749-Colouring

Finishing the iteration calculations for Kim's

$|L'| = |L| \cdot \text{keep}$, where $\text{keep} \approx e^{-\frac{p\Delta}{|L|}}$

$\Delta' \approx \Delta \cdot \text{keep}(1 - p\,\text{keep})$

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} \cdot \frac{\text{keep}(1-p\,\text{keep})}{\text{keep}}$$

$k = p\frac{\Delta}{|L|}$

$$= \frac{\Delta}{|L|} - \frac{\Delta p}{|L|}\text{keep} = \frac{\Delta}{|L|} - ke^{-k}$$

How many iterations can we do (until we run out of colours?)

Let $|L_\tau|$ be the size of L after τ iterations:

$$|L_\tau| = |L_0| \cdot \text{keep}^\tau$$
$$\approx |L_0| \cdot e^{-k\tau}$$

We need $|L_\tau| \geq 1$. $\Rightarrow \ln|L_0| - k\tau \geq \ln 1 = 0$

$$\Rightarrow \tau \leq \frac{\ln|L_0|}{k}$$

Remark.

We actually stop when $|L_\tau| \geq$ some fixed constant to ensure that the necessary inequalities hold for the LLL in every iterative step. This only means that $\tau \leq \frac{\ln|L_0|}{k} - c$ for some constant c (so this is fine)

To finish the colouring, we need $\frac{\Delta_\tau}{|L_\tau|} \leq \frac{1}{2e}$ to apply Aharoni/Haxell.

Note: $\frac{\Delta_0}{|L_0|} \approx \frac{\Delta_\tau}{|L_\tau|} + ke^{-k}\tau$. So this works if

$$\frac{\Delta_0}{|L_0|} \lesssim \frac{1}{2e} + ke^{-k}\left(\frac{\ln|L_0|}{k} - c\right)$$

$$= \text{constant} + e^{-k}\ln(L_0)$$

$\Leftrightarrow$

i.e. $\Delta_0 \le e^{-k} |L_0| \ln |L_0|$

$$\Rightarrow |L_0| \approx \frac{e^k \Delta_0}{\ln(e^k \Delta_0)} \le \frac{e^k \Delta_0}{\ln \Delta_0}$$

So, if $k=1$, we get $|L_0| \approx e \frac{\Delta_0}{\ln \Delta_0}$, $\rightarrow$ Recall that $k = p\frac{\Delta}{|L|}$, so when $k=1$, $p = \frac{|L|}{\Delta}$ and $k \to 0$ means $p \to 0$.

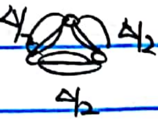
and if instead, we let $k \to 0$, we get $(1+o(1)) \frac{\Delta_0}{\ln \Delta_0}$, as desired $\square$

Remark: Why was colour degree ~~necessary~~ for the proof?
Note that in $\delta' \approx \Delta\, \text{keep}(1 - p\,\text{keep})$, the first keep only comes up in colour degree. And this keep helps cancel out some values and drive concentrations down.

Edge-Colouring:
Interested in properly colouring edges of graphs, i.e. ~~$\chi(G)$~~ $\chi(L(G))$ (the line graph)

Greedy Colouring: $\Delta(L(G)) \le 2\Delta(G) - 2$
$$\Rightarrow \chi(L(G)) \le 2\Delta(G) - 1$$
(of course $\omega(L(G)) = \Delta(G)$, if $\Delta(G) \ge 3$)

Theorem (Vizing, indep. Gupta, 1960s)
$$\chi(L(G)) \le \Delta(G) + 1.$$
NP-hard to decide if $\chi(L(G)) = \Delta$ or $\Delta+1$

Also, if G is a multigraph
$$\chi(L(G)) \le \Delta + \mu \quad (\text{where } \mu \text{ is the max. multiplicity of an edge})$$
$$\le \left\lceil \tfrac{3}{2}\Delta \right\rceil \quad (\text{Shannon}) \rightarrow \text{tight for}$$


Goldberg-Seymour Conjecture: (Independently in late 70's)

$\chi(L(G)) \le \max\{\Delta(G)+1, \chi_f(L(G))\}$

↳ fractional chromatic #.

Actually, via Edmond's matching polytope:

$$\chi_f(L(G)) = \max_{\substack{H \subseteq G \\ v(H) \text{ odd}}} \frac{e(H)}{\left(\frac{v(H)-1}{2}\right)}$$ → i.e. Defn of Hall ratio for line graphs.

Known Results:

- Kahn (90's) proved $\chi(L(G)) \le \max\{\Delta+1, (1+o(1))\chi_f(L(G))\}$.
- Two groups in 2008: If $\chi_f(L(G)) \ge \Delta + \sqrt{\Delta}$, then $\chi(L(G)) = \chi_f(L(G))$.
- Another group in 2017: —— " —— $\ge \Delta + \sqrt[3]{\Delta}$, —— " ——.
- Plantholt, '11: $\chi(L(G)) \le \chi_f(L(G)) + \log \chi_f(L(G))$

List-Colouring Conjecture:

If G is a simple graph, then

$$\chi_\ell(L(G)) = \chi(L(G)).$$

(i.e. There is no such thing as list-edge-colourings)

Two big results on LCC from 90's:

Theorem (Galvin, 1995)

If G is bipartite, then $\chi_\ell(L(G)) = \chi(L(G))$.

→ Proved Dinitz's conjecture: If for every square in an $n \times n$ matrix, I give you a list of integers, can you complete the matrix so that all entries in a row (or col) are distinct

(Equivalently, $\chi_\ell(K_{n,n}) = n$)

→ Related to Latin Squares, since the completion of a Latin Square is simply an n-edge-colouring of $K_{n,n}$.

And these exist by König's theorem: if G is bip, $\chi_\ell(L(G)) = \Delta(G)$.

Theorem (Kahn, 1996)

If G is simple, $\chi_\ell(L(G)) = (1+o(1))\chi(L(G))$

i.e. $= (1+o(1))\Delta(G)$.

Molloy + Reid (Early 2000s) improved error to $\Delta + 4\sqrt{\Delta}\log^4\Delta$.

Kahn extended this to k-uniform linear hypergraphs

↳ (Hyperedges intersect in ≤ 1 vertex)

Proof Sketch (of Kahn's)

We'll use the Naive Colouring Procedure (i.e. Only remove colours from neighbours if retained by a vertex) and Nibble.

Also, since $|L|$ is on the order of Δ, we won't need activation probabilities (i.e. Set $p=1$)

We in fact prove a colour-degree version of Kahn's theorem as follows:

If L is a list assignment for $E(G)$ (equiv. $V(L(G))$), we define for $v \in V(G)$, a colour $c : \wedge(ELC(v)) \to L(v) := \bigcup_{e \sim v} L(e)$ for $v \in V(G)$

$d_L(v,c) := |\underbrace{\{e \in E(G) : e \sim v,}_{N_{L,G}(v,c)} c \in L(e)\}|$

$d_L(v) := \max_c d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

→

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So, restating the thm:

Stronger Kahn's Theorem (Kahn, 1996)

If L is a list assignment for $E(G)$ s.t.

$$|L| \geq (1 + o(1))\Delta_L(G)$$

then G has an (edge) L-coloring.

~~(Note: For notation, let $L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$).~~

Clearly, still interested in $|L|/\Delta_L(G)$

Expectations:

(Note: Assume all color degrees and list sizes are regular via equalizing coin flips (since regularization of line graphs seems odd)).

i.e. It was deleted since no inc. edge received $\phi(e)$

Retain := Pr[an edge e is not in G']

$$= \left(1 - \frac{1}{|L|}\right)^{d(e)} = \left(1 - \frac{1}{|L|}\right)^{2\Delta - 2} \approx e^{-2\frac{\Delta}{|L|}} \approx e^{-2}$$

Vertex-keep := Pr[$c \in L(v)$ is not retained by any edge around vertex v]

$$\approx 1 - \frac{1}{e^2}$$

Edge-keep := Pr[$c \in L(e^{\text{"}uv\text{"}})$ retained]

$\approx$ Pr[c not retained around u] $\times$ Pr[c —— " —— v]

b/c mostly independent $= (\text{Vertex-keep})^2$

Then,

$$\mathbb{E}[|L'(e)|] \approx |L|\left(1-\frac{1}{e^2}\right)^2.$$

Colour-Degree:

$$\mathbb{E}[|N^{edge}_{LG}(v,c) \cap E(G')|] = |N^{edge}_{LG}(v,c)| \cdot (1-\text{Retain})$$
$$= \Delta \cdot \left(1-\frac{1}{e^2}\right)$$

But, how many of these e keep $c \in L'(c)$?
It's not Edge-keep, ~~since~~ if we assume $c \in L'(v)$, b/c then by assumption ~~we~~ no edge around v retains c

Under this:
$Pr[e \text{ keeps } c] = Pr[\text{no edge} \longrightarrow] X.$