CO342 - Introduction to Graph Theory

(Notes Scans)

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Contents

1 Connectivity

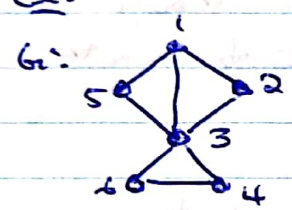
Connectivity.

In this course, we assume a graph $G=(V,E)$ is simple unless otherwise stated*.

Def'n (Subgraph)
Let S be a set of vertices in G. Then, the subgraph induced by S, denoted $G[S]$, consists of S as the vertex set, and all edges in G joining 2 vertices in S.

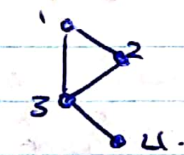
Ex:

G:



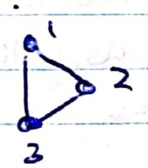
$S = \{1,2,3,4\}$

$G[S]$:



$T = \{1,2,3\}$

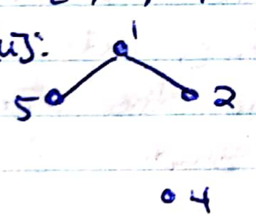
$G[T]$:



We call $G[T]$ an induced cycle.

$U = \{1,2,4,5\}$

$G[U]$:



$G[U]$ is not an induced cycle

Def'n (Min/Max degree)
We use $\delta(G)$ to denote the minimum degree among all vertices in G, and $\Delta(G)$ for the maximum degree

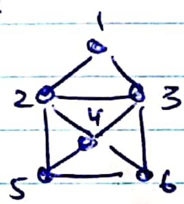
Ex: With G above $\delta(G)=2$, and $\Delta(G)=5$.

Edge Connectivity:

Recall:
A **bridge** or **cut-edge** in G is an edge whose removal increases the # of components in G.

Def'n (Disconnecting Set)
A set of edges F is a disconnecting set if removing all edges in F from G results in a disconnected graph. We use $G-F$ to denote this removal.

Ex:
G:



Some disconnecting sets are:
$\{12, 13\}$
$\{12, 23, 36, 24, 45, 46\}$
↳ Set of all edges

Note:
- If a graph is already disconnected, even the empty set is a disconnecting set.
- In general, we want to minimize the size of a disconnecting set in a graph.

Def'n (k-edge-connected)
A graph is **k-edge-connected** if there does not exist a disconnecting set of size at most $k-1$. In other words, any disconnecting set must have size at least k.

Def'n (Edge Connectivity)
The edge connectivity of G is the largest k for which G is k-edge-connected. (i.e. G is not $(k+1)$-edge-connected), We write $K'(G)$ to denote this.

Ex: For G above, it is 2-edge-connected, since it does not have a cut-edge. However, it is not 3-edge-connected since it has a disconnecting set of size 2.
So, $K'(G) = 2$.

Remarks:
- If G is not connected, then it is 0-edge-connected
- A connected graph is 1-edge-connected. If it has a bridge, then the edge connectivity is 1
- A connected graph with no cut-edges is 2-edge-connected
- If G is k-edge-connected, then G is $(k-1)$-edge-connected
- If $K'(G) = k$, then there exists a disconnecting set of size k, and it is a minimum disconnecting set.
- A graph with a single vertex has no disconnecting set. We set its edge connectivity to be 0.

Proposition 2.1:
$K'(G) \leq \delta(G)$

[Proof]
Assume G is non-trivial, so let v be a vertex of degree $\delta(G)$. Let E be the set of all edges incident with v. Then $G-E$ has no path from v to any other vertices. So, E is a disconnecting set of size $\delta(G)$, so G is not $(\delta(G)+1)$-edge-connected, and so $K'(G) \leq \delta(G)$ $\square$.

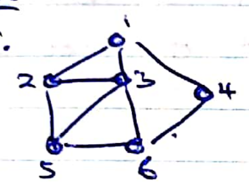
Cuts and Bonds:

Def'n (Cut)

For a set $S \subseteq V(G)$, the cut induced by S (or a cut), is the set of all edges with one end in S, and one end not in S, denoted $\delta(S)$. A cut is non-trivial if $\delta \neq 0$ and $\delta \neq V(G)$

Ex:

G:



$S = \{1, 2, 3, 5\}$

$\delta(S) = \{56, 14, 36\}$.

Recall (from M239):

G is connected if and only if every non-trivial cut is non-empty.

So, we get the following:

Corollary 2.1:

Any non-trivial cut is a disconnecting set.

Q: Is the converse of this true?

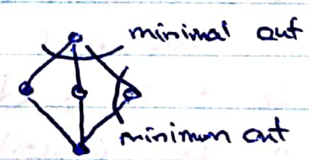
A: No! For example, you could take out all the edges of the graph.

Proposition 2.2:

For any connected graph G with at least 2 vertices, any minimal disconnecting set F is a cut, and G−F has exactly 2 components.

Note:

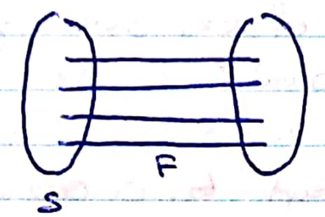
A minimal set means there is no proper subset with the same properties.



. Cr Proof

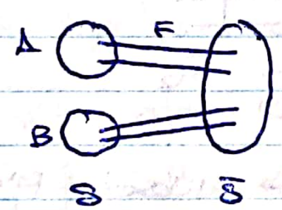
[Proof]

Let F be any minimal disconnecting set, we want to show a set $S \subseteq V(G)$ exists such that $\delta(S) = F$.
F will split G into at least 2 components:



Take S to be the vertices of one component of $G-F$. Consider the edges of $\delta(S)$. We see that $\delta(S) \subseteq F$, since no edge is in $\delta_{G-F}(S)$ because $G-F$ is a component, and so any edge in $\delta_G(S)$ must be in F.
Now, since $\delta(S)$ is a disconnecting set and F is minimal, $\delta(S) = F$, so F is a cut.

Now, suppose $F = \delta(S)$, and $G-F$ has at least 3 components.
WLOG, suppose $(G-F)[S]$ has at least 2 components



Let A and B be the vertex sets of 2 of the component
Now, $\delta_G(A)$ and $\delta_G(B)$ are non-empty (since G connected), so using a similar argument as above $\delta_G(A) \subsetneq F$ and $\delta_G(B) \subsetneq F$, contradicting the minimality of F. So therefore there must be only 2 components $\square$

Def'n (Bond)

A **bond** is a minimal non-empty cut.

Q: Suppose you have a set of edges, that, if you remove out of G, gives you 2 components. Is that a bond?

Theorem 2.1:

In a connected graph G, a cut $\delta(S)$ is a bond if and only if $G-F$ has 2 components.

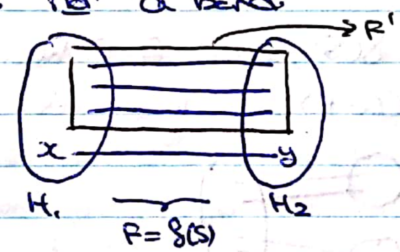
(This is a characterization of bonds in terms of cuts)

[Proof]

($\Rightarrow$) See Prop'n 2.2.

($\Leftarrow$)

Suppose $G-F$ has 2 components H_1, H_2, and suppose F is not a bond.



Then, there is a proper subset $F' \subsetneq F$ that is a cut. Let $e = xy \in F \setminus F'$, where $x \in V(H_1)$ and $y \in V(H_2)$. So e is in $G-F'$. Each vertex in H_1 has a path to x, and each vertex in H_2 has a path to y, and they can reach each other via e. So $G-F'$ is connected, contradicting the fact that F' is a cut.

So, F must be a bond $\square$.

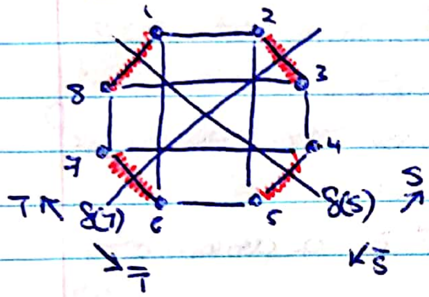
Def'n (Symmetric Difference)

The symmetric difference of 2 sets A, B is:

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

(i.e. The elements that appear exactly once)

Ex:



Let $S = \{1, 2, 3, 4\}$,
$T = \{1, 2, 7, 8\}$

What is $\delta(S) \triangle \delta(T)$?

$\delta(S) \triangle \delta(T) = \{18, 23, 67, 45\}$, all edges that only appear once.

Q: Is $\delta(S) \triangle \delta(T)$ a cut of some set?

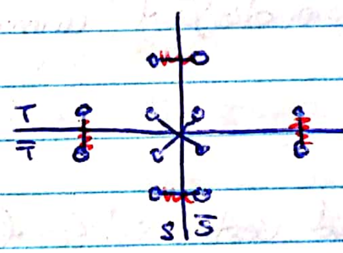
A: Yes! Let $U = \{3, 4, 7, 8\}$, then $\delta(S) \triangle \delta(T) = \delta(U)$, in fact notice that $\{3, 4, 7, 8\} = S \triangle T$.

Theorem 3.1:

$\delta(S) \triangle \delta(T) = \delta(S \triangle T)$

[Sketch]

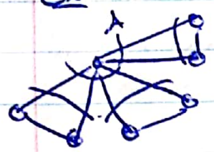
There are six types of edges



But, only the edges in red will appear in the symmetric difference. These are precisely the edges in $\delta(S \triangle T)$. $\square$

Theorem 3.2:
Every Cut is a disjoint union of bonds

Ex:



Cut A is a union of the 3 other cuts (bonds)

[Proof]

We will prove this by induction on the number of edges in the cut.

Base Case: If $|F|=0$, then F is a union of no bonds.

IH: Now, suppose F is any nonempty cut

IC:

If F is a bond, then we're done

So, assume F is not a bond. Then, there exists a proper subset $F' \subsetneq F$ that is also a nonempty cut. Then $F'' = F \setminus F' = F \Delta F'$, and by previous result, F'' is also a nonempty cut. By induction, F' and F' are both disjoint union of bonds.

Hence $F = F' \cup F''$ is a disjoint union of bonds $\square$

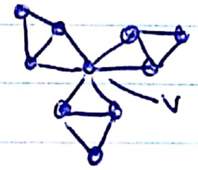
Vertex Connectivity

Def'n (Cut-vertex)

A vertex v is a cut-vertex in G if removing v and its incident edges from G increases the number of components. (We write $G-V$)

Note: The major difference between a cut-vertex and cut-edge is that removing a cut vertex may increase the number of components by more than one

Ex:

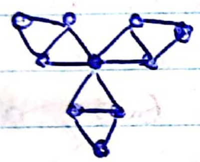


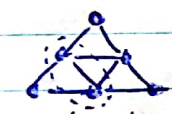
The vertex labelled v is a cut-vertex.

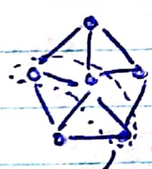
Def'n (Seperating set)
A subset S of vertices is a seperating set if removing all vertices in S from G results in a disconnected graph. (We write $G-S$)

Def'n (k-connected, connectivity)
A graph is k-connected if does not have any seperating sets of size at most $k-1$. The connectivity of G is the largest k for which G is k-connected, denoted $K(G)$.

Ex:

(i)  here $K(G)=1$

(ii)  is 2-connected, but not 3-connected
seperating set of size 2, $K(G)=2$

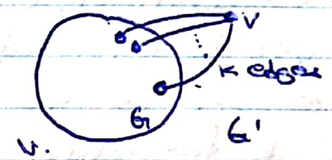
(iii)  is 3-connected, but not 4-connected
seperating set of size 3, $K(G)=3$

Note:
K_n has no seperating set, so we define $K(K_n)=n-1$

CP Expansion
Lemma

Expansion lemma:
Let G be a k-connected graph, and let G' be obtained from G by adding a new vertex v that is joined to at least k vertices in G. Then, G' is k-connected.

[Proof]



Let X be any set of $k-1$ vertices in G'.

Case 1: X doesn't include v.
Then, $G-X$ is connected since G is k-connected and at least one neighbor of v is not in X (v is incident to k edges and we are removing $k-1$) So v is connected to $G-X$, and so $G'-X$ is connected.

Case 2: X does include v.
Then, $G'-X$ is the same as removing $k-2$ vertices from G, and since G is k-connected $G'-X$ is connected.

Therefore, G' is k-connected $\square$

Theorem (Whitney)

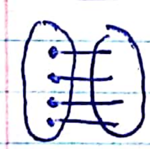
$\kappa(G) \leq \kappa'(G) \leq \delta(G)$

vertex connectivity

edge connectivity

min. degree of graph

Caution:



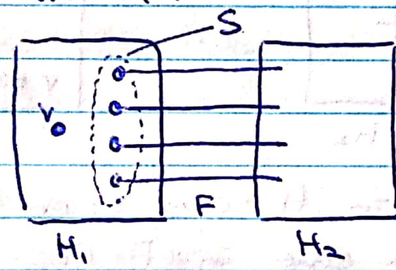
Suppose we have a min. cut, we can't just take out all the vertices, because what if they are the only vertices in the partition?
This motivates the cases in the proof.

Go [Proof]

[Proof]

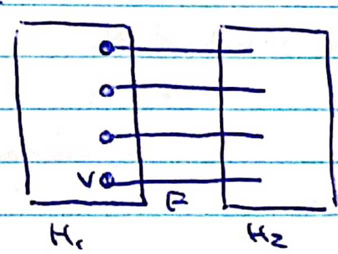
We've already proved that $\kappa'(G) \le \delta(G)$ (Prop'n 2.1).
So, we only need to prove that $\kappa(G) \le \kappa'(G)$.
For $G = K_n$, $\kappa(G) = \kappa'(G) = n-1$, so we're done.
Now, assume G is not complete. Let F be a minimum cut, so $|F| = \kappa'(G)$.
(Goal: Find a separating set of size at most $|F|$)
We have 2 cases:

Case 1:
Suppose there is a vertex v not incident with any edge in F.



Suppose $G-F$ has 2 components H_1, H_2 and WLOG suppose $v \in V(H_1)$. Let S be the set of vertices in H_1 incident with some edge in F. Then $|S| \le |F|$, and it is [redacted] a separating set since v is separated from H_2 in $G-S$. So, $\kappa(G) \le \kappa'(G)$.

Case 2:
Now, suppose that every vertex is incident with an edge in F.



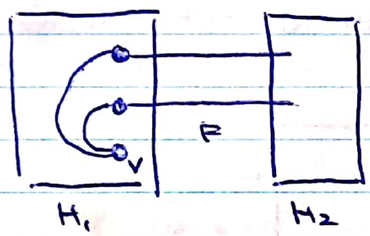
Cont

[Proof] (Con't)

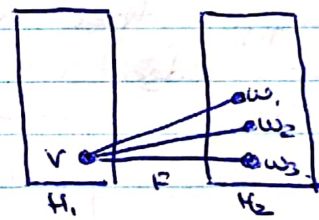
Case 2: (Con't)

Since G is not complete, there exist $v, w \in V(G)$ that are not adjacent. WLOG, suppose $v \in V(H_1)$, and let X be the set of all neighbours of v. This is a seperating set since it separates v from w. The neighbours of v can be in H_1 or H_2.

In H_1:



In H_2:



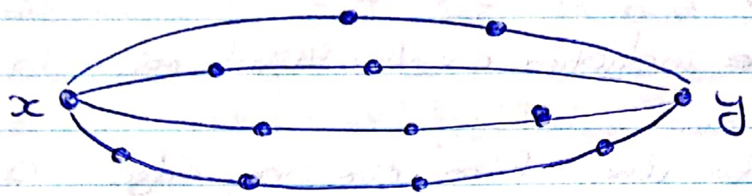
Neighbours of v in H_1 are each [redacted] incident with an edge in F (by assumption.) Neighbours of v in H_2 also each contribute one edge to F.

These must all be distinct edges, and hence $\deg(v) \leq |F|$, so $|X| \leq |F|$, and $\kappa(G) \leq \kappa'(G)$ $\square$.

Menger's Theorem:

Motivation:

Suppose we have the following graph, how many connections do we remove to destroy the connection between x and y?



In this case, we need at least 4 vertices (one for each path)

Now. Suppose you cannot destroy the connection if you remove v vertices, does that mean there are at least $v+1$ paths from x to y?

Def'n (x,y-separating set)

An *x,y-separating set* X is a separating set where x,y are in different components of $G-X$.

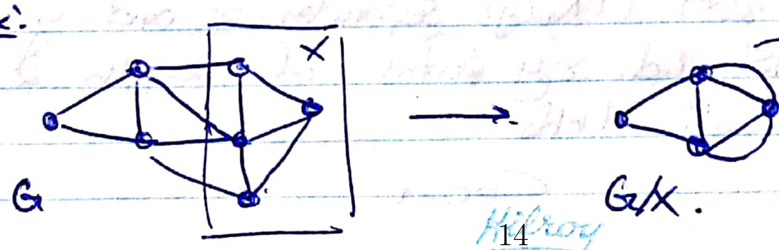
Def'n (Internally-disjoint)

A set of x,y-paths is *internally disjoint* if no two such paths share any vertices or edges except x,y.

Def'n (Shrinking/Contraction)

For a set X of vertices where $G[X]$ is connected, we denote G/X to be the graph obtained by removing all edges in $G[X]$, and identifying all vertices of X into one vertex.

Ex:



→ We might have multiple edges to one vertex.

Theorem (Menger's Theorem)
Let x, y be 2 non-adjacent distinct vertices in G. Then, the minimum size of an x,y-separating set is equal to the maximum number of internally-disjoint x,y-paths.

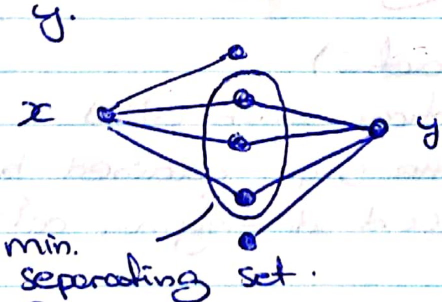
[Proof]

We use induction on the # of edges in G.

Base Case: When there are no edges, any minimum x,y-seperating set has size 0, and no x,y-paths exist.

IH: Let G be any graph with at least one edge. Let x, y be 2 non-adjacent vertices. Suppose a minimum x,y-seperating set has size k.
(Goal: Find k internally-disjoint x,y-paths)

IC:
We have 2 cases.
Case 1: (Easy)
Assume that every edge is incident with x or y.



Then, $S = N_G(x) \cap N_G(y)$ separates x and y, and we have $|S|$ i.d. x,y-paths of length 2, so the result holds

Cont

[Proof] (con't)

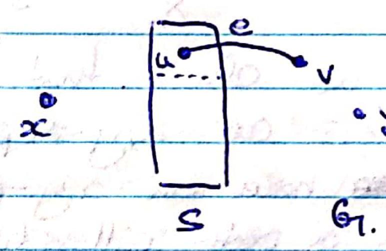
Case 2:

Let $e = uv$ be an edge in G not incident with x or y. Let $H = G - e$ and let S be a minimum x,y-seperating set in H.

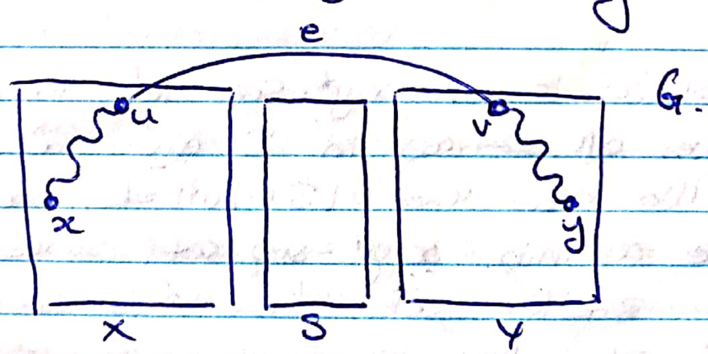
If $|S| = k$, then by induction, H contains k i.d x,y-paths which are also in G, and so we're done.

So, assume $|S| \leq k-1$. We know that $S \cup \{u\}$ is an x,y-seperating set in G.

But, any x,y-seperating set in G has size of least k. So,
$|S \cup \{u\}| \geq k \Rightarrow |S| \geq k-1$,
but we assumed that
$|S| \leq k-1$, so $\boxed{|S| = k-1.}$



Now, let $S = \{v_1, v_2, \ldots, v_{k-1}\}$, and let X, Y be components containing x and y in $H-S$ respectively



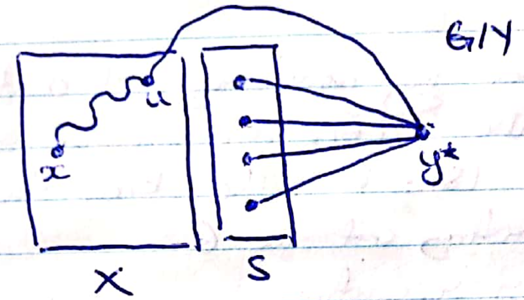
Since $|S| < k$, S is not an x,y-seperating set in G, so $G-S$ has an x,y-path that must use e

$\hookrightarrow$ con't

[Proof] (con't)

Suppose $u \in V(X)$ and $v \in V(Y)$.

Consider G/Y, and let y^* be the contracted vertex.



First, we claim that any x,y^*-sep set T in G/Y is an x,y-sep set in G. Why? Suppose we have an x,y-path that avoids T in G, such a path will still exist in G/Y, since we have done nothing but contract the graph. (i.e. we haven't removed paths). So, any x,y-separating set in G/Y have size at least k.

Now, $S \cup \{u\}$ is an x,y^*-sep set in G/Y since it removes all vertices in S and the edge $e = uv$. We know that $|S \cup \{u\}| = k$, so it must also be a min. x,y^*-sep set, since if there were a smaller set, then this would imply that G also has a smaller x,y-sep set.

We have contracted the v,y-path in G/Y, so G/Y has fewer edges than G. Hence, by induction, there exist k i.d.* x,y^*-paths in G/Y.

$\hookrightarrow$ con't

[Proof] (Con't)

And we can describe what the paths are.
Let $P_1, \ldots, P_{k-1}$ be part of the paths that start at x and end at $v_1, \ldots, v_{k-1}$ respectively, and let P_k be the x,u-path.
Similarly, we can consider G/X and find k paths $Q_1, \ldots, Q_{k-1}$ from v_i to y and Q_k from u to y. Then $P_i + Q_i$ for $i = 1, \ldots, k-1$, and $P_k + e + Q_k$ are k i.d. x,y-paths in G. □.

Note:

Why do we handle the 2 cases separately?
Notice that contracting vertices in Case 1 does nothing! Hence, induction cannot be used properly, so we must deal with that case by itself.

Corollary 4.1

G is k-connected iff there exist k i.d x,y-paths for any distinct $x,y \in V(G)$.

[Proof]

($\Rightarrow$) We break this into 2 cases!

Case 1: (x,y not adjacent)
Since G is k-connected, any x,y-sep set has size at least k. Then, by Menger's thm, there exist k i.d. x,y-paths.

Case 2: (x,y adjacent)
$G - xy$ is $(k-1)$-connected. Then, by Menger's thm, $G - xy$ have $k-1$ i.d. x,y-paths and together with xy, they form k i.d. x-y paths.

Con't

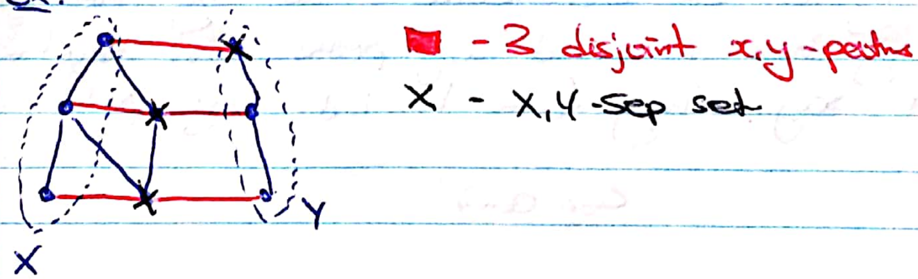
Hilroy

[Proof] (con't)
($\Leftarrow$) Let X be any set of $k-1$ vertices, and let x and y be any 2 vertices in $G-X$. Since there are k i.d x,y-paths (by assumption), at least one path remains in $G-X$, (as we are only removing $k-1$ vertices) So $G-X$ is still connected, and hence G is k-connected. ☐.

Variations (of Menger's Thm)
We now look at different versions of Menger's thm to be used in different contexts.

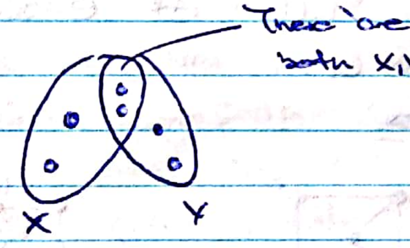
Def'n (X,Y-separating set)
For sets of vertices X, Y, an X,Y-seperating set is a set S, where $G-S$ has no path from any vertex in X to any vertex in Y
Note! S may itself include vertices in X or Y, and S could also be the entire set X or Y, i.e. $S=X$ or $S=Y$.

Def'n (disjoint X,Y-paths)
A set of disjoint X,Y-paths are paths that start with a vertex in X, ends with a vertex in Y, no other vertices are in X or Y, and no paths share any vertices.
Ex.

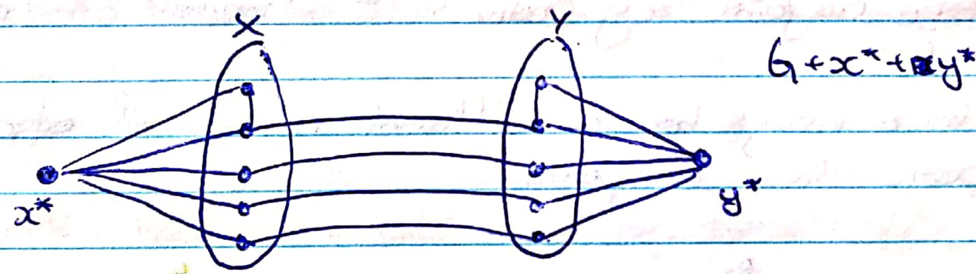


Theorem 6.1: (Set Version of Menger's)
For any $X, Y \subseteq V(G)$, the minimum size of an X,Y-separating set is equal to the maximum number of disjoint X,Y-paths.

Note:
If X, Y share vertices, then the vertex itself is an X,Y-path.



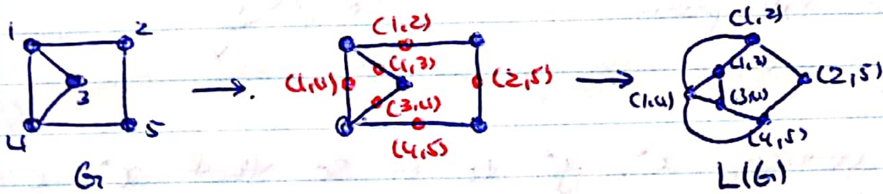
[Proof]
Add vertices x^*, y^* to G so that x^* is adjacent to all vertices in X and y^* is adjacent to all vertices in Y. Since x^* and y^* are not adjacent, by Menger's thm, the max # of i.d x^*,y^*-paths is equal to the min size of an x^*,y^*-sep set. Such a sep set is an X,Y-sep set in G.



We have to be careful when choosing an disjoint X,Y-paths. For each x^*,y^*-path, we pick the part from the last vertex in X to the first vertex in Y. This gives us the same # of disjoint X,Y-paths. □

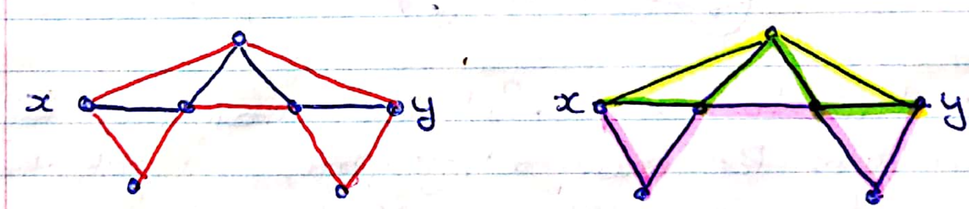
Def'n (Line Graph)
Let $G=(V,E)$. The line graph of G is $L(G)$, where the [redacted] [redacted] vertices are $\{v_e \mid e \in E(G)\}$, and two vertices v_e and v_f are adjacent in $L(G)$ iff e and f share a common endpoint in G.

Ex!



Theorem 6.2 (Edge Version of Menger's)
Let x,y be 2 distinct vertices in G. Then, the minimum size of an x,y-cut (disconnecting set) is equal to the maximum number of edge-disjoint x,y-paths.

Note!
There may be a different number of edge-disjoint paths than i.d. paths.



2 i.d. paths. 3 edge-disjoint paths

(Proof.

[Proof]

Consider $L(G)$. Let X, Y be the set of vertices in $L(G)$ corresponding to edges incident with x, y in G respectively. Notice that all edge disjoint paths in G are vertex disjoint in $L(G)$. (An edge only being used once in G results in a vertex only being crossed once in $L(G)$). So, an edge-disjoint x,y-path in G corresponds to disjoint X,Y-paths in $L(G)$, and x,y-cuts in G correspond to X,Y-sep sets in $L(G)$. The result then follows from Thm 6.1. (Set Version of Menger's)

$\square$

Corollary 6.1:

G is k-edge-connected iff there exist at least k edge-disjoint x,y-paths for any distinct $x,y \in V(G)$

[Proof]

Similar to Corollary 4.1, but with the edge version of Menger's.

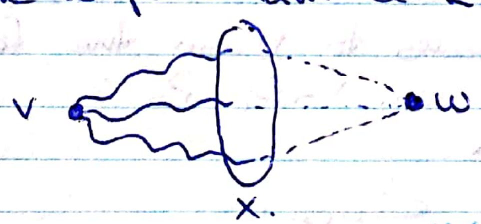
$\square$

Theorem 7.1 (Fan Lemma):

Let G be k-connected, $v \in V(G)$ and $X \subseteq V(G) \setminus \{v\}$, where $|X| \geq k$. Then, there exist a k-fan from v to X in G.

[Proof]

Obtain H from G by adding a vertex w, and join w to all vertices in X. Since $|X| \geq k$, by expansion lemma, H is k-connected. By corollary of Menger's, there are k i.d. v,w-paths in H. For each such path, we keep the part from v to the first vertex in X on that path. These k paths form a k-fan from v to X $\square$



Hilroy

Corollary 7.1':
Let G be k-connected where $k \geq 2$, let X be any set of k vertices in G. Then, there is a cycle in G containing at least all vertices in X.

[Proof]

We prove by induction on k.

Base Case: When $k=2$, for any $X = \{u, v\}$, there exist 2 internally-disjoint u,v-paths. Since G is 2-connected. These 2 paths form a cycle containing u, v

IH: Let G be k-connected, where $k \geq 3$, let X be any set of k vertices in G.

IC:

Since G is k-connected, it is also $(k-1)$-connected. Let $v \in X$, then $X \setminus \{v\}$ have size $k-1$. By induction, there is a cycle containing all vertices in $X \setminus \{v\}$, say it is C.

If C contains v, then we are done.

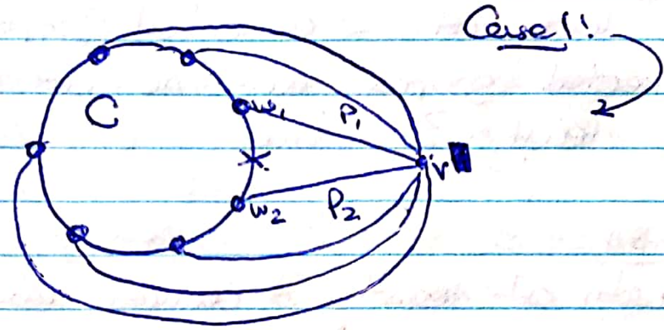
Otherwise, assume v is not in C. We have 2 cases:

Case I: (C has $k-1$ vertices)

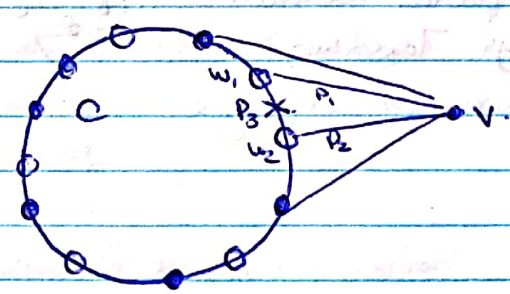
So, C only contains $X \setminus \{v\}$. By the fan lemma, there exist a $(k-1)$-fan from v to $V(C)$. Take any consecutive vertices w_1, w_2 on C, let P_1, P_2 be the paths joining w_1, w_2 to v in the fan, respectively. Then $(C - w_1 w_2) + P_1 + P_2$ is a cycle containing X.

Con't

[Proof] (cont.)



Case 2: (C has $\geq k$ vertices)
By fan lemma, there exists a k-fan from v to $V(C)$. The $k-1$ vertices in $X \setminus \{v\}$ partitions C into $k-1$ i.d. paths which we call segments. By pigeonhole principle, at least 2 of the k paths in the fan must end at vertices in the same segment. Let w_1, w_2 be these two end vertices, let P_3 be the $w_1 w_2$-path in the segment. Let P_1, P_2 be the $w_1 v$-path and $w_2 v$-path in the fan. Now P_3 has no internal vertices in X, so $(C - P_3) + P_1 + P_2$ is a cycle containing X.



O - additional vertices
• - vertices in $X \setminus \{v\}$.

$\square$.

2-connected graphs:

We now look at 2-connected graphs, which are connected graphs with no cut vertices, and at least 3 vertices.

Theorem 8.1:
For G with at least 3 vertices, and $\delta(G) \geq 1$, the following are equivalent:

(1) G is 2-connected,

(2) For any distinct $x, y \in V(G)$, there is a cycle containing x, y

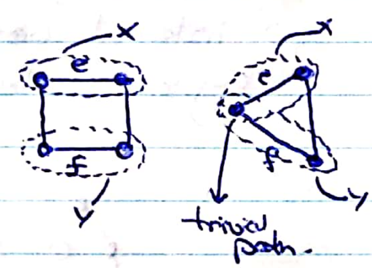
(3) For any distinct $e, f \in E(G)$, there is a cycle containing e, f

[Proof]

(1) $\Leftrightarrow$ (2): Proved by corollary of Menger's

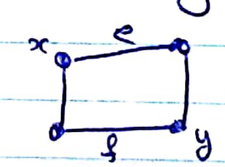
(1), (2) $\Rightarrow$ (3):

Let $e = uv$, $f = xy$. By the set version of Menger's thm, there are 2 disjoint paths between $\{u, v\}$ and $\{x, y\}$. Together, with e and f, we get a cycle containing e, f.



(3) $\Rightarrow$ (2):

Let e, f be edges incident with x, y respectively. If e, f are distinct, then, by assumption, there is a cycle which also contains x, y. Notice that e, f cannot be the same, since if they are, then there must also be a 3rd vertex AND the edge adjacent to it cannot be in the cycle. But this is a contradiction. $\square$



Ear Decomposition:

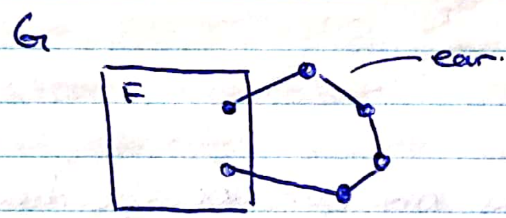
Def'n (Ear / Ear Decomposition)

Let F be a subgraph of G. An ear of F in G is a non-trivial path in G where only the 2 endpoints of the path are in F.

An ear decomposition of G is a sequence of graphs $(G_0, G_1, \ldots, G_k)$ where:

- G_0 is a cycle in G,
- For $0 \le i \le k-1$, $G_{i+1} = G_i + P_i$, where P_i is an ear of G_i in G,
- $G_k = G$.

Ex.



Theorem 8.2:

A graph G is 2-connected iff G has an ear decomposition. Furthermore, any cycle in G can be the initial cycle of an ear decomposition, and all intermediate graphs in the ear decomposition are 2-connected.

[Proof]

($\Rightarrow$) Suppose G is 2-connected. Let C be any cycle in G, and let $G_0 = C$. For $i \ge 0$, we inductively construct G_{i+1} from G_i as follows.

If $G_i = G$, then we're done.

Otherwise, there is an edge $e \in E(G) \setminus E(G_i)$. Say $e = uv$.

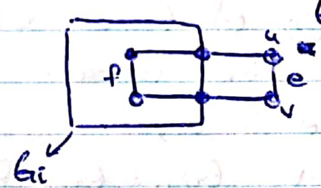
By Menger's thm for sets, there are 2 disjoint paths between $\{u, v\}$ and $V(G_i)$

$\hookrightarrow$ Con't

[Proof] (Con't)

(⇒) (Con't)

These 2 paths only meet G_i at one of their endpoints, so together with e, we obtain an ear P_i of G_i.



Let $G_{i+1} = G_i + P_i$. Since G_{i+1} has more edges than G_i, this process will terminate. (Notice that we could also have used thm 8.1)

(⇐)

Suppose that G has an ear-decomposition $(G_0, G_1, \ldots, G_t)$, where $G_{i+1} = G_i + P_i$.
We will prove that G is 2-connected by induction on i (i.e. Each G_i is 2-connected).

<u>Base Case</u>: G_0 is a cycle, which is 2-connected

<u>IH</u>: Assume that G_i is 2-connected

<u>IC</u>:

Let $x, y \in V(G_{i+1})$. If both $x, y \in V(G_i)$, then they are in a cycle since G_i is connected. There are 2 more cases. Suppose the 2 endpoints of P_i are u, v. If $x, y \in V(P_i)$, then P_i plus a u,v-path in G_i form a cycle containing x, y.
If $x \in V(P_i)$, $y \in V(G_i)$, then by fan lemma, there is a 2-fan in G_i from y to $\{u, v\}$. Together with P_i, we obtain a cycle containing x, y. Since there is a cycle containing x, y for all cases, G_{i+1} is 2-connected.

□

Strong Orientation: (and 2-connected graphs)

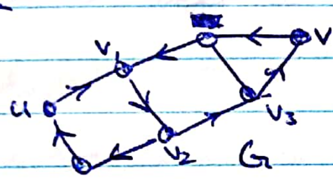
Def'n (Orientation)

An orientation of G can be obtained by adding a direction to each edge uv, either $u \to v$ or $v \to u$.

Def'n (Directed u,v-path)

A directed u,v-path has the form $u, v_1, v_2, \dots, v_k, v$, where the edges are directed $u \to v_1, v_1 \to v_2, \dots, v_k \to v$.

Ex:



This orientation of G gives a directed u,v-path u, v_1, v_2, v_3, v.

Def'n (Strong Orientation)

A strong orientation is one where there is a directed u,v-path for any ordered pairs $u,v \in V(G)$

Theorem 9.1

If G is 2-connected, then G has a strong orientation

Note:

The converse of this statement is not true. For example, we could consider the "bowtie" graph:

[Proof]

Consider an ear decomposition of G, $(G_0, \dots, G_k)$. We will inductively orient G_i so that they all have a strong orientation.

Cont

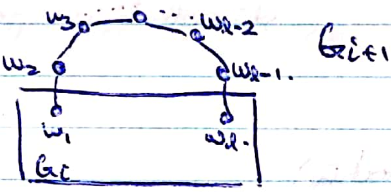
Hilroy

[Proof] (con't)

Base Case: G_0 is a cycle $v_1, v_2, \ldots, v_k, v_1$.
We will orient this $v_1 \to v_2$, $v_2 \to v_3$, $\ldots$, $\to v_k \to v_1$.
This is a strong orientation.

IH: Suppose G_i has a strong orientation.

IC:
Let $w_1, w_2 \ldots w_\ell$ be the ear we add to G_i to obtain G_{i+1}.



Now, orient the path $w_1 \to w_2$, $w_2 \to w_3$, $\ldots$, $w_{\ell-1} \to w_\ell$, and let $u, v \in V(G_{i+1})$. We have multiple cases:

Case 1 ($u, v \in V(G_i)$)
By IH, there is a directed u,v-path in G_i since it is strongly oriented.

Case 2 ($u = w_j$ and $v \in V(G_i)$).
u,v-path: Take $w_j, \ldots, w_\ell$ followed by a directed w_ℓ, v-path in G_i.
v,u-path: Take a v, w_1-path in G_i followed by $w_1, w_2, \ldots, w_j$.

Case 3 ($u = w_j$, $v = w_p$, $j < p$).
u, v-path: Take $w_j, w_{j+1}, \ldots, w_p$.
v, u-path: Take $w_j, \ldots, w_\ell$, followed by w_ℓ, w_1-path in G_i, and then $w_1, \ldots, w_j$.

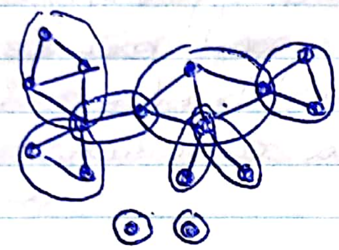
$\square$

Blocks: (and 2-connected graphs)

Def'n (Block)

A block is a maximal connected subgraph with no cut vertices.

Ex:



O-are the blocks of this graph.

Note:

1) Each block is an induced subgraph,
2) Each block is either 2-connected, a single edge, or an isolated vertex.
3) Some results may be obtained by focusing on the individual blocks (ex. A graph has a strong orientation iff each of its blocks has a strong orientation) [Theorem 10-1].

Proposition 9.1:

Two blocks in a graph share at most one vertex. Such a vertex must be a cut-vertex.

[Proof]

Suppose blocks B_1, B_2, share at least 2 vertices, so we may also assume B_1, B_2 each have at least 3 vertices, so it is 2-connected.

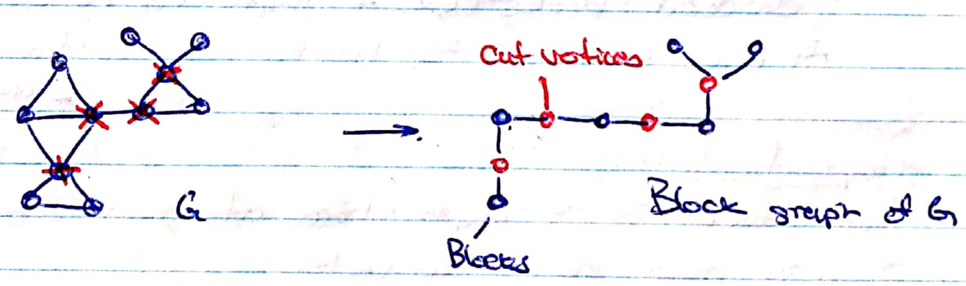
Our strategy is to show that $B_1 \cup B_2$ is 2-connected, which contradicts that B_1 and B_2 are blocks.

Con't

[Proof] (con't)
Let $x \in V(B_1 \cup B_2)$, and since $|V(B_1) \cap V(B_2)| \geq 2$, there exists $y \in V(B_1 \cap B_2)$, where $y \neq x$. Since B_1 and B_2 are 2-connected, $B_1 - x$ and $B_2 - x$ are 2-connected. Since $y \in V(B_1) \cap V(B_2)$, there is a path between y and each vertex in $V(B_1) \setminus \{x\}$ and $V(B_2) \setminus \{x\}$. Therefore $(B_1 \cup B_2) - x$ is connected, a contradiction. □

Def'n (Block graph)
The block graph of G is a bipartite graph with bipartition (B, C), where B is the set of all blocks and C is the set of all cut-vertices. $b \in B$ is adjacent to $c \in C$ iff the block b contains vertex c.

Ex:



Proposition 10.1:
The block graph is always a forest
[Proof]
Since each component of G is connected, then each component of the block graph will also be connected.
Now, we show that the block graph has no cycles. Suppose a cycle exists

→ con't

[Proof] (con't)
The cycle must alternate between cut vertices and blocks. Let $v_0, \dots, v_k$ be the cut vertices in this cycle, and B be a block. $v_0 \dots v_k$ is a cycle in G, let's call C. Then, $B \cup C$ is a block since C is 2-connected, B is 2-connected, and removing a cut vertex does not disconnect the graph. But, this is a contradiction that B is a block. $\square$.

Proposition 10.2:
If u, v are in the same block B, then every u,v-path is entirely in B.

[Proof]
Suppose there is a u,v-path that includes edges not in B. Follow the path and consider the corresponding blocks and cut vertices visited in the block graph. This is a closed walk that visits each vertex in C at most once (since you can't visit the same cut vertex more than once in a path). Since we start and end in the same block, we must have a cycle in the block graph, which contradicts Prop'n 10.1. $\square$.

~~G~~ Theorem

Theorem 10.1:
A connected graph has a strong orientation iff each of its blocks has a strong orientation.
[Proof]
($\Rightarrow$) Suppose G has a strong orientation. We keep the same orientation for each block. Let u, v be 2 vertices in a block B. We know there is a directed u,v-path in G, and by previous prop'n (10.2), such a path must be entirely in B. So, B has a strong orientation.

($\Leftarrow$) Suppose each of the blocks of G have a strong orientation. We keep the same orientation of each block for G. Let $u, v \in V(G)$, we have 2 cases:
Case 1: (u, v are in the same block)
The strong orientation of the block gives a directed u,v-path.
Case 2: (u, v not in the same block)
Let $u \in B_1$, $v \in B_k$ and consider the B_1, B_k-path in the block graph: $B_1 v_1 \ldots v_{k-1} B_k$. (The path exists since G is connected). Since each block is strongly oriented, we can find directed paths: u to v_1 in B_1, v_1 to v_2 in B_2, etc. Together, these form a directed u,v-path in G.
$\square$.

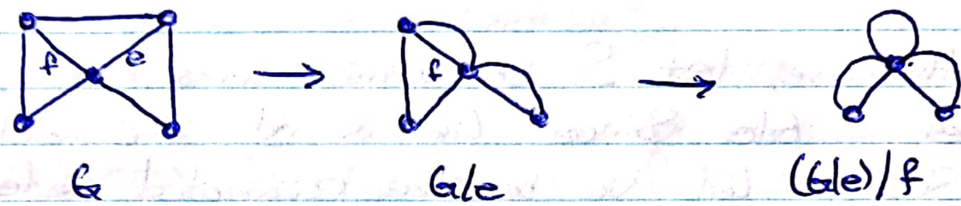
Theorem 10.2 (Robbin's Theorem)
G has a strong orientation iff G is 2-edge-connected.

3-connected graphs:

Def'n (Edge Contraction)

Suppose e is an edge in G. The contraction of e in G, denoted G/e is a graph obtained by removing e and identifying (merging) the 2 endpoints of e.

Ex:



Note: You may encounter multiple edges or loops in contractions, but since they do not affect vertex connectivity, we will remove them (for now)

Theorem 11.1:

Let G be a 3-connected graph with at least 5 vertices. Then, G has an edge e, where G/e is 3-connected.

[Proof]

We need a lemma:

Lemma 11.1:

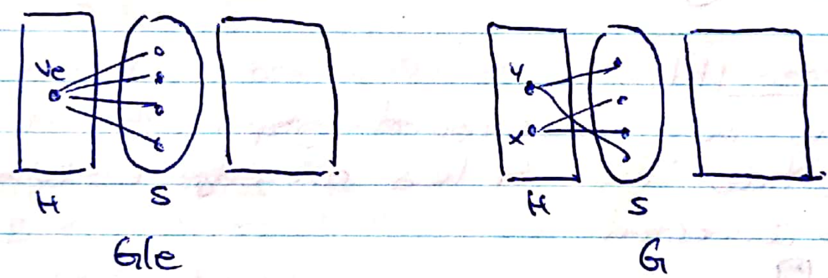
Let $k \geq 2$. If G is k-connected with at least $k+2$ vertices and $e=xy \in E(G)$ where G/e is not k-connected, then G is a separating set of size k that includes x and y.

Go [Proof] (Lemma)

[Proof] (con't)
[Proof] (Lemma)
We first deal with the case where G/e is complete, then it must have at least $k+1$ vertices. But then, it is k-connected, which contradicts our assumption. So, this case never happens.

Otherwise, let S be a minimum seperating set in G/e. Since G/e is not k-connected, $|S| \leq k-1$. Let V_e be the contracted vertex in G/e. There are 2 cases.

Case 1: ($V_e \notin S$)



Then $G/e - S$ has at least 2 components, and V_e is in one of them, say H. Neighbors of V_e are in $V(H)$ or S. The other component of $G/e - S$ remains a component in $G-S$. Then, S is a separating set in G of size at most $k-1$, a contradiction. So, this case cannot occur.

Con't

[Proof] (Con't)
[Proof] (Lemma) (Con't)
Case 2: ($v_e \in S$)
Then, $G/e - S$ is the same as $G - (S \setminus \{v_e\} \cup \{x, y\})$
So, $S \setminus \{v_e\} \cup \{x, y\}$ is a separating set in G of size k that includes x, y.

$\square$ Lemma.

We now prove the theorem. Suppose by way of contradiction that G/e is not 3-connected for all $e \in E(G)$. By Lemma 11.1, if $e = xy$, then G has a separating set of the form $\{x, y, z\}$, where $z \in V(G)$. We choose e and z so that the # of vertices in a component H of $G - \{x, y, z\}$ is minimized.

Let $S = \{x, y, z\}$. Since S is a minimal separating set, we know that every vertex in S must have at least one adjacent vertex in $G - S$. (Proof in A003, by contradiction). Let $w \in V(G)$ be a neighbour of z in H.

Now, G/zw is also not 3-connected, so there is a separating set $S' = \{z, w, v\}$ in G. And, since x and y are adjacent, there is a component J in $G - S'$ not containing x nor y. Notice that S' is also a minimum separating set. Let $u \in J$ be a neighbor of w, and so every vertex in J has a path to w via u. Such a path will not include x or y (since they are in another component), and hence this path also exists in $G - S$

Go con't

[Proof] (con't)
Further, we notice that every vertex in J is also in H. (Since we can reach w from z, and u from w, and all vertices in J through w and u). So $V(J) \subseteq V(H)$.
However, $w \in V(H)$, but $w \notin V(J)$, and so $V(J) \subsetneq V(H)$. But we assumed that H was minimal, and so we have a contradiction
□

The reverse operation of contraction (i.e. splitting a vertex v into v_1 and v_2) from a 3-connected graph also results in a 3-connected graph. However, we must be slightly careful.

Theorem 11.2:
Let G be 3-connected. Let v be a vertex of degree at least 4. Obtain H from G by splitting v into v_1, v_2, adding edge $e = v_1v_2$, and distributing edges incident with v to v_1, v_2 so that $\deg_H(v_1); \deg_H(v_2) \geq 3$. Then H is 3-connected

[Proof]
Suppose H is not 3-connected. Let S be a minimum separating set in H ($|S| \leq 2$)
We have 3 cases:

Case 1: ($v_1, v_2 \in S$)
Then, $G - v = H - \{v_1, v_2\}$ and so v is a cut vertex in G, contradicting our assumptions on G.

Cont

[Proof] (Con't)
Case 2 ($v_1, v_2 \notin S$)
v_1, v_2 must be in the same component of $H-S$, but then S is still a separating set in G, seperating v from the other component of $H-S$.

Case 3: ($v_1 \in S$, $v_2 \notin S$)
If the component containing v_2 has no other vertices, then all neighbors of v_2 are in S, but this contradicts that $\deg_H(v_2) \geq 3$.
So, there is at least one other vertex in the component containing v_2. But, then, this vertex will be seperated from the other component of $H-S$ by the set $(S \setminus \{v_1\}) \cup \{v\}$ in G, a contradiction

□.

Tutte's characterization of 3-connected graphs
G is 3-connected iff there exists a sequence $G_0, G_1, \ldots, G_k$ of graphs such that:
1) $G_0 = K_4$, $G_K = G$, and
2) For each $0 \leq i \leq k-1$, there exist an edge $xy \in G_{i+1}$ such that $\deg_{G_{i+1}}(x) \geq 3$, $\deg_{G_{i+1}}(y) \geq 3$, and $G_i = G_{i+1} / xy$.

[Proof]
Proof follows by induction on both sides, and by using Theorem 11.1 and 11.2.

□.

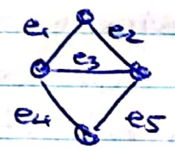
2 Vector Spaces for Graphs

Vector Spaces for Graphs.

Def'n (Edge Space)

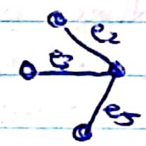
The edge space of G, denoted $\mathcal{E}(G)$, is the set of vectors of the form $GF(2)^{E(G)}$ (Note: $GF(2) \cong \mathbb{Z}_2$). We are indexing each entry with an edge in G. Each such vector represents a subgraph of G. (i.e. $e \in E(G)$ is in the subgraph iff the entry for e is 1 in the vector)

Ex:



$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Corresponds to:



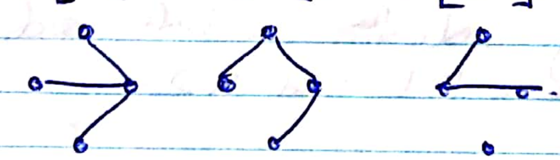
Vector Addition in $\mathcal{E}(G)$

(Note: Scalar multiplication is uninteresting)

Vector addition corresponds to the symmetric difference of the edge sets.

Ex:

$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



Note:

(i) The zero vector corresponds to the subgraph with no edges

(ii) The fundamental basis is $\{e_1, \dots, e_n\}$, so $\dim \mathcal{E}(G) = |E(G)|$.

Cut Spaces:

Def'n (Subspace)

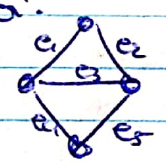
A subspace S is a subset that contains the zero vector and is closed under addition

Def'n (Cut Space)

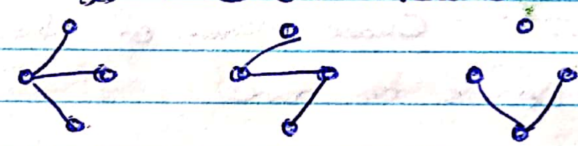
The cut space of G, denoted $C^*(G)$, is the set of all edge cuts in G (which includes the empty set)

Ex:

G:



Things in $C^*(G)$ include:



$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Proposition 12.1:

$C^*(G)$ is a subspace of $\mathcal{E}(G)$

[Proof]

This follows from the fact that the symmetric difference of a cut is still a cut □

Proposition 12.2:

Let B be the set of all bonds of G. Then B spans $C^*(G)$.

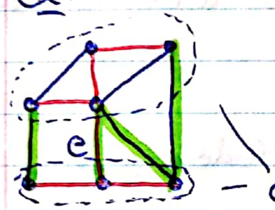
[Proof]

We proved that every cut is a disjoint union of bonds □.

Note: However the set of all bonds may not necessarily give a basis

Def'n (Fundamental Cuts)
Let G be a connected graph, and let T be a spanning tree of G. For each $e \in E(T)$, $T-e$ has 2 components. Let S be vertices of one such component. Then, $\delta_G(S)$ is the fundamental cut in G with respect to e, T. We write D_e.

Ex!



■ - Spanning Tree (T)
■ - Fundamental Cut wrt e, T (D_e)

- Choose either of these to be S.

Note: D_e is a bond

Theorem 13.1
Let G be connected and T a spanning tree of G. The set of all fundamental cuts in G with respect to T form a basis for the cut space $C^*(G)$.

[Proof]
For each edge $e = xy \in E(T)$, e is in D_e, but not in any $D_{e'}$ for $e' \in E(T) \setminus \{e\}$, since x and y are adjacent in $T - e'$, and hence must be in the same component.
If a linear combination of D_e's give 0, then the coefficient of the D_e's must be 0, since only one D_e contains e (out of all the D_e's). So, the set of fundamental cuts is independent

Con't

[Proof] (Con't)

Let F be a cut. Consider the sum $S = F + \sum_{e \in F \cap E(T)} D_e$.
Every edge $e' \in F \cap E(T)$ appears twice, once in F, once in $D_{e'}$. So $e' \notin S$, since the sum will be 0.
Any other edges in T do not appear in S. So S does not contain any edge in T. So, $S = \emptyset$.
Then,

$$F = -\sum_{e \in F \cap E(T)} D_e = \sum_{e \in F \cap E(T)} D_e$$

So, the set of fundamental cuts span $C^*(G)$ □

Corollary B.1

If G is connected, then $\dim C^*(G) = |V(G)| - 1$.

[Proof]

This is exactly the number of edges in a spanning tree □

Corollary B.2:

If G has k components, then $\dim C^*(G) = |V(G)| - k$.

Cycle Spaces:

Def'n

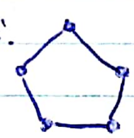
The cycle space of G, denoted $C(G)$ is the set of all cycles in G. We write:

$$C(G) = \text{Span}\{1_C\}_{C \subseteq E(G),\ C \text{ cycle}}$$

formally (where 1_C is the characteristic vector for a cycle C). Commonly we write $C(G) = \text{Span}\{C\}$.

Ex:

G:



Then, $C(G) = \text{Span}\left\{\begin{bmatrix}\vdots\end{bmatrix}\right\}$.

(there is only one cycle).

Recall: Every connected even graph (all vertices have even degree) has an Eulerian circuit.

Lemma 14.1:
Suppose we have 2 even edge sets Z_1, Z_2, then $Z_1 + Z_2$ is even.
[Proof]
Let $v \in V(G)$. Let D_1, D_2 be the edges incident to v in Z_1, Z_2 respectively.
Let D be the edges incident to v in $Z_1 + Z_2$.
Then,

$$|D| = \underbrace{|D_1|}_{\substack{\text{even,} \\ \text{since } Z_1 \text{ even}}} + \underbrace{|D_2|}_{\text{even}} - \underbrace{2|D_1 \cap D_2|}_{\text{even.}}$$

So, $|D|$ is even, and hence $Z_1 + Z_2$ is even $\square$

Proposition 14.1:
Elements of $C(G)$ are precisely even subgraphs of G.
[Proof]
($\Rightarrow$) Let $F \in C(G)$
Consider:

$$F = C_1 + \ldots + C_i + \ldots + C_k$$

We proceed by induction on the # of cycles.

Base Case: C_1 is even (it is a cycle).

IH: Assume $C_1 + \ldots + C_i$ is even

Con't

[Proof] (Con't)

($\Rightarrow$) Con't)

IC: We know that $C_1 + \dots + C_i$ is even and C_{i+1} is even. Hence, by lemma 14-1, $(C_1 + \dots + C_i) + C_{i+1}$ is also even. So the claim follows by induction.

($\Leftarrow$) Let F be an even subgraph, we want to show that F is a sum of cycles. We will proceed by induction on the # of edges in F.

Base Case: F is empty, so $F \in C(G)$ trivially.

IH: Assume that every non-trivial component has min. degree 2. thus it has a cycle C

IC: Now,

$$\underbrace{F - C}_{\text{By IH, sum of cycles.}} = \underbrace{F + C}_{\text{By lemma, this is even}} \quad \text{(Since we are in } GF(2))$$

$\Rightarrow (F - C) + C = F$

So, we've shown that F is a sum of cycles $\square$

Def'n (Fundamental Cycle)

Let T be a spanning tree of G. For any $e \notin E(T)$, the unique cycle of $T + e$ is a fundamental cycle, denoted C_e

$\hookrightarrow$ Theorem

Theorem 14.1:
Let T be a spanning tree of G, then, the fundamental cycles, with respect to T, form a basis for $C(G)$.

[Proof]

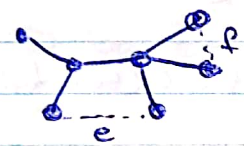
(Independence)

We claim that:

$$\{C_e\}_{e \in E(G) \setminus E(T)}$$

is linearly independent

Notice that:



Each $e, f \in E(G) \setminus E(T)$ are only part of one fundamental cycle.

In other words, each C_e has a unique non-zero entry that is 0 for all other fundamental cycles. So $\{C_e\}_{e \in E(G) \setminus E(T)}$ must be linearly independent.

(Spanning)

We want to show that any even $F \in C(G)$ is a sum of C_e's. Let $F' = F \setminus E(T)$

Consider:

$$S = \underbrace{F}_{\text{even}} + \underbrace{\sum_{e \in F'} C_e}_{\text{sum of even cycles, so even by lemma}}$$

So, by lemma, S is even, so $S \in C(G)$. And S can't contain any tree edges, so $S = 0$. Then,

$$F = -\sum_{e \in F'} C_e = \sum_{e \in F'} C_e$$

$\square$

Corollary 15.1:
For a connected graph G:

$$\dim C(G) = |E(G)| - |V(G)| + 1$$

[Proof]
There are $|V(G)| + 1$ edges in the spanning tree □

Corollary 15.2:
If G has k components, then

$$\dim C(G) = |E(G)| - |V(G)| + k$$

Orthogonal Complements:
We will show that the cut space is the orthogonal complement of the cycle space.

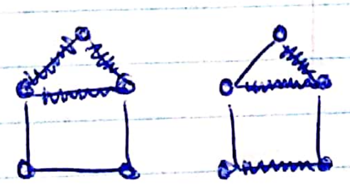
Recall: In $\mathbb{R}^N$, $\vec{v}, \vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

Def'n (Dot product in $\mathcal{E}(G)$)
The dot product in $\mathcal{E}(G)$ gives the # of edges in common. In particular, given H_1, H_2:

$$H_1 \cdot H_2 = \begin{cases} 0, & \text{when \# of edges is even} \\ 1, & \text{— " — odd} \end{cases}$$

Note: This is NOT an inner product space since $\langle \vec{x}, \vec{x} \rangle = 0 \not\Leftrightarrow \vec{x} = 0$. However, this is a bilinear form!

Ex:



H_1 H_2

$$H_1 \cdot H_2 = 0 + 1 + 1 + 0 = 0.$$

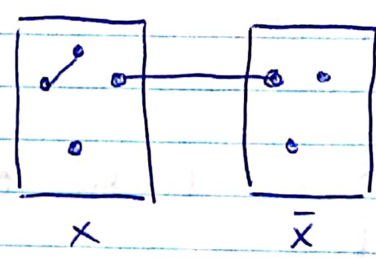
Def'n (Orthogonal Complement)
For a vector space V with subspace S, the orthogonal complement of S is the set of all vectors in V that are orthogonal to all vectors in S, denoted $S^\perp$.

$$S^\perp = \{v \in V \mid v \cdot w = 0, \ \forall w \in S\}$$

Remark:
1) $S^\perp$ is also a subspace
2) $(S^\perp)^\perp = S$
3) $\dim S + \dim S^\perp = \dim V$

Theorem 15-1
$C^*(G)^\perp = C(G)$
[Proof]
($\Rightarrow$) Let $F \in C^*(G)^\perp$, so F is orthogonal to every cut in G. In particular, F is orthogonal to cuts of the form $\delta(\{v\})$ for each vertex $v \in V(G)$. So, F has even degree at every vertex, hence $F \in C(G)$.

($\Leftarrow$) Now, suppose $F \in C(G)$. Let $D = \delta(X)$ be any cut. We want to show that $|F \cap D|$ is even.



(Two types of edges)

Con't

[Proof] (con't)
Consider the sum $S = \sum_{v \in X} \deg_F(v)$

Let F' be the set of edges with both endpoints in X. Every edge contribute 2 to S. (one for each end). Edges in $F \cap D$ contribute 1 to the sum.
So,

$$S = 2|F'| + |F \cap D|$$

Since $F \in C(G)$, $\deg_F(v)$ is even for all v. So, S is even, hence

$$|F \cap D| = S - 2|F'|$$

is even. So $F \in C^*(G)^\perp$. $\square$

Note:
1) $C(G)^\perp = C^*(G)$
2) $\dim C^*(G) + \dim C(G)$
$= |V(G)| - k + |E(G)| - |V(G)| + k$
$= |E(G)|$
$= \dim E(G)$

3 Planarity

Planarity

Def'n (Planar graph, planar embedding)
Planar graphs are ones with a drawing where no edges cross each other. Such a drawing is called a planar embedding.

Def'n (Plane Graph)
We call a planar embedding of a graph a plane graph. We will use points and lines to refer to the representation of vertices and edges in the embeddings, respectively.

Def'n (Curve)
A curve is the continuous image of a closed unit line segment

Def'n (Closed Curve)
A closed curve is the continuous image of a unit circle

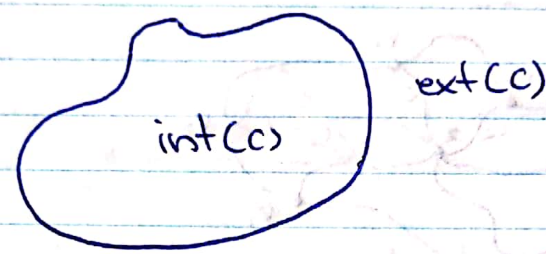
Note: Such a curve is simple if it doesn't intersect itself (mapping is one-to-one)

Def'n (Arcwise-Connected)
A set of points is arcwise-connected if any 2 of its points form the endpoints of a curve that is entirely inside the set

Def'n (Region)
A region is a maximal arcwise-connected set

Jordan Curve Theorem:

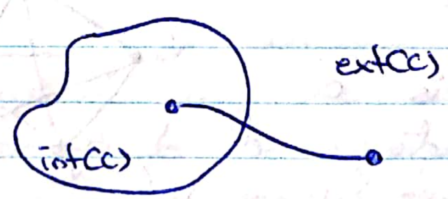
Any simple closed curve in the plane partitions the rest of the plane into 2 disjoint arcwise-connected open sets: one inside, one outside.



Def'n (Interior, Exterior)
The open set inside a simple closed curved C is its interior, denoted int(C). The open set outside of C is its exterior, denoted ext(C). (These do not include C).
Together, with C, we call them Int(C), Ext(C) respectively.

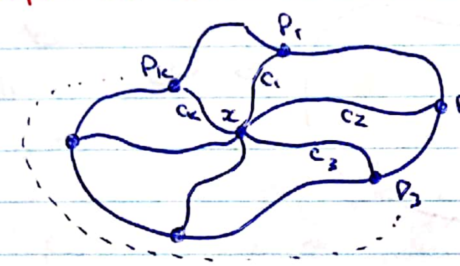
Some assumptions:

1) Any curve between a point in int(C) and a point in ext(C) must intersect C.

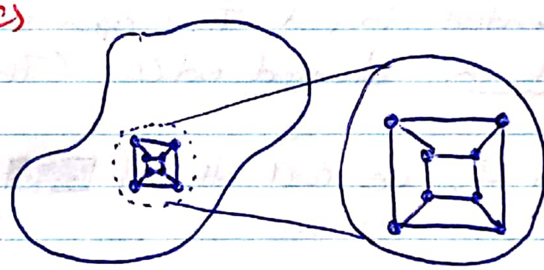


Con't

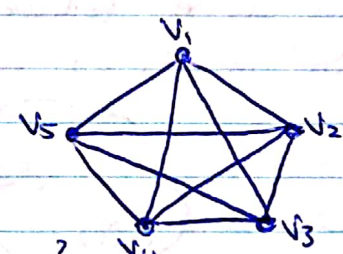
2) If $p_1, \dots, p_k$ are distinct points on C, and $x \in \text{int}(C)$, then there exist curves $c_1, \dots, c_k$ where c_i joins x to p_i. $c_i \subseteq \text{Int}(C)$, and they do not intersect except at x.



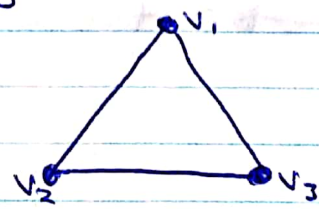
3) Any planar embedding can be redrawn in $\text{int}(C)$



Theorem 16.1:
K_5 is not planar



[Proof]
Suppose there is a planar embedding of K_5. Let $V(K_5) = \{v_1, \dots, v_5\}$.
Let C be the simple closed curve, formed by the cycle $v_1 v_2 v_3 v_1$.



cont

[Proof] (cont)

The point v_4 is either in $int(C)$ or $ext(C)$. WLOG, say $v_4 \in int(C)$. Then, the lines v_4v_1, v_4v_2, v_4v_3 are entirely in $int(C)$, except at v_1, v_2, v_3.

Let C_1, C_2, C_3 be the closed curves from the cycles $v_4v_2v_3v_4$, $v_4v_1v_3v_4$, $v_4v_1v_2v_4$. Note that C_1, C_2, C_3 do not include v_1, v_2, v_2 respectively. (So, $v_i \in ext(C_i)$ for each i, since $int(C_i) \subseteq int(C)$) Since there is a line joining v_5 to v_i for each i, v_5 is in $ext(C_i)$ for each i. So $v_5 \in ext(C)$. But there is a line joining $v_4 \in int(C)$ and $v_5 \in ext(C)$, which is a contradiction. $\square$.

Theorem: 16.2:
$K_{3,3}$ is not planar.

[Proof] Similar argument, as above, can be made $\square$

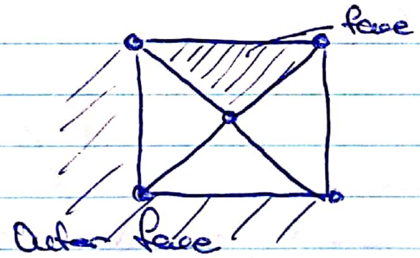
Def'n (Subdivision)
A subdivision of G is obtained by replacing each edge with a new path of length at least 1. (i.e. Introducing new vertices of deg 2 to the edges)

Proposition 16.1:
G is planar iff every subdivision of G is planar.

Corollary 16.1:
Any graph containing a subdivision of K_5 or $K_{3,3}$ is not planar.

Kuratowski's Theorem:
G is planar iff G does not contain any subdivision of K_5 or $K_{3,3}$.
[Proof] (See Theorem 23.1)

Faces:
Def'n (face, outer face, incident, adjacent)
A face in a planar embedding is a maximal subset of points that are arcwise-connected, and do not include any part of the embedding. Every embedding has one unbounded face called the outer face.



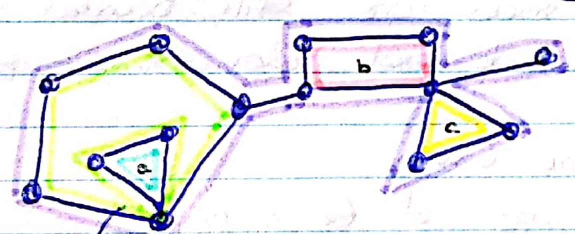
Each face is incident with the vertices and edges on its boundary. Two faces are adjacent if they share at least one edge in their boundaries.

Proposition 16.2:
A graph is embeddable on a sphere iff it is embeddable on a plane.
(We use stereographic projection from sphere to plane)

Proposition 16.3:
Let f be a face of a plane graph G. Then, there is a planar embedding of G, where the boundary of f is the boundary of the outer face

Note: These 2 propositions together tell us that we can make any face the outer face.

We now look at the boundary walk.
Ex:

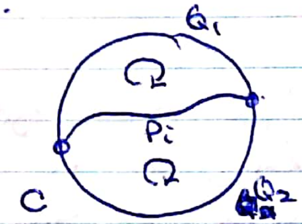


Notice that faces a, b, c are bounded by a cycle!

Theorem 17.1
In a 2-connected plane graph G, each face is bounded by a cycle.
(i.e. The boundary walks are all cycles)
[Proof]
Let $(G_0, \ldots, G_k)$ be an ear decomposition of G, where G_{i+1} is obtained from G_i by adding the ear P_i.
Now, G_0 is a cycle, which divides the plane into 2 faces (Jordan Curve Thm), and so G_0 is the cycle that bounds both faces

Cont'

[Proof] (con't)
Suppose each face of G_i is bounded by a cycle. Since G_{i+1} is a plane graph, the curve representing P_i must be entirely inside one face f, which is bounded by a cycle C.



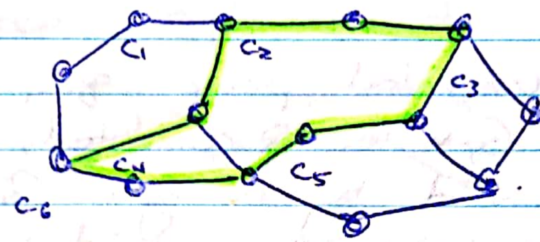
The 2 endpoints of P_i divides C into 2 paths Q_1 and Q_2. Then f is divided into 2 faces, one bounded by $Q_1 + P_i$ and the other bounded by $Q_2 + P_i$, both are cycles.
The remaining faces are still bounded by cycles since we do not touch them $\square$.

Def'n (Cycle Double Cover)
A <u>cycle double cover</u> of G is a set of cycles where every edge of G is in exactly 2 such cycles.
<u>Note</u>: 2-connected planar graphs have cycle double covers; we take the set of all facial cycles in a planar embedding

<u>Conjecture</u> 17.1:
Every 2-connected graph has a cycle double cover

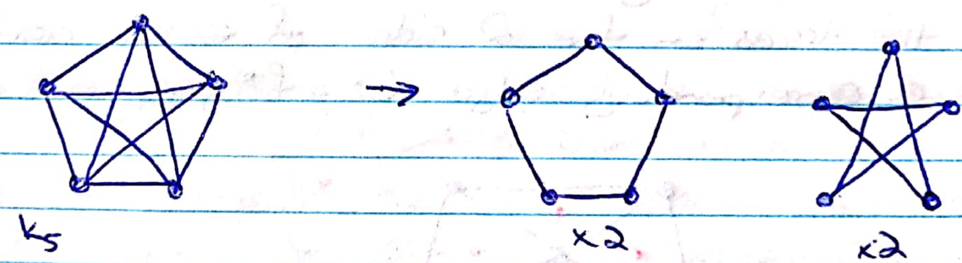
Cont. Remark

Remark:
In a 2-connected plane graph, every cycle is a sum of the facial cycles in its interior.
Ex:



The cycle in green is $C_2 + C_4$ (Interior) or $C_1 + C_3 + C_5 + C_6$ (but C_6 is exterior)

So, the cycle space is spanned by the facial cycles, which is a cycle double cover. However, this doesn't happen in nonplanar graphs! (i.e. No cycle double cover spans the cycle space if the graph is nonplanar). This is a consequence of Maclane's Theorem. (Theorem 26.1)
Ex:



is a cycle double cover, but does not span the cycle space.

→ Corollary

Corollary 17.1:
In a 3-connected plane graph, all neighbours of a vertex lie on a common cycle.

[Proof]

Let G be a 3-connected plane graph. Let $v \in V(G)$. Then, $G-v$ is 2-connected, so every face is bounded by a cycle. Consider the face that contains the point v. Then, all neighbors of v must be in the face, hence they are in a cycle. $\square$

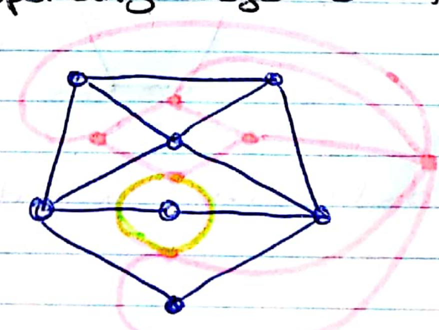
Dual Graphs:

Def'n

Given a plane graph G, define the dual G^* as follows:

- Each face f of G corresponds to a vertex f^* in G^*
- For each edge e in G, if f, g are the faces on the 2 sides of e, we add a corresponding edge $e^* = f^* g^*$.

Ex:

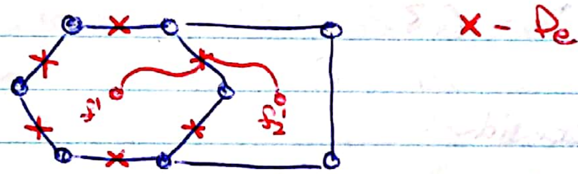


CbNotes

Note(s):
- Multiple edges may occur when 2 faces share more than one common edge. (See green in previous example)
- Loops occur when there is a cut edge

Proposition 18.1:
If G is a plane graph, then G^* is a plane graph.

[Proof]

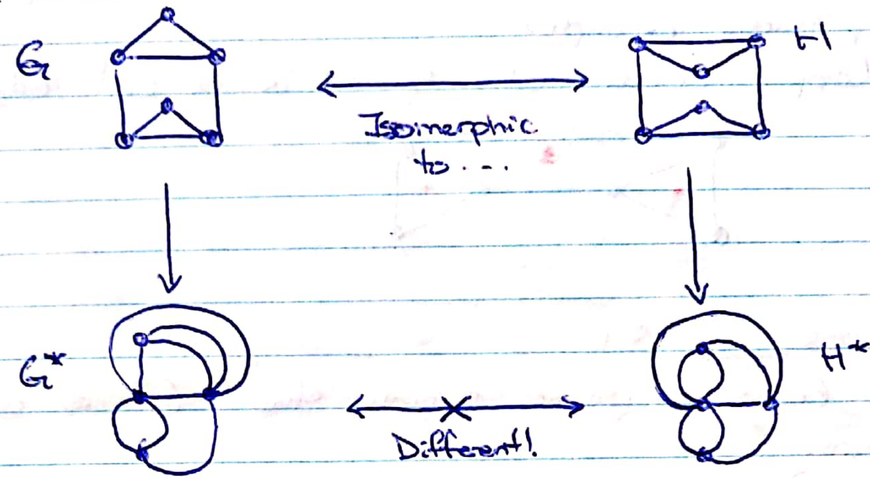


For each edge e in G, select a point p_e on the line for e. For each face f, we place f^* in the interior of the boundary of f. For each e in the boundary of f, draw a curve joining f^* to p_e that is in f. We can do so such that all curves from f^* to p_e do not cross. Joining the curves of either side of e creates a line for e^* in G. □

Con't

Dual graphs are dependent of the embedding

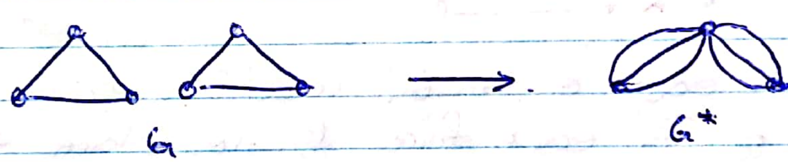
Ex!



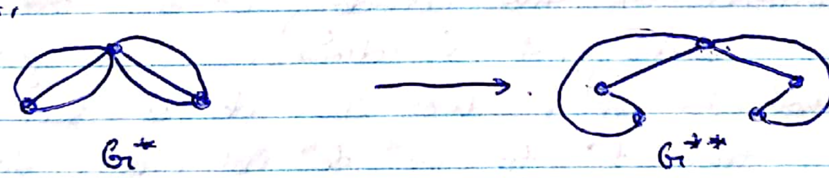
Q: Is $G^{**} = G$?

A!

We can consider:



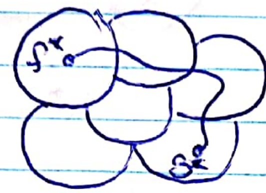
But,



So, $G^{**} \neq G$ always. Can we impose a stronger condition so that this is always true?

→ Proposition

Proposition 18.2
The dual of any plane graph is connected
[Proof]
Let G be a plane graph, let G^* be a plane dual of G. Let f^*, g^* be 2 vertices of G^*



We alternate between faces and edges.

There is a curve in the plane connecting f^* and g^*, which avoids any vertex (in G). Consider the sequence of faces and edges of G that this curve intersects. This corresponds to a walk from f^* to g^* in G^*. So, G^* is connected □.

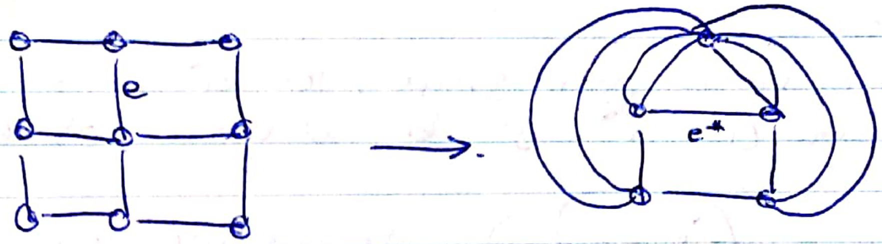
Corollary 18.1:
If G is a connected plane graph, $G^{**} = G$.
(i.e. Our question holds true if G connected).

More properties:
Assume G is a plane graph, G^* dual, $F(G)$ is the set of faces of G
1) $|V(G)| = |F(G^*)|$ (G connected
$|F(G)| = |V(G^*)|$
$|E(G)| = |E(G^*)|$
2) $\deg(f)$ in G is the same as $\deg(f^*)$ in G^*

Hilroy

Deletion and contraction:

Ex:

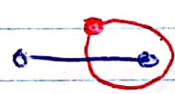


G $\qquad\qquad\qquad\qquad$ G^*

Notice that:

- Deleting e is equivalent to contracting the edge between the 2 vertices in the dual.
- Contracting e is equivalent to deleting the edge between the 2 vertices in the dual.

Exception: What if e is a cut edge?



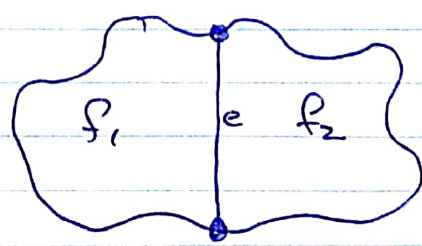
→ The loop disappears!

Proposition 18.2

Let G be a connected plane graph, let e be an edge. If e is not a cut-edge then $(G-e)^* = G^*/e^*$. And, if e is not a loop, then $(G/e)^* = G^* - e^*$.

[Proof]

Since e is not a cut-edge, the 2 sides of e are different faces, say f_1, f_2



Cont'

[Proof] (Con't)

In $G-e$, f_1, f_2 merge into a new face f. The boundary of f consists of boundaries of f_1 and f_2, except e. In G^*, this corresponds to a new vertex f^* replacing f_1^*, f_2^* with f^* incident with the same set of edges as f_1^*, f_2^* except e^*. This is the same as G^*/e^*.

Since G is connected, $G^{**} = G$. We see that e^* is not a [redacted] cut-edge (since e is not a loop). Then $(G^* - e^*)^* = G^{**}/e^{**} = G/e$.

$\square$

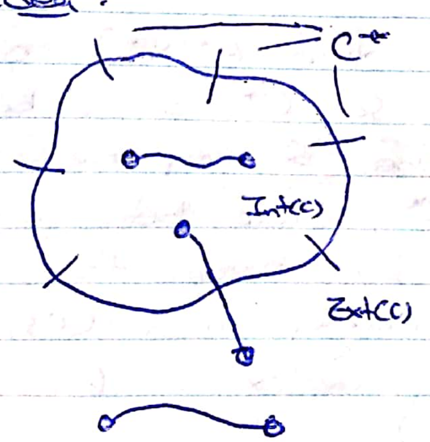
Remark: If e is a cut-edge, then e^* is a loop in G^*, so $(G-e)^* = G^* - e^*$.

Theorem 14.1':

Let G be a connected plane graph. If C is a cycle of G, then C^* is a bond in G^*. If B is a bond in G, then B^* is a cycle in G^*.

Notation: If F is a set of edges in G, we write $F^* = \{e^* \mid e \in F\}$, which is a subset of $E(G^*)$

Idea:

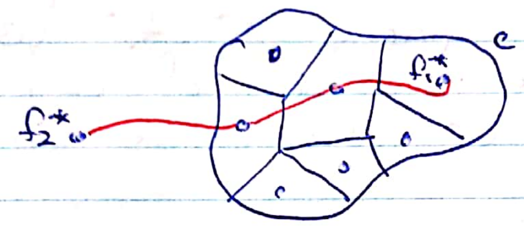


We first show that C^* is a cut. (i.e. splits into interior and exterior). Then, we can show that it is a bond because [redacted] there are exactly 2 components.

Proof

[Proof]

Suppose C is a cycle in G.
Let $X^* = \{f^* \mid f$ is a face in $\text{int}(C)\}$.
(i.e. A subset of vertices in G^*)



Let $\bar{X}^* = \{f^* \mid f$ is a face in $\text{ext}(C)\}$. Both must be nonempty since there is at least one face inside and one face outside. Pick $f_1^* \in X^*$ and $f_2^* \in \bar{X}^*$. Consider a curve from f_1^* to f_2^* that does not cross any vertices in G. The sequence of faces and edges the curve intersects in G corresponds to a f_1^*, f_2^*-walk in G^*. By Jordan Curve Theorem, the curve must intersect an edge e in C, which corresponds to using e^* in a walk. Since $e^* \in C^*$, any walk from X^* to $\bar{X}^*$ must use an edge in C^*, so C^* is a disconnecting set, and in fact $C^* = \delta(X^*)$.

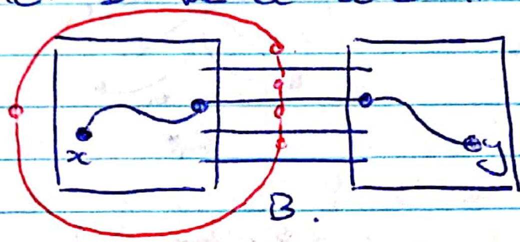
Now, we want to show that $G^*[X^*]$ and $G^*[\bar{X}^*]$ are connected, so C^* is a bond since there will only be 2 components. For any f_3^*, f_4^* in X^*, since both points are in $\text{int}(C)$, there is a curve from f_3^* to f_4^* contained entirely in $\text{int}(C)$, without intersecting any vertex in G.

(con't)

[Proof] (con't)

The sequence of faces and edges the curve intersects gives an f_3^*, f_4^*-walk in $G^*[X^*]$. So $G^*[X^*]$ is connected. Similarly, we can show that $G^*[\bar{X}^*]$ is also connected, so C^* is a bond.

Now, let B be a bond in G.



Let x, y be vertices in different components of $G - B$. Any x,y-walk uses an edge in B, so any curve from x to y must intersect an edge in B^*. So B^* must contain a cycle in order to separate x from y.

If B^* contains additional edges, then $C \subsetneq B$ (Since $C^{**} = C$ is a bond in G). But, this contradicts that B is a bond. So, B^* is a cycle. □

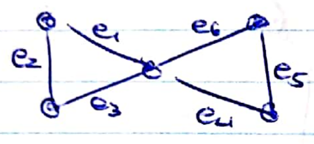
Corollary 19.1:
For a connected plane graph G, $C(G) = C^*(G^*)$. Also, $C^*(G) = C(G^*)$.

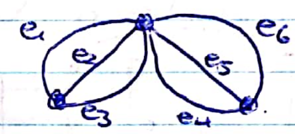
[Proof]

$C(G)$ is spanned by the cycles of G. They correspond to bonds in G^*, which span $C^*(G^*)$. Also, bonds in G^* correspond to cycles in G. So $C(G) = C^*(G^*)$. And, by taking the orthogonal complement on both sides, we get $C^*(G) = C(G^*)$ □

Def'n (Abstract Dual)
Let G be a graph. Then, G^* is an <u>abstract dual</u> of G if $E(G) = E(G^*)$
(i.e. There is a bijection between the two),
and the set of all cycles in G is the same as the set of all bonds in G^*

Ex:





Cycles:
$\{e_1, e_2, e_3\}$
$\{e_4, e_5, e_6\}$

Bonds:
$\{e_1, e_2, e_3\}$
$\{e_4, e_5, e_6\}$

Theorem 20.1:
If G^* is an abstract dual of G then
$C(G) = C^*(G^*)$ and $C^*(G) = C(G^*)$
[Proof]
See proof of Corollary 19.1, however, no mention of embeddings are relevent here
□

Corollary 20.1:
If G^* is an abstract dual of G, then G is an abstract dual of G^*
(i.e. If cycles of G are the same as the bonds of G^*, then bonds of G are the same as the cycles of G^*)

↳ [Proof]

[Proof]
Let F be a bond of G, so $F \in C^*(G)$. Then, by Theorem 20.1, $F \in C(G^*)$.
If F is a cycle, then we're done.
Otherwise, $F \in C(G^*)$, so F is even, and hence must contain some cycle $C \subsetneq F$. But $C \in C(G)^* = C^*(G)$, then C is a cut that is a proper subset of a bond F, a contradiction. Hence, F must be a cycle.

Now, let C be a cycle in G^*, so $C \in C(G^*) = C^*(G)$, is a cut in G. If C is not a bond, then a proper subset of C is a bond that is in $C(G^*)$, which is not possible since no proper non-empty subset of a cycle is in the cycle space $\square$.

Q: What types of graphs have abstract duals?
A: We've seen that planar graphs have planar duals, in fact, we will see that ONLY planar graphs have abstract duals. This is a consequence of Whitney's Theorem (See Theorem 27.1).

Bridges:
Goal: We want to prove that 3-connected planar graphs have a unique planar embedding, and hence a unique dual. (By unique, we mean that all faces will have the same face boundaries).
This will lead up to Kuratowski's Theorem!

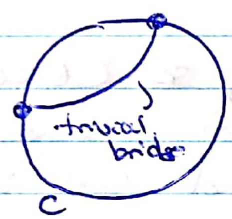
Def'n (Bridge)

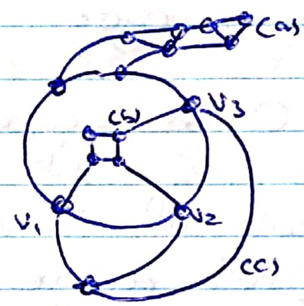
Let C be a cycle of G. A bridge of C is either:

(i) A component H of $G - V(C)$ along with the edges H to C (include the endpoints of these edges), or

(ii) An edge, not in C, joining 2 vertices of C (called trivial bridges)

Ex:





All of (a), (b), (c) are bridges.

Def'n (Vertices of attachment, Internal Vertices, k-bridge, Equivalent)

For a bridge B of C, the vertices in both B and C are vertices of attachment (VOA). The remaining vertices of B are the internal vertices. The bridge with k VOA is called a k-bridge. Two bridges with the same VOA are equivalent.

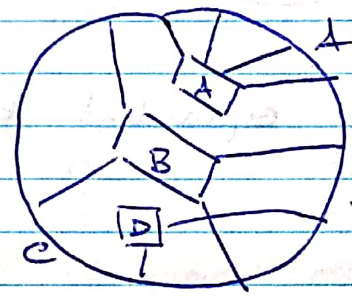
Ex:

V_1, V_2, V_3 are VOA of (b), and the remaining are internal vertices. It is also a 3-bridge, and is equivalent with (c).

Def'n (Segment, avoid, overlap)
The k VOA of a k-bridge partitions C into k segments. Two bridges avoid each other if the VOA of one bridge is entirely within a segment of another bridge. Otherwise, they overlap. (So, the bridge spreads across more than one segment)

Ex!

B partitions C into 4 segments.

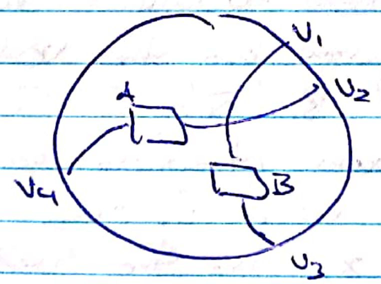


A avoids B.

D overlaps B

Def'n (Skew)
Two bridges are skew if there is a sequence of distinct vertices v_1, v_2, v_3, v_4 in C in cyclic order such that v_1, v_3 are VOA of one bridge and v_2, v_4 are VOA of the other.

Ex!



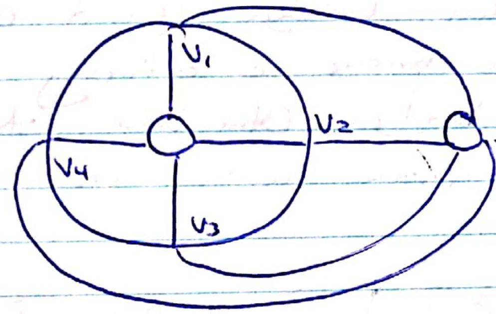
A and B are skew.

Theorem 21.1:
Overlapping bridges of a cycle C are either skew or equivalent 3-bridges.

$\hookrightarrow$ [Proof], and Note:

Note!

What about equivalent 4 (or higher) - bridges



So, we see that equivalent k-bridges ($k \geq 4$) are skew.

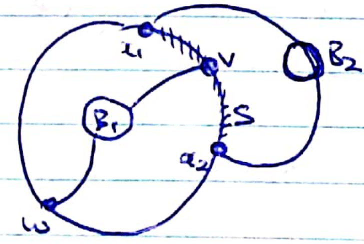
[Proof] (of Theorem)

Let B_1, B_2 be overlapping bridges, then they both need at least 2 VOA each.

We have 2 cases:

Case I: (B_1, B_2 are not equivalent)

WLOG, there is a VOA v in B_1, that is not a VOA of B_2. v must be in a segment S generated by the VOA of B_2. If all VOA of B_1 are in S, then B_1 and B_2 avoid each other, a contradiction. So, there is a VoA w of B_1 that is outside S. Let x_1, x_2 be the vertices of B_2 that create the segment S. Then, w, x_1, v, x_2 create the skew.



Con't

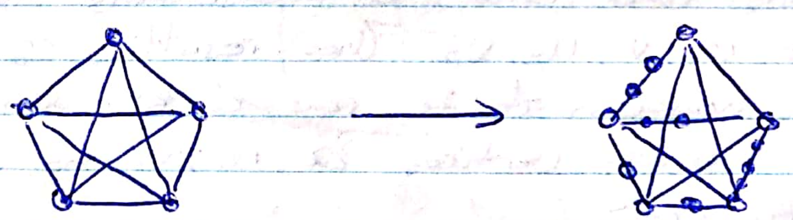
[Proof] (con't)

Case 2: (B_1, B_2 are equivalent k-bridges)
We know k is at least 2.
If k=2, they do not overlap each other, a contradiction.
If k≥4, as described in the note, they are skew, and v_1, v_2, v_3, v_4 form the skew.
So k=3.

□

Note: Given a planar embedding and a cycle C, each bridge of C is either in Int(C) or Ext(C). They are called inner bridges or outer bridges respectively.

We now relate bridges to planarity.
Recall:
We call an edge subdivision of G, the graph obtained by replacing edges of G by paths of length 2 or more.
Ex:



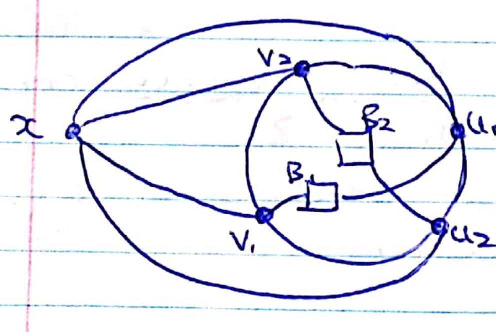
Def'n (Branch Vertices)
In a subdivision, the branch vertices are the ones with degree at least 3.
We will describe our subdivisions by giving the set of branch vertices.

Theorem 21.2:
If G is a plane graph and C is a cycle of G, then all inner and outer bridges of C avoid each other

[Proof]
Let B_1, B_2 be 2 overlapping bridges of C. By Thm 21.1, B_1, B_2 are either skew or equivalent k-bridges.

Case I: (B_1, B_2 are skew)
Then, there exist VOA u_1, v_1 of B_1 and VOA u_2, v_2 of B_2 s.t. u_1, u_2, v_1, v_2 are in cyclic order in C.

Since B_1, B_2 are connected, there exist a u_1, v_1-path P_1 in B_1 and a u_2, v_2-path P_2 in B_2, and both paths are ~~in~~ $Int(C)$. (We can assume B_1, B_2 are inner bridges WLOG) Let H be the subgraph that consists of $C \cup P_1 \cup P_2$. Add a vertex x to $ext(C)$ to H, and draw non-intersecting lines joining x to u_1, v_1, u_2, v_2. The resulting graph is plane, however it is now a K_5 subdivision with branch vertices $\{x, u_1, v_1, u_2, v_2\}$, a contradiction.

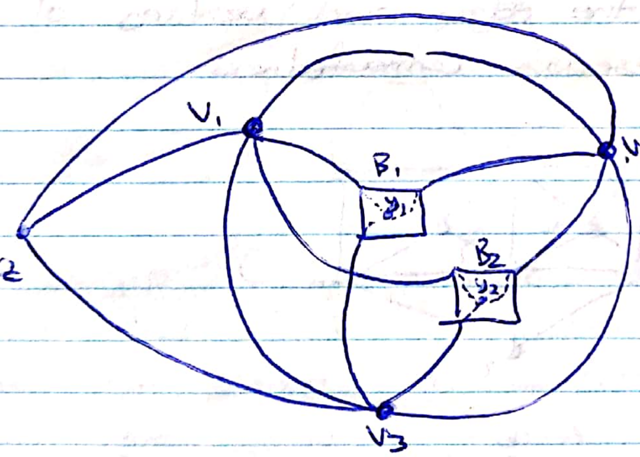


Ex. v_2 reaches u_1 by a path in the direction of the cyclic order, v_1 by a path in the reverse direction, u_2 by P_2, and x by the newly formed edge xv_2.

Cont

[Proof] (con't)
Since x was introduced so that no edge crossings would appear in $\text{ext}(C)$, the edge crossings must occur in $\text{Int}(C)$.

Case 2: (B_1, B_2 are equivalent 3-bridges)
Let v_1, v_2, v_3 be the v.o.a. of B_1 and B_2. There exist a 3-fan F_1 in B_1 from y_1 to $\{v_1, v_2, v_3\}$, and a 3-fan F_2 in B_2 from y_2 to $\{v_1, v_2, v_3\}$. Let H be $C \cup F_1 \cup F_2$, and we add a vertex z in $\text{ext}(C)$ to H, and draw non-intersecting lines from z to $\{v_1, v_2, v_3\}$. The resulting graph is a $K_{3,3}$ subdivision with branch vertices $\{v_1, v_2, v_3\}$ and $\{z, y_1, y_2\}$ forming the bipartition, a contradiction.



Ex: v_1 is adjacent to z, and there is a v_1, y_1-path from F_1 and a v_1, y_2-path from F_2.

$\square$

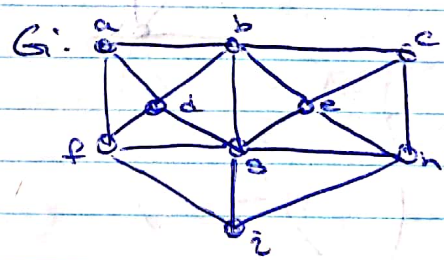
Unique embeddings of 3-connected planar graphs

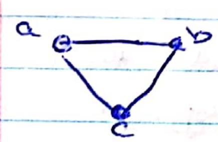
Def'n (Equivalent)
Two planar embeddings are equivalent if they have the same set of edges as face boundaries

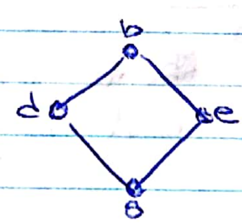
Def'n (Unique embedding)
A planar graph has a unique embedding if every embedding is equivalent.

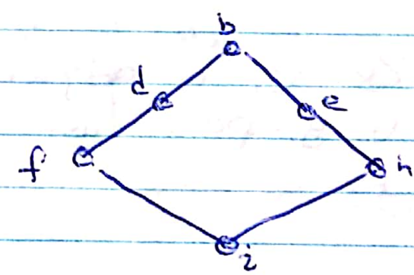
Def'n (Induced Cycle, Non-separating)
An induced cycle is a cycle that is an induced subgraph. It is non-separating if removing the edges and vertices of the cycle preserves connectedness

Ex:









Induced,
Non-separating.

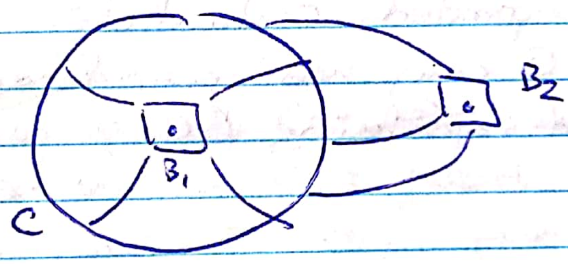
Not induced

Induced,
Separating

Theorem 22.1:
A cycle C in a 3-connected plane graph G is a facial cycle iff C is an induced non-separating cycle in G

[Proof]

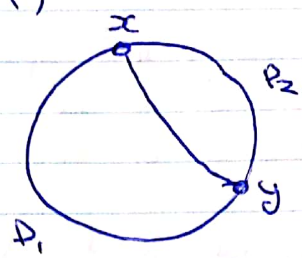
($\Leftarrow$) Suppose C is not a facial cycle, then there exist an inner bridge B_1 and an outer bridge B_2. If either bridge is trivial, then C is not an induced cycle, so this case never happens.
Otherwise, B_1, B_2 each contain a vertex not in C, which are in different components of $G - V(C)$, but then C separates these 2 vertices, contradiction.



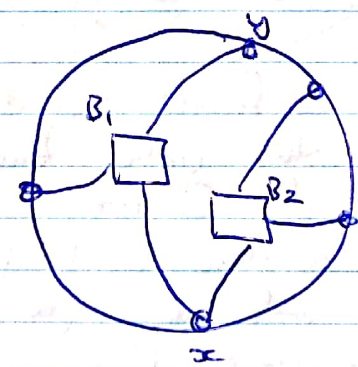
($\Rightarrow$) Suppose C is a facial cycle. We may redraw G so that C is the outer face (Prop'n 16.2, 16.3), then all bridges of C are inner bridges.
If there exists a trivial bridge xy, then $\{x, y\}$ partitions C into two paths P_1 and P_2. Since G is 3-connected, there must be a path between the internal vertices of P_1 and P_2. But, such a path is a bridge that overlaps xy. Contradiction, since G is plane. So, C is induced

Con't

[Proof] (con't)



Now, suppose there are 2 nontrivial bridges in C, B_1 and B_2. G is plane, so B_1, B_2 must avoid each other. Suppose x, y are the 2 ends of a segment formed by VoA of B_1 that contains all VoA of B_2. But, then $\{x, y\}$ is a separating set of size 2, as internal vertices of B_1 and B_2 are separated, contradiction.
Hence, C only has one bridge, so it is non-separating (C is also the outer cycle). ■



□

Theorem 22.2:
Every 3-connected planar graph has a unique embedding
[Proof]
The facial cycles of a 3-connected planar graph are the induced non-separating cycles, which is an abstract structure of G that is not dependent on the embedding. So, the facial cycles are the same for all embeddings of G, hence the embedding is unique. □

Corollary 22.1:
Every 3-connected planar graph has a unique dual.

Kuratowski's Theorem:
We previously proved (in M239) that if a graph has a K_5 or $K_{3,3}$ subdivision, then it is not planar, we will now prove:

Theorem 23.1 (Kuratowski's)
If G is not planar, then G has a subdivision of K_5 or $K_{3,3}$.

Def^n (Kuratowski Subgraph)
We call subdivisions of K_5 or $K_{3,3}$ Kuratowski subgraphs (K5)

C→Outline

Outline of proof:
We will prove the theorem by induction on the # of edges.

Base Case: We can check that the theorem holds for all graphs with at most 6 edges (They are all planar)

Then, we will have 4 cases:

(I) If G is not connected, then one component is nonplanar, we consider only this component

(II) G is 1-connected (i.e. Has a cut vertex). We consider the nonplanar blocks of G.

(III) G is 2-connected. We consider (roughly) the components of $G - \{x, y\}$. And there are 2 subcases
i) At least One component is nonplanar
ii) All components are planar

(IV) G is 3-connected. Then, $\exists e$ st. G/e is 3-connected. Again, there are 2 subcases
i) G/e is not planar
ii) G/e is planar, then either G is planar (Contradiction) or uncontracting e produces a K_5.

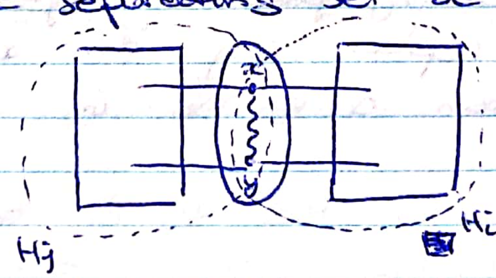
[Proof] (23.1)
Let G be any nonplanar graph

(I) If G not connected, then at least one component of G is nonplanar, and only consider that component. If the # of edges of the component is less than that of G, then by IH, that component has a KS. Otherwise, we fall into case (II), (III), or (IV).

Now, we can also assume that G is connected

(II) Suppose G has a cut vertex.
Then, G has at least 2 blocks, and at least one block B is nonplanar. Since all blocks of connected graphs are nontrivial, B has fewer edges than G. By induction, B has a KS, which is also a KS in G.

(III) Suppose G has a separating set of size 2, and let the separating set be $\{x, y\}$.



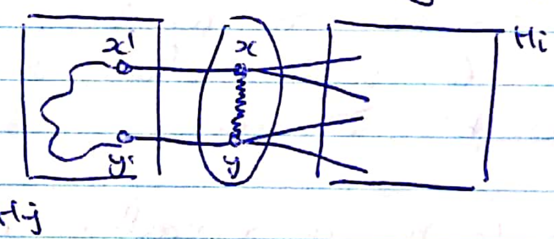
Let H_i consist of a component of $G - \{x, y\}$ together with all edges joining the component to $\{x, y\}$ and the edge xy. (We add the edge in if it is not in G).

→ Con't

[Proof] (cont)

Case I: (There is a subgraph H_i that is nonplanar)
There is at least one different H_j which must contain at least 2 edges from $\{x, y\}$. So H_i has fewer edges than G. (Note that even when we add xy, we still remove that and at least one other edge in H_j) So, by induction, H_i contains a KS, say J.

If J does not contain xy or xy is in G, then J is a KS in G. Otherwise, there are vertices x', y' in H_j that are adjacent to x and y, respectively.



There is an x', y'-path P in $H_j - \{x, y\}$, and so we can replace xy with $xx' + P + yy'$ in J to obtain a KS for G.

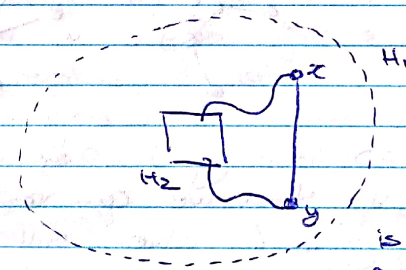
Cont

[Proof] (Con't)

Case 2: (All H_i's are planar)
We will find an embedding for G.
Start with an embedding of H_1, and now inductively create an embedding for $H_1 + H_2 + \dots + H_{i-1}$.

Consider an embedding of H_i where xy is on edge of the outer face (We can do this because H_i is planar) Now, pick a face in the embedding of $H_1 + H_2 + \dots + H_{i-1}$ that contains xy and put H_i in that face. This creates an embedding of $G + xy$, which is a contradiction, since G is non-planar.

Ex:



is a planar embedding of H_1 and H_2.

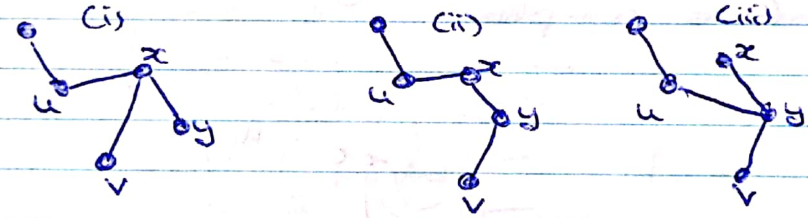
(IV) Suppose G is 3-connected.
Then, there is an edge e such that G/e is 3-connected. Let z be the contracted vertex and $e = xy$.

→ Con't

[Proof] (con't)

Case I: (G/e is not planar)
Since G/e has fewer edges than G, by induction, G/e has a KS H. If H does not contain z, then H is a KS of G and we are done.

Otherwise, H does contain z, and z can be a branch vertex or not.
Suppose z is not a branch vertex, then let u, v be the 2 neighbors of z in H. Uncontracting z gives one of 3 cases:

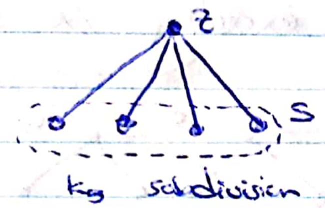


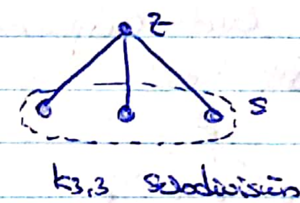
In G, if x or y is adjacent to to both u, v, then we can replace z with x or y in G, which still gives us a KS. (This is case (i) and (iii)).
Otherwise, WLOG, u is adjacent to x and v is adjacent to y. Then, we can replace u, z, v in H with u, x, y, v in G to obtain a KS in G.

Con't

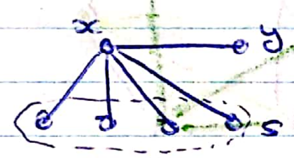
[Proof] (con't)

Now, suppose z is a branch vertex.
Let S be the set of neighbours of z in H.

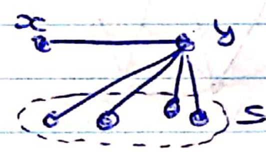




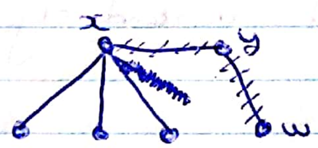
In G, if x or y is adjacent to all vertices in H, then we can replace z with such a vertex to get a KS for G:

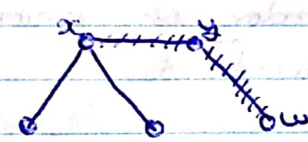


[OR]



Suppose WLOG that x is adjacent to all but one vertex in S. So, yw is an edge in G. We can replace z with x, and edge zw with the path xyw to obtain a subdivision of H, which is now a KS in G.

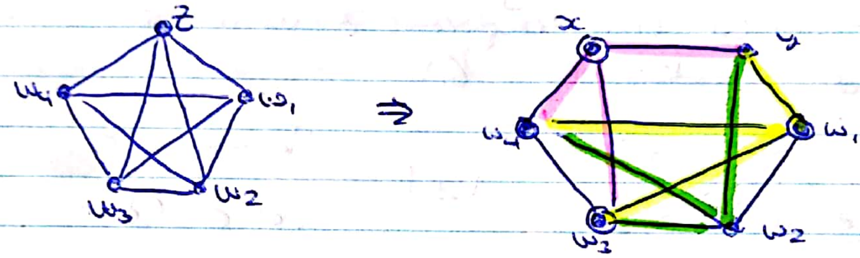




Con't

[Proof] (Con't)
The only case remaining is when H is a K_5 subdivision, and both x, y are adjacent to exactly 2 vertices in S, i.e.:

Notice that:

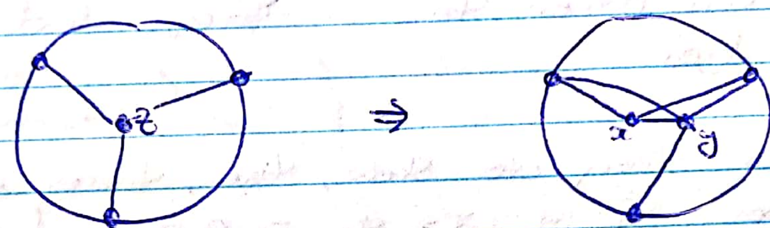


If $\{w_1, w_2, w_3, w_4\}$ are the other 4 branch vertices of H, then we can find a $K_{3,3}$ subdivision in G using $\{x, w_1, w_3\}$ and $\{y, w_2, w_4\}$ as branch vertices. (Note that we can rearrange the w_i's, depending on which vertices x and y are adjacent to)

Con't

[Proof] (cont)

Case 2: (G/e is planar)
We know that G/e is 3-connected, so $(G/e)-z$ is 2-connected. The boundary of the face containing the point z in $(G/e)-z$ is a cycle C. In G/e, all neighbours of z are in C, and so in G, all neighbours of xy are also in C

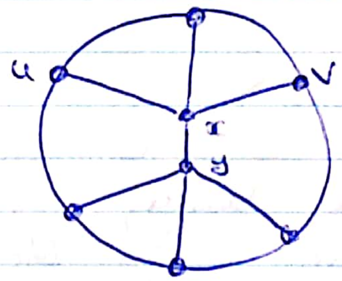


Let H_x, H_y be two subgraphs of G that consist of x, y along with their neighbours in C respectively. So, both H_x and H_y are bridges of C in $G-e$.

Suppose H_x, H_y avoid each other. We show that we can create a planar embedding. First, embed H_x by placing x inside C, and draw a fan from x to all its VOA. Since H_x and H_y avoid each other, all VOA of H_y are inside a segment of H_x, say from u to v. This segment, along with xu, xv form a face, and we can embed H_y by placing y inside the face and joining y to all its VOA and x. This is a planar embedding, so contradiction. This case never happens

Cont

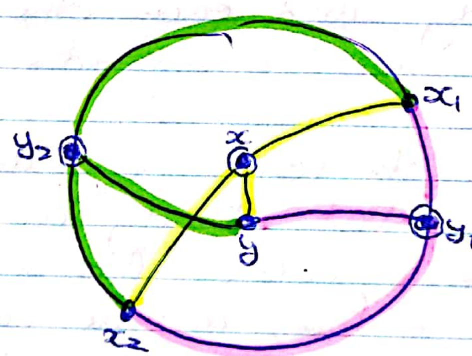
[Proof](Cont)



planar embedding if H_x, H_y avoid each other

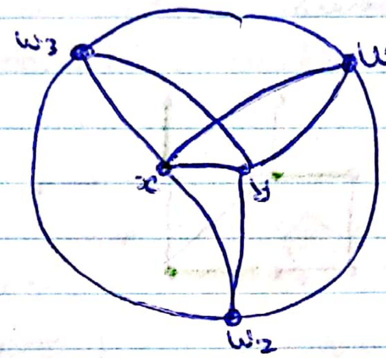
Now we can assume H_x, H_y overlap, then they are either skew or equivalent 3-bridges.

Suppose they are skew, then, there exist vertices x_1, y_1, x_2, y_2 in cyclic order where x is adjacent to x_1, x_2 and y is adjacent to y_1, y_2. Then C together with xy and the 2 paths $x_1 x x_2$, $y_1 y y_2$ form a $K_{3,3}$ subdivision with bipartition $\{x, y_1, y_2\}$ and $\{y, x_1, x_2\}$.



Cont

[Proof](cont)
Suppose H_x, H_y are equivalent 3-bridges with Vat $\{w_1, w_2, w_3\}$. Then $C \cup H_x \cup H_y \cup \{xy\}$ is a K_5 subdivision in G, with branch vertices $\{x, y, w_1, w_2, w_3\}$.



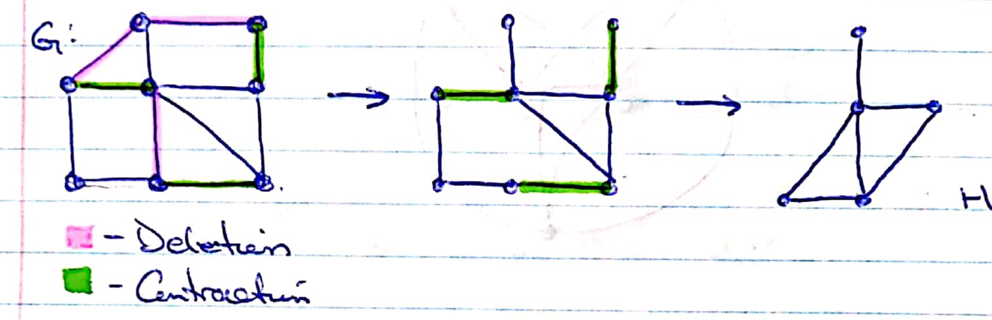
For graphs with connectivity 4 (or more), we are guaranteed at least a K_5 subdivision, since 4-connected implies # of vertices ≥ 5, with min. degree of 4, which is precisely a K_5 subdivision.
$\blacksquare$

Other characterizations of planar graphs:
We will look at 3 other characterizations of planar graphs:
1) Wagner's Theorem (25.1) - Minors
2) Maclane's Theorem (26.1) - 2-basis
3) Whitney's Theorem (27.1) - Duals

(a) Wagner's

Wagner's Theorem:

Def'n (minor)
A graph H is a minor of G, if H can be obtained from G through a series of deletions and edge contractions.
Ex:



Lemma 25.1:
If G contains a subdivision of H, then H is a minor of G.
[Proof] Contract the subdivided edges. □

Theorem 25.1: (Wagner's Thm)
A graph G is planar iff G does not contain a K_5 or $K_{3,3}$ minor.
[Proof] Use the next theorem (25.2) and Kuratowski's theorem □

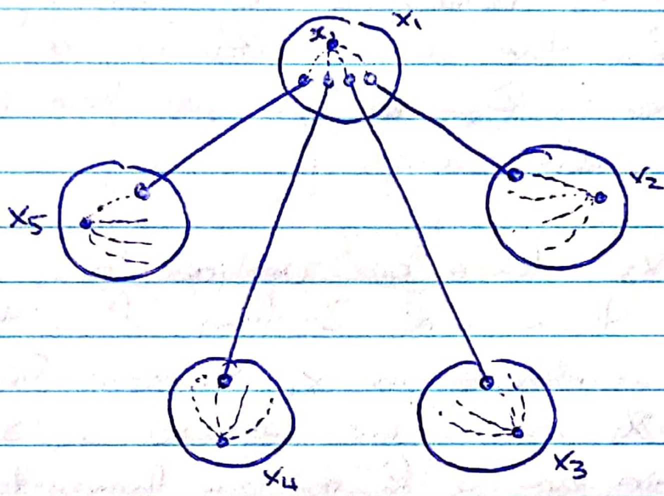
Theorem 25.2:
G contains a subdivision of K_5 or $K_{3,3}$ iff G contains a K_5 or $K_{3,3}$ minor

→ Proof

[Proof]

($\Rightarrow$) Use previous lemma

($\Leftarrow$) G has a K_5 or $K_{3,3}$ minor.
Suppose G has a K_5 minor. Then, there exist 5 connected vertex-disjoint subgraphs $X_1, \dots, X_5$ whose contractions result in K_5.



For each X_i, there are 4 edges joining it to the other subgraphs. Let V_i be the set of endpoints whose edges are outside of X_i.

If there exists a 4 fan for some vertex x_i to V_i in each X_i, then this forms the K_5 subdivision, and so we are done.

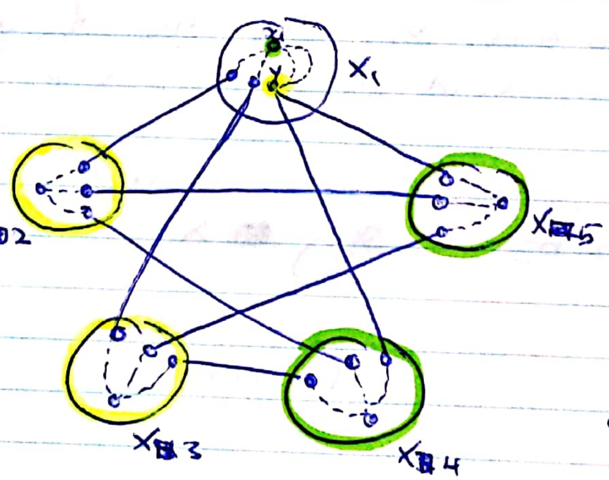
Otherwise, suppose such a 4-fan does not exist for one subgraph, say X_1.

$\Rightarrow$ con't

[Proof] (cont)

Suppose $V_1 = \{w_2, w_3, w_4, w_5\}$, where w_i is in X_i. We know that there exists a vertex x_1 in X_1 such that there is a 3-fan from x_1 to $\{w_2, w_3, w_4\}$, and P_i is an x_1, w_i-path. Since X_1 induces a connected subgraph, there is a x_1, w_5-path P_5 which is not internally disjoint from the 3-fan. Let y be the first vertex in P_5 from w_5 that is in another path, say P_4

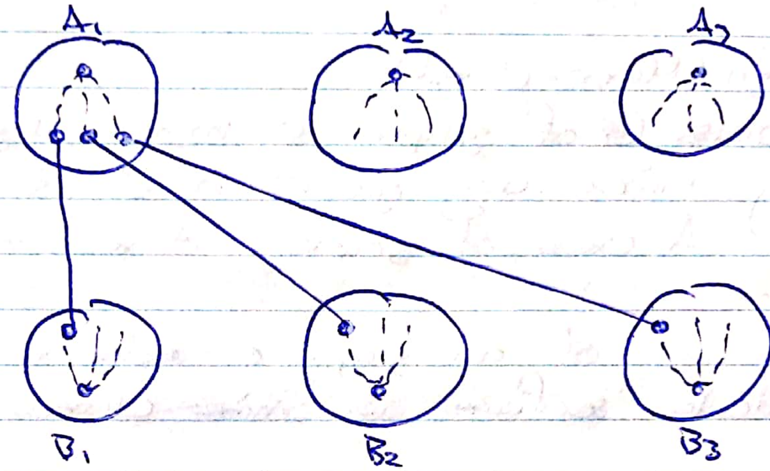
From X_2, X_3, there are vertices x_2, x_3 for which there are 3-fans from them to their neighbors in X_1, X_4, X_5. Similarly, from X_4, X_5, there are vertices x_4, x_5 for which there are 3 fans from them to their neighbors in X_1, X_2, X_3. Combine these fans, along with the paths P_2, P_3 and x_1 to y in P_4, and the paths from y to w_4, w_5 in P_4, P_5. This is a subdivision of $K_{3,3}$ with $\{x_1, x_4, x_5\}$ and $\{y, w_2, w_3\}$ as the branch vertices:



→ con't

[Proof] (con't)

Now, suppose G has a $K_{3,3}$ minor. Then, there exist connected vertex-disjoint subgraphs $A_1, A_2, A_3, B_1, B_2, B_3$, whose contractions result in a $K_{3,3}$ with bipartition $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$.



In A_1, there are 3 neighbors, one in each of B_1, B_2, B_3 in the minor. There exists a vertex x_{a_1} in A_1, such that there is a 3-fan from x_1 to these 3 vertices. Do this for all 6 subgraphs to obtain a $K_{3,3}$ subdivision

$\square$

Q: Why is Wagner's Theorem useful?

A: We can use it to look at planar embeddings on non-plane surfaces. In particular, we cannot have theorems of the form:

"G can be embedded on ___(surface)___ iff G does not contain ______ as a subdivision"

beyond a plane/sphere.

↳ con't

Theorem 25.3 (Graph Minor Theorem)
Any infinite sequence of finite graphs: $G_1, G_2, \dots,$ includes 2 graphs G_i, G_j with $i<j$ s.t. G_i is a minor of G_j.

Def'n (Minor-closed)
A class $\mathcal{G}$ of graphs is minor-closed if any minor any graph in $\mathcal{G}$ is in $\mathcal{G}$
(Note. A class of graphs = A set of graphs)

Remark 25.1:
The set of all graphs embeddable on a particular surface is minor-closed

Def'n (Minor-minimal)
For a minor-closed family of graphs $\mathcal{G}$, a graph G is minor-minimal if $G \in \mathcal{G}$, but any proper minor of G is in $\mathcal{G}$.

Theorem 25.4 (Wagner's Thm-Reworded)
For the class of planar graphs, K_5, $K_{3,3}$ are minor-minimal.

Ex: (Minor-minimal graph).
Let F be the set of all forests, it is a minor-closed class, since deleting anything still results in a forest. The 3-cycle is the only minor-minimal graph.

Theorem 25.5:
For any minor-closed family of graphs $\mathcal{G}$, the set of all minor-minimal graphs is finite.

[Proof]

Suppose the set is infinite, by graph minor thm (Thm 25.3), one is a minor of the other, contradiction $\square$

Summary:

Wagner's Thm can be extend:

"G can be embedded on a certain surface iff G does not have a finite set of graphs as minors."

Ex. The torus has ≥ 16629 minor-minimal graphs.

Maclane's Theorem:

Def'n (2-basis)

A basis B of a subspace of $\mathcal{E}(G)$ is a 2-basis if every edge of G is in at most 2 elements of B.

Theorem 26.1 (Maclane's Thm)

A graph is planar iff the cycle space has a 2-basis.

[Proof].

($\Rightarrow$) Suppose G is planar. Notice that $\mathcal{C}(G)$ has a 2-basis iff the cycle space of each block has a 2-basis. Also, recall that G is planar iff each block of G is planar. Now, we may assume that G is 2-connected. Consider any embedding of G, each face is bounded by a cycle.

$\Rightarrow$ Con't

[Proof] (Con't)
Let B be the set of all facial cycles of the embedding. Each edge is in 2 elements of B (the two faces on either side). [redacted] We want to show that B spans $C(G)$. Let C be any cycle, and F be the set of all face boundaries of faces inside C. Consider the sum:

$$S = \sum_{C' \in F} C'$$

Each edge in C is counted once in S, since only the interior of C is counted. Each edge in the interior of C is counted twice, as both sides are in the interior. So $S = C$, and we see that B spans $C(G)$. So, any subset of B is a basis for $C(G)$, hence a 2-basis. (In particular, we could remove the outer cycle)

($\Leftarrow$) [redacted], Suppose G is not planar.
We first prove that if $C(G)$ has a 2-basis, and H is a minor of G, then $C(H)$ has a 2-basis.

Let $\{Z_1, \ldots, Z_k\}$ be a 2-basis for $C(G)$, where $k = |E(G)| - |V(G)| + c$. Let e be an edge. If e is an edge, then, it is not in any Z_i. Removing e, we get $k = (|E(G)| - 1) - (|V(G)|) + (c+1)$, so $\{Z_1, \ldots, Z_k\}$ is still a 2-basis for $C(G-e)$

Con't

[Proof] (con't)
If e is in exactly on set, say Z_k, then $\{Z_1, ..., Z_{k-1}\}$ is a 2-basis for $C(G-e)$,
Since $k = (|E(G)| - 1) - V(G) + c$. decreases by 1.
If e is in 2 sets, say Z_{k-1} and Z_k, then $\{Z_1, ..., Z_{k-1} + Z_k\}$ is a 2-basis for $C(G-e)$,
Since $k = (|E(G)| - 1) - V(G) + c$, $Z_{k-1} + Z_k$ is still even, and any other edge will only appear at most twice, as $Z_{k-1} + Z_k$ does not "uncover" edges.
Now, suppose we want to contract e. The dimension doesn't change, but we remove e from any Z_i containing e. The resulting set is still even a linearly independent, and hence a 2-basis for $C(G/e)$
Finally removing a vertex v is the same as removing all edges incident with v (which falls into the above cases), and then removing the isolated vertex itself, which does not change the dimension of the 2-basis.
So, we conclude that $C(H)$ has a 2-basis.

We now want to show that K_5, $K_{3,3}$ do not have 2-bases of their cycle space.
Suppose $C(K_5)$ has a 2-basis, then $\dim C(K_5) = 10 - 5 + 1 = 6$. Let $\{Z_1, ..., Z_6\}$ be such as 2-basis, and let $Z_7 = \sum_{i=1}^{6} Z_i$. Since $Z_1, ..., Z_6$ are linearly independent, $Z_7 \neq \emptyset$, and contains edges that appear exactly one in the basis.

Con't

[Proof] (cont)

Then, in $\{Z_1, \ldots, Z_7\}$, each edge appears at most twice (if exactly twice, this means we have no cut edges). There are 10 edges in K_5, so there are 20 times they appear in $\{Z_1, \ldots, Z_7\}$, but each $|Z_i| \geq 3$ since they are/contain a cycle. So edges appear at least 21 times, contradiction.

Similarly, a cycle space of $K_{3,3}$ does not have a 2-basis. Hence, by Wagner's Thm, G must be planar, and we've shown that any non-planar graph does not have a 2-basis for its cycle space. ☐.

Theorem 27.1 (Whitney's)
A graph G is planar iff G has an abstract dual.

[Proof]

($\Rightarrow$) If G is planar, its (geometric) dual is its abstract dual

($\Leftarrow$) Let G^* be the abstract dual. So, $C(G) = C^*(G^*)$. The set $\{\delta(v) \mid v \in V(G^*) \setminus \{w\}\}$ is a basis for $C^*(G^*)$ for some $w \in V(G^*)$. Each edge appears at most twice, once for each end, so this is a 2-basis for $C^*(G^*)$, and consequently also a 2-basis for $C(G^*)$. Then, by MacLane's Theorem, G is planar.

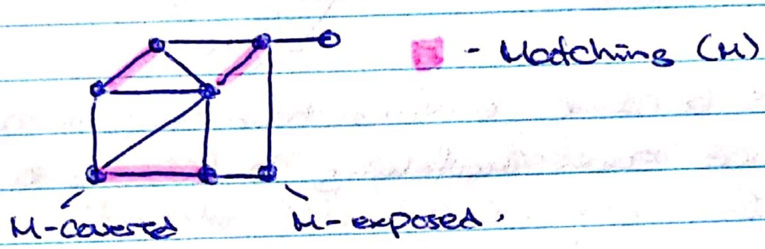
☐

4 Matchings

Matchings.

Def'n (Matching, covered/M-covered, exposed/M-exposed)
A matching M of G is a set of edges that do not share any vertices. A vertex v is covered by M (or M-covered) if v is incident with an edge in M. Otherwise, v is exposed (or M-exposed)

Ex:



Def'n (Deficiency)
We denote $\nu(G)$ to be the size of a maximum matching in G. The deficiency of G is:

$$\mathrm{def}(G) = |V(G)| - 2\nu(G)$$

(i.e. The min. # of vertices "missed" by a matching)

Def'n (Maximal)
A matching is maximal if no edge has both endpoints exposed by the matching.

Def'n (Perfect Matching / 1-factor)
A perfect matching (or 1-factor) is a matching that covers all vertices.

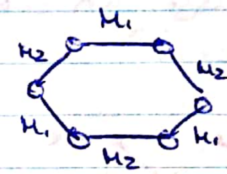
Theorem 27.1
Let M be a maximal matching and M^* be a maximum matching. Then $|M^*| \leq 2|M|$

[Proof] Follows from Prop'n 27.1 (let M^* be a maximal matching) $\square$

Q: Let M_1, M_2 be 2 matchings. What does $M_1 \Delta M_2$ look like?

A:

We get components that are paths and cycles.



This is because vertices have degree at most 2, since each matching contributes at most 1.

Furthermore, notice that you can't have odd cycles, since edges alternate between M_1 and M_2.

For any component K,

$$||K \cap M_1| - |K \cap M_2|| \le 1$$

with equality holding iff K is an odd-length path.

Proposition 27.1

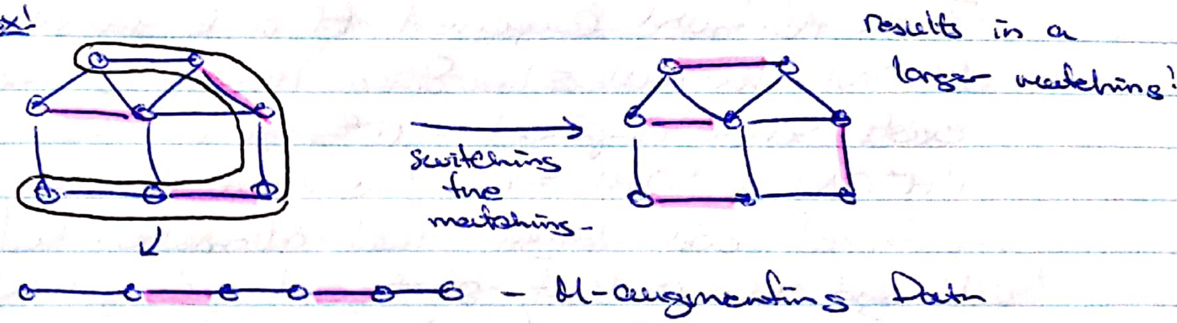
If M_1, M_2 are maximal matchings, then $|M_1| \le 2|M_2|$ and $|M_2| \le 2|M_1|$.

[Proof]

Suppose $H_1, \dots, H_k$ are the nontrivial components of $M_1 \Delta M_2$. Then $||H_i \cap M_1| - |H_i \cap M_2|| \le 1$, and since the # of edges in M_1, M_2 and not in $M_1 \Delta M_2$ are the same $|M_1| - |M_2| \le k$. So, we get that $|M_1| \le k + |M_2|$ and claim that $k \le |M_2|$. Suppose not, then there is H_i without an edge in M_2, so it must be a single edge $e = xy$ in M_1, but then both x and y are both M_2-exposed, contradicting maximality of M_2. So $|M_1| \le 2|M_2|$, and by symmetry $|M_2| \le 2|M_1|$
$\square$

Augmenting Paths:

Ex:



Def'n (M-alternating, M-augmenting)
Given a matching M, a path is <u>M-alternating</u> if edges alternate b/w in M and out of M in the path. A path is <u>M-augmenting</u> if it is M-alternating, and starts and ends with M-exposed vertices.

<u>Remark 28.1</u>: There are more edges not in M than in M, so "switching" the matching increases the size of the matching. (For M-augmenting paths)

<u>Theorem 28.1</u>:
A matching is maximum iff there are no M-augmenting paths.

[Proof]

($\Rightarrow$) (Contrapositive)
Suppose there is an M-augmenting xy-path P. Then, $P \triangle M$ is also a matching, but it covers x, y in addition to M-covered vertices. So, M is not maximum

($\Leftarrow$) (Contrapositive)
Suppose M is not maximum. Then, there is a matching N s.t. $|N| > |M|$. Consider $K = M \triangle N$.

$\hookrightarrow$ Con't

[Proof] (Con't)

Each nontrivial component H in K satisfies $||H \cap N| - |H \cap M|| \leq 1$. Since $|N| > |M|$, there exists some component H^*, where: $||H^* \cap N| - |H^* \cap M|| = 1$. This must be a path of odd length, that alternates between N and M, and it starts and ends with edges in N. So, the 2 ends of the path are M-exposed. Hence, it is an M-augmenting path. $\square$

Bipartite Matchings:

Defn (Neighbor Set)

For a subset $S \subseteq V(G)$, the neighbour set $N_G(S)$ or $N(S)$ is the set of all vertices in $V(G) \setminus S$ that is adjacent to some vertex in S.

Theorem 28.2: (Hall's Theorem)

Let G be bipartite with bipartition (A, B). Then G has a matching that covers A iff for all $S \subseteq A$, $|N_G(S)| \geq |S|$.

[Proof]

($\Rightarrow$) Any matching that covers A must cover any subset $S \subseteq A$, and their matching neighbours must be distinct in $N(S)$.

So, $|N(S)| \geq |S|$.

Con't

[Proof] (con't)

($\Leftarrow$) We prove by induction on $|A|$.
This is trivially true when $|A|=0$. Next, we break the proof into 2 cases.

Case 1: ($|N_G(S)|>|S|$ for all $\emptyset \neq S \subset A$)
Consider any edge xy, where $x \in A$, $y \in B$. Let $G' = G - \{x,y\}$, and $S' \subseteq A \setminus \{x\}$. By assumption, $|N_G(S')| > |S'|$, but since we removed y from B in G, then $|N_{G'}(S')| \geq |N_G(S')| - 1 \geq |S'|$. By induction, there is a matching in G' that covers $A \setminus \{x\}$. Together, with xy, we get a matching in G that covers A.

Case 2: ($\exists \emptyset \neq S \subset A$, where $|N_G(S)| = |S|$)
Let G_1 be the subgraph induced by $S \cup N_G(S)$ and $G_2 = G - G_1$. In G_1, for any $S' \subseteq S$, $|N_{G_1}(S')| \geq |S'|$ since all neighbours in G of S' are in $N_G(S')$ by def'n. By induction, there is a matching that covers S in G_1.
Now, consider G_2. Let $T \subseteq A \setminus S$ in G. Suppose $|N_{G_2}(T)| < |T|$. Then $N_G(S \cup T) = N_G(S) \cup N_{G_2}(T)$.
So, we get:

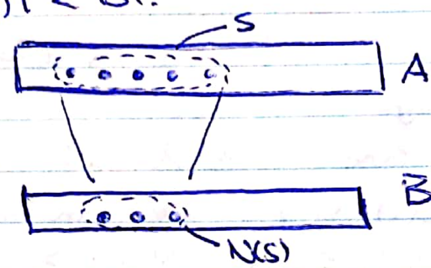
$$\begin{aligned} |N_G(S\cup T)| &= |N_G(S)| + |N_{G_2}(T)| \\ &= |S| + |N_{G_2}(T)| \\ &< |S| + |T| = |S \cup T| \end{aligned}$$

Contradicting that $|N_G(S\cup T)| \geq |S \cup T|$. So, $|N_{G_2}(T)| \geq |T|$ for all $T \subseteq A \setminus S$, and by induction, there is a matching in G_2 that covers $A \setminus S$. Together with the matching in G_1, we get a matching that covers A in G. $\square$

Q: Suppose no M covers A. What is the size of the maximum matching?

Ex:

By Hall's Thm, there must be $S \subseteq A$ s.t. $|N(S)| < |S|$.



Notice that, in this example, we miss at least 2 vertices.

Def'n (Deficiency)

In a bipartite graph, with bipartition (A,B), a subset $S \subseteq A$ has deficiency
$$\mathrm{def}(S) = |S| - |N(S)|$$

Corollary 20.1

For a bipartite G, with bipartition (A,B).
$$\nu(G) = |A| - \max\{\mathrm{def}(S) : S \subseteq A\}$$

Note: $\mathrm{def}(\emptyset) = 0$, so this max is nonnegative

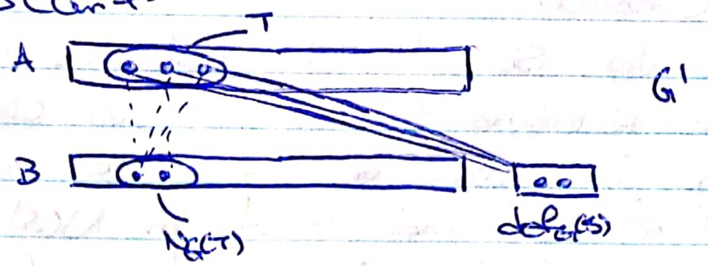
[Proof]

Let $S \subseteq A$ be a set with maximum deficiency. Any matching must miss at least $\mathrm{def}(S)$ vertices. So, $\nu(G) \le |A| - \mathrm{def}(S)$.

Now, we obtain G' from G by adding $\mathrm{def}_G(S)$ new vertices to B and join every one to all vertices in A.

→ Con't

[Proof] (Con't)



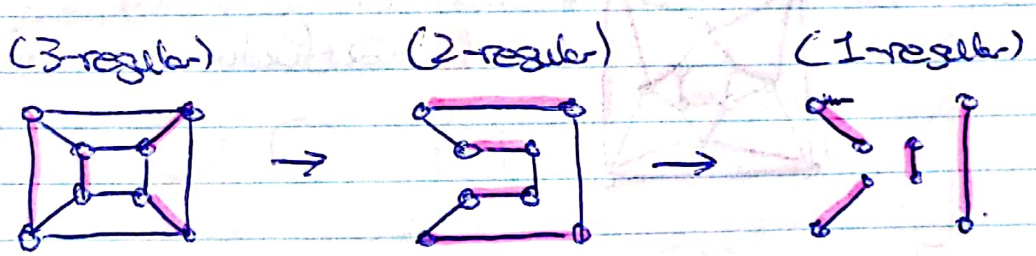
For any $T \subseteq A$,

$$|N_{G'}(T)| = |N_G(T)| + def_G(S)$$
$$= |T| \underbrace{- def_G(T) + def_G(S)}_{\geq 0, \text{ since } S \text{ has max. deficiency}}$$
$$\geq |T|$$

So, by Hall's Theorem, there is a matching M' that covers A in G', where at most there are $def_G(S)$ many edges in M' that use the newly added vertices.
So, $\nu(G) \geq |A| - def(S)$, and equality holds ∎

Corollary 29.2
If G is a k-regular bipartite graph ($k \geq 1$), then G has a perfect matching. Moreover, the edges of G can be partitioned into k perfect matchings.

Ex:



So, we get 3 perfect matchings.

⟶ [Proof]

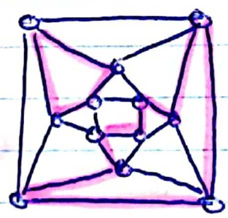
[Proof] (Sketch)

G is k-regular, so $|A| = |B|$. Let $S \subseteq A$.
there are k edges coming out of each vertex in S, so we get that $k|S|$ edges come out of S and land in $N(S)$. Since each vertex in $N(S)$ cover at most k vertices, $|N(S)| \geq |S|$.
Then, by Hall's Thm, there exists a matching that covers A, and is a perfect matching since $|A| = |B|$.
Removing such a matching results in a $(k-1)$-regular bipartite graph. And so, by induction, we get k perfect matchings $\square$.

Note: This is sometimes called the 1-factorization of the graph

Defn (2-factor)
A 2-factor of G is a 2-regular spanning subgraph

Ex)



- 2-factor

Theorem 30.1:
Any 2k-regular graph with $k \geq 1$ has a 2-factor. Moreover, the edges can be partitioned into k 2-factors.

$\hookrightarrow$ [Proof]

[Proof]

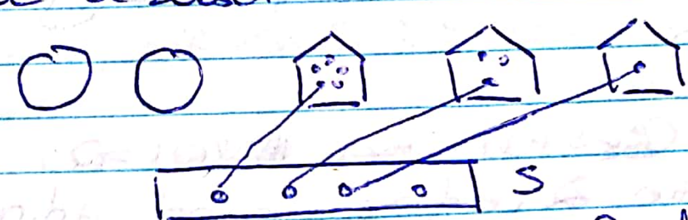
Assume G is connected. Since each vertex has even degree, it has an eulerian circuit.
Say the walk is: $v_0 e_0 v_1 e_1 \cdots v_{k-1} e_{k-1} v_k = v_0$

We create a new graph H from G by splitting each vertex into v^- (v_{out}), v^+ (v_{in}) in H, and for each edge $e_i = v_i v_{i+1}$ (as in the walk) in G, we add the edge $v_i^- v_{i+1}^+$ in H. Then H is also bipartite with bipartition $\{v^- \mid v \in V(G)\}$ and $\{v^+ \mid v \in V(G)\}$. Since each $v \in V(G)$ has degree $2k$, the circuit will visit v k times, using k edges to go in, and k edges to go out. So, H is k-regular. Then, by previous corollary (20.2), H has a perfect matching. By merging v^-, v^+, we get a 2-factor in G.

The 2nd result is done by induction □

General Matchings:

Ex:
Suppose we have a perfect matching for G, and we remove a subset $S \subseteq V(G)$.



What can we say about the # of odd components?
- # odd components $\le |S|$, since if not, we cannot have a perfect matching with an odd # of vertices.

Is the converse of this true?

cont

Defn (Odd Component)
An odd component is a component with an odd number of vertices. Denote $o(G)$ to be the # of odd components in G.

Theorem 30.2 (Parity Lemma)
Let $S \subseteq V(G)$. Then $o(G-S) - |S| \equiv |V(G)| \pmod 2$

Theorem 30.3 (Tutte's Perfect Matching Theorem)
G has a perfect matching iff $|S| \geq o(G-S)$ for all $S \subseteq V(G)$.

[Proof]
($\Rightarrow$) If G has a perfect matching M, then any odd component of $G-S$ must have at least one vertex matched to a vertex in S, since any perfect matching will leave one vertex out. So, $|S| \geq o(G-S)$.

($\Leftarrow$) Assume $|S| \geq o(G-S)$ for all $S \subseteq V(G)$
By taking $S = \emptyset$, $|S| = 0 \geq o(G-S) = o(G)$.
So $o(G) = 0$, and G has no odd components, hence $|V(G)|$ must be even.
Let $|V(G)| = 2n$, we proceed by induction on n.

Base Case: When $n=1$, $|V(G)| = 2$.
If the 2 vertices are not adjacent, then we have 2 odd components, contradiction.
So, an edge exists and we have a perfect matching.

Now, assume $n > 1$.

Con't

[Proof](Cont)

We break into 2 cases:

<u>Case 1</u>: ($|S| > o(G-S)$ for all $S \subseteq V(G)$, where $2 \le |S| \le 2n$)

By Parity lemma, $|S| \ge o(G-S) + 2$.

Let xy be an edge and $G' = G - \{x,y\}$.

Let $T \subseteq V(G')$, and consider $G'-T$. We can write $G'-T = G-(T \cup \{x,y\})$. Then, by assumption:

$$|T \cup \{x,y\}| \ge o(G-(T\cup\{x,y\})) + 2$$

$$\Rightarrow |T| + 2 \ge o(G'-T) + 2$$

$$\Rightarrow |T| \ge o(G'-T)$$

And, by induction, G' has a perfect matching, which, together with xy, is a perfect matching for G.

<u>Case 2</u>: (There is $|S| = o(G-S)$ for some $S \subseteq V(G)$, where $2 \le |S| \le 2n$)

Among all such subsets, we pick a maximal one. We first prove that $G-S$ has no even components. Suppose there is an even component C (of $G-S$). Let $x \in V(C)$. Then $C-x$ has an odd number of vertices, so it must contain at least one odd component. So, $o(G-(S\cup\{x\})) \ge o(G-S)+1$.

Then:

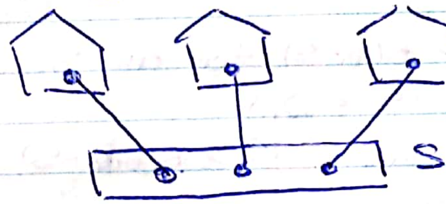
$$|S\cup\{x\}| = |S| + 1 \underset{\text{By assumption}}{=} o(G-S) + 1 \le o(G-(S\cup\{x\}))$$

But also, $|S\cup\{x\}| \ge o(G-(S\cup\{x\}))$ by assumption, so $|S\cup\{x\}| = o(G-(S\cup\{x\}))$, which contradicts the maximality of S.

Cont.

[Proof] (Con't)

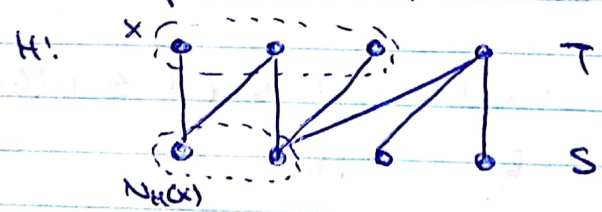
So, $G-S$ only consists of odd components



(Idea: We find a perfect matching from S to the components, using Hall's Theorem, then a P.M. within the components)

We obtain a graph H from G by contracting each odd component of $G-S$ into a vertex, and remove any edge joining 2 vertices in S. Let T be the set of all contracted vertices, then H is bipartite with bipartition (S,T). Further, since $|S| = o(G-S)$, $|S| = |T|$. We want to show that H has a perfect matching.

Let $X \subseteq T$, and suppose $|N_H(X)| < |X|$. Then in G, the odd components of $G - N_H(X)$ are those represent by X and possibly other odd components. So, $o(G - N_H(X)) > |N_H(X)|$



which is a contradiction, so $|N_H(X)| \geq |X|$

So, by Hall's Theorem, H has a perfect matching which in G, is a matching from each vertex in S to a distinct ...

Con't

[Proof] (con't)
... odd component of $G-S$. Call this matching M.

Now, consider any odd component C of $G-S$. Suppose $a \in V(C)$ is covered by M. We want to show that there is a P.M. in $C-a$.

Suppose there is $Y \subseteq V(C) \setminus \{a\}$, where $o((C-a)-Y) > |Y|$. Since $C-a$ has an even # of vertices, by Parity Lemma: $o((C-a)-Y) \geq |Y| + 2$.
Let $Z = S \cup \{a\} \cup Y$ so that the odd components of $G-Z$ are: the odd components of $C-a-Y$, and any other odd components of $G-S$.
Then:

$$o(G-Z) = (o(G-S) \underbrace{-1}_{\text{since we lose component } C}) + o(C-a-Y)$$
$$\geq (o(G-S) - 1) + |Y| + 2$$
$$= |S| - 1 + |Y| + 2$$
$$= |S| + |Y| + 1 = |Z|$$

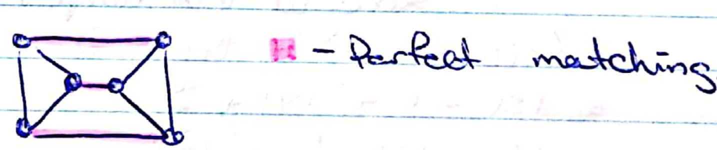
But $o(G-Z) \leq |Z|$ by assumption, so $o(G-Z) = |Z|$, contradicting maximality of S.

So, $o(C-a-Y) \leq |Y|$ for all $Y \subseteq V(C) \setminus \{a\}$, then, by induction $C-a$ has a matching. This, along with a P.M. for each odd component and M gives a P.M. for G. $\blacksquare$

Def'n (Deficiency)
For $S \subseteq V(G)$, the deficiency of S is
$$\text{def}(S) = o(G-S) - |S|$$
Note: Any matching must miss at least def(S) vertices.

Corollary 32.1 (Tutte-Berge Formula)
For any graph G,
- $\text{def}(G) = \max\{\text{def}(S) \mid S \subseteq V(G)\}$
- $\nu(G) = \dfrac{|V(G)| - \text{def}(G)}{2}$

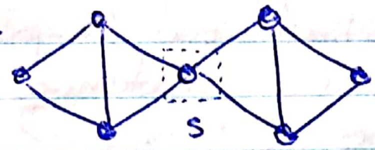
Corollary 32.2 (Petersen)
A 3-regular graph with no cut edges has a perfect matching
Ex:



[Proof]
Let G be 3-regular with no cut edges, and $S \subseteq V(G)$. Let $C_1, \dots, C_k$ be the odd components of $G-S$, so $k = o(G-S)$. For each C_i, the sum of vertex degrees in C_i is odd since $|V(C_i)|$ is odd and each has deg 3. Each edge in C_i contributes 2 to this total degree, so the number of edges joining $V(C_i)$ with S is odd. Further, this number cannot be 1, since such an edge would be a cut edge. So, there must be at least 3 edges between C_i and S. Over all k components, there are at least $3k$ edges with one edge in S. Since each vertex in S has deg 3, $|S| \geq 3k/3 = k = o(G-S)$. By Tutte's Thm, G has a perfect matching $\square$

Def'n (Tutte Set)
A tutte set is a set of vertices with max. deficiency
Ex!

G:



S is a Tutte set, since $o(G-S) - |S| = 2 - 1 = 1$.
Note: $T = \emptyset$ is also a Tutte Set.

Def'n (Essential, avoidable)
A vertex v is essential if every maximum matching
G covers v (or $\nu(G-v) = \nu(G) - 1$). It is avoidable
if some max. matching exposes v (or $\nu(G-v) = \nu(G)$).

Lemma 32.1:
Every vertex in a Tutte set is essential
[Proof]
Let S be a Tutte set, and suppose $x \in S$ is avoidable
Let $G' = G - x$, $S' = S \setminus \{x\}$, then $G - S = G' - S'$.
Then,

$$\begin{aligned} \operatorname{def}(G') &\geq o(G'-S') - |S'| \\ &= o(G-S) - |S| + 1 \\ &= \operatorname{def}(G) + 1 \quad (\text{since } S \text{ is a Tutte set}) \end{aligned}$$

So, $\nu(G') < \nu(G)$, and hence x must be essential
$\square$.

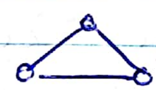
$\hookrightarrow$ Theorem

Hilroy

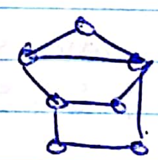
Theorem 32.2:
Let S be a Tutte set of G.
(1) Every vertex of S is essential, and is matched to a distinct odd component of $G-S$ by any maximum matching
(2) The even components of $G-S$ have perfect matchings
(3) The odd components of $G-S$ have near-perfect matchings (a matching that covers all but one vertex).

Defn (Factor-Critical)
G is factor-critical if $G-v$ has a perfect matching for $v \in V(G)$.
Note: Such a graph must have an odd number of vertices

Ex:



- and -



are both factor-critical.

Remark 33.1: Factor Critical graphs are connected

Theorem 33.1:
Let G be a connected graph. Then, G is factor-critical iff every vertex of G is avoidable

Proof

[Proof]
($\Rightarrow$) If G is factor-critical, then $G-v$ has a perfect matching, which must also be a maximum matching for G. So, v is avoided by this max. matching

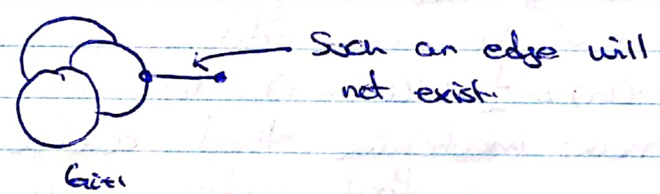
($\Leftarrow$) Suppose all vertices are avoidable. Let T be a Tutte set. Since any vertex in T is essential, $T=\phi$. So, $def(G) = o(G-T) - |T| = o(G)$, which is 1 or 0, since G is connected.
If G is an even component, then $o(G)=0$, and so $def(G)=0$, and G has a perfect matching, contradicting that every vertex is avoidable.
So, G must be an odd component, $def(G)=1$, and $\nu(G) = \frac{1}{2}(|V(G)|-1)$. For any $v \in V(G)$, there is a max. matching that avoids v. So, then $G-v$ has a perfect matching, as the max. matching has size $\nu(G)$. So, G is factor-critical. $\square$

Theorem 33.2
If T is a maximal Tutte set, then each odd component of $G-T$ is factor-critical.
[Proof] Similar to proof for Tutte's Thm (Thm 30.7) $\square$

Defn (odd closed-ear decomposition)
An odd closed-ear decomposition of G is a sequence of graphs $(G_0, \ldots, G_k)$, where:
(1) G_0 is a vertex, $G_k = G$.
(2) For each $0 \leq i \leq k-1$, $G_{i+1} = G_i + P_i$, where P_i is either an ear of G_i of odd length, or an odd cycle, where $|V(G_i) \cap V(P_i)| = 1$.

Theorem 33.3 (Lovász)
A graph G is factor-critical iff G has an odd closed-ear decomposition.
[Proof]
($\Leftarrow$) Induction + Case Analysis.

($\Rightarrow$) Suppose G is factor-critical. Let G_0 consist of one vertex v in G. Let M_v be a P.M. in $G-v$. We will iteratively build G_{i+1} from G_i by adding P_i so that no edge in M_v is in the cut induced by $V(G_{i+1})$ (called <u>the invariant</u>) i.e.



Suppose we have built G_i. If $G_i = G$, then we're done.
If there is an edge xy in G, not in G_i, where $x, y \in V(G_i)$, then $G_{i+1} = G_i + xy$, and no edge in M_v is in $\delta(V(G_{i+1}))$.
Now, suppose $x \in V(G_i)$, $y \notin V(G_i)$. By the invariant, $xy \notin M_v$. Let M_y be a P.M. of $G-y$.
Consider $M_v \triangle M_y$. The only vertices of degree 1 are v and y, since they are covered by exactly one of M_v, M_y, and all other vertices are covered by both. So, v, y are the endpoints of a path in $M_v \triangle M_y$.

Cont'd

[Proof] (Con't)

Say this path P is $v_0 v_1 \ldots v_m$, where $v_0 = y$, $v_m = v$. Since $v_0 \notin V(G_i)$ and $v_m \in V(G_i)$, there is a smallest index j, where $v_j \in V(G_i)$. Let Q be $v_0 \ldots v_j$. By the invariant, $v_{j-1}v_j \in M_y$ and $v_0 v_1 \in M_v$. Since the edges alternate between M_v and M_y, Q has even length. Then, $xy + Q$ is either an odd length ear or an odd cycle.

Set $G_{i+1} = G_i + xy + Q$.

Now we still need to prove the invariant! Note that every vertex $v_0, \ldots, v_{j-1}$ is covered by an edge in M_v, which is also in the path Q. So, no edge in M_v is in $\delta(V(G_{i+1}))$, so the invariant holds. $\square$

Theorem 35.1 (Gallai-Edmonds Structure Theorem)

Given G, let D be the set of all avoidable vertices in G, let A be the set of all vertices in $V(G) \setminus D$, that is adjacent to some vertex in D, let $C = V(G) \setminus (D \cup A)$.

Then,

(1) The subgraph induced by D are odd components, which are factor-critical.

(2) The subgraph induced by C are even components, that have perfect matchings.

(3) A is a Tutte set

(4) Each non-empty subset S of A is adjacent to at least $|S|+1$ odd components in D.

(5) If M is a maximum matching then it contains a near-perfect matching for each component of D, a perfect matching for C, and matches each vertex of A to a distinct odd component in D.

Con't