CO450 - Combinatorial Optimization

Contents

1 Introduction

1.1 Overview of the Course

Combinatorial optimization leverages tools from: combinatorics, linear programming theory and algorithms to *efficiently* solve optimization problems on discrete structures (e.g. graphs)

The course will covering the following topics:

- Spanning frees
- Max flow, Min cut
- Matroids and matroid optimization
- Matchings and related problems
- Approximation algorithms

1.2 Review of LP theory

A linear program (LP) is an optimization problem of the form:

$$\begin{aligned} \max \quad & c^\intercal x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $A \in M_{m\times n}(\mathbb{R})$, and the objective function and constraints are linear. We must also require that:

- There are a finite number of variables and constraints
- The inequalities are non-strict

Any LP has 3 possible outcomes:

1. The LP is infeasible
2. The LP is unbounded, i.e. We can achieve feasible solutions of arbitrarily "good" objective value. (For Equation (1.1), this means that $\forall v \in \mathbb{R}$ there exists a feasible solution x s.t $c^\intercal x > v$)
3. The LP has an optimal solution. (For Equation (1.1), this means there is a feasible solution x^* such that $c^\intercal x^* \geq c^\intercal x$ $\forall$ feasible solutions x)

Theorem 1.1: Fundamental Theorem of Linear Programming

There are only these 3 possible outcomes

Theorem 1.2

LPs can be solved efficiently

1.2.1 Duality

Question: How do we prove bounds on the optimal value OR justify that a solution is optimal?

Idea: We can prove bounds by taking a suitable linear combination of constraints of the LP.

For example, we can multiply the constraints of Equation (1.1) by some vector $y \geq 0$. This gives:

$$y^\intercal Ax \leq y^\intercal b \quad (y \geq 0)$$

Notice that if we require that $c^\intercal \leq y^\intercal A$, then since $x \geq 0$, we get the following chain of inequalities:

$$c^\intercal x \leq y^\intercal Ax \leq y^\intercal b$$

And this is precisely the dual LP.

$$\begin{aligned} \min \quad & b^\intercal y \\ \text{s.t.} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned} \tag{1.2}$$

We call the original LP the primal:

- Every constraint of the primal is a variable in the dual
- Every variable of the dual is a constraint in the dual

Remark 1.1. The dual of a dual gives us back the primal

1.2.2 Duality Theorems

We'll use (P) and (D) to denote the primal and dual, respectively.

Weak Duality: If x is feasible for (P) and y is feasible for (D), then $c^\intercal x \leq b^\intercal y$

Note. We can already infer things from this. For example, if (D) is unbounded, then $c^\intercal x$ is unable to obtain any solution. So, it must be infeasible

Strong Duality: If (P) has an optimal solution, then so does (D).

Suppose x^* is a feasible solution for (P) and y^* is a feasible solution for (D).

x^*, y^* are optimal for (P), (D) respectively if and only if $c^\intercal x = b^\intercal y$ if and only if

- $x_j^* \neq 0 \Rightarrow$ corresponding dual constraint is tight at y^* (i.e. $(A^\intercal y^*)_j = c_j$)
- $y_i^* \neq 0 \Rightarrow$ corresponding primal constraint is tight at y^* (i.e. $(Ax^*)_i = b_i$)

These two conditions are known as the complementary slackness (CS) conditions.

1.3 Geometry of LPs

A feasible region of an LP is called a polyhedron, i.e. $P \subseteq \mathbb{R}^n$ is a polyhedron if it can be written as $\{x \in \mathbb{R}^n \,:\, Ax \leq b\}$

A polyhedron is a convex set. (A convex set is a a set $S \subseteq \mathbb{R}^n$ where $\forall x, y \in S, \forall \lambda \in [0,1], \lambda x + (1-\lambda)y \in S$)

We say that $x \in \mathbb{R}^n$ is a convex combination of points $p^{(1)}, \ldots, p^{(k)} \in \mathbb{R}^k$ if $\exists \lambda_1, \ldots, \lambda_k \geq 0, \sum_{i=1}^{k} = 1$, such that:

$$x = \sum_{i=1}^{k} \lambda_i p^{(i)}$$

An extreme point of a convex set $S \subseteq \mathbb{R}^n$ is a point $\hat{x}$ such that $\hat{x}$ cannot be written as a convex combination of 2 distinct points of S.

A polyhedron has a finite number (possibly zero) of extreme points.

$\hat{x}$ is an extreme point of a polyhedron $P \subseteq \mathbb{R}^n$ if and only if $\exists c \in \mathbb{R}^n$ such that $\hat{x}$ is a unique optimal solution to the LP: $\{\max c^\intercal x \text{ s.t. } x \in P\}$

Theorem 1.3

Consider the (LP): $\{\max c^\intercal x \text{ s.t. } x \in P\}$ where $P \subseteq \mathbb{R}^n$ is a polyhedron. If (LP) has an optimal solution and P has extreme points, then there is always an optimal solution that is an extreme point of P.

Definition 1.1: Convex Hull

Let $S \subseteq \mathbb{R}^n$, the convex hull of S, denoted $conv(S)$ is the smallest convex set containing S. Equivalently:

$$conv(S) := \{x \in \mathbb{R}^n \,:\, x \text{ is a convex combination of a } \underline{\text{finite}} \text{ number of points of } S\}$$

Definition 1.2: Polytope

A polytope is a bounded polyhedron, i.e. $\exists \gamma \in \mathbb{R}$ such that $\forall x \in$ polytope, $|x_i| \leq \gamma \, \forall$ coordinates i.

Remark 1.2.

1. A polytope is the convex hull of its extreme points
2. $P \subseteq \mathbb{R}^n$ is a polytope if and only if $P = conv(S)$ for a finite set $S \subset \mathbb{R}^n$