7401 - Colourings

Jan 21st

Recall:

We were proving Reed-Sudakov, and had the following expected values:

$\text{keep} := \left(1-\frac{p}{2}\right)^{\Delta_L(G)} \approx e^{-\frac{pD}{L}}$ $\quad (= \Pr(c \in L'(v) \mid c \in L(v)))$

$\mathbb{E}[|L'(v)|] = \text{keep} \cdot |L|$

$\mathbb{E}[d_{L',G'}(v,c)] = d_{L,G}(v,c)\Big((1-p)\cdot \text{keep} + p\Big(\text{keep}\cdot\frac{1}{|L|} + \underbrace{\Big(1-\frac{1}{|L|}\Big)}_{\phi(u)\neq c \text{ and keeps } c}\Big(1-\frac{2}{|L|}\Big)^{D}\Big)\Big)$

not activated; keeps c; if $\phi(u)=c$ and keep c; keeping $\phi(u)$ and c

But ~~the~~ ~~and~~ it turns out that this variable doesn't concentrate well, so we'll use another.

Picking a better variable:

$d_{L',G'}(v,c) = |N_{L',G'}(v,c)| = |\{u \in \underbrace{N_{L,G}(v,c) \cap V(G')}_{G^2 \to G'} : \underbrace{c \in L'(u)}_{L^2 \to L'}\}|$

But, it turns out we don't need to care about the lists, so instead let's look at $|N_{L,G}(v,c) \cap V(G')|$.

Clearly

$d_{L',G'}(v,c) \leq |N_{L,G}(v,c) \cap V(G')|$.

$\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]$

$= \sum_{u \in N_{L,G}(v,c)} \Pr[u \in V(G')]$ (lin. of exp.)

$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \notin V(G')]$ These events independent.

$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \in A \text{ and } \phi(u) \text{ is kept}] = d_{L,G}(v,c)(1-p\cdot\text{keep})$ ☺

On to some iterative calculations:

- Now, if we could show that with "high enough" probability every $|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$ are close enough to their expectations to apply the LLL, then we'll be happy via the following calculation:

$$\frac{|L'|}{\Delta_{L'}(G)} \gtrsim \frac{\mathbb{E}[|L'(v)|]}{\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]} = \frac{\text{Keep} \cdot |L|}{(1 - p\,\text{keep}) \cdot \Delta_L(G)}$$

Some ~~ratio~~ factor → we will argue this is close to 1

→ Old ratio

$$\geq \frac{|L|}{\Delta_L(G)} \cdot \frac{1 - \frac{p\Delta}{|L|}}{1 - p\left(1 - \frac{p\Delta}{|L|}\right)}$$

{ using that $\left(1+\frac{x}{n}\right)^n \geq 1+x$ for ~~n≥~~ $n \geq 1$ and $|x| \leq n$.

Note: $\frac{\Delta}{|L|} = \frac{1}{1+\varepsilon}$ (some assumption we made)

$$= \frac{|L|}{\Delta_L(G)} \cdot \underbrace{\frac{1 - p/(1+\varepsilon)}{1 - p(1 - p/(1+\varepsilon))}}_{= 1 + \frac{p\varepsilon}{1+\varepsilon} - \frac{\varepsilon^2}{1+\varepsilon}}$$

Now if we choose $p = \varepsilon/2$, then we get

$$= 1 + \frac{\varepsilon^2}{4(1+\varepsilon)}$$

{ So, we get some improvement in each iteration.

How to show a variable is close to its expectation:

Concentration Inequalities:

Markov's Inequality:
If $X \geq 0$ is a random variable,
$$\Pr[X \geq k E[X]] \leq 1/k. \quad \forall k \geq 1$$

Chebyshev's Inequality:
The variance of a random variable X, denoted $Var[X]$ is
$$Var[X] = E[(X - E[X])^2]$$
The standard deviation σ of X is $\sqrt{Var[X]}$.

Let $\mu = E[X]$, then
$$\Pr[X \geq \mu + k\sigma] \leq 1/k^2.$$
Proof: Use Markov's inequality.

Chernoff Bounds:
Let $X = \sum_{i=1}^{n} X_i$ where X_i is a Bernoulli r.v. (i.e. only takes value of 0 and 1). and all independent:
$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}} \quad (0 \leq \delta \leq 1)$$
$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{3}} \quad \text{if } 0 \leq \delta \leq 1$$
$$e^{-\frac{\delta \mu}{3}} \quad \text{if } \delta \geq 1$$
(i.e. Exponentially small if δ constant and gives sth meaningful if $\delta \gg \frac{1}{\sqrt{\mu}}$!

Example:

$\mathbb{E}[|A \cap N_{L,G}(v,c)|] = p\Delta(G)$

$\Pr[|A \cap N_{L,G}(v,c)| \geq p\Delta(G)(1+\delta)] \leq e^{-\frac{\delta^2(p\Delta_L(G))}{3}}$ (By Chernoff).

i.e. $|A \cap N_{L,G}(v,c)| = p\Delta(G) \pm \sqrt{\Delta_L(G)\, \mathrm{polylog}\, \Delta_L(G)}$

with prob. $1 - \frac{1}{\Delta^c}$ for any c.

So, by LLL, this holds for all v with some positive prob.

But, we need to concentrate:

$|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$

Depends on color $c \in L(v)$ being in $L'(v)$ which depends on activation flips and color-assignments for $N_{L,G}(v,c)$ (But not independent!)

→ Even worse! (Stronger interactions with neighbors + 2nd neighbors)

"Simple" Concentration Bound (Book of Molloy & Reed).

Let X be a r.v. that depends only on the outcome of a set of independent trials $T_1, \ldots, T_n$. Suppose that changing the outcome of any one trial changes X by at most c (c constant) (This is called c-Lipschitz).

Then,

$$\Pr[|X - \mathbb{E}[X]| \geq t + 8c\sqrt{\mathbb{E}[X]}] \leq e^{\frac{-t^2}{3c^2 n}}$$

Remark: Note that the denominator has the # of trials (not exponential)

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Concentrating $|L'(v)|$:

Trials: activating flips & colourings of $N_G(v,c)$ $\forall c \in L(v)$

- 1-lipschitz: Since changing any one trial (activ. or colouring) changes $|L'(v)|$ by at most 1.

Recall: $\mathbb{E}[|L'(v)|] = K_{exp} \cdot \Delta_L(G) = \Theta(\Delta_L(G))$

What is r?

- If not doing colour degree, then 2Δ (Good!)
- But, w/ colour degree it's:

$2\Delta_L(G) \cdot |L|$ (If every color the neighbours holds are disjoint).

$= 2(1+\varepsilon)\Delta_L(G)^2$

(This is Bad! The simple concentration bound is only meaningful for $t > \frac{\Delta_L(G)}{\omega}$)

Talagrand's Inequality (Versions)

(A) Combinatorial version: (There is also a probability one)

Let $X \geq 0$ depend on independent trials $T_1, \ldots, T_n$. If X is c-lipschitz and r-verifiable, then for any $t > 96c\sqrt{rc^2\mathbb{E}[X]} + 128rc^2$,

then:

$$\Pr[|X - \mathbb{E}[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}}$$

Remark: If r, c constant, then get ~~expo~~ exponentially small in t, if $t = \Theta(\mathbb{E}[X])$ and still meaningful for $t \gg \sqrt{\mathbb{E}[X]}$.

r-verifiable: For every $s>0$, if $X \geq s$, then there exists a set Z of at most rs trials that "verify" that $X \geq s$, i.e. changing any trial outside of Z still results in $X \geq s$.

So, of course, counting # of heads in coin flips is 1-verifiable.
Similarly, $|A \cap N_{L,G}(v,c)|$ is 1-verifiable
(We just exhibit the heads or the activated set).

What about $|L'(v)|$?
Example: How do we show $|L'(v)| \geq 1$ (i.e. One color kept).
For all $u \in N_{L,G}(v,c)$ show either $u \notin A$ or $\Phi(u) \neq c$.
i.e. Need $\Delta_{L,G}(v,c)$ trials to verify
i.e. Need $r \geq \Delta_{L,G}(v,c)$ (This is bad!)

Pick a better variable:
$|L(v)| - |L'(v)|$ = # of colors lost.
$E[|L(v)| - |L'(v)|] = |L| - \text{keep}|L|$
$= |L|(1-\text{keep})$
$= (1+\varepsilon)\Delta_L(G)(1-\text{keep}) = \Theta(\Delta_L(G))$

Obviously, this variable is 1-Lipschitz.
What do we need to verify a color is lost?
→ Need a neighbor $v \in N_{L,G}(v,c)$ to be activated and $\Phi(v) = c$
i.e. Need 2 trials
More generally, if $|L(v)| - |L'(v)| \geq s$, need $2s$, so $r=2$ works.