ON LOGICAL ASPECTS OF EXTENSIONALITY AND CONTINUITY FOR SET-VALUED OPERATORS WITH APPLICATIONS TO NONLINEAR ANALYSIS

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Abstract. We discuss the logical principle of extensionality for set-valued operators and its relation to mathematical notions of continuity for these operators in the context of systems of finite types as used in proof mining. Concretely, we initially exhibit an issue that arises with treating full extensionality in the context of the prevalent intensional approach to set-valued operators in such systems. Motivated by these issues, we discuss a range of useful fragments of this full extensionality statement where these issues are avoided and discuss their interrelations. Further, we study the continuity principles associated with these fragments of extensionality and show how they can be introduced in the logical systems via a collection of axioms that do not contribute to the growth of extractable bounds from proofs. In particular, we place an emphasis on a variant of extensionality and continuity formulated using the Hausdorff-metric and, in the course of our discussion, we in particular employ a tame treatment of suprema over bounded sets developed by the author in previous work to provide the first proof-theoretically tame treatment of the Hausdorff metric in systems geared for proof mining. To illustrate the applicability of these treatments for the extraction of quantitative information from proofs, we provide an application of proof mining to the Mann-iteration of set-valued mappings which are nonexpansive w.r.t. the Hausdorff metric and extract highly uniform and effective quantitative information on the convergence of that method.

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1. Introduction

At least since the emergence of the fundamental correspondence between mathematical proofs and programs, it has been one of the main driving interests of proof theory to describe the computational content, and by that measuring the strength, of a mathematical theorem. In that vein, the research program of *proof mining* emerged in the 1990s through the work of Kohlenbach (following the spirit of Kreisel's program of *unwinding of proofs*, see [33, 34]) which aims at providing this content by analyzing the (prima facie) noneffective proofs of mathematical theorems as they are found in the usual literature. While this is a highly nontrivial task through the prevalent use of classical logic and infinitary set-theoretical (sometimes called ideal) principles in mainstream mathematics, this research program of proof mining is nevertheless substantiated by a firm logical basis developed using central proof-theoretic tools like Gödel's functional interpretation (see [11]) and Howard's majorizability (see [13]), and their variants, and has since its inception lead to hundreds of novel applications in core mathematics and computer science. We refer to the monograph [19] for a detailed exposition of proof mining up to 2008 and to the surveys [20, 22, 31] for further details on the theoretical developments of the field

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as well as on applications.

In more detail, the central results of the logical foundation of proof mining are the so-called *general logical metatheorems* which comprise an underlying logical system together with a theorem about that system so that, for one, this corresponding system is suitably designed so that it facilitates (relatively) easy applications to large classes of objects and proofs from the core literature of the intended area of application and, for another, the associated logical metatheorem guarantees the extractability of tame and highly uniform computational information from large classes of non-effective proofs carried out in this system, the complexity of which corresponds to the logical strength of the principles used in the proof. Further, the proofs of the logical metatheorems even provide algorithms to (in principle) extract this information.[1]

In the context of this enterprise of extractive proof theory, one of the prime (logical) issues actually arises in connection with the, from a mathematical perspective perhaps trivial, principle of extensionality. Concretely, working over the higher-type system $\mathcal{A}^\omega[X,\|\cdot\|]$ for classical analysis over an abstract normed space X defined as in the seminal works [10, 17] (see Section 2 for further details), the prevalent system used in proof mining for extracting programs from proofs pertaining to the theory of normed spaces, the extensionality of an operator $T : X \to X$ for the normed space (represented by) X is naturally formulated as

$$\forall x^X, y^X \, (x =_X y \to Tx =_X Ty)$$

where equality in X is internally defined using the norm of the space represented by X via

$$x =_X y := \|x -_X y\|_X =_{\mathbb{R}} 0_{\mathbb{R}},$$

utilizing a suitable representation of the real numbers in the underlying language. This principle, if provable in a system (say, extending $\mathcal{A}^\omega[X,\|\cdot\|]$) that is amenable to proof mining metatheorems, would immediately entail (see e.g. the discussion in [19]) the extractability of a (computable) functional $\omega : \mathbb{N}^3 \to \mathbb{N}$ such that

$$\forall k, b \in \mathbb{N} \forall x, y \in \overline{B}_b(0) \left(\|x - y\| < 2^{-\omega_B(k,b)} \to \|Tx - Ty\| < 2^{-k}\right)$$

holds for all B-bounded mappings $T : X \to X$ (i.e. $\|Tx\| \leq B$ for all $x \in X$ with $B \in \mathbb{N}$) and all normed spaces $(X, \|\cdot\|)$ axiomatized by the system,[2] where $\overline{B}_b(0) := \{x \in X \mid \|x\| \leq b\}$. So one could directly derive the uniform continuity on bounded sets for bounded operators T from its associated extensionality statement. Therefore, if discontinuous objects should be treated, one has to have issues with (and therefore has to restrict) extensionality as a principle in formal systems used in proof mining. In the practice of applying methods from proof mining, especially in the context of nonlinear analysis and fixed point theory, this has previously, more often than not, had relatively little relevance for operators of that type as for most single-valued operators considered in the respective applications, their defining properties (like e.g. nonexpansivity) immediately entail the uniform continuity and hence extensionality for these maps (as centrally also discussed in [10, 17]).

In the case of set valued operators $T : X \to 2^X$, this situation changes as first highlighted in [52] where, for one, already fragments of the extensionality principle give rise to very strong

[1]Examples of such metatheorems can in particular be found in [7, 10, 12, 17, 19, 30, 35, 36, 43, 45, 56, 50, 52, 53, 59].

[2]In fact, a more general statement holds for which the above is just a special case. Concretely, in general the result holds for all mappings T which are majorizable, i.e. bounded on bounded sets in this case, and the modulus ω in this case depends on such a majorant instead of B. See [19] for further details on this.

uniform continuity principles excluding a wide range of natural instances of such operators and where, for another, it has been shown that the key defining properties of some of the central classes of such operators considered in the literature actually are already equivalent to the associated extensionality principle, creating an a priori dire situation for extending methods from proof mining to such objects. This issue is made even more pressing by the fact that these set-valued operators have become one of the prime foci of proof mining applications in the recent years, as exemplified by the the many case studies carried out utilizing these objects (see e.g. the many works on the seminal proximal point algorithm and its variants as in [5, 6, 23, 24, 25, 27, 37, 38, 46, 51, 57] as well as case studies on nonlinear semigroups and their relation to accretive set-valued operators as in [8, 26, 47, 55] as well as other central considerations on iterations featuring these operators like in [21, 32, 49, 60, 61] among others).

It is therefore even more surprising that, contrary to these theoretical limitations, this apparent proof-theoretic strength is rarely observed in practice. In particular, essentially none of the case studies mentioned above (besides a central illustrative example [49]) require a quantitative treatment of extensionality at all if they did not feature a uniform continuity assumption in the first place. As first outlined in [52], this can be explained from a proof-theoretical perspective by the empirical fact that in many proofs from the mainstream literature of m-accretive or maximally monotone operator theory, the areas where these case studies are situated in (see [1, 65] for canonical textbooks on these subjects), one does not require the full extensionality of the operator in question but it actually suffices to have a certain so-called intensional treatment thereof together with access to the so-called resolvent which in turn is proof-theoretically tame and can be utilized to design applicable systems with accompanying metatheorems in the usual style of proof mining for these areas (see the discussions in [52] for further information).

If, however, the proof is not of that nature and really requires the extensionality of the operator, then a quantitative treatment of such will be necessary (as was e.g. the case in the previously mentioned application from [49]). This might in some situations further hinder a proof-theoretic treatment as some of the central uniform continuity principles for set-valued operators, which crucially feature in many proofs in that area and naturally imply an associated extensionality statement, are not immediately recognized as proof-theoretically tame statements and instead seem to carry computational strength already due to the use of apparently logically complicated objects like e.g. the Hausdorff metric.

The purpose of this paper is now twofold:

(1) We discuss some central issues with treating the full extensionality statement in the context of an intensional approach to set-valued operators, similar to the approach towards accretive and monotone operators taken in [52] (see also [50]). In particular, we show that, in a way, no bound extraction result akin to the metatheorems of proof mining exists for intensional systems treating suitable classes of set-valued operators and which prove the associated full extensionality principle for the operator. This in particular puts strong emphasis on extensionality as a central logical issue for proof mining in the context of set-valued operator theory.
(2) Motivated by these negative results of item (1), we discuss a range of fragments of the full extensionality principle, which arise by considering said principle from a more mathematically motivated perspective, and study the relations among them, highlighting a certain robustness. Contrary to the negative results on the rather "naive" and logically motivated full extensionality principles, we illustrate how these fragments all represent

the extensionality of the operator in a mathematically fruitful, and essentially equivalent, way. In particular, they allow for a computational interpretation which generates useful uniform continuity statements for set-valued operators that can be introduced in the logical systems via a collection of axioms that do not contribute to the growth of extractable bounds from proofs. In particular, we in that context illustrate how the most prominent uniform continuity principle for set-valued operators as formulated using the Hausdorff metric can be treated in a logically tame way in the context of an intensional approach to these operator, which presents the first proof-theoretically tame approach to the Hausdorff metric and hence for the first time enables proof mining applications utilizing this mapping in an essential way. This is then in particular illustrated in the later half of the paper by a case study where we extract quantitative information on the convergence of an iterative method devised in [62] for the approximation of fixed points of set-valued maps that are nonexpansive relative to the Hausdorff metric.

With these two contributions, we therefore provide highly necessary information for the practice of proof mining regarding proofs featuring the extensionality of set-valued operators as it is carried out using these intensional approaches, highlighting with (1) and (2) the subtlety of expressing mathematically meaningful notions of extensionality and uniform continuity in respective formal systems, where we in particular illustrate that even complicated uniform continuity statements using the Hausdorff metric can be approached in a proof-theoretically tame way, a fact that in this paper, as mentioned above, also immediately leads to novel applications.

2. Logical aspects of full extensionality principles for set-valued operators

In this section, we discuss the main aspects of the first of the previously mentioned objectives of this paper, i.e. the issues with extensionality in the context of an intensional treatment of set-valued operators $T : X \to 2^X$ over a normed vector space X.[3] In the context of these set-valued operators, we write

$$\mathrm{dom}T := \{x \in X \mid Tx \neq \emptyset\},$$

for the domain of T and

$$\mathrm{ran}T := \bigcup_{x\in X} Tx$$

for the range of T. As we are dealing with objects on normed spaces, the main system for proof mining over abstract normed spaces $\mathcal{A}^\omega[X, \|\cdot\|]$ as introduced in [17] (see also [10]) consequently forms a logical basis for these investigations. While this system is central for the present paper, we nevertheless only rely on a handful of key properties of it which we shortly discuss in the following. For any other background on this system, we refer to the presentation in [19].

Concretely, the system $\mathcal{A}^\omega[X, \|\cdot\|]$ extends $\mathcal{A}^\omega = \text{WE-PA}^\omega + \text{QF-AC} + \text{DC}$, i.e. a weakly-extensional variant of Peano arithmetic in all finite types together with the principle of quantifier-free choice in all types and the principle of dependent choice (see [19] and [68] for further details), with an additional abstract base type X and additional constants and universal axioms utilizing this type to axiomatize that X is a normed space. As such, the system $\mathcal{A}^\omega[X, \|\cdot\|]$ operates over an extended set of types T^X defined by

$$0, X \in T^X, \quad \xi, \tau \in T^X \Rightarrow \tau(\xi) \in T^X,$$

[3]We want to note that the discussion given here extends also to operators $T : X \to 2^Y$ for a second space Y, e.g. the dual space X^* of X as considered in [50, 53].

with pure types abbreviated via natural numbers through recursively defining $n+1 := 0(n)$. To induce a normed linear structure on X, one adds the constants $0_X, 1_X$ of type X, $+_X$ of type $X(X)(X)$, $-_X$ of type $X(X)$, $\cdot_X$ of type $X(X)(1)$ and $\|\cdot\|_X$ of type $1(X)$ together with suitable axioms stating that X with these operations is a real normed vector space with 1_X representing a unit vector and $-_X$ producing the additive inverse of its argument (see [10, 17, 19] for further details). In any way, equality at type 0, i.e. on the natural numbers, is the only primitive relation and equality at higher types is treated as a defined notion by setting

$$x^X =_X y^X := \|x -_X y\|_X =_{\mathbb{R}} 0,$$

using a suitable representation of the real numbers as objects of type 1 (see e.g. [19]) and by extending this to higher types via

$$s =_{\sigma(\tau)} t := \forall x^\tau (sx =_\sigma tx).$$

An intended model of this language arises from the full set-theoretic type structure $\mathcal{S}^{\omega,X}$ defined by

$$S_0 := \mathbb{N}, \quad S_X := X, \quad S_{\sigma(\tau)} := S_\sigma^{S_\tau}$$

for a given normed space $(X, \|\cdot\|)$ by suitably interpreting the additional constants present in $\mathcal{A}^\omega[X, \|\cdot\|]$ (see [10] for further details).

Crucially, this system is suitably designed so that by an application of a negative translation together with a monotone variant of Gödel's functional interpretation arising through a combination with Howard's majorizability (due to the seminal work of Kohlenbach [15], see also already [14]), the following logical metatheorem in the style of proof mining can be established for that system:

Theorem 1 ([10])**.** *Let ρ be admissible*[4] *and let $B_\forall(x,u)/C_\exists(x,v)$ be purely universal/existential, respectively, where the types of the internal quantifiers are admissible and such that they only contain $x, u/x, v$ freely. Assume that*

$$\mathcal{A}^\omega[X, \|\cdot\|] \vdash \forall x^\rho \left(\forall u^0 B_\forall(x,u) \to \exists v^0 C_\exists(x,v)\right).$$

Then there exists a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ which is defined on all strongly majorizable elements of $S_{\widehat{\rho}}$ (see [10]*), where the corresponding restriction to these elements is bar-recursively computable and where the following holds for any model $\mathcal{S}^{\omega,X}$ defined by a non-trivial real normed vector space $(X, \|\cdot\|)$: for all $x \in S_\rho$ and $x^* \in S_{\widehat{\rho}}$, if $x^* \gtrsim x$, then*

$$\mathcal{S}^{\omega,X} \models \forall u \leq_0 \Phi(x^*) B_\forall(x,u) \to \exists v \leq_0 \Phi(x^*) C_\exists(x,v).$$

Here, $\gtrsim$ is the extension due to [10, 17] *of the strong majorizability relation of Bezem and $\widehat{\rho} \in T$ is the type of the majorants of objects of type $\rho \in T^X$.*

By an intensional approach to a set-valued operator T over X, we now understand that T is treated formally via its graph as coded by its characteristic function which is an object of type $0(X)(X)$.[5] To generically talk about such systems here, we assume that the language of the system $\mathcal{A}^\omega[X, \|\cdot\|]$ is extended with a new constant χ_T of type $0(X)(X)$. We write $y \in Tx$, $(x,y) \in T$ or $(x,y) \in \mathrm{gra}T$ for the formal statement $\chi_T xy =_0 0$ in the extended language and we write $x \in \mathrm{dom}T$ for $\exists y^X (y \in Tx)$. Note that inclusions in the graph of T are in particular

[4] A type is called admissible if it is of the form $X(\sigma_k)\dots(\sigma_1)$ or $0(\sigma_k)\dots(\sigma_1)$ where each σ_i is a so-called simple type, i.e. each σ_i is of the form $X(0)\dots(0)$ or $0(0)\dots(0)$.

[5] This approach to treating set-valued operators was first employed in [52] and is by now a staple in the logical approaches to such objects in the context of systems used for proof mining.

quantifier-free. We denote the extension of the system $\mathcal{A}^\omega[X, \|\cdot\|]$ by this constant χ_T together with the characteristic function axiom

$$(\chi)_T \qquad \forall x^X, y^X \, (\chi_T xy \leq_0 1)$$

by $\mathcal{A}^\omega[X, \|\cdot\|, T]$. Naturally, an intended model $\mathcal{S}^{\omega,X}_T$ for this extended system arises from a normed space $(X, \|\cdot\|)$ and a set-valued operator $T : X \to 2^X$ by extending the induced model $\mathcal{S}^{\omega,X}$ for the system $\mathcal{A}^\omega[X, \|\cdot\|]$ by interpreting χ_T via

$$[\chi_T]_{\mathcal{S}^{\omega,X}_T} := \lambda x, y \in X. \begin{cases} 0 & \text{if } (x, y) \in T, \\ 1 & \text{otherwise.} \end{cases}$$

It rather immediately follows[6] that this simple extension $\mathcal{A}^\omega[X, \|\cdot\|, T]$ of $\mathcal{A}^\omega[X, \|\cdot\|]$ also satisfies a proof mining metatheorem akin to that presented in Theorem 1.

By the (full) extensionality axiom for T, we now mean the following formal statement in the corresponding language of $\mathcal{A}^\omega[X, \|\cdot\|, T]$:

$$(E)^\chi_T \qquad \forall x^X, y^X, z^X, w^X \, (x =_X y \wedge z =_X w \wedge z \in Tx \to w \in Ty) \, .$$

Naturally, a system like the above might now serve as the basis for further extensions with additional constants and axioms in order to axiomatize certain specific classes of set-valued operators, like e.g. done in [52] for treating (m-)accretive and (maximally) monotone operators on Hilbert spaces and in [50] for (maximally) monotone operators on Banach spaces, but this approach is not limited by these classes of objects and rather is immediately applicable for any extension of this system by additional constants, as long as these are majorizable, and suitable axioms, as long as these have a monotone functional interpretation (see [19] for further details on both of these aspects).

In the following, we however want to study the behavior of a (suitably) generic but fixed extension of that very minimal base $\mathcal{A}^\omega[X, \|\cdot\|, T]$ which we in the following denote by $\mathcal{C}^\omega$. Crucially, we only assume for $\mathcal{C}^\omega$ that is satisfies the following two properties:

(1) The system $\mathcal{C}^\omega$ satisfies a metatheorem in the style of proof mining, i.e. akin to Theorem 1, where the conclusion is (of course) only true for a certain class of intended models $\mathcal{S}^{\omega,X}_T$, which we here fix to arise only from spaces X of a certain non-empty class C_{Sp} of normed spaces and from set-valued operators $T : X \to 2^X$ of an associated non-empty class $\mathsf{C}_{\mathrm{Op}}(X)$.
(2) The system $\mathcal{C}^\omega$ axiomatizes a class of non-empty and closed set-valued operators, i.e. $X \in \mathsf{C}_{\mathrm{Sp}}$ and $T \in \mathsf{C}_{\mathrm{Op}}(X)$ implies that T is closed in $X \times X$ and that $\mathrm{dom} T \neq \emptyset$.

It should be emphasized that this in particular holds true for most systems considered for proof mining applications, in particular for the systems devised for (nonempty) m-accretive and maximally monotone operators in [50, 52] (and even for operators continuous w.r.t. the Hausdorff metric as will be discussed later on). In fact, for these classes of m-accretive or maximally monotone set-valued operators, the closedness of them in $X \times X$ is even actually equivalent to the extensionality of these mappings over respective suitable intensional systems (akin to $\mathcal{C}^\omega$, i.e. extending $\mathcal{A}^\omega[X, \|\cdot\|, T]$ and satisfying a logical metatheorem in the style of proof mining) as shown in [50, 52].

[6]Note for this that the only additional axiom $(\chi)_T$ is purely universal and that the constant χ_T, by virtue of this axiom, is trivially majorizable, see [19] for details.

We now want to investigate what consequences there are when such a system actually proves the extensionality of T or fragments thereof. So, let us initially assume that $\mathcal{C}^\omega \vdash (E)^X_T$. Then, using the bound extraction theorem, i.e. property (1), assumed for $\mathcal{C}^\omega$, we would be able to extract a functional $\omega_{X,T} : \mathbb{N} \to \mathbb{N}$ (potentially depending on X and T) such that

$$\forall b \in \mathbb{N} \forall x, y, z, w \in \overline{B}_b(0)(\|x - y\|, \|z - w\| \leq 2^{-\omega_{X,T}(b)} \wedge z \in Tx \to w \in Ty)$$

holds for any normed space $X \in \mathsf{C}_{\mathrm{Sp}}$ and operator $T \in \mathsf{C}_{\mathrm{Op}}(X)$. Now, any such operator has to be open in $X \times X$ (in a uniform way on bounded sets): given $(x, z) \in T$ with $\|x\|, \|z\| \leq b$ and y, w such that

$$\|x - y\|, \|z - w\| \leq 2^{-\omega_{X,T}(b+1)},$$

we have $\|y\|, \|w\| \leq b + 1$ and so $(y, w) \in T$. However, this provides a semantic clash with property (2) assumed for $\mathcal{C}^\omega$ as any $T \in \mathsf{C}_{\mathrm{Op}}(X)$ is, by that assumption, closed in $X \times X$ and hence clopen and so, since X (and with that $X \times X$) is a normed space, that means any T is either equal to $X \times X$ or $\emptyset$, the latter being excluded as T is also assumed to be non-empty. Not only is this restriction already here so severe that it completely trivialized the semantically considered operators, but in the context of many of the central classes of set-valued operators studied in the literature of convex analysis, as is e.g. the case for m-accretive and maximally monotone operators, the analytical properties imposed on them often already further exclude the full operator $X \times X$. In such cases, there are therefore *no* operators $T \in \mathsf{C}_{\mathrm{Op}}(X)$. Consequently, if a system $\mathcal{C}^\omega$ with the properties (1) and (2) as above has a model based on the standard structure using spaces $X \in \mathsf{C}_{\mathrm{Sp}}$ and operators $T \in \mathsf{C}_{\mathrm{Op}}(X)$, it can not prove the extensionality of the operator T.

A kind of internalized version of the above argument can be given using the principle of uniform boundedness Σ^0_1-UB^X_- as introduced in [12] (see also [18] as well as [16], the latter being where this principle was first introduced, outside of the context of abstract types however). By the results of [12] (see also [18]), Σ^0_1-UB^X_- can be consistently added to a system that enjoys bound extraction theorems in the above sense. In particular, the system $\mathcal{A}^\omega[X, \|\cdot\|, T]+\Sigma^0_1$-$\mathrm{UB}^X_-$ enjoys the same bound extraction theorems as the system $\mathcal{A}^\omega[X, \|\cdot\|, T]$. Now, the principle Σ^0_1-UB^X_- represents a carefully defined intensional version of the usual uniform boundedness principle Σ^0_1-UB^X (see also [12] and [18]), a necessary restriction in order to stay admissible in the context of unbounded spaces. However, as shown in Lemma 6.25 in [12], Σ^0_1-UB^X_- and Σ^0_1-UB^X coincide for sentences that are extensional. Now, in our context it however in particular follows that

$$\mathcal{A}^\omega[X, \|\cdot\|, T] + (E)^X_T \vdash \mathrm{Ext}(A_\exists)$$

where, following [12], $\mathrm{Ext}(A_\exists)$ represents the extensionality of the formula $A_\exists$ defined by

$$A_\exists(x, y, z, w, j) := \|x - y\|_X, \|z - w\|_X \leq_\mathbb{R} 2^{-j} \wedge z \in Tx \to w \in Ty$$

as by $(E)^X_T$, inclusions of the form $z \in Tx$ are extensional (and since the norm is provably extensional). Hence, by Lemma 6.25 from [12], in the context of Σ^0_1-UB^X_- we can actually apply Σ^0_1-UB^X to $A_\exists$ which, by internalizing the above argument, immediately allows one to derive that T is open as before, i.e. we can thereby derive that

$$\mathcal{A}^\omega[X, \|\cdot\|, T] + \Sigma^0_1\text{-}\mathrm{UB}^X_- + (E)^X_T \vdash (\mathrm{Open})_T$$

where

$$(\mathrm{Open})_T \qquad \exists \omega^{0(0)} \forall b^0 \forall x^X, y^X, z^X, w^X \Big(\|x\|_X, \|y\|_X, \|z\|_X, \|w\|_X \leq_{\mathbb{R}} b \wedge$$
$$\|x -_X y\|_X, \|z -_X w\|_X \leq_{\mathbb{R}} 2^{-\omega(b)} \wedge z \in Tx \to w \in Ty \Big)$$

is a formalization of the fact that T is open (uniform on bounded sets) as above. Therefore, the system $\mathcal{A}^\omega[X, \|\cdot\|, T] + \Sigma^0_1\text{-UB}^X_- + (E)^X_T + (\mathrm{Clsd})_T$ proves that

$$\forall x^X, y^X (y \in Tx) \vee \forall x^X, y^X (y \notin Tx)$$

where $(\mathrm{Clsd})_T$ is some suitable formalization of the closure of T. In particular, let us now consider the systems $\mathcal{V}^\omega$ or $\mathcal{T}^\omega$ from [52] which provide a treatment of m-accretive operators in normed spaces and maximally monotone operators in inner product spaces, respectively. There, we in particular find that the conclusions $\forall x^X, y^X (y \notin Tx)$ and $\forall x^X, y^X (y \in Tx)$ are excluded as, for one, T is provably non-empty in these cases and, for another, as the total operator is provably not accretive or monotone. Further, by utilizing special properties of the operators axiomatized therein, one has (by Theorem 3.1 and Theorem 3.3 in [52]) that $(\mathrm{Clsd})_T$ is provably equivalent to $(E)^X_T$. Together, we obtain that the system $\mathcal{V}^\omega + \Sigma^0_1\text{-UB}^X_- + (E)^X_T$, and similarly the variant formulated with $\mathcal{T}^\omega$, are actually inconsistent, while $\mathcal{V}^\omega + \Sigma^0_1\text{-UB}^X_-$ and $\mathcal{T}^\omega + \Sigma^0_1\text{-UB}^X_-$ still satisfy highly meaningful bound extraction theorems.

It should be noted that similar issues persist if $(E)^X_T$ is restricted to the domain of T by considering the weakened extensionality principle

$$(E)^{X_d}_T \qquad \forall x^X, y^X, z^X, w^X, v^X (x =_X y \wedge z =_X w \wedge z \in Tx \wedge v \in Ty \to w \in Ty).$$

For, suppose that $\mathcal{C}^\omega \vdash (E)^{X_d}_T$ for the previously presumed system $\mathcal{C}^\omega$, then the bound extraction theorem, i.e. property (1), assumed for $\mathcal{C}^\omega$ would yield the existence of a functional $\omega_{X,T} : \mathbb{N} \to \mathbb{N}$ such that

$$\forall b \in \mathbb{N} \forall x, y, z, w, v \in \overline{B}_b(0)(\|x - y\|, \|z - w\| \leq 2^{-\omega_{X,T}(b)} \wedge z \in Tx \wedge v \in Ty \to w \in Ty)$$

holds for any normed space $X \in \mathsf{C}_{\mathrm{Sp}}$ and any operator $T \in \mathsf{C}_{\mathrm{Op}}(X)$. This still at least implies that Tx is open for any $x \in \mathrm{dom}T$ as if $z \in Tx$ with $\|z\|, \|x\| \leq b$ are given, and w is such that $\|z - w\| \leq 2^{-\omega_{X,T}(b+1)}$, then $w \in Tx$. Again, this provides a semantic clash with property (2) assumed for $\mathcal{C}^\omega$ by which, since such a T is closed in $X \times X$, any Tx in particular is also closed so that the only operators $T \in \mathsf{C}_{\mathrm{Op}}(X)$ are of the form

$$T : x \mapsto \begin{cases} X & \text{if } x \in \mathrm{dom}T, \\ \emptyset & \text{otherwise.} \end{cases}$$

In the special case of the previously mentioned systems for, e.g., m-accretive or maximally monotone operators, this limitation on the class of axiomatized operators is now slightly less severe as it does not necessarily render models based on $\mathcal{S}^{\omega,X}_T$ (as induced by the previously fixed classes of spaces and operators) impossible (take e.g. the normal cone operator $N_{\{x\}}$ for a given point $x \in X$ in a Hilbert space, see [1], which is maximally monotone but of the above form and so is feasible for the previously mentioned system $\mathcal{T}^\omega$, for example). Nevertheless, the class is of course still extremely restrictive, presumably making any extracted results qualitatively uninteresting and so of little practical relevance. Also this result can be internalized akin to the previous discussion.

3. Useful fragments of the extensionality principle and their formal treatment

All the observations made above clearly highlight that the "naive" extensionality principles $(E)^{\chi}_{T}$ and $(E)^{\chi_d}_{T}$, derived by requiring the extensionality of the graph of T as coded intensionally via χ_T, is unsuitable for any applied considerations. In a way, this comes at no big surprise as the principles essentially require an inherently intensional object χ_T to now act extensional again.

Now, even though the use of extensionality can often be (at least partially) avoided in practice, as discussed in the introduction, there are nontrivial cases where it nevertheless features prominently, and since $(E)^{\chi}_{T}$ and $(E)^{\chi_d}_{T}$ are not amenable in any real sense to an applied proof-theoretic treatment using the intensional approach to set-valued operators, we are inclined to look for alternative formulations of extensionality to faithfully represent that property formally in this context, meanwhile being of practical, mathematical, use. Guided by the perspective of applied proof theory, we in this section study a range of fragments of the full extensionality principles, which are motivated by uniform continuity statements for set-valued operators already prominently investigated in the literature of nonlinear analysis and which in that sense all represent the extensionality of the operator in a mathematically fruitful way.

3.1. **A refined extensionality principle and its closed variant.** We begin our investigation regarding well-behaved fragments of the full extensionality principle with a uniform continuity principle for set-valued operators based on the so-called Hausdorff-like predicate as introduced by Kohlenbach and Powell in [32]. Concretely, in [32], they introduced a form of uniform continuity for a set-valued operator $T : X \to 2^X$ on a normed space X by assuming the existence of a modulus $\omega : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall x, y \in \mathrm{dom} T \left(\|x - y\| < 2^{-\omega(k)} \to H^*[Tx, Ty, 2^{-k}]\right)$$

where H^* is the aforementioned Hausdorff-like predicate defined by

$$H^*[P, Q, \varepsilon] = \forall p \in Q \exists q \in Q \left(\|p - q\| \leq \varepsilon\right).$$

This notion was introduced in [32] by logical motivations to avoid the use made of the full Hausdorff metric H, defined by

$$H(P, Q) := \max\left\{\sup_{p\in P}\inf_{q\in Q} \|p - q\|, \sup_{q\in Q}\inf_{p\in P} \|p - q\|\right\}$$

for non-empty, closed and bounded sets $P, Q \subseteq X$, in the proofs analyzed therein, which features there in the form of a uniform continuity assumption (and hence an associated extensionality statement). Further, the uniform continuity statement also features crucially in the only other previously mentioned proof mining case study from [49] that had to resolve an extensionality statement for a set-valued operator. We here now want to argue that this uniform continuity statement already represents, or at least indicates, the correct refined extensionality principle for set-valued operators, which in particular then also indicates that the above uniform continuity statement represents the faithful uniform quantitative strengthening of the extensionality of a set-valued operator as suggested by the perspective of proof mining.

For this, we first turn to the associated extensionality principle suggested by the above uniform continuity principle relative to H^* which, following [52] where this principle was already discussed from a logical perspective (albeit embedded in the context of systems treating monotone and accretive set-valued operators), takes the following form:

$$(E)^*_T \qquad \forall x^X, y^X \left(x, y \in \mathrm{dom}T \land x =_X y \to \forall k^0 \left(H^*[Tx, Ty, 2^{-k}]\right)\right)$$
$$\equiv \forall x^X, y^X \left(x, y \in \mathrm{dom}T \land x =_X y \to \forall k^0 \forall u \in Tx \exists v \in Ty \left(\|u - v\| \leq_{\mathbb{R}} 2^{-k}\right)\right).$$

Indeed, it can be immediately recognized that the uniform continuity principle suggested by the perspective of the monotone functional interpretation of $(E)^*_T$ amounts to the above uniform continuity statement, actually in a slightly less uniform variant where ω does additionally depend on a norm upper bound b on the points from X involved. Further, as discussed in [52], this uniform continuity principle can be formalized in a proof-theoretically tame way over a system treating such operators intensionally as outline above in the following way: A "naive" first formalization of the principle, resolving in particular the hidden quantifiers in $x, y \in \mathrm{dom}T$, yields

$$\forall k^0, b^0, x^X, y^X, z^X, u^X \exists v^X \big(\|x\|_X, \|y\|_X, \|z\|_X, \|u\|_X <_{\mathbb{R}} b \land z \in Ty \land u \in Tx$$
$$\land \|x -_X y\|_X <_{\mathbb{R}} 2^{-\omega(k,b)} \to \left(v \in Ty \land \|u -_X v\|_X \leq_{\mathbb{R}} 2^{-k}\right) \big)$$

where ω is a suitable constant of type $0(0)(0)$. As any such v naturally satisfies $\|v\| \leq \|u\| + \|u - v\| \leq \|u\| + 1$, the above statement can be further specified as

$(UC)^*_T$

$$\forall k^0, b^0, x^X, y^X, z^X, u^X \exists v^X \preceq_X (\|u\|_X + 1)1_X \Big(\|x\|_X, \|y\|_X, \|z\|_X, \|u\|_X <_{\mathbb{R}} b \land z \in Ty$$
$$\land u \in Tx \land \|x -_X y\|_X <_{\mathbb{R}} 2^{-\omega(k,b)} \to \left(v \in Ty \land \|u -_X v\|_X \leq_{\mathbb{R}} 2^{-k}\right) \Big)$$

where $x^X \preceq_X y^X$ means $\|x\|_X \leq_{\mathbb{R}} \|y\|_X$. As the existential quantifier over v is now bounded in terms of the preceding universal quantifiers and the inner matrix is universal, the principle $(UC)^*_T$ can be recognized as a statement of type Δ as defined in [12] for languages involving abstract types (originally stemming from the earliest works on proof mining such as [14], see also [19]), a class of formulas with a particularly trivial monotone functional interpretation, which hence are admissible in the context of systems tailored for the extraction of bounds using the monotone functional interpretation.

Now, the above extensionality principle $(E)^*_T$ seems to suggest a further extensionality principle as follows: If H^* would be "continuous" in its last argument, we could move from $\forall k^0 \left(H^*[Tx, Ty, 2^{-k}]\right)$ to $H^*[Tx, Ty, 0]$, whereby the above statement then would in particular imply the following even more concise extensionality principle

$$(E)_T \qquad \forall x^X, y^X \left(x, y \in \mathrm{dom}T \land x =_X y \to H^*[Tx, Ty, 0]\right)$$
$$\equiv \forall x^X, y^X \left(x, y \in \mathrm{dom}T \land x =_X y \to \forall u \in Tx \exists v \in Ty \left(u =_X v\right)\right).$$

Here, compared to $(E)^*_T$, the closedness of the image sets of T is already "infused", in a way, as it does not only allow us to conclude the existence of a sequence in Ty approximating u but actually allows us to conclude the existence of an extensionally equal witness v. Further, the above principle can be thought of as an "extensionalized version" of the principle $(E)^\chi_T$ in the sense that it posits the extensional equality of the set Tx not as formalized by $u \in Tx \equiv \chi(x, u) =_0 0$ but by the "extensionalized variant" $u \in_E Tx \equiv \exists u' \in Tx \left(u =_X u'\right)$. Now, while there is certainly a subtle difference between $(E)_T$ and $(E)^*_T$, the following result makes their close relationship based on the topology of the set Tx formally precise:

Proposition 2. *Over $\mathcal{A}^\omega[X, \|\cdot\|, T]$, the principle $(E)_T$ implies $(E)^*_T$.*

Further, define the closure principle

$(\mathrm{pClsd})_T$

$$\forall x^X, z^X, y^{X(0)}_{(\cdot)} \left(x \in \mathrm{dom}T \wedge \forall n^0 (y_n \in Tx) \wedge (y_n \to_X z) \to \exists w^X \left(w =_X z \wedge w \in Tx \right) \right),$$

where $y_n \to_X z$ is some formal representation of convergence in X, say

$$\forall k^0 \exists N^0 \forall n \geq_0 N \left(\|y_n -_X z\|_X \leq_{\mathbb{R}} 2^{-k} \right),$$

expressing that Tx is closed for any $x \in \mathrm{dom}T$. Then over $\mathcal{A}^\omega[X, \|\cdot\|, T] + (\mathrm{pClsd})_T$, the principle $(E)^*_T$ implies $(E)_T$.

Proof. That $(E)_T$ implies $(E)^*_T$ is obvious. To see that $(E)^*_T$ implies $(E)_T$ under the assumption of the closure of each Tx with $x \in \mathrm{dom}T$, let $u \in Tx$ and $y \in \mathrm{dom}T$ with $y = x$ be given. By $(E)^*_T$, for any k of type 0 there exists a $v_k \in Ty$ with $\|u - v_k\| \leq 2^{-k}$. Thus we have $v_k \to_X u$ and by $(\mathrm{pClsd})_T$, there exists a $w \in Ty$ with $w = u$. Thus we have shown $(E)_T$. □

So, in essence, both $(E)_T$ and $(E)^*_T$ represent the same extensionality principle which posits the equality of Tx and Ty, seen as extensional sets, for $x = y$ in the domain of T, with the difference that $(E)^*_T$ only requires a weaker approximating sequence to witness this equality which suffices in the context of closed operators.

Remark 3. For the central classes of monotone and accretive operators with total resolvents, these fragments of the full extensionality principle are equivalent to suitable "extensionalized" variants of the closure of the graph of the operator as well as the resolvent identity and the maximality. Further, for these classes, removing the restriction to $\mathrm{dom}T$ and the dependence on the norm-bounds from ω already from the principle $(UC)^*_T$ results in a very strong uniform continuity statement which, by utilizing results of [2], implies that the operator T is actually single-valued. We refer to [54] for a further discussion of both of these aspects.

Naturally, also $(E)_T$ entails its own uniform continuity principle via the perspective of the monotone functional interpretation which takes the form

$(UC)_T$

$$\forall x^X, y^X, z^X, u^X \exists v^X \preceq_X (\|u\|_X + 1)1_X \forall k^0, b^0 \Big(\|x\|_X, \|y\|_X, \|z\|_X, \|u\|_X <_{\mathbb{R}} b \wedge z \in Ty$$
$$\wedge u \in Tx \wedge \|x -_X y\|_X <_{\mathbb{R}} 2^{-\omega(k,b)} \to \left(v \in Ty \wedge \|u -_X v\|_X \leq_{\mathbb{R}} 2^{-k} \right) \Big),$$

where we already have highlighted the natural boundedness of the quantifier over v which illustrates that $(UC)_T$, similar to $(UC)^*_T$ before, is a statement of type Δ and so is similarly admissible in the context of systems tailored for the extraction of bounds using the monotone functional interpretation.

In particular, compared to $(E)^{\chi}_T$ and $(E)^{\chi_d}_T$, the fragments $(E)_T$ and $(E)^*_T$ are now very applicable as their uniform quantitative versions as guided by the monotone functional interpretation, i.e. the above uniform continuity principles $(UC)_T$ and $(UC)^*_T$, are highly non-trivially populated. This also allows us to see formally that $(E)_T$ and hence $(E)^*_T$ are properly weaker than $(E)^{\chi_d}_T$. For that, we first consider the following result which shows that any suitable operator $T : X \to 2^X$ that is uniformly continuous in the sense of $(UC)^*_T$ is closed in $X \times X$:

Proposition 4. *Any operator $T : X \to 2^X$ such that any set Tx is closed and which is uniformly continuous in the sense of $(UC)^*_T$ is closed in $X \times X$.*

Proof. As T is uniformly continuous in the sense of $(UC)^*_T$, there exists a ω with

$$\forall k, b \in \mathbb{N} \forall x, y, z, u \in \overline{B}_b(0) \big(z \in Ty \land u \in Tx \land \|x - y\| < 2^{-\omega(k,b)} \\ \to \exists v \in X \left(v \in Ty \land \|u - v\| \leq 2^{-k} \right) \big).$$

Let $(x_n, y_n) \subseteq T$ be a sequence in T such that $(x_n, y_n) \to (x, y)$ for $n \to \infty$. As (x_n, y_n) converges, the sequence is bounded and thus, using the existence of ω, we get that for any $n \in \mathbb{N}$, there exists a $v_n \in Tx$ such that the sequence v_n converges to y. As Tx is closed, we have $y \in Tx$. Thus T is closed. $\square$

Hence the results from Section 2 apply in this context and yield that $\mathcal{A}^\omega[X, \|\cdot\|, T] + (UC)_T \not\vdash (E)_T^{\chi_d}$ as there are operators which are uniformly continuous in the sense of $(UC)_T$ that are not of the form

$$T : x \mapsto \begin{cases} X & \text{if } x \in \mathrm{dom}T, \\ \emptyset & \text{otherwise,} \end{cases}$$

the most trivial example being the operator defined by $T(x) := \{x\}$ on a space that is nontrivial. So $(E)_T$ is properly weaker than $(E)_T^{\chi_d}$.

Now, as mentioned before, the predicate H^* and the associated uniform continuity principle for set-valued operators was introduced in [32] to avoid formal considerations on the Hausdorff metric. This was in particular possible as the precise value of the Hausdorff metric was not a required quantity in the proof but was only used, by means of a uniform continuity assumption, to derive certain approximation properties of the involved sets. While this was possible in [32], there certainly are other proofs from the literature where the value of the Hausdorff metric seems to feature much more essential in the proof and where hence a quantitative treatment thereof would be desirable to allow a more direct access to those proofs as they are found in the literature. In the next section, we provide such an access here by leveraging the strengths of the intensional approach and showing that in such a context, one can indeed treat the Hausdorff metric and its associated uniform continuity principle for a set-valued operator in a proof-theoretically tame way amenable to proof mining metatheorems.

In that section, we in particular further show that the associated extensionality principle is equivalent to $(E)^*_T$ whenever $H(Tx, Ty)$ is well-defined, showing that $(E)^*_T$ and hence $(E)_T$ are very robust as extensionality principles in the sense that small perturbations yields equivalent principles. Based on this robustness and the applicability of the principles $(E)_T$ and $(E)^*_T$ and their associated uniform continuity principles $(UC)_T$ and $(UC)^*_T$ as evidenced from the previous proof mining literature together with the logical motivations of this section, we thereby want to argue in this paper that $(E)_T$ and with that this cluster of related extensionality principles are the faithful and correct representation of the notion of extensionality of a set-valued operator in the context of this intensional approach.

3.2. **An extensionality principle based on the Hausdorff-metric.** We now show how the Hausdorff metric and its associated extensionality and uniform continuity principles can be formally approached in the context of systems providing and intensional treatment of set-valued operators like $\mathcal{A}^\omega[X, \|\cdot\|, T]$ (or related systems). For that, we begin with showing that for certain sets P, Q, the Hausdorff distance $H(P, Q)$ can indeed be treated in a proof-theoretically tame way in the context of the systems considered in the context of proof mining over normed linear spaces.[7] For this, we work over the basic system $\mathcal{A}^\omega[X, \|\cdot\|]$ for now. Now, to approach the Hausdorff metric, let concretely P now be a set in a normed space X which is bounded, i.e. $\|p\| \leq c$ for all $p \in P$ with $c \in \mathbb{N}$. Then we can treat the real-valued distance function

$$d(x, P) = \inf_{p \in P} \|x - p\|$$

by adding an additional constant $d(\cdot, P)$ of type $1(X)$ with the following two axiom schemes:

$$(d_P)_1 \qquad \forall x^X, p^X \, (P(p) \to d(x, P) \leq_{\mathbb{R}} \|x -_X p\|_X)$$

as well as (writing c for the real number arising from c seen as a numeral)

$$(d_P)_2 \qquad \forall x^X, k^0 \exists p \leq_X c1_X \left(P(p) \land \|x -_X p\|_X \leq_{\mathbb{R}} d(x, P) + 2^{-k}\right)$$

where $P(p)$ is a predicate describing $p \in P$. These two axioms schemes completely characterize the facts that, for one, $d(x, P)$ is supposed to be a lower bound on the norm distance $\|x - p\|$ from x to any element $p \in P$ as governed by $P(p)$, and for another, that $d(x, P)$ is arbitrarily well approximated by any such norm distance. In other words, the two schemes exactly specify that $d(x, P)$ is the greatest lower bound of all norm distances $\|x - p\|$ for all $p \in P$.[8]

These schemes become admissible if they are instantiated with a P such that the two axioms have a monotone functional interpretation. This can in particular be guaranteed if the formula P, besides potential parameters, is quantifier-free (as is e.g. naturally the case in the context of an intensional description of a set akin to the way we previously treated set-valued operators). Concretely, in this quantifier-free case (which will actually be the only concrete case occurring in the applications given in this paper), the axiom $(d_P)_2$ is of type Δ, since the existential quantifier in $(d_P)_2$ is bounded (which crucially uses the boundedness of the set specified by P), and hence admissible in systems with bound extraction theorems in the style of proof mining.

Similarly, we can add a constant $d(\cdot, Q)$ of the same type for a second bounded set Q (w.l.o.g. also bounded by c) together with the following axioms determined as above over a (in all practical circumstances of this paper quantifier-free) predicate $Q(q)$ describing $q \in Q$:

$$(d_Q)_1 \qquad \forall x^X, q^X \, (Q(q) \to d(x, Q) \leq_{\mathbb{R}} \|x -_X q\|_X) \, ,$$

$$(d_Q)_2 \qquad \forall x^X, k^0 \exists q \leq_X c1_X \left(Q(q) \land \|x -_X q\|_X \leq_{\mathbb{R}} d(x, Q) + 2^{-k}\right) .$$

In the context of both $d(x, P)$ and $d(x, Q)$, we can then introduce the quantities

$$d(P, Q) = \sup_{p \in P} d(p, Q) \text{ and } d(Q, P) = \sup_{q \in Q} d(q, P)$$

into the system by following a dual idea as the above approach towards the treatment of the infima $d(x, P)$ and $d(x, Q)$ and hence adding corresponding constants (for simplicity also denoted by) $d(P, Q)$ and $d(Q, P)$ of type 1 into the language together with another set of similar axiom schemes, concretely taking

$$(d_{P,Q})_1 \qquad \forall p^X (P(p) \to d(P, Q) \geq_{\mathbb{R}} d(p, Q)),$$

$$(d_{P,Q})_2 \qquad \forall k^0 \exists p \leq_X c1_X \left(P(p) \land d(p, Q) \geq_{\mathbb{R}} d(P, Q) - 2^{-k}\right) ,$$

[7]Naturally, a similar approach would already work over underlying metric spaces but we here only focus on the normed case.

[8]As such, these two axioms follow the general approach to the tame treatment of infima and suprema over certain well-behaved sets using two schemes $(S)_1, (S)_2$ in systems geared for proof mining as outlined in [53].

for the quantity $d(P,Q)$ as well as

$$(d_{Q,P})_1 \qquad \forall q^X (Q(q) \to d(Q,P) \geq_{\mathbb{R}} d(q,P)),$$

$$(d_{Q,P})_2 \qquad \forall k^0 \exists q \leq_X c1_X \left(Q(q) \wedge d(q,P) \geq_{\mathbb{R}} d(Q,P) - 2^{-k}\right),$$

for the quantity $d(Q,P)$.

Again, also these axiom schemes are of the form Δ if the predicates P and Q are both quantifier-free (again making use of the fact that the existential quantifiers can be bounded as the specified sets are assumed to be bounded), and so these schemes are admissible in systems with bound extraction theorems in the style of proof mining.

Lastly, we move to the concrete Hausdorff metric which can now just be introduced by a closed term involving $d(P,Q)$ and $d(Q,P)$:

$$H(P,Q) := \max_{\mathbb{R}}\{d(P,Q), d(Q,P)\}.$$

Of course, this distance can also be introduced uniformly for a family of sets described by formulas $P(p,\underline{x}), Q(q,\underline{x})$ with parameters $\underline{x}$ of type $\underline{\sigma}$ if the sets described by $P(p,\underline{x}), Q(q,\underline{x})$ are bounded by a function $c(\underline{x})$ pointwise in the parameters.

Note that the non-emptiness of the sets P, Q is not needed to define these formulas but the non-emptyness is required on a semantic level in order for these formulas to actually have a model as the objects, mapping to type 1, have to be interpreted by a real number (or by a function mapping into real numbers, respectively).

As mentioned before, this abstract treatment is fruitful at least in the context of sets describable by quantifier-free formulas, where these constants and axioms then allow for extending a previous metatheorem of an underlying system via suitable interpretations of the constants in the model[9] since the axioms are admissible as discussed before. Crucial for this however is the majorizability of the constants. This however can be easily achieved: For $d(\cdot,P)$, via the axiom $(d_P)_1$, we have

$$d(x,P) \leq \|x-p\| \leq \|x\| + \|p\| \leq \|x\| + c$$

where p is some point witnessing that P is non-empty (and thus the non-emptyness is also important for majorization). Further, we have

$$d(Q,P) \leq d(q,P) + 1 \leq \|q\| + c + 1 \leq 2c + 1$$

for a suitable q chosen with axiom $(d_{Q,P})_2$. From this, majorants for $d(\cdot,P)$ and $d(Q,P)$ are immediate.

By a similar reasoning, $d(\cdot,Q)$ as well as $d(P,Q)$ are majorizable and this extends to any variant using additional parameters if the sets are non-empty and bounded pointwise for all parameters. Naturally, also the resulting bounding function $c(\underline{x})$ then has to be majorizable as a function of type $0(\underline{\sigma}^t)$.

We are now in particular interested in using this way of formulating the Hausdorff-distance to talk about uniform continuity formulations and extensionality principles for set-valued operators T treated as in the previous basic system $\mathcal{A}^\omega[X,\|\cdot\|,T]$. Then the sets P and Q can

[9]Concretely, the new constants $d(\cdot,P), d(\cdot,Q), d(P,Q)$ and $d(Q,P)$, which produce real numbers based on their inputs, are naturally interpreted in the respective models via a functional $(\cdot)_\circ$ canonically selecting a representing Cauchy sequence with a fast rate, see [17, 19] for details.

be taken to be of the form Tx with a parameter x of type X for a given set-valued operator T which is represented in the system by an intensional description over its graph via χ_T as discussed before. Formally, this is naturally represented by taking $P(p,x) := \chi_T(x,p) =_0 0$. As this resulting formulation of the set Tx is quantifier-free, the above axioms in particular become admissible for bound extraction results if, as discussed before, the operator T is actually such that all Tx are bounded with a bounding function c of type $0(X)$ that is majorizable. In the language of [52], the existence of such a c is equivalent to the operator T being uniformly majorizable, i.e. to being bounded on bounded sets. Thus, to treat such operators in the Hausdorff metric, we consider an additional constant T^* of type 1 together with the axiom

$$(T^*) \qquad \forall x^X, y^X, b^0 \left(y \in Tx \land \|x\|_X <_{\mathbb{R}} b \to \|y\|_X \leq_{\mathbb{R}} T^*b\right)$$

which serves as a majorant (and hence witness) to c. Then we can as above introduce constants $d(\cdot, Tx)$ and $d(Tx,Ty)$ for $x, y \in \mathrm{dom}T$ into the language using χ_T and T^* to form H such that the expression $H(Tx,Ty)$ is represented by a term for any x and y.

With this, an extensionality statement corresponding to the Hausdorff metric now indeed can be written as a formal sentence in this extended language via

$$(E)^H_T \qquad \forall x^X, y^X \left(x, y \in \mathrm{dom}T \land x =_X y \to H(Tx,Ty) =_{\mathbb{R}} 0\right).$$

In that context, this extensionality principle $(E)^H_T$ is provably equivalent to the previous principle $(E)^*_T$ as the following result shows. For that, let $\mathcal{A}^\omega[X, \|\cdot\|, T, H]$ refer to the system which results from $\mathcal{A}^\omega[X, \|\cdot\|, T]$ by adding the respective constants and axioms for the Hausdorff metric required to introduce $H(Tx,Ty)$ as detailed above.

Proposition 5. *Over $\mathcal{A}^\omega[X, \|\cdot\|, T, H]$, the principles $(E)^H_T$ and $(E)^*_T$ are equivalent.*

Proof. To show that $(E)^*_T$ implies $(E)^H_T$, let $x = y$ for $x, y \in \mathrm{dom}T$. Fixing k of type 0, let $u \in Tx$ be such that $d(u,Ty) + 2^{-(k+1)} \geq d(Tx,Ty)$, using the axioms for $d(Tx,Ty)$. Then use $(E)^*_T$ to pick $v \in Ty$ with $\|u - v\| \leq 2^{-(k+1)}$. We now have

$$d(Tx,Ty) \leq d(u,Ty) + 2^{-(k+1)} \leq \|u - v\| + 2^{-(k+1)} \leq 2^{-k}$$

using the axioms for $d(u,Ty)$. As k was arbitrary, we have $d(Tx,Ty) = 0$, and similarly we can show $d(Ty,Tx) = 0$. This yields $H(Tx,Ty) = 0$ and we have shown $(E)^H_T$.

To show that $(E)^H_T$ implies $(E)^*_T$, again let $x = y$ for $x, y \in \mathrm{dom}T$ and fix k of type 0 as well as $u \in Tx$. As $(E)^H_T$ implies $H(Tx,Ty) = 0$, we have $d(Tx,Ty) = 0$. Using the axioms for $d(Tx,Ty)$, we have $d(u,Ty) = 0$ and so using the axioms for $d(u,Ty)$, we have that there exists a $v \in Ty$ with

$$\|u - v\| \leq d(u,Ty) + 2^{-k} = 2^{-k}.$$

Thus we have shown $(E)^*_T$. ∎

Further, the monotone functional interpretation then associates to this a corresponding uniform continuity principle for set-valued operators. Further, this principle is actually the usual notion of uniform continuity for set-valued operators w.r.t. the Hausdorff metric (as commonly used in the analytic literature, see e.g. [41]). Concretely, the monotone functional interpretation posits the existence of a modulus ω of type $0(0)(0)$ satisfying[10]

$$(UC)^H_T \qquad \forall x^X, y^X, u^X, v^X, k^0, b^0((x,u),(y,v) \in T \wedge \|x\|_X, \|y\|_X, \|u\|_X, \|v\|_X <_{\mathbb{R}} b \wedge \|x -_X y\|_X <_{\mathbb{R}} 2^{-\omega(k,b)} \to H(Tx,Ty) \leq_{\mathbb{R}} 2^{-k}).$$

As this statement $(UC)^H_T$ is now universal based on our treatment of the Hausdorff-metric, it can thus be freely added to a system with bound extraction theorems in the style of proof mining together with an accompanying constant ω and the preceding treatment of H so that, for this extension, one retains the bound extraction results. Even further, a quantitative analysis of Proposition 5 immediately yields that if $\omega(k,b)$ is a modulus of uniform continuity for T in the sense of $(UC)^H_T$, we have that $\omega(k+1,b)$ is a corresponding modulus for the uniform continuity of T in the sense of $(UC)^*_T$ and similarly we conversely have that if $\omega(k,b)$ is a modulus of uniform continuity for T in the sense of $(UC)^*_T$, then $\omega(k+1,b)$ is a corresponding modulus for the uniform continuity of T in the sense of $(UC)^H_T$ (in this context where $H(Tx,Ty)$ is well-defined).

In the next section, we will illustrate the applicability of this approach towards the Hausdorff metric by analyzing iterative methods related to set-valued mappings which are uniformly continuous w.r.t. the Hausdorff metric.

4. An application: Quantitative results on Mann-iterations for nonexpansive set-valued mappings in Banach spaces

In this section, we illustrate the applicability of the treatment of the continuity principle based on the Hausdorff metric developed formally in a framework for proof mining for the first time in this paper by providing quantitative results on a Mann-type iteration of set-valued mappings which are nonexpansive w.r.t. the Hausdorff metric.

Concretely, let X be a Banach space and denote by $CB(X)$ the collection of nonempty, closed and bounded subsets of X. We still write $H(A,B)$ for the Hausdorff metric for $A,B \in CB(X)$ which is well-defined and real-valued and we write

$$d(x,A) = \inf_{a\in A} \|x-a\|$$

for a given set $A \in CB(X)$ as before. A set-valued map $T : D \subseteq X \to CB(X)$ is called nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|$$

for any $x,y \in D$. We say that a point x is a fixed point of T if $x \in Tx$ and we denote the set of fixed points of T by $F(T)$.

The following is a rather immediate consequence of the definition of the Hausdorff metric:

Lemma 6 (see e.g. [42])**.** *Let $A,B \in CB(X)$. For any $a \in A$ and $\varepsilon > 0$, there exists some $b \in B$ with*

$$\|a-b\| \leq H(A,B) + \varepsilon.$$

[10] As already discussed in the context of $(UC)^*_T$, while the following principle stipulates uniform continuity on bounded subsets, the literature often even considers situations where the continuity is uniform over the whole space, i.e. with ω independent of b.

Based on this lemma, it is immediately clear that given a nonempty convex set K and starting points $x_0 \in K$, $y_0 \in Tx_0$ together with scalars $\alpha_n \in [0,1]$ and $\gamma_n \in (0,\infty)$, one can inductively define an iteration

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n \tag{†}$$

where $y_{n+1} \in Tx_{n+1}$ is chosen such that $\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$. This iteration defined in that way was studied in [62] and in the case that the set K is additionally compact, the authors obtained the following convergence result:

Theorem 7 ([62]). *Let $K \subseteq X$ be nonempty, convex and compact. Let $T : K \to CB(K)$ be a set-valued map that is nonexpansive and suppose that $F(T) \neq \emptyset$ as well as $T(p) = \{p\}$ for each $p \in F(T)$. Let (x_n) be defined as in (†) with starting points $x_0 \in K$, $y_0 \in Tx_0$ and scalars $(\alpha_n) \subseteq [0,1]$ and $(\gamma_n) \subseteq (0,\infty)$ such that*
(1) $\lim_{n\to\infty} \gamma_n = 0$,
(2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then (x_n) converges strongly to a fixed point of T.

The main feature of the sequence exploited in the proof is that it is Fejér monotone (see in particular [3, 4]). This well-studied class of sequences possesses very general convergence theorems which guarantee the weak convergence of such sequences under very mild asymptotic regularity assumptions. In compact (metric) spaces, like in the above result, the convergence is in particular strong.

These general convergence results for Fejér monotone sequences from compact sets were analyzed through the lens of proof mining in [27] where, under the assumption of the existence of moduli which witness uniform quantitative reformulations of the central properties involved, a construction of a rate of metastability for the sequence in question is presented, i.e. a bound on the n in the expression

$$\forall k \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n+g(n)] \left(\|x_i - x_j\| \leq \frac{1}{k+1} \right)$$

in terms of k and g. This noneffectively equivalent phrasing of the Cauchy property is particularly useful for more uniform and finitary considerations on convergence, as in particular also highlighted in [67, 66], and such a bound is in general the most one can hope for if one aims at computable information for Fejér monotone sequences as already in the most simple cases of ordinary Fejér monotonicity, there, in general, are no computable rates of convergence as one can show using methods from computability theory (essentially reducing to the seminal paper [63], see also [44], and see [27] for a more detailed discussion of this). However, aiming for computable rates of convergence, in [29], a general principle of metric regularity is studied (encompassing various forms of well-known regularity assumptions from nonlinear analysis and optimization like metric subregularity, weak sharp minima, error bounds, etc.) and under the assumption of such a metric regularity principle, the authors then provide a construction for a computable as well as highly uniform full rate of convergence for a given Fejér monotone iteration which moreover holds in the absence of any compactness assumptions.

These general but abstract proof mining results were previously successfully instantiated for many different situations in which Fejér monotone sequences occur to derive rates of metastability and rates of convergence. In particular, we want to mention the applications in the context of the composition of two firmly nonexpansive mappings in nonlinear spaces from [28], the proximal point algorithm in uniformly convex Banach spaces from [24] and in CAT(0)-spaces as in [39, 40] as well as algorithms for finding zeros of differences of monotone operators from [49] and Korpelevich's extragradient method as in [48].

It is also here that we apply the results from [27, 29] to derive rates of metastability and rates of convergence (under a metric regularity assumption) for the above iteration which are, as before, not only computable in their parameters but also highly uniform. For that, we need to extract the previously mentioned moduli witnessing uniform quantitative versions of the Fejér monotonicity and asymptotic regularity which themselves arise from an application of proof mining to the respective proofs of these properties given in the course of the proof of Theorem 7 in [62]. As these proofs in particular rely on the utilization of the Hausdorff metric, this application given here is in particular to be seen as a case study to illustrate the applicability of the treatment of the Hausdorff metric discussed in the previous section.

4.1. **The central assumptions and their quantitative content.** In this section, we now first discuss the central assumptions present in Theorem 7 and in particular discuss (using the underlying logical methodology) what kind of quantitative assumptions they entail to potentially feature in the analysis of the main theorem.

The first important assumption present in Theorem 7 is the compactness of the set K. This compactness assumption on K is witnessed in the following by a quantitative modulus of compactness introduced in [9] under the name of a *modulus of total boundedness*[11] which takes the form of a function $\gamma : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$ and for any $(x_n) \subseteq K$:

$$\exists 0 \leq i < j \leq \gamma(k) \left(\|x_i - x_j\| \leq \frac{1}{k+1} \right).$$

Such a modulus exists if, and only if, K is compact and we refer to [27] for various discussions on the construction of such moduli for certain concrete classes of compact sets and spaces.

As a second assumption, we find the non-emptyness of the fixed point set $F(T)$ which will be represented by a concrete witness p_0 (i.e. $p_0 \in K$ and $p_0 \in Tp_0$) in the following. As follows by the perspective of majorization, the bounds extracted later will of course only depend on an upper bound on the norm of p_0, which by the compactness and therefore the boundedness of K, is in particular represented by any upper bound on the diameter of K.

One of the most crucial assumptions, in some sense, is the single-valuedness of T on actual fixed points, i.e. the assumption that $Tp = \{p\}$ if $p \in F(T)$. This implication is equivalent to

$$(*) \qquad \forall p \in K \, (d(p, Tp) = 0 \to H(\{p\}, Tp) = 0)$$

which in turn unravels into

$$\forall p \in K \forall k \in \mathbb{N} \exists j \in \mathbb{N} \left(d(p, Tp) \leq \frac{1}{j+1} \to H(\{p\}, Tp) \leq \frac{1}{k+1} \right)$$

[11]In [27], the name *II-modulus of total boundedness* is used but we here follow the conventions from [9] where such a modulus is just called a *modulus of total boundedness*.

and in that way the logical methodology induces[12] a modulus $\theta : \mathbb{N} \to \mathbb{N}$ bounding (and thus witnessing) such a j in terms of k, i.e. such that[13]

$$\forall p \in K \forall k \in \mathbb{N} \left(d(p, Tp) \leq \frac{1}{\theta(k)+1} \to H(\{p\}, Tp) \leq \frac{1}{k+1} \right).$$

Note that by a simple compactness argument, possessing such a modulus is equivalent to the property $(*)$ in compact spaces:

Lemma 8. *Let K be compact and let $T : K \to CB(K)$ be a nonexpansive operator. Then T satisfies $(*)$ if, and only if,*

$$(**) \qquad \forall k \in \mathbb{N} \exists j \in \mathbb{N} \forall p \in K \left(d(p, Tp) \leq \frac{1}{j+1} \to H(\{p\}, Tp) \leq \frac{1}{k+1} \right).$$

Proof. Clearly, $(**)$ implies $(*)$. Conversely, suppose that $(**)$ fails, i.e. suppose there exists a $k \in \mathbb{N}$ such that for any $j \in \mathbb{N}$:

$$\exists p_j \in K \left(d(p_j, Tp_j) \leq \frac{1}{j+1} \wedge H(\{p_j\}, Tp_j) > \frac{1}{k+1} \right).$$

Then $d(p_j, Tp_j) \leq \frac{1}{j+1}$ implies that for any $j \geq 1$, there exists a $q_j \in Tp_j$ such that $\|p_j - q_j\| \leq 1/j$. Further, $H(\{p_j\}, Tp_j) > \frac{1}{k+1}$ now implies that there exists a $q'_j \in Tp_j$ such that $\|p_j - q'_j\| > \frac{1}{k+1}$.

We now pick subsequences p_{j_i}, q_{j_i} and q'_{j_i} such that $p_{j_i} \to p$, $q_{j_i} \to q$ and $q'_{j_i} \to q'$ with $p, q, q' \in K$. Then $\|p - q\| = 0$ and $H(Tp_{j_i}, Tp) \to 0$ for $i \to \infty$ as T is nonexpansive. Thus in particular $d(q_{j_i}, Tp), d(q'_{j_i}, Tp) \to 0$ which yields

$$d(q, Tp) \leq \|q - q_{j_i}\| + d(q_{j_i}, Tp) \to 0$$

and thus $d(p, Tp) = d(q, Tp) = 0$. Similarly $d(q', Tp) = 0$ and thus $q' \in Tp$. However, we have $\|p - q'\| \geq \frac{1}{k+1}$ and so $H(\{p\}, Tp) \geq \|p - q'\| \geq \frac{1}{k+1}$. This is a contradiction to $(*)$. ☐

In that way, the existence of such a modulus is implied already by the assumptions in Theorem 7.

At last, we consider the assumptions on the auxiliary sequences γ_n and α_n. For γ_n, where it is assumed that

$$\lim_{n \to \infty} \gamma_n = 0,$$

we will later rely on a rate of convergence τ witnessing this property, i.e. on a τ satisfying

$$\forall k \in \mathbb{N} \forall n \geq \tau(k) \left(\gamma_n \leq \frac{1}{k+1} \right).$$

[12]To formalize the above statement in the language of the previous systems, we have to represent the set $\{p\}$ using an additional constant χ_s of type $0(X)(X)$ together with two axioms expressing that $\chi_s(p, \cdot)$ intensionally codes the singleton $\{p\}$ for all p:

$$\forall p^X \left(\chi_s(p, p) =_0 0 \right),$$
$$\forall p^X, x^X \left(\chi_s(p, x) =_0 0 \to x =_X p \right).$$

In that way, the treatment of $\{p\}$ is intensional as we can not prove that for $x = p$, we also have $x \in \{p\}$ in the sense that $\chi_s(p, x) =_0 0$. Then $H(\{p\}, Tx)$ can be introduced using χ_s and some χ_T coding T as discussed in the first part of this paper. In particular, this utilizes that T is bounded since it maps into $CB(K)$ and K is bounded.

[13]Note that the (full) independence on p is suggested by the logical methodology as the set K is in particular bounded.

For α_n, the assumption that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$$

is witnessed by a value $a \in \mathbb{N}^*$ with the property

$$\forall n \geq a \left(\frac{1}{a} \leq \alpha_n \leq 1 - \frac{1}{a} \right)$$

in similarity to [5].

Remark 9. For the previous treatment of the Hausdorff metric, it was crucial that the sets come equipped with a modulus witnessing their boundedness. Note again that the existence of such a modulus is immediate for sets of the form Tx as $Tx \in CB(K)$ and thus $Tx \subseteq K$ which is bounded as K is compact. In that way, for the quantitative results, we will later rely on a bound on the diameter of K (as mentioned before). Note that such a bound can not be computed from the modulus of total boundedness γ for K as this modulus is only non-effectively equivalent to the total boundedness of K in the usual sense and thus only implies the boundedness of K non-effectively (see [27] for a further discussion of this).

4.2. **Suzuki's lemma and its analysis.** The main analytical ingredient of the convergence proof from [62] is a well-known lemma from [64]:

Lemma 10 ([64])**.** *Let $(x_n), (y_n)$ be bounded sequences in a Banach space X and let $(\alpha_n) \subseteq [0,1]$ be such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\in\mathbb{N}} \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$ as well as*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

This lemma was analyzed quantitatively in [5] and we will rely in the following on this analysis:

Lemma 11 ([5])**.** *Let $(x_n), (y_n)$ be sequences in a Banach space X with $\|x_n\|, \|y_n\| \leq b$ for some $b \in \mathbb{N}^*$ and let $(\alpha_n) \subseteq [0,1]$ be such that there exists a $a \in \mathbb{N}^*$ with the property*

$$\forall n \geq a \left(\frac{1}{a} \leq \alpha_n \leq 1 - \frac{1}{a} \right).$$

Suppose that $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$ as well as that there exists a monotone function $\tau : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall n \geq \tau(k) \left((\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq \frac{1}{k+1} \right).$$

Then for any $k \in \mathbb{N}$ and any $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \varphi_{a,\tau,b}(k,g) \forall m \in [n; n+g(n)] \left(\|x_m - y_m\| \leq \frac{1}{k+1} \right),$$

where $\varphi_{a,\tau,b}(k,g) = \max\{a, \tau(t(2t+1)a^t(k+1)-1)\} + (bt(2t+1)a^t(k+1)-1)t + r_0$ for

$$r_i := \begin{cases} 0 & \text{if } i = b(k+1), \\ t + r_{i+1} + \widehat{g}(\max\{a, \tau(t(2t+1)a^t(k+1)-1)\} + it + r_{i+1}) & \text{if } i < b(k+1). \end{cases}$$

where $\widehat{g}(m) = t + g(m)$ and $t = 2ba(k+1)$.

4.3. **Fejér monotonicity and metastability.** We now present the extractions of the quantitative versions of Fejér monotonicity and asymptotic regularity.

For this, we first need to define an appropriate notion of an approximate solution (i.e. of an approximate fixed point) as the results given in [27] rely on uniform reformulations of the respective properties in terms of such approximate solutions. For our concrete situation here, note that p is a fixed point of T if, and only if, $d(p, Tp) = 0$ (as Tp is closed since $Tp \in CB(K)$). In that vein, we call p a $\frac{1}{k+1}$-approximate fixed point of T if

$$d(p, Tp) \leq \frac{1}{k+1}$$

and define correspondingly

$$AF_k = \left\{ p \in K \mid d(p, Tp) \leq \frac{1}{k+1} \right\}$$

as the set of approximate solutions which extend the set of full solutions

$$F = \{p \in K \mid d(p, Tp) = 0\} = F(T).$$

Now, for the Fejér monotonicity of (x_n), we concretely strive to establish the existence of the following modulus relative to the chosen AF_k:

Definition 12 ([27])**.** A function $\chi : \mathbb{N}^3 \to \mathbb{N}$ is a modulus of uniform Fejér monotonicity for (x_n) w.r.t. (AF_k) if for any $n, m, r \in \mathbb{N}$, any $p \in AF_{\chi(k,m,r)}$ and any $l \leq m$:

$$\|x_{n+l} - p\| < \|x_n - p\| + \frac{1}{r+1}.$$

For this, we can now extract the following from the proof of Fejér monotonicity given in [62] for the sequence (x_n) defined as in (†).

Lemma 13. *Let θ be such that*

$$\forall p \in K \forall k \in \mathbb{N} \left(d(p, Tp) \leq \frac{1}{\theta(k)+1} \to H(\{p\}, Tp) \leq \frac{1}{k+1} \right).$$

Then sequence (x_n) defined as in (†) is uniformly Fejér monotone w.r.t. (AF_k) with a modulus

$$\chi(n, m, r) = \theta(m(r+1)+1).$$

Proof. Let p be given with $d(p, Tp) \leq \frac{1}{\chi(n,m,r)+1}$. Then

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|y_n - p\| \\
&\leq (1-\alpha_n) \|x_n - p\| + \alpha_n d(y_n, Tp) + \alpha_n(\|y_n - p\| - d(y_n, Tp)) \\
&\leq (1-\alpha_n) \|x_n - p\| + \alpha_n H(Tx_n, Tp) + \alpha_n(\|y_n - p\| - d(y_n, Tp)) \\
&\leq \|x_n - p\| + (\|y_n - p\| - d(y_n, Tp))
\end{aligned}$$

and by induction we get

$$\|x_{n+l} - p\| \leq \|x_n - p\| + \sum_{i=0}^{l-1} (\|y_{n+i} - p\| - d(y_{n+i}, Tp))$$

for any $l \geq 1$. It is rather immediate to see that in general, for non-empty sets $Y, Z \subseteq X$ and a point x, we have $d(x, Y) \leq d(x, Z) + H(Y, Z)$ and instantiating this yields

$$\|y_{n+i} - p\| = d(y_{n+i}, \{p\}) \leq d(y_{n+i}, Tp) + H(\{p\}, Tp)$$

and thus $\|y_{n+i} - p\| - d(y_{n+i}, Tp) \leq H(\{p\}, Tp)$. As now $p \in AF_{\chi(n,m,r)}$, we get

$$H(\{p\}, Tp) < \frac{1}{m(r+1)}.$$

In particular, in that case we have

$$\begin{aligned}\|x_{n+l} - p\| &\leq \|x_n - p\| + mH(\{p\}, Tp)\\ &< \|x_n - p\| + \frac{1}{r+1}\end{aligned}$$

for $l \leq m$. □

Remark 14. Note that if T satisfies $(*)$, the sequence is Fejér monotone w.r.t. $F(T)$ in the usual sense as can be shown by following the proof of the above Lemma 13. In particular, this result holds without any compactness assumption for K.

For the asymptotic behavior, we are interested in the following type of quantitative information:

Definition 15 ([27])**.** A function Φ is an approximate F-point bound for (x_n) w.r.t. (AF_k) if for any $k \in \mathbb{N}$:

$$\exists n \leq \Phi(k)\,(x_n \in AF_k)\,.$$

The construction of such a Φ for the sequence studied here relies on analyzing the proof of the statement $d(x_n, Tx_n) \to 0$ from [62] which relies on Suzuki's lemma. Concretely, we get the following:

Lemma 16. *Let b be a bound on the diameter of K and let $(\alpha_n) \subseteq [0,1]$ be such that there exists an $a \in \mathbb{N}^*$ with the property*

$$\forall n \geq a\left(\frac{1}{a} \leq \alpha_n \leq 1 - \frac{1}{a}\right).$$

Let τ be a monotone rate of convergence for $\gamma_n \to 0$. Let $\varphi_{a,\tau,b}$ be defined as in Lemma 11. Then (x_n) defined as in (†) has approximate F-points w.r.t. (AF_k) with an approximate F-point bound

$$\Phi(k) = \varphi_{a,\tau,b}(k, 0).$$

Proof. As in [62], we can derive

$$\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n \leq \|x_{n+1} - x_n\| + \gamma_n$$

which yields that

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \gamma_n$$

and thus τ satisfies the assumption of Lemma 11. Applying Lemma 11, we get that for any $k \in \mathbb{N}$ and any $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \varphi_{a,\tau,b}(k, g)\forall m \in [n; n + g(n)]\left(\|x_m - y_m\| \leq \frac{1}{k+1}\right).$$

In particular, we get for any $k \in \mathbb{N}$ that

$$\exists n \leq \varphi_{a,\tau,b}(k, 0)\left(\|x_n - y_n\| \leq \frac{1}{k+1}\right)$$

which yields that for this n, we have

$$d(x_n, Tx_n) \leq \|x_n - y_n\| \leq \frac{1}{k+1},$$

i.e. $x_n \in AF_k$. $\square$

Remark 17. While the full function $\varphi_{a,\tau,b}$ is rather complex, in the above special case of considering the constant-0 function, it simplifies considerably to

$$\varphi_{a,\tau,b}(k,0) = \max\{a, \tau(t(2t+1)a^t(k+1)-1)\} + (bt(2t+1)a^t(k+1)-1)t + 2b(k+1)t$$

for $t = 2ba(k+1)$.

Lastly, we show that $F(T)$ is not only closed but that it is even sufficiently uniformly closed respective to the approximations AF_k in a concrete way introduced in [27]:

Definition 18 ([27])**.** The solution set F is called uniformly closed w.r.t. (AF_k) with moduli δ, ω if for any $k \in \mathbb{N}$, any $q \in AF_{\delta(k)}$ and any p with $\|p-q\| \leq 1/(\omega(k)+1)$, we have $p \in AF_k$.

Lemma 19. *The set $F = F(T)$ is uniformly closed w.r.t. (AF_k) with moduli*

$$\begin{cases} \delta(k) = 2k+1, \\ \omega(k) = 4k+3. \end{cases}$$

Proof. Note that we have

$$\begin{aligned} d(p,Tp) &\leq d(p,Tq) + H(Tp,Tq) \\ &\leq \|p-q\| + d(q,Tq) + \|q-p\| \end{aligned}$$

and thus if $q \in AF_{2k+1}$ and $\|p-q\| \leq \frac{1}{4(k+1)}$, then $d(p,Tp) \leq \frac{1}{k+1}$, i.e. $p \in AF_k$. $\square$

Combined, we can now apply the general result from [27] to get the following quantitative version of Theorem 7:

Theorem 20. *Let γ be a modulus of total boundedness for K. Let b be a bound on the diameter of K and let $(\alpha_n) \subseteq [0,1]$ be such that there exists an $a \in \mathbb{N}^*$ with the property*

$$\forall n \geq a \left(\frac{1}{a} \leq \alpha_n \leq 1 - \frac{1}{a} \right).$$

Let $(\gamma_n) \subseteq (0,\infty)$ be such that $\gamma_n \to 0$ and let τ be a monotone rate of convergence for $\gamma_n \to 0$. Let θ be such that

$$\forall p \in K \forall k \in \mathbb{N} \left(d(p,Tp) \leq \frac{1}{\theta(k)+1} \to H(\{p\},Tp) \leq \frac{1}{k+1} \right).$$

Let $\varphi_{a,\tau,b}(k,0)$ be defined as in Remark 17, i.e.

$$\varphi_{a,\tau,b}(k,0) = \max\{a, \tau(t(2t+1)a^t(k+1)-1)\} + (bt(2t+1)a^t(k+1)-1)t + 2b(k+1)t$$

for $t = 2ba(k+1)$. Then (x_n) defined as in (†) is Cauchy and moreover, for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Psi(k,g) \forall i,j \in [N; N+g(N)] \left(\|x_i - x_j\| \leq \frac{1}{k+1} \wedge x_i \in AF_k \right)$$

where $\Psi(k,g) = \Psi_0(P,k,g)$ for $P = \gamma(4k+3)$ and with

$$\begin{cases} \Psi_0(0,k,g) = 0, \\ \Psi_0(n+1,k,g) = \varphi_{a,\tau,b}(\chi^M_{k,g}(\Psi_0(n,k,g), 8k+7), 0), \end{cases}$$

and where

$$\chi(n,m,r) = \theta(m(r+1)+1),$$
$$\chi_k(n,m,r) = \max\{2k+1, \chi(n,m,r)\},$$
$$\chi^M_{k,g}(n,r) = \max\{\chi_k(i,g(i),r) \mid i \leq n\}.$$

Proof. The result rather immediately follows from Theorem 5.3 in [27] (which itself builds on Theorem 5.1 in [27]) by instantiating the bound given there with the moduli obtained in Lemmas 13, 16, 19. Concretely, χ in [27] is instantiated by χ as above and Φ in [27] is instantiated by $\varphi_{a,\tau,b}(\cdot,0)$. Further, δ_F and ω_F in [27] are instantiated by δ and ω as in Lemma 19 and we have $G = H = \mathrm{id}$ and thus $\alpha_G(k) = \beta_H(k) = k$. Note lastly that as τ is monotone, so is $\varphi_{a,\tau,b}(\cdot,0)$ as follows by Remark 17. The bounds given here result from the ones given in [27] only by immediate simplifications. □

Remark 21. Theorem 20 is a full finitization of Theorem 7 in the sense of Tao as it only references finite segments of the iteration (x_n) but it trivially implies back the original formulation of Theorem 7 as all the moduli naturally exist and since metastability is (non-effectively) equivalent to convergence (see also Remark 5.5 in [27]).

4.4. **Moduli of regularity and rates of convergence.** In this section, using the results from [29], we give constructions for rates of convergence based on the assumption of a (very general) kind of regularity notion as discussed in the introduction.

The central notion here is consequently the following instantiation of the abstract notion of a modulus of regularity from [29]:

Definition 22. Let $z \in F(T)$ and $r > 0$. A function $\phi : (0,\infty) \to (0,\infty)$ is called a modulus of regularity for T w.r.t $\overline{B}_r(z)$ if for all $\varepsilon > 0$ and all $x \in \overline{B}_r(z)$:

$$d(p,Tp) < \phi(\varepsilon) \to \mathrm{dist}(x,F(T)) < \varepsilon.$$

If there is a $z \in F(T)$ such that ϕ is a modulus of regularity w.r.t. $\overline{B}_r(z)$ for all $r > 0$, then ϕ is just called a modulus of regularity for T.

Remark 23. Note that the work [29] is written in the context of a formal setup where instead of using sets F/AF_k as above to formulate the solutions and approximative solutions, a function $F : X \to [0,+\infty]$ is employed and the roles of the sets F/AF_k are (conceptually) replaced by $\mathrm{zer}F/\{x \mid F(x) \leq \varepsilon\}$ for $\varepsilon > 0$. The above notion arises from the general definition given in [29] by using $F(x) := d(x,Tx)$ but we in the following suppress this whole setup from [29].

Note that the function $d(p,Tp)$ is continuous in p if T is nonexpansive as

$$\begin{aligned} d(p,Tp) &\leq d(p,Tq) + H(Tp,Tq) \\ &\leq \|p-q\| + d(q,Tq) + \|q-p\| \end{aligned}$$

and thus

$$|d(p,Tp) - d(q,Tq)| \leq 2\,\|p-q\|\,.$$

It follows from Proposition 3.3 of [29] that any such nonexpansive map T has a modulus of regularity (albeit in general being uncomputable) if K is compact.

Under the assumption of such a modulus, we now get the following result on rates of convergence:

Theorem 24. *Let $z \in F(T) \neq \emptyset$ and let b be a bound on the diameter of K. Assume that K is closed. Let (x_n) be defined as in (†). Assume that T satisfies $(*)$. Let (α_n) with a and (γ_n) with τ as well as $\varphi_{a,\tau,b}(k,0)$ be as in Theorem 20 (and Remark 17). Let ϕ be a modulus of regularity for T w.r.t. $\overline{B}_b(z)$. Then (x_n) is Cauchy with*

$$\forall \varepsilon > 0 \forall i, j \geq \varphi_{a,\tau,b}\left(\left\lceil \frac{1}{\phi(\varepsilon/2)} \right\rceil, 0\right) (\|x_i - x_j\| < \varepsilon).$$

and further (x_n) converges to a fixed point of T with a rate of convergence

$$\varphi_{a,\tau,b}\left(\left\lceil \frac{1}{\phi(\varepsilon/2)} \right\rceil, 0\right).$$

Proof. The result is a straightforward instantiation of the general abstract Theorem 4.1 from [29], using the previous Lemma 16 by which we have that

$$\forall \varepsilon > 0 \exists n \leq \varphi_{a,\tau,b}\left(\left\lceil \frac{1}{\varepsilon} \right\rceil, 0\right) (d(x_n, Tx_n) < \varepsilon).$$

Note for this that the sequence (x_n) is Fejér monotone w.r.t. $F(T)$ by Remark 14 since T satisfies $(*)$. That (x_n) converges to a fixed point of T with the given rate follows from Theorem 4.1, (i) in [29] for which we need that K is complete (which follows as X is a Banach space and as K is closed) and that $F(T)$ is closed which follows from the fact that $d(p, Tp)$ is uniformly continuous in p and $F(T) = (d(\cdot, T\cdot))^{-1}(0)$. □

Remark 25. Note that the above Theorem 24 holds without any compactness assumptions on K. Thus, in the presence of a modulus of regularity, the convergence result from Theorem 7 immediately holds for any closed, bounded and non-empty set K and any nonexpansive mapping T with $F(T) \neq \emptyset$ that satisfies $(*)$.

Finally, we look at a notion for multi-valued mappings where simple instances of such moduli of regularity can be derived. Following [58], a multivalued mapping $T : K \to \mathrm{CB}(K)$ is said to satisfy Condition I if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ and

$$d(x, Tx) \geq f(d(x, F(T)))$$

for all $x \in K$. If the property that $f(r) > 0$ for $r \in (0, \infty)$ is witnessed in a uniform and quantitative way by a function $\phi : (0, \infty) \to (0, \infty)$ with

$$f(r) < \phi(\varepsilon) \to r < \varepsilon$$

for any $r, \varepsilon > 0$, then such a ϕ is clearly already a modulus of regularity for T. This in particular is true for mappings that satisfy Condition II of [58], i.e. where there exists a real $\alpha > 0$ such that

$$d(x, Tx) \geq \alpha d(x, \mathrm{F}(T))$$

where then ϕ can be given by $\phi(\varepsilon) = \alpha\varepsilon$. Examples of mappings which satisfy Condition II are for instance discussed in [58] and for these, the above rates of convergence therefore instantiate immediately.

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References

[1] H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* CMS Books in Mathematics. Springer, Cham, 2nd edition, 2017.

[2] C.E. Chidume and C.H. Morales. Accretive operators which are always single-valued in normed spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 67(12):3328–3334, 2007.

[3] P.L. Combettes. Quasi-Fejérian Analysis of Some Optimization Algorithms. *Studies in Computational Mathematics*, 8:115–152, 2001.

[4] P.L. Combettes. Fejér monotonicity in convex optimization. In C.A. Floudas and P.M. Pardalos, editors, *Encyclopedia of Optimization*, pages 1016–1024. Springer, New York, 2009.

[5] B. Dinis and P. Pinto. Metastability of the multi-parameters proximal point algorithm. *Portugaliae Mathematica*, 77(3):345–381, 2020.

[6] B. Dinis and P. Pinto. Quantitative results on the multi-parameters proximal point algorithm. *Journal of Convex Analysis*, 28(3):729–750, 2021.

[7] F. Ferreira, L. Leuştean, and P. Pinto. On the removal of weak compactness arguments in proof mining. *Advances in Mathematics*, 354, 2019. 106728.

[8] R. Findling and U. Kohlenbach. Rates of convergence and metastability for Chidume's algorithm for the approximation of zeros of accretive operators in Banach spaces. *Numerical Functional Analysis and Optimization*, 45(3):216–233, 2024.

[9] P. Gerhardy. Proof mining in topological dynamics. *Notre Dame Journal of Formal Logic*, 49:431–446, 2008.

[10] P. Gerhardy and U. Kohlenbach. General logical metatheorems for functional analysis. *Transactions of the American Mathematical Society*, 360:2615–2660, 2008.

[11] K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958.

[12] D. Günzel and U. Kohlenbach. Logical metatheorems for abstract spaces axiomatized in positive bounded logic. *Advances in Mathematics*, 290:503–551, 2016.

[13] W.A. Howard. Hereditarily majorizable functionals of finite type. In A.S. Troelstra, editor, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*, pages 454–461. Springer, New York, 1973.

[14] U. Kohlenbach. Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. *The Journal of Symbolic Logic*, 57:1239–1273, 1992.

[15] U. Kohlenbach. Analysing proofs in analysis. In W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, editors, *Logic: From Foundations to Applications. European Logic Colloquium*, pages 225–260. Oxford University Press, Oxford, 1996.

[16] U. Kohlenbach. Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. *Archive of Mathematical Logic*, 36:31–71, 1996.

[17] U. Kohlenbach. Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357(1):89–128, 2005.

[18] U. Kohlenbach. A logical uniform boundedness principle for abstract metric and hyperbolic spaces. *Electronic Notes in Theoretical Computer Science*, 165:81–93, 2006.

[19] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics.* Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2008.

[20] U. Kohlenbach. Recent progress in proof mining in nonlinear analysis. *IFCoLog Journal of Logics and their Applications*, 10(4):3361–3410, 2017.

[21] U. Kohlenbach. A polynomial rate of asymptotic regularity for compositions of projections in Hilbert space. *Foundations of Computational Mathematics*, 19:83–99, 2019.

[22] U. Kohlenbach. Proof-theoretic Methods in Nonlinear Analysis. In B. Sirakov, P. Ney de Souza, and M. Viana, editors, *Proc. ICM 2018*, volume 2, pages 61–82. World Scientific, 2019.

[23] U. Kohlenbach. Quantitative analysis of a Halpern-type Proximal Point Algorithm for accretive operators in Banach spaces. *Journal of Nonlinear and Convex Analysis*, 21(9):2125–2138, 2020.

[24] U. Kohlenbach. Quantitative results on the Proximal Point Algorithm in uniformly convex Banach spaces. *Journal of Convex Analysis*, 28(1):11–18, 2021.

[25] U. Kohlenbach. On the Proximal Point Algorithm and its Halpern-type variant for generalized monotone operators in Hilbert space. *Optimization Letters*, 16:611–621, 2022.

[26] U. Kohlenbach and A. Koutsoukou-Argyraki. Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators. *Journal of Mathematical Analysis and Applications*, 423:1089–1112, 2015.

[27] U. Kohlenbach, L. Leuştean, and A. Nicolae. Quantitative Results on Fejér Monotone Sequences. *Communications in Contemporary Mathematics*, 20(2):42pp., 2018.

[28] U. Kohlenbach, G. López-Acedo, and A. Nicolae. Quantitative Asymptotic Regularity Results for the Composition of Two Mappings. *Optimization*, 66:1291–1299, 2017.

[29] U. Kohlenbach, G. López-Acedo, and A. Nicolae. Moduli of regularity and rates of convergence for Fejér monotone sequences. *Israel Journal of Mathematics*, 232:261 – 297, 2019.

[30] U. Kohlenbach and A. Nicolae. A proof-theoretic bound extraction theorem for CAT(κ)-spaces. *Studia Logica*, 105:611–624, 2017.

[31] U. Kohlenbach and P. Oliva. Proof Mining: A Systematic Way of Analysing Proofs in Mathematics. *Proceedings of the Steklov Institute of Mathematics*, 242:136–164, 2003.

[32] U. Kohlenbach and T. Powell. Rates of convergence for iterative solutions of equations involving set-valued accretive operators. *Computers & Mathematics with Applications*, 80:490–503, 2020.

[33] G. Kreisel. On the Interpretation of Non-Finitist Proofs–Part I. *The Journal of Symbolic Logic*, 16(4):241–267, 1951.

[34] G. Kreisel. On the Interpretation of Non-Finitist Proofs–Part II. Interpretation of Number Theory. Applications. *The Journal of Symbolic Logic*, 17(1):43–58, 1952.

[35] L. Leuştean. Proof Mining in $\mathbb{R}$-trees and Hyperbolic Spaces. *Electronic Notes in Theoretical Computer Science*, 165:95–106, 2006.

[36] L. Leuştean. An application of proof mining to nonlinear iterations. *Annals of Pure and Applied Logic*, 165(9):1484–1500, 2014.

[37] L. Leuştean, A. Nicolae, and A. Sipoş. An abstract proximal point algorithm. *Journal of Global Optimization*, 72(3):553–577, 2018.

[38] L. Leuştean and P. Pinto. Quantitative results on the Halpern type proximal point algorithm. *Computational Optimization and Applications*, 79(1):101–125, 2021.

[39] L. Leuştean and A. Sipoş. An application of proof mining to the proximal point algorithm in CAT(0) spaces. In A. Bellow, C. Calude, and T. Zamfirescu, editors, *Mathematics Almost Everywhere. In Memory of Solomon Marcus*, pages 153–168. World Scientific, 2018.

[40] L. Leuştean and A. Sipoş. Effective strong convergence of the proximal point algorithm in CAT(0) spaces. *Journal of Nonlinear and Variational Analysis*, 2(2):219–228, 2018.

[41] C. Moore and B.V.C. Nnoli. Iterative solution of nonlinear equations involving set-valued uniformly accretive operators. *Computers & Mathematics with Applications*, 42(1–2):131–140, 2001.

[42] S.B. Nadler. Multi-valued contraction mappings. *Pacific Journal of Mathematics*, 30:475–487, 1969.

[43] M. Neri and N. Pischke. Proof mining and probability theory. 2024. 49pp., submitted manuscript available at `https://arxiv.org/abs/2403.00659v2`.

[44] E. Neumann. Computational Problems in Metric Fixed Point Theory and their Weihrauch Degrees. *Logical Methods in Computer Science*, 11(4), 2015. 20, 44pp.

[45] L. Paunescu and A. Sipoş. A proof-theoretic metatheorem for tracial von Neumann algebras. *Mathematical Logic Quarterly*, 69(1):63–76, 2023.

[46] P. Pinto. A rate of metastability for the Halpern type Proximal Point Algorithm. *Numerical Functional Analysis and Optimization*, 42(3):320–343, 2021.

[47] P. Pinto and N. Pischke. On Computational Properties of Cauchy Problems generated by Accretive Operators. *Documenta Mathematica*, 28(5):1235–1274, 2023.

[48] N. Pischke. On Korpelevich's Extragradient Algorithm. *Zeitschrift für Analysis und ihre Anwendungen*, 42(1/2):117–129, 2023.

[49] N. Pischke. Quantitative Results on Algorithms for Zeros of Differences of Monotone Operators in Hilbert Space. *Journal of Convex Analysis*, 30(1):295–315, 2023.

[50] N. Pischke. A proof-theoretic bound extraction theorem for monotone operators in Banach spaces. 2024. 38pp., submitted manuscript available at `https://nicholaspischke.github.io`.

[51] N. Pischke. Generalized Fejér monotone sequences and their finitary content. *Optimization*, 2024. to appear, 68pp., doi:10.1080/02331934.2024.2390114.

[52] N. Pischke. Logical Metatheorems for Accretive and (Generalized) Monotone Set-Valued Operators. *Journal of Mathematical Logic*, 24(2), 2024. 2350008, 59pp.
[53] N. Pischke. Proof mining for the dual of a Banach space with extensions for uniformly Fréchet differentiable functions. *Transactions of the American Mathematical Society*, 377(10):7475–7517, 2024.
[54] N. Pischke. *Proof-Theoretical Aspects of Nonlinear and Set-Valued Analysis*. PhD thesis, TU Darmstadt, 2024. Thesis available at `https://doi.org/10.26083/tuprints-00026584`.
[55] N. Pischke. Rates of convergence for the asymptotic behavior of second-order Cauchy problems. *Journal of Mathematical Analysis and Applications*, 533(2), 2024. 128078, 15pp.
[56] N. Pischke. A proof-theoretic metatheorem for nonlinear semigroups generated by an accretive operator and applications. *Selecta Mathematica*, 31(2), 2025. 32, 67pp.
[57] N. Pischke and U. Kohlenbach. Effective rates for iterations involving Bregman strongly nonexpansive operators. *Set-Valued and Variational Analysis*, 32(4), 2024. 33, 58pp.
[58] H.F. Senter and W.G. Dotson. Approximating fixed points of nonexpansive mappings. *Proceedings of the American Mathematical Society*, 44(2):375–380, 1974.
[59] A. Sipoş. Proof mining in L^p spaces. *The Journal of Symbolic Logic*, 84(4):1612–1629, 2019.
[60] A. Sipoş. Quantitative inconsistent feasibility for averaged mappings. *Optimization Letters*, 16(6):1915–1925, 2022.
[61] A. Sipoş. The computational content of super strongly nonexpansive mappings and uniformly monotone operators. *Israel Journal of Mathematics*, 2023. to appear, 17pp., arXiv:2303.02768.
[62] Y. Song and H. Wang. Convergence of iterative algorithms for multivalued mappings in Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 70(4):1547–1556, 2009.
[63] E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *Journal of Symbolic Logic*, 14:145–158, 1949.
[64] T. Suzuki. Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. *Journal of Fixed Point Theory and Applications*, 2005(1):103–123, 2005.
[65] W. Takahashi. *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama, Japan, 2000.
[66] T. Tao. Norm Convergence of Multiple Ergodic Averages for Commuting Transformations. *Ergodic Theory and Dynamical Systems*, 28:657–688, 2008.
[67] T. Tao. *Structure and Randomness: Pages from Year One of a Mathematical Blog*, chapter Soft analysis, hard analysis, and the finite convergence principle. American Mathematical Society, Providence, RI, 2008.
[68] A.S. Troelstra, editor. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*. Springer, New York, 1973.