

CO749 - Random Graph Theory

(Lecture Summaries ... sort of)

University of Waterloo
Nicholas Pun
Winter 2020

Contents

Lecture 1 Introduction	2
1.1 Probability Primer	2
Lecture 2 Concentration Inequalities, Coupling, Connection Theorem	4
2.1 Concentration Inequalities	4
2.2 Coupling	4
2.3 Connection Theorem	5
Lecture 3 Threshold, First Order Logic of Graphs	7
3.1 Threshold	7
3.2 First Order Logic of Graphs	7
Lecture 4 Evolution of Graphs and Theorem E	9
4.1 Small Subgraphs	9
Lecture 5 Cycles, Degrees of Vertices, Critical Window Analysis	10
5.1 Degrees of Vertices	10
5.2 Critical Window Analysis	10
Lecture 6	13
Lecture 7	18
Lecture 8	24
Lecture 9	31
Lecture 10	37
Lecture 11	43
Lecture 12	48
Lecture 13	55
Lecture 14	61
Lecture 15	67
Lecture 16	72
References	78

Lecture 1: Introduction

Definition 1.1. The probability space we'll work in is denoted with the triple (G, \mathbb{P}, F) , where G is a class of graphs, \mathbb{P} a probability measure and F a sigma algebra.

Normally, G is a finite set, \mathbb{P} is a discrete probability measure and $F = 2^G$

Definition 1.2 (Erdős-Rényi Random Graph Model).

- The $\mathcal{G}(n, p)$ model: A graph with vertex set $[n]$ is constructed randomly by including each edge in $k_{[n]}$ with probability p
- The $\mathcal{G}(n, m)$ model: A graph is chosen uniformly at random from all graphs with vertex set $[n]$ and has m edges.

(Aside: We can think of $\mathcal{G}(n, m)$ as labelling the edges)

Other models:

- $\mathcal{G}(n, d)$ is the model of random d -regular graphs
- $\mathcal{G}(n, \tilde{d})$ where $\tilde{d} = (d_1, \dots, d_n)$ is a vector representing the degrees of vertices. (This is a generalization of $\mathcal{G}(n, d)$)
- $\mathcal{G}(n, r)$ is the model of random geometric graphs. The construction is as follows: Pick n points uniformly in the unit square, then, add an edge if and only if the distance between two points is $\leq r$
- Random trees. A tree is chosen uniformly at random from the n^{n-2} trees on n vertices.

In this class, we will primarily focus on the Erdős-Rényi Model.

1.1 Probability Primer

Definition 1.3. A discrete probability space consists of a countable set Ω and a probability function $\mathbb{P} : \Omega \rightarrow [0, 1]$ such that $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

A subset of Ω is called an event. The probability of $A \subseteq \Omega$ is $\sum_{\omega \in A} \mathbb{P}(\omega)$, denoted $\mathbb{P}(A)$.

Proposition 1.1 (Inclusion-Exclusion). For events A, B :

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

and, in general:

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i_1 < i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \dots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^n A_i\right)$$

Corollary 1.1. $\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i)$

Definition 1.4. Two events are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$

Definition 1.5. A random variable (r.v) X is a function $X : \Omega \rightarrow \mathbb{R}$. In a discrete probability space, the expectation of X is defined by: $\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega)$

Proposition 1.2 (Linearity of Expectation). $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Proof. $\mathbb{E}(X + Y) = \sum_{\omega \in \Omega} (X + Y)(\omega) \mathbb{P}(\omega) = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega) + \sum_{\omega \in \Omega} Y(\omega) \mathbb{P}(\omega) = \mathbb{E}(X) + \mathbb{E}(Y)$ \square

Lemma 1.1.

- For any $n \geq k \geq 1$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$$

- (Stirling's Formula)

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \mathcal{O}(n^{-2})\right)$$

- For every $t \in \mathbb{R}$, $e^t \geq 1 + t$

Lemma 1.2. Assume $k = o(\sqrt{n})$. Then, $\binom{n}{k} \sim \frac{n^k}{k!}$

Proof.

$$\begin{aligned} \binom{n}{k} &= \frac{1}{k!} \prod_{i=0}^{k-1} (n-i) \\ &= \frac{n^k}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \\ &= \frac{n^k}{k!} \prod_{i=0}^{k-1} e^{\mathcal{O}(i/n)} \quad (\log(1-x) = \mathcal{O}(x)) \\ &= \frac{n^k}{k!} \exp\left(\mathcal{O}\left(\frac{1}{n} \sum_{i=0}^{k-1} i\right)\right) \\ &= \frac{n^k}{k!} \exp\left(\mathcal{O}\left(\frac{k^2}{n}\right)\right) \\ &= (1 + o(1)) \frac{n^k}{k!} \quad (\text{as } k = o(\sqrt{n})) \end{aligned}$$

\square

Remark 1.1. $k = o(n^{\frac{2}{3}})$, then $\binom{n}{k} \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$

Lecture 2: Concentration Inequalities, Coupling, Connection Theorem

Definition 2.1. Given a sequence of probability spaces $(\Omega_n, P_n)_{n \geq 1}$. We say that A_n holds asymptotically almost surely (a.a.s) if $P_n(A_n) \rightarrow 1$ as $n \rightarrow \infty$

2.1 Concentration Inequalities

Theorem 2.1 (Markov's Inequality). Let X be a nonnegative random variable. Then, for any real $t > 0$, $\Pr(X \geq t) \leq \frac{\mathbb{E} X}{t}$

Proof. Let I_t be the indicator r.v. that $X \geq t$. Then, $X \geq t \cdot I_t$, so:

$$\mathbb{E} X \geq t \cdot \mathbb{E} I_t = t \cdot \mathbb{P}(X \geq t)$$

□

Theorem 2.2 (Chebyshev's Inequality). For any $t \geq 0$

$$\mathbb{P}(|X - \mathbb{E} X| \geq t) \leq \frac{\text{Var } X}{t^2}$$

Example 2.1. Let X be the number of edges in $\mathcal{G}(n, p)$, $N = \binom{n}{2}$. $X \sim \text{Bin}(N, p)$ so $\mathbb{E} X = Np$ and $\text{Var } X = p(1-p)N$.

Further, by Chevyshev's Inequality, for all $t > 0$:

$$\mathbb{P}(|X - \mathbb{E} X| \geq t) \leq \frac{p(1-p)N}{t^2}$$

△

This leads us to the following proposition:

Proposition 2.1. Let $f_n \rightarrow \infty$ as $n \rightarrow \infty$. Then,

$$\mathbb{P}(|X - Np| \geq f_n \sqrt{p(1-p)N}) \leq \frac{1}{f_n^2} = o(1)$$

So a.a.s:

$$pN - f_n \sqrt{p(1-p)N} \leq X \leq pN + f_n \sqrt{p(1-p)N}$$

2.2 Coupling

Definition 2.2. Given 2 r.v.s X, Y , a coupling of X and Y is a construction of a joint distribution of (\hat{X}, \hat{Y}) into the probability space such that marginally $\hat{X} \sim X$ and $\hat{Y} \sim Y$

Lemma 2.1.

(a) Let $0 \leq m_1 < m_2 \leq N$ and $0 \leq p_1 < p_2 \leq 1$. There exist couplings such that:

$$\mathcal{G}(n, m_1) \subseteq \mathcal{G}(n, m_2) \quad \text{and} \quad \mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, p_2)$$

where by $\mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, p_2)$ (and respectively, m_1 and m_2), we mean that there exists a coupling (G_1, G_2) such that:

- Marginally, $G_1 \sim \mathcal{G}(n, p_1)$, $G_2 \sim \mathcal{G}(n, p_2)$, and
- jointly, $G_1 \subseteq G_2$ always

(b) Let $m_1 = pN - f\sqrt{p(1-p)N}$, $m_2 = pN + f\sqrt{p(1-p)N}$ ($f = f(n)$ as before). Then, there exists a coupling (G_1, H, G_2) such that:

- $G_1 \sim \mathcal{G}(n, m_1)$, $G_2 \sim \mathcal{G}(n, m_2)$, $H \sim \mathcal{G}(n, p)$
- $\mathbb{P}(G_1 \subseteq H \subseteq G_2) = 1 - o(1)$

Proof.

(a) Let $G_1 \sim \mathcal{G}(n, p_1)$. For G_2 , include every non-edge in G_1 , include it independently with probability $q = 1 - \frac{1-p_2}{1-p_1}$. Clearly, $G_1 \subseteq G_2$. Then, check the probability that an edge is not included in G_2 :

$$(1 - p_1)(1 - q) = (1 - p_1) \left(1 - \left(1 - \frac{1 - p_2}{1 - p_1} \right) \right) = 1 - p_2$$

For $G_1 \sim \mathcal{G}(n, m_1)$, $G_2 \sim \mathcal{G}(n, m_2)$, we choose permutation and the first m_1, m_2 edges

□

2.3 Connection Theorem

Definition 2.3. Let Ω be the set of graphs on $[n]$. $Q \subseteq \Omega$ is a graph property if it is invariant under graph isomorphism. We say Q is monotone increasing if:

$$G \in Q \Rightarrow H \in Q \quad \forall H \supseteq G$$

Further, we say Q is convex if:

$$G_1, G_2 \in Q, G_1 \subseteq G_2 \Rightarrow H \in Q \quad \forall G_1 \subseteq H \subseteq G_2$$

Theorem 2.3. Suppose Q is monotone. Then, for any $0 \leq m_1 \leq m_2 \leq N$, $0 \leq p_1 \leq p_2 \leq 1$:

$$\begin{aligned} \mathbb{P}(\mathcal{G}(n, m_1) \in Q) &\leq \mathbb{P}(\mathcal{G}(n, m_2) \in Q) \\ \mathbb{P}(\mathcal{G}(n, p_1) \in Q) &\leq \mathbb{P}(\mathcal{G}(n, p_2) \in Q) \end{aligned}$$

Theorem 2.4 (Connection Theorem). Let Q be a graph property:

- (i) Given $p = p(n)$. Suppose for all $m = pN + \mathcal{O}(\sqrt{p(1-p)N})$ we have $\mathcal{G}(n, m) \in Q$ a.a.s.
Then, a.a.s $\mathcal{G}(n, p) \in Q$
- (ii) Suppose Q is convex. Given $m = m(n)$ and suppose $\mathcal{G}(n, m/N) \in Q$ a.a.s. Then, a.a.s $\mathcal{G}(n, m) \in Q$

Proof Sketch.

- (i) Write $\mathbb{P}(\mathcal{G}(n, p) \in G)$ in terms of the number of edges in the graph, i.e. $\mathbb{P}(\mathcal{G}(n, p) \in G) = \sum_{m=0}^N \mathbb{P}(X = m, \mathcal{G}(n, p) \in Q)$ (Law of total probability). Then, use Proposition 2.1.
- (ii) Condition on the number of edges, and analyze the probabilities of having a graph with: less edges than 1 standard deviation (m_1), more edges than 1 standard deviation (m_2), and a number of edges within 1 standard deviation (m). Then construct graphs from $\mathcal{G}(n, m_1)$ and $\mathcal{G}(n, m_2)$ and use convexity to show $\mathbb{P}(\mathcal{G}(n, m) \in Q) = 1 - o(1)$

□

Lecture 3: Threshold, First Order Logic of Graphs

3.1 Threshold

Definition 3.1. We say a property Q has a threshold p_0 if:

$$\mathbb{P}(\mathcal{G}(n, p) \in Q) \rightarrow \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}$$

Theorem 3.1 (Bollobás & Thomason, 1987). Every non-trivial monotone property has a threshold

Definition 3.2. We say a property Q has a sharp threshold p_0 if $\forall \epsilon > 0$:

$$\mathbb{P}(\mathcal{G}(n, p) \in Q) \rightarrow \begin{cases} 0 & \text{if } p \leq (1 - \epsilon)p_0 \\ 1 & \text{if } p \geq (1 + \epsilon)p_0 \end{cases}$$

Definition 3.3. The window of a threshold is $\delta(\epsilon) = p_{1-\epsilon} - p_\epsilon$

3.2 First Order Logic of Graphs

Example 3.1.

$$\forall x \forall y \exists z (x = y \vee x \sim y \vee (x \sim z \wedge y \sim z))$$

is the statement characterizing the graphs of diameter ≤ 2 \triangle

Fix $k > 0$. Let P_k be the property that for any disjoint sets W and V of order at most k , there exists a vertex $x \in V(G) \setminus (W \cup V)$ such that x is adjacent to all vertices in W and is adjacent to none of V

Lemma 3.1. Suppose $m(n), p(n)$ satisfy the following:

For every fixed $\epsilon > 0$

$$\begin{aligned} mn^{-2+\epsilon} &\rightarrow \infty, & (N-m)n^{-2+\epsilon} &\rightarrow \infty \\ pn^\epsilon &\rightarrow \infty, & (1-p)n^\epsilon &\rightarrow \infty \end{aligned}$$

For every fixed $k > 0$, a.a.s $\mathcal{G}(n, p) \in P_k$ and $\mathcal{G}(n, m) \in P_k$

Theorem 3.2 (0-1 law of the 1st order logic of random graphs). Suppose $m(n), p(n)$ satisfy the conditions of the lemma. Suppose Q is a graph property given by a 1st order sentence. Then, either Q holds a.a.s or does not hold a.a.s.

Proof Sketch. We play a k -round Ehrenfeucht-Fraïssé Game. Player 1 chooses vertices from either graph and Player 2 must choose vertices from the other graph.

After k rounds, this produces two sequences v_1, v_2, \dots, v_k in G_1 and v'_1, v'_2, \dots, v'_k in G_2 . Player 2 wins if $v_i \mapsto v'_i \forall 1 \leq i \leq k$ is an isomorphism between $G_1[v_1, v_2, \dots, v_k]$ and $G_2[v'_1, v'_2, \dots, v'_k]$, and Player 1 wins otherwise.

The idea is that if G_1, G_2 are similar, then player 2 will win, but if they are not similar, then player 1 can exploit the dissimilarity.

Claim: Let $Th_k(G)$ be the set of graph properties of G expressible by 1st-order logic sentences with quantifier depth at most k . Player 2 has a winning strategy if and only if $Th_k(G_1) = Th_k(G_2)$ (i.e. They share the same set of properties) \square

Lecture 4: Evolution of Graphs and Theorem E

Theorem 4.1 (Theorem E).

- (a) Fix $k \geq 2$ integer. If $n^{\frac{k-2}{k-1}-2} \ll p \ll n^{\frac{k-1}{k}-2}$, then a.a.s $\mathcal{G}(n, p)$ is a forest and the largest component is of order k .
- (b) If $p \ll \frac{1}{n}$, then a.a.s $\mathcal{G}(n, p)$ is a forest and the largest component is of order $o(\log n)$
- (c) If $p = \frac{c}{n}$, $0 < c < 1$, then a.a.s every component of $\mathcal{G}(n, p)$ is a tree or unicyclic and the largest component has order $\Theta(\log n)$
- (d) If $p = \frac{c}{n}$, $c > 1$, then a.a.s $\mathcal{G}(n, p)$ contains a unique component of linear order and all other components of order $\mathcal{O}(\log n)$
- (e) When $p \geq \frac{\log n + \log \log n \omega(1)}{2n}$, a.a.s $\mathcal{G}(n, p)$ has a giant component and a few isolated vertices
- (f) When $p \geq \frac{\log n + \omega(1)}{n}$, a.a.s $\mathcal{G}(n, p)$ connected and has a perfect matching if n even, or a matching of size $\frac{n-1}{2}$ if n odd.
- (g) When $p \geq \frac{\log n + \log \log n + \omega(1)}{n}$, a.a.s $\mathcal{G}(n, p)$ is Hamiltonian

4.1 Small Subgraphs

Lemma 4.1. If $p = o(1/n)$, then a.a.s $\mathcal{G}(n, p)$ has no cycles

Proof. Use Markov's Inequality □

Proof. (of Theorem 4.1 (a))

That $\mathcal{G}(n, p)$ is a forest is directly implied by above. It remains to show that every tree has order $\leq k$ and there is one tree of order k .

Let X_t be the number of trees of order t in $\mathcal{G}(n, p)$. First, we show that $\mathbb{E}(\sum_{t \geq k+1} X_t) = o(1)$, so a.a.s $\sum_{t \geq k+1} X_t = 0$ by Markov's inequality. This tells us that we don't have trees of order $> k$

Then, we show the existence of a tree of order k by using the 2nd moment method. □

Lecture 5: Cycles, Degrees of Vertices, Critical Window Analysis

Let $X \sim \text{Po}(\lambda)$ and recall that $\mathbb{P}(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \forall i \geq 0$ and $\mathbb{E}(X)_r = \lambda^r \forall r \geq 0$.

Theorem 5.1. Given a sequence of random variables $(X_n)_{n \geq 1}$ and $\lambda > 0$, suppose $\mathbb{E}(X_n)_r \rightarrow \lambda^r \forall r \geq 0$. Then, $X_n \rightarrow \text{Po}(\lambda)$ as $n \rightarrow \infty$

Let Y_k denote the number of k -cycles in $\mathcal{G}(n, p)$.

Theorem 5.2. Let $p = \frac{c}{n}$, where c fixed, then for every $k \geq 3$, $Y_k \rightarrow \text{Po}\left(\frac{c^k}{2k}\right)$

$$\text{Proof. } \mathbb{E}[Y_k] = \binom{n}{k} \frac{(k-1)!}{2} p^k \sim \frac{n^k}{k!} \frac{(k-1)!}{2} p^k = \frac{n^k p^k}{2k} = \frac{c^k}{2k}$$

Now, fix $r \geq 2$ and consider:

$$(Y_k)_r = |\{C_1, \dots, C_r : C_i \text{ is a } k\text{-cycle}\}|$$

So, $(Y_k)_r$ can be treated like the number of ordered lists of k -cycles. First, consider r k -cycles that are all vertex-disjoint, then consider the case where there are vertex intersections. Apply Theorem 5.1 to finish the proof. \square

5.1 Degrees of Vertices

Theorem 5.3. Let $p = \frac{c \log n}{n}$ ($c > 0$ fixed).

- (a) If $c < 1$, a.a.s \exists isolated vertices
- (b) If $c > 1$, a.a.s there are no isolated vertices

(Note: This is a sharp threshold)

Proof. Let X be the number of isolated vertices.

$$\mathbb{E}[X] = n \cdot (1 - p)^{n-1} \sim n^{1-c}$$

Then, if $c > 1$, then use Markov's inequality to find (b). If $c < 1$, then use Chebyshev's inequality to find (a). \square

5.2 Critical Window Analysis

Theorem 5.4. Let $p = \frac{\log n + c}{n}$, c fixed. Then

$$\mathbb{P}(X = 0) \sim e^{-e^{-c}} \tag{5.1}$$

Corollary 5.1. Let $p = \frac{\log n + c(n)}{n}$. If $c(n) \rightarrow -\infty$, then a.a.s $\mathcal{G}(n, p)$ has isolated vertices. And, if $c(n) \rightarrow +\infty$, then a.a.s $\mathcal{G}(n, p)$ has no isolated vertices

Let X_k be the number of vertices with degree k

Theorem 5.5. Let $\epsilon > 0$ be fixed, $\epsilon n^{-\frac{3}{2}} \leq p \leq 1 - \epsilon n^{-\frac{3}{2}}$. Let $k = k(n)$ be a nonnegative integer and $\lambda_k(n) = n \cdot \mathbb{P}(\text{Bin}(n-1, p) = k)$. Then,

- (i) If $\lambda_k(n) = o(1)$, then a.a.s $X_k = 0$
- (ii) If $\lambda_k(n) \rightarrow o(1)$, then a.a.s $X_k \geq t$ for any fixed t .
- (iii) If $0 < \lambda_k := \lim_{n \rightarrow \infty} \lambda_k(n) < \infty$, then $\mathbb{P}(X_k = t) \sim e^{\lambda_k} \cdot \frac{\lambda_k^t}{t!}$

Theorem 5.6. Let $p = \frac{\log n + c}{n}$, c fixed. Then, $\mathbb{P}(\mathcal{G}(n, p) \text{ is connected}) \rightarrow e^{-e^{-c}}$

In a feat of ultimate laziness (as the true computer scientist I am), the rest of the notes are going to be scans.

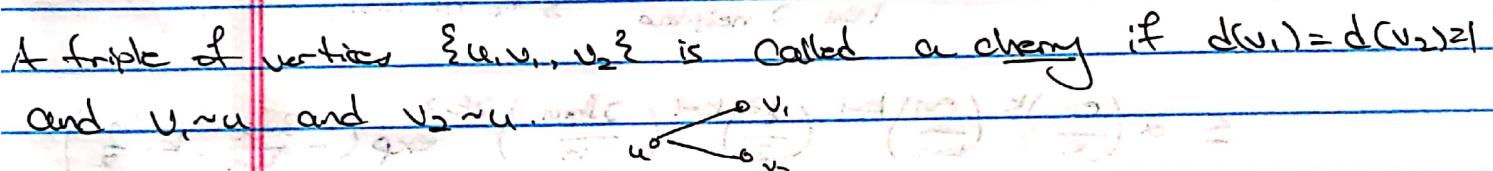
Lecture 6:

$G(n, n, p)$ denotes the random bipartite graph with $V = U \cup U_2$, $|U| = |U_2| = n$ and uv is an edge with prob p for $u \in U$, $v \in U_2$.

Exercise: let $p = \frac{\log n + c}{n}$. let X denote # of isolated vertices in $G(n, n, p)$. Then,

- $P(X=0) \sim e^{-2e^{-c}}$
- $P(G(n, n, p) \text{ is connected}) \sim e^{-2e^{-c}}$

A triple of vertices $\{u, v_1, v_2\}$ is called a cherry if $d(v_1) = d(v_2) \geq 1$ and $v_1 \sim u$ and $v_2 \sim u$.



Exercise: let $p \sim c \log n / n$ where $c > 3/5$. Then show there are no cherries in $G(n, n, p)$.

Theorem (Hoff's theorem)

A bipartite graph G with vertex set $V \cup U_2$ has a PML if $|V| = |U_2|$ and $|N(S)| \geq |S| \forall S \subseteq V$.

Exercise: Suppose G fails Hoff's condition. Then, there exists a set $S \subseteq U_2$ such that $S \subseteq V_1$ or $S \subseteq V_2$.

- (i) $|N(S)| = |S| - 1$
- (ii) $|S| \leq \lceil \frac{n}{2} \rceil$
- (iii) Every vertex in $N(S) \geq 2$ neighbors in S

(let X_k denote a k -set S satisfying (i), (ii), (iii))

X_1 : # isolated vertices

X_2 : # cherries

Lemma:

Let $(\exists)_k \frac{\log n}{n} < p < \frac{2\log n}{n}$. Then, $\sum_{k=3}^{\lceil \frac{n}{2} \rceil} \mathbb{E} X_k = o(1)$

Proof:

For each $3 \leq k \leq \lceil \frac{n}{2} \rceil$

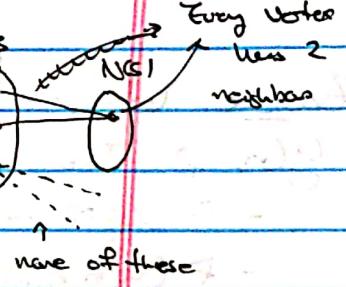
$$\mathbb{E} X_k \leq 2 \cdot \binom{n}{k} \cdot \binom{n}{k-1} \left(\left(\frac{k}{2} \right) p^2 \right)^{k-1} (1-p)^{k(n-k+1)}$$

Choose from
either V_1 or V_2

Each vertex
in $V_1 \cup V_2$
has 2 neighbors

no edges
going from
 s to t in $V_1 \cup V_2$

No edges



none of these

$$\leq 2 \left(\frac{en}{k} \right)^k \left(\frac{en}{k-1} \right)^{k-1} \left(\frac{k^2}{2} \right)^{k-1} \left(\frac{2\log n}{n} \right)^{2k-2} \exp \left(- \frac{3\log n}{4n} \cdot k \cdot \frac{n}{2} \right)$$

$$\text{Since } n-k+1 \leq \frac{n}{2}$$

$$\leq 2n \left(\frac{k}{k-1} \right)^{k-1} \left(\frac{2e^2 \log^2 n}{n^{3p}} \right)^k$$

$$= O(n \alpha^k), \text{ where } \alpha = \frac{3e^2 \log n}{n^{3p}}$$

So:

$$\sum_{k=3}^{\lceil \frac{n}{2} \rceil} \mathbb{E} X_k = o(1)$$

Theorem

Let $p = \frac{\log n + c}{n}$, c fixed. Then $\Pr(G(n, n, p) \text{ has a PM}) \sim e^{-2^{-c}}$

Proof:

Let C_1 denote the event that $X_1 = 0$, i.e. no isolated vertices.

$$C_2 \quad \text{---} \quad \text{---} \quad X_k = 0 \quad \forall 2 \leq k \leq \lceil \frac{n}{2} \rceil$$

Hoeffding's Theorem and Exercise $\Rightarrow C_1 \cap C_2 \Rightarrow G(n, n, p) \text{ has a PM}$

So, $\Pr(G(n, n, p) \text{ has PM}) \geq \Pr(C_1 \cap C_2)$

$$\begin{aligned} &= \Pr(C_1) - \Pr(C_1 \cap \bar{C}_2) \\ &\geq \Pr(C_1) - \Pr(\bar{C}_2) = \Pr(C_1) - o(1) \end{aligned}$$

Proof (cont'd)

So, $G(n, n, p)$ has a PM $\Rightarrow C_1$. ①

And,

$\Pr(G(n, n, p) \text{ has a PM}) \leq \Pr(C_1)$ ②

So:

$$\Pr(G(n, n, p) \text{ has a PM}) = \Pr(C_1) + o(1) \approx e^{-2e^{-c}}$$

Let T_k denote the number of components that are trees of order k .

$$(let T = \sum_{k=1}^n kT_k)$$

$$ET_k = \binom{n}{k} k^{k-2} p^{k-1} ((1-p))^{(\frac{n}{k}) - (k-1) + k(n-k)}$$

Edges in tree
↓
k vertices # of trees -
no other edges

↓
edges coming out of component
except for tree edges

$$ET = \sum_{k=1}^n k ET_k$$

Theorem: Let $p = \frac{c}{n}$, $c > 0$ fixed.

(a) If $0 < c < 1$, then $E[T(G(n, p))] = n + o(1)$

(b) If $c > 1$, then $E[T(G(n, p))] = t(c)n + o(1)$,

$$\text{where } t(c) = \frac{1}{c} \sum_{k=1}^{c-1} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

$$\text{Note: } t(c) \leq \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{ek}{k}\right)^k (ce^{-c})^k = \sum_{k=1}^{\infty} \frac{1}{k} (ce^{1-c})^k$$

Can show this is < 1 for $c > 1$.

So, $t(c)$ is well-defined.

$C \Rightarrow$

Exercise: (a) $S(c) = ct(c)$ is the unique solution of $se^{-s} = Ce^{-c}$ in the range $0 < c \leq 1$.

(b) $t(c) \approx 1$ for $0 < c \leq 1$.

Note: $\mathbb{E}[T(S(n,p))]$ is cf. in \mathbb{P} and $t(c) \rightarrow 1$ as $c \rightarrow 1$.

So, $\mathbb{E}[T(G(n,p))] = n + o(n)$. (Since $t(c)n = n + \text{"some error"}$)

Proof:

Since $t(c) \approx 1$ for $0 < c \leq 1$, it's sufficient to show that

$$\mathbb{E}T = t(c)n + o(n) \quad \forall c \in \mathbb{R}.$$

$$\mathbb{E}T_k = \binom{n}{k} k^{k-1} \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{n-k} e^{-\frac{k(c+3)}{2}} + 1.$$

Consider $1 \leq k \leq n$. (In this range $(2) \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$)

... uh... actually we'll use!

$$(2) = \frac{n^k}{k!} \exp\left(-\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right)\right).$$

Then,

$$\mathbb{E}T_k = n \cdot \frac{\frac{k^{k-1}}{n!} c^{k-1} e^{-ck} \exp\left(-\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right)\right)}{k!} + O\left(\frac{k}{n}\right).$$

Hence,

$$\begin{aligned} \left| \sum_{k=1}^n k \mathbb{E}T_k - \sum_{k=1}^n n \cdot \frac{\frac{k^{k-1}}{n!} c^{k-1} e^{-ck}}{k!} \right| \\ = \sum_{k=1}^n O\left(\frac{k^2}{n}\right) n \cdot \frac{\frac{k^{k-1}}{n!} c^{k-1} e^{-ck}}{k!} \\ = \sum_{k=1}^n O(k \cdot e^{k-ck} c^k) \\ = \sum_{k=1}^n O(k \underbrace{(ce^{-c})^k}_{< 1}) = O(1). \end{aligned}$$

$$\frac{\mathbb{E}(\text{Class}[T_{k+1}])}{\mathbb{E}[T_{k+1}]} = \underbrace{(n-k)(1+\frac{1}{k})^{k-2}}_{\leq (1-\frac{k}{n})} \cdot \frac{c}{n} (1-\frac{c}{n})^{n-k-2}$$

$$\leq (1-\frac{k}{n}) \cdot e^c e^{-\frac{cn}{n-k}} \cdot (1-\frac{c}{n})^{-2}$$

$$= (1-\frac{k}{n}) e^c e^{-c(1-\frac{k}{n})} (1-\frac{c}{n})^{-2}$$

$$\text{Let } \eta_U = (1-\frac{k}{n}).$$

$$= \underbrace{\eta_U e^{-cn}}_{cn e^{1-cn} \leq 1} (1-\frac{c}{n})^{-2}$$

$$\leq (1-\frac{c}{n})^{-2} := \lambda.$$

If it is easy to verify (using formulas we've shown for $\mathbb{E}[T_k]$)

that $\mathbb{E}[T_{k+1}] = o(n^{-1})$ for any fixed $M > 0$, where $k = n^{1/3}$

$$\sum_{k=1}^n k \mathbb{E}[T_k] \leq k_1 \mathbb{E}[T_{k_1}] \sum_{i=0}^n \lambda^i$$

$$\leq O(n) \cdot k_1 \mathbb{E}[T_{k_1}]$$

$$= O(n^{-3})$$

Since $\lambda^i \leq \lambda^n = O(1)$

If it is easy to check that $\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} c^{k-1} e^{-ck} = o(1) \Rightarrow O(n^{-1})$

$$|\mathbb{E}[T - tcn]| \leq \left| \sum_{k=1}^{\infty} k \mathbb{E}[T_k] - n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck} \right|$$

$$\leq \sum_{k=n}^{\infty} k \mathbb{E}[T_k] + \sum_{k=n}^{\infty} n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck}$$

$$= O(1) + o(1) + o(1) = O(1)$$

□

Clarification on thresholds:

A property Q has a threshold $P_0 = P_0(n)$ if:

$$\Pr(G(n,p) \in Q) \rightarrow \begin{cases} 0 & \text{if } p = o(P_0) \Leftrightarrow p < P_0 \\ 1 & \text{if } p = \omega(P_0) \Leftrightarrow p > P_0 \end{cases}$$

\Leftrightarrow Q has a threshold $P_0 = P_0(n)$ if

$\exists \varepsilon > 0, \exists c > 0, n_0 > 0$ s.t. for all $n > n_0$

$$\Pr(G(n,p) \in Q) < \varepsilon \text{ if } p < c \frac{1}{n}$$

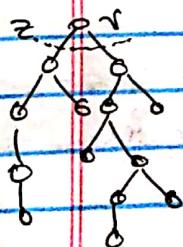
$$\& \Pr(G(n,p) \in Q) > 1 - \varepsilon \text{ if } p > c \frac{1}{n}$$

Recall: (Theorem 2(d))

If $p = c/n$ and $c \geq 1$, then a.s. $G(n,p)$ contains a unique component of linear order ($\ln n$), and all other components are of order $O(\log n)$.

Intuition for proof:

Branching (Exploration Process)



$$\deg(v) \sim \text{Bin}(n-1, p) \stackrel{\text{approx.}}{\sim} \text{Po}(c)$$

$$z \sim \text{Po}(c)$$

The out degree distribution

approximately Poisson (same theorem)

- Start at a vertex v ,

some and explore the graph
perimeter by doing a BPS(DFS on
its children).

- Degrees (# of children)

is roughly the binomial
distribution.

Formalizing this idea:

A Galton-Watson branching process:

Let Z be a random variable on nonnegative integers. At time $t=0$, a single particle comes into existence. At each $t \geq 1$, each existing particle gives birth to a random number of new particles and then it dies. These random numbers are i.i.d. copies of Z .

Let X_t denote the number of particles born during step $t \geq 0$.

Thus, $X_0 = 1$, and X_t has the same distribution as Z .

Let Σ denote the event "that $X_t = 0$ for some $t \geq 0$ ", i.e. the event is the process terminates after a finite t of steps.

Let $P_Z = \Pr(\Sigma)$

Exercise: For every $t \geq 1$, $\mathbb{E} X_t = (\mathbb{E} Z)^t$

Theorem

(i) If $\mathbb{E} Z < 1$, then $P_Z = 1$.

(ii) If $\mathbb{E} Z = 1$, and $\Pr(Z=0) > 0$, then $P_Z = 1$.

(iii) If $\mathbb{E} Z > 1$ and $\Pr(Z=0) = 0$, then $0 < P_Z < 1$

to avoid triviality (if $\Pr(Z=0) = 0$, then we always give birth to new children, so $P_Z = 0$)

→ Proof

Proof:

$$(i) \lim_{t \rightarrow \infty} \Pr(X_t > 0) \leq \lim_{t \rightarrow \infty} (\mathbb{E} Z)^t = 0$$

Exercise +erton

(ii) and (iii):

Let P_j be the probability that the process becomes extinct after j steps. This block shows let v_1, \dots, v_k denote the set of children born at step 1. The process terminates after j steps iff the X_i processes starting from v_1, \dots, v_k , all terminate after $j-1$ steps. These X_i processes are independent.

Then:

$$P_j = \sum_{i=0}^{\infty} \Pr(Z=i) P_{j-i}$$

\downarrow i children \hookrightarrow processes terminate in $j-1$ steps

$$\text{Let } G(x) = \mathbb{E} X^x = \sum_{i=0}^{\infty} \Pr(Z=i) x^i, \text{ then } P_j = G(P_{j+1})$$

Note:

$$\cdot G(1) = 1$$

$$\cdot G(0) > 0 \text{ since } G(0) = \Pr(Z \geq 0) > 0$$

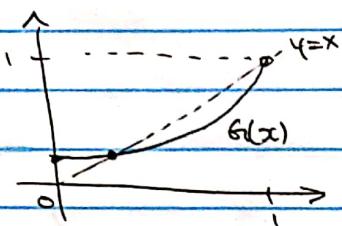
• G convex (since G is a probability generating function and they are always convex)

$$\cdot G'(1) = \sum_{i=1}^{\infty} i x^{i-1} \Pr(Z=i) \Big|_{x=1} = \mathbb{E} Z \geq 1.$$

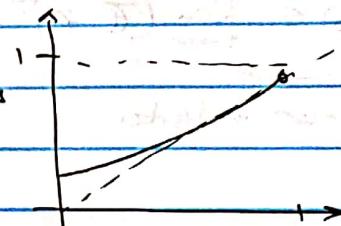
By assumption.



These facts can help us visualize G .



*Shape comes
from that
 G is convex

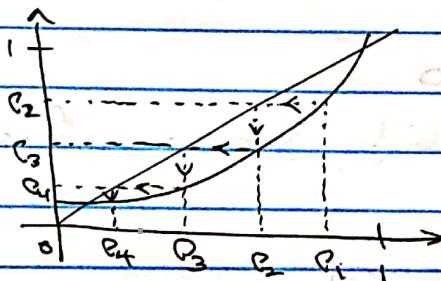


If $\exists z > 1$, then
 $G(z)=z$ has 2 roots,
one at x_1 and
the other $0 < x_2 < 1$

If $\exists z = 1$, then $G(z)=z$
has a unique root at x_1
(the line $y=x$ is tangent to
 G at $\frac{1}{2}x=1$)

For (ii) (and (iii) is similar), it is sufficient to show that
 $P_t \rightarrow P_2$ for $t \rightarrow \infty$. OKP, $\angle \leq 90^\circ$, and the conclusion follows
from the properties of G .

Proof by picture!



$P_2 = G(P_1)$, then find P_2 on the x-axis
by following $y=x$. Then $P_3 = G(G(P_2))$,
and continue until we converge
at the fixed point $G(P_2) = P_2$.

Theorem 1. (Chernoff Bound)

Let X_1, \dots, X_n be independent $\{0, 1\}$ -valued random variables. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}X$. Then the following probability bounds hold:

(a) For any $\delta > 0$:

$$\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^\mu.$$

→ (a) \Rightarrow (b)

(b) For any $0 < \delta \leq 1$

$$\Pr(X \geq (1+\delta)\mu) \leq \exp(-\frac{\mu\delta^2}{2})$$

(c) For any $0 < \delta \leq 1$

$$\Pr(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}} \right)^\mu$$

$$\Rightarrow \Pr(X \leq (1-\delta)\mu) \leq \exp(-\frac{\mu\delta^2}{2})$$

Proof:

Let $X_i \sim \text{Bernoulli}(p_i)$

→ holds for any $t > 0, t \in \mathbb{R}$

$$\Pr(X \geq (1+\delta)\mu) = \Pr(e^{tX} \geq e^{t(1+\delta)\mu})$$

$$\leq \frac{\mathbb{E}e^{tX}}{e^{t(1+\delta)\mu}}$$

$$= \frac{\sum_{i=1}^n \mathbb{E}e^{tX_i}}{e^{t(1+\delta)\mu}} \quad \text{as } X_1, \dots, X_n \text{ are independent}$$

$$= \frac{\sum_{i=1}^n (1-p_i + p_i e^t)}{e^{t(1+\delta)\mu}}$$

$$\leq \frac{\sum_{i=1}^n e^{p_i(t-1)}}{e^{t(1+\delta)\mu}} = \frac{e^{t(\mu - 1)}}{e^{t(1+\delta)\mu}}$$

$$\sum p_i = \mu$$

For part (a), take $t = \log(1+\delta) > 0$

→

Proof: (cont)

Part (b) follows by noting that $\frac{e^\delta}{(1+\delta)^{1+\delta}} \leq e^{-\delta/3}$ for $0 < \delta \leq 1$.

For part (c), For ~~t > 0~~ $t < 0$

$$\begin{aligned} \Pr(X \leq (1-\delta)\mu) &= \Pr(e^{tx} \geq e^{t(1-\delta)\mu}) \\ &\leq \frac{\mathbb{E}(e^{tx})}{e^{t(1-\delta)\mu}} \quad \text{Proceed as before} \\ &\leq \frac{e^{t(\mu t - 1)}}{e^{t(1-\delta)\mu}} \end{aligned}$$

And part (d) follows by taking $t = \log(1-\delta) < 0$.

The last inequality follows by $\frac{e^{-\delta}}{(1-\delta)^{1-\delta}} \leq e^{-\delta/2}$ for $0 < \delta \leq 1$. \square

Remark:

The Chernoff bounds hold for any x_1, x_2, x_n that are independent $[0, 1]$ -valued as well.

Modifications for the proof:

At the step

$$\frac{\mathbb{E}(e^{tx_i})}{e^{t(1-\delta)\mu}} \text{ we cannot split this to } \frac{\prod_{i=1}^n (1-p_i + p_i e^t)}{e^{t(1-\delta)\mu}}$$

But we will use that e^x is a convex function and $e^x \leq x + \beta$ for all $x \in [0, 1]$ where $\beta = e^1 - 1$ and $\beta \geq 1$.

Then,

$$\mathbb{E}(e^{tx_i}) \leq \mathbb{E}(\alpha(tx_i) + \beta) = p_i e^t + (1-p_i)$$

and the remaining steps are the same.

Lecture 8:

Thm E(8): Let $c > 1$. a.s. $G(n, c/n)$ has a unique giant component and all other components are of order $O(\log n)$

Still need a bit more probability theory before we can prove this theorem!

Stochastic Dominance:

Given 2 real r.v.s X and Y . We say that X stochastically dominates Y if:

$$P(X \geq x) \geq P(Y \geq x) \quad \forall x$$

We can couple X and Y such that $X \geq Y$ always. Then X stochastically dominates Y always. (Proof below)

Converse: If we have 2 random graphs S_1 and S_2 such that S_1 and S_2 can be coupled so that $S_1 \leq S_2$, then for any monotone increasing property \mathcal{G} :

$$\Pr(S_2 \in \mathcal{G}) \geq \Pr(S_1 \in \mathcal{G})$$

Proof:

$$\text{H.c. let } S^X(x) = \{\omega \in \Omega \text{ s.t. } X(\omega) \geq x\}$$

$$S^Y(x) = \{\omega \in \Omega \text{ s.t. } Y(\omega) \geq x\}$$

Since in Ω we have $X(\omega) \geq Y(\omega)$ thus Ω

$$\Rightarrow S^X(x) \geq S^Y(x) \quad \forall x$$

So:

$$\Pr(X \geq x) = \int_{\omega \in S^X(x)} dP(\omega) \geq \int_{\omega \in S^Y(x)} dP(\omega) = \Pr(Y \geq x)$$

* The p_i 's are P 's (rhs's)

Proof (of T(d))

Let $k_0 = \lceil 2 \log n \rceil$ and $k_1 = \lceil n^{1/3} \rceil$ where c_0 is a sufficiently large constant. We will prove that:

(i) There are no components of $G(n, p_n)$ with order between k_0 and k_1 .

(ii) There is only ≤ 1 component of order greater than k_1 .

(iii) The total # of vertices in small components (order $\leq k_0$) is asymptotic to $\frac{c_0}{k_0!}$, where $P^{\text{ex}} = e^{-c_0}$, or $p_c = \frac{c_0}{n}$.

(Compare P with $t(c)$ in the theorem of tree component
⇒ "P = t(c)" (Check yourself!).

* Your Intuition about P :

Take a uniformly random vertex v and explore its neighbors, its 2nd neighbor, etc. This mimics the branching process with $Z \sim P_0(c)$. Recall that P_2 is the probability of extinction. Then, P_2 is the root of $P = G(P)$, or $P \leq 1$. (whose $G(x)$ is the ZPGF for this distribution), and $G(x) = e^{-c + cx}$ (this is the PGF for Poisson).

(Aside:

$$\begin{aligned} G(x) &= \sum_{j=0}^{\infty} \Pr(Z=j) x^j = \sum_{j=0}^{\infty} \frac{e^{-c} c^j}{j!} x^j \\ &= e^{-c} \sum_{j=0}^{\infty} \frac{c^j}{j!} x^j \\ &= e^{-c+cx}. \end{aligned}$$

and P_2 is the fixed point for this PPF.

Draft: (can't)

Parts (i) - (iii) says that $G(n, p)$ contains a unique component of linear order and all other components are small. Intuitively, the prob. that v lies in a small component coincides with P_2 .

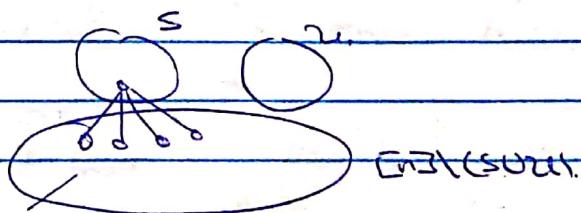
Consider the following graph exploration process. It starts with $t=0$ and $S = \{v\}$, $U = \emptyset$. In each step $t \geq 1$, the process takes an arbitrary vertex $u \in S$, adds all neighbors of u in $G(S \cup U)$ into S , and then moves u from S to U . The process terminates when $S = \emptyset$ and U contains all vertices in the same component as v and the size of U equals the # of steps the growth exploration process takes.

Let X_t be the size of S after step t . $X_0 = 1$

By the construction of the process, $|U| = t$ and $|U \cup S| = X_t + 1$ after t steps of the process.

(let $Z_t = X_t - X_{t-1} + 1$ (# of new vertices added))

Then, ~~assuming~~ $Z_t \sim \text{Bin}(n - (X_{t-1} + 1), p)$



Z_t counts all the new vertices, ~~and there are~~.
This is a binomial distribution since an edge appears in $\text{Ent}(S \cup U)$ with probability p .

Proof:

To prove part (i), it is sufficient to prove that with probability $\alpha(1/n)$ X_t reaches 0 for some $k_0 < t < k_1$, as by union bound, with probability $\alpha(n)$, there is a component of order between k_0 and k_1 .

How many vertices add the last ζ ?

Let $k_0 < \tau < k_1$ and E_τ be the event that $(X_t)_{t \geq 0}$ becomes 0 after step τ .

- Given E_τ , define $W_\tau := \sum_{t=1}^{\tau} Z_t = X_\tau - X_0 + 1 = \tau - 1$
- $Z_t \geq 0 \Rightarrow X_t \geq X_{t-1}$ (In the worst case, we add no new vertex and only remove ^{itself}).

If for some $t' < \tau$ we have $X_{t'} > k_1$, then consequently $X_t > 0$ for all $t < t'$, contradicting event E_τ .
So $E_\tau \Rightarrow X_t \leq k_1$ for all $t' < \tau$.

In Summary:

$$E_\tau \Rightarrow \begin{cases} W_\tau = \tau - 1 & \textcircled{1} \\ X_t \leq k_1 \text{ for all } t' < \tau & \textcircled{2} \end{cases}$$

② implies that $W_\tau := \sum_{t=1}^{\tau} Z_t$ stochastically dominates $\sum_{t=1}^{2k_1} \tilde{Z}_t$ where \tilde{Z}_t are i.i.d. copies of $Z \sim \text{Bin}(n-1, p)$

Since both $X_t \leq k_1$, this r.v. will be easier to analyze since Z changes with even step'.

Exercise 22:

Task! $\text{Bin}(n, p)$ dominates $\text{Bin}(n_2, p)$ if $n \geq n_2$

Proof (cont.)

\rightarrow Since even $\sum_{k=0}^n$ binomial.

$$W_r \sim \text{Bin}(r(n-2k), p)$$

$$\mathbb{E} W_r = r(n-2k) \cdot \frac{c}{n} = cr \quad (\text{where } c > 1)$$

Let k_0 be a constant such that

$\forall k \geq k_0 \Rightarrow \frac{2k}{n} \leq \epsilon$ (This is true for large n).

Let $\sigma = cr > 0$ and let $\varepsilon = \sigma/c$. Then $(1-\varepsilon)c = 1 + \frac{\sigma}{2}$

By Chernoff Bound:

$$\Pr(G_\varepsilon) \leq \Pr(\textcircled{1}) \text{ and } \textcircled{2}$$

$$\leq \Pr(W_r \leq 1)$$

Since $r \leq (1-\varepsilon)cr \leq c\left(1 - \frac{2k_0}{n}\right)$

$$= \Pr(W_r < (1-\varepsilon)cr\left(1 - \frac{2k_0}{n}\right)) = \left(1 + \frac{\sigma}{2}\right)^r \left(1 - \frac{2k_0}{n}\right)$$

$$\leq \exp(-\kappa \varepsilon^2 r) \quad (\text{for some constant } \kappa > 0)$$

$$= o\left(\frac{1}{n}\right) \quad \text{since } r \geq k_0 = \text{const} \text{ and } c \text{ is sufficiently large}$$

And part (i) follows by taking union bound over W_r .



③

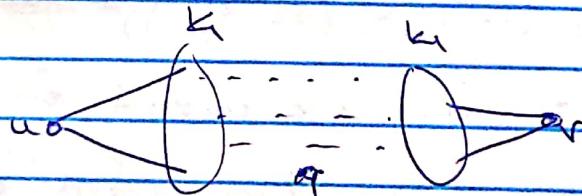
Proof (cont'd)

Part (ii). For any 2 vertices u and v , we prove that with prob $\Omega(n^{-2})$, they lie in 2 different components, each of order greater than k .

Let $x_t(u)$ and $x_t(v)$ denote the size of $S(u)$ and $S(v)$ with respect to the graph exploration processes starting from u and v respectively.

We may assume that $x_t(u)$ and $x_t(v)$ are positive for all $t < k$. (Since otherwise one of them lies in order k , i.e. the component has order $< k$.)

Take $\epsilon = \frac{1}{k^2}$ and let $W_{t,i} = \sum_{j=1}^{k_i} z_{t,j} = x_{t,i}(j) - x_0(i) + k_i$ for $i \in \{u, v\}$.



We want to show w.h.p. that if after k steps, we still have a large # of vertices to explore there must be an edge b/w the component containing u and the component containing v .

Proof (cont)

By Chernoff Bound, (and similar argument as (i)).

$$\Pr(W_k \geq i) \leq (1-\epsilon)k e^{-(\frac{\epsilon k}{n})} = o(n^{-2})$$

Hence with prob $> 1-o(n^{-2})$, $X_{k,W}, X_{k,V} \leq (\frac{5}{n})k$

If $S(u) \cap S(v) \neq \emptyset$ after k_1 steps, then we're done.

else, we may assume $|S(u)|, |S(v)| \geq (\frac{5}{n})k_1$ and

$\Pr(\text{no edge b/w } S(u) \text{ and } S(v))$

$$\leq (1-p)^{(\frac{5}{n}k_1)^2}$$

$$\leq o(n^{-2})$$

\Rightarrow Rest (ii) (By union Bound)

Lecture 9:

* All P 's are there.

Recall:

$k_0 = \text{deg}_n \rightarrow k_0 = n^{2/3}$. Prove this.

(i) No components of order (k_0, k_1)

(ii) Only one component of order $\leq k_0$.

(iii) # vertices in small component $\xrightarrow{n \rightarrow \infty} P^c$, where $P^c = e^{P^c}$ ($0 < P < 1$)

We will prove (iii) this class.

(Let Y denote the # of vertices lying in small components (i.e. of order $\leq k_0$). We will prove that the probability for a random uniformly random vector to be in a small component is $P + o(1)$.)

Lemma: Fix $c > 1$ and let $Z_n \sim \text{Bin}(n, p)$ where $p \xrightarrow{n \rightarrow \infty} c/n$

Then $P_{Z_n} \rightarrow P$ as $n \rightarrow \infty$ where $P^c = e^{P^c}$, ($0 < P < 1$)

Proof:

$$(\text{Let } G(x) = E(x^{Z_n}) = ((1-p+px)^n)^{-c+cx+O(\frac{p}{n}) \cdot o(1)})$$

which converges to e^{-c+cx} pointwise as $n \rightarrow \infty$.

By the proof of the theorem for the branching process,
Part (iii):

P is the unique solution of $G(x) = x$, where $0 < x < 1$.

As $G(x) \xrightarrow{x \rightarrow 0} e^{-c+cx}$ pointwise and P is the unique solution of $G(x) = x$

$x = P(x)$, $0 < x < 1$, it follows immediately $P_{Z_n} \rightarrow P$ as $n \rightarrow \infty$.

Lemma $EY = (1 + \epsilon)P_n$.

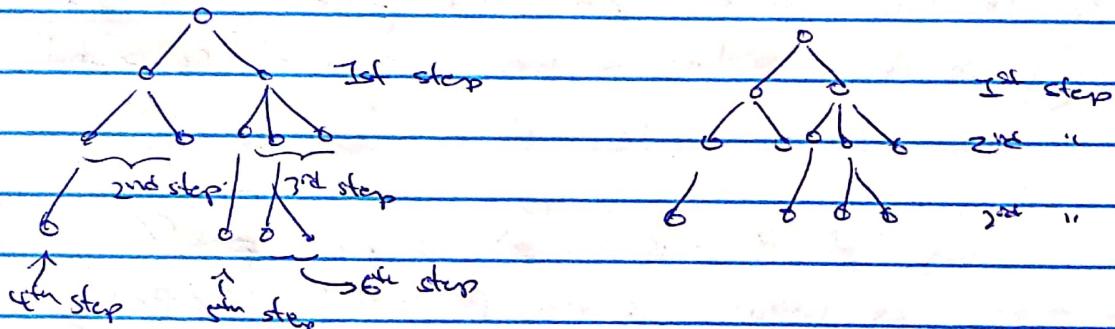
Proof:

Consider the graph exploration process starting from v , \hat{v}_v .

Let \mathcal{E}_v denote the event that \hat{v}_v lasts k steps.

Consider the slowed-down version of the branching process where $Z_n \sim \text{Bin}(n, (1 + \xi) \frac{c}{n})$, where $\xi \geq 0, \xi = o(1)$.

(Slowed-down Branching Process.)



i.e. Parent and Children are child at a time.)

$\text{Bin}(n, (1 + \xi) \frac{c}{n})$ stochastically dominates $\text{Bin}(n - (\frac{\xi}{2} + \epsilon)t, \frac{c}{n})$,
so we can couple (\hat{v}_v, \hat{v}_w) such that \hat{v}_v terminates no later than \hat{v}_w . (Exercise!)

Let P_t denote the probability that \hat{v}_v terminates during the first t steps. We know $P_t \rightarrow P_{\infty}$.

That is $\lim_{t \rightarrow \infty} P_t \geq \lim_{t \rightarrow \infty} P_{t+1}$: $P_{\infty} \geq P_{\infty} - \epsilon$

And also, $P_{\infty} \rightarrow P$ as $n \rightarrow \infty \Rightarrow P_{\infty} \geq P - \epsilon$.

$$\Rightarrow P_{\infty} \geq P - 2\epsilon$$

Cox

By the Coupling (\hat{P}_v, ψ), we have $P(Z_v) \geq P(\psi$ terminates before k_0 steps) = $P_{\text{no}} > P-\epsilon \Rightarrow P(Z_v) \geq P-\alpha_1$

For the upper bound, consider the stand-dead process with branching with Z_{n-k} , $(1-\varphi) \in \mathbb{N}$. Z_n is stochastically dominated by $Z_n(\hat{P}-(\epsilon_1+\epsilon_2), \frac{\epsilon_1}{n})$ for $n < k_0$.

Again, we can couple these two processes such that during the first k_1 steps the graph exploration process will terminate no earlier than the branching process.

$$\Rightarrow P(Z_v) \leq P(\text{Branching Process} \text{ last} \leq k_0 \text{ steps})$$

$= P + \alpha_1$, using similar argument as above

$$\Rightarrow P(Z_v) = P + \alpha_1$$

$$\Rightarrow EY = (1 + \alpha_1)P$$

□

$EY(N-1)$: Fix a pair of vertices u and v .

$$Pr(u \text{ is in a small component}) = P + \alpha_1$$

Condition on that: The graph exploration process only "exposes" $\leq k_0$ vertices.

(Exercise: Try to show that u, v is not exposed)

→ i.e. In the

same component
that u is in.

In the graph exploration process starting from u , condition on C_u , the component containing u . This is equivalent to running the graph exploration process in $G(n, p)$ where $n' = n - O(k_1)$.

Repeat the proof as for EY . We have, \square

We have $\Pr(V \text{ is in a small component} | u \text{ is in a small component}) = P + o(1)$.

$$\mathbb{E}Y(Y-1) = n \mathbb{E}Y \cdot (\mathbb{P}^2 + o(1)) \sim (\mathbb{E}Y)^2$$

By Chebyshev's Inequality: $\forall \varepsilon > 0$

$$\Pr(|Y - \mathbb{E}Y| \geq \varepsilon \mathbb{E}Y) \leq \frac{\text{Var}Y}{\varepsilon^2 (\mathbb{E}Y)^2} = \frac{o(1) \cdot (\mathbb{E}Y)^2}{\varepsilon^2 (\mathbb{E}Y)^2} = o(1)$$

So we have $\mathbb{E}n \mathbb{E}Y = (P + o(1))n$. \square

Hamiltonian Cycle in $G(n, p)$:
 $\frac{1}{p}$'s are new
 p 's

Theorem:

$$\text{Let } p = \frac{\log n + \log \log n + o(\log n)}{n}$$

(a) If $\frac{x(n)}{n} \rightarrow \infty$, then a.a.s $G(n, p)$ is Hamiltonian

(b) If $\frac{x(n)}{n} \rightarrow -\infty$, then a.a.s $G(n, p)$ is not Hamiltonian.

(Note: When degree 1 vertices disappear, then our graph becomes Hamiltonian (w.h.p)).

Properties:

Let D denote the property that min. degree is ≥ 2
HALL \implies Graph is Hamiltonian

$G \in \text{HALL} \Rightarrow G \in D$.



Lemmas (Exercise)

- (a) If $p = \log n + \log \log n \rightarrow -\infty$ then a.a.s $G(n,p) \notin D_2$
(b) If $p = (\log n + \log \log n) \rightarrow \infty$ then a.a.s $G(n,p) \in D_2$.

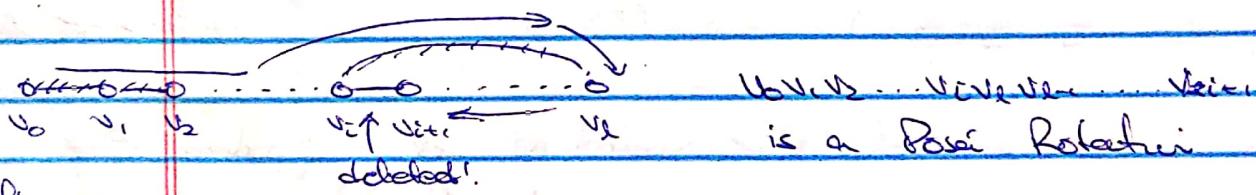
Lemmas \Rightarrow Part (b) of the theorem.

Idea for proof:

We try to build a layer and layer path until we build a Hamilton cycle.

What is the probability of us creating a layer path? \rightarrow But we can only build from the 2 end

Rosé Rotation:



Def'n:

Let $P = v_0, \dots, v_k$ be a longest path in G . So assume $v_i v_k$ is an edge. Then, the operation of deleting edge $v_i v_k$ from P and adding edge $v_{i+1} v_k$ to P is called a Rose Rotation.

The Rose Rotation converts P to another longest path



②

Given a longest path $P = v_0, \dots, v_l$. Let $\mathbb{P} P$ be the set of longest paths obtained by fixing end v_0 and repeatedly performing Poss Rotations.

Let $\text{End}(v_0)$ be the set of ends of paths in \mathbb{P} other than v_0 .

Given a subset S of vertices, let

$$N(S) = \{v \in S : \exists u \in S \text{ s.t. } v \sim u\}$$

Lemma: $|N(\text{End}(v_0))| < 2|\text{End}(v_0)|$

Proof:

It is sufficient to show that if $v_i \in N(\text{End}(v_0))$, then one of v_{i-1}, v_{i+1} must be in $\text{End}(v_0)$
 (Extremal Case).

$$\begin{array}{c} \square \square \square \\ v_0 \quad \square \end{array}$$

Consider a sequence of Poss rotations which produces P' with $v_i \sim x$, where $x \neq v_0$ is an end of P' . Let y and z be the left and right neighbors of v_i on P' . If $\{y, z\} = \{v_{i-1}, v_{i+1}\}$, the one of them can be added to $\text{End}(v_0)$ by performing a Poss Rotation on P' . Otherwise, one of the edges $v_i v_{i-1}, v_i v_{i+1}$ has been deleted in a previous Poss Rotation \Rightarrow one of v_{i-1}, v_{i+1} has been added to $\text{End}(v_0)$.

□

Lecture 10:

Recall:

- ① Want to prove if $p_n - (\log n + \log \log n) \rightarrow \infty$, then a.a.s $G(n,p) \in \text{HAM}$
- ② Past Relation: $|N(\text{End}(v_0))| < 2|E_{\text{end}}(v_0)|$

Lemma: If $p_n - (\log n + \log \log n) \rightarrow \infty$ then there exists a constant $\epsilon > 0$.

Assume $p_n - (\log n + \log \log n) \rightarrow \infty$ and $p_n \leq \log n + \epsilon$. a.a.s there is no $S \subseteq [n]$, $|S| \leq \epsilon n$ and $|N(S)| < 2|S|$.

Proof: later. (We'll need this for the upcoming proof)

Back to the proof of hamiltonicity of $G(n,p)$:

Let $\epsilon > 0$ be the constant in the lemma.

Let Exp denote the property that $H \subseteq [n]$ where $|H| \leq \epsilon n$, $|N(H)| \geq |H|$.

$C_{n,T}$ denote the property that G is connected

Let $f = p_n - (\log n + \log \log n)$

By our assumption, $f \rightarrow \infty$ as $n \rightarrow \infty$.

Let $p_* = \log n + \log \log n + f/2$ (let $G' \sim G(n, p_*)$). Let G_r be the random graph by adding $f/8$ uniformly random edges to G' .

We can show (Exercise!) that G_r and $G(n,p)$ can be coupled so that a.a.s $G_r \subseteq G(n,p)$. So, it is sufficient to show that.

a.a.s G_r is Hamiltonian



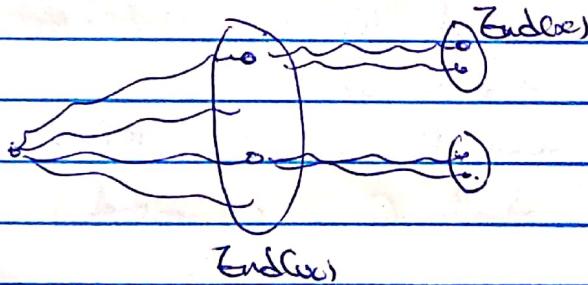
Proof (cont)

We know that G' is G 's EXPONENT.

Take a longest path $P = v_0, \dots, v_l$ in G' . Let P be the set of paths obtained from P via rotations with v_0 fixed.
Let $\text{End}(v_0)$ denote the other ends of paths in P .

Since G' is EXP and $|\text{N}(\text{End}(v_0))| <_2 |\text{End}(v_0)| \Rightarrow |\text{End}(v_0)| \geq \epsilon_{in}$.

Now for each $x \in \text{End}(v_0)$ let P_x denote the set of paths obtained via P via rotations with x fixed and let $\text{End}(x)$ be the set of the other ends of pf P_x .



Let $\mathcal{E} = \{(x, y) \mid x \in \text{End}(v_0), y \in \text{End}(x)\}$. \mathcal{E} is called the set of breasts.

As before, we have $|\text{End}(x)| \geq \epsilon_{in}$ for any $x \in \text{End}(v_0)$.

It follows now that $|\mathcal{E}| \geq \epsilon_{in}^2 n^2$. Consider \mathcal{E} . We will sprinkle the \mathcal{E} /s edges in sequence. We call each sprinkled edge a "trial". A trial is successful if it is in \mathcal{E} and fails otherwise. If a trial is successful, then since G' is connected, it will result in finding a path longer than P or finding a Hamiltonian cycle if P is a Hamiltonian path.

Proof (Cont)

Each trial is successful with probability $\frac{c}{n} \geq ce^{-c}$ (for some constant $c > 0$). Only at most n successful trials are needed to guarantee a Hamiltonian cycle in G .

$$\Pr(G \text{ RHTW}) \leq \Pr(\text{Bin}(n, ce^{-c}) \leq n) = e^{-\frac{n ce^{-c}}{2}} = o(1)$$

□

Borders/bottleneck lemma $\Rightarrow \log S \leq 2 \log n$.

Lemma: Assume probability $p = 1/n$:

(a) Each set $S \subseteq V$ with $|S| \leq \frac{n}{\log n}$ induces at most $3|S|$ edges (Sublinear-sized subgraphs are sparse)

(b) No two vertices with degree ≤ 100 are within distance 5

(c) No vertex with degree ≤ 100 lies in a cycle of length ≤ 5

Proof: Exercise!

Back to λ -expander lemma:

Proof:

Let $n^{3/4} \leq S \leq \varepsilon n$ where ε_0 is a sufficiently small constant

Let $E(S, W)$ be the expected number of pairs of disjoint sets (S, W) s.t. $|S|=s$, $|W|=w$ and $W=N(S)$

$$E(S, W) = \binom{n}{s} \binom{n-s}{w} (1 - (1-p)^s)^w$$

every vertex in W seeds a vertex
 in S
 ↓ ↓ ↓
 Choose S Choose W Don't see any vertex in S Disjoint S cannot see any other vertices
 ↓
 Complement! See a vertex in S

$$Z(S, \omega) \leq \left(\frac{e^n}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega (ps)^{\omega} \exp(-ps(n-\omega))$$

$$= \left(\frac{e^n}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega (ps)^{\omega} (\exp(-pn(1 - \frac{\omega}{n})))^{S-}$$

(Since $p_n > \log n$, $\frac{ps}{S} < \log n$, if $\varepsilon_0 < \frac{1}{2}$)

$$\leq (\log n)^S \left(\frac{e}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega \exp(ps(n-\omega)) \\ < \log n.$$

$$\leq (\log n)^\omega \left(\frac{e}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega \exp(pn \cdot \frac{S-\omega}{S} \log n)$$

$$\leq (\log n)^\omega \left(\frac{e}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega n^{2\varepsilon_0 S \log n}$$

Then,

$$\sum_{S=1}^{\infty} \sum_{\omega=1}^{\infty} Z(S, \omega) \leq \sum_{S=n^3}^{\infty} \sum_{\omega=1}^{\infty} (\log n)^\omega \left(\frac{e}{S}\right)^S \left(\frac{e^n}{\omega}\right)^\omega n^{2\varepsilon_0 S \log n}$$

$$\leq \sum_{S=n^3}^{\infty} (\log n)^{\frac{2\varepsilon_0 S}{S}} \left(\frac{e}{S}\right)^S \left(\frac{e^n}{S}\right)^{\frac{2\varepsilon_0 S}{S}} n^{2\varepsilon_0 S \log n}$$

$$= \sum_{S=n^3}^{\infty} \left(\log n \cdot \frac{e}{S} \cdot \frac{e^n}{S} \cdot n^{6\varepsilon_0}\right)^S$$

$$\leq C \sum_{S=n^3}^{\infty} \left(\frac{C \log n}{S^2} \cdot n^{2+6\varepsilon_0}\right)^S$$

$$\leq C \sum_{S=n^3}^{\infty} \left(\frac{C \log n}{S^2} n^{2+6\varepsilon_0}\right)^S$$

Choose ε_0 small so that $2+6\varepsilon_0 < \frac{9}{4}$. Therefore,

$$\sum_{S=n^3}^{\infty} \sum_{\omega=1}^{\infty} Z(S, \omega) \leq \sum_{S=n^3}^{\infty} n^{\alpha S} = \infty \text{ for some } \alpha >$$

Now for $S \subseteq \mathbb{N}^n$. Assume G_S is a graph satisfying properties (a)-(cc) from the lemma. and $G \in \text{EDZ}$. (min. degree of G_S is ≥ 2)

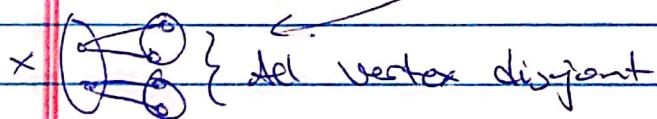
We know that $G_{\{m,p\}}$ has these properties.
It is sufficient to show that any graph G_S satisfying these above properties ~~satisfies~~ satisfies

$\forall S \subseteq \mathbb{N}^n$ where $|S| \leq n^{34}$, $|N(S)| \geq 2|S|$

Let's call a vertex "light" if its degree is at most ∞ , otherwise call it "heavy". For any $S \subseteq \mathbb{N}^n$. let $X \subseteq S$ be the set of light vertices in S and let $Y = S \setminus X$.

Case 1: $Y = \emptyset$ (i.e. all vertices in S are light)

Then $|N(S)| = |N(X)| \geq 2|X| = 2|S|$. by $G \in \text{EDZ}$ and property (b).



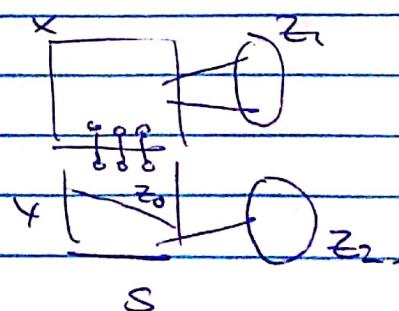
Case 2: $Y \neq \emptyset$.

Assume $|N(S)| < 2|S|$

$$Z_0 = N(X) \cap Y$$

$$Z_1 = N(S) \setminus Y$$

$$Z_2 = N(Y) \setminus (X \cup Z_1)$$



Prob (cont.)

By assumption

$$\text{We have } |N(S)| = |Z_1| + |Z_2| + |Z_3| \leq 2|S| = 2(|X| + |Y|)$$

Since GCD2 and by properties (b) and (c), we will have.

$$|Z_1| + |Z_2| \geq 2|X|$$

$$|Z_3| = e(X, Y) \leq |Y|$$

$$e(Y, Z_1) \leq |Y|$$

(We will continue after reading next!)

Lecture 11:

Recall:

Lemma: $p_{n-1}(\deg v - \bar{\deg} \deg v) \rightarrow \infty$ and $p_n \ll \deg v$. Then, $\exists \varepsilon_0 > 0$ s.t. a.a.s $S \subseteq [n]$, $|S| \leq \varepsilon_0 n$ and $|N(S)| \geq 2|S|$

Proof:

Ass., no S with $n^{3/4} \leq |S| \leq \varepsilon_0 n$, $|N(S)| < 2|S|$ (last class)

And we've also previously proved the following properties:

a.a.s in $G(n,p)$:

(a) $S \subseteq [n]$ with $|S| \leq \frac{n}{\log n}$ induces $\leq 3|S|^2$ edges

(b) No path of length at most 5 joining 2 light vertices

(c) No cycle of length ≤ 5 has a light vertex

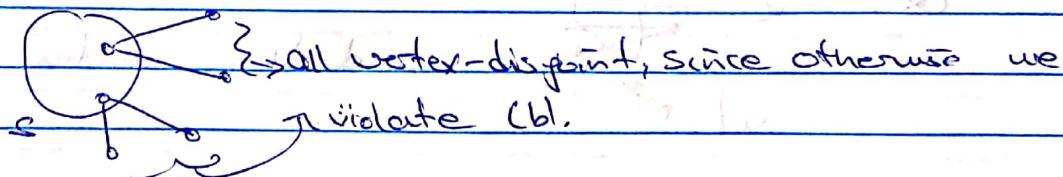
We will now complete the proof in the lemma.

For $S \leq n^{3/4}$. Assume G satisfies (a)-(c) and $G \models D2$. Assume

$S \subseteq [n]$ with $s = |S| < n^{3/4}$. and $|N(S)| < 2|S|$

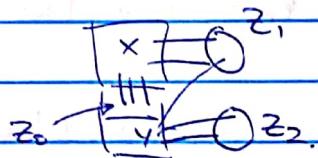
Let X be the set of light vertices in S and $Y = S \setminus X$.

If $Y = \emptyset$, then $|N(S)| \geq 2|S|$ (by (b) and D2)



Proof: (cont.)

If $Y \neq \emptyset$. Let $Z_0 = N(X) \cap Y$, $Z_1 = N(X) \setminus Y$, $Z_2 = N(Y) \setminus (X \cup Z_1)$



$$|N(S)| = |Z_0| + |Z_1| < 2|S| = 2(|X| + |Y|)$$

↑
By assumption

$e(A, B) : \# \text{ of edges}$
b/w A and B.

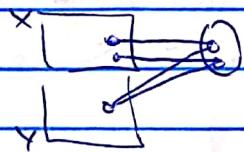
By (c) of D2 and (b), (c)

$$(i) |Z_0| + |Z_1| \geq 2|X| \quad (\text{D2 and (b)})$$

$$(ii) |Z_0| = e(X, Y) \leq |Y|$$

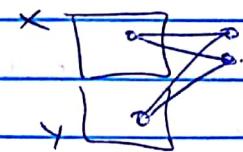
$$(iii) e(Y, Z_1) \leq |Y|$$

→ Every vertex in Y can only see one vertex in Z_1 .
Suppose otherwise, then we get a contradiction.



But this is a path of length 5,
that joins 2 vertices in X
(violating (b))

CR



and this violates (c).

C

Hence,

$$2|X_1 - (Z_{01} + Z_{-1})| \leq |Z_1| + |Z_{-1}| < 2(|X_1| + |Y_1|)$$

By (ii)

By assumption

Thus:

$$|Z_2| \leq |Z_{01}| + 2|Y_1| \leq 3|Y_1|.$$

By (iii)

$$\text{By (a), } e(Y, Z_2) \leq e(Y \cup Z_2) \leq 3(|Y| + |Z_2|) \leq 12|Y_1|.$$

Since every vertex in Y has degree ≥ 100 , we have:

$$\underbrace{e(Y, Z_2)}_{\leq |Y_1|} + \underbrace{2e(Y)}_{\leq 6|Y_1|} + \underbrace{e(X, Y)}_{\leq |Y_1|} - e(Y, Z_2) \geq 100|Y_1|$$

(By (a)) (By (iii)) (By (ii)) (By (a))

$$\Rightarrow 100|Y_1| \leq 20|Y_1| \Rightarrow |Y_1| = 0$$

Contradicting that $Y \neq \emptyset$

Note: The 100 was arbitrarily chosen, we could have chosen any large #.

A103:

Show that if $p = o\left(\frac{\log n}{n}\right)$ a.a.s $\Delta(G_{n,p})$ has a 2-part concentration.

Let $\lambda_k = n \cdot P(Bin(n-1, p) = k) = \text{Expected # of vertices with degree } k$

① Prove that λ_k monotonically decreasing on $k > np$.

Let k^* be the integer $\gg np$ which minimizes $\max\{\lambda_k, \frac{1}{\lambda_k}\}$.
But, we have to show that this is actually well-defined.

Instead, we could try:

② Let $k^* \geq np$ be the max. integer s.t. $\lambda_{k^*} \geq 1$. (And let $k^* = n-1$ if $\lambda_{n-1} \geq 1$)

③ Prove that \forall constant $c > 0$, $\lambda_{k^*+np} = o(1)$
 $\Rightarrow k^* = o(np)$

④ $\frac{\lambda_k}{\lambda_{k+1}} \sim \frac{np}{K} \Rightarrow \lambda_{k^*} \geq 1 \text{ and } \lambda_{k^*+1} < 1$

Case 1: $\lambda_{k^*} = o(1)$

$\lambda_{k^*+2} = o(1)$ and $\sum_{j \geq k^*+2} \lambda_j = o(1)$ (By (4))

\Rightarrow a.a.s $\Delta \in \{k^*, k^*+1\}$

Case 2: $\lambda_{k^*} = o(1)$, $\lambda_{k^*-1} = o(1)$, $\lambda_{k^*+1} = o(1)$

Then, $\sum_{j \geq k^*} \lambda_j = o(1)$ (By (4)).

\Rightarrow a.a.s $\Delta \in \{k^*-1, k^*\}$

Subsequence Principle:

If for every subsequence of a sequence there is a subsequence converging to a limit α , then the entire sequence must converge to the same limit.
 (True for any sequence in a topological space)

Suppose you want to prove a limit theorem for \mathbb{R}^n .
 You just need to prove such a limit theorem for p_n .
 s.t. p_n has an expression that converges to a limit
 (e.g. $n^{1/p} \rightarrow c$ for some a, b)
 (e.g.:
 $p = \Theta(n^{-2/3})$, then $n^{2/3} \rightarrow c$)

In the example of the 2-point concentration:

We have $p = o(\log n)$. Assume the 2-point concentration is not true. Then, there is a subsequence $S_i \in p_n$ s.t.
 $\exists \varepsilon > 0, \Pr(\exists \{i, j\} : \text{dist}(x_i, x_j) < 1 - \varepsilon) \rightarrow 0$.

In the subsequence:

However we can show that there is a subsequence of S (so a subsubsequence of p_n), s.t. 2-point concentration holds in this subsubsequence, a contradiction.

Lecture 12:

Martingales and Azuma-Hoeffding Inequality:

Suppose X, Y are r.v.s in a probability space. Let $Z = E[Y|X]$ (the expectation of Y conditioned on X) is a random variable defined by:

$$\int_A Z dP = \int_A Y dP$$

for every X -measurable set A .

Alternative definition for "nice" X and Y :

$$E(Y|X)(\omega) = E(Y|X=x(\omega)) \text{ for every } \omega \in \Omega. \quad (*)$$

Whenever the above $E(Y|X=x(\omega))$ is well defined.

$$\text{Recall: } E(X|A) = \frac{E(X \cdot I_{\{A\}})}{P(A)} = \frac{1}{P(A)} \int_A X(\omega) dP(\omega)$$

where A is an event and $I_{\{A\}}$ is the indicator variable for A .

When is $(*)$ well-defined?

For instance,

① If X is a discrete random variable, then

$$E(Y|X=x(\omega)) = \frac{E(Y|X=x(\omega))}{P(X=x(\omega))}$$

② If the conditional distribution of Y given X is a continuous distribution and the conditional density function is denoted $f_{Y|X}(y|x)$, then $E(Y|X=x) = \int y f_{Y|X}(y|x) dy$.

Example:

Rolling 2 dice independently with X_1, X_2 denoting the points shown. (Let $X = X_1 + X_2$. What is $E(Y|X_1)$?

Condition on $X_1 = x$, we have

$$\begin{aligned} E(X|X_1=x) &= E(X_2+x|X_1=x) \quad \rightarrow \text{Since } X_1, X_2 \\ &= x + E(X_2) \quad \text{are independent} \\ &= x + 7/2 \end{aligned}$$

This holds for every x so $E(X|X_1) = x + 7/2$

Example:

~~Uniform distribution.~~ Find the mean of $f(x)$.

Let X be a real number uniformly distributed in $[0, 1]$ and Y be a r.v. uniformly chosen from $[X, 1]$. What is $E(Y|X)$?

Conditional Density Function $f_{Y|X}(y|x) = \frac{1}{1-x}$

Hence,

$$E(Y|X=x) = \int_x^1 \frac{y}{1-x} dy = \frac{(1-x)^2}{2}$$

This holds for all $x \in [0, 1]$ so $E(Y|X) = \frac{(1-x)^2}{2}$

Defn Assume (Y_0, Y_1, Y_2, \dots) is a random process. (We say)
 X_0, X_1, X_2, \dots is a martingale with respect to $(Y_t)_{t \geq 0}$
if for every $t \geq 0$:

$$E(X_t | Y_0, Y_1, \dots, Y_t) = X_t \quad \text{Decreasing sequence}$$

$(X_t)_{t \geq 0}$ is called a super-martingale if the above " $=$ " is replaced with " \leq ", and a sub-martingale if instead we have " \geq ".

↗ Increasing sequences

Example (Dob's Martingale)

Consider a set of random variables Y_1, \dots, Y_n each taking values in a domain A . Let $f: A^n \rightarrow \mathbb{R}$. Define $(X_t)_{t \leq n}$ as follows:

$$X_0 = E(f(Y_1, \dots, Y_n))$$

$$X_t = E(f(Y_1, \dots, Y_n) | Y_1, \dots, Y_t) \quad \text{for all } 1 \leq t \leq n$$

(Note: An example of the Y 's is $Y_1, \dots, Y_{(1)}$ - the edges of a random graph; and we can have f be any statistic, such as the chromatic number. Then X_0 is the expected value of the total chromatic number, and X_t is the chromatic number given the first t edges. Finally, X_n is the actual chromatic number.)

Let's verify that this is martingale

Car

②

Aside: Tower Property

Suppose $G_1 \subseteq G_2 \subseteq \mathcal{F}$ (sigma algebras)

Then

$$\mathbb{E}(\mathbb{E}(X|G_1)|G_2) = \mathbb{E}(X|G_1), \text{ and}$$

$$\mathbb{E}(\mathbb{E}(X|G_2)|G_1) = \mathbb{E}(X|G_1)$$

i.e. "Smaller" sigma field wins.

Example (cont'd)

For every $0 \leq t \leq n-1$

$$\mathbb{E}(X_{t+1}|Y_1, \dots, Y_t)$$

$$= \mathbb{E}(\mathbb{E}(f(Y_1, \dots, Y_{t+1})|Y_1, \dots, Y_t))$$

$$= \mathbb{E}(f(Y_1, \dots, Y_t))$$

$$= X_t.$$

By Tower Property

Example:

$\Omega = \{\text{Canadian Citizens}\}$

P : Uniform Distribution

$\mathcal{F} : 2^{\Omega}$

$f : \Omega \rightarrow \mathbb{R}$ (height of citizen)

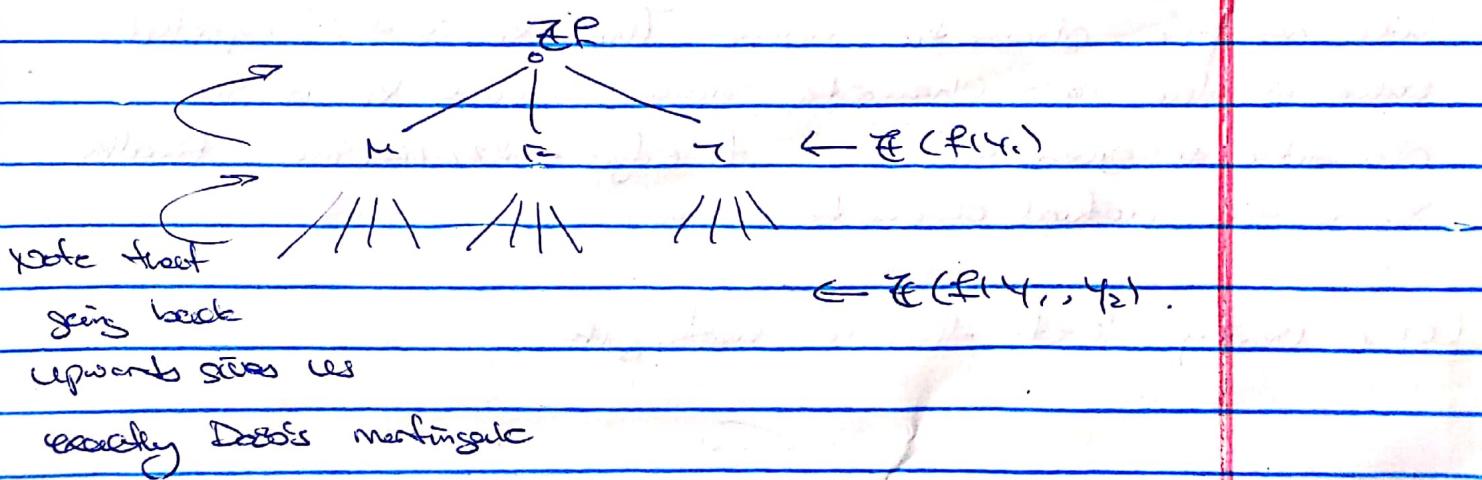
$Y_1 : \Omega \rightarrow \{\text{M}, \text{F}, \text{T}\}$ (Gender)

$Y_2 : \Omega \rightarrow \mathbb{N}$ (Age)

$X_0 = \mathbb{E} f \rightarrow$ average height of Canadian citizens

$X_1 = \mathbb{E}(f|Y_1) \rightarrow$ average height ~~not~~ segregated by gender

$X_2 = \mathbb{E}(f|Y_1, Y_2) \rightarrow$ ~~average height~~ segregated by gender AND age



Example (Edge Exposure Martingale)

Consider $G(n,p)$ and an ~~arbitrary~~ arbitrary order of the $\binom{n}{2}$ edges of $K_{n,2}$. Let y_i be the indicator variable such that the i -th edge in $K_{n,2}$ is in $G(n,p)$. Let f be an graph function (e.g. The chromatic #).

Define a martingale $X_0, \dots, X_{\binom{n}{2}}$ by

$$X_0 = \mathbb{E}(f(G(n,p)))$$

$$X_t = \mathbb{E}(f(G(n,p)) | Y_1, \dots, Y_t) \text{ for every } 1 \leq t \leq \binom{n}{2}$$

Note that this is a special case of the Doob's martingale

(We can expose the graph quicker though!)

Example (Vertex Exposure Martingale)

Consider $G(n,p)$ and any graph function f . Define $(X_t)_{0 \leq t \leq \binom{n}{2}}$ by:

$$X_0 = \mathbb{E}(f(G(n,p)))$$

$$X_t(\omega) = \mathbb{E}(f(G(n,p)) | G_{t,2} = H_{t,2}) \text{ for all } 1 \leq t \leq n.$$

Where $G_{t,2}$ is the subgraph induced by the vertex set

$$[t] = \{1, \dots, t\}$$

C

③

Example:

$G(4, 4)$. Consider the vertex exposed Martingale $(X_i)_{i \in \mathbb{N}}$ with respect to $f(g)$ as the chromatic number of g (g is a graph).

Let $g = \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array}$. What is $X_2(g)$ and $X_4(g)$?

$X_4(g) = \chi(g) = 2 \rightarrow$ This is just the chromatic number since we've exposed all the information \mathcal{E} on g .

$$X_2(g) \subset \{f(P(g)) \mid \{1, 2\}, \{2, 3\} \in g, \{1, 3\} \notin g\}$$

$$\begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array} = 2 \left(\left(\frac{3}{4}\right)^3 + 3 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) \right)$$

$$\begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array} \quad \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array} \quad \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array}$$

$$+ 3 \left(2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) - \left(\frac{1}{4}\right)^3 \right) = \frac{125}{64}$$



Example:

Let Y_1, \dots, Y_n be ind. r.v.s. Let $X_t = \sum_{j=1}^t (Y_j - E[Y_j])$

Claim: $(X_t)_{t \geq 0}$ is a martingale w.r.t. $(Y_t)_{t \geq 0}$.

Proof:

To prove, we want to verify $E(X_t | Y_1, \dots, Y_t) = X_t$.

$$\begin{aligned} E(X_t | Y_1, \dots, Y_t) &= E\left(\sum_{j=1}^t (Y_j - E[Y_j]) | Y_1, \dots, Y_t\right) \\ &= \sum_{j=1}^t (Y_j - E[Y_j]) + E(Y_{t+1} - E[Y_{t+1}] | Y_1, \dots, Y_t) \\ &= X_t \end{aligned}$$

Independent from Y_1, \dots, Y_t ,
so $E[Y_{t+1}]$ cancels out.

Theorem (Borel's Inequality).

Let $(X_i)_{i \geq 0}$ be a martingale w.r.t. $(Y_i)_{i \geq 0}$. Assume for all $k \geq 1$

$$|X_k - X_{k-1}| \leq C_k$$

Then,

$$Pr(|X_n - X_0| \geq t) \leq 2 \exp\left(-\frac{t^2}{2 \sum_{k=1}^n C_k^2}\right).$$

Lecture 13:

Theorem (Azuma's Inequality)

Proof.

Let $Z_i = X_i - Y_{i-1}$. WLOG we may assume $X_0 = 0$.

Then, $X_n = \sum_{i=1}^n Z_i$.

By assumption (Lipschitz condition):

$$-C_i \leq Z_i \leq C_i \quad \forall i \quad \text{since } E(X_i | Y_0, \dots, Y_{i-1}) = X_{i-1}.$$

and $E(Z_i | Y_0, \dots, Y_{i-1}) = E(X_i | Y_0, \dots, Y_{i-1}) - X_{i-1} = 0$.

For $\alpha > 0$,

$$\Pr(X_n \geq t) = \Pr(e^{\alpha X_n} \geq e^{\alpha t})$$

$$\leq e^{-\alpha t} E\left(\prod_{i=1}^n e^{\alpha Z_i}\right)$$

Martingale

→ Here, we used independence in

→ Martingale property → Chernoff Bound proof now

$$= e^{-\alpha t} E(E\left(\prod_{i=1}^n e^{\alpha Z_i} | Y_0, \dots, Y_{i-1}\right))$$

$$= e^{-\alpha t} \left(\prod_{i=1}^n E(e^{\alpha Z_i} | Y_0, \dots, Y_{i-1})\right)$$

→ Convexity Property

We use the following inequalities:

$$\circ e^x \leq \frac{\sinh(x)}{c} x + \text{cosh}(c) \quad \forall x \in [-c, c], c > 0$$

$$\circ \text{cosh}(x) \leq e^{x^2/2} \quad \forall x > 0$$

Now, $E(Z_i | Y_0, \dots, Y_{i-1}) = 0$

$$\begin{aligned} \Pr(X_n \geq t) &\leq e^{-\alpha t} E\left(\prod_{i=1}^n e^{\alpha Z_i} \cdot E\left(\frac{\sinh(\alpha Z_i)}{\alpha Z_i} \cdot \alpha Z_i | Y_0, \dots, Y_{i-1}\right)\right) \\ &\leq e^{-\alpha t} E\left(\prod_{i=1}^n e^{\alpha Z_i}\right) \cdot e^{\alpha^2 c^2/2} \\ &\leq e^{-\alpha t + \frac{\alpha^2 c^2}{2}} \quad (\text{Induction}) \end{aligned}$$



Proof (contd)

and putting $x = \frac{t}{\frac{\sum C_k^2}{B_{k1}}}$ above gives $P(X_n \geq t) \leq \exp\left(-\frac{t^2}{2\sum C_k^2}\right)$

The same argument works for $P(X_n \leq -t)$ by considering $(-X_i)_{i \geq 0}$. This gives Azuma's inequality as desired \Rightarrow

Exercise: Deduce Chernoff Bound type result using Azuma's Inequality.

Theorem: Let $(X_t)_{t \geq 0}$ be a supermartingale w.r.t. $(Y_t)_{t \geq 0}$ s.t.
 $|X_t - X_{t-1}| \leq C \quad \forall t \geq 1$.

Then,

$$P(X_n - X_0 \geq t) \leq \exp\left(-\frac{t^2}{2\sum C_k^2}\right)$$

(and we get the other tail with submartingale)

Proof: Same as above.

A step function f is said to satisfy the edge Lipschitz condition with constant c if $|f(H) - f(H')| \leq c$ whenever H and H' differ by only one edge.

We say f satisfies the vertex Lipschitz condition with constant c if $|f(v) - f(v')| \leq c$ whenever $v(H) > v(H')$ are incident with only one vertex.



Theorem:

If f satisfies the edge vertex Lipschitz condition with constant c , then the corresponding edge (vertex) exposure Martingale satisfies $|X_{t+1} - X_t| \leq c$.

Proof:

$$X_{t+1} = \mathbb{E}(f(Y_1, \dots, Y_{t+1}))$$

Let $\bar{g}(\omega) = (Y_1, \dots, Y_t)(\omega)$ (ω is a specific graph)

$$X_{t+1}(\omega) = \mathbb{E}(f | \bar{g}(\omega), (Y_1, \dots, Y_{t+1})) = (\underbrace{Y_1(\omega), \dots, Y_t(\omega)}, Y_{t+1}(\omega)) = \bar{g}(\omega)$$

$$X_t(\omega) = \mathbb{E}(f | (Y_1, \dots, Y_t)) = (\underbrace{Y_1(\omega), \dots, Y_t(\omega)}_{= \bar{g}(\omega)}).$$

$$X_{t+1}(\omega) - X_t(\omega) = \sum_{\vec{z} \in \mathbb{F}_{0,1,2}^{(2)-(t+1)}} f(\bar{g}(\omega), y_{t+1}(\omega), \vec{z}) P_{\vec{z}}(A_{t+1}, \dots, A_{t+1}) = \vec{z}.$$

$$= \sum_{\vec{z} \in \mathbb{F}_{0,1,2}^{(2)-(t+1)}} \sum_{y \in \mathbb{F}_{0,1,2}} f(\bar{g}(\omega), y, \vec{z}) P_{\vec{z}}(\vec{z}) P_y(y_{t+1} = y).$$

$$= \sum_{\vec{z} \in \mathbb{F}_{0,1,2}^{(2)-(t+1)}} P(\vec{z}) \left(f(\bar{g}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \mathbb{F}_{0,1,2}} P_y(f(\bar{g}(\omega), y, \vec{z})) \right)$$

$$|X_{t+1}(\omega) - X_t(\omega)| = \sum_{\vec{z} \in \mathbb{F}_{0,1,2}^{(2)-(t+1)}} P(\vec{z}) |f(\bar{g}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \mathbb{F}_{0,1,2}} f(\bar{g}(\omega), y, \vec{z}) P_y|$$

$$\leq c \cdot \text{for every } \omega.$$

Theorem (Shourin & Spencer, 1987)

$$P(\{X(G_{n,p}) - \mathbb{E}(X(G_{n,p})) \geq t\}) \leq 2e^{-t^2/2n}.$$

$\Rightarrow \forall \epsilon > 0, \exists c > 0$

$$P(\{X(G_{n,p}) - \mathbb{E}(X(G_{n,p})) \leq cn\}) \geq 1 - \epsilon$$

Proof:

Consider the voter exposure martingale w.r.t f being the characteristic #. Obviously, f is 1-Lipschitz since modifying a set of edges incident with only one voter can change f by ± 1 . So,

$$P(\{X(G_{n,p}) - \mathbb{E}(X(G_{n,p})) \geq t\})$$
$$= P(\{X_n \geq t\}) \leq e^{-t^2/2n}.$$

by Azuma's Inequality.

Theorem (Bollobás, 1988)

$$\text{a.s } X(G_{n,\frac{1}{2}}) \sim \frac{n}{\log n}$$

Note: Since we are in $G_{n,\frac{1}{2}}$,

clusters and induced sets are the same since b_i and \bar{b}_i both in $G_{n,\frac{1}{2}}$.

Proof:

We first establish a lower bound. Let $w(G)$ denote the size of a maximum cluster (or induced set) of G . Obviously $X(G) \geq w(G)$. Consider $G_{n,\frac{1}{2}}$.

The expected # of k -clusters in $G_{n,\frac{1}{2}}$ is $f(k) = \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}}$

Let $k_0 = \min \{k \geq 2 : f(k) \leq 1\}$ (This is well defined since $f(k)$ is decreasing on $k \geq 2$).

(Exercise: knowing n , $f(k_0+1) = o(1)$ and $f(k_0-1) \approx w(n)$).

Proof: (a)

Hence, α is $\omega(C_n) \leq k+1$ (And since $\alpha(\bar{C}_n) = \omega(\bar{C}_n)$ and $\bar{G} \sim G(n, \frac{1}{2})$, we know α is $\alpha(G) \leq k+1$).

$$\Rightarrow \chi(G) \cdot \alpha(C_n) \geq n.$$

$$\Rightarrow \alpha \text{ is } \chi(\bar{G} \circ G(n, \frac{1}{2})) \geq (1 + \alpha(1)) \frac{n}{\log n}.$$

Now, we establish our upper bound:

Set $k = k_0 - 1$. We knew $f(k) = \omega(n^3)$ and we want to prove that α is $\omega(G_{np}) \geq k$. We want a concentration on the # of k -cliques.

(Follows for following proof.)

Consider a random variable Y that denotes the maximal size of a family of edge disjoint cliques of size k .

We will prove that EY is large.

Key lemma: $EY \geq (1 - o(1))(\frac{n^k}{k!})$. \leftarrow Will show next class.

Let Y_0, Y_1, \dots, Y_m be the edge exposure martingale on $G(n, \frac{1}{2})$ with $Y_0 = EY$.

It is easy to see that $|Y_m - Y_{m-1}| \leq 1$ because every edge is contained in at most 1 k -clique in the maximal family of edge disjoint k -cliques.

By Azuma's Inequality



Proof Case 4

$$\Pr(Y_{(n)} - \mathbb{E}Y_1 \geq t) \leq \exp\left(-\frac{t^2}{2\mathbb{E}Y_1}\right)$$

Choosing $t = \frac{n^2}{2\log n}$ yields

$$\begin{aligned}\Pr(\alpha(G_m, b) < k) &= \Pr(Y_{(n)} < k) \\ &\leq \Pr(Y_{(n)} - \mathbb{E}Y_1 \geq t)\end{aligned}$$

$$\leq \exp\left(-\frac{n^2}{2\log n}\right) \quad \text{Use key lemma}$$

Set $m = \lfloor n/\log n \rfloor$. For every m -set $S \subseteq [n]$, $G_{S,S}$ is distributed as $G(m, b)$. Set $k_0 = k_0(n)$ and let $k = k_0^{-1}$. Then,

$$k \log m \sim 2 \log n$$

Same argument as above applied to $G(m^{-1}, k)$ shows

$$\Pr(\alpha(G_{S,S}) < k) = \exp\left(-\frac{m^2}{2\log m}\right)$$

$$= \exp\left(-\frac{n^2}{2\log n}\right).$$

Lecture 14:

Recall in $G(n, 1/2)$

$$\cdot \mathbb{E}(\# \text{k-cliques}) = f(k) = \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\cdot k_0 = \min\{k \geq 2 : f(k) \leq 1\}$$

$$\cdot k_0 \sim 2\log n, f(k_0+1) = o(1), f(k_0-1) = \omega(n^3)$$

$$\cdot \text{Let } k_0 = k_0 - 1$$

χ : Size of a maximum family of edge disjoint k -cliques

Key Lemma: $\mathbb{E} \chi \geq C(\epsilon) \frac{n^2}{2^{k_0}}$

$$\cdot \Pr(\chi(G(n, 1/2)) < k) = \exp(-\Sigma(n^2/\log^2 n))$$

$$\cdot \text{Let } m = \lfloor n/\log^2 n \rfloor$$

$$\cdot \Pr(\chi(G(m)) < k) = \exp(-\Sigma(m^2/\log^2 m)) = \exp(-\Sigma(n^2/\log^2 n))$$

There are $\binom{n}{m} < 2^n$ sets of size m .

$\Pr(\exists \text{ an } m\text{-set } S \text{ s.t. } \chi(G[S]) < k)$

$$\leq 2^n \cdot \exp(-\Sigma(n^2/\log^2 n))$$

$$= o(1)$$

Now we colour $G(n, 1/2)$ as follows: repeatedly pick out independent sets of order k and give a colour to that independent set until the # of the remaining colors is (\ll that m). Then, give distinct colors to them.

In total, we use $\leq \frac{n}{k} + m \approx \frac{n}{k} = \frac{n}{2\log n}$.

$$\text{So, } \chi(G(n, 1/2)) \leq (\epsilon + o(1)) \frac{n}{2\log n}$$

□

Proof of key lemma:

The expected # of k -cliques is $f(k)$

Let \mathcal{C} be a family of k -cliques obtained by including each of the k -cliques of $G(n, p)$ independently at prob. p .

Let $\mathbb{E}(\mathcal{W})$ of pairs of k -cliques (α, β) in \mathcal{C} such that

$$\mathcal{E}(\alpha) \cap \mathcal{E}(\beta) = \emptyset$$

Let \mathcal{C}' be obtained by deleting one of α, β if $\{\alpha, \beta\} \in \mathcal{W}$.

Then, \mathcal{C}' is a family of edge-disjoint k -cliques.

So:

$$\mathbb{E}(\mathcal{C}' \mid \mathcal{W}).$$

$$\mathbb{E}(Y) \geq \mathbb{E}(\mathcal{C}) - \mathbb{E}(\mathcal{C}' \mid \mathcal{W})$$

Expected # of k -cliques

$$\mathbb{E}(\mathcal{C}) = p \cdot f(k)$$

Probability of being included.

Permutation

Exercise: The expected # of pairs of k -cliques in $G(n, p)$

that intersect edges is asymptotically $f(k)^2 \cdot \frac{k^4}{2n^2}$

$$\text{Then, } \mathbb{E}(\mathcal{W}) \sim p^2 f(k)^2 \frac{k^4}{2n^2}$$

$$\mathbb{E}(Y) \geq p f(k) - (1 + o(1)) p^2 f(k)^2 k^4 / (2n^2)$$

$$\text{Choosing } p = \frac{n^2}{k^4 f(k)} \text{ gives } \mathbb{E}(Y) \geq (1 + o(1)) \frac{n^2}{2k^4}.$$

II.

Theorem:

(Let $p = n^{-\alpha}$ where $\alpha > \frac{1}{6}$ fixed.) Then, there exists an integer $u = u(n, p)$ s.t. α -a.s. $n \leq \chi(g_{n,p}) \leq 2u+3$.

Proof:

We have shown (Assignment question) that for any fixed $c > 0$, c -a.s. every subgraph induced by $\lceil c\sqrt{n} \rceil \times \lceil c\sqrt{n} \rceil$ vertices can be 3-colored.

Fix $\varepsilon > 0$, and let $u \geq u(n, p, \varepsilon)$ be the smallest integer s.t. $\Pr(\chi(g_{n,p}) \leq u) > \varepsilon$.

(Let Y be the size of the smallest subset S s.t. $G-S$ is u -colorable. Consider the vertex excess martingale w.r.t Y , then Y satisfies the 1-Lipschitz condition (Revealing one more vertex changes X by at most 1).

By Azuma's inequality:

$$\Pr(Y - \mathbb{E}Y \geq t) \leq e^{-t^2/2}, \quad \Pr(Y - \mathbb{E}Y \leq -t) \leq e^{-t^2/2}$$

$$\Rightarrow \Pr(Y \geq \mathbb{E}Y + \lambda\sqrt{n}) \leq e^{-\lambda^2/2}, \quad \Pr(Y \leq \mathbb{E}Y - \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$$

Choose λ s.t. $e^{-\lambda^2/2} = \varepsilon$, then

$$\Pr(Y \leq \mathbb{E}Y - \lambda\sqrt{n}) \leq \varepsilon. \quad \textcircled{1}$$

The event $Y = u$ is equivalent to $g_{n,p}$ being u -colorable.

i.e. $\{\mathcal{Y} = u\} = \{g_{n,p}\}$ is u -colorable

$$\Rightarrow \Pr(Y = u) = \Pr(\chi(g_{n,p}) \leq u) \quad (\text{by } \textcircled{1})$$

But $\Pr(\chi(g_{n,p}) \leq u) > \varepsilon$ and $\Pr(Y \leq \mathbb{E}Y - \lambda\sqrt{n}) \leq \varepsilon$, so

$$\mathbb{E}Y - \lambda\sqrt{n} < 0 \Rightarrow \mathbb{E}Y < \lambda\sqrt{n}$$

Q.E.D

Proof (c)

Now,

$$\Pr(Y \geq 2\lambda\sqrt{n}) < \Pr(Y \geq \mathbb{E}Y + \lambda\sqrt{n}) < \varepsilon.$$

So, with probability $1-\varepsilon$, $Y \leq 2\lambda\sqrt{n}$.

i.e. There exists a set S with $|S| \leq 2\lambda\sqrt{n}$ s.t.

$G(n, p) - S$ is ~~a~~ λ -colorable

And, we also know that a.c.s S is 3-colorable

$$\Pr(X(G(n, p)) \leq n-1 \text{ OR } X(G(n, p)) \geq n+4) \leq \varepsilon$$

$$\leq \varepsilon + \underbrace{\Pr(Y \geq 2\lambda\sqrt{n})}_{\leq \varepsilon} + \underbrace{\Pr(|S| = |S| \text{ not 3-colorable})}_{\text{as } \varepsilon \rightarrow 0}$$

$$\leq 3\varepsilon$$

And the assertion holds by ε letting $\varepsilon \rightarrow 0$. 17

What happens when $p = cn$, where c is fixed?

Theorem (Achlioptas and Friedgut, 2009)

For any $k \geq 3$ fixed there is a sharp threshold sequence $c_k = c_k(c)$ such that $\exists \varepsilon$:

$$\lim_{n \rightarrow \infty} \Pr(G(n, cn) \text{ } k\text{-colorable}) = \begin{cases} 0 & \text{if } c < c(1-\varepsilon)c_k(c) \\ 1 & \text{if } c > c(1+\varepsilon)c_k(c) \end{cases}$$

(Determining c_k remains open)

Differential Equation Method:

Let's say $c = n$ is fixed. Suppose n balls are thrown into n bins sequentially uniformly at random. Let X_i be the # of empty bins after i balls are thrown. (Heuristically)

$$\circ X_0 = n$$

$$\circ X_{i+1} = X_i - I_{\{X_i \geq 1\}} - \text{the ball thrown into empty bin}.$$

\uparrow
of empty bins stay the same

$$\circ \Pr(E_{i+1}) = \frac{X_i}{n}. \quad \text{Formally } \mathbb{E}(I_{\{X_{i+1} \geq 1\}} | X_0, \dots, X_i) = \frac{X_i}{n}.$$

(Heuristically)

$$\circ \mathbb{E}(X_{i+1} - X_i | H_i) = -\frac{X_i}{n}.$$

\uparrow
history of first
 i balls

We want to find the trajectory of $(X_i)_{i=0}^n$

↳ what does this mean? What is the trajectory of $c n \cdot u$?

We will formalize this later. For now ...

... let's assume that all X_i are concentrated around $\mathbb{E}X_i$. So, we find the trajectory of $(\mathbb{E}X_i)_{i=0}^n$.

Then,

$$\mathbb{E}(X_{i+1}) - \mathbb{E}(X_i) = -\frac{\mathbb{E}X_i}{n}.$$

We write,

$$\mathbb{E}X(t) = \frac{\mathbb{E}(X_0)}{n}.$$



and the above becomes:

$$\mathbb{E}X_{it+1} - \mathbb{E}X_i = -\frac{\mathbb{E}X_i}{n} \rightarrow n(\underbrace{X(t+\frac{1}{n}) - X(t)}_{\xrightarrow{\text{now}} X'(t)}) = -\mathbb{E}X(t)$$

This tells us that

the solution of $X'(t) = -X$ gives (by scaling) the
trajectory of $(\mathbb{E}X_i)_{i=0}^n$:

$$\frac{dx}{dt} = -x, \int \frac{1}{x} dx = -\int dt \Rightarrow x(t) = ce^{-t}$$

$$X_0 = n \Rightarrow \mathbb{E}X_0(0) = 1 \Rightarrow c=1, \text{ so } X(t) = e^{-t} \Rightarrow \mathbb{E}X_i = ne^{-in}$$

Suppose we want to show the concentration of X_j .

For $j \geq 0$, define $Z_j = \mathbb{E}(X_j | \mathcal{H}_j)$

\hookrightarrow The σ -field generated by the random allocation of the first j balls

We also have $|Z_j - Z_{j-1}| \leq 1$, so by Azuma's Inequality,

$$\Pr(|X_j - \mathbb{E}X_j| \geq \sqrt{\alpha j}) \leq 2e^{-\alpha j/2}$$

$\Rightarrow X_j$ with probability $\geq 1 - e^{-n^{1/2}}$, $X_j = ne^{-jn} + o(n)$

Lecture 15:

A discrete-time random process is a probability space which can be denoted by $(\Omega, \mathcal{Q}, Q_1, Q_2, \dots)$, where each Q_i takes values in some set S . The elements of the space are sequences (q_0, q_1, q_2, \dots) where each $q_i \in S$. We use h_t to denote (q_0, q_1, \dots, q_t) , which is called the history up to step t of the process. Let H_t be the random counterpart of h_t .

We care about a sequence of random processes indexed by n ($n=1, 2, \dots$). Thus, $q_i = q_i^{(n)}$ and $S = S^n$ (We drop n from the notation if it's unambiguous). Asymptotic rates to $n \rightarrow \infty$.

S^{out} denotes the set of all $h = (q_0, \dots, q_t)$ where each $q_i \in S^n$

We say a function $f(u_1, \dots, u_j)$ satisfies a Lipschitz condition on $\mathbb{D} \subseteq \mathbb{R}^j$ if there exists a constant L s.t.

$$|f(u_1, \dots, u_j) - f(v_1, \dots, v_j)| \leq L \max_{1 \leq i \leq j} |u_i - v_i|$$

We are interested in the trajectories of a set of r.v. Y_1, Y_n induced by a random process.

Suppose $\mathbb{D} \subseteq \mathbb{R}^m$, define the stopping time $T_\mathbb{D}(Y_1, Y_n)$ by the minimum integer $t \geq n$ such that

$$\left(\frac{Y_1(t)}{n}, \frac{Y_2(t)}{n}, \dots, \frac{Y_m(t)}{n} \right) \notin \mathbb{D}$$

Theorem (DE Method)

This is our r.v.

For $1 \leq l \leq a$, where a is fixed. Let $\gamma_l: \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_l: \mathbb{R}^n \rightarrow \mathbb{R}$. Such that for some constant C_0 and δ_l

~~This is the basic function~~

all l ,

$$|\gamma_l(t)| < C_0,$$

for all $t \in \mathbb{R}^n$ for all n .

Assume the following conditions hold above in (i) and (ii)

D is some bounded, connected, open set containing the closure of:

$$\{(0, z_1, \dots, z_a)\}; \text{Pr}(\gamma_l(t)) = z_{ln}, 1 \leq l \leq a \text{ for some } n$$

(i) (Boundedness Hypothesis)

For some fixed $c > 0$,

$$\max_{1 \leq l \leq a} |\gamma_l(t_e) - \gamma_l(t)| \leq c \text{ for all } t < t_0$$

(ii) (Trend Hypothesis)

For all $l \leq a$,

$$|t \frac{d}{dt}(\gamma_l(t_e) - \gamma_l(t))|_{(t_e)} - f_l(t, \frac{\gamma_l(t)}{n}, \dots, \frac{\gamma_l(t)}{n})| = o(1) \text{ for } t < t_0$$

(iii) (Open Lipschitz Hypothesis)

Then function f_l is continuous and satisfies a Lipschitz

Condition on $D \cap \{(t, z_1, \dots, z_a)\}; t \geq 0$

Then, the following are true:

(a) For $(0, \tilde{z}_1, \dots, \tilde{z}_a) \in D$, the system of ODEs.

$$\frac{dz_l}{dx} = f_l(t, z_1, \dots, z_a) \quad l=1, \dots, a$$

has a unique solution in D for $z_0: \mathbb{R} \rightarrow \mathbb{R}$ passing through

$z_0(0) = \tilde{z}_l, 1 \leq l \leq a$ which extends to pass arbitrarily close to the boundary of D :

Theorem (cont)

(b). For every $\epsilon > 0$, a.s.

$$Y_n(t) = n Z_n(t/n) + o(n)$$

uniformly for $0 \leq t \leq n$. and for every $1 \leq l \leq n$, where

$Z_l(\omega)$ is the solution in (a) with $\tilde{Z}_l = \frac{1}{n} Y_n(0)$, and
 $\sigma = \sigma(n) = \omega^{(l)}$ is the supremum of x to which the solution
can be extended before reaching ~~within~~ within l^∞ -distance
 ϵ of the boundary of D .

Back to the toy example.

We proved that $E(X_{t+1} - X_t | X_t) = -\frac{X_t}{n}$ (a)

Define $f(x, z) = -\frac{x}{n} z$

We also have $0 \leq X_t \leq 1$ a.s. for all $t \leq n$ (b)

$|X(t+1) - X(t)| \leq 1$ for all $0 \leq t \leq n$ (c)

Let $\Omega \subseteq \mathbb{R}^2$ such that $D = \{(x, z) : -2 < x < 2, -2 < z < 2\}$

(d) \Rightarrow Boundedness hypothesis

(e) \Rightarrow Total Hypothesis

(f) $\Rightarrow D$ contains all points (ω, z) such that $P(X(0) = z) \neq 0$
for some n . Moreover, D is bounded, connected & open

(g) $\Rightarrow |f(x, z)| \leq C$ for some constant $C > 0$. It is easy to
see that $f(x, z) = -\frac{x}{n} z$ is Lipschitz on D .

By theorem, $\frac{dz}{dx} = -z$ has a unique solution satisfying
 $Z_0 = 1$, which extends to point arbitrarily close to boundary
of D . Indeed, we knew $Z(x) = e^{-x}$

By part (b) of the DE method, we have

$$\begin{aligned}x(i) &= n \gamma(i/n) + o(n) \\&= n \cdot e^{-i/n} + o(n)\end{aligned}$$

uniformly for all i .

Min-degree graph process.

Start with an empty graph on vertex set $\{n\}$. In each step, we can choose a vertex with minimum degree, call it u , and then we choose a vertex not yet adjacent to u , call it v . Add an edge between u and v . Partition the graph process into phases. For $k \geq 0$, let phase k consist of the steps where $d(u) = k$.

We will consider phase 0 first. Let G_t be the graph obtained after step t and $y_t(t)$ be the # of vertices of degree 1 in G_t , again, let $H_t = (G_0, G_1, \dots, G_t)$.

In each step of phase 0: (let u_t, v_t denote the vertices chosen in step t)

$$\begin{aligned}\mathbb{E}(\#\text{ edges}) &= \mathbb{E}[\sum_{t=0}^{\infty} \mathbb{I}(d(v_t) = 1)] \\&= \frac{\sum_{t=0}^{\infty} y_t(t)}{n-1} \leftarrow \text{Total vertices} \sum_{t=0}^{\infty} d(v_t) = i\end{aligned}$$

$$\text{Let } x_i(t+1) = x_i(t+1) - \mathbb{I}(d(v_i) = 1)$$

$$y_0(t+1) = y_0(t) - (1 - x_0(t+1))$$

$$y_1(t+1) = y_1(t) + 1 + x_0(t+1) - x_1(t+1)$$

$$y_i(t+1) = y_i(t) + x_{i-1}(t+1) - x_i(t+1) \quad \text{for } i \geq 2$$

$$\Rightarrow y_i(t+1) = y_i(t) - \delta_{i0} + \delta_{i1} + x_{i-1}(t+1) - x_i(t+1) \quad \text{for all } i \geq 2$$

by setting $x_{i-1}(t) = 0$ for all t .

$$\Rightarrow E(Y_i(t+1) - Y_i(t) | H_t) = -\bar{\delta}_0 + \bar{\delta}_{ii} + \underbrace{\frac{Y_{i-1}(t)}{n-1}}_{\substack{\sim \\ X_{i-1}(t+1)}} - \underbrace{\frac{Y_i(t)}{n-1}}_{\substack{\sim \\ X_i(t+1)}}$$

Lecture 16:

Recall the min-degree graph process.

$Y_i(t)$: # of vertices of degree i in G_t

$H_t := (G_0, \dots, G_t)$: History up to step t

We showed last class'

$$\mathbb{E}(Y_i(t+1) - Y_i(t) | H_t) = -\delta_{i0} + \delta_{ii} + \frac{Y_{i-1}(t) - Y_i(t)}{n-1}$$

$$= -\delta_{i0} + \delta_{ii} + \frac{Y_{i-1}(t) - Y_i(t)}{n} + o(1)$$

(By setting $Y_i(t) = 0$)

Let $Z'_i = -\delta_{i0} + \delta_{ii} + Z_{i-1} - Z_i$ for any $i \geq 0$, with $Z_i(\omega) = 0$ for all ω ,

with initial condition: $Z_0(0) = 1$ and $Z_i(0) = 0 \quad \forall i \geq 1$.

For $i=0$: $Z'_0 = -1 - Z_0$ which gives $Z_0(x) = 2e^{-x} - 1$.

which gives $Z_0(x) = 2e^{-x} - 1$.

For $i=1$: $Z'_1 = 1 + (2e^{-x} - 1) - Z_1 = 2e^{-x} - Z_1$ which gives $Z_1(x) = 2xe^{-x}$.

which gives $Z_1(x) = 2xe^{-x}$.

In general, $Z_i(x) = \sum_{j=0}^i e^{-x}$

Theorem: Fix $a \in \mathbb{R}$ and suppose $0 < s < d + \alpha^2$ fixed. Then

c.c.s

$$Y_i(t) = nZ_i(t/n) + o(n)$$

(uniformly for all $0 \leq i \leq a$ for all $0 \leq t \leq n$).

Proof: By checking cell hypotheses of DE Theorem (Exercise!).

Theorem

Fix $\alpha \in \mathbb{Z}$, $1 \leq l \leq a$, $Y_l : S^{l+1} \rightarrow \mathbb{R}$ and $f_l : \mathbb{R}^{a+l} \rightarrow \mathbb{R}$

\mathbb{D} : Bounded, connected open set containing the closure
of $\{(0, z_1, \dots, z_a) : P_l(Y_l(0) = z_i, 1 \leq i \leq a) \neq 0\}$ for
some $z_i \in \mathbb{R}$

(i) (Boundedness) $|Y_l(t+1) - Y_l(t)| \leq c$ for all $1 \leq l \leq a$

(ii) (Trend Hypothesis) $E(Y_l(t+1) - Y_l(t) | H_t) = f_l(\frac{t}{n}, \frac{Y_{1(t)}}{n}, \dots, \frac{Y_{a(t)}}{n}) + o(n)$

(iii) (Lipschitz) f_l is C β and Lipschitz on $\mathbb{D} \cap \{(t, z_1, \dots, z_a) : t \geq 0\}$

Then,

(a) Uniqueness and existence of the sol'n to the ODEs.

(b) For every $\varepsilon > 0$, there is $Y_l(t) = n Z_l(t/n) + o(n)$ uniformly
for all $t \geq 0$ such that for all $1 \leq l \leq a$, where σ is the
supremum of x to which the sol'n of the ODEs
can be extended before within ε L^∞ -distance of
the boundary of \mathbb{D} .

Proof:

We are only going to prove the case where $a=1$.

Let $\lambda = o(n)$ be the specified factor. Let $w = \lambda n$.

First, we prove convergence of $Y_l(t+\lambda) - Y_l(t)$.

Assume that $(t/n, \frac{Y_{1(t)}}{n})$ is in L^∞ -distance at least 2ε
from the boundary of \mathbb{D} .

For all $k=1, \dots, w$:

$$Y_l(t+k\lambda) = Y_l(t) + o(\lambda) \quad (\text{By (i)})$$

Hence, $(\frac{t+k\lambda}{w}, \frac{Y_{1(t+k\lambda)}}{w}) \in \mathbb{D}$ for any $k \leq w$.



Proof (Cont'd)

By (ii) and (iii)

$$\mathbb{E}(Y(t_{k+1}) - Y(t_k)) = 0 \mid \mathcal{H}_{t_k}$$

$$= f\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) + o(n)$$

$$= f\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) + O\left(\frac{k}{n}\right) + o(n)$$

Since $Y(t_{k+1}) = Y(t_k) + O(k)$,
↳ f Lipschitz.

There exists $g(n) = o(1)$ such that

$$\mathbb{E}(Y(t_{k+1}) - Y(t_k)) \leq f\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) + g(n)$$

Hence,

$$(Y(t_{k+1}) - Y(t_k) - kf\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) - kg(n))_{k \geq 0}$$

is a supermartingale in \mathcal{K} w.r.t. $\mathcal{H}_t, \dots, \mathcal{H}_{t_n}$

We need to check Lipschitz:

$$|Y(t_{k+1}) - Y(t_k) - kf\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) - kg(n)|$$

$$= |Y(t_{k+1}) - Y(t_k) - kf\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right) - kg(n)|$$

$$\leq |Y(t_{k+1}) - Y(t_k)| + |f\left(\frac{t_k}{n}, \frac{Y(t_k)}{n}\right)| + |g(n)|$$

$$\leq C \quad \leq C \quad \text{each}$$

Since Lipschitz
on D .

$$\leq \text{Some sufficiently large constant} \cdot (\text{cell } \#)$$

So, by Azuma's Inequality

$$\Pr(Y(t_n) - Y(t) - wF\left(\frac{t_n}{n}, \frac{Y(t_n)}{n}\right) \geq ws(n) + c\sqrt{2ws} \mid \mathcal{H}_t)$$

$$\leq \exp\left(-\frac{c^2 - 2ws}{2ws^2}\right)$$

$$= e^{-\alpha}$$

Proof: (Contd)

The lower tail can be bounded using exactly the same argument, but using a submartingale.

$$\Rightarrow \Pr(|Y(t_{\text{low}}) - Y(t) - w f(t_n, Y(t_n))| \geq w \gamma n) + \Pr(Y(t) \leq t_n) \leq 2e^{-\alpha}.$$

(We will fix w later). ↑ Both tails.

(So, we have shown concentration of Y)

But, we also need to show that Y follows the trajectory of t .

Next, we are going to compare $Y(t)$ with $nZ(t_n)$.

Define $k_i = iw$, $i=0, 1, \dots, i_0$, where $i_0 = \lceil \frac{t}{w} \rceil$.

(k_i is the i^{th} "check" $\in \mathbb{N}$ (step)). ↑ We break up the function and show at each check, the error is small.

We will show, by induction, that:

$$\Pr(|Y(k_i) - Z(\frac{k_i}{n})| \geq B_i) = O(ie^{-\alpha})$$

where $B_i = Bw(\lambda + \frac{w}{n})(((1 + \frac{Bw}{n})^i - 1))$

and $\forall i, B_i < 1$ and B is a constant

Base case: $i=0$: $Z(0) = \frac{Y(0)}{n}$, so there is no error.

Induction: Assume it holds for the first i checks



Proof: (cont.) $\|u_i - z(\frac{k_i}{n}) \cdot n\|$ is approximated by F .

$$\|u_i - z(\frac{k_i}{n}) \cdot n\| = |A_1 + A_2 + A_3 + A_4|.$$

where A_1 is approximated by derivative

$$A_1 = u_i - n \cdot z(\frac{k_i}{n})$$

$$A_2 = u_i - u_i - wF(\frac{k_i}{n}, \frac{u_i}{n})$$

$$A_3 = wz'(\frac{k_i}{n}) + nz(\frac{k_i}{n}) - nz(\frac{k_{i+1}}{n})$$

$$A_4 = wF(\frac{k_i}{n}, \frac{u_i}{n}) - wz'(\frac{k_i}{n})$$

And, we will bound each A_i to show that the sum is banded.

By induction $|A_i| \leq B_i$ with prob $\geq 1 - O(\epsilon^{-2})$

We've shown that

$$|A_1| \leq \omega \epsilon n + O(\sqrt{\omega} \epsilon) \text{ with prob } \geq 1 - 2\epsilon^{-2}$$

by choosing $\alpha, \tilde{\lambda}$ such that $\alpha n = O(\tilde{\lambda})$, $\tilde{\lambda} = O(\epsilon)$
and $\epsilon = O(\omega \tilde{\lambda}^2)$.

Since z is the solution of the ODE and F is α -Lipschitz:

$$|z(\frac{k_{i+1}}{n}) - z(\frac{k_i}{n}) - \frac{\omega}{n} z'(\frac{k_i}{n})| \leq C \frac{\omega^2}{n} \text{ for some constant } C > 0.$$

To By Taylor expansion

$$\text{Thus, } |A_3| \leq C \frac{\omega^2}{n^2}$$

C

Proof (Cont.)

Finally, for Δ_n .

Induction + Lipschitz

$$|\Delta_n| \leq \omega |f\left(\frac{k_i}{n}, \frac{y(k_i)}{n}\right) - f\left(\frac{k_i}{n}, \frac{z(k_i)}{n}\right)|$$

$$+ \omega |f\left(\frac{k_i}{n}, \frac{z(k_i)}{n}\right) - z\left(\frac{k_i}{n}\right)|.$$

$= 0$, by defn.

$$\leq C'' \cdot \omega \left| \frac{y(k_i)}{n} - z\left(\frac{k_i}{n}\right) \right|$$

$$\leq C''' \frac{\omega}{n} \cdot B_i \quad \text{with prob } \geq 1 - O(ie^{-\alpha})$$

So, with prob $\geq 1 - O(ie^{-\alpha}) - O(e^{-\alpha})$

$$= 1 - O((i+1)e^{-\alpha}).$$

$$|y(k_i) - n z\left(\frac{k_i}{n}\right)| \leq B_{i+1}$$

by summing $\sum_{i=1}^n$ and choosing sufficiently large B .

We still want to check that B_{i+1} is not too big!

$$B_{i+1} = B_0 \left(\tilde{\lambda} + \frac{\omega}{n} \right) \left(\left(1 + \frac{B_0}{n} \right)^i - 1 \right)$$

$$= \tilde{\lambda} \underbrace{\left(1 + \frac{B_0}{n} \right)^i}_{= B\lambda} - 1$$

$$= B_0 \omega \left(\tilde{\lambda} + \lambda \right) \left(\left(1 + B\lambda \right)^i - 1 \right).$$

$$\leq \tilde{\lambda}$$

since $\lambda \ll \omega \ll \tilde{\lambda} = O(1)$

$$\leq B_0 \left(\tilde{\lambda} + \lambda \right) \left(C \left(1 + B\lambda \right)^i - 1 \right)$$

$$= 2B_0 \tilde{\lambda} \left(C \left(1 + B\lambda \right)^i - 1 \right)$$

$$= O(\tilde{\lambda}^2) \leq e^{B\lambda i} = e^{B\lambda \frac{\omega}{n}} = e^{B\lambda \frac{\omega}{B}} = e^{B\omega} = O(1)$$

$$= o(n)$$

(We will finish the rest on Monday!).

References

- [1] Bollobás Béla. *Random graphs*. Academic Press, 1985.
- [2] Alan Frieze and Michał Karoński. *Introduction to Random Graphs*. Cambridge University Press, 2015.