749 – Colorings

Feb 27th

Local Fractional Coloring:

For low ω theorems:

- Kim's theorem: If G is girth 5 then $\chi \le (1+o(1))\frac{\Delta}{\ln \Delta}$
- Johansson / Molloy: Triangle-free $\Rightarrow \chi \le (1+o(1))\frac{\Delta}{\ln \Delta}$
- Johansson: $\chi \le 200\omega \frac{\Delta \ln\ln \Delta}{\ln \Delta}$
- Bonamy, Kelly, Nelson, Postle: $\chi \le 200\Delta\sqrt{\frac{\ln \omega}{\ln \Delta}}$

Q: Are there local fractional versions of all these thms?

E.g. If G has girth ≥ 5 or triangle free, does there exist a $(1-o(1))\frac{\ln d(v)}{d(v)}$ - coloring?

Sub-question: Do any of the ind. set versions hold?

E.g. If G has girth ≥ 5 or triangle-free, is $\alpha(G) \ge \sum_v (1-o(1))\frac{\ln d(v)}{d(v)}$

$\rightarrow$ Thm (Shearer, 1984)

If G is triangle free and avg deg d, then
$$\alpha(G) \ge (1-o(1))\frac{\ln d}{d} v(G)$$

Remark: Avg degree, independent set versions trivially exist at the cost of a constant

$\rightarrow$ Observation: At most ½ of the vertices have deg $\ge$ twice the average (Markov's Inequality)

$\Rightarrow \exists G'$ induced subgraph of G with $v(G') \ge \frac{v(G)}{2}$ and $\Delta(G') \le 2\cdot \text{avg}(G)$

By Molloy, $\chi(G') \le (1+o(1))\frac{\Delta(G')}{\ln \Delta(G')} \le (1+o(1))\frac{2\,\text{avg}(G)}{\ln 2\,\text{avg}(G)}$

$$\Rightarrow \alpha(G') \ge \frac{v(G')}{\chi(G')} \ge (1-o(1))\frac{\ln \text{avg}(G)}{2\,\text{avg}(G)}\left(\frac{v(G)}{2}\right) = \frac{(1-o(1))}{4}\frac{\ln \text{avg}(G)}{\text{avg}(G)} v(G)$$

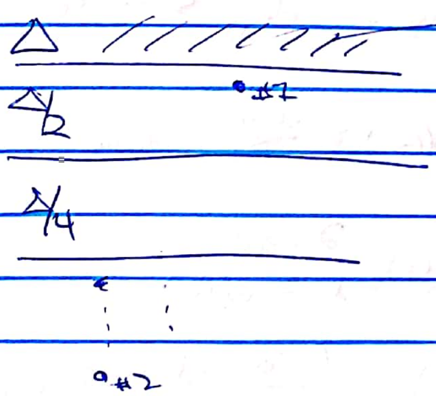
And hence this works for other things, depending on what you assume on ω.

Simpler Q: If G is triangle-free, then does G have an f-coloring where $f(v) \gg \frac{1}{d(v)}$?

(or even better than $\frac{1}{d(v)+1}$?) $\to$ Brooks

Theorem (Kelly, Postle)
Yes we can (Actual statement in a bit)

Proof Ingredients:
1) We'll use the max. degree fractional coloring theorems as black boxes
2) Stratification: Divide the vertices into strata according to their degree; so each strata will have all degrees in the same multiplicative order (E.g. Δ to $\frac{\Delta}{2}$, ~~$\frac{\Delta}{4}$~~ then $\frac{\Delta}{2}$ to $\frac{\Delta}{4}$, etc.)
3) We color by highest degree strata first
(Intuition: High degree vertices demand so little, it is okay to have them pick first)



$\to$ Apply Molloy's to give $\frac{\ln \Delta}{\Delta}$ amount of color to this section

Problem 1: Vertex on the boundary may run out of ~~___~~ color

Problem 2: May run out of color ~~lower strata~~ on lower strata

Let's change the strata:

Δ ______

$\frac{\Delta}{2\sqrt{\ln\Delta}}$ ______ Access $\frac{\ln\Delta}{\Delta}$ → But we have an issue here!

$\frac{\Delta}{\sqrt{\ln\Delta}} \times \frac{\ln\Delta}{\Delta} = \sqrt{\ln\Delta} > 1.$

$\frac{\Delta/\sqrt{\ln\Delta}}{\sqrt{4/\sqrt{\ln\Delta}}}$ ______ So the 2nd strata has already lost all its colors.

Idea, try give 1/2 the colors to even strata and 1/2 the colors to odd strata

This argument gives $\Omega\left(\frac{\sqrt{\ln d(v)}}{d(v)}\right)$ for triangle free

We can do even better:
Try an arbitrary # of different ~~typ~~ types of strata and figure it out!
M = # of types, and the $i \bmod M$ layers have access to $1/M$th segment of $[0,1]$

So we stratify:

$$\begin{matrix} \Delta, & \frac{\Delta}{(2r(\Delta))^{1/M}}, & \frac{\Delta}{(2r(\Delta))^{2/M}}, & \dots \\ \shortparallel & \shortparallel & \shortparallel & \\ \Delta_0 & \Delta_1 & \Delta_2 & \end{matrix}$$

A strata receives $\frac{r(\Delta_i)}{\Delta_i} \cdot \frac{1}{M}$ color. However, there are vertices in strata of deg $\frac{\Delta_i}{(2r(\Delta_i))^{1/m}}$, so each vertex gets

$$\frac{2(d(v))^{1-1/M}}{2M\, d(v)} = \frac{r(d(v))}{2(d(v))} = \frac{1}{Mr(d(v))^{1/M}}.$$

So, what M minimizes $Mx^{1/M}$? $M = \ln x$

$\Rightarrow Mx^{1/M} = e \ln x$.

Theorem (~~[illegible]~~ Kelly, Postle)

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

Let $r : \mathbb{N} \to \mathbb{R}$ be non-decreasing s.t. $\forall \varepsilon > 0$, if d is sufficiently large then:

$r(d \cdot r d(1)$

... uh... just find the paper

Corollary

If G is triangle-free, then G has a $(1-o(1))\frac{1}{2e}\frac{\ln d(G)}{d(G)\ln\ln d(G)}$ - coloring.

Remark!

- Thm only useful if $r(\Delta) = o(\Delta)$
- $\log\log$ term!

Hadwiger's Conjecture: (1943)

If G is K_t-minor-free, then $\chi(G) \leq t-1$.

Norin-Song : $\chi \leq O(t(\log t)^{.354\ldots})$

Postle : $\chi \leq O(t(\log t)^{1/4+\varepsilon})$, $\forall \varepsilon$

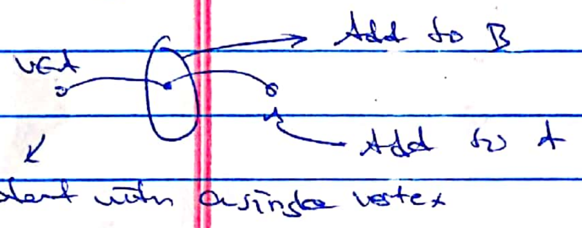
We'll need the following thm of Duchet-Meyniel ('82)

If G is K_t-minor-free, then $\alpha(G) \geq \frac{v(G)}{2t}$

Proof:

Let $A, B \subseteq V(G)$ s.t. A ind, $A \cup B$ connected and $A \cup B$ dominating (and $A \cup B$ nonempty)

Claim: This exists



Claim: $G - (A \cup B)$ is K_{t-1}-minor-free

$\rightarrow$ Get a decomposition into at most $(t-1)$ $(A-B)$'s

$\Rightarrow \sum |A_i| \geq \frac{v(G)}{2} \Rightarrow \exists i$ s.t. $|A_i| \geq \frac{v(G)}{2(t-1)}$.