July 24th, 2018

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Recall:

For $v \in \mathbb{R}^d$, $\|v\|_\infty := \max_{i=1,\dots,d} |v_i|$

Sevast'janov/Steinitz (SS) Lemma:

Given $v_1, \dots, v_n \in \mathbb{R}^d$ s.t. $\sum_{i=1}^{n} v_i = \vec{0}$, we can efficiently find a permutation σ of $1, \dots, n$ s.t.

$$\left\| \sum_{k=1}^{j} v_{\sigma(k)} \right\|_\infty \le d \cdot \max_{k=1,\dots,n} \|v_k\|_\infty \quad \forall j = 1, \dots, n.$$

Algorithm for $F||C_{max}$:

1) Increase p_{ij}'s up to p_{max} (arbitrarily) to ensure that $T_i = T_{max}$ $\forall i = 1, \dots, m$
2) Apply SS lemma to vectors:
   $v_j := (p_{1,j} - p_{2,j}, \dots, p_{m-1,j} - p_{m,j})$
   $\forall j = 1, \dots, n$,
   to get permutation σ.
3) Return permutation schedule corresponding to σ.

What's left: Prove lemma!

[Proof] (lemma)

We will construct σ by specifying all its prefixes starting in reverse order,

i.e. We will construct

$$V_n := \{v_1, \dots, v_n\} \supsetneq V_{n-1} \supsetneq \dots \supsetneq V_1$$

where $|V_j| = j$ $\forall j = 1, \dots, n$ so that $\forall j = 1, \dots, n$ we have:

$$\left\| \sum_{v \in V_j} v \right\|_\infty \le d \cdot M \qquad (*)$$

(let $M = \max_{k=1,\dots,n} \|v_k\|_\infty$)

[Proof] (Cont.)

- V_n satisfies (*) (Since $\|\sum_{v\in V_n} v\|_\infty = \vec{0}$)

*Note: Triangle inequality:
($\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$).

So, if we have constructed V_{d+1} with $|V_{d+1}| = d+1$, then we can:
- Take V_d ~~with~~ ~~any~~ to be any subset of V_{d+1} with d vectors
- Take V_{d-1} to be any subset of V_d with $d-1$ vectors
- ... and so on

This works since for any set of $j\le d$ vectors V_j, we have:

$$\Big\|\sum_{v\in V_j} v\Big\|_\infty \le j\cdot \max_{v\in V_j}\|v\|_\infty \le d\cdot M.$$

By triangle inequality.

So, our goal is: Construct $V_n \supsetneq V_{n-1} \supsetneq \dots \supsetneq V_{d+1}$ (Since once we have constructed V_{d+1} the rest follows from above).
with $|V_j| = j$ $\forall j = d+1, \dots, n$ and (*) holds $\forall j = d+1, \dots, n$

Idea: Use induction, given V_j satisfying (*), show how to get $V_{j-1} \subsetneq V_j$ satisfying (*). We will strengthen (*) as follows:

Sufficient condition implying (*):
Suppose $j\ge d$. Suppose we have a set $S\subseteq V_j$ with $|S| = j-d$ s.t. $\sum_{j\in S} v_j = 0$.
Then, we claim: V_j satisfies (*)

Proof of claim:

$$\|\sum_{v\in V_j} v\|_\infty = \|\sum_{v\in S} v + \sum_{v\in V_j\setminus S} v\|_\infty$$

$$= \|\sum_{v\in V_j\setminus S} v\|_\infty \quad \text{and } |V_j\setminus S| = d$$

$$\le |V_j\setminus S| \max_{v\in V_j\setminus S}\|v\|_\infty$$

$$\le d\mu$$

We write ~~these~~ boxed portion on back as an IP

(IP$_j$) variables $\lambda_v^j \ \forall v\in V_j$ $\equiv 1$ if $v\in S$, 0 o/w

$$\boxed{(\text{IP}_j)\ \sum_{v\in V_j}\lambda_v^j = j-d,\ \sum_{v\in V_j}\lambda_v^j v = \vec{0} \quad \lambda_v^j\in\{0,1\}\ \forall v\in V_j}$$

(IP$_j$) is feasible $\Rightarrow$ V_j satisfy $(*)_j$

Modification strategy: If $j > d+1$, and we have V_j s.t. (IP$_j$) is feasible, then we can get $V_{j-1}\subsetneq V_j$ s.t. (IP$_{j-1}$) is feasible

In fact, this is too strong an ambition. We will show that the LP-relaxation of (IP$_j$), which we will denote (LP$_j$):

$$(\text{LP}_j)\ \sum_{v\in V_j}\lambda_v^j = j-d,\ \sum_{v\in V_j}\lambda_v^j v = \vec{0},\ 0\le\lambda_v^j\le 1,\ \forall v\in V_j$$

is feasible still implies V_j satisfies $(*)$

Claim 1: (LP_j) is feasible $\Rightarrow V_j$ satisfies $(*)$.

Proof:

Let $\{\lambda_v^j\}_{v\in V_j}$ be a feasible solution to (LP_j)

$$\left\|\sum_{v\in V_j} v\right\|_\infty := \left\|\sum_{v\in V_j}\lambda_v^j v + \sum_{v\in V_j}(1-\lambda_v^j)v\right\|_\infty.$$

$$= \left\|\sum_{v\in V_j}(1-\lambda_v^j)v\right\|_\infty \quad \left(\text{Since } \sum_{v\in V_j}\lambda_v^j v = 0.\right)$$

$$\leq \sum_{v\in V_j}\|(1-\lambda_v^j)v\|_\infty \quad (\text{Triangle Inequality})$$

$$= \sum_{v\in V_j}(1-\lambda_v^j)\|v\|_\infty$$

$$\leq \sum_{v\in V_j}(1-\lambda_v^j)M.$$

$$= \left(j - \sum_{v\in V_j}\lambda_v^j\right)M = d\cdot M.$$

Lemma 2: We now want to show:
Suppose $j > d+1$. Suppose we have $V_j \subseteq V_n$ s.t. (LP_j) is feasible. Then, we can efficiently find $V_{j-1} \subsetneq V_j$ with $|V_j| = j-1$ s.t. (LP_{j-1}) is feasible

Notice that Claim 1 + Lemma 2 will finish the proof. Since:

- (LP_n) is feasible, we set $\lambda_v^n = \frac{n-d}{n}$ $\forall v\in V_n$, this gives a feasible sol'n to (LP_n).
- By repeatedly using Lemma 2, we get sets $V_n \supsetneq V_{n-1} \supsetneq \dots \supsetneq V_{d+1}$ s.t. (LP_j) feasible $\forall j = d+1, \dots, n$.

$\Rightarrow V_j$ satisfies (*) $\forall j = d+1, \dots, n$.

Then, from $d+1$, we can arbitrarily choose the remaining sets $V_d, V_{d-1}, \dots, V_1$.

Now, for the proof of Lemma 2:

<u>Recall</u>: (from LP theory - CO250 / 2017)

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$

<u>Defn</u>: $\bar{x} \in P$ is an extreme point of P if $\nexists\, x^{(1)}, x^{(2)} \in P$, $x^{(1)} \ne x^{(2)}$, $0 < \lambda < 1$ s.t.

$$\bar{x} = \lambda x^{(1)} + (1-\lambda) x^{(2)}.$$

Further:

$\bar{x} \in P$ is an extreme point of P iff

$$\operatorname{rank}\Big([A_{ij}]_{\substack{j=1,\dots,n \\ \underbrace{i : (A\bar{x})_i = b_i}_{\text{tight constraints}}}} \Big) = n.$$

Finally,

If $\operatorname{rank}(A) = n$ and $P \ne \emptyset$, then P has an extreme point. (and this can be found efficiently).

Consider the system: (P) ← Similar to (LP_{j-1}), but we have variables $\forall v \in V_j$.

$$\sum_{v \in V_j} x_v = j-1-d \quad (1), \qquad \sum_{v \in V_j} x_v v = \bar{0} \quad (2)$$

$$0 \le x_v \le 1, \quad \forall v \in V_j.$$
(3a) (3b)

Corresponds to 2 inequalities ($\le$ and $\ge$).

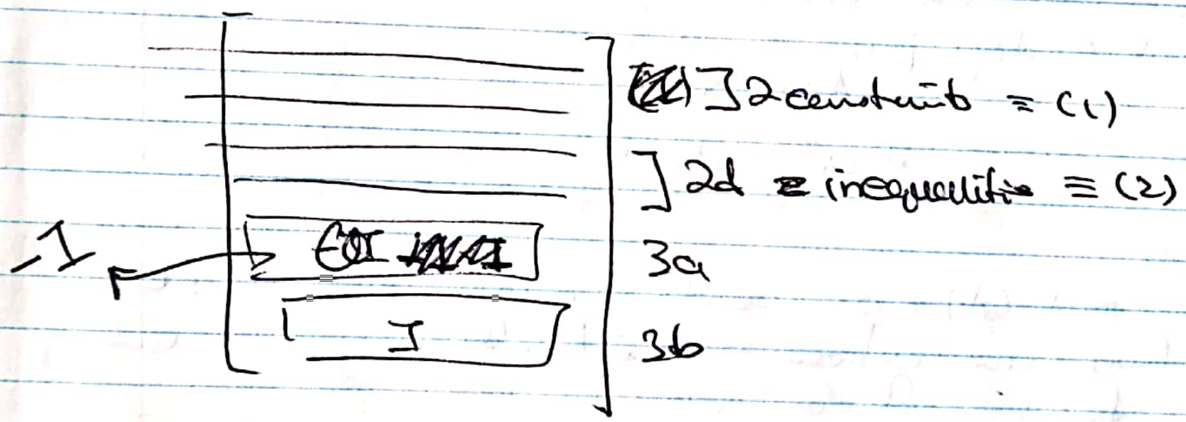
Corresponds to $2d$ $\le$-inequalities.

Will show: (LP_j) feasible $\Rightarrow$ every extreme point feasible solution to (P) has $x_v = 0$ for some $v \in V_j$ $\Rightarrow$ Taking V_{j-1} to be $V_j \setminus \{\text{vector } v \text{ s.t. } x_v = 0\}$ then $\lambda_v^{j-1} = x_v$ $\forall v \in V_{j-1}$ is a feasible solution to (LP_{j-1}).

(LP_j) is feasible: Say $\{\lambda_v^j\}_{v \in V_j}$ is feasible to (LP_j)

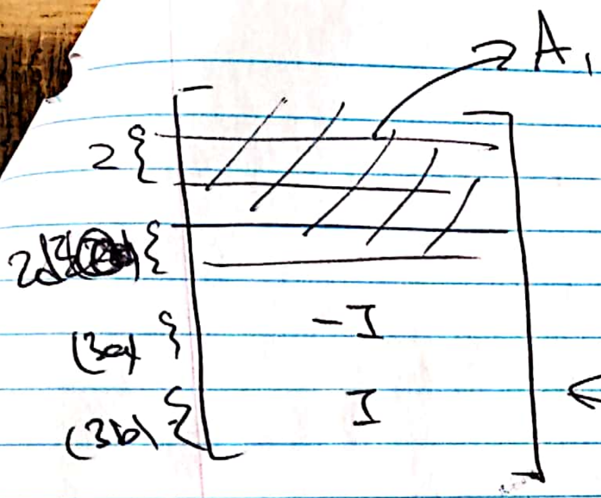
$$\Rightarrow x_v = \frac{j-1-d}{j-d} \lambda_v^j \quad \forall v \in V_j: \text{ feasible sln to } (P)$$

Constraints of (P) written as $\leq$ inequalities.



So, ~~...~~

So, the inequalities in matrix form $\geq$ has rank $= j$.



~~[illegible]~~
Constraint matrix.

Suppose $\bar{x}$: extreme point of feasible region of (P)

We want to show that at least of of the (3a) constraints is tight for $\bar{x}$ (and so one of the values are 0).

Suppose not:

Let $A' = [A_{ij}]_{j=1,\dots,n}$, $i = a^i x_i = b^i$ (tight constraints)

We know $\operatorname{rank}(A') = j$.

If B is a $j \times j$ submatrix of A' with $\operatorname{rank}(B) = j$, then B contains $\leq d+1$ rows from A_1 $\Rightarrow$ B must contain $\geq j-d-1$ rows from (3b) block of A $\Rightarrow$ $\geq j-d-1$ rows equal to I, and all x_v variables equal to 1, other x_v variables are strictly positive, but then $\bar{x}$ violates condition (i). $\square$