749 - Colorings

Feb 25th

Recall:

Thm (Caro-Wei)

If G is a graph, then G has an f-coloring where $f(v) = \frac{1}{d(v)+1}$ $\forall v$

3 proofs:

1) Min degree - Delete & Extend

2) Probabilistic

3) Max degree - Delete, "Selectively Reduce", Extend

Q: Are there local fractional versions of other coloring theorems?

A local fractional Brooks's Theorem:

Multiple Viewpoints:

- Under what conditions can you get an f-col w/ $\frac{1}{d(v)}$?
  (Or, weaker version $\frac{1}{d(v)+\varepsilon}$ for some $\varepsilon \in (0,1)$)
- Caro-Wei is tight for disjoint union of cliques. What if we "forbid this"? (What can we save?)

  Idea #1: Control local clique #'s, i.e. $\omega(v) = \omega(G[N(v)])$
  - We could assume $\omega(v) \le d(v)$ $\forall v$. ~~Recall~~ (Instead of possible $d(v)+1$ in clique) → Recall that if $\omega(v) = d(v)+1$, v is called simplicial. So, we restrict simplicial vertices

  -OR-
  - We could try a larger f for non-simplicial and $f = \frac{1}{d(v)+1}$ for simplicial.

- CR -

Strongest Version:

Show that the necessary condition:

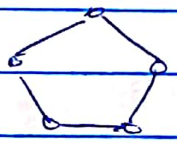
$$\sum_{v \in V(K)} f(v) \leq 1 \quad \forall \text{cliques } K$$

is sufficient

(Since if cliques demand too much color, then we're screwed)

Q: For what ε is this nec. clique condition sufficient?

If $\varepsilon < 1/2$, then no!



then, $f(v) = \frac{1}{d(v)+\varepsilon} > 2/5$,

so we get a better than $2/5$-coloring.

Theorem (Kelly, Postle)

If G is a graph and f a demand function for G such that

- $f(v) \leq \frac{1}{d(v)+1/2} \quad \forall v \in V(G)$, and
- $\sum_{v \in K} f(v) \leq 1 \quad \forall \text{clique } K \text{ in } G$,

then G has an f-coloring.

Proof Ideas:

- Min. degree type proof (Probabilistic, max-degree didn't work well)

Proof Idea (Cont)

1st Obs) Min degree vertex has many min. degree neighbors

2nd Obs) Prove that min. deg vertices decompose into cliques

i.e.

⋮

□

Corollary (Brewer)

If G is a K_{r+1}-free graph on n vertices with:

$$e(G) \geq \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{n}{4r} + 1$$

then $\chi(G) \leq r$ (r-polite)

~~Brooks~~

Local Frac. Vizings:

If G is a graph, $E(G)$ has an f-colouring where

$$f(e) = \frac{1}{\max\{d(u), d(v)\}+1} \quad \forall e \in E(G)$$

$e = uv$

Proof:

Have to prove $(f(e) : e \in E(G))$ is in the Edmund's Matching Polytope:

- $\sum_{e \ni v} f(e) \leq 1$ $\longrightarrow$ $\sum_{e \ni v} \frac{1}{d(v)+1} \leq \frac{d(v)}{d(v)+1} \leq 1$
- $\sum_{e \in G[S]} f(e) \leq \left\lfloor \frac{|S|}{2} \right\rfloor$ $\longrightarrow$ Show that $f'(e) \leq \frac{v(G)-1}{2}$, where $f(v) = \frac{1}{2}\left(\frac{1}{d(u)+1} + \frac{1}{d(v)+1}\right)$ ↳ Average of max.
- $f(e) \geq 0$

$$\sum_e \left(\frac{1}{d(u)+1} + \frac{1}{d(v)+1}\right)$$

$$= \sum_v \frac{d(v)}{d(v)+1} \leq \sum_v \frac{\Delta(G)}{\Delta(G)+1} \leq v(G) - 1.$$

Perfect Graph:

Recall:

G is perfect if $\forall$ induced subgraphs H, $\omega(H) = \chi(H)$

Strong perfect graph thm (Chudnovsky, Robertson, Seymour, Thomas, 2006)
G perfect iff neither G nor $\bar{G}$ contain an induced odd cycle of length ≥ 5

Weak Version: (Lovasz '70s)
G perfect iff $\bar{G}$ perfect

Theorem (Kelly, Postle)
If G is perfect, then G has a $\frac{1}{\omega(G)}$-coloring

Lemma:
If you copy a vertex in a perfect graph, you remain perfect.

~~Proof (AR)~~:
Blow up ~~[illegible]~~