CO749 - Graph Colourings

(Notes Scans)

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Contents

Summary	**2**
Lecture 1	**3**
Lecture 2	**8**
Lecture 3	**16**
Lecture 4	**21**
Lecture 5	**29**
Lecture 6	**35**
Lecture 7	**41**
Lecture 8	**48**
Lecture 9	**54**
Lecture 10	**62**
Lecture 11	**71**
Lecture 12	**79**
Lecture 13	**85**
Lecture 14	**89**
Lecture 16	**94**
Lecture 17	**101**
Lecture 18	**107**
References	**114**

Summary

Lecture 1, 2 - History

Lecture 3 - Probabilistic method overview, colour degree

Lecture 4 - Reed-Sudakov, Wasteful colouring procedure, expectations for variables in proof of Reed-Sudakov

Lecture 5 - Reed-Sudakov (continued), Concentration Inequalities, Talagrand's inequality

Lecture 6 - Finish Reed-Sudakov, exceptional Talagrand's, Balls & Bins

Lecture 7 - Regularization & Equalizing coin flips, Kim's theorem for girth-five graphs

Lecture 8 - Finishing Kim's, edge-colouring and Kahn's theorem

Lecture 9 - Finishing Kahn's

Lecture 10 - Reed's conjecture, upper bounds, and working towards [1]

Lecture 11 - See [2, 3]

Lecture 12 - Localized Colouring theorems, Local Fractional Colouring, Caro-Wei

Lecture 13 - More Local theorems, Perfect Graphs (Not sure why I only had 2 pages of notes for this lecture)

Lecture 14 - See [4]

Lecture 15 - Missed this one

Lecture 16, 17, 18 - Hadwiger's Conjecture, Proof of Norin-Song theorem (This is probably inaccurate. Lack of review has left me to forget what the main ideas of these lectures were.)

~~7?? - Hopf Algebra~~
749 - Graph Colouring

Jan 7th

Lecture 1:

Def'n 1: A k-colouring of a graph G is a partition of $V(G)$ into at most k independent sets.

Def'n 2: A k-colouring of a graph G is a map $f: V(G) \to [k]$ such that $\forall e = vw \in E(G)$, $f(v) \neq f(w)$

Def'n 3: A k-colouring of a graph G is a graph homomorphism to K_k.

Takeaway: There are multiple ways to view what a colouring is.

Weakenings, Generalizations, and Variants of Colouring:
Variants: ("Changing what you color")
- Edge Colouring: A k-edge-colouring of a graph G is
  Def 1: A partition of $V(G)$ into at most k matchings
  2: A k-colouring of $L(G)$ (the line graph of G)

~~"Changing what you color"~~:

- Total Colouring: A k-total-colouring of a graph G is a map $f: V(G) \cup E(G) \to [k]$ s.t.
  - $f(v) \neq f(w)$ $\forall vw \in E(G)$
  - $f(v) \neq f(e)$ $\forall v \sim e \in E(G)$
  - $f(e) \neq f(e')$ $\forall e \sim e' \in E(G)$ ($\in V(G)$)

Generalizations: ("Changing what you are allowed to color")
- List Colouring: (idea: Lists of available colours to vertices)
  Def: ~~A k-list-colouring~~ k-list-assignment of a graph G is an assignment of lists $(L(v): v \in V(G))$ such that

①②

$|L(v)| \geq k \quad \forall v \in V(G)$

An L-colouring of a graph G is a colouring ϕ of G such that $\phi(v) \in L(v)$ $\forall v \in V(G)$

~~"Changing what you are allowed to colour"~~

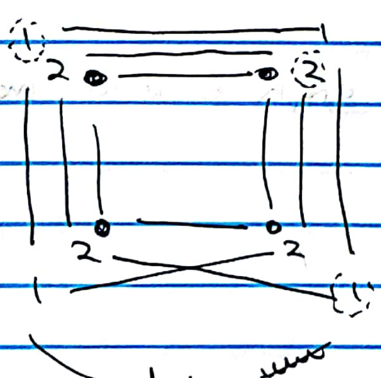
- Correspondence Colouring:

Def: A k-correspondence-~~colouring~~ assignment is a pair $((L(v) : v \in V(G)), (M_{uv} : uv \in E(G)))$, where M_{uv} is a matching from $L(u)$ to $L(v)$

An (L, M)-colouring of G is a colouring ϕ of G such that:

- $\phi(v) \in L(v)$ $\forall v \in V(G)$
- $\phi(u)$ is not matched to $\phi(v)$ in M_{uv} $\forall uv \in E(G)$

Ex: $G = C_4$



$L(v) = \{1, 2\}$ $\forall v$

Q: Is there an (L, M)-colouring of G for this L, M?

No!

And we run into trouble here

Remarks:

- We may as well assume $L(v) = \{1, \dots, |L(v)|\}$ $\forall v \in V(G)$
- Correspondence has a "local notion of colour", while list colouring / ordinary colouring have a "global notion of colour"

Weakenings: ("Change what a colour is")

- No Restrictions: Improper colourings, i.e. mappings of graphs
- d-defective colouring: Each colour has maximum degree d
- c-clustered Colouring: Every monochromatic component has size (# of vertices) $\leq c$
- No mono. path of length $> l$.
- Every colour is d-degenerate
- Every colour is triangle-free (or more generally, bounded clique #)

Fractional Colouring: ("Changing how you colour")

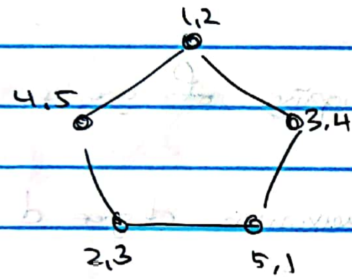
Def: An (a,b)-colouring of a graph G is a map ϕ such that $\phi(v)$ is a subset of $[a]$ of size b and $\forall uv \in E(G)$, $\phi(u) \cap \phi(v) = \emptyset$

Remark: If G has a k-colouring, then G has a (kb, b)-colouring $\forall b$

(i.e. Graph homomorphism to Kneser graph on a, b)

The fractional chromatic #: $\chi_f(G) = \inf\{\frac{a}{b} : G$ has an (a,b)-colouring$\}$.

Ex. $\chi_f(C_5) = 5/2$

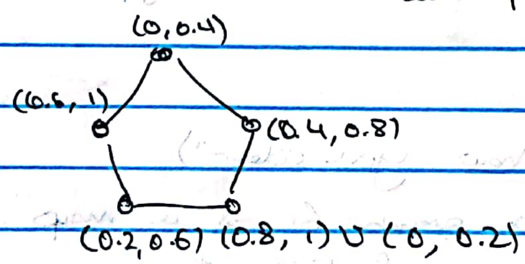


Remark:
- There is also an LP-formulation and its dual. (its dual asks about weighted independent set)
- An assignment of measurable subsets $\Phi(v)$ of $[0,1]$ $\forall v \in V(G)$ such that:

$\mu(\Phi(u) \cap \Phi(v)) = \phi$

$\chi_f(G) = 1 / \sup\{\varepsilon : G$ has a colouring as above, with $\mu(\Phi(v)) = \varepsilon \ \forall v\}$

Ex:



Let's call the above an *f-colouring* if $\mu(\phi(v)) = f(v)$ $\forall v$.

Proposition: $\chi_f(G) = k$ iff G has a $(1/k)$-colouring

Proposition: G has an f-colouring iff the vector $(f(v) : v \in V(G))$ is in the independent set polytope (i.e. $\exists$ probability distribution on independent sets such that $Pr[v \in I] = f(v)$)

Combinatorics:

- Defective Clustering ×
- List Defective ✓
- Frac. Defective ✓
- List Edge ✓
- Frac. Edge ✓
- Edge Total ×
- List Correspondence × (Correspondence is already list).
- Fractional List ✓
  - Find b-colouring from a-list-assignment. (multicolouring)

$$\chi_{f,list}(G) = \inf\left\{\frac{a}{b} : \forall \text{ a-list-assign } G \text{ has a b-colouring}\right\}$$

Theorem: $\chi_f(G) = \chi_{f,list}(G)$

- Fractional Correspondence.

Lecture 2:

Questions / Results: (on Graph Colourings)

Def'n (k-colourable)
A graph G is k-colorable if G has a k-coloring

Def'n (Chromatic Number)
The chromatic number, denoted $\chi(G)$, is the minimum k s.t. G is k-colorable

Question: Why is this a ~~natural~~ good ~~quest~~ definition?
If a graph is k-colorable, then it is $(k+1)$-colorable, so this is a natural def'n.
(Prop: If $H \subseteq G$, then $\chi(H) \leq \chi(G)$)

Remark:
A graph is 1-colorable iff no edges
— " — 2-colorable iff no odd cycles
— " — 3-colorable iff no good answer!
↳ Since NP-hard to decide if a graph is 3-col.

A graph is critical for k-coloring if it is not k-colorable, but every proper subgraph is. (also formerly known as $(k+1)$-critical)

List Colouring: → (Introduced by Erdős, Rubin, Taylor in 1979 and independently by Vizing 1976)
The list chromatic number (aka choice number or choosability) denoted $\chi_\ell(G)$ is the minimum k such that G has a L-coloring $\forall$ k-list-assignments L.

Proposition: $\chi(G) \leq \chi_\ell(G)$

Proposition: If $H \subseteq G$, $\chi_\ell(H) \leq \chi_\ell(G)$
(i.e. The list chromatic number remains monotone)

Def'n G is k-list-colorable if $\chi_\ell(G) \leq k$
(aka k-choosable)

Def'n G is critical for k-list-colouring if G is not k-list-col. but every proper subgraph is

L-critical w.r.t a list assignment L if G is not L-col, but $\forall$ proper subgraph is.

Q: How is list colouring different from colouring?

Theorem: $\chi_\ell(K_{d,d}) = \Theta(\log d)$
(But note: $\chi(K_{d,d}) = 2$)

Theorem (Alon 2000)
If G is a graph of min. degree d, then $\chi_\ell(G) = \Omega(\log d)$.

Conjecture: $\forall k$, if $\chi(G) \le k$ (and ~~min. degree~~ triangle free) then $\chi_\ell(G) = O(\log d)$

Correspondence Colouring (aka DP-colouring)

Def: Corr. Chromatic # (aka DP-chr. #), denoted $\chi_\ell(G)$ (aka $\chi_{DP}(G)$) is min k s.t. $\forall (L,M)$ k-corr.-assign., G has a (L,M)-coloring

critical for "

(L,M)-critical for "

Theorem (Bernshteyn, 2018)
If G is d-regular, then $\chi_\ell(G) = \Omega\left(\frac{d}{\log d}\right)$

Back to questions:

Types of questions:

- Chromatic # as related to other graph parameters (e.g. degree, clique #, etc.)
- Chromatic # of certain graph classes (e.g. Planar, surfaces, etc.)
- Algorithmic questions, e.g. Deciding if colouring exists, finding a colouring, can you sample a colouring uniformly at random
- How many colourings?
- Re-configuration: Can we get from one colour to another?

2

Relations to Other Parameters:-

Colouring Ones: $\max_{H \subseteq G} \frac{v(H)}{\alpha(H)} \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$

Hall ratio

(Recent result of 2019): $\nexists f$ s.t. $\chi_\ell(G) \leq f(\text{Hall ratio})$

Degree, Clique #, Girth:

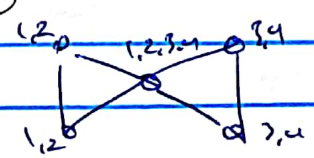
$\chi_{DP}(G) \leq \underbrace{\Delta(G)}_{\text{max degree}} + 1$ (Greedy Bound)

Brooks's Thm (1941): $\chi(G) \leq \Delta(G)$ unless G contains $K_{\Delta+1}$ or an odd cycle if $\Delta = 2$

(Other version: If connected, then $\chi \leq \Delta$ unless G is iso to clique or odd cycle if $\Delta = 2$)

E.R.T, independently Vizing: $\chi_\ell(G) \leq$ —— " ——

ERT (Borodin) If L is a list assignment of connected G such that $|L(v)| \leq d(v)$, then G is L-colorable unless every block of G is a clique or odd cycle



Girth:

Kim '95: If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\log \Delta}$

Johannsen '99: If G is triangle-free, then: $\chi(G) = O\left(\frac{\Delta}{\log \Delta}\right)$

Molloy '17: —— " —— then: $\chi(G) \leq (1+o(1))\left(\frac{\Delta}{\log \Delta}\right)$

If G is K_r-free for fixed r, then $\chi \leq O\left(\frac{\Delta}{\log \Delta}\log\log \Delta\right)$

Conj: $\log \log \Delta$ is not neccessary.

Question: Can we do better than $\frac{\Delta}{\log \Delta}$?

Erdős '59: $\exists$ graphs of arbitrary girth and chromatic #

Best known result in Ramsey Theory
$\Rightarrow \exists$ graphs of arbitrary girth and $\chi \geq \frac{1}{2}\frac{\Delta}{\log \Delta}$

$\hookrightarrow$ So, no, we can't!

Question: Is the answer 1 or 1/2 or in-between?

Reed's Conjecture (1998) $\chi(G) \leq \left\lceil \frac{\Delta+1+\omega}{2} \right\rceil$
$\Rightarrow$ True for $\chi_f(G)$ (Reed)
for if $\omega \geq .9999998\Delta$ (i.e. large ~~omega~~).

Thm (Delcourt, P.)
If Δ large enough, then $\chi \leq \left\lceil \frac{12}{13}(\Delta+1) + \frac{1}{13}\omega \right\rceil$.

Chromatic # of Graph Classes:

Four Color Theorem (Appel & Haken, 1977/79)
(Conjectured 1852)

If G is planar, $\chi(G) \leq 4$.

→ Later proof by Robertson, Sanders, Seymour, and Thomas (1994/6)

(Formally verified by prover systems in 2000s)

Grötzsch's Theorem (1959)

If G is planar, triangle-free, then $\chi(G) \leq 3$.

Surfaces:

Euler genus of a surface =

$2 \times$ # of handles + # of crosscaps

Heawood's Bound: $\forall g \geq 1$, if G is a graph embedded in a surface of genus g, then

$$\chi(G) \leq \frac{7 + \sqrt{24g + 1}}{2}$$

Ringel-Youngs Thm (1960's)

Heawood's Bound is tight for every surface, except the Klein bottle, where $\chi \leq 6$.

Hadwiger's Conjecture:

(1943) If G has no K_t-minor, then $\chi(G) \le t-1$

→ Easy for $t \le 3$ and proved by Had for $t=4$

Wagner '37 - Showed $t=5$ is equiv to 4CT

Robertson, Seymour, Thomas '94: $t=6$ equiv to 4CT

Open for all $t \ge 7$.

Weakenings:

Thm (Reed-Seymour, '99):

If G has no K_t-minor, then $\chi_f(G) \le 2t$

Thm (Edwards, Kang, Kim, Oum, Seymour, '15)

$\forall t, \exists d$ s.t. if G has no K_t-minor, then G is a d-defective, t-colorable

Thm (Dvorak-Norin, '18+)

—— " ——, G is $\exists c$-clustered, t-col.

Thm (Kostochka, Thomason '80s)

If G has no K_t-minor, then G is $O(t\sqrt{\log t})$-degenerate

Thm (Norin, Song, '19+)

$\forall \beta > 1/4$, G is $O(t(\log t)^{\beta})$-col.

4

List version of Hadwiger's is false

$\exists K_t$-minor free G w/ $\chi_\ell(G) \geq 4/3 t$

Strong perfect graph thm: (Chudnovsky, Robertson, Seymour, Thomas, '06)

G is perfect iff G has no induced C_{2k+1} or $\overline{C_{2k+1}}$ $\forall k \geq 2$

7ℓ9 - Colouring Jan 14th

Lecture 3:

The Probabilistic Method:
3 Pillars of prob. method:
- Linearity of Expectation (and basic probability)
- Lovász Local Lemma (introduced in 1975)
- Concentration Inequalities - (Markov, Chebyshev, Chernoff, Talagrand's) 1990s

Linearity of Expectation:
Expectation: Let X be a discrete $\mathbb{R}$ r.v., then: $\mathbb{E}[X] = \sum_{i \in \mathbb{R}} \mathbb{P}[X=i] \cdot i$

Linearity of Expectation: If $X = \sum_{i \in [n]} X_i$ (n finite), then $\mathbb{E}[X] = \sum_{i \in [n]} \mathbb{E}[X_i]$

Independent variables: 2 ~~random variables~~ events in a probability space A and B are independent if $\Pr[A \mid B] = \mathbb{P}[A]$ or $\Pr[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be ~~events~~ a collection of events in a probability space. Then:
- We say the events in $\mathcal{A}$ are pairwise independent if $\forall i \neq j$ in $\mathcal{A}$ A_i, A_j are independent.
- We say the events in $\mathcal{A}$ are mutually independent if $\forall i$, A_i is mutually independent of $\mathcal{A} \setminus \{A_i\}$, i.e. $\forall \mathcal{A}' \subseteq \mathcal{A} \setminus \{A_i\}$ $\Pr[A_i \mid \bigcap_{A_j \in \mathcal{A}'} A_j] = \Pr[A_i]$

Question: If $\mathcal{A}$ is pairwise independent then is it mutually independent?
A: No! Example: $n \geq 3$ coins and ~~uniform~~ space of flippings where # of heads is even and A_i is coin i is Heads. and $\mathcal{A} = \{A_1, \dots, A_n\}$.

①

Lovász Local Lemma: → often referred to as "bad events"

If $A = \{A_1, \dots, A_n\}$ is a set of events in a probability space, and for every i, there is $B_i \subseteq [n] \setminus \{i\}$ such that A_i is mutually independent of $A \setminus (\{A_i\} \cup \{A_j : j \in B_i\})$*

AND $|B_i| \le d$, $\forall i \in [n]$ ← i.e. Bad events are not too dependent

AND $Pr(A_i) \le p$, $\forall i \in [n]$ i.e Bad events are unlikely

AND $ep(d+1) \le 1$ (also $4pd \le 1$ works too!)

Then,

$$Pr[\bigcap_{i \in [n]} \bar{A}_i] > 0.$$

(i.e. with positive probability, none of the bad events occur)

*Wrong Version: If A_i is independent of at most d other events.
(Example: 1 is mutually independent of any n-2 sets of coins but not n-1 ⇒ No unique maximal set).

Remarks:

- Use the local lemma, say, if union bound fails (Recall that the union bound is: If $\sum_{i \in [n]} Pr[A_i] < 1$, then $Pr[\bigcap \bar{A}_i] > 0$)
- The "positive probability" is not very large, i.e. Usually $O(2^{-|A|})$, so could be exponentially small on the # of events.
- We use it to construct/force a good outcome.

Algorithmic Q: Can we find such a good outcome efficiently?

{Sampling is bad idea since this may take exponentially long}. (b/c of low probability)

↪

Moser-Tardos (2000): There exists an algorithm to find a good outcome of the LLL in "the variable model" that runs in time $O(|A|)$.

Every event depends on a bounded # of variables w/ bounded # of states and $B_i = \{A_j \in A \setminus \{A_i\} : A_j$ and A_i depend on at least one common variable$\}$.

→ Independent Trials

Moser-Tardos Algorithm:
- Sample all the independent trials in the probability space
- WHILE $\exists$ a bad event A_i:
  - Resample, at random, all the trials that A_i depends on
- RETURN good outcome

(Some) Applications of LLL to colourings:

Hypergraph Colouring:
Recall: Hypergraph $H=(V,E)$: V - set of vertices, E - set of hyperedges (i.e. ~~sets~~ subsets of ~~ed~~ vertices)

H is k-uniform if $|e|=k$ $\forall e \in E(H)$.

(Note: $k=2$ gives us simple graphs)

Def'n A k-colouring of a hypergraph H is a partition of $V(H)$ into at most k independent sets of H.

A set $I \subseteq V(H)$ is independent if $\nexists e \in E(H)$ s.t. $e \subseteq I$

or equivalently $\exists\ \phi: V(H) \to [k]$ such that $\forall e \in E(H)$, $\exists v_1 \neq v_2 \sim e$ such that $\phi(v_1) \neq \phi(v_2)$ (i.e. No monochromatic edges)

The <u>chromatic</u> <u>number</u> $\chi(H)$ is the min k s.t. H is k-colorable

Q: How is $\chi(H)$ related to $\Delta(H) := \max_{v \in V(H)} d_H(v)$, (where $d_H(v) := |\{e \in E(H) : e \ni v\}|$). ?

<u>Trivial Bound</u>: $\chi(H) \leq \Delta(H) + 1$ (Greedy!)

<u>Theorem</u>: If H is a k-uniform hypergraph ($k \geq 2$), then
$$\chi(H) \leq \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$$

<u>Proof</u>:

We will use LLL.

Assign every vertex v of H a colour $\phi(v)$ from $[L]$, where $L = \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$.

Define bad event:

A_e = Edge e is monochromatic in ϕ.

$\Pr(A_e) = 1/L^{k-1}$ (because color of 1st vertex can be anything, but after that, $\forall v \sim e$ needs a different color). $(= p)$

Let $B_e = \{f \neq e \in E(H) : f \sim e\}$, the by variable model, it follows that A_e is mut. ind. of $A \setminus (A_e \cup B_e)$

$|B_e| \leq \underbrace{k}_{\text{at most } k \text{ vertices}} \underbrace{(\Delta(H)}_{\text{max degree.}} - 1) (= d.) \leq k\Delta(H) - 1$

So: $ep(d+1) = e \cdot \frac{1}{L^{k-1}} k\Delta(H) = \frac{ek\Delta(H)}{((ek\Delta(H))^{\frac{1}{k-1}})^{k-1}} = 1.$

So, by LLL, $\exists$ a ϕ avoiding all A_e, i.e. a k-colouring of H ☒

Color Degree:

Def'n: Let G be a graph and L list assignment of G. Let $|L|$ denote the min. size of a list, i.e. $|L| := \min_{v \in V(G)} |L(v)|$

We define:

The color-degree of a vertex $v \in V(G)$ in color $c \in L(v)$ as:

$$d_L(v,c) := |\{u \in N(v) : c \in L(u)\}|$$

The color-degree of v, denoted $d_L(v) = \max_{c \in L(v)} d_L(v,c)$

The maximum color-degree of G w.r.t. L, is:

$$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$$

Question: Does $\exists$ function f s.t. if $|L| \geq f(\Delta_L(G))$, then G has an L-colouring? (analogous to $|L| \geq \Delta(G)+1$, then G has an L-colouring)

Theorem (Alon '88 (with constant $\geq .5$), '92)
If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring

Theorem (Haxell '00 - follows from a more general thm)
If $|L| \geq 2\Delta_L(G)$, then G has an L-colouring

Theorem (Reed-Sudakov, 2002)
If $|L| \geq (1+o(1))\Delta_L(G)$, then G has an L-colouring

3

Lecture 4:

Review: (Colour Degree)

G graph, L list assignment

$|L| := \min_{v \in V(G)} |L(v)|$

$d_L(v,c) := |\{ v \in N(v) : c \in L(v) \}|$

$d_L(v) := \max_{c \in L(v)} d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

Theorem (Alon '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring.

Proof:

(We may assume WLOG $|L(v)| = L \ \forall v$)

- Color every vertex v, $\phi(v)$, uniformly at ~~random~~ random from its list
- Bad events: $A_{\underset{=uv}{e},c}$: $\phi(u) = \phi(v) = c \quad \forall c \in L(u) \cap L(v)$
- Then $Pr[A_{e,c}] = \frac{1}{L^2} = p$.
- $A_{e,c}$ is mutually independent of $A_{e',c'}$ whose $e' \nsim e$ (and $\neq$)
- $B_{\underset{=uv}{e},c} = \{ A_{\underset{=u'v'}{e'},c'} : e' \sim e \text{ or } e' = e \text{ and } c \in L(u') \cap L(v') \}$

$= 2 \cdot |L| \cdot \Delta_L(G) \quad = d \cdot e$ ← Since we counted ourselves

2 ↓ Pick an end (u or v) Call this x

$|L|$ ↓ Pick a color c' in $L(x)$

$\Delta_L(G)$ → Pick a neighbor w of x where $c \in L(w)$.

(1)

Proof (cont'd)

Then, by LLL:

Since $ep(d+1) = e\frac{1}{|L|^2}(2|L| \cdot \Delta_L(G))$

$$= \frac{2e\Delta_L(G)}{|L|} \leq 1.$$

Then $\exists$ a coloring ϕ avoiding all $A_{e,c}$, i.e. an L-coloring of G, as desired. ∎

(Note: We can actually get rid of the "+1" in the statement of the theorem)

Theorem (Haxell)

If $|L| \geq 2\Delta_L(G)$, then G has an L-coloring.

Actually follows from the following more general theorem:

Theorem (Haxell)

Let $k \geq 1$ be an integer. If $V_1, V_2, \ldots, V_r$ is a partition of $V(G)$ into independent sets for a graph G such that:

- $|V_i| \geq 2k$ $\forall i \in [r]$, and
- $\Delta(G) \leq k$, and

Then $\exists$ an independent set I of G s.t. $\forall i \in [r]$, $I \cap V_i \neq \emptyset$.

(Called an independent ~~sevset~~ transversal)

How does it imply previous thm?
Let H be such that
$V(H) = \{(v,c) : v \in V(G),\ c \in L(v)\}$.
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c = c'\}$
Then H satisfies independent transversal theorem by assumption on G. So, by that theorem, $\exists$ an independent transversal, i.e. An L-coloring

Remark: Clearly also this implies correspondence version by letting
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c$ matched to c' in $M_{vv'}\}$.

Remark: This is tight! (i.e. The ratio of 2). For general independent transversals.

Conjecture (Reed, ~'90s)
If $|L| \geq \Delta_L(G) + 1$, then G has an L-coloring

But, Bohman and Holzman disproved this conjecture!
Still open if $|L| \geq \Delta_L(G) + 2$ works! (Has been open for about 20 years)

Theorem (Reed–Sudakov, '02)
If $|L| \geq (1+o(1))\Delta_L(G)$, then G has an L-coloring
~~[illegible]~~
(Note: Equivalent to:
$\forall \varepsilon > 0, \exists \Delta_\varepsilon$ s.t. $\forall \Delta \geq \Delta_\varepsilon$, if $\Delta(G) = \Delta$ and $|L| \geq (1+\varepsilon)\Delta$, then G has an L-coloring).

Remark:
~~Reed~~ Loh and Sudakov proved that the ratio for independent transversals is $(1+o(1))$ assuming that every vertex has at most $o(k)$ neighbors in any other partitions..

Proof (Reed-Sudakov)

Note: Uses Rödl ~~N~~ Nibble (aka semirandom) method of iteratively constructing a solution a little bit at a time. (In particular, uses the "Wasteful Coloring Procedure" to implement one such iterative procedure)

Note #2: Uses LLL and concentration inequalities to prove that there is a decent enough outcome of the WCP as to continue iterating, until the Big Finish.

What is our finish? Alon's Thm (or Haxell's), i.e. can L-color if $|L| \geq 2e\Delta_L(G)$..

How will our iteration step be improving?: Progress will be in the ratio, $|L|/\Delta_L(G)$. In fact, in our proof, it will take some constant, depending on ε, # of steps.

Proof details:

Big assumption for now:

Let's assume $\forall c \in L(v)$ that $d_L(v,c) = d(v) = \Delta(G)$

(of course, we may assume wlog $|L(v)| = |L| \ \forall v$)

Wasteful Colouring Procedure:

- Independently "activate" each vertex of G for some probability p to be fixed later.

(Remark (if the calculations are correct):

$$p = \frac{\varepsilon(1+\varepsilon)}{2}$$

works and any smaller $p \geq \text{polylog}(\Delta)$).

- Now color every activated vertex v with a color $\phi(v)$ selected uniformly at random from $L(v)$

This is the "wasteful" step!

- Remove $\phi(v)$ from the list of all v's neighbours

Uncolour any vertex which has the same colour as neighbour (or equiv. $\phi(v)$ is no longer in its list)

- Let ϕ' be the resulting colour, and

L' be the resulting list

Remark: In the "naive colouring procedure" we only remove colours from neighbour's list if they keep the colour.

Scanned with CamScanner

Here, we use WCP over NCP b/c
1) NCP is harder to analyze for concentrations, and
2) The probability that an active keeps its color will be close to 1, so in practice, much difference here

We let U be the set of vertices that do not receive a color from ϕ'

let $G' = G[U]$

Claim $L'(v) = L(v) \setminus \{\phi'(u) : u \in N(v) \cap A\}$ ← where A is activated (wasteful procedure)

→ $\setminus U$ would be for naive.

We'll prove that there is an outcome with

$$\frac{|L'|}{\Delta_{L'}(G')} > 1+\varepsilon \quad \leftarrow \underline{\text{GOAL!}}$$

(In fact, $(1+\varepsilon)(1+\varepsilon/4)$.)

~~uuuuuuuuuu~~

Expectations:

- Lists: Want to calculate $E[|L'(v)|]$ for a given v

$$E[|L'(v)|] = \sum_{c \in L(v)} \Pr[c \in L'(v)]$$

By linearity of Expectation

$$= \sum_{c \in L(v)} \Pr[\nexists u \in N(v) \cap A \text{ with } \phi(u) = c]$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{}$$

$$\prod_{u \in N(v)} \Pr[u \notin A \text{ or } \phi(u) \neq c] \leftarrow u \in N(v)$$

See note on next pg.

$$\prod_{u \in N(v)} (1 - \Pr[u \in A \text{ and } \phi(u) = c])$$

But activation is independent of colouring:

$$\prod_{u \in N(v)} 1 - \Pr[u \in A] \cdot \Pr[\phi(u) = c]$$

$$= \prod_{u \in N(v)} \left(1 - p \cdot \frac{1}{|L|}\right).$$

$u \in N(v,c)$

In fact, since $\phi(u) = c$ only happens if $c \in L(u)$, we can write $u \in N(v,c)$.

$$= \left(1 - \frac{p}{|L|}\right)^{|N_L(v,c)|}$$

$$= \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)} \leftarrow \text{by color regularity}$$

$$\approx e^{-\frac{p \Delta_L(G)}{|L|}}$$

So, the sum is:

$$\mathbb{E}[|L'(v)|] = e^{-\frac{p \Delta_L(G)}{|L|}} \cdot |L|.$$

So, note that when $p \approx 1$, the we cut the list by around a 1/3 each time. But, if p small, then we retain a decent chunk.

Color-degrees:

$$\underset{(\text{in } G')}{\mathbb{E}[d_{L'}(v,c)]} = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

(Remark: Do this whether or not $c \in L'(v)$)

$$\mathbb{E}[d_{L'}(v,c)] = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

$$= \sum_u \Pr\begin{bmatrix} (u \in A \text{ and } c \in L'(u)) \\ \text{OR } (u \in A \cap \mathcal{U} \text{ and } c \in L'(u)) \end{bmatrix}$$

$$= \sum_u \Pr[u \in A] \Pr[c \in L'(u)]$$

$$+ \Pr[\underbrace{\quad}_{c=\phi(v)} \quad \underbrace{\quad}_{c \neq \phi(v)}]$$

$c=\phi(v)$: $\Pr[u \in A \cap \mathcal{U}]$ $\qquad$ $c \neq \phi(v)$: $\le \Pr[u \in A \cap \mathcal{U}] \Pr[c \in L'(u)]$

Let's pretend that $c=\phi(v)$ does not matter:

$$\le \sum_u \left(u - p(1-\frac{p}{|L|}\right)^{\Delta_L(G)} \quad \Pr[u \in A] \Pr[u \in \mathcal{U}]$$

$$+ \Pr[u \in A \cap \mathcal{U}] \cdot \frac{1}{|L|}$$

$$+ \left(p(1 - \underbrace{\Pr[\phi(v) \in L'(v)]}_{(1-\frac{p}{|L|})^{\Delta_L(G)}}) \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right)$$

$= \Delta_L(G)$ times

Asymptotically the dominant constant is 1

Lecture 5:

Recall:
We were proving Reed-Sudakov, and had the following expected values:

$$\text{Keep} := \left(1-\frac{p}{2}\right)^{\Delta_L(G)} \approx e^{-\frac{p\Delta}{L}} \quad (= \Pr(c \in L'(v) \mid c \in L(v)))$$

$$\mathbb{E}[|L'(v)|] = \text{keep} \cdot |L|$$

$$\mathbb{E}[d_{L',G'}(v,c)] = d_{L,G}(v,c)\Big(\underbrace{(1-p)}_{\text{not activated}} \cdot \underbrace{\text{Keep}}_{\text{keeps } c} + p\Big(\underbrace{\text{keep} \cdot \frac{1}{|L|}}_{\text{if } \phi(v)=c \text{ and keep } c} + \underbrace{\left(1-\frac{1}{|L|}\right)\left(1-\frac{2}{|L|}\right)^{\Delta}}_{\text{keeping } \phi(v) \text{ and } c}\Big)\Big)$$

$\phi(v) \neq c$ and keep c.

But it turns out that this variable doesn't concentrate well, so we'll use another.

Picking a better variable:

$$d_{L',G'}(v,c) = |N_{L',G'}(v,c)| = |\{v \in \underbrace{N_{L,G}(v,c) \cap V(G')}_{G^{\varepsilon} \to G'} : \underbrace{c \in L(v)}_{L^{\varepsilon} \to L'}\}|$$

But, it turns out we don't need to care about the lists, so instead let's look at $|N_{L,G}(v,c) \cap V(G')|$.

Clearly

$$d_{L',G'}(v,c) \leq |N_{L,G}(v,c) \cap V(G')|.$$

$$\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]$$

$$= \sum_{u \in N_{L,G}(v,c)} \Pr[u \in V(G')] \quad \text{(lin. of exp.)}$$

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \notin V(G')]$$

These events independent.

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \in A \text{ and } \phi(u) \text{ is kept}] = d_{L,G}(v,c)(1-p \cdot \text{Keep})$$

On to some iterative calculations:

- Now, if we could show that with "high enough" probability every $|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$ are close enough to their expectations to apply the LLL, then we'll be happy w/ the following calculation:

$$\frac{|L'|}{\Delta_{L'}(G)} \gtrsim \frac{\mathbb{E}[|L'(v)|]}{\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]} = \frac{\text{keep} \cdot |L|}{(1-p\,\text{keep}) \cdot \Delta_L(G)}$$

Same ~~ratio~~ factor → we will argue this is close to 1.

Old ratio

$$\geq \frac{|L|}{\Delta_L(G)} \cdot \frac{1-\frac{p\Delta}{|L|}}{1-p\left(1-\frac{p\Delta}{|L|}\right)}$$

{ using that $\left(1+\frac{x}{n}\right)^n \geq 1+x$ for $n \geq 1$ and $|x| \leq n$.

Note: $\frac{\Delta}{|L|} = \frac{1}{1+\varepsilon}$ (some assumption we made)

$$= \frac{|L|}{\Delta_L(G)} \cdot \underbrace{\frac{1-p/(1+\varepsilon)}{1-p(1-p/(1+\varepsilon))}}$$

$$= 1 + \frac{p\varepsilon}{1+\varepsilon} - \frac{\varepsilon^2}{1+\varepsilon}$$

Now if we choose $p = \varepsilon/2$, then we get

$$= 1 + \frac{\varepsilon^2}{4(1+\varepsilon)}$$

{ So, we get some improvement in each iteration.

How to show a variable is close to its expectation:

Concentration Inequalities:

Markov's Inequality:
If $X \geq 0$ is a random variable,
$$Pr[X \geq k E[X]] \leq 1/k. \quad \forall k \geq 1$$

Chebyshev's Inequality:
The variance of a random variable X, denoted $Var[X]$ is
$$Var[X] = E[(X - E[X])^2]$$
The standard deviation σ of X is $\sqrt{Var[X]}$.

Let $\mu = E[X]$, then
$$Pr[X \geq \mu + k\sigma] \leq 1/k^2.$$
Proof: Use Markov's inequality.

Chernoff Bounds:
Let $X = \sum_{i=1}^{n} X_i$ where X_i is a Bernoulli r.v. (i.e. only takes value of 0 and 1). and all independent:
$$Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}} \quad (0 \leq \delta \leq 1)$$
$$Pr[X \geq (1+\delta)\mu] \leq \begin{cases} e^{-\frac{\delta^2 \mu}{3}} & \text{if } 0 \leq \delta \leq 1 \\ e^{-\frac{\delta \mu}{3}} & \text{if } \delta \geq 1 \end{cases}$$
(i.e. Exponentially small if δ constant and gives something meaningful if $\delta \gg \frac{1}{\sqrt{\mu}}$!).

Example:

$\mathbb{E}[|A \cap N_{L,G}(v,c)|] = p\Delta(G)$

$\Pr[|A \cap N_{L,G}(v,c)| \geq p\Delta_L(G)(1+\delta)] \leq e^{-\frac{\delta^2(p\Delta_L(G))}{3}}$ (By Chernoff).

i.e. $|A \cap N_{L,G}(v,c)| = p\Delta(G) \pm \sqrt{\Delta_L(G)\,\mathrm{polylog}\,\Delta_L(G)}$

with prob. $1 - \frac{1}{\Delta^c}$ for any c.

So, by LLL, this holds for all v with some positive prob.

But, we need to concentrate:

$|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$

Depends on color $c \in L(v)$ being in $L'(v)$ which depends on activation flips and color-assignments for $N_{L,G}(v,c)$ (But not independent!)

Even worse! (Stronger interactions with neighbors + 2nd neighbors)

"Simple" Concentration Bound (Book of Molloy & Reed).

Let X be a r.v. that depends only on the outcome of a set of independent trials $T_1, \ldots, T_n$. Suppose that changing the outcome of any one trial changes X by at most c (c constant) (This is called c-Lipschitz).

Then,

$$\Pr[|X - \mathbb{E}[X]| \geq t + 8c\sqrt{\mathbb{E}[X]}] \leq e^{\frac{-t^2}{3c^2 n}}$$

Remark: Note that the denominator has the # of trials. (not exponential)

Concentrating $|L'(v)|$:

Trials: activating flips & colourings of $N_{L_G}(v,c)$ $\forall c \in L(v)$

- 1-lipschitz: Since changing any one trial (actio. or colouring) changes $|L'(v)|$ by at most 1.

Recall: $E[|L'(v)|] = Keep \cdot \Delta_L(G) = \Theta(\Delta_L(G))$

What is r?

- If not doing colour degree, then 2Δ (Good!)
- But, w/ colour degree it's:

$2\Delta_L(G)\cdot|L|$ (If every colour the neighbours holds are disjoint).

$= 2(1+\varepsilon)\Delta_L(G)^2$

(This is Bad! The simple concentration bound is only meaningful for $t > \frac{\Delta_L(G)}{\omega}$)

Talagrand's Inequality (v.e.c.)

(A) Combinatorial version: (There is also a probability one)

Let $X \geq 0$ depend on independent trials $T_1, \ldots, T_n$. If X is c-lipschitz and r-verifiable, then for any $t > 96c\sqrt{rc^2 E[X]} + 128rc^2$

then:

$$Pr[|X - E[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4E[X]+t)}}$$

Remark: If r, c constant, then get ~~expo~~ exponentially small in t, if $t = \Theta(E[X])$ and still meaningful for $t \gg \sqrt{E[X]}$.

3

r-verifiable: For every $s > 0$, if $X \geq s$, then there exists a set Z of at most rs trials that "verify" that $X \geq s$, i.e. Changing any trials outside of Z still results in $X \geq s$.

So, of course, counting # of heads in coin flips is 1-verifiable.
Similarly, $|A \cap N_{L,G}(\vec{v}, c)|$ is 1-verifiable
(We just exhibit the heads or the activated set).

What about $|L'(\omega)|$?
Example: How do we show $|L'(\omega)| \geq 1$ (i.e. One color kept).
For all $u \in N_{L,G}(v, c)$ show either $u \notin A$ or $\Phi(u) \neq c$.
i.e. Need $\Delta_{L,G}(v, c)$ trials to verify
i.e. Need $r \geq \Delta_{L,G}(v, c)$ (This is bad.)

Pick a better variable:

$$|L(\omega)| - |L'(\omega)| = \# \text{ of colors lost.}$$

$$\mathbb{E}[|L(\omega)| - |L'(\omega)|] = |L| - \text{keep}|L|$$
$$= |L|(1 - \text{keep})$$
$$= (1+\varepsilon)\Delta_L(G)(1 - \text{keep}) = \Theta(\Delta_L(G))$$

Obviously, this variable is 1-Lipschitz.
What do we need to verify a color is lost?
→ Need a neighbor $u \in N_{L,G}(\omega, c)$ to be activated and $\Phi(u) = c$
i.e. Need 2 trials
More generally, if $|L(\omega)| - |L'(\omega)| \geq s$, need $2s$, so $r = 2$ works.

7149_Colorings

Jan 23rd.

Lecture 6:

Concentrating One last variable for Reed-Sudakov:

$|N_{L,G}(v,c) \cap V(G')|$

$\mathbb{E}[\quad] = \Delta_L(G)\cdot((1-p\cdot \text{keep})$

Can we concentrate this?

Is it r-verifiable for some r?

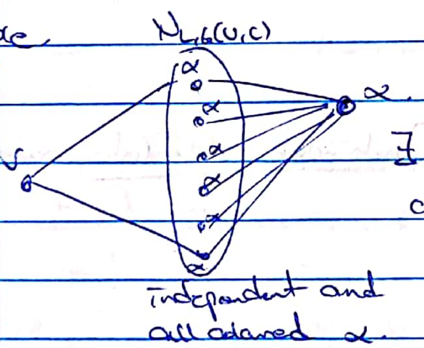
→ Note: $u \in N_{L,G}(v,c) \cap V(G')$ if either $u \notin A$ or $u \in A$ but $\exists$ $w \in N_{L,G}(u,\phi(w)) \cap A$ s.t. $\phi(u) = \phi(v)$

So, this is 4-verifiable! (at most 2 vertex flips and 2 colors)

But, is it c-Lipschitz for some c?

Not really: c could need to be at least $\Delta_L(G)$ as follows

For example,



$\exists$ exactly one neighbor (and its common to all) of these colored α.

If α changes to β, then all the colors change.

Idea: However, it's unlikely for this to happen.

i.e. Unlikely that a vertex can change the outcome by $\Omega(\deg)$

(we'll prove this later)

So, if we could somehow use Talagrand's with this "likely" Lipschitz constant, we'd find

$$\Pr(|\mathbb{E}[X]-X| > t + 60rc^2 + 8\sqrt{rc^2\mathbb{E}[X]}) \leq e^{-\frac{t^2}{8rc^2(t+\mathbb{E}[X])}}$$ } something like this.

$\mathbb{E}[X] \approx \Theta(\Delta)$ works for $t = \sqrt{\Delta}\,\text{polylog}\,\Delta$

①

So we'd get that Prob. $|L'(v)|$ or $|N_{v,G}(v,c) \cap V(G')|$ too far from $\mathbb{E}[X]$ is $\leq \frac{1}{\Delta^c}$ for any c.

Argue each event is at most independent of all but a set of at most $(\Delta \cdot L)^2 \leq \Delta^5$ events. and apply LLL to argue w/ pos. prob. that none happen

Then iterate and finish w/ Alon/Haxell when $L/\Delta \geq 2c$ or 2

□

Remark: This will only work for large enough Δ, b/c to apply the local lemma, we'd need $e^{-\frac{\log^3\Delta}{8}} \leq \frac{1}{\Delta^{10}}$. (and then extrapolate back through iterations, how large original Δ needs to be)

An Exceptional Outcome Version of Combinatorial Talagrand's Inequality.

Theorem:
Let $((\Omega_i, \Sigma_i, P_i))_{i=1}^n$ be probability spaces. Let (Ω, Σ, P) be their product space and $\Omega^* \subseteq \Omega$ be a set of "exceptional outcomes", and let $X: \Omega \to \mathbb{R}_{\geq 0}$ be a nonnegative random variable. Let $r, c \geq 0$. If X is (r,c)-certifiable w.r.t. Ω^*, then for any $t \geq 96c\sqrt{r\mathbb{E}[X]} + 128rc^2 + 8P[\Omega^*]\sup X$, then

$$\Pr[|X - \mathbb{E}[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}} + 4P[\Omega^*]$$

Remark: We need to pay some costs:
- Needing $t > 8P[\Omega^*] \sup X$, and
- An extra $4P[\Omega^*]$ in the prob. bound.

Def'n (r,c)-certifiable

- If $\omega = (\omega_1, \dots, \omega_n) \in \Omega$ and $s > 0$, an <u>(r,c)-certificate</u> (w.r.t X, ω, s, Ω^*) is an index set $I \subseteq \{1, \dots, n\}$ of size at most rs s.t. $\forall k \geq 0$, we have $X(\omega') \geq s - kc$ $\forall \omega' = (\omega_1', \dots, \omega_n') \in \Omega \setminus \Omega^*$ s.t. $\omega_i \neq \omega_i'$ for at most k values of I
- If $\forall s > 0$ and $\omega \in \Omega \setminus \Omega^*$ s.t. $X(\omega) \geq s$, $\exists$ an (r,c)-certificate, then <u>(r,c)-certifiable</u> w.r.t. Ω^*

<u>Remark:</u>

- For $k=0$, the certificate just acts as an r-verifier for non-exceptional outcomes as in normal combinatorial Talagrand's
- What this really requires is an r-verifier for which, if you change at most k of its trials (and any # of the verifier trials), you lose at most kc.
  - $\Rightarrow$ This is kind of like changing any trial in a non-exceptional outcome changes at most c, but requires more generally changing k trials changes by at most kc for every $k \geq 0$ (Bonus: Can assume you end in a non-exceptional outcome)

Back to ~~concentrating~~ concentrating $|N_{L,G}(v,c) \cap V(G')|$.

up to 2nd neighborhood

Ω^*: $\exists w \in N^{\le 2}(v)$ (Note: At most $(\Delta L)^2$ of these if we delete edges b/w vertices w/ disjoint lists)

s.t. $\exists c' \in L(w)$ and $\ge$ polylog Δ vertices in $N_{L,G}(w,c')$ with colour c'

(actually can use $N_{L,G}(w,c) \cap N_{L,G}(v,c)$ instead)

Claim: $\Pr[\Omega^*] \le \frac{1}{\Delta^c}$ for any c and Δ large enough.

(this will be enough to apply Talagrand's since $\exp|N_{L,G}(v,c) \cap V(G')| \le \Delta$) ~~and only~~

This also assumes that we show it's (r,c)-certifiable for some r,c w.r.t. Ω^*.

Proof (that $|N_{L,G}(v,c) \cap V(G')|$ is (r,c)-certifiable.

So, let ω be a non-exceptional outcome. We need $\forall s \ge 0$ a set of at most rs trials to build certificate.

So, we use the SU activation flip / colour assignments of vertices (and $\le$ neighbors) in $N_{L,G}(v,c) \cap V(G')$ (so, what we used before)

Need: $\forall k \ge 0$ changes to these trials, as well as any outside, that $|N_{L,G}(v,c) \cap V(G')| \le s - kc$

Can argue $k =$ polylog Δ works here b/c we start non-exceptional.

(b/c if value changes, had to change b/c of some critical change)

How to prove claim:

Note that $\Pr[\Omega^*] \underset{\text{by union bound}}{\le} \underbrace{|N^{\le 2}(v)|}_{\le \Delta^4} \Pr\left(\begin{array}{l} \exists c' \in L(u) \\ \text{and } 2\,\mathrm{polylog}\,\Delta \text{ in } N_{L_G}(w,c') \\ \text{colored } c' \end{array} \Big| \, w \in N^{\le 2}(v) \right)$

Suffices to show $\le \frac{1}{\Delta^c}$ for any c (and large enough Δ).

Balls and Bins:

m balls and n bins. and uniformly at random assign each ball to some bin independently.

Expected # of balls in bin i: m/n.

Harder Q: What's the max # of balls in a ~~test~~ bin?

If $m = n$ with high probability the max prob is $O\left(\frac{\log n}{\log\log n}\right)$

→ Relating this back to colouring:

Vertices $\equiv$ Balls and Colors $\equiv$ Bins

in neighbourhood $(N_{L_G}(u,c'))$ (in $L(w)$)

(Note: If ball can't go to same bin, the probabilities are right, we can take this and can only get a worse bound)

We have $\Delta_L(G)$ Balls and $|L|$ Bins so $\Delta \simeq \Theta(|L|)$ and so no bin has $\ge \mathrm{polylog}\,\Delta$ balls with high probability

③

Balls and Bins - Bounds:

Can $m = \Delta$, $n = \lceil L \rceil$

Upper bounds: By Union bound, $\Pr[\exists \text{ bin w/} \geq k \text{ balls}]$

$$\leq \Delta \cdot \Pr[\text{bin } i \text{ has} \geq k \text{ ball}]$$

$$\Pr[\text{Bin } i \text{ has} \geq k \text{ balls}] \leq \binom{\Delta}{k} \cdot \frac{1}{L^k}$$

$$\approx \left(\frac{\Delta e}{k}\right)^k \cdot \frac{1}{L^k} = \left(\frac{\Delta e}{kL}\right)^k$$

So, we would need: $k \log k > c \log \Delta$, i.e. $k > \frac{\log \Delta}{\log\log \Delta}$

~~Some constants~~.

to find $\leq \frac{1}{\Delta^c}$ for any c.

Lower Bound:

$$\Pr[\text{Bin } i = k] = \binom{\Delta}{k} \frac{1}{L^k} \left(1 - \frac{1}{L}\right)^{\Delta - k}$$

$$E\left[\begin{matrix}\#\text{ of bins w/} \\ \text{exactly } k\end{matrix}\right] = \Delta \cdot \Pr[\text{Bin } i = k]$$

$$\approx \Delta \cdot \left(\frac{\Delta e}{kL}\right)^k \cdot e^{-\frac{\Delta k}{L}}$$

By fiddling w/ constant for k

Can get $\geq \frac{1}{\Delta}$, and so $E \geq 1$,

(Use Tallgrand's w/ $c = 2$, $r = k$)

Same for upper bound (Can use $r = 1$, $c = 1$).

Lecture 7:

Back to our assumption that all color degrees are the same:
For Reed - Sudakov:

- Keeping a color is more likely if smaller color degree
- So $\mathbb{E}[|L'(v)|] \geq$ Expectation when regular
- $\mathbb{E}[d_{L,G'}(v,c)] \leq d_{L,G}(v,c)(1 - p\text{keep}) \leq$ ____ $(1 - p\text{keep})$

So keep changes with the list. $\rightarrow$ ~~color assigned~~ $\text{keep}(u, \phi(uw)) = (1 - \frac{p}{|L|})^{d_{L,G}(u, \phi(w))}$ min. keep, i.e. old #. (but we can always extend every list)

Even if it wasn't the case that expectations were only better for us in the non-regular case:

(1) Regularization: Embed our graph into a regularized version where ~~the~~ coloring the resulting graph yields a coloring of the original.

(2) Equalizing Coin Flips: Here, for every $u \in V(G)$ and $c \in L(u)$, we add a coin flip $F_{u,c}$ which keeps c for u with probability $\text{keep}/\text{keep}(u,c)$ and hence every color is kept with probability ~~keep/$F_{u,c}$~~ $\text{keep}(u,c) \cdot F_{u,c} = \text{keep}$.

Note: (2) only works if all desired ~~coin~~ coin flips have probability ≤ 1 ~~AND~~ we would have to redo all the concentrations, adding the coin flips into verifications (Lipschitz.

Regularization:
Lemma: Every graph G is an induced subgraph of a $\Delta(G)$-regular graph.

$\hookrightarrow$ Proof Sketch

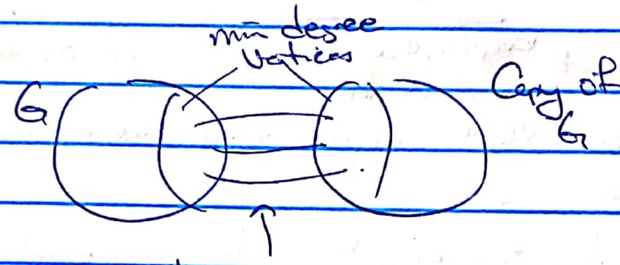
①

Proof Sketch:

By induction on $\Delta(G) - \delta(G)$ (max degree) (min degree)

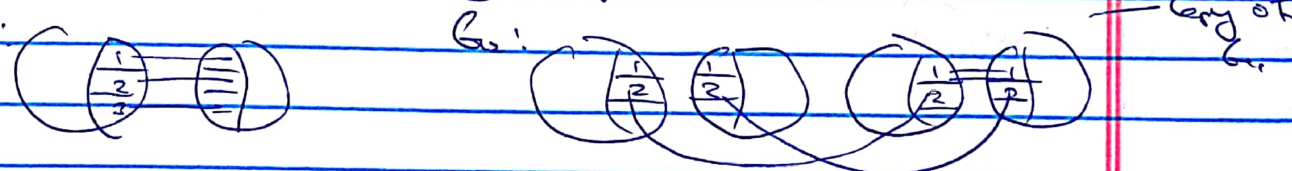
If $\Delta(G) = \delta(G)$, then G is $\Delta(G)$-regular, as desired.
So, we may assume $\delta(G) < \Delta(G)$.
Define:

$G' =$



Now $\delta(G') > \delta(G)$ and hence by IH, $\exists$ G ~~with~~ $\Delta(G)$-regular containing G' as an induced subgraph. $\square$

Note: Construction actually gives if G is triangle-free, then G' is triangle-free.
However, if G has girth ≥ 5, G' may have 4-cycles.
We may tweak the construction to preserve girth 5 (and girth 6, but not ≥ 7)
First, find a $(\Delta+1)$-coloring of G. Only add a matching b/w color i of min-deg. copies:

G_1: 

So, do $\Delta+1$ doublings to up min. deg. by 1. Start over...

For Reed-Sudakov, we need to regularize color-degree, not degree:

(1) If we pass to correspondence coloring, we can use the construction above, but we only add a conflict b/w min-deg. colors b/w copies of vertices

(2) If we're more careful with making lists for the copies, this should work for list-coloring as well.

Kim's Theorem for Girth Five Graphs:

Theorem (Kim, 1995)

If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\ln \Delta}$.

Recall: This is tight up to constant factor, in particular, in random d-regular graphs have $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$ with high probability, and so $\exists$ d-regular graphs of arbitrary girths and $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$.

(Some naive) Intuition:

- Coupon Collector Problem: Suppose there are n types of coupons and when you receive a coupon you receive a type u.a.r.

Q: How many coupons do you need to get to collect all the types? $\Theta(n \log n)$

(2)

Proof Sketch:

In the first n coupons,

$Pr[\text{Obtained} \geq 1 \text{ of coupon } i]$

$= 1 - Pr[0 \text{ of coupon } i]$

$= 1 - (1 - \frac{1}{n})^n$

$\approx 1/e.$

$E[\text{types collected after } n] \approx n(1 - \frac{1}{e})$

$E[\text{still to collect}] \approx n/e$

$E[\text{types still to collect after } n \cdot t \text{ coupons}] = n/e^t.$

Use concentration inequalities to make these concentration around expectation w.h.p. $\square$

So, we could think that:

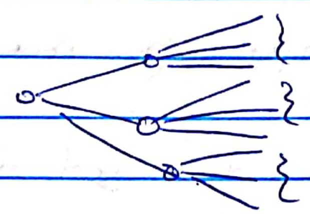
types of coupons = colors in my list $(n) = L$

coupon samples = colors neighbors receive $(n \ln n) = \Delta.$

(Solve to find $L \approx \frac{\Delta}{\ln \Delta}$).

Real coloring (randomly) is not uniformly random,

→ there's where the girth 5-ness can help since for every vertex we see:



All disjoint.

This is "somewhat uniformly random"

Proof (of Kim's)

We actually prove a stronger theorem (as Kim did):

If G has girth ≥ 5 and L is a list assignment of G such that $|L| \geq (1+o(1))\frac{\Delta_L(G)}{\ln \Delta_L(G)}$, then G has an L-colouring.

i.e. He proved this theorem for colour-degree.

We'll use Nibble and the Wasteful Colouring Procedure.

We may assume all colour degrees and list sizes are the same by Regularization / Equalizing Coin Flips.

We will be interested in tracking the ratio: $\frac{\Delta_L(G)}{|L|}$.

(Reciprocal of what we used for Reed-Sudakov)

(Note: This starts as $\approx \ln \Delta_L(G)$).

We will show that we can (after many steps) reduce to a graph with ratio $\leq \frac{1}{20}$ (so $|L| \geq 20\Delta_L(G)$), and then we apply Alon/Haxell to finish.

Expectations for one step of WCP:

$$\Pr[c \in L'(v) \mid c \in L(v)] = \left(1-\frac{p}{|L|}\right)^{\Delta(G)} =: \text{keep} \approx e^{-\frac{p\Delta}{|L|}}.$$

$$\mathbb{E}[|L'(v)|] = |L| \cdot \text{keep}.$$

$$\mathbb{E}[d_{L',G'}(v,c)] \leq d_{L,G}(v,c) \cdot \Big(\underbrace{\text{keep}}_{\text{keeping } c} \cdot \Big(\underbrace{\tfrac{1}{|L|}(1-p)}_{\text{if } c=\phi(u)\text{, where } v \in N_{L,G}(v,c)} + \underbrace{\left(1-\tfrac{1}{|L|}\right)(1-p\,\text{keep})}_{\text{keeping } \phi(u)\neq c}\Big)\Big)$$

where $\frac{1}{|L|}(1-p) + \left(1-\frac{1}{|L|}\right)(1-p\,\text{keep}) = (1-p\,\text{keep}) + \underbrace{\frac{1}{|L|}\,p(\text{keep}-1)}_{\leq 0}$

$$\leq \underbrace{d_{L,G}(v,c)}_{=\Delta_L(G)}\text{keep}\,(1-p\,\text{keep})$$

13

Remark:

In Reed-Sudakov, we threw out the keep in $E[d_{L,G}(v,c)]$ for analysis purposes to get the L/Δ ratio improving by keep/1-keep which was >1, since $|L| > \Delta$.
But, for Kim's, we'll need to keep the keep in $E[d_{L,G}(v,c)]$ which complicates concentration analysis, but everything will still concentrate b/c girth ≥ 5.

Concentrations:

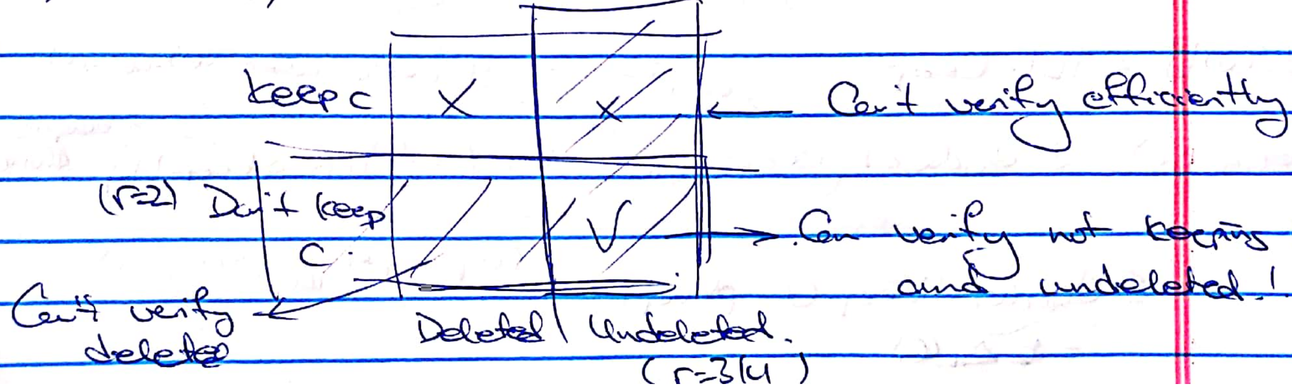
- Show $\Pr[|L'(v)| - \underbrace{E[|L'(v)|]}_{=\text{keep}\cdot|L|}] \leq \frac{1}{\Delta^{10}}$

or $|L| - |L'(v)|$

Proof same as for Reed-Sudakov, i.e. Use Talagrand's with $c=1$, $r=2$ (really 3 w/ activation flips)

- Show $\Pr[d_{L,G}(v,c) - E[d_{L,G}(v,c)]] \leq \frac{1}{\Delta^{10}}$.
There, we can use Talagrand's with $c=1$ (except for v itself, use exceptional Talagrand for v or other trick), and $r=$? (Is it possible to verify # of vertices undeleted and keeping c in dist)
↳ No, can't efficiently verify.

So, instead, concentrate other variables:



Jan 28th

Can verify all boxes indirectly here assuming all expectation of size of boxes is roughly same. ~~[illegible]~~ (i.e. The intersection shouldn't be small compared to the size of the cells)

Iterating the ratio:

Apply LLL to find:

$|L'| \approx |L| \cdot \text{keep}$

$\Delta' \approx \Delta \cdot \text{keep}(1-p\,\text{keep})$

So:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta \cdot \text{keep}(1-p\,\text{keep})}{|L| \cdot \text{keep}}$$

$$= \frac{\Delta}{|L|} - \frac{p\Delta}{|L|}\text{keep} \quad \left(\text{Recall: keep} \approx e^{-\frac{p\Delta}{|L|}}.\right)$$

So, let $k := \frac{p\Delta}{|L|}$, we get:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} - ke^{-k}$$, so for $k=1$, this gives $-1/e$.

Now run this forever to get the ratio ??

Can't do this since we'd run out of colors.

(Since $|L'| \approx |L| \cdot \text{keep}$ and $\text{keep} \approx \frac{1}{e}$, so we'd have to stop at about ~~[illegible]~~ $\ln\Delta$ steps)

Lecture 8:

Finishing the iteration calculations for Kim's

$|L'| = |L| \cdot keep$, where $keep \approx e^{-\frac{p\Delta}{|L|}}$

$\Delta' \approx \Delta \cdot keep(1-pkeep)$

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} \cdot \frac{keep(1-pkeep)}{keep} \qquad k = p\frac{\Delta}{|L|}$$

$$= \frac{\Delta}{|L|} - \frac{\Delta p}{|L|} keep = \frac{\Delta}{|L|} - ke^{-k}$$

How many iterations can we do (until we run out of colours?)

Let $|L_\tau|$ be the size of L after τ iterations:

$$|L_\tau| = |L_0| \cdot keep^\tau$$
$$\approx |L_0| \cdot e^{-k\tau}$$

We need $|L_\tau| \geq 1$. $\Rightarrow \ln|L_0| - k\tau \geq \ln 1 = 0$

$$\Rightarrow \tau \leq \frac{\ln|L_0|}{k}$$

Remark.

We actually stop when $|L_\tau| \geq$ some fixed constant to ensure that the necessary inequalities hold for the LLL in every iterative step. This only means that $\tau \leq \frac{\ln|L_0|}{k} - c$ for some constant c (so this is fine)

To finish the colouring, we need $\frac{\Delta_\tau}{|L_\tau|} \leq \frac{1}{2e}$ to apply Alon/Haxell.

Note: $\frac{\Delta_0}{|L_0|} \approx \frac{\Delta_\tau}{|L_\tau|} + ke^{-k}\tau$. So this works if

$$\frac{\Delta_0}{|L_0|} \lessapprox \frac{1}{2e} + ke^{-k}\left(\frac{\ln|L_0|}{k} - c\right)$$

$$= \text{constant} + e^{-k}\ln(L_0)$$

$\Leftrightarrow$

1

Scanned with CamScanner

i.e. $\Delta_0 \leq e^{-k}|L_0|\ln|L_0|$

$\Rightarrow |L_0| \approx \frac{e^k\Delta_0}{\ln(e^k\Delta_0)} \leq \frac{e^k\Delta_0}{\ln\Delta_0}$

Recall that $k = p\frac{\Delta}{|L|}$, so when $k=1$, $p=\frac{|L|}{\Delta}$ and $k\to 0$ means $p\to 0$.

So, if $k=1$, we get $|L_0| \approx e\frac{\Delta_0}{\ln\Delta_0}$,

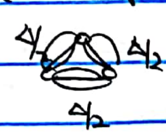
and if instead, we let $k\to 0$, we get $(1+o(1))\frac{\Delta_0}{\ln\Delta_0}$, as desired □

Remark: Why were colour degree necessary for the proof?
Note that in $\Delta' \approx \Delta\, \underline{keep}(1-p\,keep)$, the first keep only comes up in colour degree. And this keep helps cancel out some values and drive concentrations down.

Edge-Colouring:
Interested in properly colouring edges of graphs, i.e. ~~$\chi(G)$~~ $\chi(L(G))$ (the line graph)

Greedy Colouring: $\Delta(L(G)) \leq 2\Delta(G)-2$
$\Rightarrow \chi(L(G)) \leq 2\Delta(G)-1$
(of course $\omega(L(G)) = \Delta(G)$, if $\Delta(G)\geq 3$)

Theorem (Vizing, indep. Gupta, 1960s)
$\chi(L(G)) \leq \Delta(G)+1$.
NP-hard to decide if $\chi(L(G)) = \Delta$ or $\Delta+1$

Also, if G is a multigraph
$\chi(L(G)) \leq \Delta+\mu$ (where μ is the max. multiplicity of an edge)
$\leq \lceil \frac{3}{2}\Delta \rceil$ (Shannon) $\rightarrow$ tight for 

Goldberg - Seymour Conjecture! (Independently in late 70's)

$\chi(L(G)) \leq \max\{\Delta(G)+1, \chi_f(L(G))\}$

fractional chromatic #.

Actually, via Edmond's matching polytope:

$$\chi_f(L(G)) = \max_{\substack{H \subseteq G \\ v(H) \text{ odd}}} \frac{e(H)}{\left(\frac{v(H)-1}{2}\right)} \Rightarrow \text{ie Defin of Hall ratio for line graphs.}$$

Known Results:

- Kahn (90's) proved $\chi(L(G)) \leq \max\{\Delta+1, (1+o(1))\chi_f(L(G))\}$.
- Two groups in 2008: If $\chi_f(L(G)) \geq \Delta+\sqrt{\Delta}$, then $\chi(L(G)) = \chi_f(L(G))$.
- Another group in 2017+: —— " —— $\geq \Delta+\sqrt[3]{\Delta}$, —— " ——.
- Plantholdt, '11: $\chi(L(G)) \leq \chi_f(L(G)) + \log \chi_f(L(G))$.

List-Colouring Conjecture:

If G is a simple graph, then

$$\chi_\ell(L(G)) = \chi(L(G)).$$

(i.e. There is no such thing as list-edge-colouring)

Two big results on LCC from 90's:

Theorem (Galvin, 1995)

If G is bipartite, then $\chi_\ell(L(G)) = \chi(L(G))$.

→ Proved Dinitz's conjecture: If for every square in an $n\times n$ matrix, I give you a list of integers, can you complete the matrix so that all entries in a row (or col) are distinct

(Equivalently, $\chi_\ell(K_{n,n}) = n$).

→ Related to Latin Squares, since the completion of a Latin Square is simply an n-edge-colouring of $K_{n,n}$.

And these exist by König's theorem, if G is bip, $\chi_\ell(L(G)) = \Delta(G)$.

Theorem (Kahn, 1996)
If G is simple, $\chi_\ell(L(G)) = (1+o(1))\chi(L(G))$
i.e. $= (1+o(1))\Delta(G)$.

Molloy + Reid (Early 2000s) improved error to $\Delta + 4\sqrt{\Delta}\log^4\Delta$.
Kahn extended this to k-uniform linear hypergraphs
↳ (Hyperedges intersect in ≤1 vertex)

Proof Sketch (of Kahn's)
We'll use the Naive Colouring Procedure (i.e. Only remove colours from neighbours if retained by a vertex) and Nibble.
Also, since $|L|$ is on the order of Δ, we won't need activation probabilities (i.e. Set $p=1$)

We in fact prove a colour-degree version of Kahn's theorem as follows:
If L is a list assignment for $E(G)$ (equiv. $V(L(G))$), we define for $v \in V(G)$, a colour $c \in (E L(v)) \to L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$

$d_L(v,c) := |\underbrace{\{e \in E(G) : e \ni v,\ c \in L(e)\}}_{N_{L,G}(v,c)}|$

$d_L(v) := \max_c d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

Jan 30th

So, restating the thm:

Stronger Kahn's Theorem (Kahn, 1996)

If L is a list assignment for $E(G)$ s.t.

$$|L| \geq (1+o(1))\Delta_L(G)$$

then G has an (edge) L-coloring.

~~(Note: For notation, let $L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$).~~

Clearly, still interested in $|L|/\Delta_L(G)$

Expectations:

(Note: Assume all color degrees and list sizes are regular & via equalizing coin flips (since regularization of line graphs seems odd)).

Retain := Pr[an edge e is not in G'] ← i.e. It was deleted since no inc. edge received $\phi(e)$

$$= \left(1-\frac{1}{|L|}\right)^{d_L(e)} = \left(1-\frac{1}{|L|}\right)^{2\Delta-2} \approx e^{-2\frac{\Delta}{|L|}} \approx e^{-2}$$

Vertex-keep := Pr[$c \in L(u)$ is not retained by any edge around vertex v]

$$\approx 1 - \frac{1}{e^2}$$

Edge-keep := Pr[$c \in L(e^{uv})$ retained]

$\approx$ Pr[c not retained around u] $\times$ Pr[c —— " —— v]

b/c mostly independent $= (\text{Vertex-keep})^2$

3

Then,

$$\mathbb{E}[|L'(e)|] \approx |L|\left(1-\frac{1}{e^2}\right)^2.$$

Colour-Degree:

$$\mathbb{E}[|N^{edge}_{LG}(v,c) \cap E(G')|] = |N^{edge}_{LG}(v,c)| \cdot (1-\text{Retain})$$
$$= \Delta \cdot \left(1-\frac{1}{e^2}\right)$$

<u>But</u>, how many of these e <u>keep</u> $c \in L'(e)$?

It's not Edge-keep, ~~since~~ if we assume $c \in L'(v)$: b/c then by assumption ~~we~~ no edge around v retains c

Under this:

$\Pr[e \text{ keeps } c] = \Pr[\text{no edge} \longrightarrow] X.$

Lecture 9:

Molloy, Reed - Bal. Proof - Kahn.

Finishing Kahn's proof!

$\text{Retain}_i := \left(1 - \frac{1}{|L|}\right)^{2D-2} \approx e^{-2}$.

$\text{Vertex-keep}_i = 1 - \text{Retain} \approx 1 - \frac{1}{e^2}$.

$\text{Edge-keep} \approx \overset{\text{mostly independent}}{(\text{Vertex-keep})^2} \approx \left(1 - \frac{1}{e^2}\right)^2$ (Can argue that the dependence is small).

$E[|L'(e)|] \approx |L| \cdot \left(1 - \frac{1}{e^2}\right)^2$

$E[D_{L,G'}(v, c)] \approx |D| \left(1 - \frac{1}{e^2}\right)^2$

Approximates since there are small error terms.

Even assuming the errors from ~~expt~~ expectations won't hurt us too badly & that we concentrate these variables so as to have little error from concentration, unfortunately, at best these decrease at roughly the same rate - and so we won't make progress.

Need a new finish!

Reserve Colours!

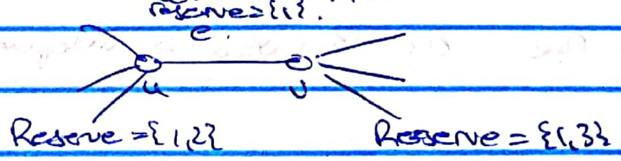
Idea: Before using nibble and naive colouring procedure, we reserve colors around vertices to be used by edges in a "final step".

Choose $\text{Reserve}_v \subseteq L(v) := \bigcup_{e \ni v} L(e)$ (uniformly at random w/ prob p)

For each edge $e = uv$, define

$L_e := L(e) - (\text{Reserve}_u \cup \text{Reserve}_v)$ → Lose colours from list (reserved)

$\text{Reserve}_e := \text{Reserve}_u \cap \text{Reserve}_v$ → Now e can use the overlapping colors safely



(1)

Lemma:

(Setting $p = \frac{\log^4\Delta}{\sqrt{\Delta}}$) $\exists$ a choice of $Reserve_v$ for $V(G)$ such that $\forall u \in V(G)$, $\forall e \in E(G)$, $\forall c \in Reserve_v$:

(a) $|L(e) \cap (Reserve_u \cup Reserve_v)| \leq 3\sqrt{\Delta}\log^4\Delta$ $\rightarrow L(e) - L_e$

(b) $|\underbrace{\cancel{L(e)}}_{L_e} \cap (Reserve_u \cap Reserve_v)| \geq \frac{1}{2}\log^8\Delta \rightarrow Reserve$

(c) $|\{e = uw \in N_{L_G}(v, c) : c \in Reserve_w\}| \leq 2\sqrt{\Delta}\log^4\Delta$ (*)

Intuition:

(a) You don't lose too many colors

(b) There is at least one color to use near the end

Proof:

$\mathbb{E}[|L(e) \cap (Reserve_u \cup Reserve_v)|]$

$\leq |L(e)| \cdot 2p \leftarrow$ Since may appear in either $Reserve_u$ OR $Reserve_v$.

$\leq \Delta \cdot 2\frac{\log^4\Delta}{\sqrt{\Delta}}$

$= 2\sqrt{\Delta}\log^4\Delta$.

$\mathbb{E}[|L(e) \cap (Reserve_u \cap Reserve_v)|]$

$\leq |L(e)| \cdot p^2 \leftarrow$ Needs to appear in both $Reserve_u$ and $Reserve_v$.

$\geq \Delta\frac{\log^8\Delta}{\Delta} = \log^8\Delta$.

$\mathbb{E}[(*)] = \Delta \cdot p \leq \sqrt{\Delta}\log^4\Delta$

Since each color $c \in Reserve_w$ is independent

You'll notice these variables are all sums of $\{0,1\}$-random variables (even more, all Bernoulli), so we can apply Chernoff Bounds:

$$\Pr[\,|L(e) \cap (\text{Reserve}_u \cup \text{Reserve}_v)| \geq 3/2\, \mathbb{E}[\ ]\,] \leq e^{-\frac{\Delta \log^3 \Delta}{3/2}}$$

$$\Pr[\quad (b) \quad] \leq e^{-\frac{(\log^8 \Delta)^2}{3/2}}$$

$$\Pr[\quad (c). \quad] \leq e^{-\frac{\Delta \log^3 \Delta}{3}}$$

So all at most $e^{-\log^8 \Delta}$ for large enough Δ which is $\leq \frac{1}{\Delta^c}$ for any c and large enough Δ.

So, we apply LLL with following bad events:

$A_e = (a) \geq 3\sqrt{\Delta} \log^4 \Delta$

$B_e = (b) \leq 1/2 \log^8 \Delta$

$C_{v,c} = (c) \geq 2\sqrt{\Delta} \log^4 \Delta.$

Probability for any bad event $\leq \frac{1}{\Delta^c}$ for any c, while each event is mutually independent of the set of events depending on edges/vertices at distance >4.

So: using $c=5$ (or 6) suffices for LLL $\square$

What is (c) saying?: If $c \in \text{Reserve}_v$, this counts $e \in N_{u,G}(v,c)$ with $c \in \text{Reserve}_e$

We define now the reserve degree of a vertex:

$$d_{res}(v,c) = |\{e \in N_{u,G}(v,c) : c \in \text{Reserve}_e\}|.$$

Idea for finishing w/ Reserved Colours:
Reserved degrees decreases rather quickly because edges in "reserved neighborhood" will be deleted from G, however Reserve never changes during Nibble.
So, if we can get $\max d_{res}(v,c) \leq \frac{1}{2}|Reserve| = \frac{1}{4}\log^{2}\Delta$, we can finish by applying ~~Haxell~~ Haxell to reserve colour assignment

---

Expectation of new reserved degree in one step of Nibble:

$\mathbb{E}[d_{res,L',G'}(v,c)]$

$= d_{res,L,G}(v,c) \cdot \underbrace{(1-\text{Retain})}_{\text{Probability an edge not deleted}}$

$\approx d_{res,L,G}(v,c) \cdot \left(1-\frac{1}{e^2}\right)$

Decreases at half the rate multiplicatively compared to $|L'|$ and Δ'

We can run Nibble for T iterations, where T is about the solution to the following:

$1 = |L| \cdot ((1-\text{Retain})^2)^T = |L|\underbrace{\left(\left(1-\frac{1}{e^2}\right)^2\right)}_{=k}^T$

$\Rightarrow T = \frac{\ln|L|}{-\ln k}$

(So a roughly logarithmic amount of steps)

Yet,

$$d_{res, L_i^*, G_i}(v,c) \approx d_{res,L,G}(v,c) \cdot \left(\left(1-\tfrac{1}{e^{\cdot}}\right)\right)^T$$
$$= d_{res,L,G}(v,c) \cdot K^{T/2}$$
$$\leq 2\sqrt{\Delta}\log^4\Delta \cdot \frac{1}{\sqrt{|L|}}$$
$$\leq 2\log^4\Delta$$
$$\leq \tfrac{1}{4}\log^8\Delta \quad \text{for large enough } \Delta$$

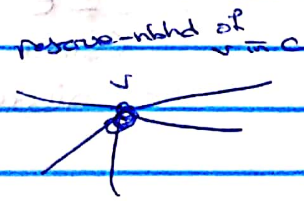
(So we don't have to run too many iterations for the reserved degree to be small enough)

<u>Remark</u>: We actually want to use $T = \frac{\ln|L|}{-\ln|K|} - \text{poly}\log\log\Delta$ since we need to concentrate d_{res} during while enough to apply LLL, so need $d_{res} \geq \text{poly}\log\Delta$

So, this gives an L-coloring if $|L| \geq \Delta + 4(\log^4\Delta)\sqrt{\Delta}$.
(Since we need $3\sqrt{\Delta}\log^4\Delta$ for reserves and an extra $\sqrt{\Delta}\,\text{poly}\log\Delta$ for concentration errors)

<u>Concentration</u> for $|L'|$, Δ', d'_{res}:

We'll do <u>d'_{res}</u> first!
d'_{res} counts # edges from d_{res} <u>not</u> deleted.



(3)

c-Lipschitz for c = ______ :

- If we change coloring on an edge not incident with v, this changes ≤ 2 of these edges
- If we change coloring incident with v, this can change at most itself and 1 other edge (they both lose the color). Otherwise, ~~there~~ if there are ≥ 3 edges colored the same color, then changing v won't matter.

Note: We can't use Chernoff b/c not sum of independent ——"—— Simple conc. bound b/c may depend on Δ^2 trials (=> Bad Bound)

So: we'll use Talagrand's and we need to verify this variable

Note: Verifying ~~retaining~~ a color requires 2Δ trials.

But we can verify not retaining using $r=2$ by showing its color $\phi(e)$ and some edge f near w/ $\gamma(f)=\phi(e)$.
So, we can apply Talagrand to show this is within about $\sqrt{\mathbb{E}[\deg]}$ with probability $\underbrace{e^{-C\mathbb{E}[\deg]}}_{\leq e^{-c\log^8\Delta}}$ (C constant)

(since $\deg \geq \log^8\Delta$ during nibble)

Concentrating Δ':

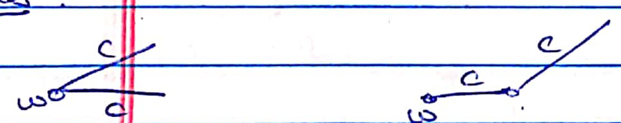
Assume c is not retained around v, this counts # of edges in $X_{L,G}(v,c)$ which are not deleted and the other end of the edge keeps c.

Again, we are 2-Lipschitz, since changing a single color can affect at most 1 other vertex. (If we have ≥3, then changing the color won't matter) (Same ~~argu~~ argument as before)

As before, not deleted is 2-verifiable, so we'd be happy if "other end keeps c" or its complement were r-verifiable for constant r since the expectation of intersection of these are on the same order as Δ.
(Note: As before, verifying edge around w retains c takes around 2Δ trials)

What about verifying the complement:
i.e. # of edges e where other end w does not have an edge retaining c?

Cases:



if 2 colored c, this is easy, show 2

To verify this one, we need to check all neighbors!

Trick: Change Variables
Define
~~X_{i}~~ $X_{j,k}$ to be # of ~~variables~~ edges e where e and other end w
$\geq j$ are assigned color c
$\geq k$ have color c subsequently removed.

So we can write the variable above (the component count) as: $X_{1,0} - \underbrace{(X_{1,1} - X_{2,1})}_{\text{counts receive exactly 1 and remove} \geq 1}$ ~~...~~.

Easy to check that $X_{j,k}$ concentrates w/ 2-Lipschitz and $(j+k)$-verifiable and we can use Talagrand's on each. This works since they have roughly same expectation and j, k constant.

For IL'1, we use the same trick, but ~~...~~ counting both ends of e.

7409 - Colouring Feb 6th

Lecture 10:

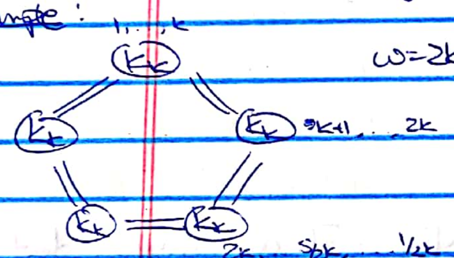
Reed's Conjecture:

Recall: Conjecture (Reed, 1998): If G is a graph, then
$$\chi(G) \geq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil.$$

Remark: If true, this is tight for some ω.

↓ Example of this.

Example:



$\omega = 2k$, $\Delta = 3k-1$ $\Rightarrow$ $\chi = \lceil \frac{5}{2}k \rceil$.

Why is this true?!

- $\exists$ a $\lceil 5/2 k \rceil$-colouring $\Rightarrow$ See labels
- We can also see this since $\alpha \leq 2$, and so the 5k colours we need can be cut in half

Reed's Conjecture has been proved for various special cases of graphs, e.g.: line graphs, claw-free graphs, etc.

Theorem (Reed, 1998)
If $\omega(G) \geq (1 - \frac{1}{10^?})(\Delta(G)+1)$ then the conjecture holds
→ So if clique # is large, then the conjecture is true.

Corollary:
$\exists\, \varepsilon > 0$ s.t. $\chi(G) \leq \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\,\omega(G) \rceil$.
→ So if we're not satisfied w/ the half we could try some convex combination.

What is ε?.

- $\varepsilon = \frac{1}{2\cdot 10^8}$ (Reed '98).
- $\varepsilon = \frac{1}{320e^6}$ (King + Reed '12)
- $\varepsilon = \frac{1}{26}$ (Bonamy, Perrett, P. '16+)
- $\varepsilon = \frac{1}{13}$ (Delcourt, P. '17+).

How do we prove these bounds?:

Idea! Combine Randomness & Structure

3-Part-Plan (King & Reed)

(1) Do something different for large ω:

(a) $\omega \geq \left(1 - \frac{1}{10^8}\right)(\Delta(G)+1)$, use Reed.

(b) King showed that $\omega \geq \frac{2}{3}(\Delta+1)$, then $\exists$ an independent set I s.t. $\omega(G-I) < \omega(G)$, and

$$\Delta(G-I) < \Delta(G)$$

Now $G-I$ has a smaller ratio of ω/Δ, repeat until $\omega' < \frac{2}{3}(\Delta'+1)$, and now we use the I's to colour

(c) Use "harder" randomness + structure (Delcourt, P.)

(2) Now, assuming a small ω. Prove that a critical graph has neighborhoods that are "somewhat" sparse

Structure

(3) Now, use random colouring + Nibble.

Randomness

Feb 6th

Formalizing (2) and (3):

Def'n

A graph G is σ-sparse if $\forall v \in V(G)$ we have that
$$e(G[N(v)]) \le (1-\sigma)\binom{\Delta}{2}.$$

Remark: Note the use of Δ instead of $d(v)$, i.e. small degree vertices trivially satisfy this condition

For (2):

Def'n $\longrightarrow$ For list colourings: $\mathrm{Save}_\ell(G) := \Delta(G)+1-|L|$

$\mathrm{Save}(G) := \Delta(G)+1-\chi(G)$

$\mathrm{Gap}(G) := \Delta(G)+1-\omega(G)$

Reed's conjecture: $\mathrm{Save}(G) \ge \left\lfloor \frac{\mathrm{Gap}(G)}{2} \right\rfloor$

Our Result: $\mathrm{Save}(G) \ge \varepsilon\,\mathrm{Gap}(G)$ ($\varepsilon = 1/3$ say).

Theorem (Delcourt, P.)

If G is L-critical, then G is σ-sparse where ~~$\sigma = \frac{\mathrm{Gap}(G)}{2\Delta} - \frac{2\,\mathrm{Save}(G)}{\Delta}$~~.

$$\sigma = \frac{\mathrm{Gap}}{2\Delta} - \frac{2\,\mathrm{Save}_\ell}{\Delta}$$

What good is sparsity?

Theorem

If G is σ-sparse and Δ is large enough, then ~~$\exists$~~

$\chi(G) \le (1-f(\sigma))(\Delta+1)$, where $f(\sigma) = \frac{1}{2e^6}\sigma$ $\leftarrow$ Molloy & Reed '02

$\approx .1827\sigma - 0.78\sigma^{3/2}$

(Bruhn & Joos '18)

$\approx .3\sigma - .25\sigma^{3/2}$

(Bonamy, Perrett, P., '16+). 2

How to get $\varepsilon = 1/13$?

We know:

$$\begin{aligned} \text{Save}(G) &\geq (.3\sigma - .125\sigma^{3/2})\Delta \\ &\gtrsim \tfrac{2}{9}\sigma\Delta \quad (\text{for } \Delta \geq 1/4) \\ &\geq \tfrac{2}{9}\Delta\left(\tfrac{\text{Gap}(G)}{2\Delta} - \tfrac{2\,\text{Save}_2(G)}{\Delta}\right) \quad \Leftarrow (\text{By thm}) \\ &= \tfrac{\text{Gap}}{9} - \tfrac{4}{9}\text{Save} \end{aligned}$$

$\Rightarrow$ Save $\geq \frac{1}{13}$ Gap.

Theorem (Erdős, Rubin, Taylor '79) $\longrightarrow$ Equivalently, show $\chi(K_n - M) = n - e(M)$

If M is a matching in G, then $\chi_\ell(G) \leq v(G) - e(M)$

Note:

- Obviously[?], $\chi(G) \leq v(G) - e(M)$ (color each edge of matching with same color)

Interesting part is it works for lists!

Proof:

By Induction on n. Let L be a list assign for G with $|L| = n - e(M)$

Case I: $\exists e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \emptyset$. Now let $c \in L(u) \cap L(v)$. Colour u, v w/ c. Remove c from the other lists; delete u, v.
Let $G' = G \setminus \{u, v\}$, $G' = K_{n-2} - (\underbrace{M - e}_{M'})$, while $|L'| \geq |L| - 1$
$= v(G') - e(M')$

By induction, $\exists L'$ colouring, done

G

Proof (con't)

Case 2:

$\nexists e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \emptyset$.

We consider an ~~aux~~ auxiliary bipartite graph H where:

$V(H) = (\underset{\text{vertices}}{V(G)}, \underset{\text{colors}}{\bigcup_{u \in V(G)} L(u)})$, and

$E(H) = \{(v, c) : c \in L(v)\}$

Claim: $\exists$ a matching of H saturating $V(G)$, equivalently an L-coloring of G where every vertex receives a unique color

Let $S \subseteq V(G)$

- If $\exists e = uv \in M$ s.t. $u, v \in S$, then $N_H(S) \geq |L(u) \cup L(v)|$

By (b) $\overset{\text{as}}{=} |L(u)| + |L(v)| \geq 2(n - e(M)) \geq n \geq |S|$

- If $\nexists$ such $e \Rightarrow |S| \leq n - e(M)$, so if $S \neq \emptyset$, then $\exists v \in S$ and $N_H(S) \geq |L(v)| \geq n - e(M) \geq |S|$ if $S = \emptyset$

$\square$

Theorem (Delcourt, P.)

If M is a matching in $\overline{G}$ and L is a list assignment for G s.t.:

(a) $|L(v)| \geq e(M)$ $\quad \forall v \in V(M)$

(b) $|L(a)| + |L(b)| \geq v(G)$ $\quad \forall a, b \in E(M)$

(c) $|L(v)| \geq v(G) - e(M)$ $\quad \forall v \in V(G) \setminus V(M)$

3

Proof:

Same as ~~[illegible]~~ proof before, except cases for Hall's.

Cases:

(1) $\exists e = uv \in H$ s.t. $u, v \in S$ (by (b))

(2) $\exists v \in V(G) \setminus V(H)$ s.t. $v \in S$ (by (c))

(3) $\omega \Rightarrow |S| \leq e(H)$ (by (a))

wat?? ??

$\square$

Theorem:

Let G be a graph, H be an induced subgraph of G, $s \geq 0$, Suppose that L is a list-assignment s.t. $|L(v)| \geq d(v) - s$ $\forall v \in V(H)$. If G is L-critical, then

$$e(H) \geq (m-s)(v(H) - m - s)$$

where

$m = \nu(\bar{H})$ (i.e. size of largest antimatching in H)

Obs: If G is a graph,

$$\nu(\bar{G}) \geq \frac{v(G) - \omega(G)}{2}$$ (follows from greedily removing antiedges)

Proof:

By induction on $v(H)$. Let M be a matching of $\bar{H}$ of size m.

If $m \le s$: trivial

So, assume $v(H) > m > s \ge 0$

Since G is L-crit., $\exists$ L-colouring ϕ of $G - V(H)$.

Let $L'(u) = L(u) \setminus \{\phi(w) : w \in N(u) \setminus V(H)\}$ $\forall u \in V(H)$

Note $\forall u \in V(H)$,

$$|L'(u)| \ge |L(u)| - |N(u) \setminus V(H)|$$
$$\ge d(u) - s - |N(u) \setminus V(H)|$$
$$= d_H(u) - s$$

Since G is not L-col, H is not L'-col. So, by thm (Delcourt), $\exists$ at least one of (a)-(c) does not hold when applied to H, L', M:

Say (a) does not hold:

$\exists v \in V(M)$ s.t. $\underset{d_H(v)-s}{\underset{\shortparallel}{|L'(v)|}} \le \overset{=m}{e(M)} - 1$

So: $d_H(v) \le m - 1 + s$

Let $H' = H - \{v\}$, $v(\bar{H}')$. So, by induction on H',

$$e(\bar{H}') \ge (m-1-s)(v(H)-1-(m-1)-s)$$
$$e(\bar{H}) \ge e(\bar{H}') + (v(H) - (d_H(v)+1))$$
$$\ge e(\bar{H}') + v(H) - m - s$$
$$= (m-s)(v(H)-m-s)$$

4

Proof: (con't)

(b) does not hold:

$\exists ab \in E(H)$ s.t. $|L'(a)| + |L'(b)| \le v(H) - 1$

$\Rightarrow d_H(a) + d_H(b) \le v(H) - 1 + 2s$

$H' = H \setminus \{a,b\}$, $v(H') \ge m-1$

By induction:

$$e(\bar{H}') \ge (m-1-s)(v(H)-2-(m-1)-s)$$

$$= v(H) - m - s - 1$$

$$= (m-s)(v(H)-m-s) + (-v(H) + 2)$$

$$e(\bar{H}) \ge e(\bar{H}') + 1 + (v(H) - 2 - d_H(a)) + (v(H) - 2 - d_H(b))$$

... (work it out... magic happens)

(c) does not hold:

$\exists v \in V(H) \setminus V(M)$ s.t. $|L'(v)| \le v(H) - m - 1$

$\Rightarrow d_H(v) \le v(H) - m - 1 + s.$

$H' = H - v$, $v(\bar{H}') \ge m.$

By induction:

$$e(\bar{H}') \ge (m-s)(v(H) - 1 - m - s)$$

$$= (m-s)(v(H) - m - s) - (m-s)$$

$$e(\bar{H}) \ge e(\bar{H}') + (v(H) - (d_H(v) + 1))$$

$$= v(H) - m + s.$$

... can't work it out

QED.

Proving Density theorem.

Apply the previous thm to $H = N[v]$ (closed neighborhood)

$$\Rightarrow e(G[\overline{N(v)}]) \geq (m-s)(v(H) - m - s)$$

where

$$s = \mathrm{Save}(G) - 1, \text{ and}$$

$$m = v(\overline{N[v]}) \geq \frac{\Delta + 1 - \omega(G)}{2} = \frac{\mathrm{Gap}(G)}{2}, \text{ and}$$

$$v(H) = \Delta + 1$$

$$\approx \left(\frac{\mathrm{Gap}}{2} - \mathrm{Save}\right)\left(\underbrace{\Delta + 1 - \frac{\mathrm{Gap}}{2}}_{\geq \frac{\Delta+1}{2}} - \mathrm{Save}\right)$$

$$\geq \frac{\mathrm{Gap} \cdot \Delta}{4} - \mathrm{Save}\,\Delta.$$

and since

$$e(\overline{H}) = \sigma \binom{\Delta}{2} \approx \sigma \frac{\Delta^2}{2},$$

So:

$$\sigma = \frac{2}{\Delta^2}\left(\frac{\mathrm{Gap} \cdot \Delta}{4} - \mathrm{Save} - \Delta\right)$$

$$= \frac{\mathrm{Gap}}{2\Delta} - \frac{2\,\mathrm{Save}}{\Delta}.$$

7409 - Coloring

Feb 11th

Lecture 11:

Sparsity Lemma:

If G is a σ-sparse graph for $\sigma \geq \frac{\text{polylog}\,\Delta}{\Delta}$ and $\Delta(G)$ sufficiently large, then:

$$\chi(G) \leq (1 - f(\sigma))(\Delta+1)$$

where:

$f(\sigma) \approx .023\sigma$ Molloy & Reed $\rightarrow$ Naive Coloring Procedure

$.18\sigma - .07\sigma^{3/2}$ Bruhn & Joos

$.3\sigma - .12\sigma^{3/2}$ Bonamy, Perrett, P.

New Ideas in Bruhn & Joos:

1) Naive is a bit "too naive", i.e. It's a bit wasteful to uncolor both vertices in a conflict

Idea: For every edge, we ~~independently~~ ~~flip a coin~~ direct the edge and independently flip a coin before the random coloring. We only uncolor the "head" of the edge in a conflict if coin flip is heads and similarly for tails.

2) Analyze the # of repeated colors in $N(v)$

(i.e. $\underbrace{|N(v) \setminus V(G')|}_{\substack{\text{"\# of neighbors (of } v) \\ \text{deleted": i.e. retained color} \\ \text{and not in } G'}} - \underbrace{(|L(v)| - |L'(v)|)}_{\text{"colors lost"}}$)

Via the # of pairs, triples, etc. of repeated colors.

Molloy + Reed observed that if this value is $\geq \Delta + 1 - |L|$, then we can greedily color G'.

Need to figure out $\mathbb{E}[\text{"this value"}]$, argue its concentrated and apply LLL.
let's call it the "savings of v"

Pairs + Triples:

$\text{Pairs}_{v}^{x} := \{ (u,w) \in \binom{N(v)}{2} : uw \notin E(G),\ \phi(u) = \phi(w) \text{ and } u,w \notin V(G') \}$

$\text{Triples}_{v}^{x} := \{ (u,v,w) \in \binom{N(v)}{3} : uv, vw, uw \notin E(G),\ \phi(u) = \phi(v) = \phi(w), \text{ and } u,v,w \notin V(G') \}$

Claim: $\text{Savings}_v \geq \text{Pairs}_v - \text{Triples}_v$.

Expectation of Pairs, Triples:

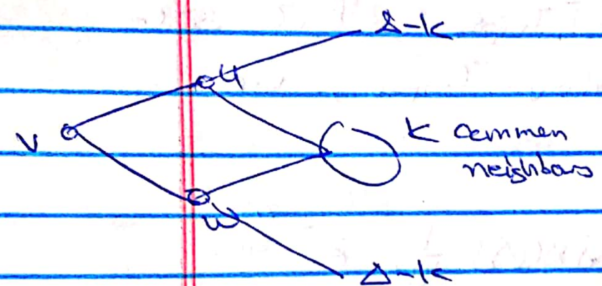
Let's assume G is Δ-regular:

$\Pr[\underbrace{v \notin V(G')}_{\text{i.e. Retains color}}] = \left(1 - \frac{1}{2 \cdot |L|}\right)^{\Delta} \approx e^{-\frac{\Delta}{2|L|}}$

Need to lose coin flip to lose color

$\mathbb{E}[\text{Pairs}_v] = \#\text{ of non-edges in } v \cdot \Pr[\text{Pairs}_v^x]$

$\Pr[\phi(u) = \phi(w)] = \frac{1}{|L|}$; $\Pr[u,w \notin V(G')]$ $(\cdot\ C_\Delta)$



$$\Pr[u,w \in V(G')] = \left(1 - \frac{1}{2|L|}\right)^{2(\Delta-k)} \left(1 - \frac{3}{4|L|}\right)^{k}$$

Neighbors not in common (Bad event is losing coin flip)

Neighbors in common (Bad event is either u or w lose coin flip).

$$= \left(1 - \frac{1}{2|L|}\right)^{2\Delta} \cdot \frac{\left(1 - \frac{3}{4|L|}\right)^{k}}{\left(1 - \frac{1}{2|L|}\right)^{2k}}$$

Can argue that $\left(1 - \frac{3}{4|L|}\right) \geq \left(1 - \frac{1}{2|L|}\right)^2$ for $|L|$ large enough, and so:

$$E[\text{Pairs}_v] \geq \underbrace{\#\text{ of non-edges of } v}_{} \cdot \left(1 - \frac{1}{2|L|}\right)^{2\Delta}$$

$$\geq \varepsilon \binom{\Delta}{2}$$

$$\approx e^{-\Delta/|L|}.$$

2

The calculation is similar for $\mathbb{E}[\text{Trips}_v]$, BUT since we want to argue $\text{Sewings}_v \geq \text{Pairs}_v - \text{Trips}_v$, we want an upper bound.

$$\mathbb{E}[\text{Trips}_v] \geq \underbrace{\#\text{ of triangles in } G[N(v)]}_{\text{But what is this?}} \cdot \frac{1}{|L|^2} \cdot e^{-\frac{7\Delta}{8|L|}}.$$

Theorem (Rödl, 2002)
If G is a graph w/ $\sigma\binom{V(G)}{2}$ edges, then G has at most $\sigma^{3/2}\binom{V(G)}{3}$ triangles.

So: $\mathbb{E}[\text{Pairs}_v - \text{Trips}_v]$

$$\geq \sigma\binom{\Delta}{2} \cdot \frac{1}{|L|} \cdot e^{-\frac{\Delta}{|L|}} - \sigma^{3/2}\binom{\Delta}{3}\frac{1}{|L|^2}e^{-\frac{7\Delta}{8|L|}}.$$

(where $\sigma\binom{\Delta}{2} = \#$ of nonedges in $N(v)$).

$$\approx \frac{\sigma}{2e} - \sigma^{3/2} \cdot \frac{1}{6e^{7/8}}$$, ~~which~~ ~~are~~ ~~the~~ ~~#'s~~ ~~we~~ ~~wanted~~ which, if worked out, will match the #'s ~~Pluralkar~~ we wanted (from Bruhn & Joos)

Then, use Exceptional Talagrand's and difference of variables to concentrate

New Idea in Bonamy, Perrett, P.:
- Why stop after one iteration?
Keep using nice procedure, i.e. use Nibble!
Issue: Only works if uncolored subgraph is still "somewhat sparse"
Idea: Uncolored subgraph will be "pseudorandom" subgraph of G

Defn: If G is a graph, we say a subgraph H of G is a μ-pseudorandom subgraph of G if $\forall u,v \in V(G)$,
$$\left| |N_G(u) \cap N_G(v) \cap V(H)| - \underbrace{\mu}_{\text{expected probability of going in } V(H)} |N_G(u) \cap N_G(v)| \right| \le 10\sqrt{\Delta}\log^4\Delta.$$

So, if we can control this #, then:

Lemma: $\forall \sigma > \sigma' \sim$ and $\mu > 0$, if $\Delta(G)$ is large enough, and H is a μ-pseudorandom subgraph of G, and G is σ-sparse, then H is σ'-sparse

So, how does this give a savings of $(.3\sigma^{2} - .20^{3/2})\Delta$?
Savings for max deg Δ graph (from Bruhn & Joos) $\approx (.18\sigma - .07\sigma^{3/2})\Delta$
Savings in 2nd step: $\approx (.18\sigma' - .07\sigma'^{3/2})\Delta'$
⋮
—"— tth " : $\approx (.18\sigma_t - .07\sigma_t)\Delta_t$.
What is Δ' (and Δ_t)?

What is $\Delta(G')$?

$E[|N(u) \cap V(G')|]$

$= \underbrace{|N(u)|}_{=\Delta} \cdot (1 - \underbrace{Pr[\exists \in N(u) ; u \notin V(G')]}_{\approx e^{-\Delta/2\Delta}})$ (we've done this b/4)

$= \Delta(1 - e^{-\Delta/2\Delta})$.

We can concentrate this (using LLL) to get:

$\Delta_t \approx \Delta_0 (1 - e^{-\Delta/2\Delta})^t$

So:

Total Savings $= \sum_t$ Savings at each step

$= (.18\sigma - .17\sigma^{3/2}) \cdot \sum_{t \geq 0} \Delta_t$

$= \quad " \quad \cdot \underbrace{\sum \Delta_0 (1 - e^{-\Delta/2\Delta})^t}$

$= \Delta_0 \cdot e^{\frac{\Delta}{2\Delta}} \approx \sqrt{e} \approx 1.6$

Feb 11th

Johansson / Molloy / Bernshteyn Theorems:

Theorem (Johansson)
If G is triangle-free then
$\chi_\ell(G) \le O\left(\frac{\Delta}{\ln \Delta}\right)$

Molloy ('12) $(1+o(1))\frac{\Delta}{\ln \Delta}$

Bernshteyn ('18) True for DP-coloring

Bernshteyn: uses LLL

Molloy: uses entropy compression

Theorem (Johansson)
If G is K_r-free for fixed r
$\chi_\ell(G) \le O\left(\frac{\Delta \ln \ln \Delta}{\ln \Delta}\right)$.

Molloy: $\chi_\ell(G) \le 200 r \frac{\Delta \ln \ln \Delta}{\ln \Delta}$ (r no longer fixed)

Bernshteyn: True for DP-coloring

In fact, Molloy/Bernshteyn gives a more general result framework:
(Bonamy, Kelly, Nelson, Postle '18+)
$\chi_\ell(G) \le 200 \Delta \sqrt{\frac{\ln \omega(G)}{\ln \Delta}}$
(even $\chi_{DP}(G)$)

In spirit this answers the following question:

What assumption do we need on $\omega(G)$ to guarantee that:

$$\chi(G) \le \frac{\Delta}{c} \qquad (c \ge 2)$$

Using BKNP: $\omega \le \Delta^{\frac{1}{200c^2}}$

This is almost tight:

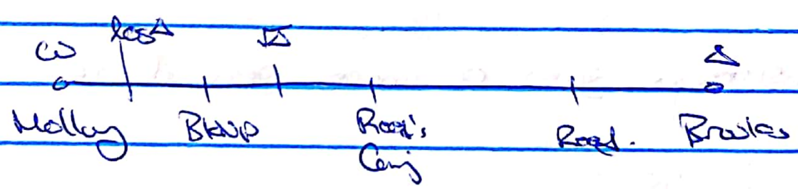
Ramsey Theory $\Rightarrow$ Need $\omega \le {}^{\frac{1}{c-1}}$.

Conjecture: linear in c is correct

implied by the following conjecture:

$$\chi_f(G) \le 200\Delta \cdot \frac{\ln \omega}{\ln \Delta}$$

This would give the complete picture:



Feb 13th

Lecture 12:

Localized Coloring Theorems: → Interpolates standard thms in coloring

1) Local List Coloring → localize how many colors are available (i.e. $|L(v)|$)
2) Local Fractional Coloring
   → Localize how much color a vertex recieves. (its demand $f(v)$)
   → Has connections with independent set bounds.

Survey of ~~recent~~ results.
- Can localize # of colors → $|L(v)|$
- Can localize maximum degree (Δ) → $d(v)$
- " clique # → $\omega(v) := \omega(G[N(v)])$
  (size of largest clique containing v)

Ranging over $\omega(v)$ vs. $d(v)$

Greedy: If $|L(v)| \geq d(v)+1$ $\forall v$, then G has an L-coloring

Local List Brooks: If $|L(v)| \geq d(v)$, then G has an L-col
(ERT '79, Borodin '78 ...) unless G ~~has~~ contains ~~an~~ a component where every block is a clique or odd cycle

Local List Reed's Conj:

If $|L(v)| \geq \left\lceil \frac{d(v)+1+\omega(v)}{2} \right\rceil$ $\forall v \in V(G)$,

then G has an L-coloring

Theorem (Kelly, Postle)

$\exists \varepsilon > 0$ ($\varepsilon \approx \frac{1}{1000}$), if $|L(v)| \geq \lceil (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v) \rceil$ $\forall v$ and $\delta(G) \geq \text{polylog}\,\Delta(G)$, then G has an L-coloring

On "edge-colouring" locations

Recall: Thm (Galvin) If G is bipartite, $\chi_\ell'(G) = \chi'(G)$.

Local List Galvin: (Borodin–Kostochka–Woodall '99)

If G is bipartite and $|L(e)| \geq \max\{d(u), d(v)\}$, then G has an L-coloring. ($e = uv$)

(Note there is no restriction on degrees)

Local list Kahn's (Bonamy, Delcourt, Lang, Postle, 20+)

If $|L(e)| \geq (1+o(1)) \max\{d(u), d(v)\}$ and $\delta(G) \geq \text{polylog}\,\Delta$ then G has an L-coloring.

Local Reed–Sudakov

Theorem (Abbasi, Kim, Postle)

$\exists\, c > 0$ such that if $|L(v)| \geq c \cdot d_\ell(v)$ $\forall v$ and $\delta(G) > \text{polylog}\,\Delta$, then G has an L-col.

Feb 13th

Local Fractional Coloring!

Recall:

Demand function $f(v)$ $f: V(G) \to [0,1] \cap \mathbb{Q}$.

"How much color they demand"

Proposition: (Dvořák, Sereni, Volec)

Let G be a graph with demand function f. TFAE:

(a) G has an f-coloring ($\phi: V(G) \to$ measurable subset of $[0,1]$ s.t. $\mu(\phi(v)) = f(v)$ $\forall v$ and $\phi(u) \cap \phi(v) = \emptyset$ $\forall e = uv \in E(G)$)

(b) There exist a common denominator N for f s.t. G has a (f,N)-coloring (i.e. $\phi(v) \subseteq [N]$ where $|\phi(v)| = f(v) \cdot N$ and $\phi(v) \cap \phi(u) = \emptyset$ $\forall e$).

(c) $\exists$ probability distribution on independent sets of G s.t. $\Pr[v \in I] \geq f(v)$ $\forall v$

(d) The vector of demands $(f(v) : v \in V(G))$ is in the stable set polytope of G

(e) $\forall$ nonnegative weight function $w: V(G) \to \mathbb{R}_+$, the graph G contains an independent set I s.t. $\sum_{v \in I} w(v) \geq \sum_{v \in V(G)} w(v) f(v)$.

Remark! $\chi_f(G) := \min\{k : \exists f\text{-col of } G \text{ w/ } f(v) = 1/k\}$.

Also,

(e) $\Rightarrow$ $\alpha(G) \geq \sum_{v \in V(G)} f(v)$. $\left(\geq \frac{V(G)}{\chi_f(G)}\right)$.

②

Results for local frac. colouring:

Local frac greedy ? $\to$ $f(v) = \frac{1}{d(v)+1}$?

$\to$ Does there exist such an analogue $\}$ would be the most natural analogue

Theorem (Caro–Wei, '91) — Independently by

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$$

In fact, the proof gives an f-col where $f(v) = \frac{1}{d(v)+1}$.

Corollary: (Turán)

$\alpha(G) \geq \frac{v(G)}{\underbrace{\mathrm{avg}(d(v))}_{\text{average degree of } v}+1}$ $\to$ By convexity of $\frac{1}{x+1}$

---

3 proofs of Caro–Wei:

Proof I: (Min-deg proof)

Let G be a minimum counterexample. Let $v \in V(G)$ be of min. deg; since G is a min. counterexample, $G-v$ has an f-col ϕ.

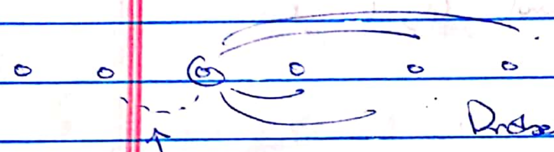
Note that $\forall u \in N(v)$, $\mu(\phi(u)) = f(u) = \frac{1}{d(u)+1} \leq \frac{1}{d(v)+1}$.

So, v "sees" at most $\frac{d(v)}{d(v)+1}$ colours and so $\mu(L_\phi(v)) \geq 1 - \frac{d(v)}{d(v)+1}$ $= \frac{1}{d(v)+1} = f(v)$

Proof #2: (Probabalistic Proof)

Choose a total ordering $<$ of $V(G)$ u.a.r. Let $v \in I$ if $v < u$ $\forall u \in N(v)$.

$$\Pr[v \in I] = \frac{1}{d(v)+1}$$



Probability v only has forward edges is $\frac{1}{d(v)+1}$.

No back edges.

Proof #3: (Max degree proof)

Let G be a min. counterexample. Let $v \in V(G)$ s.t. $f(v)$ is minimum. Let $\Phi(v) \subseteq [0,1]$ be a set of measure at least $f(v)$. Let f' be a demand function for $G-v$.

s.t.

$$f'(u) = \frac{1}{d_{G-v}(u)+1}$$

(Idea: Neighbors of v are overcolored)

Since, $G-v$ has an f'-coloring

G is min counterexample,

Note $\forall u \in N(v)$,

$$f'(u)\,\mu(L_\Phi(v)) = \frac{1-f(v)}{d_{G-v}(u)+1} \geq \frac{1-\frac{1}{d_G(u)+1}}{d_G(u)} = \frac{1}{d_G(u)+1} \geq f(u).$$

By lemma, G has a f-col

$\hookrightarrow$ See back.

Lemma

Let G be a graph with demand function f, fractional list assignment L and $g: V(G) \to \mathbb{R}$. If $g(v) \le f(v)\mu(L(v))$ $\forall v$ and $\forall S \subseteq V(G)$ s.t. $\mu(\bigcap_{v \in S} L(v)) > 0$ the graph $G[S]$ has an f-col, then G has a fractional (g, L)-col.

Frac list-assign: $L(v) \subseteq \mathbb{R}$

Frac (g, L)-col: $\phi(v) \subseteq L(v)$, $\mu(\phi(v)) \geq g(v)$

Lecture 13:

Recall:

Thm (Caro-Wei)

If G is a graph, then G has an f-coloring where $f(v) = \frac{1}{d(v)+1}$ $\forall v$

3 proofs:

1) Min degree - Delete & Extend

2) Probabilistic

3) Max degree - Delete, "Selectively Reduce", Extend

Q: Are there local fractional versions of other coloring theorems?

A local fractional Brooks's Theorem:

Multiple Viewpoints:

- Under what conditions can you get an f-col w/ $\frac{1}{d(v)}$?
  (Or, weaker version $\frac{1}{d(v)+\varepsilon}$ for some $\varepsilon \in (0,1)$)
- Caro-Wei is tight for disjoint union of cliques. What if we "forbid this"? (What can we save?)

  Idea #1: Control local clique #'s, i.e. $\omega(v) = \omega(G[N(v)])$
  - We could assume $\omega(v) \leq d(v)$ $\forall v$. ~~Recall~~ (Instead of possible $d(v)+1$ in clique) $\rightarrow$ Recall that if $\omega(v) = d(v)+1$, v is called simplicial. So, we restrict simplicial vertices

  -OR-

  - We could try a larger f for non-simplicial and $f = \frac{1}{d(v)+1}$ for simplicial.

- CR -

Strongest Version:

Show that the necessary condition:

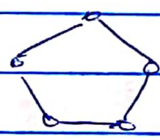
$$\sum_{v \in V(K)} f(v) \leq 1 \quad \forall \text{cliques } K$$

is sufficient

(Since if cliques demand too much color, then we're screwed)

Q: For what ε is this nec. clique condition sufficient?

If $\varepsilon < 1/2$, then no!



then, $f(v) = \frac{1}{d(v)+\varepsilon} > 2/5$,

so we get a better than $2/5$-coloring.

Theorem (Kelly, Postle)

If G is a graph and f a demand function for G such that

- $f(v) \leq \frac{1}{d(v)+1/2} \quad \forall v \in V(G)$, and
- $\sum_{v \in K} f(v) \leq 1 \quad \forall$ clique K in G,

then G has an f-coloring.

Proof Ideas:

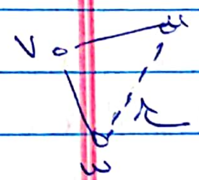
- Min. degree type proof. (Probabilistic, max-degree didn't work well)

Proof Idea (Cont)

1st Obs) Min degree vertex has many min. degree neighbors

2nd Obs) Prove that min. deg vertices decompose into cliques

i.e.



⋮

□

Corollary (Brouwer)

If G is a K_{r+1}-free graph on n vertices with:

$$e(G) \geq \left(1-\frac{1}{r}\right)\frac{n^2}{2} - \frac{n}{4r}+1$$

then $\chi(G) \leq r$ (r-partite)

Local Proof Vizing's:

If G is a graph, $E(G)$ has an f-colouring where

$$\underset{=uv}{f(e)} = \frac{1}{\max\{d(u),d(v)\}+1} \qquad \forall e \in E(G)$$

Proof:

Have to prove $(f(e) : e \in E(G))$ is in the Edmond's Matching Polytope:

- $\sum_{e \ni v} f(e) \leq 1$ $\longrightarrow$ $\sum_{e \ni v} \frac{1}{d(v)+1} \leq \frac{d(v)}{d(v)+1} \leq 1$
- $\sum_{e \in G[S]} f(e) \leq \left\lfloor \frac{|S|}{2} \right\rfloor$ $\longrightarrow$ Show that $f'(e) \leq \frac{V(G)-1}{2}$, where $f(v) = \frac{1}{2}\left(\frac{1}{d(u)+1} + \frac{1}{d(v)+1}\right)$

  ↳ Average of max.
- $f(e) \geq 0$

$$\sum_e \left(\frac{1}{d(u)+1} + \frac{1}{d(v)+1}\right) = \sum_v \frac{d(v)}{d(v)+1} \leq \sum_v \frac{\Delta(G)}{\Delta(G)+1} \leq V(G)-1.$$

(2)

Perfect Graph:

Recall:

G is perfect if $\forall$ induced subgraphs H, $\omega(H) = \chi(H)$

Strong perfect graph thm (Chudnovsky, Robertson, Seymour, Thomas, 2006)

G perfect iff neither G nor $\bar{G}$ contain an induced odd cycle of length ≥ 5

Weak Version: (Lovasz '70s)

G perfect iff $\bar{G}$ perfect

Theorem (Kelly, Postle)

If G is perfect, then G has a $\frac{1}{\omega(G)}$-coloring

Lemma:

If you copy a vertex in a perfect graph, you remain perfect.

~~Proof (1)~~:

Blow up ~~[illegible]~~ G to G

749 - Coloring

Feb 27th

Lecture 14:

Local Fractional Coloring:

For low ω theorems:

- Kim's theorem: If G is girth 5 then $\chi \le (1+o(1)) \frac{\Delta}{\ln \Delta}$
- Johannsen / Molloy: Triangle-free $\Rightarrow \chi \le (1+o(1)) \frac{\Delta}{\ln \Delta}$
- Johannsen: $\chi \le 200\omega \frac{\Delta \ln\ln \Delta}{\ln \Delta}$
- Bonamy, Kelly, Nelson, Postle: $\chi \le 200 \Delta \sqrt{\frac{\ln \omega}{\ln \Delta}}$

Q: Are there local fractional versions of all these thms?

E.g. If G has girth ≥ 5 or triangle free, does there exist a $\left((1-o(1)) \frac{\ln d(v)}{d(v)}\right)$-coloring?

Sub-question: Do any of the ind. set versions hold?

E.g. If G has girth ≥ 5 or triangle-free, is $\alpha(G) \ge \sum_v (1-o(1)) \frac{\ln d(v)}{d(v)}$

$\rightarrow$ Thm (Shearer, 1984)

If G is triangle free and avg deg d, then
$$\alpha(G) \ge (1-o(1)) \frac{\ln d}{d} v(G)$$

Remark: Avg degree, independent set versions trivially exist at the cost of a constant

$\Rightarrow$ Observation: At most $\frac{1}{2}$ of the vertices have deg $\ge$ twice the average (Markov's Inequality)

$\Rightarrow \exists G'$ induced subgraph of G with $v(G') \ge \frac{v(G)}{2}$ and $\Delta(G') \le 2\,\mathrm{avg}(G)$

By Molloy, $\chi(G') \le (1+o(1)) \frac{\Delta(G')}{\ln \Delta(G')} \le (1+o(1)) \frac{2\,\mathrm{avg}(G)}{\ln 2\,\mathrm{avg}(G)}$

$$\Rightarrow \alpha(G') \ge \frac{v(G')}{\chi(G')} \ge (1-o(1)) \frac{\ln \mathrm{avg}(G)}{2\,\mathrm{avg}(G)} \left(\frac{v(G)}{2}\right) = \frac{(1-o(1))}{4} \frac{\ln \mathrm{avg}(G)}{\mathrm{avg}(G)} v(G)$$

①

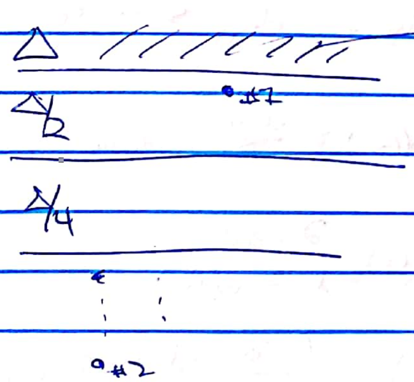
And hence this works for other things, depending on what you assume on ω.

Simpler Q: If G is triangle-free, then does G have an f-coloring where $f(v) \gg \frac{1}{d(v)}$?

(or even better, than $\frac{1}{d(v)+1}$?) → Brooks

Theorem (Kelly, Postle)
Yes we can (Actual statement in a bit)

Proof Ingredients:
1) We'll use the max. degree fractional coloring theorems as black boxes
2) Stratification: Divide the vertices into strata according to their degree; so each strata will have all degrees in the same multiplicative order (E.g. Δ to $\frac{\Delta}{2}$, $\frac{\Delta}{2}$ down $\frac{\Delta}{2}$ to $\frac{\Delta}{4}$, etc..)
3) We color by highest degree strata first
(Intuition: High degree vertices demand so little, it is okay to have them pick first)



→ Apply Molloy's to give $\frac{\ln \Delta}{\Delta}$ amount of color to this section

Problem I: Vertex on the boundary may run out of ~~...~~ color

Problem ~~#~~ 2: May run out of color at ~~lower~~ ~~strata~~ ~~...~~ in lower strata

Let's change the strata.

Δ

$\frac{\Delta}{2\sqrt{\ln\Delta}}$ Arcs in $\frac{\ln\Delta}{\Delta}$ → But we have an issue here.

$$\frac{\Delta}{\sqrt{\ln\Delta}} \times \frac{\ln\Delta}{\Delta} = \sqrt{\ln\Delta} > 1.$$

$\frac{\Delta/\sqrt{\ln\Delta}}{\sqrt{\Delta/\sqrt{\ln\Delta}}}$ So the 2nd strata has already lost all its colors.

Idea, try give $1/2$ the colors to even strata and $1/2$ the colors to odd strata

This argument gives $\Omega\left(\frac{\sqrt{\ln d(v)}}{d(v)}\right)$ for triangle free

We can do even better!
Try an arbitrary # of different ~~typ~~ types of strata and figure it out!
M = # of types, and the i mod M layers have access to $1/M$th segment of $[0,1]$

So we stratify:

$$\Delta, \frac{\Delta}{(2r(\Delta))^{1/M}}, \frac{\Delta}{(2r(\Delta))^{2/M}}, \dots$$

$$\Delta_0 \quad \Delta_1 \quad \Delta_2$$

A strata receives $\frac{r(\Delta_i)}{\Delta_i} \cdot \frac{1}{M}$ color. However, there are vertices in strata of deg $\frac{\Delta_i}{(2r(\Delta_i))^{1/m}}$, so each vertex gets

$$\frac{2(d(v))^{1-1/M}}{2M\, d(v)} = \frac{r d(v))}{2(d(v))} = \frac{1}{Mr(d(v))^{1/M}}.$$

②

So, what M minimizes $Mx^{1/M}$? $M = \ln x$

$\Rightarrow Mx^{1/M} = e \ln x$.

Theorem (~~[illegible]~~, Kelly, Postle)

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

Let $r: \mathbb{N} \to \mathbb{R}$ be non-decreasing s.t. $\forall \varepsilon > 0$, if d is sufficiently large then:

$r(d \cdot r d(1))$

... uh... just find the paper

Corollary

If G is triangle-free, then G has a $(1-o(1))\frac{1}{e}\frac{\ln d(G)}{d(G)\ln\ln d(G)}$-coloring.

Remark!

- Thm only useful if $r(\Delta) = o(\Delta)$
- $\log\log$ term!

Hadwiger's Conjecture: (1943)

If G is K_t-minor-free, then $\chi(G) \le t-1$.

Norin-Song: $\chi \le O(t(\log t)^{.354\ldots})$

Postle: $\chi \le O(t(\log t)^{1/4+\varepsilon})$ $\forall\varepsilon$

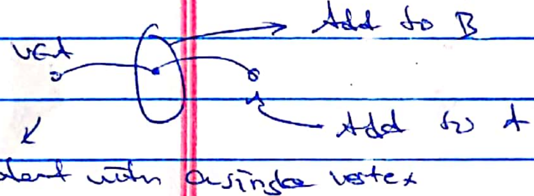
We'll need the following thm of Duchet-Meyniel ('82)

If G is K_t-minor-free, then $\alpha(G) \geq \frac{v(G)}{2t}$

Proof:

Let $A, B \subseteq V(G)$ s.t. A ind, $A \cup B$ connected and $A \cup B$ dominating (and $A \cup B$ nonempty)

Claim: This exists



Claim: $G - (A \cup B)$ is K_{t-1}-minor-free

$\rightarrow$ Get a decomposition into at most $(t-1)$ $(A-B)$'s

$\Rightarrow \sum |A_i| \geq \frac{v(G)}{2} \Rightarrow \exists i$ s.t. $|A_i| \geq \frac{v(G)}{2(t-1)}$.

6

7401-Coloring

Mon 5th

Lecture 16:

Theorem (Bollobás, Thomason '96)
If G is $22k$-connected, then G is k-linked.

(State-of-the-art: Thomas-Wollan '07: $10k$-connected $\Rightarrow$ k-linked)

Theorem (Kostochka, Thomason (indep. by) '80's)
If G has no K_t-minor, then $ad(G) \le O(t\sqrt{\log t})$
(ad = average degree)

Idea: If G has large avg. degree, then G contains a minor of large avg. degree ~~and~~ and a small # of vertices.

Lemma:
If $e(G) \ge k \cdot v(G)$, then G contains a minor H s.t. $v(H) \le 2k+1$ and $\delta(H) \ge k+1$

Proof:
We may assume that $\delta(G) \ge k+1$ o/w we can delete a min. degree vertex and apply induction.
Similarly, we may assume that $e(G) = k \cdot v(G)$, o/w we can delete edges and apply induction.

Thus, $\exists$ a vertex $v \in V(G)$ with $d(v) \le 2k$ (and $d(v) \ge v$).
Let $H = N[v]$. If $\exists u \in N(v)$ s.t. $|N(u) \cap N(v)| \le k-1$, then induct on G/uv. So $\forall u \in N(v)$, $|N(u) \cap N(v)| \ge k$ and hence $\delta(H) \ge k+1$ and $v(H) \le 2k+1$, as desired.

In fact, something stronger is true!

Lemma (Thomason, '84)

Let $0<\beta<1$ be the root of the equation $1 = \beta(1+\log(\frac{2}{\beta}))$ (Note! $\beta \approx .37$) and let $k \geq 3$ be an integer. If G is a graph s.t. $e(G) \geq kv(G)$, then G contains a minor H s.t. $v(H) \leq k+2$ and $2\delta(H) \geq v(H) + \lfloor \beta k \rfloor - 1$.

Why is this nice? The guarantee $2\delta(H) \geq v(H) + \lfloor \beta k \rfloor - 1$ means we get min. degree at least ~~50~~ ~60% of the original degree. For min. degrees higher than 50%, we get a linear amount of common neighbors!

Proof (of Kostochka-Thomason)

Idea: We will prove the contrapositive, if we have high average degree, then we can use the previous lemma to find small minors, and then we finish with probabilistic method.

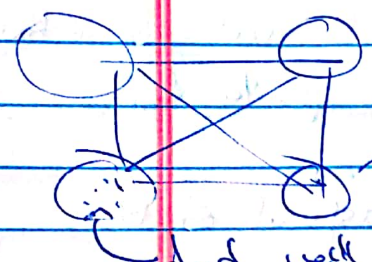
We'll prove that if c is large enough and $ad(G) \geq \underbrace{ct\sqrt{\log t}}_{:=2k}$, then G contains a K_t-minor.

Using the easier lemma, $\exists$ minor H of G s.t. $v(H) \leq 2k+1$ and $\delta(H) \geq k+1$.

(Note: We will try to bag vertices up to use to k-linked theorems, without worrying about the connectivity of the bags)

Proof (con't)

Example: We want smth like:



And we'll worry about individual connectivity in a bit

Let $p = \frac{e(H)}{v(H)}$, the density.

Let $X_1, \dots, X_t$ be vertex-disjoint subsets of H of size $\frac{c}{200}\sqrt{\log t}$ chosen u.a.r.

$$Pr[e(X_i, X_j) = 0] = (1-p)^{|X_i||X_j|}$$
$$= (1-p)^{\frac{c^2}{40000}\log t}.$$

and we claim this is $< 1/t^2$, and so:

$$\mathbb{E}[(i,j) \in \binom{[t]}{2} : e(X_i, X_j) = 0] < t^2 \cdot \frac{1}{t^2} < 1.$$

So, there exist a choice of X's s.t. $e(X_i, X_j) > 0$ $\forall i \neq j \in [t]$.

Now, we're almost done, but each X_i may not be connected! (So, we may not be able to actually contract them)

Finish #1: Use better lemma (i.e. (Thomason's lemma)) to guarantee that $|N(u) \cap N(v)| \geq .05K$ $\forall u, v \in V(H)$, then greedily pick common neighbors for each X_i outside of $\bigcup_{i=1}^{t} X_i$ to connect each X_i, avoiding previously picked common neighbors.

②

Proof (Cont)

Finish #2: Using Mader, $\exists$ subgraph H' of H with connectivity $\geq k/2$ and hence min deg. $\geq k/2$ and so $p \geq 1/4$ for each H' and $v(H') \leq v(H) \leq 2k\alpha$.
So, apply random partition (of X's) to H' instead of H.
Now, by Bollobás, Thomason, H' is $k/4$-linked. Take a matching from $\bigcup_{i} X_i$ to a set of vertices in $G - \bigcup X_i$. (Such exists since $\delta(H') \geq k/2$ and $\bigcup_i X_i \leq k/4$). Now, use the matched vertices and s.t. vertices of X as terminals to find vertex disjoint paths connecting each X_i

$\square$.

Proof (Thomason's Lemma)
By the other lemma, there exists a minor H' of G s.t. $v(H') \leq 2k$ and $\delta(H) \geq k$. (We can drop the $'+1$'s)
We define a $\mathbb{R}$-valued function $f(G)$ on graphs as

$$f(G) = \frac{\beta k v(G)}{2}\left(\log\left(\frac{v(G)}{\beta k}\right) + 1\right).$$

and let:

$$D := \{G : v(G) \geq \beta k \text{ and } e(G) \geq f(G)\}.$$

Claim: $H' \in D$
Proof: $v(H') \geq k \geq \beta k$ and $e(H') \geq \frac{k v(H')}{2} \cdot 1$

$$\geq \frac{k v(H')}{2}\left(\beta\left(1 + \log\left(\frac{2}{\beta}\right)\right)\right)$$

Proof (Con't).

$$= \frac{\beta k v(G)}{2}\left(\ln\left(\frac{2}{\beta}\right)+1\right)$$

$\hookrightarrow \geq \frac{v(H')}{\beta k}$ since $v(H') \leq 2k$.

$$\geq f(H')$$

$\square$ (Claim)

Let H'' be a minor of H' minimal w.r.t. containment in D (and such exists since $H' \in D$, by claim).

Since the complete graph of order $\lceil \beta k \rceil$ is not in D (b/c $f(G_{\lceil \beta k \rceil}) = \frac{\beta k \lceil \beta k \rceil}{2}\left(\log\left(\frac{\lceil \beta k \rceil}{\beta k}\right)+1\right) \geq \lceil \beta k \rceil^2 \geq \binom{\lceil \beta k \rceil}{2} \geq e(G_{\lceil \beta k \rceil})$),

then it follows that $v(H'') \geq \lceil \beta k \rceil + 1$.

Moreover, $e(H'') \geq \lceil f(H'') \rceil$, o/w deleting an edge stays in D, contradicting the minimality of H''.

By minimality of H'', $\forall e = uv \in E(H'')$, we have that

$$e(H''/uv) \underset{\text{max}}{\leq} f(H''/uv)$$

(since $v(H''/uv) \geq v(H'') - 1 \geq \beta k$)

Let u be a vertex in H'' of min. degree. Let $H = H''[N_{H''}(u)]$

$$v(H) \leq d_{H''}(u) \underset{\uparrow}{\leq} \frac{2e(H'')}{v(H'')} = \frac{2\lceil f(H'') \rceil}{v(H'')} = \frac{2\,\frac{\beta k v(H'')}{2}\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right)}{v(H'')}$$

b/c u has min. degree in H''

$$= \beta k\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right)$$

$\hookrightarrow$

Proof: (cont.)

And since $v(H'') \leq v(H') \leq 2k$:

$$\log\left(\frac{v(H'')}{\beta k}\right) \leq \log\left(\frac{2}{\beta}\right) \text{ and hence}$$

$$v(H) \leq \beta k\left(\log\left(\tfrac{2}{\beta}\right)+1\right) = k \qquad \left(\text{since } \beta\left(\log\left(\tfrac{2}{\beta}\right)+1\right) = 1\right).$$

It remains to show that $2\delta(H) \geq v(H) + \lfloor \beta k \rfloor - 1$.

Note:

$$\delta(H) \geq \min_{v \in N_{H''}(u)} |N_{H''}(u) \cap N_{H''}(v)|.$$

(Now, we have to show that the # of common neighbors is actually high)

By previous obs., we have that

$$e(H'') - e(H''/uv) \geq f(H'') - f(H''/uv)$$

So,

$$|N_{H''}(u) \cap N_{H''}(v)| \geq f(H'') - f(H''/uv) \underbrace{-\,1}_{\text{For } uv \text{ itself.}}$$

Then

$$2\delta(H) - v(H)$$

$$\geq 2\left(f(H'') - f(H''/uv) - 1\right) - v(H)$$

$$\geq 2\left(\frac{\beta k v(H'')}{2}\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right) - \frac{\beta k\, v(H''/uv)}{2}\left(\log\left(\frac{v(H''/uv)}{\beta k}\right)+1\right)-1\right) - v(H).$$

$$= \beta k\left(v(H'')\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right) - (v(H'')-1)\left(\log\left(\frac{v(H'')-1}{\beta k}\right)+1\right)\right) - 2 - v(H)$$

Proof (cont)

$$= \beta k \left(v(H'') \left(\log\left(\frac{v(H'')}{\beta k}\right) - \log\left(\frac{v(H'')-1}{\beta k}\right) + 1 + \log\left(\frac{v(H'')-1}{\beta k}\right)\right)\right.$$

$$-2 - \underbrace{v(H)}_{\text{Recall that } v(H) \le \beta k \left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right).}$$

Lecture 17:

Theorem (Bollobás - Thomason, '96)
If G is $22k$-connected, then G is k-linked.

Lemma (Thomason, '84)
Let $0<\beta<1$ be the root of $1=\beta+(1+\ln(\frac{2}{\beta}))$ and let $k\geq 3$ be an integer. Let G be a graph with $e(G)>k\,v(G)$. Then G contains a minor s.t.
(1) $v(H)\leq k+2$, and
(2) $2\delta(H)\geq v(H)+\lfloor\beta k\rfloor-1$.

Idea: If we can rearrange this minor so that each terminal is in a distinct "bag", then we can use "many common neighbors in H" property to do the linking.

Key Technical Lemma:
Let $d\geq 0$, $k\geq 0$, and $l\geq d+\lfloor\frac{3k}{2}\rfloor$ be integers. Let G be a graph containing vertex-disjoint non-empty connected subgraphs $C_1, C_2, \ldots, C_l$ such that each C_i is adjacent to all but at most d other C_j's. Suppose $S=\{s_1,\ldots,s_k\}$ is a set of k vertices such that there is no S-cut of order $<k$ which avoids $d+1$ of the subgraphs $C_1,\ldots,C_l$.
Then, G contains vertex-disjoint non-empty connected subgraphs $D_1,\ldots,D_m$, where $m=l-\lfloor k/2\rfloor$ such that $\forall i\in[k]$, the subgraph D_i contains s_i and is adjacent to all but at most d of the subgraphs $D_{k+1},\ldots,D_m$.

Note: In fact, for every $i \in [m]$, D_i is adjacent to all but at most d of $D_1, \ldots, D_m$.

Proof:

In fact, we'll prove the lemma under a weaker assumption on the C_i's (and hence a stronger version of the lemma)

Each C_i is at least one of the following:

- Connected (as before), or
- Not connected and each of its components contains a vertex in S

And is adjacent to all but at most d of the other C_j's ($j \neq i$) that do not contain a vertex in S.

Let $G, C_1, \ldots, C_\ell, S$ be a minimum counterexample.

Obs: $\nexists$ an isolated vertex v of G.

Proof: If $v \notin S$, then $G-v$ is a smaller counterexample b/c if $v \in C_i$, then $C_i = \{v\}$ but then C_i is non-adjacent to too many other C_i's. So say $v \notin$ any C_i, but then deleting it does not affect the C_i's or connectivity.

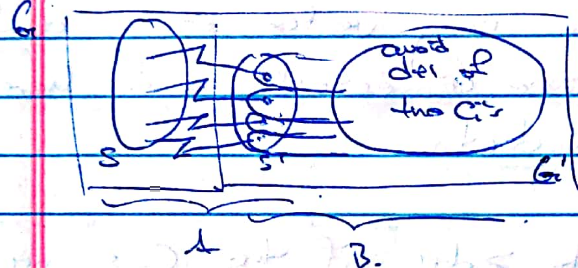
If $v \in S$, then $S-v$ is an S-set of order $\leq k-1$ which avoids at least $\ell - k \geq d + \lfloor k/2 \rfloor \geq d+1$, a contradiction (This is the S-cut assumption) $\square$ (Obs)

$\Rightarrow$ Proof continued.

Proof (cont)

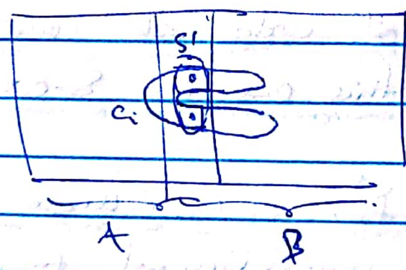
Claim: The only S-cut of order k avoiding $\geq d\ell$ of the C_i's is S itself

Proof: Suppose not, that is, suppose $\exists S' \subseteq V(G)$, $|S'|=k$, and $S' = A \cap B$ for an S-cut (A, B), where $S \subseteq A$ and A avoids $\geq d\ell$ of the C_i's.



Let $G' = G[B]$ $\forall i \in [\ell]$ and let $C_i' = C_i \cap G'$

Remark: If a component of C_i "splits" into multiple components in G', then each such component in G' contains a vertex of S'



Obviously, C_i' are vertex-disjoint since the C_i are; and further, they are non-empty since each C_i has to contain a vertex in B ~~that~~ as it is adjacent to at least 1 of the $d\ell$ subgraphs avoided by A, by assumption.

Proof: (Cont)

Subclaim: C_i are non-empty, vertex-disjoint and satisfy the weaker assumptions

Proof.

We've established non-empty, vertex-disjoint. Now, suppose C_i is not connected, and have a component J not containing a vertex of S'. But, then J is a component of C_i and does not contain any vertex of S' and hence any vertex of A (since connected) and hence any vertex of S, a contradiction.

So, WMA. $\exists$ C_i non-adj to $\geq d+1$ of the C_js not containing a vertex of S', but those C_j's must be equal to C_j and hence C_i is not adjacent to them, a contradiction.

$\square$ (Subclaim)

Further, $\nexists$ an S'-cut in G' of order $<k$ avoiding $\geq d+1$ C_i's b/c also we can use this as our S-cut.

Now, since $S' \neq S$, $G' \subsetneq G$, $\exists$ $D_1', \dots, D_m'$ where $\forall i \in [k]$, $s_i' \in D_i'$ and $\forall i \in [m]$ D_i' is non-adjacent to at most d of $D_{k+1}', \dots, D_m'$. Let $D_i = D_i'$ if $i \in [k+1, m]$. Since A avoided $d+1$ of the C_i's then $\exists$ a cut in $G[A]$ of order $<k$, separating S from S'. By Menger's theorem, $\exists$ k vertex-disjoint paths $P_1, \dots, P_k$ connecting S to S' in $G[A]$. Let $D_i = D_i' \cup P_i$ and now, after relabeling, D_i is as desired.

$\square$ (Claim)

Proof (Con't)

~~Claim~~

Claim: $\forall e = uv \in E(G)$, $\exists i \neq j$ s.t. $u \in C_i$, $v \in C_j$

Proof:

Case 1: $v \in S$, then G/uv gives rise to new C_i that satisfy ~~…~~ weaker assumptions and connectivity, b/c of Claim (i.e. we would be able to find another S-cut of order $<k$)

Case 2: $u, v \in S$, let $G' = G - uv$. Connectivity is preserved b/c uv doesn't affect S-cut. Non-neighbors okay b/c not in distinct C_is. and component D of C_i containing u, v splits into ≤ 2 components, each containing a vertex of S □(Claim)

Corollary: $V(G) = \bigcup C_i$ ← Hahn. what?

Cor: Every C_i is an ind. set. (and hence $|C_i| = 1$ or $C_i \subseteq S$)

Remark: We'd be done by letting $C_i = D_i$ if every $|C_i| = 1$.

Also, if $|C_i| \geq 2$ let $\mathcal{C} = \{v \in S, C_i \text{ containing } v \text{ has } \geq 2 \text{ vertices}\}$.

Claim: Exist matching M from $\mathcal{C}$ to $V(G) - S$ in G

Proof: By Hall's theorem, $\exists X \subseteq \mathcal{C}$ such that $Y = N(X) \cap (V(G) - S)$ has $|Y| < |X|$

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Proof (Con't)

Proof (Claim - Con't)

But then $S' = S - X \cup Y$ has $|S'| < |S| = k$ and is an S-cut avoiding $|V(G) - S - S'| = l - k - k/2 = l - 3k/2 \geq d+1$ C_i's.

$\square$ (Claim)

And that completes the proof $\square$.

Def'n A graph G is (k,n)-knit if $1 \leq n \leq k \leq |V(G)|$ and $\forall S \subseteq V(G)$, $|S| = k$ and partition S into $S_1, \ldots, S_t$ where $t \geq n$ non-empty parts, then G contains vertex-disjoint, connected subgraphs $F_1, \ldots, F_t$ s.t. $S_i \subseteq V(F_i)$ $\forall i \in [t]$.

Obs: If G is $(2k,k)$-knit, then G is k-linked.

(. . .

Lecture 18:

Recall!
We finished proving the key technical lemma (Bollobás - Thomason):

Let $d \geq 0$, $k \geq 2$, $l \geq d + 3k/2$ be integers. Let G be a graph with $\kappa(G)$ (connectivity) $\geq k$ containing vertex-disjoint, non-empty, connected subgraphs $C_1, \ldots, C_l$ such that each of them is non-adj. to at most d others. Let $S = \{s_1, \ldots, s_k\} \subseteq V(G)$. Then G contains vertex-disjoint, non-empty, connected subgraphs $D_1, \ldots, D_m$, where $m = l - \lfloor k/2 \rfloor$ such that

- $\forall i \in [k]$, $s_i \in D_i$
- $\forall i \in [m]$, D_i is non-adj. to at most d of $\{D_{k+1}, \ldots, D_m\}$.

(We technically proved a stronger version - that we avoid S-cut, but we'll only need this statement)

Def'n G is (k,n)-knit if $1 \leq n \leq k \leq V(G)$ and $\forall S \subseteq V(G)$, $|S| = k$ and partition $S_1, \ldots, S_t$ of S (non-empty), $t \geq n$, then $\exists$ vertex-disjoint, connected subgraphs $F_1, \ldots, F_t$ s.t. $S_i \subseteq V(F_i)$ $\forall i \in [t]$

And, $(2k,k)$-knit $\Rightarrow$ k-linked.

Theorem: Let G be a graph with $\kappa(G) \geq k$ such that G has a minor H with $2\delta(H) \geq v(H) + \lfloor \frac{5k}{2} \rfloor - 2 - n$, then G is (k,n)-knit. In particular, if $\kappa(G) \geq 2k'$ and $2\delta(H) \geq v(H) + 4k' - 2$, then G is k-linked.

Proof:

Let the bags in the model of H in G be the C_i's. Hence let $l = v(H)$, $d := v(H) - 1 - \delta(H)$. Let $S = \{S_1, \dots, S_k\}$, then $l = 2d + 2 - v(H) + 2\delta(H) \geq 2d + \lfloor \frac{5k}{2} \rfloor - n$, ~~and the~~ conditions to apply key technical lemma.

So, $\exists D_1, \dots, D_m$, $\forall i \in [k]$, $s_i \in D_i$ and $\forall i \in [m]$, non-adj $\leq d$ of $D_{k+1}, \dots, D_m$.

Obs: $\forall$ pair D_i, D_j $(i, j \in [m])$ has at least $m - k - 2d \geq l - \lfloor \frac{3k}{2} \rfloor - 2d \geq k - n$ subgraphs in $D_{k+1}, \dots, D_m$ to which they are both adj. Now greedily connect S_1, then S_2, etc., to S_t by ordering each $S_i = \{S_{i,1}, S_{i,2}, \dots, S_{i,r}\}$. Find common neighbor of $S_{i,1}$ and $S_{i,j}$ disjoint from previous common neighbor. Let F_i be the appropriate unions of the D_i's

$\square$

Corollary: $\exists c > 0$ s.t. $\forall k$, if G is a graph w/ $\kappa(G) \geq ck$, then G is (k,n)-knit $\forall n \leq k$.

Corollary: If $\kappa(G) \geq 22k$, then G is k-linked.

Corollary: Let H be a graph with vertices $v_1, \dots, v_m$. Let G be a graph with $\kappa(G) \geq 22e(H) + v(H)$ and let $u_1, \dots, u_m$ be distinct vertices of G. Then G contains $e(H)$ pairwise vertex-disjoint paths P_{ij} joining u_i to u_j whenever $v_iv_j \in E(H)$.

Corollary: If $\kappa(G) \ge 11p^2$ or $e(G) \ge 22p^2 v(G)$, then G contains a K_p-subdivision.

Back to Hadwiger's Conjecture:

Theorem (Norin, Song, '19+)
If G has no K_t-minor, then ~~G is~~ $\chi(G) \le O(t(\log t)^{.354})$.

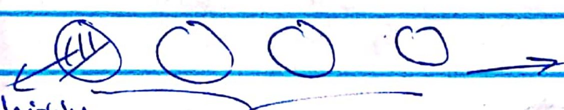
Theorem (Postle, '19)
" — " — , then $\chi(G) \le O(t(\log t)^{\beta})$ $\forall \beta > 1/4$.

Big Picture: no K_t-minor $\Rightarrow \alpha(G) \ge \frac{v(G)}{2(t-1)}$

- Small ($v(G) \le t\,\mathrm{poly}\log t$) graphs: Use the Duchet-Meyniel independence result iteratively to color w/ few colors.
- Large graphs:
  - We may assume large min. degree by criticality.
  - (Even further) — " — large connectivity by Kawarabayashi.

Case I:
$\exists$ many (i.e. $\ge \sqrt{\log t}$) vertex-disjoint small (as above), dense subgraphs $\Rightarrow$ we can find K_t-minor.

$\ge t(\log t)^{1/4}$



Dense $\Rightarrow$ highly connected subgraphs

$\ge \sqrt{\log t}$

If there are many dense subgraphs, we can always find a K_t-minor.

(2)

Case 2:

many ....

Then, we can partition G into:

- A few of these small subgraphs.
  This can be colored by our previous argument.
- And, the remainder will induce a sparse graph (density $\leq O(f(\log f)^P)$) and hence can be colored w/ separate set of colors by greedy.

Why would this be true? ((We've removed all small "dense" subgraphs, and claim remaining set is sparse).

~~Small graphs~~:

Things we still need to argue:

- Small graphs
- Case 1
- Sparsity in case 2.

~~Sketch~~

Small graphs:

In fact, Seymour (2016) noted that the Duchet-Meyniel implies:

Theorem: If G has no K_t-minor, then $\exists X \subseteq V(G)$ with $|X| \geq v(G)/2$ and $\chi(G[X]) \leq t-1$.

Cor: If G has no K_t-minor, then $\chi(G) \leq \left(\log\left(\frac{v(G)}{t}\right)+2\right)t$

Proof: By previous thm, $\forall$ integer $s \geq 0$, $\exists$ disjoint $X_1, \ldots, X_s \subseteq V(G)$ s.t. $|V(G) - \bigcup_{i=1}^{s} X_i| \leq \frac{v(G)}{2^s}$. Let $s = \lceil \log_2\left(\frac{v(G)}{t}\right) \rceil$.

then $\frac{v(G)}{2^s} \leq t$ and so $\chi(G \setminus \bigcup_{i=1}^{s} X_i) \leq v(G \setminus \bigcup_{i=1}^{s} X_i) \leq t$

So,

$$\chi(G) \leq t + \sum_{i=1}^{s} \chi(G[X_i]) \leq t + (st - 1) \leq (s+1)t$$ ■

Cor: If $v(G) \leq t(\log t)^c$ and G has no K_t-minor, then

$$\chi(G) \leq t(c \log\log t) + 2$$

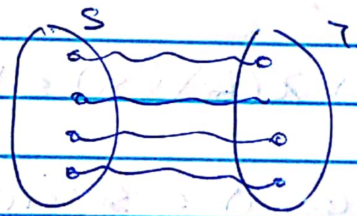
Remark: The tight examples for Kostochka-Thomason deg. bounds are random graphs on $t\sqrt{\log t}$ vertices and hence small.

Onto Case I!

Lemma (Norin, Song)

$\exists\, c>0$ s.t.:

Let G be a graph and let $\ell \ge s \ge 2$ be positive integers. Let $s_1,\dots,s_\ell, t_1,\dots,t_\ell, r_1,\dots,r_s \in V(G)$ be distinct. If $\kappa(G) \ge C\cdot\max\{\ell, s\sqrt{\log s}\}$, then $\exists$ a K_s-model M rooted at $\{r_1,\dots,r_s\}$ (i.e. each r_i is in a different bag of M) and an $\{(s_i,t_i)\}_{i\in[\ell]}$-linkage P in G s.t. M and P are vertex-disjoint.

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Proof:

Let $d = d(G) := \frac{e(G)}{v(G)}$ be the density (Note that $d(G) \ge \frac{\kappa(G)}{2} \ge \frac{C\ell}{2}$). By better minor lemma, $\exists$ a model H of graph H in G, where $v(H) \le d\ell$ and every vertex in H has at $\le \frac{v(H)}{2} - \frac{d}{10}$ non-neighbors

Now need to key technical lemma

Pf Proof: (cont)

Let the C_i's be the bags of H.

Let $S = \{s_1, \ldots, s_{k_2}, t_1, \ldots, t_2, r_1, \ldots, r_s\}$

$\to$ Send to key technical ~~changes~~!

Got lazy. Idea for rest of proof:

- Split bags not containing r_i's be used for linking and minor creation
- How? Do this randomized $\to$ Use Chernoff Bound to bound low common neighbors.
- Then, use these ~~sets~~ bags to link r_i's to create K_t-minor

$\square$

Theorem (Norin, Song)

$\exists\, c > 1$ s.t. let G be a graph w/ $\chi(G) \geq ct(\log t)^{1/4}$,

Let $r \geq \frac{\sqrt{\log t}}{2}$ be an integer. If $\exists$ ~~disjoint~~ pairwise p.w. vertex disjoint subgraphs $H_1, \ldots, H_r$ of G s.t. $d(H_i) \geq ct(\log t)^{1/4}$ $\forall i \in [r]$, then G has a K_t-minor.

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[3] Marthe Bonamy, Tom Kelly, Peter Nelson, and Luke Postle. Bounding χ by a fraction of Δ for graphs without large cliques, 2018.

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