749 - Random

Feb 2th

Recall:

① Want to prove if $pn - (\log n + \log\log n) \to \infty$, then a.a.s $\mathcal{G}(n,p) \in HAM$

② Posa Rotation:

$|N(End(v_0))| < 2|End(v_0)|$.

Lemma: Then there exists a constant $\varepsilon_0 > 0$.
Assume $pn - (\log n + \log\log n) \to \infty$ and $pn \le 2\log n$ s.t. a.a.s there is no $S \subseteq [n]$, $|S| \le \varepsilon_0 n$ and $|N(S)| < 2|S|$.

Proof: Later! (We'll need this for the upcoming proof)

Back to the proof of hamiltonicity of $\mathcal{G}(n,p)$:

Let $\varepsilon_0 > 0$ be the constant in the lemma.

Let Exp denote the property that $\forall S \subseteq [n]$ where $|S| \le \varepsilon_0 n$ $|N(S)| \ge 2|S|$

CNT denote the property that G is connected

Let $f = pn - (\log n + \log\log n)$

By our assumption, $f \to \infty$ as $n \to \infty$.

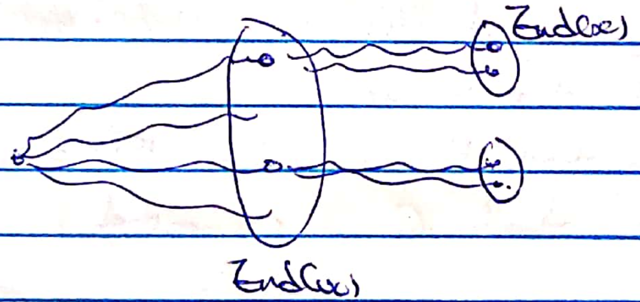
Let $p_1 = \dfrac{\log n + \log\log n + f/2}{n}$. Let $G' \sim \mathcal{G}(n,p_1)$. Let G be the random graph by adding $fn/8$ uniformly random edges to G'.

We can show (exercise!) that G and $\mathcal{G}(n,p)$ can be coupled so that a.a.s $G \subseteq \mathcal{G}(n,p)$. So, it is sufficient to show that a.a.s G is Hamiltonian

Proof (con't)
We know that a.a.s. $G' \in \mathrm{Exp} \cap \mathrm{CNT}$.
Take a longest path $P = v_0 v_1 \ldots v_\ell$ in G'. Let $\mathcal{P}$ be the set of paths obtained from Pósa rotations with v_0 fixed.
Let $\mathrm{End}(v_0)$ denote the other ends of paths in $\mathcal{P}$.

Since $G' \in \mathrm{Exp}$ and $|N(\mathrm{End}(v_0))| < 2|\mathrm{End}(v_0)|$. $\Rightarrow$ $|\mathrm{End}(v_0)| \geq \varepsilon_0 n$.

Next, for each $x \in \mathrm{End}(v_0)$ let $\mathcal{P}_x$ denote the set of paths obtained via Pósa rotations with x fixed and let $\mathrm{End}(x)$ be the set of the other ends of $p \in \mathcal{P}_x$.



Let $\xi = \{\{x,y\}, x \in \mathrm{End}(v_0), y \in \mathrm{End}(x)\}$. ξ is called the set of boosters.

As before, we have $|\mathrm{End}(x)| \geq \varepsilon_0 n$ for any $x \in \mathrm{End}(v_0)$.
It follows now that $|\xi| \geq \varepsilon_0^2 n^2$. Consider G. We will sprinkle the $\delta n/2$ edges in sequence. We call each sprinkled edge a "trial". A trial is successful if it is in ξ and fails otherwise. If a trial is successful, then since G' is connected, it will result in finding a path longer than P or finding a Hamiltonian ~~pa~~ cycle if P is a Hamiltonian path.

Proof (cont)

Each trial is successful with probability $\geq c\varepsilon_0^2$ (for some constant $c>0$). Only at most n successful trials are needed to guarantee a Hamiltonian cycle in G.

$$\Pr(G \notin HAM) \leq \Pr(\mathrm{Bin}(\tfrac{n}{\varepsilon_0}, c\varepsilon_0^2) \leq n) = e^{-\Omega(\frac{n}{\varepsilon_0})} = o(1)$$

$\square$

$\frac{1}{2}\log n \leq pn \leq 2\log n$.

Lemma: Assume ~~$pn \geq \ldots \log n$~~. Aas:

(a) Each $S \subseteq [n]$ with $|S| \leq \frac{n}{\log^2 n}$ induces at most $3|S|$ edges (Sublinear-sized subgraphs are sparse)

(b) No two vertices with degree <100 are within distance 5

(c) No vertex with degree <100 lies in a cycle of length ≤ 5

Proof: Exercise!

Back to 2-expander lemma:

Proof:

Let $n^{3/4} \leq s \leq \varepsilon_0 n$ where ε_0 is a sufficiently small constant

Let $E(s,w)$ be the expected number of pairs of disjoint sets (S,W) s.t. $|S|=s$, $|W|=w$ and $W=N(S)$

every vertex in w sees a vertex in S

$$E(s,w) = \binom{n}{s}\binom{n-s}{w}(1-(1-p)^s)^w (1-p)^{s(n-s-w)}$$

Choose S — Choose w — Don't see any vertex in S — Complement: See a vertex in S — S cannot see any other vertices

$\longrightarrow$ Since $1-(1-p)^s \le ps$

$$E(s,w) \le \left(\frac{en}{s}\right)^s \left(\frac{en}{w}\right)^w (ps)^w \exp(-ps(n-s-w))$$

$$= \left(\frac{en}{s}\right)^s \left(\frac{en}{w}\right)^w (ps)^w \left(\exp(-pn(1-\frac{s+w}{n}))\right)^s.$$

(Since $pn \ge \log n$, $ps \le p\varepsilon_0 n < \log n$, if $\varepsilon_0 < \frac{1}{2}$)

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w \exp(\underbrace{ps}_{<\log n}(s+w))$$

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w \exp(\underbrace{pn}_{\le 2\log n}.s.\frac{s+w}{n})$$

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w n^{2s(s+w)/n}$$

Then

$$\sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} E(s,w) \le \sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w n^{2s(s+w)/n}.$$

$$\le \sum_{s=n^{3/4}}^{\varepsilon_0 n} (\log n)^{2s} \left(\frac{e}{s}\right)^s \left(\frac{en}{2s}\right)^{2s} n^{2s.3s/n}.$$

$$= \sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\log^2 n \cdot \frac{e}{s} \cdot \frac{en^2}{4s^2} \cdot n^{6s/n}\right)^s$$

$$\le C\sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\frac{c\log^2 n}{s^3} \cdot n^{2+\frac{6s}{n}}\right)^s$$

$$\le C\sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\frac{c\log^2 n}{s^3} n^{2+6\varepsilon_0}\right)^s.$$

Choose ε_0 small so that $2+6\varepsilon_0 < 9/4$. Therefore,

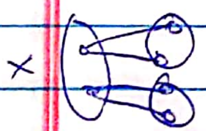
$$\sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} E(s,w) \le \sum_{s=n^{3/4}}^{\varepsilon_0 n} n^{-\alpha s} = o(1) \text{ for some } \alpha > 0$$

Now for $S < n^{3/4}$. Assume G is a graph satisfying properties (a)-(c) from the lemma and $G \in D2$ (min. degree of G is ≥ 2)

We know that a.a.s $G(n,p)$ has these properties. It is sufficient to show that any graph G satisfying these above properties satisfies
$\forall S \subseteq [n]$ where $|S| \leq n^{3/4}$, $|N(S)| \geq 2|S|$

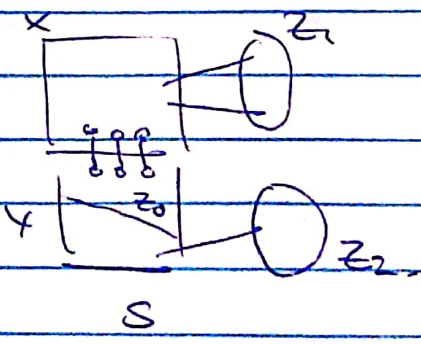
We call a vertex "light" if its degree is at most 100, otherwise call it "heavy". For any $S \subseteq [n]$, let $X \subseteq S$ be the set of light vertices in S and let $Y = S \setminus X$.

Case I: $Y = \phi$ (i.e. All vertices in S are light)
Then $|N(S)| = |N(X)| \geq 2|X| = 2|S|$. by $G \in D2$ and property (b).



{ All vertex disjoint

Case 2: $Y \neq \emptyset$.
Assume $|N(S)| < 2|S|$
$Z_0 = N(X) \cap Y$
$Z_1 = N(X) \setminus Y$
$Z_2 = N(Y) \setminus (X \cup Z_1)$



Proof (cont)

We have $|N(S)| = |Z_1| + |Z_2| + |Z_3| \overbrace{< 2|S|}^{\text{By assumption}} = 2(|X| + |Y|)$

Since $G \in D2$ and by properties (b) and (c), we will have.

$|Z_0| + |Z_1| \geq 2|X|$

$|Z_0| = e(X, Y) \leq |Y|$

$e(Y, Z_1) \leq |Y|$.

(We will continue after reading week!)