The Probabalistic Method:

3 Pillars of prob. method:

- Linearity of Expectation (and basic probability)
- Lovász Local Lemma (introduced in 1975)
- Concentration Inequalities - (Markov, Chebyshev, Chernoff, Talagrand's) 1990s

Linearity of Expectation:

Expectation: let X be a discrete $\mathbb{R}$ r.v., then: $\mathbb{E}[X] = \sum_{i \in \mathbb{R}} \Pr[X=i] \cdot i$

Linearity of Expectation: If $X = \sum_{i \in [n]} X_i$ (n finite), then $\mathbb{E}[X] = \sum_{i \in [n]} \mathbb{E}[X_i]$

Independent variables: 2 events in a probability space ~~random variables~~ A and B are independent if $\Pr[A \mid B] = \Pr[A]$ or $\Pr[A \cap B] = \Pr[A]\Pr[B]$.

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of events ~~events~~ in a probability space. Then:

- We say the events in $\mathcal{A}$ are pairwise independent if $\forall i \neq j$ in $\mathcal{A}$ A_i, A_j are independent.
- We say the events in $\mathcal{A}$ are mutually independent if $\forall i$, A_i is mutually independent of $\mathcal{A} \setminus \{A_i\}$, i.e. $\forall \mathcal{A}' \subseteq \mathcal{A} \setminus \{A_i\}$ $\Pr[A_i \mid \bigcap_{A_j \in \mathcal{A}'} A_j] = \Pr[A_i]$

Question: If $\mathcal{A}$ is pairwise independent then is it mutually independent?

A: No! Example: $n \geq 3$ coins and space ~~weight~~ of flipping where # of heads is even and A_i is coin i is Heads. and $\mathcal{A} = \{A_1, \ldots, A_n\}$.

Lovász Local Lemma: → often refered to as "bad events"

If $A = \{A_1, \dots, A_n\}$ is a set of events in a probability space, and for every i, there is $B_i \subseteq [n] \setminus \{i\}$ such that A_i is mutually independent of $A \setminus (\{A_i\} \cup \{A_j : j \in B_i\})$.*

AND $|B_i| \le d$, $\forall i \in [n]$ ← i.e. Bad events are not too dependent

AND $Pr(A_i) \le p$, $\forall i \in [n]$ i.e Bad events are unlikely

AND $ep(d+1) \le 1$ (also $4pd \le 1$ works too!)

Then,

$$Pr[\bigcap_{i \in [n]} \bar{A}_i] > 0.$$

(i.e. with positive probability, none of the bad events occur)

*Wrong Version: If A_i is independent of at most d other events.

(Example: 1 is mutually independent of any $n-2$ sets of coins but not $n-1$ ⇒ No unique maximal set).

Remarks:

- Use the local lemma, say, if union bound fails (Recall that the union bound is: If $\sum_{i \in [n]} Pr[A_i] < 1$, then $Pr[\bigcap \bar{A}_i] > 0$)
- The "positive probability" is not very large, i.e. Usually $O(2^{-|A|})$, so could be exponentially small in the # of events.
- We use it to construct/force a good outcome.

Algorithmic Q: Can we find such a good outcome efficiently?

{Sampling is bad idea since this may take exponentially long}. (b/c of low probability)

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Moser-Tardos (2000): There exists an algorithm to find a good outcome of the LLL in "the variable model" that runs in time $O(|A|)$.

Every event depends on a bounded # of variables w/ bounded # of states and $B_i = \{A_j \in A \setminus \{A_i\} : A_j$ and A_i depend on at least one common variable$\}$.

Independent Trials

Moser-Tardos Algorithm:
- Sample all the independent trials in the probability space
- WHILE $\exists$ a bad event A_i:
  - Resample, at random, all the trials that A_i depends on
- RETURN good outcome

(Some) Applications of LLL to colourings:

Hypergraph Colouring:
Recall: Hypergraph $H=(V,E)$: V - Set of vertices, E - Set of hyperedges (i.e. ~~Sets~~ subsets of ~~ed~~ vertices)

H is k-uniform if $|e|=k$ $\forall e \in E(H)$.

(Note: $k=2$ gives us simple graphs)

Def'n A k-colouring of a hypergraph H is a partition of $V(H)$ into at most k independent sets of H.

A set $I \subseteq V(H)$ is independent if $\nexists e \in E(H)$ s.t. $e \subseteq I$

2

or equivalently $\exists\ \phi : V(H) \to [k]$ such that $\forall e \in E(H)$, $\exists v_1 \neq v_2 \sim e$ such that $\phi(v_1) \neq \phi(v_2)$ (i.e. No monochromatic edges)

The chromatic number $\chi(H)$ is the min k s.t. H is k-colorable

Q: How is $\chi(H)$ related to $\Delta(H) := \max_{v \in V(H)} d_H(v)$, (where $d_H(v) := \{e \in E(H) : e \sim v\}$). ?

Trivial Bound: $\chi(H) \leq \Delta(H) + 1$ (Greedy!)

Theorem: If H is a k-uniform hypergraph ($k \geq 2$), then
$$\chi(H) \leq \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$$

Proof:

We will use LLL.

Assign every vertex v of H a colour $\phi(v)$ from $[L]$, where $L = \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$.

Define bad event:

A_e = Edge e is monochromatic in ϕ.

$Pr(A_e) = \underbrace{1/L^{k-1}}_{(=p)}$ (because color of 1st vertex can be anything, but after that, $\forall v \sim e$ needs a different color).

Let $B_e = \{f \neq e \in E(H) : f \sim e\}$, the by variable model, it follows that A_e is mut. ind. of $\mathcal{A} \setminus (A_e \cup B_e)$

$|B_e| \leq \underbrace{k}_{\text{at most } k \text{ vertices}} \underbrace{(\Delta(H) - 1)}_{\text{max degree.}} (= d.) \leq k\Delta(H) - 1$

So: $ep(d+1) = e \cdot \frac{1}{L^{k-1}} k\Delta(H) = \frac{ek\Delta(H)}{((ek\Delta(H))^{\frac{1}{k-1}})^{k-1}} = 1.$

So, by LLL, $\exists$ a ϕ avoiding all A_e, i.e. a k-colouring of H ☒

Color Degree:

Def'n: Let G be a graph and L list assignment of G. Let $|L|$ denote the min. size of a list, i.e. $|L| := \min_{v \in V(G)} |L(v)|$

We define:

The color-degree of a vertex $v \in V(G)$ ~~is~~ in color $c \in L(v)$ as:

$$d_L(v,c) := |\{u \in N(v) : c \in L(u)\}|$$

The color-degree of v, denoted $d_L(v) = \max_{c \in L(v)} d_L(v,c)$

The maximum color-degree of G w.r.t. L, ~~denoted~~ is:

$$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$$

Question: Does $\exists$ function f s.t. if $|L| \geq f(\Delta_L(G))$, then G has an L-colouring? (analogous to $|L| \geq \Delta(G)+1$, then G has an L-colouring)

Theorem (Alon '88 (with constant $\geq .5$), '92)
If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring

Theorem (Haxell '00 - follows from a more general thm)
If $|L| \geq 2\Delta_L(G)$, then G has an L-colouring

Theorem (Reed-Sudakov, 2002)
If $|L| \geq (1+o(1))\Delta_L(G)$, then G has an L-colouring