1 Graph Theory Primer

Let $G = (V, E)$ be a graph, where V is the vertex set and E is the edge set.

Definition 1.1

The **degree** of a vertex $v \in V$ (denoted $\deg(v)$) is the number of edges with one end in v. (i.e. The size of the set $\{va \mid a \in V,\ a \neq v\}$).

A **walk** is a sequence of vertices $v_1 v_2 \ldots v_k$ where $v_i v_{i+1}$ is an edge. A **path** is a walk where all vertices are distinct. A **cycle** is a walk where $v_1 = v_k$ and $v_1, \ldots v_{k-1}$ are distinct.

Finally, we say a graph is **connected** if there exists a path between any two vertices in G.

Definition 1.2

For $S \subset V$, the **cut** induced by S is the set of all edges with one end in S and one end not in S, denoted $\delta(S) = \{uv \in E \mid u \in S,\ v \notin S\}$. Given two vertices $s, t \in V$ with $s \in S, t \notin S$, we call $\delta(S)$ an s,t**-cut**

An s,t**-path** is a path with starting vertex s and ends on t.

Theorem 1.1

There exists an s,t-path if and only if every s,t-cut is nonempty

Definition 1.3

A **tree** is a connected graph with no cycles. A **spanning tree** is a subgraph that is a tree and has vertex set V

Note the following:

- A tree on n vertices contains $n - 1$ edges.
- If T is a tree, then adding an edge $uv \notin T$ creates exactly one cycle C. Moreover, if xy is an edge in C, then $T + uv - xy$

Let $D = (N, A)$ be a directed graph. N is a set of nodes and A is a set of ordered pairs of nodes (called arcs).

Definition 1.4

For an arc (u, v), we call u the **tail** and v the **head**.

The **out-degree** of node u (denoted $d(u)$ or $d^{\text{out}}(u)$) is the number of arcs with tail u. The **in-degree** of node u (denoted $d(\overline{u})$ or $d^{\text{in}}(u)$) is the number of arcs with head u.

A **diwalk** is a sequence of nodes $v_1 v_2 \ldots v_k$ where (v_i, v_{i+1}) is an arc. **Dipaths** and **Dicycle** are defined analgous to simple graphs but with arcs instead of edges.

For $S \subset N$, the **cut** induced by S is denoted $\delta(S) = \{(u, v) \in A \mid u \in S, v \notin S\}$. (sometimes written as $\delta^{\text{out}}(S)$) This is the set of arcs with tail in S. We denote the complement of S by $\overline{S}$, and define $\delta(\overline{S}) = \{(u, v) \in A \mid u \notin S, v \in S\}$ (sometimes written as $\delta^{\text{in}}(S)$)) to be the set of arcs with head in S. Finally, if $s \in S, t \notin S$, then $\delta(S)$ is an s, t-cut.

Theorem 1.2

There is an s, t-dipath if and only if every s, t cut is non-empty

Proof. ($\Rightarrow$) Suppose there exists an empty s, t-cut $\delta(S)$. This partitions the graph into two sets of nodes S and $N \setminus S$, with $s \in S$ and $t \in N \setminus S$ and no outgoing edges from S to $N \setminus S$. As such, an s, t-dipath cannot exist.

($\Leftarrow$) Suppose every s, t-cut is non-empty and let S be the set of nodes $v \in A$ where a s, v-dipath exists. If $t \in S$, then we're done, so suppose $t \notin S$. Then, $\delta(S)$ is an s, t-cut. By assumption, $\delta(S)$ is non-empty and so there is an arc $(x, y) \in \delta(S)$ with $x \in S$ and $y \in N \setminus S$. Since $x \in S$, an s, x-dipath P exists and since $y \notin S$, there does not exist an s, y-dipath, but $P + (x, y)$ is one. This is a contradiction $\square$

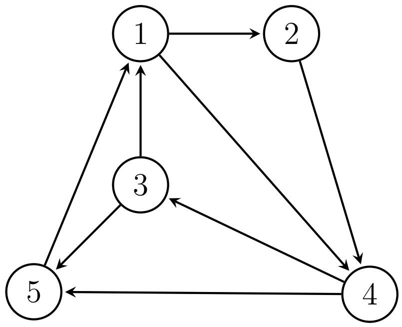
Definition 1.5

The **node-arc incidence matrix** M of a digraph $D = (N, A)$ is a matrix of $|N|$ rows and $|A|$ columns, such that:

- The rows correspond to the nodes of D,
- The columns correspond to the arcs of D,
- And the entry for node u and arc (i, j), denoted $m_{u,ij}$ is given by:

$$m_{u,ij} = \begin{cases} 0 & \text{if } u \neq i \text{ and } u \neq j, \\ +1 & \text{if } u = j \text{ and,} \\ -1 & \text{if } u = i \end{cases}$$

Example 1.1
An example of a digraph $D = (N, A)$ (on the left) and its corresponding node-arc incidence matrix on the right.



$$\begin{array}{cc} & \begin{array}{cccccccc} 12 & 14 & 24 & 31 & 35 & 43 & 45 & 51 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$