BH - Combinatorial Design Jan 12th

Last time:

(v,b,r,k,λ)-BIBD def'n and

① $v/k = b/r$

② $v(v-1)/k(k-1) = b/\lambda$

③ $v-1/k-1 = r/\lambda$.

(Cross out def'n of order ~~in notes!~~ last class's.)

Corollary: (Necessary Conditions)

If a (v,k,λ)-BIBD exists, then $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $\lambda(v-1) \equiv 0 \pmod{k-1}$

Proof:

Since b is an integer, and $b = \frac{\lambda v(v-1)}{k(k-1)}$, then $k(k-1)$ divides $\lambda v(v-1)$.

Similarly, since r is an integer and $r = \frac{\lambda(v-1)}{k-1}$, then $k-1$ divides $\lambda(v-1)$. ☒

Ex: A Steiner Triple System is a $(v,3,1)$-BIBD. Determine all values of v satisfying the necessary conditions.

→ Needs to satisfy $v(v-1) \equiv 0 \pmod 6$

$v-1 \equiv 0 \pmod 2$

which gives us the 2 possibilities:

$v \equiv 1 \pmod 6$

$v \equiv 3 \pmod 6$

Question: Are these sufficient?

Isomorphism:

Let $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ be designs. We say that these are isomorphic if there exists a bijection $f: V \to V'$ such that [redacted] $\mathcal{B}' = \{\{f(x) \mid x \in \alpha\} \mid \alpha \in \mathcal{B}\}$.

Other ways to specify a design:

- Incidence Structure:

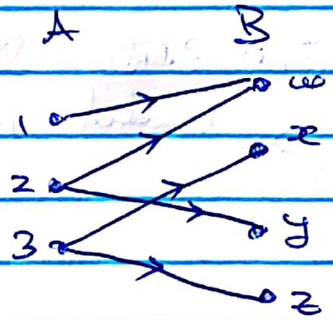
Def'n An incidence structure is a relation "$\to$" on $(A, \mathcal{B})$ where A and $\mathcal{B}$ are sets. For any pair $(a,b) \in A \times \mathcal{B}$ either $a \to b$ or $a \not\to b$. (We call this "$\to$" an incidence or incidence relation)

Given an incidence relation $\to$ on $(A, \mathcal{B})$, we construct a design $(V, \mathcal{B})$:

$V = A$

$\mathcal{B} = \{\{a \to b \mid a \in A\} \mid b \in \mathcal{B}\}$

Ex.



$V = \{1,2,3\}$

$\mathcal{B} = \{\{1,2\}$, (from w)

$\{3\}$, (from x)

$\{2\}$, (from y)

$\{3\}$, (from z) $\}$.

Note: $\{3\}$ exists in the design twice.

Note: This incidence structure is effectively a bipartite graph, and 2 designs are isomorphic iff their incidence structures ~~bipartite graphs~~ are isomorphic.

- Incidence Matrix:

Given a design $(V, \mathcal{B})$, write $V = \{x_1, x_2, \ldots, x_v\}$, $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_b\}$. The incidence matrix N is the $v \times b$ matrix (one row for every point, one column for every block), where

$$N_{ij} = \begin{cases} 1 & \text{if } x_i \in \alpha_j \\ 0 & \text{o/w.} \end{cases}$$

Each column of the matrix specifies a block of the design.

Ex:

$$N = \begin{array}{c|cccc} & w & x & y & z \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \end{array}$$

Proposition

$(V, \mathcal{B})$ is a (v, b, r, k, λ)-BIBD iff its incidence matrix N satisfies the following conditions:

(a) $N \mathbb{1}_b = r \mathbb{1}_v$

(b) $\mathbb{1}_v^T N = k \mathbb{1}_b^T$

(c) $NN^T = (r-\lambda) I_v + \lambda J_v$.

$\Longrightarrow$ Proof

Proof:

$$N\mathbb{1}_b = \begin{pmatrix} \sum_{j=1}^{b} N_{1j} \\ \vdots \\ \sum_{j=1}^{b} N_{vj} \end{pmatrix}, \text{ so } N\mathbb{1} = r\mathbb{1} \text{ iff } \sum_{j=1}^{b} N_{ij} = r \text{ for}$$

all $i = 1, \ldots, v$. and $\sum_{j=1}^{b} N_{ij}$ is the # of blocks containing the blocks the point x_i. ~~and~~ ~~x_i~~ ~~set~~ ~~that~~ is N_i

so $\sum_{j=1}^{b} N_{ij} = r \;\forall i$ iff x_i lies in r blocks $\forall i$

Similarly, $\mathbb{1}^{T} N = k\mathbb{1}^{T}$ iff each block has k points.

Finally, consider NN^T.

$$(NN^T)_{ii} = \sum_{j=1}^{b} N_{ij}(N^T_{ji}) = \sum_{j=1}^{b} N_{ij} \cdot N_{ij}$$

$$= \sum_{j=1}^{b} N_{ij} \quad \text{since } N_{ij} = 0 \text{ or } 1.$$

Hence $(NN^T)_{ii} = r$ iff x_i is in r blocks.

(Aside: $(r-\lambda) I_v + \lambda J_v = \begin{pmatrix} r-\lambda & & 0 \\ & \ddots & \\ 0 & & r-\lambda \end{pmatrix} + \begin{pmatrix} \lambda & \lambda & \cdots & \lambda \\ \vdots & & \ddots & \vdots \\ \lambda & \cdots & \cdots & \lambda \end{pmatrix}$.

$$= \begin{pmatrix} r & & \lambda \\ & \ddots & \\ \lambda & & r \end{pmatrix}.$$

)

$\Leftarrow$

Proof (cont)

And, for $i \neq j$,

$$(NN^T)_{ij} = \sum_{l=1}^{v} N_{il} N^T_{lj} = \sum_{l=1}^{v} (N_{il} N_{jl})$$

$$= \begin{cases} 1 & \text{if } x_i \in x_l \text{ and } x_j \in x_l. \\ 0 & \text{o/w} \end{cases}$$

This is the number of blocks containing both x_i and x_j.
So, $(NN^T)_{ij} = \lambda \ \forall i \neq j$ iff every pair of points is in λ blocks. ☒

Example:
Let's do the exercise on complements using incidence matrices: If (V, B) is a (v, b, r, k, λ)-BIBD, let $(V, \bar{B})$ be the complement design. Let N and $\bar{N}$ be the incidence matrices

$\bar{N} = K - N$ where $K = J_{v \times b} = \mathbb{1}_v \mathbb{1}_b^T$.

Check (a), (b), (c) for $\bar{N}$:

(a) $\bar{N}\mathbb{1}_b = (K-N)\mathbb{1}_b = K\mathbb{1}_b - N\mathbb{1}_b = b\mathbb{1}_v - r\mathbb{1}_v = (b-r)\mathbb{1}_v$

(b) $\mathbb{1}_v^T \bar{N} = \dots = (v-k)\mathbb{1}_b$

(c) $\bar{N}\bar{N}^T = (K-N)(K-N)^T$

$= KK^T - NK^T - KN^T + NN^T$

$= \mathbb{1}_v\mathbb{1}_b^T\mathbb{1}_b\mathbb{1}_v^T - N\mathbb{1}_b\mathbb{1}_v^T - \mathbb{1}_v\mathbb{1}_b^T N^T + NN^T$

$= b\mathbb{1}_v\mathbb{1}_v^T - r\mathbb{1}_v\mathbb{1}_v^T - r\mathbb{1}_v\mathbb{1}_v^T + (r-\lambda)I_v + \lambda J_v$

$= (r-\lambda)I_v + (b-2r+\lambda)J_v$

Exercise (which will be discussed next class)
Is $(r-\lambda)I_v + \lambda J$ invertible?

Cor: In any non-trivial BIBD, the rows of N are linearly independent

Proof:
Since $NN^T = (r-\lambda)I_v + \lambda J_v$ invertible, rank(N) = rank$(NN^T) = v$
& So, ~~the~~ N ~~rows~~ has v rows $\Rightarrow$ v rows of N is linearly independent.

Cor: (Fisher's Inequality)
In any nontrivial (v,b,r,k,λ)-BIBD, $b \geq v$.