July 5th, 2018

CO454

Recall:

Def'n (TSP)

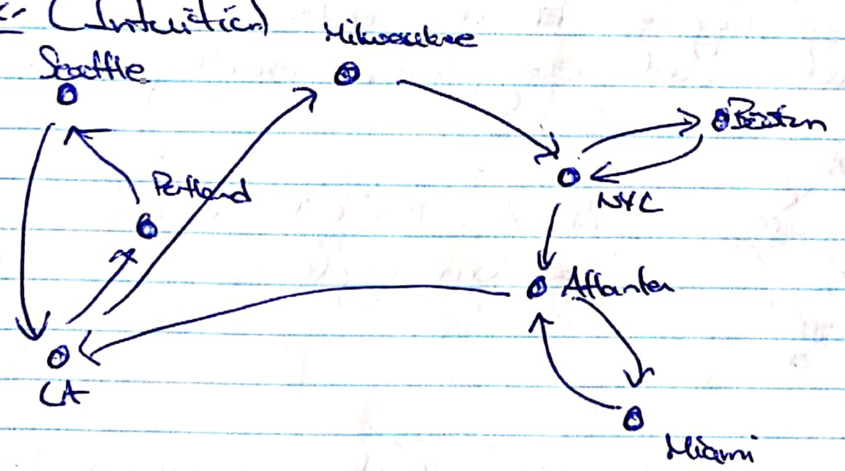
GIVEN: Complete graph $G=(V,E)$ and edge costs $c_e \geq 0$ for all $e \in E$

FIND: A simple cycle T (called TSP Tour) that visits every vertex of min. total cost (i.e. $c(T) = \sum_{e \in T} c_e$)

GOAL TODAY: Design approx. algorithms for metric instances. (i.e. edge costs satisfy triangle inequality $\forall u,v,w \in V$, $c_{uv} \leq c_{uw} + c_{wv}$

Claim I: If edge costs satisfy triangle inequality for all $u,v \in V$ and uv-paths P
$c_{uv} \leq c(P)$

Ex: (Intuition)



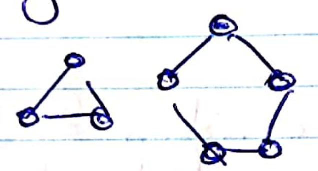
Google Flights gives us the above

Problem: This is not a TSP tour

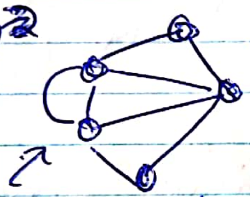
How can we fix this? → Remove duplicates (i.e. Approximate in order of visited time).

Graph Theory

Def'n (Eulerian)
A (multi)-graph is called Eulerian if all the nodes have even degree
Ex:
1) All Cycles are Eulerian:

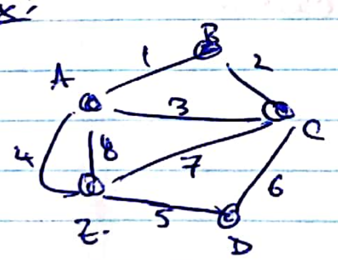


2)



Edge is duplicated, this is a multigraph example

Def'n (Eulerian Cycle)
An Eulerian Cycle in a graph is a closed walk that visits every edge exactly once.
Ex:



ABCAEDCEA is an Eulerian cycle.

Lemma: Every connected Eulerian graph has an Eulerian cycle.
(Note: - Eulerian cycle will visit all vertices.
- We can find this in poly time)
[Proof]
Exercise - Do DFS. ("Fleury's"); Glue cycles together
□

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Strategy:

1) Find an Eulerian subgraph H of G s.t. $C(H) \le 2\cdot OPT_{TSP}$.
2) From H, obtain a tour T of cost $C(T) \le C(H)$

Then, we have:

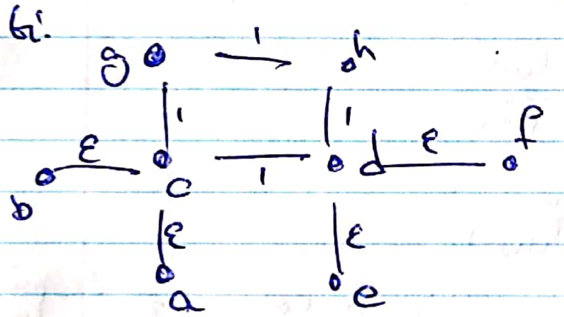
$$C(T) \le C(H) \le 2\cdot OPT_{TSP}$$

E' can take multiple copies of edges of E

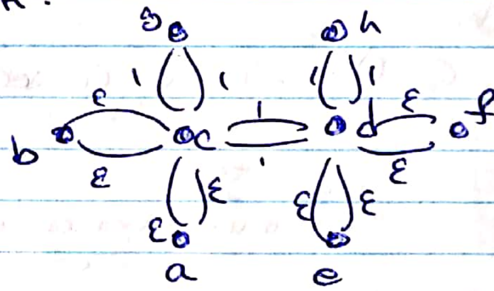
Claim 2:
From a connected Eulerian "subgraph" $H=(V,E')$ of G we can find a TSP tour T with $c(T) \le c(H) = \sum_{e\in E'} c_e$.

Idea behind proof: (Edges not drawn take on weights of the shortest path)

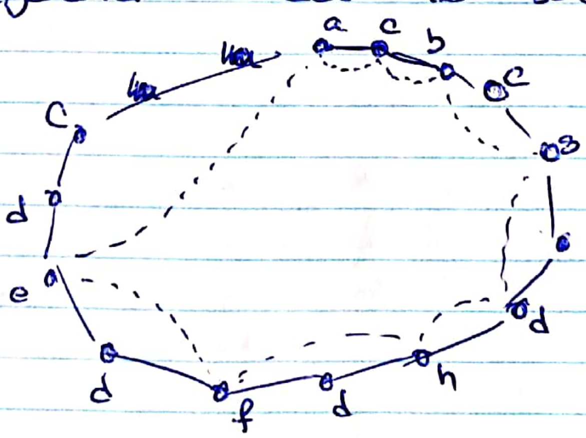
G:



H:



Step I: Find an Eulerian cycle
acbcgcdhdfdedca is such a cycle.



Step 2: Go around the cycle and skip vertices if we have already seen them (dotted lines)

Tour T: acbgdhfea.

[Proof] (Sketch)

From Eulerian H, obtain Eulerian cycle $\tilde{\tau}$ by DFS. Obtain tour τ from $\tilde{\tau}$ by "shortcutting" edges.

i.e. let τ be obtained from $\tilde{\tau}$ by taking vertices in order of first appearance on $\tilde{\tau}$ starting from an arbitrary vertex r.

If u, v consecutive on τ, then

$c_{uv} \le$ cost of $\underbrace{\text{[portion of } \tilde{\tau} \text{ from first appearance of } u \text{ to the first appearance of } v\text{]}}_{\tilde{\tau}_{uv}}$.

Then,

$$c(\tau) = \sum_{uv \text{ consecutive on } \tau} c_{uv}$$

$$\le \sum_{uv \text{ consecutive on } \tau} c(\tilde{\tau}_{uv})$$

$$= c(\tilde{\tau})$$

$$= c(H)$$

Since $\tilde{\tau} = \bigcup_{uv \text{ consecutive on } \tau} \tilde{\tau}_{uv}$ (disjoint union)

$\square$

This completes the proof of step 2.

Recall:
Step 1: Finding "cheap" Eulerian subgraphs.
GOAL:
Find H, connected Eulerian, s.t.

$$c(H) \leq 2 \cdot OPT_{TSP}$$

Observation:
(Spanning tree bound)
Let T^* be an optimal TSP tour.
Then,

$$OPT_{TSP} = c(T^*) \geq c(T^* - uv) \text{ for any edge } uv \in T^*$$

Since $T^* - uv$ is a spanning tree $\longrightarrow \geq MST(G, c)$

min cost spanning tree in G under edge weight c.

$$\Rightarrow 2 \cdot OPT_{TSP} \geq 2 \cdot MST(G, c)$$

Algorithm: (2-Approx.)
1. Find MST $F \subseteq E$ of G.
2. Take 2 copies of every edge in F to obtain H.
3. Apply claim 2 to obtain tour T.

Claim: H is Eulerian
[Proof] Exercise.

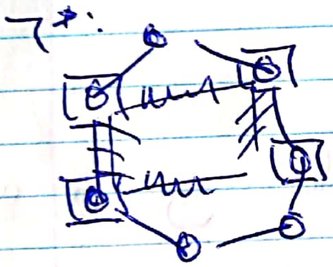
3/2-Approx:

Idea: Fix odd degree vertices by matching them with each other

Claim: Let $U \subseteq V$ with $|U|$ even. Let $P_{PM}(U)$ be the min cost of a perfect matching in $G[U]$. Then, $OPT_{TSP} \geq 2P_{PM}(U)$.

Picture:

T^*:



□ - U

M - Matching 1

ııı - Matching 2

[Proof]

Take optimal tour T^* and shortcut to obtain T^*_U containing only the nodes of U.

This cycle can be decomposed into two U perfect matchings M_1 and M_2.

Then,

$$OPT_{TSP} = c(T^*) \geq c(T^*_U) = c(M_1) + c(M_2)$$
$$\geq 2P_{PM}(U)$$

☒

Christofedes Algorithm:

1) Find MST F
2) Let $\mathcal{U}$ = { odd degree vertices in F }
   (Remark: $|\mathcal{U}|$ is even - Handshake lemma)
3) Find min-cost perfect matching
   $M_{\mathcal{U}}^* \subseteq E$ in $G[\mathcal{U}]$
   ~~Claim:~~
4) Let $H = F \cup M_{\mathcal{U}}^*$
   Remark: H is connected and Eulerian.
   [Proof - Exercise]
5) Apply claim 2 to obtain T.

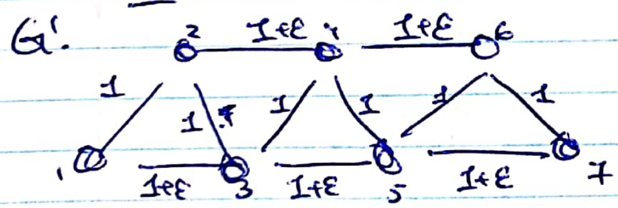
Then,

$$c(T) \leq c(H) = c(F) + c(M_{\mathcal{U}}^*)$$
$$\leq OPT_{TSP} + \tfrac{1}{2} OPT_{TSP}$$
$$= 3/2 \cdot OPT_{TSP}.$$

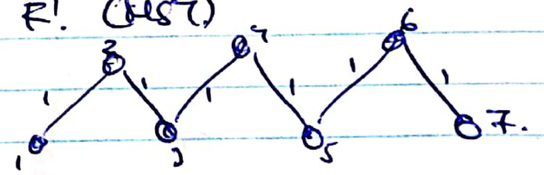
$\emptyset$

Ex:

G:



[1] F: (MST)



[2] {1, 7} are vertices w/ odd degree.

[3] {17} = $M^*_{\{1,7\}}$

[4] H = F ∪ {17}.

[5] We're done!

Note: The min tour is 1246 7531