

# 1 Graph Theory Primer

Let  $G = (V, E)$  be a graph, where  $V$  is the vertex set and  $E$  is the edge set.

## Definition 1.1

The **degree** of a vertex  $v \in V$  (denoted  $\deg(v)$ ) is the number of edges with one end in  $v$ . (i.e. The size of the set  $\{va \mid a \in V, a \neq v\}$ ).

A **walk** is a sequence of vertices  $v_1v_2 \dots v_k$  where  $v_iv_{i+1}$  is an edge. A **path** is a walk where all vertices are distinct. A **cycle** is a walk where  $v_1 = v_k$  and  $v_1, \dots, v_{k-1}$  are distinct.

Finally, we say a graph is **connected** if there exists a path between any two vertices in  $G$ .

## Definition 1.2

For  $S \subset V$ , the **cut** induced by  $S$  is the set of all edges with one end in  $S$  and one end not in  $S$ , denoted  $\delta(S) = \{uv \in E \mid u \in S, v \notin S\}$ . Given two vertices  $s, t \in V$  with  $s \in S, t \notin S$ , we call  $\delta(S)$  an  $s, t$ -**cut**.

An  $s, t$ -**path** is a path with starting vertex  $s$  and ends on  $t$ .

## Theorem 1.1

There exists an  $s, t$ -path if and only if every  $s, t$ -cut is nonempty

## Definition 1.3

A **tree** is a connected graph with no cycles. A **spanning tree** is a subgraph that is a tree and has vertex set  $V$ .

Note the following:

- A tree on  $n$  vertices contains  $n - 1$  edges.
- If  $T$  is a tree, then adding an edge  $uv \notin T$  creates exactly one cycle  $C$ . Moreover, if  $xy$  is an edge in  $C$ , then  $T + uv - xy$

Let  $D = (N, A)$  be a directed graph.  $N$  is a set of nodes and  $A$  is a set of ordered pairs of nodes (called arcs).

#### Definition 1.4

For an arc  $(u, v)$ , we call  $u$  the **tail** and  $v$  the **head**.

The **out-degree** of node  $u$  (denoted  $d(u)$  or  $d^{\text{out}}(u)$ ) is the number of arcs with tail  $u$ .

The **in-degree** of node  $u$  (denoted  $d(\bar{u})$  or  $d^{\text{in}}(u)$ ) is the number of arcs with head  $u$ .

A **diwalk** is a sequence of nodes  $v_1 v_2 \dots v_k$  where  $(v_i, v_{i+1})$  is an arc. **Dipaths** and **Dicycle** are defined analgous to simple graphs but with arcs instead of edges.

For  $S \subset N$ , the **cut** induced by  $S$  is denoted  $\delta(S) = \{(u, v) \in A \mid u \in S, v \notin S\}$ . (sometimes written as  $\delta^{\text{out}}(S)$ ) This is the set of arcs with tail in  $S$ . We denote the complement of  $S$  by  $\bar{S}$ , and define  $\delta(\bar{S}) = \{(u, v) \in A \mid u \notin S, v \in S\}$  (sometimes written as  $\delta^{\text{in}}(S)$ ) to be the set of arcs with head in  $S$ . Finally, if  $s \in S, t \notin S$ , then  $\delta(S)$  is an  $s, t$ -cut.

#### Theorem 1.2

There is an  $s, t$ -dipath if and only if every  $s, t$  cut is non-empty