Alg-Random

Mar 2nd

Theorem (Azuma's Inequality)

Proof:

Let $Z_i = X_i - X_{i-1}$. WLOG we may assume $X_0 = 0$.

Then, $X_n = \sum_{i=1}^{n} Z_i$.

By assumption (Lipschitz condition):

$-c_i \le Z_i \le c_i \quad \forall i$ ← Since $E(X_i \mid Y_0, \dots, Y_{i-1}) = X_{i-1}$.

and $E(Z_i \mid Y_0 \dots Y_{i-1}) = E(X_i \mid Y_0, \dots, Y_{i-1}) - X_{i-1} = 0$

For $\alpha > 0$,

$$Pr(X_n \ge t) = P(e^{\alpha X_n} \ge e^{\alpha t})$$
$$\le e^{-\alpha t} E\left(\prod_{i=1}^{n} e^{\alpha Z_i}\right) \quad \text{Markov.}$$

→ Here, we used independence in the Chernoff Bound proof now we use the Martingale property,

$$= e^{-\alpha t} E\left(E\left(\prod_{i=1}^{n} e^{\alpha Z_i} \mid Y_0, \dots Y_{n-1}\right)\right)$$

← Tower Property

$$= e^{-\alpha t} E\left(\prod_{i=1}^{n-1} e^{\alpha Z_i}\, E\left(e^{\alpha Z_n} \mid Y_0, \dots, Y_{n-1}\right)\right).$$

We use the following inequalities:

- $e^{x} \le \frac{\sinh(c)}{c} x + \cosh(c) \quad \forall x \in [-c, c],\ c > 0$
- $\cosh(x) \le e^{x^2/2} \quad \forall x > 0$

Now.

← $E(Z_n \mid Y_0, \dots Y_{n-1}) = 0$.

$$Pr(X_n \ge t) \le e^{-\alpha t} E\left(\prod_{i=1}^{n-1} e^{\alpha Z_i}\, E\left(\frac{\sinh(\alpha c_n)}{\alpha c_n} \cdot \alpha Z_n + \cosh(\alpha c_n) \mid Y_0, \dots Y_{n-1}\right)\right)$$
$$\le e^{-\alpha t} E\left(\prod_{i=1}^{n-1} e^{\alpha Z_i}\right) \cdot e^{\alpha^2 c_n^2 / 2}$$
$$\le e^{-\alpha t + \frac{\alpha^2}{2} \sum_{i=1}^{n} c_i^2} \quad \text{(Induction)}$$

Proof (cont)

And putting $x = \frac{t}{\sum_{k=1}^{n} c_k^2}$ above gives $\Pr(X_n \geq t) \leq \exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right)$

The same argument works for $\Pr(X_n \leq -t)$ by considering $(-X_t)_{t\geq 0}$. This gives Azuma's inequality as desired. ⇗

Exercise: Deduce Chernoff Bound type result using Azuma's Inequality.

Theorem: Let $(X_t)_{t\geq 0}$ be a ~~Martingale~~ Super-Martingale w.r.t $(Y_i)_{i\geq 0}$ s.t. $|X_t - X_{t-1}| \leq c_t \ \forall t \geq 1$.

Then,

$$\Pr(X_n - X_0 \geq t) \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right)$$

(and we get the other tail with submartingale)

Proof: Same as above.

A graph function f is said to satisfy the edge Lipschitz condition with constant c if $|f(H) - f(H')| \leq c$ whenever H and H' differ by only one edge.

We say f satisfies the vertex Lipschitz condition with constant c if $|f(H) - f(H')| \leq c$ whenever $E(H) \Delta E(H')$ are incident with only one vertex.

Theorem:
If f satisfies the edge (vertex) Lipschitz condition with constant c, then the corresponding edge (vertex) exposure Martingale satisfies $|X_{t+1} - X_t| \le c \quad \forall t$.

Proof:

$X_{t+1} = E(f \mid Y_1, \ldots, Y_{t+1})$

Let $\vec{y}(\omega) = (Y_1, \ldots, Y_t)(\omega)$ (ω is a specific graph)

$X_{t+1}(\omega) = E(f \mid (Y_1, \ldots, Y_{t+1}) = \underbrace{(Y_1(\omega), \ldots, Y_t(\omega)}_{=\vec{y}(\omega)}, Y_{t+1}(\omega)))$

$X_t(\omega) = E(f \mid (Y_1, \ldots, Y_t) = \underbrace{(y_1(\omega), \ldots, y_t(\omega))}_{=\vec{y}(\omega)})$

$$X_{t+1}(\omega) - X_t(\omega) = \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) \Pr((Y_{t+2}, \ldots, Y_{\binom{n}{2}}) = \vec{z}) .$$

$$- \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} \sum_{y \in \{0,1\}} f(\vec{y}(\omega), y, \vec{z}) \Pr(\vec{z}) \Pr(Y_{t+1} = y) .$$

$$= \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} P(\vec{z}) \left(f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \{0,1\}} P(y) f(\vec{y}(\omega), y, \vec{z}) \right)$$

$$|X_{t+1}(\omega) - X_t(\omega)| = \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} P(\vec{z}) \left| \underbrace{f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \{0,1\}} f(\vec{y}(\omega), y, \vec{z}) P(y)}_{\le c} \right|$$

$\le c$ for every ω.

$\square$

Theorem (Shamir & Spencer, 1987)

$\Pr(|\chi(G(n,p)) - \mathbb{E}(\chi(G(n,p)))| \geq t) \leq 2e^{-t^2/2n}$.

$\Rightarrow \forall \varepsilon > 0, \exists c > 0$

$\Pr(|\chi(G(n,p)) - \mathbb{E}(\chi(G(n,p)))| \leq c\sqrt{n}) \geq 1-\varepsilon$

Proof:

Consider the vertex exposure martingale w.r.t f being the chromatic #. Obviously, f is 1-Lipschitz since modifying a set of edges incident with only one vertex can change f by ≤ 1. So,

$\Pr(|\chi(G(n,p)) - \mathbb{E}\chi(G(n,p))| \geq t)$

$= \Pr(|X_n - X_0| \geq t) \leq e^{-t^2/2n}$.

by Azuma's Inequality. ☒

Theorem (Bollobás, 1988)

A.a.s. $\chi(G(n,1/2)) \sim \dfrac{n}{2\log_2 n}$.

Note: Since we are in $G(n,1/2)$, cliques and ind. sets are the same since G and $\bar{G}$ both in $G(n,1/2)$.

Proof:

We first establish a lower bound. Let $\omega(G)$ denote the size of a maximum clique (or ind set) of G. Obviously $\chi(G) \geq \omega(G)$.

Consider $G(n,1/2)$.

The expected # of k-cliques in $G(n,1/2)$ is $f(k) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$.

Let $k_0 = \min\{k \geq 2 : f(k) \leq 1\}$ (This is well defined since $f(k)$ is decreasing on $k \geq 2$).

(Exercise: $k_0 \sim 2\log_2 n$, $f(k_0+1) = o(1)$ and $f(k_0-4) = \omega(n^3)$).

Proof: (cont)

C_n: Conjugate #.

Hence, a.a.s $\omega(C_n) \le k_0+1$ (And since $\alpha(G_n) = \omega(\bar{C_n})$ and $\bar{G_n} \sim \mathcal{G}(n, 1/2)$, we know a.a.s $\alpha(G) \le k_0+1$).

As $\chi(G) \cdot \alpha(C_n) \ge n$.

$\Rightarrow$ a.a.s $\chi(\mathcal{G}(n, 1/2)) \ge (1+o(1))\frac{n}{2\log_2 n}$.

Now, we establish an upper bound:
Set $k = k_0 - 4$. We know $f(k) = \omega(n^3)$ and we want to prove that a.a.s $\omega(\mathcal{G}(n,p)) \ge k$. We want a concentration on the # of k-cliques.

(Bollobás ~~did~~ gave the following proof:)
Consider a random variable Y that denotes the maximal size of a family of edge disjoint cliques of size k.
We will prove that $\mathbb{E}Y$ is large.

Key Lemma: $\mathbb{E}Y \ge (1+o(1))(n^2/2k^4)$. $\leftarrow$ Will show next class.

Let $Y_0, Y_1, \ldots, Y_{\binom{n}{2}}$ be the edge exposure martingale on $\mathcal{G}(n, 1/2)$ with $Y_0 = \mathbb{E}Y$.
It is easy to see that $|Y_m - Y_{m-1}| \le 1$ because every edge is contained in at most 1 k-clique in the maximal family of edge disjoint k-cliques
By Azuma's Inequality

Proof (cont.)

$$\Pr(|Y_{\binom{n}{2}} - \mathbb{E}Y| \geq t) \leq 2\exp\left(-\frac{t^2}{2\binom{n}{2}}\right)$$

Choosing $t = \frac{n^2}{\log^4 n}$ yields

$$\Pr(\omega(G(n, \tfrac{1}{2})) < k) = \Pr(Y = 0)$$

$$\leq \Pr(|Y_{\binom{n}{2}} - \mathbb{E}Y| \geq t)$$

$$\leq 2\exp\left(-\Omega\left(\frac{n^2}{\log^8 n}\right)\right).$$

Use key lemma

Set $m = \lfloor n/\log^2 n \rfloor$. For every m-set $S \subseteq [n]$, $G[S]$ is distributed as $G(m, \tfrac{1}{2})$. Set $k_0 = k_0(m)$ and let $k = k_0 - 4$. Then,

$k \sim 2\log_2 m \sim 2\log_2 n$

Same argument as above applied to $G(m, \tfrac{1}{2})$ shows

$$\Pr(\alpha(G[S]) < k) = \exp\left(-\Omega\left(\frac{m^2}{\log^8 m}\right)\right)$$

$$= \exp\left(-\Omega\left(\frac{n^2}{\log^{12} n}\right)\right).$$