CO351 - Network Flow Theory

Contents

1 Graph Theory Primer

Let $G = (V, E)$ be a graph, where V is the vertex set and E is the edge set.

Definition 1.1

The **degree** of a vertex $v \in V$ (denoted $\deg(v)$) is the number of edges with one end in v. (i.e. The size of the set $\{va \mid a \in V,\ a \neq v\}$).
A **walk** is a sequence of vertices $v_1 v_2 \ldots v_k$ where $v_i v_{i+1}$ is an edge. A **path** is a walk where all vertices are distinct. A **cycle** is a walk where $v_1 = v_k$ and $v_1, \ldots v_{k-1}$ are distinct.
Finally, we say a graph is **connected** if there exists a path between any two vertices in G.

Definition 1.2

For $S \subset V$, the **cut** induced by S is the set of all edges with one end in S and one end not in S, denoted $\delta(S) = \{uv \in E \mid u \in S,\ v \notin S\}$. Given two vertices $s, t \in V$ with $s \in S, t \notin S$, we call $\delta(S)$ an s,t**-cut**
An s,t**-path** is a path with starting vertex s and ends on t.

Theorem 1.1

There exists an s,t-path if and only if every s,t-cut is nonempty

Definition 1.3

A **tree** is a connected graph with no cycles. A **spanning tree** is a subgraph that is a tree and has vertex set V

Note the following:

- A tree on n vertices contains $n - 1$ edges.
- If T is a tree, then adding an edge $uv \notin T$ creates exactly one cycle C. Moreover, if xy is an edge in C, then $T + uv - xy$

Let $D = (N, A)$ be a directed graph. N is a set of nodes and A is a set of ordered pairs of nodes (called arcs).

Definition 1.4

For an arc (u,v), we call u the **tail** and v the **head**.
The **out-degree** of node u (denoted $d(u)$ or $d^{\text{out}}(u)$) is the number of arcs with tail u.
The **in-degree** of node u (denoted $d(\overline{u})$ or $d^{\text{in}}(u)$) is the number of arcs with head u.
A **diwalk** is a sequence of nodes $v_1v_2\dots v_k$ where (v_i,v_{i+1}) is an arc. **Dipaths** and **Dicycle** are defined analgous to simple graphs but with arcs instead of edges.
For $S \subset N$, the **cut** induced by S is denoted $\delta(S) = \{(u,v) \in A \mid u \in S, v \notin S\}$. (sometimes written as $\delta^{\text{out}}(S)$) This is the set of arcs with tail in S. We denote the complement of S by $\overline{S}$, and define $\delta(\overline{S}) = \{(u,v) \in A \mid u \notin S, v \in S\}$ (sometimes written as $\delta^{\text{in}}(S)$)) to be the set of arcs with head in S. Finally, if $s \in S, t \notin S$, then $\delta(S)$ is an s,t-cut.

Theorem 1.2

There is an s,t-dipath if and only if every s,t cut is non-empty