CO463 - Convex Optimization

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Contents

Lecture 1: Introduction

Definition 1.1. Let $x, y \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$, $z(\lambda) = \lambda x + (1-\lambda)y$ is a convex combination of x and y

This simple definition leads to many strong algebraic and topological results.

For this course, we will work in Euclidean space $\mathbb{E}^n$ with inner product $\langle x, y\rangle$ and norm $||x|| = \sqrt{\langle x, y\rangle}$.

On $\mathbb{R}^n$, we will use the familiar dot product $\langle x, y\rangle = x^\intercal y = \sum_{i=1}^n x_i y_i$ and norm $||x|| = \sqrt{\sum_{i=1}^n x_i^2}$.

Definition 1.2. $C \subseteq \mathbb{E}^n$ is convex set if $x, y \in C$ $0 \leq \lambda \leq 1 \Rightarrow \lambda x + (1-\lambda)y \in C$ (or equivalently, $x, y \in C \Rightarrow [x, y] \subseteq C$)

Note. We can write $z(\lambda)$ in the following ways:

$$\begin{aligned} z(\lambda) &= \lambda x + (1-\lambda)y \\ &= y + \lambda(x - y) \\ &= x + (1-\lambda)(y - x) \end{aligned}$$

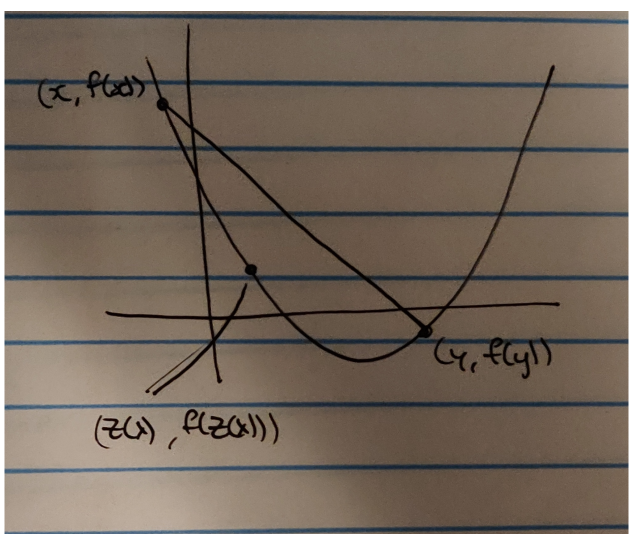
The second equation can be interpreted as: "Beginning at y and moving towards x". And, the third equation can be interpreted as: "Beginning at x and moving towards y"

Definition 1.3. Let $C \subseteq \mathbb{E}^n$, C convex set. Then $f : C \to \mathbb{R}$ is a convex function if:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

$\forall x, y \in C,\ \forall 0 \leq \lambda \leq 1$

Example 1.1.



The line between $(x, f(x))$ and $(y, f(y))$ is called the secant line. For convex function, the graph lies below the secant line.

The region <u>above</u> the graph is called the <u>epigraph</u>, denoted epi f. It is defined as:

$$\operatorname{epi} f = \{(r, x) \in \mathbb{R} \times \mathbb{E}^n \ : \ x \in C, f(x) \subseteq r\}$$

for $f : C \to \mathbb{R}$, C convex.

We will see that:

Theorem 1.1. f is a convex function if and only if epi f is a convex set

Suppose we have the following minization problem:

$$\begin{aligned} p^* = &\min f(x) \\ &\text{s.t } x \in C \end{aligned}$$

where $C \in \mathbb{E}^n$ convex, $f : C \to \mathbb{R}^n$ convex function.

We will show that: f convex $\Rightarrow$ (local min <u>iff</u> global min)

<u>Applications:</u>

- Linear Programming
- Convex Relaxation