| + bN_0 \\
&\leq \left(1 + \frac{\gamma_n}{\beta}\right)\|x_n - x_0\| + bN_0 \\
&\leq 2\|x_n - x + x - x_0\| + bN_0 \\
&\leq 2\|x_n - x\| + bN_0 + 2d \\
&\leq 2L + bN_0 + 2d,
\end{aligned}$$

where we have used Lemma 3.3 to infer $\|x_n - x\| \leq L$. Combining this with the above yields the result. $\square$

The next lemma is then a quantitative version of the quasi-Fejér monotonicity of the sequence established as (2.13) in [3] (recall Lemma 3.3), i.e. that

$$\|x_{n+1} - x\| \leq \|x_n - x\| + \|e_n\|$$

for all $n \in \mathbb{N}$ and $x \in \mathrm{zer}(A + B)$.

Lemma 3.8. *Let $x \in \mathrm{zer}(A + B)$ and $\|x_0 - x\| \leq d$ and define $L := d + (4b + 2)R$ as well as $H := 2L + bN_0 + (2 + bk)M_0 + 3d$. Let $(x^*, y^*) \in A + B$ be given such that $\|x - x^*\| \leq H$ as well as*

$$\|y^*\| < \frac{\varepsilon^2}{4bH}$$

for $\varepsilon > 0$. Then, for any $n \in \mathbb{N}$, we have

$$\|x_{n+1} - x^*\| < \|x_n - x^*\| + \|e_n\| + \varepsilon.$$

Proof. Let $n \in \mathbb{N}$ be given. As $(x^*, y^*) \in A + B$, we have $(x^*, \gamma_n y^*) \in \gamma_n A + \gamma_n B$ with

$$\|\gamma_n y^*\| \leq b\,\|y^*\| < \frac{\varepsilon^2}{4H}.$$

In particular, we get $\gamma_n y^* = \gamma_n z^* + \gamma_n B x^*$ for some $z^* \in A x^*$ and so

$$\gamma_n y^* - \gamma_n B x^* = \gamma_n (y^* - B x^*) \in \gamma_n A x^*.$$

By the definition of the resolvent, we further have $\tilde{y}_n - \tilde{z}_n = \tilde{y}_n - J_{\gamma_n A}\tilde{y}_n \in \gamma_n A \tilde{z}_n$ and so in particular

$$\begin{aligned}
u_n &= \gamma_n^{-1}(x_n - \tilde{z}_n) + B\tilde{z}_n - B x_n \\
&= \gamma_n^{-1}(x_n - \gamma_n B x_n - \tilde{z}_n) + B\tilde{z}_n \\
&= \gamma_n^{-1}(\tilde{y}_n - \tilde{z}_n) + B\tilde{z}_n \in (A + B)\tilde{z}_n.
\end{aligned}$$

By the monotonicity of $\gamma_n A$, we have $\langle \tilde{z}_n - x^*, \tilde{z}_n - \tilde{y}_n - \gamma_n Bx^* + \gamma_n y^*\rangle \leq 0$, and by the monotonicity of $\gamma_n B$, we have $\langle \tilde{z}_n - x^*, \gamma_n Bx^* - \gamma_n B\tilde{z}_n\rangle \leq 0$. Adding these inequalities, we get

$$\begin{aligned}\langle \tilde{z}_n - x^*, \tilde{z}_n - \tilde{y}_n - \gamma_n B\tilde{z}_n\rangle &\leq \|\tilde{z}_n - x^*\| \, \|\gamma_n y^*\| \\ &\leq (\|\tilde{z}_n - x\| + \|x - x^*\|) \, \|\gamma_n y^*\| \\ &\leq 2H \, \|\gamma_n y^*\| \\ &< \varepsilon^2/2\end{aligned}$$

using Lemma 3.7. Now, we further have (reasoning similarly as in [3]) that[3]

$$\begin{aligned}2\gamma_n\langle \tilde{z}_n - x^*, Bx_n - B\tilde{z}_n\rangle =&2\langle \tilde{z}_n - x^*, \tilde{z}_n - \tilde{y}_n - \gamma_n B\tilde{z}_n\rangle \\ &+ 2\langle \tilde{z}_n - x^*, \gamma_n Bx_n + \tilde{y}_n - \tilde{z}_n\rangle \\ <&2\langle \tilde{z}_n - x^*, \gamma_n Bx_n + \tilde{y}_n - \tilde{z}_n\rangle + \varepsilon^2 \\ =&2\langle \tilde{z}_n - x^*, x_n - \tilde{z}_n\rangle + \varepsilon^2 \\ =&\|x_n - x^*\|^2 - \|\tilde{z}_n - x^*\|^2 - \|x_n - \tilde{z}_n\|^2 + \varepsilon^2.\end{aligned}$$

This yields

$$\begin{aligned}\|x_n - \tilde{y}_n + \tilde{r}_n - x^*\|^2 &= \|\gamma_n Bx_n + \tilde{z}_n - \gamma_n B\tilde{z}_n - x^*\|^2 \\ &= \|\tilde{z}_n - x^*\|^2 + 2\gamma_n\langle \tilde{z}_n - x^*, Bx_n - B\tilde{z}_n\rangle + \gamma_n^2 \|Bx_n - B\tilde{z}_n\|^2 \\ &< \|x_n - x^*\|^2 - \|x_n - \tilde{z}_n\|^2 + \gamma_n^2 \|Bx_n - B\tilde{z}_n\|^2 + \varepsilon^2 \\ &\leq \|x_n - x^*\|^2 - \left(1 - \frac{\gamma_n^2}{\beta^2}\right) \|x_n - \tilde{z}_n\|^2 + \varepsilon^2 \\ &\leq \|x_n - x^*\|^2 + \varepsilon^2 \\ &\leq (\|x_n - x^*\| + \varepsilon)^2\end{aligned}$$

as $\gamma_n \leq \left(1 - \frac{1}{k}\right)\beta$ and so $0 \leq 1 - \frac{\gamma_n^2}{\beta^2}$. Combined, we thus have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|x_n - y_n + r_n - x^*\| \\ &\leq \|x_n - \tilde{y}_n + \tilde{r}_n - x^*\| + \|e_n\| \\ &< \|x_n - x^*\| + \|e_n\| + \varepsilon.\end{aligned}$$

□

Lemma 3.9. *Let $(a_n) \subseteq [0, \infty)$ be given such that $\sum_{n=0}^{\infty} a_n \leq K$. Then for any $\varepsilon > 0$, $N \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$:*

$$\exists n_0 \in \left[N; \widehat{g}^{(\lceil \frac{K}{\varepsilon} \rceil)}(N)\right] \ \forall n \in [n_0; n_0 + g(n_0)] \, (a_n < \varepsilon)$$

where $\widehat{g}(n) := n + g(n) + 1$.

Proof. Suppose on the contrary that there are $\varepsilon > 0$, $N \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n_0 \in \left[N; \widehat{g}^{(\lceil \frac{K}{\varepsilon} \rceil)}(N)\right] \ \exists n \in [n_0; n_0 + g(n_0)] \, (a_n \geq \varepsilon) \, .$$

Define $\tilde{g}(n) := n + g(n)$. Then, as

$$\widehat{g}^{(i)}(N) \in \left[N; \widehat{g}^{(\lceil \frac{K}{\varepsilon} \rceil)}(N)\right]$$

[3]The last equality here follows from the Hilbert space identity $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ (cf. Lemma 2.12 in [1]) by taking $x = \tilde{z}_n - x^*$ and $y = x_n - \tilde{z}_n$.

for all $i \leq \lceil \frac{K}{\varepsilon} \rceil$, we have

$$\sum_{n=0}^{\infty} a_n \geq \sum_{i=0}^{\lceil \frac{K}{\varepsilon} \rceil} \sum_{n=\widehat{g}^{(i)}(N)}^{\tilde{g}(\widehat{g}^{(i)}(N))} a_n \geq \sum_{i=0}^{\lceil \frac{K}{\varepsilon} \rceil} \varepsilon > \left\lceil \frac{K}{\varepsilon} \right\rceil \cdot \varepsilon \geq K,$$

which is a contradiction. ☐

Lemma 3.10. *Let $x \in \mathrm{zer}(A+B)$ and $d \geq \|x_0 - x\|$. Then, for any $\varepsilon > 0$, $N \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$:*

$$\exists n_0 \in [N; \varphi_N(\varepsilon, g)] \ \ \forall n \in [n_0; n_0 + g(n_0)] \, (\|x_n - \tilde{z}_n\|, \|u_n\| < \varepsilon)$$

where

$$\varphi_N(\varepsilon, g) := \widehat{g}^{\left(\left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil\right)}(N)$$

for $\widehat{g}(n) := n + g(n) + 1$ and $L := d + (4b+2)R$ as well as $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$.

Proof. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \|u_n\| &\leq \gamma_n^{-1} \|x_n - \tilde{z}_n\| + \|B\tilde{z}_n - Bx_n\| \\ &\leq \left(\gamma_n^{-1} + \beta^{-1}\right) \|x_n - \tilde{z}_n\| \\ &\leq (k + k(1 - k^{-1})) \|x_n - \tilde{z}_n\| \\ &= (2k-1) \|x_n - \tilde{z}_n\| \end{aligned}$$

since $1/k \leq (1 - 1/k)\beta$ and so $\beta^{-1} \leq k(1 - k^{-1})$. Using Lemma 3.4, we get

$$\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \frac{1}{k^2} \|x_n - \tilde{z}_n\|^2 + \varepsilon_n$$

for $\varepsilon_n = 2L\|e_n\| + \|e_n\|^2$. Also using Lemma 3.4, we have

$$\sum_{n=0}^{\infty} \varepsilon_n \leq 2L(4b+2)R + (4b+2)^2R^2.$$

Therefore, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \|x_n - \tilde{z}_n\|^2 &\leq k^2 \|x_0 - x\|^2 + k^2 \sum_{n=0}^{\infty} \varepsilon_n \\ &\leq k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2) = K. \end{aligned}$$

Take $\varepsilon > 0$ and $N \in \mathbb{N}$ as well as $g : \mathbb{N} \to \mathbb{N}$. By Lemma 3.9, there exists an $n_0 \in [N; \varphi_N(\varepsilon, g)]$ such that

$$\forall n \in [n_0; n_0 + g(n_0)] \left(\|x_n - \tilde{z}_n\|^2 < \varepsilon^2/(2k-1)^2\right)$$

and so for any such an n, we have $\|x_n - \tilde{z}_n\| < \varepsilon/(2k-1) \leq \varepsilon$ and by the above $\|u_n\| < \varepsilon$. ☐

As a corollary, we immediately get the following lemma (by setting $g(n) := 0$ in the above Lemma 3.10):

Lemma 3.11. *Let $x \in \mathrm{zer}(A+B)$ and $d \geq \|x_0 - x\|$. Then, for any $\varepsilon > 0$ and $N \in \mathbb{N}$:*

$$\exists n \in [N; \varphi'(\varepsilon, N)] \, (\|x_n - \tilde{z}_n\|, \|u_n\| < \varepsilon)$$

where

$$\varphi'(\varepsilon, N) := N + \left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil$$

and $L := d + (4b+2)R$ as well as $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$.

3.2. **Rates of metastability under a relative compactness assumption.** We now give our first quantitative rendering of Theorem 3.1 in the form of a joint rate of metastability for the sequences $(x_n), (z_n)$ under the assumption that the set $\{\tilde{z}_n \mid n \in \mathbb{N}\}$ is totally bounded. As this set is in particular bounded, this assumption is in particular satisfied if $\mathcal{H}$ is finite dimensional.

As we are concerned with quantitative results, we will also rely on a quantitative rendering of this total boundedness assumption which we express here using the notion of a modulus of total boundedness introduced in [5] (called a II-modulus of total boundedness in [11]).

Definition 3.12 (essentially [5])**.** For $A \subseteq \mathcal{H}$, a function $\gamma : (0, \infty) \to \mathbb{N}$ is called a *modulus of total boundedness for A* if for all $\varepsilon > 0$ and any $(x_n) \subseteq A$:

$$\exists 0 \leq i < j \leq \gamma(\varepsilon) \left(\|x_i - x_j\| < \varepsilon \right).$$

Such a modulus can for example be easily given for any bounded set in a finite dimensional Hilbert space $\mathcal{H}$ (cf. Example 2.8 in [11]): If $\mathcal{H}$ is finite dimensional with dimension d and $A \subseteq \mathcal{H}$ is bounded with $b > 0$ such that $\|a\| \leq b$ for any $a \in A$, then

$$\gamma(\varepsilon) := \left\lceil 2(\lceil 2/\varepsilon \rceil + 1)\sqrt{d} b \right\rceil^d$$

is a modulus of total boundedness for A.

Our first main quantitative result then takes the form of the next theorem. Note for this, that under the assumptions of Theorem 3.1, the operator $A + B$ is actually maximally monotone (as B is a total, single-valued and maximally monotone mapping, see e.g. Corollaries 20.28 and 25.5 in [1]) and so the corresponding resolvent J_{A+B} is single-valued, total, nonexpansive and satisfies $\mathrm{Fix}(J_{A+B}) = \mathrm{zer}(A + B)$.

Theorem 3.13. *Assume that* $\mathrm{zer}(A + B) \neq \emptyset$ *and let* $x \in \mathrm{zer}(A + B)$*. Let* $d \in \mathbb{N}$ *be such that* $\|x_0 - x\| \leq d$*. Assume that* $\overline{B}_H(x)$ *is totally bounded with a modulus of total boundedness* γ*. Then for any* $\varepsilon > 0$ *and* $g : \mathbb{N} \to \mathbb{N}$*:*

$$\exists n \leq \Delta(\varepsilon, g)\ \forall i, j \in [n; n + g(n)] \left(\|x_i - x_j\|, \|x_i - J_{A+B} x_i\| < \varepsilon \right)$$

where $\Delta(\varepsilon, g) := \Delta_0(P_{\varepsilon/3}, \varepsilon/3, g)$ *for* $P_\varepsilon := \gamma(\varepsilon/8) + 1$ *and*

$$\begin{cases} \Delta_0(0, \varepsilon, g) := 0, \\ \Delta_0(n + 1, \varepsilon, g) := \varphi'\left(\chi_g^M(\Delta_0(n, \varepsilon, g), \varepsilon/8), \alpha(\varepsilon/8)\right) \end{cases}$$

with

$$\varphi'(\varepsilon, N) := N + \left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil \text{ and } \chi(n, m, \varepsilon) := \min\left\{ \frac{\varepsilon}{4}, \frac{\varepsilon^2}{16m^2 bH} \right\}$$

as well as

$$\chi_g(n, \varepsilon) := \chi(n, g(n), \varepsilon) \text{ and } \chi_g^M(n, \varepsilon) := \min\{\chi_g(i, \varepsilon) \mid i \leq n\}$$

and with $L := d + (4b + 2)R$ *and* $K := k^2(d^2 + 2L(4b + 2)R + (4b + 2)^2 R^2)$ *as well as* $H := 2L + bN_0 + (2 + bk)M_0 + 3d$*.*

Proof. We apply the general and abstract macros for deriving rates of metastability for the convergence of quasi-Fejér monotone sequences in totally bounded metric spaces as developed in [11] (see in particular Theorem 6.4 therein). While the notions therein actually work with natural numbers as representatives for errors ε via expressions of the form $1/(k + 1)$, we here follow the general theme of the paper and present all bounds and moduli using generic reals ε as errors, which of course naturally arise through suitable modifications, such as e.g. converting maxima to minima, of the bounds resulting from [11] (the calculations of which we do not spell out any further as they are rather trivial, but perhaps tedious). Now, to instantiate said results, we define sets of approximate solutions $AF_\varepsilon \subseteq \overline{B}_H(x)$, given $\varepsilon > 0$, via

$$v \in AF_\varepsilon :\equiv \exists w \in \overline{B}_H(x)\ \exists u \in (A+B)(w)\ (\|v-w\|, \|u\| < \varepsilon)$$

for $v \in \overline{B}_H(x)$. By Lemma 3.11 (and using Lemmas 3.7 and 3.3 as well), φ' is a (monotone) so-called lim inf bound for (x_n) relative to AF_ε in the sense of [11], that is for all $\varepsilon > 0$ and all $N \in \mathbb{N}$:

$$\exists n \in [N; \varphi'(\varepsilon, N)]\, (x_n \in AF_\varepsilon).$$

Lemma 3.8 implies that χ is a modulus of uniform quasi-Fejér monotonicity for (x_n) relative to AF_ε in the sense of [11], that is for all $\varepsilon > 0$ and all $n, m \in \mathbb{N}$:

$$\forall v \in AF_{\chi(n,m,\varepsilon)}\ \forall l \leq m \left(\|x_{n+l} - v\| < \|x_n - v\| + \sum_{i=n}^{n+l-1} \|e_i\| + \varepsilon \right).$$

To see this, note that for $v \in AF_{\chi(n,m,\varepsilon)}$ there exists a $w \in \overline{B}_H(x)$ and $u \in (A+B)(w)$ such that

$$\|v - w\| < \frac{\varepsilon}{4} \text{ and } \|u\| < \frac{\varepsilon^2}{16m^2bH}.$$

For $l \leq m$, Lemma 3.8 then implies that

$$\|x_{n+l} - w\| < \|x_n - w\| + \sum_{i=n}^{n+l-1} \|e_i\| + \frac{\varepsilon}{2}$$

which in turn yields

$$\begin{aligned}
\|x_{n+l} - v\| &\leq \|x_{n+l} - w\| + \frac{\varepsilon}{4} \\
&< \|x_n - w\| + \sum_{i=n}^{n+l-1} \|e_i\| + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&\leq \|x_n - v\| + \sum_{i=n}^{n+l-1} \|e_i\| + \varepsilon.
\end{aligned}$$

Theorem 6.4 of [11], over the space $X := \overline{B}_H(x)$, then yields the existence of an $n \leq \Delta_0(P_\varepsilon, \varepsilon, g)$ such that $\|x_i - x_j\| < \varepsilon$ for all $i, j \in [n; n+g(n)]$. To see that for the corresponding $n \leq \Delta(\varepsilon, g) = \Delta_0(P_{\varepsilon/3}, \varepsilon/3, g)$, we also have $\|x_i - J_{A+B}x_i\| < \varepsilon$ for all $i \in [n; n+g(n)]$ for that n, note that as in the proof of Theorem 6.4 in [11], the point n can actually be chosen such that $x_n \in AF_{\chi_g^M(m,\varepsilon/24)}$ for some m, i.e. there are $w \in \overline{B}_H(x)$ and $u \in (A+B)(w)$ such that in particular $\|x_n - w\|, \|u\| < \varepsilon/9$. Thereby, we have

$$\begin{aligned}
\|J_{A+B}x_n - x_n\| &\leq \|J_{A+B}x_n - J_{A+B}w\| + \|J_{A+B}w - w\| + \|w - x_n\| \\
&\leq \|u\| + 2\|w - x_n\| < \varepsilon/3.
\end{aligned}$$

This in turn implies

$$\|J_{A+B}x_i - x_i\| \leq \|J_{A+B}x_n - x_n\| + 2\|x_n - x_i\| < \varepsilon.$$

for all $i \in [n; n+g(n)]$. □

Remark 3.14. It should be noted that if we are only interested in deriving a rate of metastability for the sequence (x_n), then the maximal monotonicity of $A+B$ is not required in the above proof and it in fact applies, mutatis mutandis, to the exact version of Tseng's method as commonly formulated (and where B is only assumed to be defined on some suitable convex subset of $\mathcal{H}$).

A similar observation that the maximal monotonicity of $A+B$ can be avoided for the exact version of Tseng's method, in the context of a uniform monotonicity assumption, has already been made in [19] (see the discussion at the beginning of Section 2 therein).

A rate of metastability for (z_n) can then be derived by "joining" a rate of metastability for $\|x_n - z_n\| \to 0$ (as e.g. given by a combination of Lemma 3.10 and Lemma 3.5).

Theorem 3.15. *Assume that* $\mathrm{zer}(A+B) \neq \emptyset$ *and let* $x \in \mathrm{zer}(A+B)$. *Let* $d \in \mathbb{N}$ *be such that* $\|x_0 - x\| \leq d$. *Assume that* $\overline{B}_H(x)$ *is totally bounded with a modulus of total boundedness* γ. *Then for any* $\varepsilon > 0$ *and* $g : \mathbb{N} \to \mathbb{N}$*:*

$$\exists n \leq \widehat{\Delta}(\varepsilon, g)\ \forall i, j \in [n; n+g(n)]\, (\|x_i - x_j\|, \|x_i - z_i\|, \|x_i - J_{A+B}x_i\| < \varepsilon)$$

where

$$\widehat{\Delta}(\varepsilon, g) := \max\left\{\Delta(\varepsilon, \tilde{g}_n), \psi(\varepsilon, \bar{g}) \mid n \leq \psi(\varepsilon, \bar{g})\right\},$$

with $\tilde{g}_n(m) := \max\{n, m\} + g(\max\{n, m\})$ *and*

$$\bar{g}(n) := \max\{n, \Delta(\varepsilon, \tilde{g}_n)\} + \max\{g(\max\{n, m\}) \mid m \leq \Delta(\varepsilon, \tilde{g}_n)\}.$$

Here, $\Delta(\varepsilon, g)$ *is as in Theorem 3.13 and* $\psi(\varepsilon, g) := \varphi_{\alpha(\varepsilon/2)}(\varepsilon/2, g)$ *with* φ *as in Lemma 3.10.*

Proof. The proof, together with the bound given above, is similar to that of Theorem 5.8 in [10].[4] Concretely, first note that Lemma 3.10 yields that there exists an $n \leq \psi(\varepsilon, g)$ such that $n \geq \alpha(\varepsilon/2)$ and $\|x_i - \tilde{z}_i\| < \varepsilon/2$ for any $i \in [n; n+g(n)]$. As $n \geq \alpha(\varepsilon/2)$, Lemma 3.5 implies that $\|z_i - \tilde{z}_i\| < \varepsilon/2$ for all $i \geq n$ so that we have combined that $\|x_i - z_i\| < \varepsilon$ for all $i \in [n; n+g(n)]$. We now get that $\widehat{\Delta}(\varepsilon, g)$ is a joint rate of metastability as follows: Let $n_0 \leq \psi(\varepsilon, \overline{g})$ be such that

$$\forall i \in [n_0; n_0 + \bar{g}(n_0)]\, (\|x_i - z_i\| < \varepsilon)$$

and let $n_1 \leq \Delta(\varepsilon, \tilde{g}_{n_0})$ be such that

$$\forall i, j \in [n_1; n_1 + \tilde{g}_{n_0}(n_1)]\, (\|x_i - x_j\|, \|x_i - J_{A+B}x_i\| < \varepsilon).$$

Define $n := \max\{n_0, n_1\}$. Then clearly $n \geq n_0, n_1$. Further, we have

$$\begin{aligned} n_0 + \bar{g}(n_0) &\geq \bar{g}(n_0) \\ &= \max\{n_0, \Delta(\varepsilon, \tilde{g}_{n_0})\} + \max\{g(\max\{n_0, m\} \mid m \leq \Delta(\varepsilon, \tilde{g}_{n_0})\} \\ &\geq \max\{n_0, n_1\} + g(\max\{n_0, n_1\}) = n + g(n) \end{aligned}$$

as well as

$$n_1 + \tilde{g}_{n_0}(n_1) \geq \tilde{g}_{n_0}(n_1) = \max\{n_0, n_1\} + g(\max\{n_0, n_1\}) = n + g(n).$$

This shows that

$$[n; n+g(n)] \subseteq [n_0; n_0 + \bar{g}(n_0)] \cap [n_1; n_1 + \tilde{g}_{n_0}(n_1)]$$

and so $\|x_i - x_j\|, \|x_i - J_{A+B}x_i\|, \|x_i - z_i\| < \varepsilon$ for all $i, j \in [n; n+g(n)]$ follows immediately. $\square$

Remark 3.16. The above Theorem 3.15 then in particular provides a fully finitary version in the sense of Tao [18] of the convergence result from Theorem 3.1, (1), in finite dimensional spaces since metastability is elementarily (albeit noneffectively) equivalent to the Cauchy property of the sequence in question. In that way, Theorem 3.15 immediately yields that (x_n) is Cauchy and hence convergent as well as that (z_n) converges to the same limit and that this limit is a fixed point of J_{A+B} (as that mapping is in particular nonexpansive and hence continuous), i.e. it is a zero of $A+B$.

[4] Note that [10] is the arXiv-version of the paper [11], containing further supplementary material.

Remark 3.17. An alternative route for deriving a separate rate of metastability for the sequences $(\tilde{z}_n)$ and (z_n) is to employ a novel generalized variant of Fejér monotonicity together with abstract results for deriving rates of metastability (similar to [11]) as developed in [13]. Concretely, note that for any $\varepsilon > 0$, $g : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$ as well as $(x^*, y^*) \in A + B$ such that $\|x^* - x\| \leq H$ and

$$\|y^*\| < \frac{\varepsilon^2}{16(g(n))^2 bH}$$

for a suitable H defined relative to a solution $x \in \mathrm{zer}(A + B)$ as in Lemma 3.7, we get

$$\begin{aligned}
\|\tilde{z}_{n+l} - x^*\| &\leq \|x_{n+l} - x^*\| + \|x_{n+l} - \tilde{z}_{n+l}\| \\
&< \|x_n - x^*\| + \sum_{i=n}^{n+l-1} \|e_i\| + \|x_{n+l} - \tilde{z}_{n+l}\| + \frac{\varepsilon}{2} \\
&\leq \|\tilde{z}_n - x^*\| + \sum_{i=n}^{n+l-1} \|e_i\| + \|x_n - \tilde{z}_n\| + \|x_{n+l} - \tilde{z}_{n+l}\| + \frac{\varepsilon}{2}
\end{aligned}$$

for any $l \leq g(n)$ using Lemma 3.8. If then n is such that $n \geq \alpha(\varepsilon/6)$, so that $\sum_{i=n}^{n+l-1} \|e_i\| < \varepsilon/6$ by Lemma 3.2, and

$$\forall i \in [n; n + g(n)] \left(\|x_i - \tilde{z}_i\| < \varepsilon/6\right),$$

we get $\|\tilde{z}_{n+l} - x^*\| < \|\tilde{z}_n - x^*\| + \varepsilon$ for any such ε, g, n as well as suitable $(x^*, y^*) \in A + B$ as above. Hence, if we define

$$x^* \in AF_\varepsilon :\equiv \exists y^* \in (A + B)(x^*) \left(\|y^*\| < \varepsilon\right)$$

for $x^* \in \overline{B}_H(x)$ similar to before as well as

$$S(n, \varepsilon, g) :\equiv \forall i \in [n; n + g(n)] \left(\|x_i - \tilde{z}_i\| < \varepsilon\right) \text{ and } n \geq \alpha(\varepsilon)$$

we actually have shown that

$$\begin{gathered}
\forall \varepsilon > 0 \ \forall g \in \mathbb{N}^{\mathbb{N}} \ \forall x^* \in X \\
\left(x^* \in AF_{\chi(n,\varepsilon,g)} \text{ and } S(n, \varepsilon/6, g) \to \forall l \leq g(n) \left(\|\tilde{z}_{n+l} - x^*\| < \|\tilde{z}_n - x^*\| + \varepsilon\right)\right)
\end{gathered}$$

for $\chi(n, \varepsilon, g) := \varepsilon^2/16(g(n))^2 bH$, i.e. we have shown that $(\tilde{z}_n)$ is uniformly locally S-relativized metastable Fejér monotone w.r.t. the approximate solution sets AF_ε in the sense of Definition 4.6 of [13].[5] Further, note that by Lemma 3.10, there exists a function $\varphi_N(\varepsilon, g)$ such that

$$\forall \varepsilon > 0 \ \forall g \in \mathbb{N}^{\mathbb{N}} \ \exists n_0 \leq \varphi_{\alpha(\varepsilon)}(\varepsilon, g) \ \forall n \in [n_0; n_0 + g(n_0)] \left(\tilde{z}_n \in AF_\varepsilon \text{ and } S(n, \varepsilon, g)\right).$$

So the approach followed in [13] can be applied to construct a rate of metastability for $(\tilde{z}_n)$ also in this case. In particular, a rate of metastability for (z_n) can then be derived as before.

3.3. **Rates of convergence under a uniform monotonicity assumption.** We now give our second quantitative rendering of Theorem 3.1 in the form of a rate of convergence for the sequence (x_n) under the additional assumption that the operator $A + B$ is uniformly monotone at a zero $x \in \mathrm{zer}(A + B)$, which in particular follows already if A or B are uniformly monotone.

To phrase this uniform monotonicity assumption in a suitable, quantitative, way, we here rely on the notion of a modulus of uniform monotonicity at zero as developed in [9].

[5] Note that, similarly as in [11], the notions therein actually work with natural numbers as representatives for errors ε via expressions of the form $1/(k + 1)$.

Definition 3.18 (Compare Definition 10 in [9])**.** Let $x \in \mathrm{zer}(A+B)$ and let $D \subseteq \mathcal{H}$ be a bounded set with $x \in D$ such that $\|x-y\| \leq K$ for all $y \in D$. Then, a function $\Theta : (0,\infty) \to (0,\infty)$ is a modulus of uniform monotonicity at the zero x for the operator $A+B$ on the bounded subset D if

$$\forall \varepsilon > 0 \ \forall y \in D \ \forall u \in (A+B)(y) \left(\|x-y\| \in [\varepsilon, K] \to \langle y-x, u\rangle \geq \Theta(\varepsilon)\right).$$

This latter result extends the previous quantitative analysis of Tseng's method under uniform monotonicity assumptions carried out in [19] to this inexact and parametrized version (where it again should be noted however that the assumption of the totality of the operator B is relaxed in [19]).

Theorem 3.19. *Assume that* $\mathrm{zer}(A+B) \neq \emptyset$ *and let* $x \in \mathrm{zer}(A+B)$*. Let* $d \in \mathbb{N}$ *be such that* $\|x_0 - x\| \leq d$*. Assume we have some modulus* Θ *of uniform monotonicity at the zero* x *for the operator* $A+B$ *on* $\overline{B}_H(x)$*, i.e.*

$$\forall \varepsilon > 0 \ \forall y \in \overline{B}_H(x) \ \forall u \in (A+B)(y) \left(\|x-y\| \in [\varepsilon, H] \to \langle y-x, u\rangle \geq \Theta(\varepsilon)\right).$$

Then (x_n) *converges to* x *with a rate of convergence* ρ*, i.e.*

$$\forall \varepsilon > 0 \, \forall n \geq \rho(\varepsilon) \left(\|x_n - x\| < \varepsilon\right),$$

where $\rho(\varepsilon) := \varphi'(\mu(\varepsilon/2), \alpha(\varepsilon/2))$ *for*

$$\varphi'(\varepsilon, N) := N + \left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil \text{ and } \mu(\varepsilon) := \min\left\{\frac{\varepsilon}{2}, \frac{\Theta(\varepsilon/2)}{H}\right\}.$$

Here: $L := d + (4b+2)R$ *and* $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$ *as well as* $H := 2L + bN_0 + (2+bk)M_0 + 3d$*.*

Proof. We apply the general and abstract macros for deriving rates of convergence for quasi-Fejér monotone sequences under a metric regularity assumption as developed in [12, 15] (with [12] phrased "only" for Fejér monotone sequences, while [15] provides the rather trivial modifications required to treat quasi-Fejér monotone sequences). Concretely, to instantiate said results, we define

$$F : \overline{B}_H(x) \to [0, +\infty], v \mapsto \inf_{w \in \overline{B}_H(x)} \max\{\|v-w\|, \inf_{u \in (A+B)(w)} \|u\|\}.$$

Note then first that $\mathrm{zer}F = \mathrm{zer}(A+B)$ as clearly for $v \in \mathrm{zer}(A+B)$, we have $F(v) = 0$ and conversely, if $F(v) = 0$, there are sequences $(w_n) \subseteq \overline{B}_H(x)$ and (u_n) with $u_n \in (A+B)(w_n)$ such that $w_n \to v$ and $\|u_n\| \to 0$. As A is maximally monotone, the graph of A is closed and as B is continuous and single-valued on a closed domain, its graph is also closed and so $A+B$ has a closed graph. We hence have $0 \in (A+B)(v)$. Further, by Lemma 3.11 (together with Lemmas 3.7 and 3.3), φ' satisfies

$$\exists n \in [N; \varphi'(\varepsilon, N)] \left(F(x_n) < \varepsilon\right)$$

for all $\varepsilon > 0$ and all $N \in \mathbb{N}$. Also, Lemma 3.3 yields that (x_n) is quasi-Fejér monotone w.r.t. $\mathrm{zer}F$. Lastly, μ is a modulus of regularity for F w.r.t. $\mathrm{zer}F$ and $\overline{B}_H(x)$ in the sense of [12] (and even a modulus of uniqueness, see Definition 1.1 in [12] and the references given there), that is for all $\varepsilon > 0$ and $v \in \overline{B}_H(x)$:

$$F(v) < \mu(\varepsilon) \text{ implies } \|v-x\| < \varepsilon.$$

To see that, let $F(v) < \mu(\varepsilon)$. Thus, there exists a $w \in \overline{B}_H(x)$ and a $u \in (A+B)(w)$ such that $\|v-w\| < \varepsilon/2$ and $\|u\| < \Theta(\varepsilon/2)/H$. In particular, we have

$$\langle w-x, u\rangle \leq \|w-x\| \, \|u\| \leq H \, \|u\| < \Theta(\varepsilon/2)$$

and so by the properties of Θ we have $\|w - x\| < \varepsilon/2$. In particular, we have $\|v - x\| < \varepsilon$. The rates then follow from (the proof of) Theorem 3.2 in [15], over the space $X = \overline{B}_H(x)$. □

Bootstrapped on this rate of convergence for (x_n), we can now also give a rate of convergence for the auxiliary sequence (z_n).

Theorem 3.20. *Assume that* $\mathrm{zer}(A+B) \neq \emptyset$ *and let* $x \in \mathrm{zer}(A+B)$. *Let* $d \in \mathbb{N}$ *be such that* $\|x_0 - x\| \leq d$. *Assume we have some modulus* Θ *of uniform monotonicity at the zero* x *for the operator* $A + B$ *on* $\overline{B}_H(x)$, *i.e.*

$$\forall \varepsilon > 0 \ \forall y \in \overline{B}_H(x) \ \forall u \in (A+B)(y) \left(\|x - y\| \in [\varepsilon, H] \to \langle y - x, u \rangle \geq \Theta(\varepsilon) \right).$$

Then (z_n) *converges to* x *and further*

$$\forall \varepsilon > 0 \ \forall n \geq \rho'(\varepsilon) \left(\|z_n - x\| < \varepsilon \right),$$

where

$$\rho'(\varepsilon) := \max\{\rho(\varepsilon/3\sqrt{2}k), \alpha'(\varepsilon^2/18k^2), \alpha(\varepsilon/3)\})$$

with $\alpha'(\varepsilon) := \max\{\alpha(\varepsilon/4L), \alpha(\sqrt{\varepsilon}/\sqrt{2})\}$ *and* ρ *as well as the other constants as in Theorem 3.19.*

Proof. Let $\varepsilon > 0$ be given. Note that α' is a rate for ε_n converging to 0, with $\varepsilon_n = 2L\|e_n\| + \|e_n\|^2$ as in Lemma 3.4. Lemma 3.4 then yields that

$$\|\tilde{z}_n - x_n\|^2 \leq k^2 \left(\|x_n - x\|^2 + \varepsilon_n \right) < k^2(\varepsilon^2/18k^2 + \varepsilon^2/18k^2)$$

and so $\|\tilde{z}_n - x_n\| < \varepsilon/3$ for any $n \geq \rho'(\varepsilon)$. By Lemma 3.5 and Theorem 3.19, we get

$$\begin{aligned} \|z_n - x\| &\leq \|z_n - \tilde{z}_n\| + \|\tilde{z}_n - x_n\| + \|x_n - x\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for any such n. □

Remark 3.21. Similar as in Remark 3.17, also the rates of convergence for (z_n) presented in the above Theorem 3.20 could actually have been, alternatively, obtained by applying the novel generalized variant of Fejér monotonicity together with abstract results for deriving rates of convergence (similar to [12]) as developed in [13]. Concretely, as follows by the discussions in Remark 3.17, the sequence $(\tilde{z}_n)$ satisfies

$$\|\tilde{z}_{n+l} - x\| \leq \|\tilde{z}_n - x\| + \sum_{i=n}^{n+l-1} \|e_i\| + \|x_n - \tilde{z}_n\| + \|x_{n+l} - \tilde{z}_{n+l}\|$$

for any $n, l \in \mathbb{N}$ and any $x \in \mathrm{zer}(A+B)$. If then n is such that $\sum_{i=n}^{\infty} \|e_i\| < \varepsilon/3$ and $\|x_{n+l} - \tilde{z}_{n+l}\| < \varepsilon/3$ for any $l \in \mathbb{N}$, then $\|\tilde{z}_{n+l} - x\| < \|\tilde{z}_n - x\| + \varepsilon$ for any $l \in \mathbb{N}$. Hence, if we define

$$S(n, \varepsilon) :\equiv n \geq \max\{\rho(\varepsilon/\sqrt{2}k), \alpha'(\varepsilon^2/2k^2), \alpha(\varepsilon)\}$$

for ρ as in Theorem 3.19, we actually have shown (recall also the argument in the proof of Theorem 3.20) that

$$\begin{gathered} \forall \varepsilon > 0 \ \forall x \in \mathrm{zer}(A+B) \ \forall n \in \mathbb{N} \\ \left(S(n, \varepsilon/3) \to \forall l \in \mathbb{N} \left(\|\tilde{z}_{n+l} - x\| < \|\tilde{z}_n - x\| + \varepsilon \right) \right), \end{gathered}$$

i.e. we have shown that $(\tilde{z}_n)$ is uniformly locally S-relativized Fejér monotone w.r.t. $\mathrm{zer}(A+B)$ in the sense of Definition 3.1 of [13].[6] As further discussed in Remark 3.17, $(\tilde{z}_n)$ contains

[6] Again, note that the notion therein actually works with natural numbers as representatives for errors via expressions of the form $1/(k+1)$.

approximate zeros of the operator $A+B$ and together with a modulus of uniform monotonicity at the zero x for the operator $A+B$, which can be used to construct a modulus of regularity in the sense of [12] as before, the corresponding results on rates from [13] (see in particular Theorem 3.4) can be applied to get a rate of convergence for $(\tilde{z}_n)$. A rate of convergence for (z_n) then follows using Lemma 3.5.

4. Applications to primal-dual splitting methods for monotone inclusions

In this section, we now apply our abstract and general results established before to the primal-dual algorithm devised by Combettes and Pesquet [4] to solve monotone inclusions for operators which are composed of set-valued and Lipschitzian operators generated by a mix of various operations such as (parallel) sums and linear compositions. In particular, the method achieves a "wide" split of the problem where all the Lipschitzian components are processed in a forward style and all the set-valued operators are processed in a backward style via their resolvents.

Concretely, the problem approached in the work of Combettes and Pesquet is the following: Given a Hilbert space $\mathcal{H}$, let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and let $C : \mathcal{H} \to \mathcal{H}$ be a monotone and μ-Lipschitzian operator for some $\mu \in (0, \infty)$. Further, let $\mathcal{G}_i$ for $i = 1, \dots, m$, $m \geq 1$, be m-many further Hilbert spaces and, for each such i, let $r_i \in \mathcal{G}_i$, $B_i : \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone and let $D_i : \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be monotone such that D_i^{-1} is ν_i-Lipschitzian for some $\nu_i \in (0, \infty)$. Lastly, let $L_i : \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator for each i. Then, we want to solve the primal inclusion problem

$$\text{(P)} \qquad \text{find } \overline{x} \in \mathcal{H} \text{ s.t. } z \in A\overline{x} + \sum_{i=1}^{m} L_i^*((B_i \Box D_i)(L_i\overline{x} - r_i)) + C\overline{x}$$

together with the dual inclusion problem

$$\text{(D)} \qquad \text{find } \overline{v}_1 \in \mathcal{G}_1, \dots, \overline{v}_m \in \mathcal{G}_m \text{ s.t. } \exists x \in \mathcal{H} \begin{cases} z - \sum_{i=1}^{m} L_i^*\overline{v}_i \in Ax + Cx \\ \forall i \in \{1, \dots, m\}\, (\overline{v}_i \in (B_i \Box D_i)(L_i x - r_i)) . \end{cases}$$

We refer to the comprehensive discussion in [4] for how this very general problem formulation unifies and extends various well-known problems involving composite operators in the literature.

To solve the above problem, the following method was then devised in [4]: given

$$z \in \operatorname{ran}\left(A + \sum_{i=1}^{m} L_i^*(B_i \Box D_i)(L_i(\cdot) - r_i) + C\right)$$

and starting points $x_0 \in \mathcal{H}$ and $(v_{1,0}, \dots, v_{m,0}) \in \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$, iteratively define

$$(+) \qquad \begin{cases} y_{1,n} := x_n - \gamma_n \left(Cx_n + \sum_{i=1}^{m} L_i^* v_{i,n} + a_{1,n}\right), \\ p_{1,n} := J_{\gamma_n A}\left(y_{1,n} + \gamma_n z\right) + b_{1,n}, \\ \text{For } i = 1, \dots, m\text{:} \begin{cases} y_{2,i,n} := v_{i,n} + \gamma_n \left(L_i x_n - D_i^{-1} v_{i,n} + a_{2,i,n}\right), \\ p_{2,i,n} := J_{\gamma_n B_i^{-1}}(y_{2,i,n} - \gamma_n r_i) + b_{2,i,n}, \\ q_{2,i,n} := p_{2,i,n} + \gamma_n \left(L_i p_{1,n} - D_i^{-1} p_{2,i,n} + c_{2,i,n}\right), \\ v_{i,n+1} := v_{i,n} - y_{2,i,n} + q_{2,i,n}, \end{cases} \\ q_{1,n} := p_{1,n} - \gamma_n \left(Cp_{1,n} + \sum_{i=1}^{m} L_i^* p_{2,i,n} + c_{1,n}\right), \\ x_{n+1} := x_n - y_{1,n} + q_{1,n}, \end{cases}$$

where $(a_{1,n})$, $(b_{1,n})$ and $(c_{1,n})$ are absolutely summable sequences in $\mathcal{H}$, $(a_{2,i,n})$, $(b_{2,i,n})$ and $(c_{2,i,n})$ are absolutely summable sequences in $\mathcal{G}_i$ for each $i = 1, \dots, m$, and $(\gamma_n) \subseteq (0, \infty)$ is a sequence of parameters.

As highlighted in [4], the advantage of the above method is that it presents a fully split algorithm for solving the above problem in the sense that it only employs the operators A, C and $(L_i)_{1\leq i\leq m}$, $(B_i)_{1\leq i\leq m}$, $(D_i)_{1\leq i\leq m}$ separately and in a parallel fashion, making use of the single-valued operators C and $(L_i)_{1\leq i\leq m}$, $(D_i^{-1})_{1\leq i\leq m}$ through explicit steps.

The main result established about the algorithm given by (+) is the following:

Theorem 4.1 (Theorem 3.1 in [4]). *Given a Hilbert space $\mathcal{H}$, let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and let $C : \mathcal{H} \to \mathcal{H}$ be a monotone and μ-Lipschitzian operator for some $\mu \in (0, \infty)$. Let $\mathcal{G}_i$ for $i = 1, \dots, m$, $m \geq 1$, be m-many further Hilbert spaces and, for each such i, let $r_i \in \mathcal{G}_i$, $B_i : \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone and let $D_i : \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be monotone such that D_i^{-1} is ν_i-Lipschitzian for some $\nu_i \in (0, \infty)$. Lastly, let $L_i : \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator for each i.*

Given

$$z \in \mathrm{ran}\left(A + \sum_{i=1}^{m} L_i^*(B_i \square D_i)(L_i(\cdot) - r_i) + C\right),$$

consider the sequences defined by (+) with absolutely summable sequences $(a_{1,n})$, $(b_{1,n})$ and $(c_{1,n})$ in $\mathcal{H}$ and $(a_{2,i,n})$, $(b_{2,i,n})$ and $(c_{2,i,n})$ in $\mathcal{G}_i$ for each $i = 1, \dots, m$ and a parameter sequence $(\gamma_n) \subseteq [\varepsilon, (1 - \varepsilon)/\beta']$ for some $\varepsilon \in (0, 1/(\beta' + 1))$ where

$$\beta' := \max\{\mu, \nu_1, \dots, \nu_m\} + \sqrt{\sum_{i=1}^{m} \|L_i\|^2}.$$

Then there are solutions $\overline{x}$ to (P) *and $(\overline{v}_1, \dots, \overline{v}_m)$ to* (D) *such that*

(1) the sequences (x_n) and $(p_{1,n})$ weakly converge to $\overline{x}$,
(2) the sequences $(v_{i,n})$ and $(p_{2,i,n})$ weakly converge to $\overline{v}_i$,
(3) if A or C is uniformly monotone at $\overline{x}$, then (x_n) and $(p_{1,n})$ strongly converge to $\overline{x}$,
(4) if, for some $i = 1, \dots, m$, B_i^{-1} or D_i^{-1} is uniformly monotone at $\overline{v}_i$, then $(v_{i,n})$ and $(p_{2,i,n})$ strongly converge to $\overline{v}_i$.

The crucial observation made in [4] is that the above algorithm given by (+) reduces to an instance of ($*$), i.e. the extension of Tseng's splitting method with error terms and variable parameters, by moving to a suitable composite higher space together with a suitable choice of instantiating operators.

Concretely, over the composite space $\mathcal{K} = \mathcal{H} \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$, we define $M : \mathcal{K} \to 2^{\mathcal{K}}$ by

$$M(x, v_1, \dots, v_m) = (-z + Ax) \times (r_1 + B_1^{-1}v_1) \times \cdots \times (r_m + B_m^{-1}v_m)$$

and $Q : \mathcal{K} \to \mathcal{K}$ by

$$Q(x, v_1, \dots, v_m) = (Cx + L_1^*v_1 + \cdots + L_m^*v_m, -L_1x + D_1^{-1}v_1, \dots, -L_mx + D_m^{-1}v_m).$$

In particular, it holds that

$$\begin{aligned}(M + Q)(x, v_1, \dots, v_m) =&(Cx + L_1^*v_1 + \cdots + L_m^*v_m - z + Ax)\\ &\times (r_1 - L_1x + D_1^{-1}v_1 + B_1^{-1}v_1)\\ &\times \dots\\ &\times (r_m - L_mx + D_m^{-1}v_m + B_m^{-1}v_m).\end{aligned}$$

As shown in [4] (see (3.11) therein), the operator M is maximally monotone and Q is monotone and β'-Lipschitzian for β' as in Theorem 4.1.

Crucially now, setting

$$\begin{cases} \mathbf{x}_n := (x_n, v_{1,n}, \dots, v_{m,n}), \\ \mathbf{y}_n := (y_{1,n}, y_{2,1,n}, \dots, y_{2,m,n}), \\ \mathbf{p}_n := (p_{1,n}, p_{2,1,n}, \dots, p_{2,m,n}), \\ \mathbf{q}_n := (q_{1,n}, q_{2,1,n}, \dots, q_{2,m,n}), \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{a}_n := (a_{1,n}, a_{2,1,n}, \dots, a_{2,m,n}), \\ \mathbf{b}_n := (b_{1,n}, b_{2,1,n}, \dots, b_{2,m,n}), \\ \mathbf{c}_n := (c_{1,n}, c_{2,1,n}, \dots, c_{2,m,n}), \end{cases}$$

the assumptions on the sequences $(a_{1,n}), (b_{1,n})$ and $(c_{1,n})$ as well as $(a_{2,i,n})$, $(b_{2,i,n})$ and $(c_{2,i,n})$ imply that $(\mathbf{a}_n)$, $(\mathbf{b}_n)$ and $(\mathbf{c}_n)$ are absolutely summable and that the iteration (+) reduces to

$$\begin{cases} \mathbf{y}_n := \mathbf{x}_n - \gamma_n \left(Q\mathbf{x}_n + \mathbf{a}_n\right), \\ \mathbf{p}_n := J_{\gamma_n M}\mathbf{y}_n + \mathbf{b}_n, \\ \mathbf{q}_n := \mathbf{p}_n - \gamma_n \left(Q\mathbf{p}_n + \mathbf{c}_n\right), \\ \mathbf{x}_{n+1} := \mathbf{x}_n - \mathbf{y}_n + \mathbf{q}_n, \end{cases}$$

which is an instance of $(*)$ for the operators M and Q. Further, as shown as (3.20) in [4], we get that

$$\begin{aligned} \mathrm{zer}(M+Q) &= \{(\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \mid \overline{x} \text{ solves (P) and } \overline{v}_1, \dots, \overline{v}_m \text{ solve (D) with } x = \overline{x}\} \\ &\subseteq \{(\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \mid \overline{x} \text{ solves (P) and } \overline{v}_1, \dots, \overline{v}_m \text{ solve (D)}\}. \end{aligned}$$

We can now utilize this compositionality to derive a rate of metastability in the context of items (1) and (2) from Theorem 4.1, under a suitable relative compactness assumption, as well as rates of convergence and metastability in the context of items (3) and (4) from Theorem 4.1, under the uniform monotonicity assumption.

4.1. **Rates of metastability under a relative compactness assumption.** We begin with the former, i.e. a rate of metastability in the context of items (1) and (2) in Theorem 4.1.

Theorem 4.2. *Let $\overline{\mathbf{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \mathrm{zer}(M+Q)$. Take $d \geq \|\mathbf{x}_0 - \overline{\mathbf{x}}\|$. Let $b \in \mathbb{N} \setminus \{0\}$ be such that $b \geq 1/\beta'$ for β' as in Theorem 4.1 and let $M_0, N_0 \in \mathbb{N}$ be such that*

$$\|\mathbf{x}_0 - \tilde{\mathbf{p}}_0\|, \|\tilde{\mathbf{y}}_0 - \tilde{\mathbf{p}}_0\| \leq M_0 \text{ and } \|Q\mathbf{x}_0\| \leq N_0$$

with $\tilde{\mathbf{p}}_n, \tilde{\mathbf{y}}_n$ defined similarly as $\tilde{z}_n, \tilde{y}_n$ in Section 3. Further, assume $(\gamma_n) \subseteq [1/k, (1-1/k)/\beta']$. Also, let R_0 be such that

$$\sum_{n=0}^{\infty} \|a_{1,n}\|, \sum_{n=0}^{\infty} \|b_{1,n}\|, \sum_{n=0}^{\infty} \|c_{1,n}\|, \sum_{n=0}^{\infty} \|a_{2,l,n}\|, \sum_{n=0}^{\infty} \|b_{2,l,n}\|, \sum_{n=0}^{\infty} \|c_{2,l,n}\| \leq R_0$$

for all $l \in [1;m]$ and let α_0 be a joint Cauchy modulus for all these series. Lastly, let $\overline{B}_H(\overline{\mathbf{x}})$ be totally bounded with a modulus of total boundedness γ.

Then for any $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Delta(\varepsilon, g)\ \forall i, j \in [n; n+g(n)]\ \forall l \in [1;m]\ (\|x_i - x_j\|, \|v_{l,i} - v_{l,j}\| < \varepsilon)$$

where $\Delta(\varepsilon, g) := \Delta_0(P_{\varepsilon/3}, \varepsilon/3, g)$ for $P_\varepsilon := \gamma(\varepsilon/8) + 1$ and

$$\begin{cases} \Delta_0(0, \varepsilon, g) := 0, \\ \Delta_0(n+1, \varepsilon, g) := \varphi'\left(\chi_g^M(\Delta_0(n, \varepsilon, g), \varepsilon/8), \alpha(\varepsilon/8)\right) \end{cases}$$

with

$$\varphi'(\varepsilon, N) := N + \left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil \text{ and } \chi(n, m, \varepsilon) := \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon^2}{16m^2bH}\right\}$$

as well as

$$\chi_g(n, \varepsilon) := \chi(n, g(n), \varepsilon) \text{ and } \chi_g^M(n, \varepsilon) := \min\{\chi_g(i, \varepsilon) \mid i \leq n\}$$

and with $L := d + (4b+2)R$ *and* $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$ *as well as* $H := 2L + bN_0 + (2+bk)M_0 + 3d$. *Further,* $R := (m+1)R_0$ *and* $\alpha(\varepsilon) := \alpha_0(\varepsilon/6b(m+1))$.

Proof. We apply Theorem 3.13 by appropriately instantiating the moduli. As most of the instantiations are immediate, we here just note the following: By the properties of R_0 we have

$$\begin{aligned}\sum_{n=k}^{\infty} \|\mathbf{a}_n\| &= \sum_{n=k}^{\infty} \sqrt{\|a_{1,n}\|^2 + \sum_{l=1}^{m} \|a_{2,l,n}\|^2} \\ &\leq \sum_{n=k}^{\infty} \left(\|a_{1,n}\| + \sum_{l=1}^{m} \|a_{2,l,n}\| \right) \\ &= \sum_{n=k}^{\infty} \|a_{1,n}\| + \sum_{l=1}^{m} \sum_{n=k}^{\infty} \|a_{2,l,n}\|\end{aligned}$$

and similarly for $\|\mathbf{b}_n\|$, $\|\mathbf{c}_n\|$. By that, we get

$$\sum_{n=0}^{\infty} \|\mathbf{a}_n\|, \sum_{n=0}^{\infty} \|\mathbf{b}_n\|, \sum_{n=0}^{\infty} \|\mathbf{c}_n\| \leq (m+1)R_0 = R$$

and further we have

$$\sum_{n=\alpha(\varepsilon)}^{\infty} \|\mathbf{a}_n\|, \sum_{n=\alpha(\varepsilon)}^{\infty} \|\mathbf{b}_n\|, \sum_{n=\alpha(\varepsilon)}^{\infty} \|\mathbf{c}_n\| < \varepsilon/6b$$

for all $\varepsilon > 0$ which entails (as $b \geq 1$) that α is a Cauchy modulus for $\sum_{n=0}^{\infty} d_n < \infty$ with $d_n := 3b\|\mathbf{a}_n\| + 2\|\mathbf{b}_n\| + b\|\mathbf{c}_n\|$. Applying Theorem 3.13 then yields the existence of an $n \leq \Delta(\varepsilon, g)$ such that $\|\mathbf{x}_i - \mathbf{x}_j\| < \varepsilon$, for all $i, j \in [n; n+g(n)]$ and this yields

$$\|x_i - x_j\|, \|v_{l,i} - v_{l,j}\| < \varepsilon$$

for all $l = 1, \ldots, m$ and $i, j \in [n; n+g(n)]$ (recall that the product norm bounds the individual norms, as discussed in Section 2). □

As in Section 3, we can also derive a joint rate of metastability which incorporates the sequences $(p_{1,n})$ and $(p_{2,l,n})$, $l = 1, \ldots, m$. As the result rather immediately follows from Theorem 3.15, we just state it and omit any further details on its proof.

Theorem 4.3. *Let* $\overline{\mathbf{x}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m) \in \mathrm{zer}(M+Q)$. *Take* $d \geq \|\mathbf{x}_0 - \overline{\mathbf{x}}\|$. *Let* $b \in \mathbb{N} \setminus \{0\}$ *be such that* $b \geq 1/\beta'$ *for* β' *as in Theorem 4.1 and let* $M_0, N_0 \in \mathbb{N}$ *be such that*

$$\|\mathbf{x}_0 - \tilde{\mathbf{p}}_0\|, \|\tilde{\mathbf{y}}_0 - \tilde{\mathbf{p}}_0\| \leq M_0 \text{ and } \|Q\mathbf{x}_0\| \leq N_0$$

with $\tilde{\mathbf{p}}_n, \tilde{\mathbf{y}}_n$ *defined similarly as* $\tilde{z}_n, \tilde{y}_n$ *in Section 3. Further, assume* $(\gamma_n) \subseteq [1/k, (1-1/k)/\beta']$. *Also, let* R_0 *be such that*

$$\sum_{n=0}^{\infty} \|a_{1,n}\|, \sum_{n=0}^{\infty} \|b_{1,n}\|, \sum_{n=0}^{\infty} \|c_{1,n}\|, \sum_{n=0}^{\infty} \|a_{2,l,n}\|, \sum_{n=0}^{\infty} \|b_{2,l,n}\|, \sum_{n=0}^{\infty} \|c_{2,l,n}\| \leq R_0$$

for all $l \in [1; m]$ *and let* α_0 *be a joint Cauchy modulus for all these series. Lastly, let* $\overline{B}_H(\overline{\mathbf{x}})$ *be totally bounded with a modulus of total boundedness* γ.

Then for any $\varepsilon > 0$ *and* $g : \mathbb{N} \to \mathbb{N}$*:*

$$\exists n \leq \widehat{\Delta}(\varepsilon, g)\ \forall i, j \in [n; n + g(n)]\ \forall l \in [1; m]$$
$$(\|x_i - x_j\|, \|v_{l,i} - v_{l,j}\|, \|p_{1,i} - x_i\|, \|p_{2,l,i} - v_{l,i}\| < \varepsilon)$$

where

$$\widehat{\Delta}(\varepsilon, g) := \max\{\Delta(\varepsilon, \tilde{g}_n), \psi(\varepsilon, \bar{g}) \mid n \leq \psi(\varepsilon, \bar{g})\},$$

with $\tilde{g}_n(m) := \max\{n, m\} + g(\max\{n, m\})$ *and*

$$\bar{g}(n) := \max\{n, \Delta(\varepsilon, \tilde{g}_n)\} + \max\{g(\max\{n, m\}) \mid m \leq \Delta(\varepsilon, \tilde{g}_n)\}.$$

Here, $\Delta(\varepsilon, g)$ *and* α *are as in Theorem 4.2 and* $\psi(\varepsilon, g) := \varphi_{\alpha(\varepsilon/2)}(\varepsilon/2, g)$ *with* φ *as in Lemma 3.10.*

4.2. **Rates of convergence and metastability under a uniform monotonicity assumption.** In this last section, we are now concerned with providing quantitative information for the convergence of the iteration $(+)$ in the context of uniform monotonicity assumptions. While this iteration is reduced to the extended form of Tseng's splitting method $(*)$ as discussed and utilized before, the quantitative implications regarding rates of convergence resulting from this are reasonably subtle. Concretely, while we were able to obtain rates of convergence for the iteration $(*)$ in Theorem 3.19, this rested on the quasi-Fejér monotonicity of the iteration as a whole regarding the solution and while that form of monotonicity still holds for the composite package $\mathbf{x}_n$, each component sequence however does not, to our knowledge, satisfy that monotonicity. As a consequence, we are "only" able to obtain rates of metastability for the component sequences in the context of partial uniform monotonicity assumptions for the operators $A + B$ and $B_i^{-1} + D_i^{-1}$. However, if all these sums $A + B$ and $B_i^{-1} + D_i^{-1}$ of the component operators are uniformly monotone at the same time, then this transfers to the uniform monotonicity of the defined operator $M + Q$ and so rates of convergence for the sequences $(\mathbf{x}_n)$ and $(\mathbf{p}_n)$ towards the solution tuple can be derived nonetheless. For that, we begin with the following results which illustrate the relationship between $M + Q$ and the constituting operators $A + B$ and $B_i^{-1} + D_i^{-1}$.

Lemma 4.4. *Let* $((x, \underline{v}), (\alpha, \underline{w})) \in M + Q$ *and* $((x', \underline{v'}), (\alpha', \underline{w'})) \in M + Q$ *with*

$$\alpha = Cx + a + L_1^* v_1 + \cdots + L_m^* v_m - z \text{ and } w_i = r_i - L_i x + D_i^{-1} v_i + b_i$$

for $a \in Ax$ *and* $b_i \in B_i^{-1} v_i$ *and similarly for* α' *and* w_i' *for* $a' \in Ax'$ *and* $b_i' \in B_i^{-1} v_i'$*. Then it holds that*

$$\langle (x - x', \underline{v} - \underline{v'}), (\alpha - \alpha', \underline{w} - \underline{w'}) \rangle \geq \langle x - x', a + Cx - (a' + Cx') \rangle$$

and

$$\langle (x - x', \underline{v} - \underline{v'}), (\alpha - \alpha', \underline{w} - \underline{w'}) \rangle \geq \langle v_i - v_i', b_i + D_i^{-1} v_i - (b_i' + D_i^{-1} v_i') \rangle$$

for any $i = 1, \ldots, m$.

Proof. By definition of $M + Q$ and the inner product on the product space $\mathcal{K}$, we have

$$\begin{aligned}
&\langle (x - x', \underline{v} - \underline{v'}), (\alpha - \alpha', \underline{w} - \underline{w'}) \rangle \\
&\quad \geq \langle x - x', a + Cx - (a' + Cx') \rangle + \sum_{i=1}^m \langle x - x', L_i^* v_i - L_i^* v_i' \rangle \\
&\quad + \sum_{i=1}^m \langle v_i - v_i', L_i x' - L_i x \rangle + \sum_{i=1}^m \langle v_i - v_i', b_i + D_i^{-1} v_i - (b_i' + D_i^{-1} v_i') \rangle.
\end{aligned}$$

As L_i^* is the adjoint of L_i, we get that

$$\sum_{i=1}^{m} \left(\langle x - x', L_i^* v_i - L_i^* v_i' \rangle + \langle v_i - v_i', L_i x' - L_i x \rangle \right) = 0$$

so that

$$\begin{aligned} &\langle (x - x', \underline{v} - \underline{v'}), (\alpha - \alpha', \underline{w} - \underline{w'}) \rangle \\ &\qquad \geq \langle x - x', a + Cx - (a' + Cx') \rangle + \sum_{i=1}^{m} \langle v_i - v_i', b_i + D_i^{-1} v_i - (b_i' + D_i^{-1} v_i') \rangle. \end{aligned}$$

Now, both $A + C$ and all $B_i^{-1} + D_i^{-1}$ are monotone operators so that we obtain $\langle x - x', a + Cx - (a' + Cx') \rangle \geq 0$ as well as $\langle v_i - v_i', b_i + D_i^{-1} v_i - (b_i' + D_i^{-1} v_i') \rangle \geq 0$ for any $i = 1, \dots, m$ from which the claimed inequalities immediately follow. □

The following lemma, which allows us to transfer moduli of uniform monotonicity from the composing operators to the operator $M + Q$ is then immediate.

Lemma 4.5. *Let $\overline{\mathbf{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \mathrm{zer}(M + Q)$. Assume that $A + C$ is uniformly monotone at $\overline{x}$ on the set $\overline{B}_H(\overline{x})$ with modulus Θ_0, i.e.*

$$\begin{gathered} \forall \varepsilon > 0 \ \forall v \in (A + C)(\overline{x}) \ \forall y \in \overline{B}_H(\overline{x}) \ \forall u \in (A + C)(y) \\ (\|\overline{x} - y\| \in [\varepsilon, H] \to \langle y - \overline{x}, u - v \rangle \geq \Theta_0(\varepsilon)) \end{gathered}$$

and, for each $i = 1, \dots, m$, let $B_i^{-1} + D_i^{-1}$ be uniformly monotone at $\overline{v}_i$ on the set $\overline{B}_H(\overline{v}_i)$ with modulus Θ_i, i.e.

$$\begin{gathered} \forall \varepsilon > 0 \ \forall v \in (B_i^{-1} + D_i^{-1})(\overline{x}) \ \forall y \in \overline{B}_H(\overline{v}_i) \ \forall u \in (B_i^{-1} + D_i^{-1})(y) \\ (\|\overline{v}_i - y\| \in [\varepsilon, H] \to \langle y - \overline{v}_i, u - v \rangle \geq \Theta_i(\varepsilon)) \,. \end{gathered}$$

Then $M + Q$ is uniformly monotone at the zero $\overline{\mathbf{x}}$ on the set $\overline{B}_H(\overline{\mathbf{x}}) \subseteq \overline{B}_H(\overline{x}) \times \overline{B}_H(\overline{v}_1) \times \cdots \times \overline{B}_H(\overline{v}_m)$ with a modulus

$$\Theta(\varepsilon) := \min\{\Theta_0(\varepsilon/\sqrt{m+1}), \Theta_1(\varepsilon/\sqrt{m+1}), \dots, \Theta_m(\varepsilon/\sqrt{m+1})\},$$

i.e.

$$\forall \varepsilon > 0 \ \forall \mathbf{y} \in \overline{B}_H(\overline{\mathbf{x}}) \ \forall \mathbf{u} \in (M + Q)(\mathbf{y}) \left(\|\overline{\mathbf{x}} - \mathbf{y}\| \in [\varepsilon, H] \to \langle \mathbf{y} - \overline{\mathbf{x}}, \mathbf{u} \rangle \geq \Theta(\varepsilon) \right).$$

Remark 4.6. In the above Lemma 4.5, and in the following, we are utilizing the notion of a modulus of uniform monotonicity at a point (cf. Definition 1.6 in [19]), which is slightly more general than the notion of a modulus of uniform monotonicity at a zero given in Definition 3.18. For example, the modulus Θ_0 witnessing that $A + C$ is uniformly monotone at $\overline{x}$ on the set $\overline{B}_H(\overline{x})$ now also quantifies over all $v \in (A + C)(\overline{x})$ as $\overline{x}$ is not necessarily a zero of $A + C$ anymore. A similar remark applies to the other moduli Θ_i.

We now begin with the result where we get a rate of metastability for each composite iteration individually under the respective uniform monotonicity assumption for the operators corresponding to that component.

Theorem 4.7. *Let $\overline{\mathbf{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \mathrm{zer}(M + Q)$. Take $d \geq \|\mathbf{x}_0 - \overline{\mathbf{x}}\|$. Let $b \in \mathbb{N} \setminus \{0\}$ be such that $b \geq 1/\beta'$ for β' as in Theorem 4.1 and let $M_0, N_0 \in \mathbb{N}$ be such that $\|\mathbf{x}_0 - \tilde{\mathbf{p}}_0\|, \|\tilde{\mathbf{y}}_0 - \tilde{\mathbf{p}}_0\| \leq$*

M_0 and $\|Q\mathbf{x}_0\| \leq N_0$ with $\tilde{\mathbf{p}}_n, \tilde{\mathbf{y}}_n$ defined similarly as $\tilde{z}_n, \tilde{y}_n$ in Section 3. Further, assume $(\gamma_n) \subseteq [1/k, (1-1/k)/\beta']$. Also, let R_0 be such that

$$\sum_{n=0}^{\infty} \|a_{1,n}\|, \sum_{n=0}^{\infty} \|b_{1,n}\|, \sum_{n=0}^{\infty} \|c_{1,n}\|, \sum_{n=0}^{\infty} \|a_{2,l,n}\|, \sum_{n=0}^{\infty} \|b_{2,l,n}\|, \sum_{n=0}^{\infty} \|c_{2,l,n}\| \leq R_0$$

for all $l \in [1;m]$ and let α_0 be a joint Cauchy modulus for all these series.

(1) *If $A+C$ is uniformly monotone at $\overline{x}$ with modulus Θ, i.e.*

$$\forall \varepsilon > 0\ \forall v \in (A+C)(\overline{x})\ \forall y \in \overline{B}_H(\overline{x})\ \forall u \in (A+C)(y)$$
$$(\|\overline{x} - y\| \in [\varepsilon, H] \to \langle y - \overline{x}, u - v\rangle \geq \Theta(\varepsilon))$$

then (x_n) and $(p_{1,n})$ converge to $\overline{x}$ and further for any $\varepsilon > 0$ and any $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n_0 \leq \rho(\varepsilon, g)\ \forall n \in [n_0; n_0 + g(n_0)]\ (\|x_n - \overline{x}\|, \|p_{1,n} - \overline{x}\| < \varepsilon).$$

(2) *For any $l \in [1;m]$, if $B_l^{-1} + D_l^{-1}$ is uniformly monotone at $\overline{v}_l$ with modulus Θ, i.e.*

$$\forall \varepsilon > 0\ \forall v \in (B_l^{-1} + D_l^{-1})(\overline{x})\ \forall y \in \overline{B}_H(\overline{v}_l)\ \forall u \in (B_l^{-1} + D_l^{-1})(y)$$
$$(\|\overline{v}_l - y\| \in [\varepsilon, H] \to \langle y - \overline{v}_l, u - v\rangle \geq \Theta(\varepsilon)).$$

then $(v_{l,n})$ and $(p_{2,l,n})$ converge to $\overline{v}_l$ and further for any $\varepsilon > 0$ and any $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n_0 \leq \rho(\varepsilon, g)\ \forall n \in [n_0; n_0 + g(n_0)]\ (\|v_{l_n} - \overline{v}_l\|, \|p_{2,l,n} - \overline{v}_l\| < \varepsilon).$$

In any of these cases, we have

$$\rho(\varepsilon, g) := \widehat{g}^{\left(\left\lceil \frac{K(2k-1)^2}{\tilde{\varepsilon}^2}\right\rceil\right)}(\alpha(\varepsilon/3))$$

for $\widehat{g}(n) := n + g(n) + 1$ and $\tilde{\varepsilon} := \min\{\Theta(\varepsilon/6)/H, \varepsilon/6\}$. Here: $L := d + (4b+2)R$ and $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$ as well as $H := 2L + bN_0 + (2+bk)M_0 + 3d$. Further, $R := (m+1)R_0$ and $\alpha(\varepsilon) := \alpha_0(\varepsilon/6b(m+1))$.

Proof. We only discuss (1) as (2) follows similarly. Fix $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$. We have

$$\exists n_0 \in [\alpha(\varepsilon/3); \rho(\varepsilon, g)]\ \ \forall n \in [n_0; n_0 + g(n_0)]\ (\|\mathbf{x}_n - \tilde{\mathbf{p}}_n\|, \|\mathbf{u}_n\| < \tilde{\varepsilon})$$

using Lemma 3.10, where $\mathbf{u}_n$ is defined analogous to Section 3. Applying the Cauchy-Schwartz inequality yields

$$\langle \tilde{\mathbf{p}}_n - \overline{\mathbf{x}}, \mathbf{u}_n\rangle \leq \|\tilde{\mathbf{p}}_n - \overline{\mathbf{x}}\|\|\mathbf{u}_n\| \leq H\|\mathbf{u}_n\| < \Theta(\varepsilon/6)$$

for any such n. As we have $0 \in (M+Q)(\overline{\mathbf{x}})$ and $\mathbf{u}_n \in (M+Q)(\tilde{\mathbf{p}}_n)$ (which follows simply by definition of $\mathbf{u}_n, \tilde{\mathbf{p}}_n$ and M, Q), we get

$$\langle s_n - \overline{x}, a_n + Cs_n - (a' + C\overline{x})\rangle \leq \langle \tilde{\mathbf{p}}_n - \overline{\mathbf{x}}, \mathbf{u}_n\rangle < \Theta(\varepsilon/6)$$

for any such n by Lemma 4.4, where s_n is the first component of $\tilde{\mathbf{p}}_n$ and a_n, a' are such that $a_n + Cs_n \in (A+C)(s_n)$ and $a' + C\overline{x} \in (A+C)(\overline{x})$ as in Lemma 4.4. As Θ is a modulus of uniform monotonicity for $A+C$, we get $\|s_n - \overline{x}\| < \varepsilon/6$ for any such n as $\|s_n - \overline{x}\| \leq \|\tilde{\mathbf{p}}_n - \overline{\mathbf{x}}\| \leq H$. This in particular yields

$$\|x_n - \overline{x}\| \leq \|x_n - s_n\| + \|s_n - \overline{x}\| < \|\mathbf{x}_n - \tilde{\mathbf{p}}_n\| + \varepsilon/6 < \tilde{\varepsilon} + \varepsilon/6 \leq \varepsilon/3 \leq \varepsilon$$

as well as

$$\begin{aligned}
\|p_{1,n} - \overline{x}\| &\leq \|p_{1,n} - s_n\| + \|s_n - x_n\| + \|x_n - \overline{x}\| \\
&\leq \|\mathbf{p}_n - \tilde{\mathbf{p}}_n\| + \|\tilde{\mathbf{p}}_n - \mathbf{x}_n\| + \|x_n - \overline{x}\| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

for any such n, as in particular $n \geq n_0 \geq \alpha(\varepsilon/3)$ and so $\|\mathbf{p}_n - \tilde{\mathbf{p}}_n\| < \varepsilon/3$ by Lemma 3.5 (noting that $z_n, \tilde{z}_n$ therein correspond to $\mathbf{p}_n, \tilde{\mathbf{p}}_n$ here). □

As mentioned before, if all these assumptions are satisfied simultaneously, we get the following result on a rate of convergence:

Theorem 4.8. *Let $\overline{\mathbf{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \mathrm{zer}(M+Q)$. Take $d \geq \|\mathbf{x}_0 - \overline{\mathbf{x}}\|$. Let $b \in \mathbb{N} \setminus \{0\}$ be such that $b \geq 1/\beta'$ for β' as in Theorem 4.1 and let $M_0, N_0 \in \mathbb{N}$ be such that $\|\mathbf{x}_0 - \tilde{\mathbf{p}}_0\|, \|\tilde{\mathbf{y}}_0 - \tilde{\mathbf{p}}_0\| \leq M_0$ and $\|Q\mathbf{x}_0\| \leq N_0$ with $\tilde{\mathbf{p}}_n, \tilde{\mathbf{y}}_n$ defined similarly as $\tilde{z}_n, \tilde{y}_n$ in Section 3. Further, assume $(\gamma_n) \subseteq [1/k, (1-1/k)\beta']$. Also, let R_0 be such that*

$$\sum_{n=0}^{\infty} \|a_{1,n}\|, \sum_{n=0}^{\infty} \|b_{1,n}\|, \sum_{n=0}^{\infty} \|c_{1,n}\|, \sum_{n=0}^{\infty} \|a_{2,l,n}\|, \sum_{n=0}^{\infty} \|b_{2,l,n}\|, \sum_{n=0}^{\infty} \|c_{2,l,n}\| \leq R_0$$

for all $l \in [1;m]$ and let α_0 be a joint Cauchy modulus for all these series. Lastly, let $A+C$ be uniformly monotone at $\overline{x}$ with modulus Θ_0 and, for each $l = 1, \dots, m$, let $B_l^{-1} + D_l^{-1}$ be uniformly monotone at $\overline{v}_l$ with modulus Θ_l, all as in the previous Theorem 4.7.

Then (x_n) converges to $\overline{x}$ and $(v_{l,n})$ to $\overline{v}_l$ with a rate of convergence ρ, i.e.

$$\forall \varepsilon > 0\ \forall n \geq \rho(\varepsilon)\ \forall l \in [1;m]\ (\|x_n - \overline{x}\|, \|v_{l,n} - \overline{v}_l\| < \varepsilon),$$

where $\rho(\varepsilon) := \varphi'(\mu(\varepsilon/2), \alpha(\varepsilon/2))$ for

$$\varphi'(\varepsilon, N) := N + \left\lceil \frac{K(2k-1)^2}{\varepsilon^2} \right\rceil \text{ and } \mu(\varepsilon) := \min\left\{\frac{\varepsilon}{2}, \frac{\Theta(\varepsilon/2)}{H}\right\}.$$

with

$$\Theta(\varepsilon) := \min\{\Theta_0(\varepsilon/\sqrt{m+1}), \Theta_1(\varepsilon/\sqrt{m+1}), \dots, \Theta_m(\varepsilon/\sqrt{m+1})\}.$$

Here: $L := d + (4b+2)R$ for $R := (m+1)R_0$ and $\alpha(\varepsilon) := \alpha_0(\varepsilon/6b(m+1))$ as well as $K := k^2(d^2 + 2L(4b+2)R + (4b+2)^2R^2)$ and $H := 2L + bN_0 + (2+bk)M_0 + 3d$.

Further, $(p_{1,n})$ converges to $\overline{x}$ and $(p_{2,l,n})$ to $\overline{v}_l$ with a rate of convergence ρ', i.e.

$$\forall \varepsilon > 0\ \forall n \geq \rho'(\varepsilon)\ \forall l \in [1;m]\ (\|p_{1,n} - \overline{x}\|, \|p_{2,l,n} - \overline{v}_l\| < \varepsilon),$$

where

$$\rho'(\varepsilon) := \max\{\rho(\varepsilon/3\sqrt{2}k), \alpha'(\varepsilon^2/18k^2), \alpha(\varepsilon/3)\})$$

with ρ and the other constants as before and $\alpha'(\varepsilon) := \max\{\alpha(\varepsilon/4L), \alpha(\sqrt{\varepsilon}/\sqrt{2})\}$.

Proof. We apply Theorems 3.19 and 3.20 by appropriately instantiating the moduli given therein. For that, we mainly proceed similarly to the proof of Theorem 4.2 and hence here only focus on the assumption of a modulus of uniform monotonicity. For that, note that the assumptions on the moduli $\Theta_0, \Theta_1, \dots, \Theta_m$ imply, by Lemma 4.5, that $M+Q$ is uniformly monotone at the zero $\overline{\mathbf{x}}$ on $\overline{B}_H(\overline{\mathbf{x}})$ with

$$\forall \varepsilon > 0\ \forall \mathbf{y} \in \overline{B}_H(\overline{\mathbf{x}})\ \forall \mathbf{u} \in (M+Q)(\mathbf{y})\ (\|\overline{\mathbf{x}} - \mathbf{y}\| \in [\varepsilon, H] \to \langle \mathbf{y} - \overline{\mathbf{x}}, \mathbf{u}\rangle \geq \Theta(\varepsilon)).$$

In particular, Theorem 3.19 then yields $\|\mathbf{x}_n - \overline{\mathbf{x}}\| < \varepsilon$ for all $n \geq \rho(\varepsilon)$ and $\varepsilon > 0$ and this yields

$$\|x_n - \overline{x}\|, \|v_{l,n} - \overline{v}_l\| < \varepsilon$$

for all such n and for all $l = 1, \dots, m$. Further, Theorem 3.20 yields $\|\mathbf{p}_n - \overline{\mathbf{x}}\| < \varepsilon$ for all $n \geq \rho'(\varepsilon)$ which implies that

$$\|p_{1,n} - \overline{x}\|, \|p_{2,l,n} - \overline{v}_l\| < \varepsilon$$

for all such n and for all $l = 1, \dots, m$. □

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