Theorem (Bollobas - Thomason, '96)
If G is $22k$-connected, then G is k-linked.

Lemma (Thomason, '84)
Let $0<\beta<1$ be the root of $1=\beta+(1+\ln(\frac{2}{\beta}))$ and let $k\geq 3$ be an integer. Let G be a graph with $e(G)>kv(G)$.
Then G contains a minor s.t.
(1) $v(H)\leq k+2$, and
(2) $2\delta(H)\geq v(H)+\lfloor\beta k\rfloor-1$.

Idea: If we can rearrange this minor so that each terminal is in a distinct "bag", then we can use "many common neighbors in H" property to do the linking

Key Technical Lemma:
Let $d\geq 0$, $k\geq 0$, and $l\geq d+\lfloor\frac{3k}{2}\rfloor$ be integers. Let G be a graph containing vertex-disjoint non-empty connected subgraphs $C_1, C_2, \ldots, C_l$ such that each C_i is adjacent to all but at most d other C_j's. Suppose $S=\{s_1,\ldots,s_k\}$ is a set of k vertices such that there is no S-cut of order $<k$ which avoids $d+l$ of the subgraphs $C_1,\ldots,C_l$.
Then, G contains vertex-disjoint non-empty connected subgraphs $D_1,\ldots,D_m$, where $m=l-\lfloor k/2\rfloor$ such that $\forall i\in[k]$, the subgraph D_i contains S_i and is adjacent to all but at most d of the subgraphs $D_{k+1},\ldots,D_m$.

Note: In fact, for every $i \in [m]$, D_i is adjacent to all but at most d of $D_{i+1}, \ldots, D_m$.

Proof:

In fact, we'll prove the lemma under a weaker assumption on the C_i's (and hence a stronger version of the lemma)

Each C_i is at least one of the following:

- Connected (as before), or
- Not connected and each of its components contains a vertex in S

And is adjacent to all but at most d of the other C_j's ($j \neq i$) that do not contain a vertex in S.

Let $G, C_1, \ldots, C_\ell, S$ be a minimum counterexample.

Obs: $\nexists$ an isolated vertex v of G.

Proof: If $v \notin S$, then $G-v$ is a smaller counterexample b/c if $v \in C_i$, then $C_i = \{v\}$ but then C_i is non-adjacent to too many other C_i's. So say $v \notin$ any C_i, but then deleting it does not affect the C_i's or connectivity.

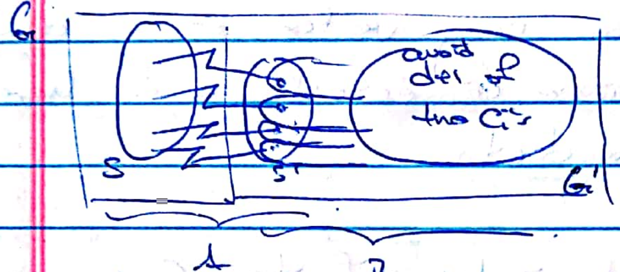
If $v \in S$, then $S-v$ is an S-set of order $\leq k-1$ which avoids at least $\ell - k \geq d + \lfloor k/2 \rfloor \geq d+1$, a contradiction (This is the S-cut assumption) $\square$ (Obs)

Proof continued.

Proof (cont)

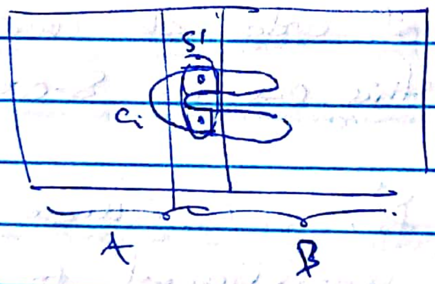
Claim: The only S-cut of order k avoiding $\geq d+1$ of the C_i's is S itself

Proof: Suppose not, that is, suppose $\exists S' \subseteq V(G)$, $|S'| = k$, and $S' = A \cap B$ for an S-cut (A, B), where $S \subseteq A$ and A avoids $\geq d+1$ of the C_i's.



Let $G' = G[B]$ $\forall i \in [l]$ and let $C_i' = C_i \cap G'$

Remark: If a component of C_i "splits" into multiple components in G', then each such component in G' contains a vertex of S'



Obviously, C_i' are vertex-disjoint since the C_i are, and further, they are non-empty since each C_i has to contain a vertex in B as it is adjacent to at least 1 of the $d+1$ subgraphs avoided by A, by assumption.

Proof: (Cont'd)

Subclaim: C_i are non-empty, vertex-disjoint and satisfy the weaker assumptions

Proof.

We've established non-empty, vertex-disjoint. Now, suppose C_i is not connected, and have a component J not containing a vertex of S'. But, then J is a component of C_i and does not contain any vertex of S' and hence any vertex of A (since connected) and hence any vertex of S, a contradiction.

So, WLOG, $\exists$ C_i non-adj to $\geq d+1$ of the C_j's not containing a vertex of S', but those C_j's must be equal to C_j and hence C_i is not adjacent to them, a contradiction.

$\square$ (Subclaim)

Further, $\nexists$ an S'-cut in G' of order $<k$ avoiding $\geq d+1$ C_i's b/c also we can use this as our S-cut.

Now, since $S' \neq S$, $G' \subsetneq G$, $\exists$ $D_1', \ldots, D_m'$ where $\forall i \in [k]$, $S_i \in D_i'$ and $\forall i \in [m]$ D_i' is non-adjacent to at most d of $D_{k+1}', \ldots, D_m'$. Let $D_i = D_i'$ if $i \in [k+1, m]$. Since A avoided $d+1$ of the C_i's then $\nexists$ a cut in $G[A]$ of order $<k$, separating S from S'. By Menger's theorem, $\exists$ k vertex-disjoint paths $P_1, \ldots, P_k$ connecting S to S' in $G[A]$. Let $D_i = D_i' \cup P_i$ and now, after relabeling, D_i is as desired.

$\square$ (Claim)

Proof (Con't)

~~Claim~~

Claim: $\forall e = uv \in E(G)$, $\exists i \neq j$ s.t. $u \in C_i$, $v \in C_j$

Proof:

Case 1: $v \notin S$, then G/uv gives rise to new C_i that satisfy ~~...~~ weaker assumptions and connectivity, b/c of Claim (i.e. we would be able to find another S-cut of order $<k$)

Case 2: $u, v \in S$, let $G' = G - uv$. Connectivity is preserved b/c uv doesn't affect S-cut. Non-neighbors okay b/c not in distinct C_is. and component D of C_i containing u, v splits into ≤ 2 components, each containing a vertex of S. $\square$ (Claim)

Corollary: $V(G) = \bigcup V(C_i)$ ← Haha. what?

~~Pro~~

Cor: Every C_i is an ind. set. (and hence $|C_i| = 1$ or $C_i \subseteq S$)

Remark: We'd be done by letting $C_i = D_i$ if every $|C_i| = 1$.

Also, if $|C_i| \geq 2$ let $\mathcal{C} = \{v \in S,\ C_i$ containing v has ≥ 2 vertices$\}$.

Claim: Exist matching of from $\mathcal{C}$ to $V(G) - S$ in G

Proof: By Hall's theorem, $\exists X \subseteq \mathcal{C}$ such that $Y = N(X) \cap (V(G) - S)$ has $|Y| < |X|$

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Proof (Con't)

Proof (Claim - con't)

But then $S' = S - X \cup Y$ has $|S'| < |S| = k$ and is an S-cut avoiding $|V(G) - S - S'| = l - k - k/2 = l - 3k/2 \geq d+1$ C_i's.

$\square$ (Claim)

And that completes the proof $\square$.

Defn A graph G is (k,n)-knit if $1 \leq n \leq k \leq |V(G)|$ and $\forall S \subseteq V(G)$, $|S| = k$ and partition S into $S_1, \ldots, S_t$ where $t \geq n$ non-empty parts, then G contains vertex-disjoint, connected subgraphs $F_1, \ldots, F_t$ s.t. $S_i \subseteq V(F_i)$ $\forall i \in [t]$.

Obs: If G is $(2k, k)$-knit, then G is k-linked.