CO749 - Graph Colourings

(Notes Scans)

University of Waterloo
Nicholas Pun

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Summary

Lecture 1, 2 - History

Lecture 3 - Probabilistic method overview, colour degree

Lecture 4 - Reed-Sudakov, Wasteful colouring procedure, expectations for variables in proof of Reed-Sudakov

Lecture 5 - Reed-Sudakov (continued), Concentration Inequalities, Talagrand's inequality

Lecture 6 - Finish Reed-Sudakov, exceptional Talagrand's, Balls & Bins

Lecture 7 - Regularization & Equalizing coin flips, Kim's theorem for girth-five graphs

Lecture 8 - Finishing Kim's, edge-colouring and Kahn's theorem

Lecture 9 - Finishing Kahn's

Lecture 10 - Reed's conjecture, upper bounds, and working towards [1]

Lecture 11 - See [2, 3]

Lecture 12 - Localized Colouring theorems, Local Fractional Colouring, Caro-Wei

Lecture 13 - More Local theorems, Perfect Graphs (Not sure why I only had 2 pages of notes for this lecture)

Lecture 14 - See [4]

Lecture 15 - Missed this one

Lecture 16, 17, 18 - Hadwiger's Conjecture, Proof of Norin-Song theorem (This is probably inaccurate. Lack of review has left me to forget what the main ideas of these lectures were.)

~~7?? - Hopf Algebra~~
749 - Graph Colouring

7th
Jan 7th

Lecture 1:

Def'n 1: A k-colouring of a graph G is a partition of $V(G)$ into at most k independent sets.

Def'n 2: A k-colouring of a graph G is a map $f: V(G) \to [k]$ such that $\forall e = vw \in E(G)$, $f(v) \neq f(w)$

Def'n 3: A k-colouring of a graph G is a graph homomorphism to K_k.

Takeaway: There are multiple ways to view what a colouring is.

Weakenings, Generalizations, and Variants of Colouring:
Variants: ("Changing what you color")
- Edge Colouring: A k-edge-colouring of a graph G is
  Def 1: A partition of $V(G)$ into at most k matchings
  2: A k-colouring of $L(G)$ (the line graph of G)

~~"Changing what you color"~~:

- Total Colouring: A k-total-colouring of a graph G is a map $f: V(G) \cup E(G) \to [k]$ s.t.
  - $f(v) \neq f(w) \quad \forall vw \in E(G)$
  - $f(v) \neq f(e) \quad \forall v \sim e$ ($e \in E(G)$, $v \in V(G)$)
  - $f(e) \neq f(e') \quad \forall e \sim e' \in E(G)$

Generalizations: ("Changing what you are allowed to color")
- List Colouring: (idea: Lists of available colours to vertices)
  Def: ~~A k-list-colouring~~ k-list-assignment of a graph G is ~~an~~ assignment of lists $(L(v): v \in V(G))$ such that

① ②

$|L(v)| \geq k \quad \forall v \in V(G)$

An *L-colouring* of a graph G is a colouring ϕ of G such that $\phi(v) \in L(v) \quad \forall v \in V(G)$

"~~Changing what you are allowed to colour~~"

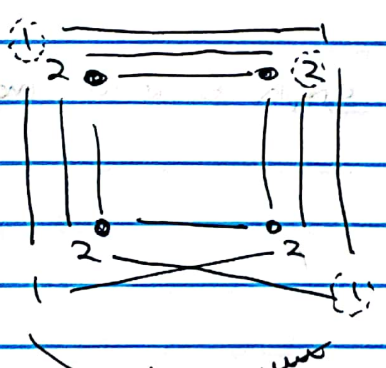
- Correspondence Colouring:

Def: A k-*correspondence-assignment* is a pair $((L(v) : v \in V(G)), (M_{uv} : uv \in E(G)))$, where M_{uv} is a matching from $L(u)$ to $L(v)$

An (L,M)-*colouring* of G is a colouring ϕ of G such that:

- $\phi(v) \in L(v) \quad \forall v \in V(G)$
- $\phi(u)$ is not matched to $\phi(v)$ in $M_{uv} \quad \forall uv \in E(G)$

Ex: $G = C_4$



$L(v) = \{1,2\} \quad \forall v$

Q: Is there an (L,M)-colouring of G for this L,M?

No!

And we run into trouble here

Remarks:

- We may as well assume $L(v) = \{1, \ldots, |L(v)|\}$ $\forall v \in V(G)$
- Correspondence has a "local notion of colour", while list colouring / ordinary colouring have a "global notion of colour"

Weakenings: ("Change what a colour is")

- No Restrictions: Improper colourings, i.e. mappings of graphs
- d-defective colouring: Each colour has maximum degree d
- c-clustered Colouring: Every monochromatic component has size (# of vertices) $\leq c$
- No mono. path of length $> l$.
- Every colour is d-degenerate
- Every colour is triangle-free (or more generally, bounded clique #)

- Fractional Colouring: ("Changing how you colour")

Def: An (a,b)-colouring of a graph G is a map ϕ such that $\phi(v)$ is a subset of $[a]$ of size b and $\forall uv \in E(G)$, $\phi(u) \cap \phi(v) = \phi$

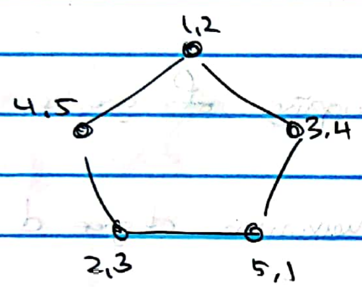
Remark: If G has a k-colouring, then G has a (kb, b)-colouring $\forall b$

1

(i.e. Graph homomorphism to Kneser graph on a, b)

The fractional chromatic #: $\chi_f(G) = \inf\{\frac{a}{b} : G \text{ has an } (a,b)\text{-colouring}\}$.

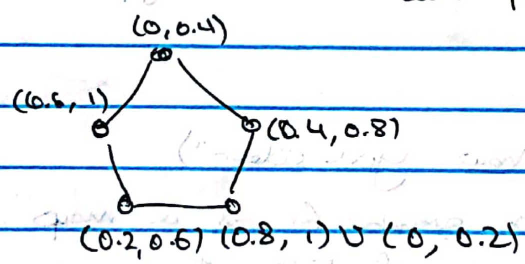
Ex. $\chi_f(C_5) = 5/2$



Remark:
- There is also an LP-formulation and its dual. (its dual asks about weighted independent set)
- An assignment of measurable subsets $\Phi(v)$ of $[0,1]$ $\forall v \in V(G)$ such that:

$m(\Phi(u) \cap \Phi(v)) = \phi$

$\chi_f(G) = 1 / \sup\{\varepsilon : G \text{ has a colouring as above, with } \mu(\Phi(v)) = \varepsilon \ \forall v\}$

Ex: 

Let's call the above an *f-colouring* if $\mu(\phi(v)) = f(v)$ $\forall v$.

Proposition: $\chi_f(G) = k$ iff G has a $(1/k)$-colouring

Proposition: G has an f-colouring iff the vector $(f(v) : v \in V(G))$ is in the independent set polytope (i.e. $\exists$ probability distribution on independent sets such that $Pr[v \in I] = f(v)$)

Combinatorics:

- Defective Clustering ×
- List Defective ✓
- Frac. Defective ✓
- List Edge ✓
- Frac. Edge ✓
- Edge Total ×
- List Correspondence × (correspondence is already list).
- Fractional List ✓
  - Find b-colouring from a-list-assignment. (multicolourings)

$$\chi_{f,list}(G) = \inf\left\{\frac{a}{b} : \forall \text{ a-list-assign } G \text{ has a b-colouring}\right\}$$

Theorem: $\chi_f(G) = \chi_{f,list}(G)$

- Fractional Correspondence.

Lecture 2:

Questions / Results: (on Graph Colourings)

Def'n (k-colourable)
A graph G is k-colorable if G has a k-coloring

Def'n (Chromatic Number)
The chromatic number, denoted $\chi(G)$, is the minimum k s.t. G is k-colorable

Question: Why is this a ~~natural~~ good ~~defn~~ definition?
If a graph is k-colorable, then it is $(k+1)$-colorable, so this is a natural def'n.
(Prop: If $H \subseteq G$, then $\chi(H) \leq \chi(G)$)

Remark:
A graph is 1-colorable iff no edges
— " — 2-colorable iff no odd cycles
— " — 3-colorable iff no good answer!
↳ Since NP-hard to decide if a graph is 3-col.

A graph is critical for k-coloring if it is not k-colorable, but every proper subgraph is. (also formerly known as $(k+1)$-critical)

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List Colouring: → (Introduced independently by Erdős, Rubin, Taylor in 1979 and Vizing 1976)
The list chromatic number (aka choice number or choosability) denoted $\chi_\ell(G)$ is the minimum k such that G has a L-coloring $\forall$ k-list-assignments L.

Proposition: $\chi(G) \leq \chi_\ell(G)$

Proposition: If $H \subseteq G$, $\chi_\ell(H) \leq \chi_\ell(G)$
(i.e. The list chromatic number remains monotone)

Def'n G is k-list-colorable if $\chi_\ell(G) \leq k$
(aka k-choosable)

Def'n G is critical for k-list-colouring if G is not k-list-col. but every proper subgraph is

L-critical w.r.t a list assignment L if G is not L-col, but $\forall$ proper subgraph is.

Q: How is list colouring different from colorings?

Theorem: $\chi_\ell(K_{d,d}) = \Theta(\log d)$
(But note: $\chi(K_{d,d}) = 2$)

Theorem (Alon 2000)
If G is a graph of min. degree d, then $\chi_\ell(G) = \Omega(\log d)$.

Conjecture: $\forall k$, if $\chi(G) \le k$, (and ~~min. degree~~ triangle free) then $\chi_\ell(G) = O(\log d)$

Correspondence Colouring (aka DP-colouring)

Def: Corr. Chromatic # (aka DP-chr. #), denoted $\chi_c(G)$ (aka $\chi_{DP}(G)$) is min k s.t. $\forall (L,M)$ k-corr.-assign., G has a (L,M)-coloring

critical for "

(L,M)-critical for "

Theorem (Bernshteyn, 2018)
If G is d-regular, then $\chi_c(G) = \Omega\left(\frac{d}{\log d}\right)$

Back to questions:

Types of questions:
- Chromatic # as related to other graph parameters (E.g. degree, clique #, etc.)
- Chromatic # of certain graph classes (E.g. Planar, surfaces, etc.)
- Algorithmic questions, e.g. Deciding if colouring exists, finding a colouring, can you sample a colouring uniformly at random
- How many colourings?
- Re-configuration: Can we get from one colour to another?

Relations to Other Parameters:-

Colouring Ones: $\max_{H \subseteq G} \frac{v(H)}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \chi_\ell(G) \le \chi_{DP}(G)$

Hall ratio

(Recent result of 2019): $\nexists f$ s.t. $\chi_\ell(G) \le f(\text{Hall ratio})$

Degree, Clique #, Girth:

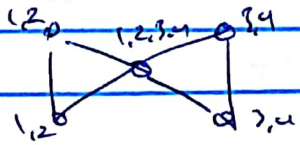
$\chi_{DP}(G) \le \underbrace{\Delta(G)}_{\text{max degree}} + 1$ (Greedy Bound)

Brooks's Thm (1941): $\chi(G) \le \Delta(G)$ unless G contains $K_{\Delta+1}$ or an odd cycle if $\Delta = 2$

(Other version: If connected, then $\chi \le \Delta$ unless G is iso to clique or odd cycle if $\Delta = 2$)

ERT, independently Vizing: $\chi_\ell(G) \le$ ——"——

ERT (Borodin) If L is a list assignment of connected G such that $|L(v)| \le d(v)$, then G is L-colorable unless every block of G is a clique or odd cycle



Girth:

Kim '95: If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\log \Delta}$

Johannsen '99: If G is triangle-free, then: $\chi(G) = O\left(\frac{\Delta}{\log \Delta}\right)$

Molloy '17: —— " —— then: $\chi(G) \leq (1+o(1))\left(\frac{\Delta}{\log \Delta}\right)$

If G is K_r-free for fixed r, then $\chi \leq O\left(\frac{\Delta}{\log \Delta}\log\log \Delta\right)$

Conj: $\log\log \Delta$ is not neccessary.

Question: Can we do better than $\frac{\Delta}{\log \Delta}$?

Erdős '59: $\exists$ graphs of arbitrary girth and chromatic #

Best known result in Ramsey Theory

$\Rightarrow \exists$ graphs of arbitrary girth and $\chi \geq \frac{1}{2}\frac{\Delta}{\log \Delta}$

↳ So, no, we can't!

Question: Is the answer 1 or 1/2 or in-between?

Reed's Conjecture (1998) $\chi(G) \leq \left\lceil \frac{\Delta+1+\omega}{2} \right\rceil$

$\Rightarrow$ True for $\chi_f(G)$ (Reed)

for if $\omega \geq .9999948\Delta$ (i.e. large ~~omega~~).

Thm (Delcourt, P.)

If Δ large enough, then $\chi \leq \left\lceil \frac{11}{12}(\Delta+1) + \frac{1}{12}\omega \right\rceil$.

Chromatic # of Graph Classes:

Four Color Theorem (Appel & Haken, 1977/79)
(Conjectured 1852)

If G is planar, $\chi(G) \leq 4$.

→ Later proof by Robertson, Sanders, Seymour, and Thomas (1994/6)
(Formally verified by proof systems in 2000s)

Grötzsch's Theorem (1959)
If G is planar, triangle-free, then $\chi(G) \leq 3$.

Surfaces:
Euler genus of a surface =
$2 \times$ # of handles + # of crosscap

Heawood's Bound: $\forall g \geq 1$, if G is a graph embedded in a surface of genus g, then

$$\chi(G) \leq \frac{7 + \sqrt{24g + 1}}{2}$$

Ringel-Youngs Thm (1960's)
Heawood's Bound is tight for every surface, except the Klein bottle, where $\chi \leq 6$.

Hadwiger's Conjecture:
(1943) If G has no K_t-minor, then $\chi(G) \le t-1$
→ Easy for $t \le 3$ and proved by Had for $t=4$

Wagner '37 - Showed $t=5$ is equiv to 4CT
Robertson, Seymour, Thomas '94 : $t=6$ equiv to 4CT

Open for all $t \ge 7$.

Weakenings:
Thm (Reed-Seymour, '99):
If G has no K_t-minor, then $\chi_f(G) \le 2t$

Thm (Edwards, Kang, Kim, Oum, Seymour, '15)
$\forall t, \exists d$ s.t. if G has no K_t-minor, then G is a d-defective, t-colorable

Thm (Dvorak - Norin, '18+)
— " —, G is $\exists c$-clustered, t-col.

Thm (Kostochka, Thomason '80s)
If G has no K_t-minor, then G is $O(t\sqrt{\log t})$-degenerate

Thm (Norin, Song, '19+)
$\forall \beta > 1/4$, G is $O(t(\log t)^{\beta})$-col.

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List version of Hadwiger's is false
i.e. $\exists K_t$-minor free G w/ $\chi_\ell(G) \geq 4/3\, t$

Strong perfect graph thm: (Chudnovsky, Robertson, Seymour, Thomas, '06)
G is perfect iff G has no induced C_{2k+1} or $\overline{C_{2k+1}}$ $\forall k \geq 2$

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Lecture 3:

The Probabilistic Method:

3 Pillars of prob. method:

- Linearity of Expectation (and basic probability)
- Lovász Local Lemma (introduced in 1975)
- Concentration Inequalities - (Markov, Chebyshev, Chernoff, Talagrand's) 1990s

Linearity of Expectation:

Expectation: Let X be a discrete $\mathbb{R}$ r.v., then: $\mathbb{E}[X] = \sum_{i \in \mathbb{R}} \mathbb{P}[X=i] \cdot i$

Linearity of Expectation: If $X = \sum_{i \in [n]} X_i$ (n finite), then $\mathbb{E}[X] = \sum_{i \in [n]} \mathbb{E}[X_i]$

Independent variables: 2 ~~random variables~~ events in a probability space A and B are independent if $Pr[A|B] = \mathbb{P}[A]$ or $Pr[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

Let $\mathcal{A} = \{A_1, \dots, A_\ell\}$ be ~~events~~ a collection of events in a probability space. Then:

- We say the events in $\mathcal{A}$ are pairwise independent if $\forall i \neq j$ in ~~$\mathcal{A}$~~ A_i, A_j are independent
- We say the events in $\mathcal{A}$ are mutually independent if $\forall i$, A_i is mutually independent of $\mathcal{A} \setminus \{A_i\}$, i.e. $\forall \mathcal{A}' \subseteq \mathcal{A} \setminus \{A_i\}$ $Pr[A_i \mid \bigcap_{A_j \in \mathcal{A}'} A_j] = Pr[A_i]$

Question: If ~~$\mathcal{A}$~~ $\mathcal{A}$ is pairwise independent then is it mutually independent?

A: No! Example: $n \geq 3$ coins and ~~event~~ space of flippings where # of heads is even and A_i is coin i is Heads. and $\mathcal{A} = \{A_1, \dots, A_n\}$.

Lovász Local Lemma: → often referred to as "bad events"

If $A = \{A_1, \dots, A_n\}$ is a set of events in a probability space, and for every i, there is $B_i \subseteq [n] \setminus \{i\}$ such that A_i is mutually independent of $A \setminus (\{A_i\} \cup \{A_j : j \in B_i\})$ *

i.e. Bad events are not too dependent

AND $|B_i| \leq d$, $\forall i \in [n]$

AND $Pr(A_i) \leq p$, $\forall i \in [n]$ i.e Bad events are unlikely

AND $ep(d+1) \leq 1$ (also $4pd \leq 1$ works too!)

Then,

$$Pr[\bigcap_{i \in [n]} \bar{A}_i] > 0.$$

(i.e. with positive probability, none of the bad events occur)

*Wrong Version: If A_i is independent of at most d ~~other~~ events.
(Example: 1 is mutually independent of any $n-2$ sets of coins but not $n-1$ ⇒ No unique maximal set).

Remarks:

- Use the local lemma, say, if union bound fails (Recall that the union bound is: If $\sum_{i \in [n]} Pr[A_i] < 1$, then $Pr[\bigcap \bar{A}_i] > 0$)
- The "positive probability" is not very large, i.e. Usually $O(2^{-|A|})$, so could be exponentially small in the # of events.
- We use it to construct/force a good outcome.

Algorithmic Q: Can we find such a good outcome efficiently?
{Sampling is bad idea since this may take exponentially long}. (b/c of low probability)

→

Moser-Tardos (2000): There exists an algorithm to find a good outcome of the LLL in "the variable model" that runs in time $O(|A|)$.

→ Independent Trials

Every event depends on a bounded # of variables w/ bounded # of states and $B_i = \{A_j \in A \setminus \{A_i\} : A_j$ and A_i depend on at least one common variable$\}$.

Moser-Tardos Algorithm:
- Sample all the independent trials in the probability space
- WHILE $\exists$ a bad event A_i:
    - Resample, at random, all the trials that A_i depends on
- RETURN good outcome

(Some) Applications of LLL to colourings:

Hypergraph Colouring:
Recall: Hypergraph $H=(V,E)$: V - Set of vertices, E - Set of hyperedges (i.e. ~~Sets~~ subsets of ~~Ed~~ vertices)

H is k-uniform if $|e|=k$ $\forall e \in E(H)$.

(Note: $k=2$ gives us simple graphs)

Def'n A k-colouring of a hypergraph H is a partition of $V(H)$ into at most k independent sets of H.

A set $I \subseteq V(H)$ is independent if $\nexists e \in E(H)$ s.t. $e \subseteq I$

or equivalently $\exists\, \phi : V(H) \to [k]$ such that $\forall e \in E(H)$, $\exists v_1 \neq v_2 \sim e$ such that $\phi(v_1) \neq \phi(v_2)$ (i.e. No monochromatic edges)

The chromatic number $\chi(H)$ is the min k s.t. H is k-colorable

Q: How is $\chi(H)$ related to $\Delta(H) := \max_{v \in V(H)} d_H(v)$, (where $d_H(v) := |\{e \in E(H) : e \sim v\}|$). ?

Trivial Bound: $\chi(H) \leq \Delta(H) + 1$ (Greedy!)

Theorem: If H is a k-uniform hypergraph ($k \geq 2$), then
$$\chi(H) \leq \lceil (e k \Delta(H))^{\frac{1}{k-1}} \rceil$$

Proof!

We will use LLL.

Assign every vertex x of H a colour $\phi(x)$ from $[L]$, where $L = \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$.

Define bad event:

A_e = Edge e is monochromatic in ϕ.

$\Pr(A_e) = \frac{1}{L^{k-1}}$ (because color of 1st vertex can be anything, but after that, $\forall v \sim e$ needs a different color).
$(=p)$

Let $B_e = \{f \neq e \in E(H) : f \sim e\}$, the by variable model, it follows that A_e is mut. ind. of $\mathcal{A} \setminus (A_e \cup B_e)$

$|B_e| \leq \underbrace{k}_{\text{at most } k \text{ vertices}} \underbrace{(\Delta(H) - 1)}_{\text{max degree.}} (= d.) \leq k\Delta(H) - 1$

So: $ep(d+1) = e \cdot \frac{1}{L^{k-1}} k\Delta(H) = \frac{ek\Delta(H)}{((ek\Delta(H))^{\frac{1}{k-1}})^{k-1}} = 1.$

So, by LLL, $\exists$ a ϕ avoiding all A_e, i.e. a k-coloring of H ☒

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Color Degree:

Def'n: Let G be a graph and L list assignment of G. Let $|L|$ denote the min. size of a list, i.e. $|L| := \min_{v \in V(G)} |L(v)|$

We define:

The color-degree of a vertex $v \in V(G)$ in color $c \in L(v)$ as:

$$d_L(v,c) := |\{u \in N(v) : c \in L(u)\}|$$

The color-degree of v, denoted $d_L(v) = \max_{c \in L(v)} d_L(v,c)$

The maximum color-degree of G w.r.t. L, is:

$$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$$

Question: Does $\exists$ function f s.t. if $|L| \geq f(\Delta_L(G))$, then G has an L-colouring? (analogous to $|L| \geq \Delta(G)+1$, then G has an L-colouring)

Theorem (Alon '88 (with constant $\geq .5$), '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring

Theorem (Haxell '00 - follows from a more general thm)

If $|L| \geq 2\Delta_L(G)$, then G has an L-colouring

Theorem (Reed-Sudakov, 2002)

If $|L| \geq (1+o(1))\Delta_L(G)$, then G has an L-colouring

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 Jan 18th

Lecture 4:

Review: (Colour Degree)

G graph, L list assignment

$|L| := \min_{v\in V(G)} |L(v)|$

$d_L(v,c) := |\{v \in N(v) : c \in L(v)\}|$

$d_L(v) := \max_{c\in L(v)} d_L(v,c)$

$\Delta_L(G) := \max_{v\in V(G)} d_L(v)$

Theorem (Alon '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring.

Proof:

(We may assume WLOG $|L(v)| = L \ \forall v$)

- Color every vertex $\phi(v)$ v, uniformly at ~~random~~ random from its list
- Bad events: $A_{e,c}$ $\underset{=uv}{}$: $\phi(u) = \phi(v) = c$ $\forall c \in L(u) \cap L(v)$

- Then $Pr[A_{e,c}] = \frac{1}{L^2} = p$.

- $A_{e,c}$ is mutually independent of $A_{e',c'}$ whose $e' \not\sim e$ (and $\neq$)

- $B_{e,c} = \{A_{e',c'} : e' \sim e$ or $e' = e$ and $c \in L(u') \cap L(v')\}$ ($e = uv$, $e' = u'v'$)

$= 2 \cdot |L| \cdot \Delta_L(G) = d$ ← Since we counted ourselves

↓ Pick an end (u or v) call this x

↓ Pick a color c' in $L(x)$

→ Pick a neighbor w of x where $c \in L(w)$.

Proof (cont'd)

Then, by LLL:

Since $ep(d+1) = e \frac{1}{|L|^2}(2|L| \cdot \Delta_L(G))$

$$= \frac{2e\Delta_L(G)}{|L|} \le 1.$$

Then $\exists$ a coloring ϕ avoiding all $A_{e,c}$, i.e. an L-coloring of G, as desired. ∎

(Note: We can actually get rid of the "+1" in the statement of the theorem)

Theorem (Haxell)

If $|L| \ge 2\Delta_L(G)$, then G has an L-coloring.

Actually follows from the following more general theorem:

Theorem (Haxell)

Let $k \ge 1$ be an integer. If $V_1, V_2, \ldots, V_r$ is a partition of $V(G)$ into independent sets for a graph G such that:

- $|V_i| \ge 2k$ $\forall i \in [r]$, and
- $\Delta(G) \le k$, ~~and~~

Then $\exists$ an independent set I of G s.t. $\forall i \in [r]$, $I \cap V_i \ne \emptyset$.

(Called an independent ~~set~~ transversal)

Go

How does it imply previous thm?
Let H be such that
$V(H) = \{(v,c) : v \in V(G),\ c \in L(v)\}$.
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c = c'\}$
Then H satisfies independent transversal theorem by assumption on G. So, by that theorem, $\exists$ an independent transversal, i.e. An L-coloring

Remark: Clearly also this implies correspondence version by letting
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c$ matched to c' in $M_{vv'}\}$.

~~Result~~ Remark: This is tight! (i.e. The ratio of 2). For general independent transversals.

Conjecture (Reed, ~90s)
If $|L| \geq \Delta_L(G) + 1$, then G has an L-coloring

But, Bohman and Holzman disproved this conjecture!
Still open if $|L| \geq \Delta_L(G) + 2$ works! (Has been open for about 20 years)

Theorem (Reed - Sudakov, '02)
If $|L| \geq (1+o(1))\Delta_L(G)$, then G has an L-coloring
(Note: Equivalent to:
$\forall \varepsilon > 0, \exists \Delta_\varepsilon$ s.t. $\forall \Delta \geq \Delta_\varepsilon$, if $\Delta(G) = \Delta$ and $|L| \geq (1+\varepsilon)\Delta$, then G has an L-coloring).

Remark:
~~Reed~~ Loh and Sudakov proved that the ratio for independent transversals is $(1+o(1))$ assuming that every vertex has at most $o(k)$ neighbors in any other partitions.

Proof (Reed-Sudakov)

Note: Uses Rödl ~~Nibble~~ (aka semirandom) method of iteratively constructing a solution a little bit at a time. (In particular, uses the "Wasteful Coloring Procedure" to implement one such iterative procedure)

Note #2: Uses LLL and concentration inequalities to prove that there is a decent enough outcome of the WCP as to continue iterating, until the Big Finish.

What is our finish? Alon's Thm (or Haxell's), i.e. can L-color if $|L| \geq 2e\Delta_L(G)$.

How will our iteration step be improving?: Progress will be in the ratio, $|L|/\Delta_L(G)$. In fact, in our proof, it will take some constant, depending on ε, # of steps.

Proof details:

Big assumption for now:

Let's assume $\forall c \in L(v)$ that $d_L(v,c) = d(v) = \Delta(G)$

(of course, we may assume wlog $|L(v)| = |L| \ \forall v$)

Wasteful Colouring Procedure:

- Independently "activate" each vertex of G for some probability p to be fixed later.

(Remark (if the calculations are correct):

$p = \frac{\varepsilon(1+\varepsilon)}{2}$

works and any smaller $p \geq \text{polylog}(\Delta)$).

- Now color every activated vertex v with a color $\phi(v)$ selected uniformly at random from $L(v)$

This is the "wasteful" step!

- Remove $\phi(v)$ from the list of all v's neighbours. Uncolour any vertex which has the same color as neighbour (or equiv. $\phi(v)$ is no longer in its list)

- Let ϕ' be the resulting colour, and L' be the resulting list

Remark: In the "naive colouring procedure" we only remove colours from neighbour's list if they keep the colour.

Here, we use WCP over NCP b/c
1) NCP is harder to analyze for concentrations, and
2) The probability that an active keeps its color will be close to 1, so in practice, much difference here

We let U be the set of vertices that do not receive a color from Φ'

let $G' = G[U]$

Update $L'(v) = L(v) \setminus \{\Phi'(u) : u \in N(v) \cap A\}$ ← where A is activated (wasteful procedures)

$\setminus U$ would be for naive.

We'll prove that there is an outcome with

$$\frac{|L'|}{\Delta_{L'}(G')} > 1+\varepsilon \quad \leftarrow \underline{\text{GOAL!}}$$

(In fact, $(1+\varepsilon)(1+\varepsilon/4)$.)

~~uuuuuuuuuuu~~

Expectations:

- Lists: Want to calculate $E[|L'(v)|]$ for a given v

$$E[|L'(v)|] = \sum_{c \in L(v)} \Pr[c \in L'(v)]$$

By linearity of expectation

$$= \sum_{c \in L(v)} \Pr[\nexists u \in N(v) \cap A \text{ with } \Phi(u) = c]$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$\prod_{u \in N(v)} \Pr[u \notin A \text{ or } \Phi(u) \neq c]$

See note on next pg. ← $u \in N(v)$

$= \prod_{u \in N(v)} (1 - \Pr[u \in A \text{ and } \Phi(u) = c])$

But activation is independent of colouring:

$$\prod_{u \in N(v)} 1 - \Pr[u \in A] \Pr[\phi(u) = c]$$

$$= \prod_{u \in N(v)} \left(1 - p \cdot \frac{1}{|L|}\right).$$

$u \in N_L(v,c)$

In fact, since $\phi(u) = c$ only happens if $c \in L(u)$, we can write $u \in N_L(v,c)$.

$$= \left(1 - \frac{p}{|L|}\right)^{|N_L(v,c)|}$$

$$= \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)} \longleftarrow \text{by color regularity}$$

$$\approx e^{-\frac{p\Delta_L(G)}{|L|}}$$

So, the sum is:

$$\mathbb{E}[|L'(v)|] = e^{-\frac{p\Delta_L(G)}{|L|}} \cdot |L|.$$

So, note that when $p \approx 1$, the we cut the list by around a 1/3 each time. But, if p small, then we retain a decent chunk.

Color-degrees:

$$\mathbb{E}[d_{L'}(v,c)] = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

(in G')

(Remark: Do this whether or not $c \in L'(v)$)

④

$$\mathbb{E}[d_{L'}(v,c)] = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

$$= \sum_u \Pr\left[\begin{matrix} (u \in A \text{ and } c \in L'(u)) \\ \text{OR } (u \in A \cap U \text{ and } c \in L'(u)) \end{matrix}\right]$$

$$= \sum_u \Pr[u \in A] \Pr[c \in L'(u)]$$

$$+ \Pr[\qquad]$$

$c = \varphi(v)$ $\qquad$ $c \neq \varphi(v)$

$\Pr[u \in A \cap U]$ $\qquad$ $\leq \Pr[u \in A \cap U]\, \Pr[c \in L'(u)]$

Let's pretend that $c = \varphi(v)$ does not matter:

$$\leq \sum_u \left(u - p(1 - \frac{p}{|L|}\right)^{\Delta_L(G)} \qquad \Pr[u \in A] \Pr[u \in U]$$

$$+ \Pr[u \in A \cap U] \cdot \frac{1}{|L|}$$

$$+ \left(p(1 - \underbrace{\Pr[\varphi(v) \in L'(v)]}_{\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}}) \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right)$$

$= \Delta_L(G)$ times

Asymptotically the dominant constant is 1

Lecture 5:

Recall:
We were proving Reed-Sudakov, and had the following expected values:

$$\text{Keep} := \left(1-\frac{p}{2}\right)^{\Delta_L(G)} \approx e^{-\frac{p\Delta}{L}} \quad (= \Pr(c \in L'(v) \mid c \in L(v)))$$

$$\mathbb{E}[|L'(v)|] = \text{keep} \cdot |L|$$

$$\mathbb{E}[d_{L',G'}(v,c)] = d_{L,G}(v,c)\Big((1-p)\cdot \text{Keep} + p\Big(\text{keep}\cdot\frac{1}{|L|} + \underbrace{\Big(1-\frac{1}{|L|}\Big)\Big(1-\frac{2}{|L|}\Big)^{\Delta}}\Big)\Big)$$

not activated; keeps c; if $\phi(v)=c$ and keep c; $\phi(v)\neq c$ and keep c.; keeping $\phi(v)$ and c

But it turns out that this variable doesn't concentrate well, so we'll use another.

Picking a better variable:

$$d_{L',G'}(v,c) = |N_{L',G'}(v,c)| = |\{u \in \underbrace{N_{L,G}(v,c) \cap V(G')}_{G^2 \to G'} : \underbrace{c \in L'(u)}_{L^2 \to L'}\}|$$

But, it turns out we don't need to care about the lists, so instead let's look at $|N_{L,G}(v,c) \cap V(G')|$.

Clearly

$$d_{L',G'}(v,c) \leq |N_{L,G}(v,c) \cap V(G')|.$$

$$\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]$$

$$= \sum_{u \in N_{L,G}(v,c)} \Pr[u \in V(G')] \quad \text{(lin. of exp.)}$$

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \notin V(G')] \quad \text{These events independent.}$$

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \in A \text{ and } \phi(u) \text{ is kept}] = d_{L,G}(v,c)(1 - p\cdot\text{Keep})$$

On to some iterative calculations:

- Now, if we could show that with "high enough" probability every $|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$ are close enough to their expectations to apply the LLL, then we'll be happy via the following calculations:

$$\frac{|L'|}{\Delta_{L'}(G)} \gtrsim \frac{\mathbb{E}[|L'(v)|]}{\mathbb{E}[|N_{L',G}(v,c) \cap V(G')|]} = \frac{\text{Keep} \cdot |L|}{\underbrace{(1-p\,\text{Keep})}_{\text{Same factor}} \cdot \underbrace{\Delta_L(G)}_{\text{Old ratio}}}$$

~~ratio~~ factor → we will argue this is close to 1.

$$\geq \frac{|L|}{\Delta_L(G)} \cdot \frac{1-\frac{p\Delta}{|L|}}{1-p\left(1-\frac{p\Delta}{|L|}\right)}$$

{ using that $\left(1+\frac{x}{n}\right)^n \geq 1+x$ for $n \geq 1$ and $|x| \leq n$.

Note: $\frac{\Delta}{|L|} = \frac{1}{1+\varepsilon}$ (Same assumption we made)

$$= \frac{|L|}{\Delta_L(G)} \cdot \underbrace{\frac{1-p/1+\varepsilon}{1-p(1-p/(1+\varepsilon))}}_{= 1 + \frac{p\varepsilon}{1+\varepsilon} - \frac{\varepsilon^2}{1+\varepsilon}}.$$

Now if we choose $p = \varepsilon/2$, then we get

$$= 1 + \frac{\varepsilon^2}{4(1+\varepsilon)}$$

{ So, we get some improvement in each iteration.

How to show a variable is close to its expectation:

Concentration Inequalities:

Markov's Inequality:
If $X \geq 0$ is a random variable,
$$Pr[X \geq k E[X]] \leq 1/k. \quad \forall k \geq 1$$

Chebyshev's Inequality:
The variance of a random variable X, denoted $Var[X]$ is
$$Var[X] = E[(X - E[X])^2]$$
The standard deviation σ of X is $\sqrt{Var[X]}$.

~~Theorem~~, Let $\mu = E[X]$, then
$$Pr[X \geq \mu + k\sigma] \leq 1/k^2.$$
Proof: Use Markov's inequality.

Chernoff Bounds:
Let $X = \sum_{i=1}^{n} X_i$ where X_i is a Bernoulli r.v. (i.e. only takes value of 0 and 1). and all independent:
$$Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}} \quad (0 \leq \delta \leq 1)$$
$$Pr[X \geq (1+\delta)\mu] \leq \begin{cases} e^{-\frac{\delta^2 \mu}{3}} & \text{if } 0 \leq \delta \leq 1 \\ e^{-\frac{\delta \mu}{3}} & \text{if } \delta \geq 1 \end{cases}$$
(i.e. Exponentially small if δ constant and gives something meaningful if $\delta \gg \frac{1}{\sqrt{\mu}}$!.

2

Example:

$\mathbb{E}[|A \cap N_{L,G}(v,c)|] = p\Delta_L(G)$

$\Pr[|A \cap N_{L,G}(v,c)| \geq p\Delta_L(G)(1+\delta)] \leq e^{-\frac{\delta^2(p\Delta_L(G))}{3}}$ (By Chernoff).

i.e. $|A \cap N_{L,G}(v,c)| = p\Delta_L(G) \pm \sqrt{\Delta_L(G)\,\mathrm{polylog}\,\Delta_L(G)}$

with prob. $1 - \frac{1}{\Delta^c}$ for any c.

So, by LLL, this holds for all v with some positive prob.

But, we need to concentrate:

$|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$

Depends on color $c \in L(v)$ being in $L'(v)$ which depends on activation flips and color-assignments for $N_{L,G}(v,c)$ (But not independent!)

Even worse! (Stronger interactions with neighbors + 2nd neighbors)

"Simple" Concentration Bound (Book of Molloy & Reed).

Let X be a r.v. that depends only on the outcome of a set of independent trials $T_1, \ldots, T_n$. Suppose that changing the outcome of any one trial changes X by at most c (c constant) (This is called c-Lipschitz).

Then,

$$\Pr[|X - \mathbb{E}[X]| \geq t + 8c\sqrt{\mathbb{E}[X]}] \leq e^{\frac{-t^2}{3c^2 n}}$$

Remark: Note that the denominator has the # of trials. (not exponential)

Concentrating $|L'(v)|$:

Trials: activating flips & colourings of $N_{L,G}(v,c)$ $\forall c \in L(v)$

- 1-lipschitz: Since changing any one trial (activ. or colouring) changes $|L'(v)|$ by at most 1.

Recall: $\mathbb{E}[|L'(v)|] = K_{exp} \cdot \Delta_L(G) = \Theta(\Delta_L(G))$

What is r?

- If not doing colour degree, then 2Δ (Good!)
- But, w/ colour degree it's:

$2\Delta_L(G) \cdot |L|$ (If every color the neighbours holds are disjoint).

$= 2(1+\varepsilon)\Delta_L(G)^2$

(This is Bad! The simple concentration bound is only meaningful for $t > \frac{\Delta_L(G)}{\omega}$)

Talagrand's Inequality (version)

(A) Combinatorial version: (There is also a probability one)

Let $X \geq 0$ depend on independent trials $T_1, \ldots, T_n$. If X is c-lipschitz and r-verifiable, then for any $t > 96\sqrt{rc^2 \mathbb{E}[X]} + 128rc^2$

then:

$$\Pr[|X - \mathbb{E}[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}}$$

Remark: If r, c constant, then get ~~[illegible]~~ exponentially small in t, if $t = \Theta(\mathbb{E}[X])$ and still meaningful for $t \gg \sqrt{\mathbb{E}[X]}$.

r-verifiable: For every $s>0$, if $X \geq s$, then there exists a set Z of at most rs trials that "verify" that $X \geq s$, i.e. Changing any trial outside of Z still results in $X \geq s$.

So, of course, counting # of heads in coin flips is 1-verifiable.
Similarly, $|A \cap N_{i,G}(\vec{v}, c)|$ is 1-verifiable.
(We just exhibit the heads or the activated set).

What about $|L'(\omega)|$?
Example: How do we show $|L'(\omega)| \geq 1$ (i.e. One color kept).
For all $u \in N_{i,G}(v,c)$ show either $u \notin A$ or $\Phi(u) \neq C$.
i.e. Need $\Delta_{i,G}(v,c)$ trials to verify
i.e. Need $r \geq \Delta_{i,G}(v,c)$ (This is bad.)

Pick a better variable:

$|L(\omega)| - |L'(\omega)| = $ # of colors lost.

$\mathbb{E}[|L(\omega)| - |L'(\omega)|] = |L| - \text{keep}|L|$

$= |L|(1-\text{keep})$

$= (1+\varepsilon)\Delta_i(G)(1-\text{keep}) = \Theta(\Delta_i(G))$

Obviously, this variable is 1-Lipschitz.
What do we need to verify a color is lost?
→ Need a neighbor $v \in N_{i,G}(\omega, c)$ to be activated and $\Phi(\omega) = c$
i.e. Need 2 trials
More generally, if $|L(\omega)| - |L'(\omega)| \geq s$, need $2s$, so $r=2$ works.

749_Coloring Jan 23rd.

Lecture 6:

Concentrating One last variable for Reed-Sudakov:

$|N_{L,G}(v,c) \cap V(G')|$

$\mathbb{E}[\quad] = \Delta_L(G)\cdot((1-p\cdot keep)$

Can we concentrate this?

Is it r-verifiable for some r?

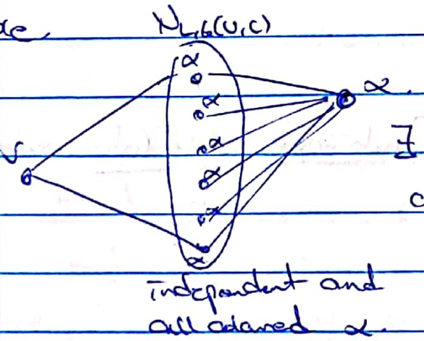
$\to$ Note: $u \in N_{L,G}(v,c) \cap V(G')$ if either $u \notin A$ or $u \in A$ but $\exists$ $w \in N_{L,G}(u,\phi(w)) \cap A$. s.t. $\phi(u)=\phi(v)$

So, this is 4-verifiable! (at most 2 coin flips and 2 colors)

But, is it c-Lipschitz for some c?

Not really: c could need to be at least $\Delta_L(G)$ as follows

For example



$\exists$ exactly one neighbor (and its common to all) of these colored α.

If α changes to β, then all the colors change.

Idea: However, it's unlikely for this to happen.

i.e. Unlikely that a vertex can change the outcome by $\Omega(\deg G)$

(we'll prove this later)

So, if we could somehow use Talagrand's with this "likely" Lipschitz constant, we'd find.

$$\Pr(|\mathbb{E}[X]-X| > t+ 61rc^2+8\sqrt{rc^2\mathbb{E}[X]}) \leq e^{-\frac{t^2}{8rc^2(t+\mathbb{E}[X])}} \Big\} \text{ sth like this.}$$

$\mathbb{E}[X] \approx \Theta(\Delta)$ works for $t = \sqrt{\Delta}\,\mathrm{polylog}\,\Delta$

1

So we'd get that Prob. $|L'(v)|$ or $|N_{v,G}(v,c) \cap V(G')|$ too far from $\mathbb{E}[X]$ is $\le \frac{1}{\Delta^c}$ for any c.

Argue each event is at most independent of all but a set of at most $(\Delta \cdot L)^2 \le \Delta^5$ events and apply LLL to argue w/ pos. prob. that none happen

Then iterate and finish w/ Alon/Haxell when $L/\Delta \ge 2c$ or 2

□

Remark: This will only work for large enough Δ, b/c to apply the local lemma, we'd need $e^{-\log^2 \Delta} \le \frac{1}{\Delta^{10}}$. (and then extrapolate back through iterations, how large original Δ needs to be)

An Exceptional Outcome Version of Combinatorial Talagrand's Inequality.

Theorem:
Let $((\Omega_i, \Sigma_i, P_i))_{i=1}^n$ be probability spaces. Let (Ω, Σ, P) be their product space and $\Omega^* \subseteq \Omega$ be a set of "exceptional outcomes", and let $X : \Omega \to \mathbb{R}_{\ge 0}$ be a nonnegative random variable. Let $r, c \ge 0$. If X is (r,c)-certifiable w.r.t. Ω^*, then for any $t > 96c\sqrt{r\mathbb{E}[X]} + 128rc^2 + 8P[\Omega^*]\sup X$, then

$$\Pr[|X - \mathbb{E}[X]| > t] \le 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}} + 4P[\Omega^*]$$

Remark: We need to pay some costs:

- Needing $t \geq 8P[\Omega^*]\sup X$, and
- An extra $4P[\Omega^*]$ in the prob. bound.

Def'n ~~(r,c)-verifiable~~ ((r,c)-certifiable)

- If $\omega = (\omega_1, \dots, \omega_n) \in \Omega$ and $s > 0$, an (r,c)-certificate (w.r.t. X, ω, s, Ω^*) is an index set $I \subseteq \{1, \dots, n\}$ of size at most rs s.t. $\forall k \geq 0$, we have $X(\omega') \geq s - kc$ $\forall \omega' = (\omega_1', \dots, \omega_n') \in \Omega \setminus \Omega^*$ s.t. $\omega_i \neq \omega_i'$ for at most k values of I
- If $\forall s > 0$ and $\omega \in \Omega \setminus \Omega^*$ s.t. $X(\omega) \geq s$, $\exists$ an (r,c)-certificate, then (r,c)-certifiable w.r.t. Ω^*

Remark:

- For $k=0$, the certificate just acts as an r-verifier for non-exceptional outcomes as in normal combinatorial Talagrand's
- What this really requires is an r-verifier for which, if you change at most k of its trials (and any # of the verifier trials), you lose at most kc.
  - ⇒ This is kind of like changing any trial in a non-exceptional outcome changes at most c, but requires more generally changing k trials changes by at most kc for every $k \geq 0$ (Bonus: Can assume you end in a non-exceptional outcome)

Back to ~~concentr~~ concentrating $|N_{L,G}(v,c) \cap V(G')|$.

Ω^*: $\exists\, w \in N^{\le 2}(v)$ (Note: At most $(\Delta L)^2$ of these if we delete edges b/w vertices w/ disjoint lists)

($\overset{\text{up to 2nd neighborhood}}{\le 2}$)

s.t. $\exists\, c' \in L(w)$ and $\ge$ polylog Δ vertices in $N_{L,G}(w,c')$ with colour c'

(actually can use $N_{L,G}(w,c) \cap N_{L,G}(v,c)$ instead)

Claim: $\Pr[\Omega^*] \le \frac{1}{\Delta^c}$ for any c and Δ large enough

(this will be enough to apply Talagrand's since $E_p|N_{L,G}(v,c) \cap V(G')| \le \delta$) ~~and only~~

This also assumes that we show it's (r,c)-certifiable for some r, c w.r.t. Ω^*.

Proof (that $|N_{L,G}(v,c) \cap V(G')|$ is (r,c)-certifiable.

So, let ω be a non-exceptional outcome. We need $\forall s>0$ a set of at most rs trials to build certificate.

So, we use the ≤ 4 activation flip/colour assignments of vertices (and $\le$ neighbors) in $N_{L,G}(v,c) \cap V(G')$ (so, what we used before)

Need: $\forall k \ge 0$ changes to these trials, as well as any outside, that $|N_{L,G}(v,c) \cap V(G')| \le s - kc$

Can argue $k =$ polylog Δ works here b/c we start non-exceptional. (b/c if value changes, had to change b/c of some critical change)

How to prove claim:

Note that $\Pr[\mathcal{E}^*] \underset{\text{by union bound}}{\le} \underbrace{|N^{\le 2}(v)|}_{\le \Delta^4} \Pr\left(\begin{array}{l} \exists c' \in L(w) \\ \text{and } 2\,\text{polylog}\Delta \text{ in } N_{L_c}(w, c') \\ \text{colored } c' \end{array} \,\middle|\, w \in N^{\le 2}(v) \right)$

Suffices to show $\le \frac{1}{\Delta^c}$ for any c (and large enough Δ).

Balls and Bins:

m balls and n bins. and uniformly at random assign each ball to some bin independently.

Expected # of balls in bin i: m/n.

Harder Q: What's the max # of balls in a bin?

If $m = n$ with high probability the max prob is $O\left(\frac{\log n}{\log\log n}\right)$

→ Relating this back to coloring:

Vertices $\equiv$ Balls and Colors $\equiv$ Bins

in neighborhood $(N_{L_c}(w, c'))$ (in $L(w)$)

(Note: If ball can't go to same bin, the probabilities are right, we can take this and can only get a worse bound)

We have $\Delta_L(G)$ Balls and $|L|$ Bins so $\Delta \approx \Theta(|L|)$ and so no bin has $\ge$ polylog Δ balls with high probability

(5)

Balls and Bins - Bounds:

Can $= \Delta$, $n = \lceil L \rceil$

Upper bounds: By Union bound, $\Pr[\exists \text{ bin w/} \geq k \text{ balls}]$
$$\leq \Delta \cdot \Pr[\text{bin } i \text{ has} \geq k \text{ ball}]$$

$$\Pr[\text{Bin } i \text{ has} \geq k \text{ balls}] \leq \binom{\Delta}{k} \cdot \frac{1}{L^k}$$

$$\approx \left(\frac{\Delta e}{k}\right)^k \cdot \frac{1}{L^k} = \left(\frac{\Delta e}{kL}\right)^k$$

So, we would need: $k \log k > c \log \Delta$, i.e. $k > \frac{\log \Delta}{\log\log \Delta}$

~~constants~~.

to find $\leq \frac{1}{\Delta^c}$ for any c.

Lower Bound:

$$\Pr[\text{Bin } i = k] = \binom{\Delta}{k} \frac{1}{L^k} \left(1 - \frac{1}{L}\right)^{\Delta - k}$$

$$E\left[\begin{matrix}\# \text{ of bins w/} \\ \text{exactly } k\end{matrix}\right] = \Delta \cdot \Pr[\text{Bin } i = k]$$

$$\approx \Delta \cdot \left(\frac{\Delta e}{kL}\right)^k \cdot e^{-\frac{\Delta k}{L}}$$

By fiddling w/ constant for k

Can get $\geq \frac{1}{\Delta}$, and so $E \geq 1$,

(Use Talagrand's w/ $c = 2$, $r = k$)

Same for upper bound (Can use $r = 1$, $c = 1$).

Lecture 7:

Back to our assumption that all color degrees are the same:
For Reed-Sudakov:

- Keeping a color is more likely if smaller color degree
- So $E[|L'(v)|] \geq$ Expectation when regular
- $E[d_{L',G'}(v,c)] \leq d_{L,G}(v,c)(1-p\,keep) \leq \_\_\_\_ (1-p\,keep)$

So keep changes with the list. $\rightarrow$ ~~color~~ ~~assigned~~ $keep(u,\phi(uw)) = (1-\frac{p}{|L|})^{d_{L,G}(u,\phi(w))}$; min. keep, i.e. old #. (but we can always extend every list)

Even if it wasn't the case that expectations were only better for us in the non-regular case:

(1) Regularization: Embed our graph into a regularized version where ~~the~~ coloring the resulting graph yields a coloring of the original.

(2) Equalizing Coin Flips: Here, for every $u \in V(G)$ and $c \in L(u)$, we add a coin flip $F_{u,c}$ which keeps c for u with probability $keep/keep(u,c)$ and hence every color is kept with probability ~~keep/F_{u,c}~~ $keep(u,c) \cdot F_{u,c} = keep$.

Note: (2) only works if ordered ~~coin~~ coin flips have probability ≤ 1 ~~AND~~ we would have to redo all the concentrations, adding the coin flips into verifications / Lipschitz.

Regularization:

Lemma: Every graph G is an induced subgraph of a $\Delta(G)$-regular graph.

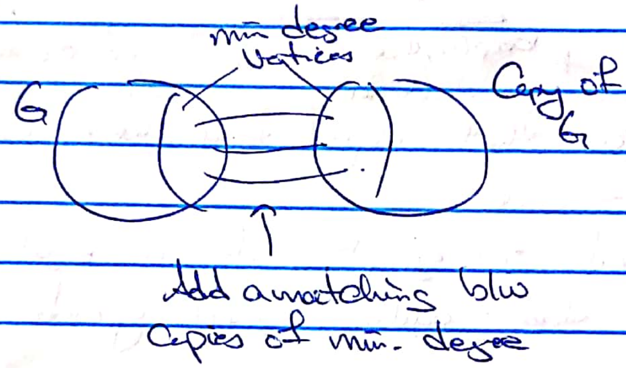
$\hookrightarrow$ Proof Sketch

Proof Sketch:

By induction on $\Delta(G) - \delta(G)$ (min degree) (max degree)

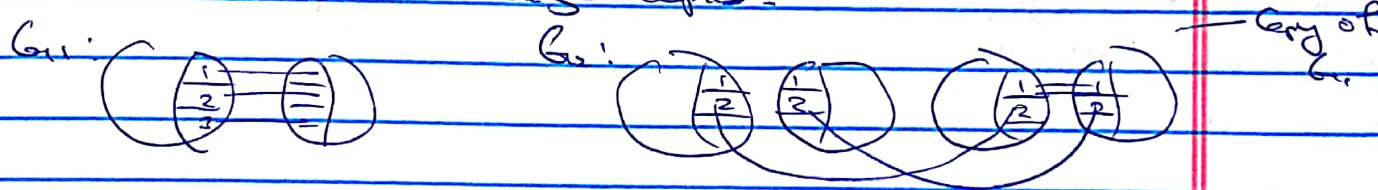
If $\Delta(G) = \delta(G)$, then G is $\Delta(G)$-regular, as desired.
So, we may assume $\delta(G) < \Delta(G)$.
Define:

$G' =$



Now $\delta(G') > \delta(G)$ and hence by IH, $\exists$ G ~~with~~ $\Delta(G)$-regular containing G' as an induced subgraph. $\square$

Note: Construction actually gives if G is triangle-free, then G' is triangle-free.
However, if G has girth ≥ 5, G' may have 4-cycles.
We may tweak the construction to preserve girth 5 (and girth 6, but not ≥ 7)
First, find a $(\Delta+1)$-coloring of G. Only add a matching b/w color i of min-deg. copies:



So, do $\Delta+1$ doublings to up min. deg. by 1. Start over...

For Reed-Sudakov, we need to regularize color-degree, not degree:

(1) If we pass to correspondence coloring, we can use the construction above, but we only add a conflict b/w min-deg. colors b/w copies of vertices

(2) If we're more careful with making lists for the copies, this should work for list-coloring as well.

Kim's Theorem for Girth Five Graphs:

Theorem (Kim, 1995)

If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\ln \Delta}$.

Recall: This is tight up to constant factor, in particular, in random d-regular graphs have $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$ with high probability, and so $\exists$ d-regular graphs of arbitrary girth and $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$.

(Some naive) Intuition:

- Coupon Collector Problem: Suppose there are n types of coupons and when you receive a coupon you receive a type u.a.r.

Q: How many coupons do you need to get to collect all the types? $\Theta(n\log n)$

Proof Sketch:

In the first n coupons,

$\Pr[\text{obtained} \geq 1 \text{ of coupon } i]$

$= 1 - \Pr[0 \text{ of coupon } i]$

$= 1 - (1 - \frac{1}{n})^n$

$\approx 1/e.$

$\mathbb{E}[\text{types collected after } n] \approx n(1 - \frac{1}{e})$

$\mathbb{E}[\text{still to collect}] \approx n/e$

$\mathbb{E}[\text{types still to collect after } n \cdot t \text{ coupons}] = n/e^t.$

Use concentration inequalities to make these concentration around expectation w.h.p.

$\square$

So, we could think that:

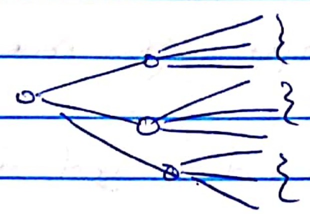
types of coupons = colors in my list $(n) = L$

coupon samples = colors neighbors receive $(n \ln n) = \Delta.$

(Solve to find $L \approx \frac{\Delta}{\ln \Delta}$).

But, coloring (randomly) is not uniformly random,

→ there's where the girth 5-ness can help. since for every vertex we see:



All disjoint.

This is "somewhat uniformly random"

Proof (of Kim's)
We actually prove a stronger theorem (as Kim did):
If G has girth ≥ 5 and L is a list assignment of G such that $|L| \geq (1+o(1))\frac{\Delta_L(G)}{\ln \Delta_L(G)}$, then G

has an L-colouring.
i.e. He proved this theorem for colour-degree.

We'll use Nibble and the Wasteful Colouring Procedure.
We may assume all colour degrees and list sizes are the same by Regularization / Equalizing Coin Flips.

We will be interested in tracking the ratio: $\frac{\Delta_L(G)}{|L|}$.
(Reciprocal of what we used for Reed-Sudakov)
(Note: This starts as $\approx \ln \Delta_L(G)$).
We will show that we can (after many steps) reduce to the graph with ratio $\leq \frac{1}{20}$ (so $|L| \geq 20\Delta_L(G)$).
and then we apply Alon/Haxell to finish.

Expectations for one step of WCP:
$\Pr[c \in L'(v) \mid c \in L(v)] = \left(1-\frac{p}{|L|}\right)^{\Delta(G)} =: \text{keep} \approx e^{-\frac{p\Delta}{|L|}}$.

$\mathbb{E}[|L'(v)|] = |L| \cdot \text{keep}$.

$\mathbb{E}[d_{L',G'}(v,c)] \leq d_{L,G}(v,c) \cdot \left(\text{keep}\cdot\left(\overset{\text{if } c=\phi(u), \text{ where } u\in N_{L,G}(v,c)}{\frac{1}{|L|}(1-p)} + \left(1-\frac{1}{|L|}\right)\overset{\text{keeping } \phi(u)\neq c}{(1-p\,\text{keep})}\right)\right)$

keeping c

$\leq d_{L,G}(v,c)\,\text{keep}\,(1-p\,\text{keep})$ $\quad = (1-p\,\text{keep}) + \underbrace{\frac{1}{|L|}p(\text{keep}-1)}_{\leq 0}$

$= \text{keep}\,\Delta_L(G)$.

Remark:

In Reed-Sudakov, we threw out the keep in $\mathbb{E}[d_{L',G'}(v,c)]$ for analysis purposes to get the L/Δ ratio improving by keep/1-keep which was >1, since $|L| \geq \Delta$.
But, for Kim's, we'll need to keep the keep in $\mathbb{E}[d_{L',G'}(v,c)]$ which complicates concentration analysis, but everything will still concentrate b/c girth ≥ 5.

Concentrations:

- Show $\Pr[\,|L'(v)| - \underbrace{\mathbb{E}[|L'(v)|]}_{=\text{keep}\cdot|L|}\,] \leq \frac{1}{\Delta^{10}}$ or $|L| - |L'(v)|$

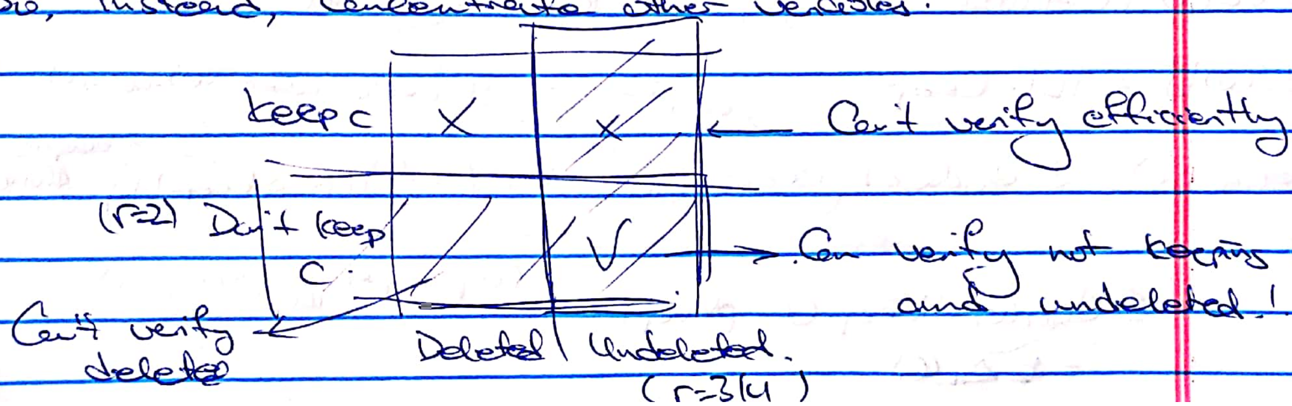
  Proof same as for Reed-Sudakov, i.e. Use Talagrand's ^ with $c=1$, $r=2$ (really 3 w/ activation flips)

- Show $\Pr[\,d_{L',G'}(v,c) - \mathbb{E}[d_{L',G'}(v,c)]\,] \leq \frac{1}{\Delta^{10}}$.

  Here, we can use Talagrand's ^ (Exc. Tal.) with $c=1$ (except for v itself, use exceptional Talagrand for v or other trick), and $r=?$ (Is it possible to verify # of vertices undeleted and keeping c in dist?)

  ↳ No, can't efficiently verify.

So, instead, concentrate other variables:



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Can verify all boxes indirectly here assuming all expectation of size of boxes is roughly same. ~~...~~ (i.e. the intersection shouldn't be small compared to the size of the set)

Iterating the ratio:
Apply LLL to find:
$|L'| \approx |L| \cdot \text{keep}$
$\Delta' \approx \Delta \cdot \text{keep}(1 - p\,\text{keep})$
So:
$$\frac{\Delta'}{|L'|} \approx \frac{\Delta \cdot \text{keep}(1-p\,\text{keep})}{|L| \cdot \text{keep}}$$
$$= \frac{\Delta}{|L|} - \frac{p\Delta}{|L|}\text{keep} \quad \left(\text{Recall: keep} \approx e^{-\frac{p\Delta}{|L|}}.\right)$$

So, let $k := \frac{p\Delta}{|L|}$, we get:
$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} - ke^{-k}$$, so for $k=1$, this gives $-1/e$.

Now run this forever to get the ratio ??
Can't do this since we'd run out of colors.
(Since $|L'| \approx |L| \cdot \text{keep}$ and $\text{keep} \approx \frac{1}{e}$, so we'd have to stop at about ~~...~~ $\ln \Delta$ steps)

Lecture 8:

Finishing the iteration calculations for Kim's

$|L'| = |L| \cdot keep$, where $keep \approx e^{-\frac{p\Delta}{|L|}}$

$\Delta' \approx \Delta \cdot keep(1-p\,keep)$

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} \cdot \frac{keep(1-p\,keep)}{keep} \qquad k = p\frac{\Delta}{|L|}.$$

$$= \frac{\Delta}{|L|} - \frac{\Delta p}{|L|} keep = \frac{\Delta}{|L|} - ke^{-k}$$

How many iterations can we do (until we run out of colours?)

Let $|L_T|$ be the size of L after T iterations:

$$|L_T| = |L_0| \cdot keep^T$$
$$\approx |L_0| \cdot e^{-kT}$$

We need $|L_T| \geq 1. \Rightarrow \ln|L_0| - kT \geq \ln 1 = 0$

$$\Rightarrow T \leq \frac{\ln|L_0|}{k}$$

Remark.

We actually stop when $|L_T| \geq$ some fixed constant to ensure that the necessary inequalities hold for the LLL in every iterative step. This only means that $T \leq \frac{\ln|L_0|}{k} - c$ for some constant c (so this is fine)

To finish the colouring, we need $\frac{\Delta_T}{|L_T|} \leq \frac{1}{2e}$ to apply Alon/Haxell.

Note: $\frac{\Delta_0}{|L_0|} \approx \frac{\Delta_T}{|L_T|} + ke^{-k}T$. So this works if

$$\frac{\Delta_0}{|L_0|} \lessapprox \frac{1}{2e} + ke^{-k}\left(\frac{\ln|L_0|}{k} - c\right)$$

$$= \text{constant} + e^{-k}\ln(L_0)$$

$\Leftrightarrow$

1

i.e. $\Delta_0 \le e^{-k}|L_0|\ln|L_0|$

$\Rightarrow |L_0| \approx \frac{e^k\Delta_0}{\ln(e^k\Delta_0)} \le \frac{e^k\Delta_0}{\ln\Delta_0}$ Recall that $k = p\frac{\Delta}{|L|}$, so when $k=1$, $p=\frac{|L|}{\Delta}$ and $k \to 0$ means $p \to 0$.

So, if $k=1$, we get $|L_0| \approx e\frac{\Delta_0}{\ln\Delta_0}$,

and if instead, we let $k \to 0$, we get $(1+o(1))\frac{\Delta_0}{\ln\Delta_0}$, as desired $\square$

Remark: Why were colour degree necessary for the proof?
Note that in $\Delta' \approx \Delta\,\text{keep}(1-p\,\text{keep})$, the first keep only comes up in colour degree. And this keep helps cancel out some values and drive concentrations down.

Edge-Colouring:

Interested in properly colouring edges of graphs, i.e. ~~$\chi(G)$~~ $\chi(L(G))$ (the line graph)

Greedy Colouring: $\Delta(L(G)) \le 2\Delta(G)-2$

$\Rightarrow \chi(L(G)) \le 2\Delta(G)-1$

(of course $\omega(L(G)) = \Delta(G)$, if $\Delta(G) \ge 3$)

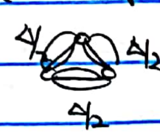
Theorem (Vizing, indep. Gupta, 1960s)

$\chi(L(G)) \le \Delta(G)+1$.

NP-hard to decide if $\chi(L(G)) = \Delta$ or $\Delta+1$

Also, if G is a multigraph

$\chi(L(G)) \le \Delta + \mu$ (where μ is the max. multiplicity of an edge)

$\le \lceil \frac{3}{2}\Delta \rceil$ (Shannon) $\rightarrow$ tight for 

Goldberg - Seymour Conjecture: (Independently in late 70's)

$\chi(L(G)) \leq \max\{\Delta(G)+1, \chi_f(L(G))\}$

fractional chromatic #.

Actually, via Edmond's matching polytope:

$$\chi_f(L(G)) = \max_{\substack{H \subseteq G \\ v(H) \text{ odd}}} \frac{e(H)}{\left(\frac{v(H)-1}{2}\right)} \Rightarrow \text{ie. Defn of Hall ratio for line graphs.}$$

Known Results:

- Kahn (90's) proved $\chi(L(G)) \leq$ ~~# tends~~ $\max\{\Delta+1, (1+o(1))\chi_f(L(G))\}$.
- Two groups in 2008: If $\chi_f(L(G)) \geq \Delta + \sqrt{\Delta}$, then $\chi(L(G)) = \chi_f(L(G))$.
- Another group in 2017+: —— " —— $\geq \Delta + \sqrt[3]{\Delta}$, —— " ——.
- Plantholdt, '11: $\chi(L(G)) \leq \chi_f(L(G)) + \log \chi_f(L(G))$.

List-Colouring Conjecture:

If G is a simple graph, then

$$\chi_\ell(L(G)) = \chi(L(G)).$$

(i.e. There is no such thing as list-edge-colouring).

Two big results on LCC from 90's:

Theorem (Galvin, 1995)

If G is bipartite, then $\chi_\ell(L(G)) = \chi(L(G))$.

→ Proved Dinitz's conjecture: If for every square in an $n \times n$ matrix, I give you a list of integers, can you complete the matrix so that all entries in a row (or col) are distinct

(Equivalently, $\chi_\ell(K_{n,n}) = n$).

→ Related to Latin Squares, since the completion of a Latin Square is simply an n-edge-colouring of $K_{n,n}$.

②

And these exist by König's theorem, if G is bip, $\chi_\ell(L(G)) = \Delta(G)$.

Theorem (Kahn, 1996)

If G is simple, $\chi_\ell(L(G)) = (1+o(1))\chi(L(G))$

i.e. $= (1+o(1))\Delta(G)$.

Molloy + Reid (early 2000s) improved error to $\Delta + 4\sqrt{\Delta}\log^4 \Delta$.

Kahn extended this to k-uniform linear hypergraphs

↳ (Hyperedges intersect in ≤1 vertex)

Proof Sketch (of Kahn's)

We'll use the Naive Colouring Procedure (i.e. Only remove colours from neighbours if retained by a vertex) and Nibble.

Also, since $|L|$ is on the order of Δ, we won't need activation probabilities (i.e. Set $p=1$)

We in fact prove a colour-degree version of Kahn's theorem as follows:

If L is a list assignment for $E(G)$ (equiv. $V(L(G))$), we define for $v \in V(G)$, a colour $c \in L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$

$$d_L(v,c) := |\underbrace{\{e \in E(G) : e \ni v, c \in L(e)\}}_{N_{L,G}(v,c)}|$$

$$d_L(v) := \max_c d_L(v,c)$$

$$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$$

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So, restating the thm:

Strong~~er~~ Kahn's Theorem (Kahn, 1996)

If L is a list assignment for $E(G)$ s.t.

$$|L| \geq (1 + o(1))\Delta_L(G)$$

then G has an (edge) L-coloring

~~(Note: For notation, let $L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$).~~

Clearly, still interested in $|L|/\Delta_L(G)$

Expectations:

(Note: Assume all color degrees and list sizes are regular & via equalizing coin flips (since regularization of line graphs seems odd)).

Retain := Pr[an edge e is not in G'] ← i.e. If was deleted since no inc. edge received $\phi(e)$

$$= \left(1 - \frac{1}{|L|}\right)^{d_L(e)} = \left(1 - \frac{1}{|L|}\right)^{2\Delta - 2} \approx e^{-2\frac{\Delta}{|L|}} \approx e^{-2}$$

Vertex-keep := Pr[$c \in L(v)$ is not retained by any edge around vertex v]

$$\approx 1 - \frac{1}{e^2}$$

Edge-keep := Pr[$c \in L(e^{uv})$ retained]

$\approx$ Pr[c not retained around u] $\times$ Pr[c — " — v] ← b/c mostly independent

$= (\text{Vertex-keep})^2$

3

Then,

$$\mathbb{E}[|L'(e)|] \approx |L| \left(1 - \frac{1}{e^2}\right)^2$$

Colour-Degree:

$$\mathbb{E}[|N^{edge}_{LG}(v,c) \cap E(G')|] = |N^{edge}_{LG}(v,c)| \cdot (1 - \text{Retain})$$

$$= \Delta \cdot \left(1 - \frac{1}{e^2}\right)$$

<u>But</u>, how many of these e <u>keep</u> $c \in L'(e)$?

It's not Edge-keep, ~~since~~ if we assume $c \in L'(v)$ b/c then by assumption ~~we~~ no edge around v retains c

Under this:

$\Pr[e \text{ keeps } c] = \Pr[\text{no edge} \longrightarrow] X.$

Lecture 9:

Molloy & Reed - Bal.. Proof - Kahn.

Finishing Kahn's proof:

$\text{Retain}_c := \left(1 - \frac{1}{|L|}\right)^{2D-2} \approx e^{-2}$.

$\text{Vertex-keep}_c := 1 - \text{Retain}_c \approx 1 - \frac{1}{e^2}$

$\text{Edge-keep} \approx \underbrace{(\text{Vertex-keep})^2}_{\text{mostly independent}} \approx \left(1 - \frac{1}{e^2}\right)^2$ (Can argue that the dependence is small).

$E[|L'(e)|] \approx |L| \cdot \left(1 - \frac{1}{e^2}\right)^2$

$E[D_{L,G'}(v,c)] \approx |\Delta| \left(1 - \frac{1}{e^2}\right)^2$

Approximates since there are small error terms.

Even assuming the errors from ~~expt~~ expectations won't hurt us too badly & that we concentrate these variables so as to have little error from concentration, unfortunately, at best these decrease at roughly the same rate - and some won't make progress.

Need a new finish!

Reserve Colours:

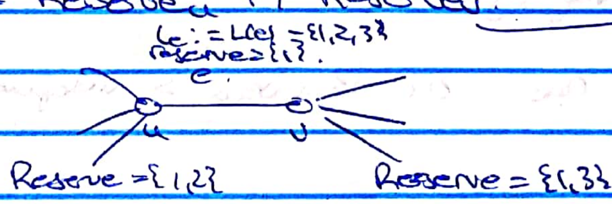
Idea: Before using nibble and naive colouring procedure, we reserve colours ~~and~~ around vertices to be used by edges ~~during the~~ in a "final step".

Choose $\text{Reserve}_v \subseteq L(v) := \bigcup_{e \ni v} L(e)$ (uniformly at random w/ prob p)

For each edge $e = uv$, define

$L_e' := L(e) - (\text{Reserve}_u \cup \text{Reserve}_v)$ → Lose colours from list (reserved)

$\text{Reserve}_e := \text{Reserve}_u \cap \text{Reserve}_v$ → Now e can use the overlapping colours safely



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Lemma:

(Setting $p = \frac{\log^4\Delta}{\sqrt{\Delta}}$) $\exists$ a choice of $Reserve_v$ for $V(G)$ such that $\forall u \in V(G)$, $\forall e \in E(G)$, $\forall c \in Reserve_v$:

(a) $|L(e) \cap (Reserve_u \cup Reserve_v)| \leq 3\sqrt{\Delta}\log^4\Delta$ $\rightarrow L(e) - L_e$

(b) $|\underbrace{\cancel{L(e)}}_{L_e} \cap (Reserve_u \cap Reserve_v)| \geq \frac{1}{2}\log^8\Delta$ $\rightarrow$ Reserve

(c) $|\{e = uv \in N_{L_G}(v,c) : c \in Reserve_w\}| \leq 2\sqrt{\Delta}\log^4\Delta$ $(*)$

Intuition:

(a) You don't lose too many colors

(b) There ~~one~~ is ~~many~~ at least one color to use near the end

Proof:

$\mathbb{E}[|L(e) \cap (Reserve_u \cup Reserve_v)|]$

$\leq |L(e)| \cdot 2p$ ← Since ~~must~~ may appear in either $Reserve_u$ or $Reserve_v$.

$\leq \Delta \cdot 2\frac{\log^4\Delta}{\sqrt{\Delta}}$

$= 2\sqrt{\Delta}\log^4\Delta$.

$\mathbb{E}[|L(e) \cap (Reserve_u \cap Reserve_v)|]$

$\leq |L(e)| \cdot p^2$ ← Needs to appear in both $Reserve_u$ and $Reserve_v$.

$\geq \Delta\frac{\log^8\Delta}{\Delta} = \log^8\Delta$.

$\mathbb{E}[(*)] = \Delta \cdot p \leq \sqrt{\Delta}\log^4\Delta$

Since each color $c \in Reserve_w$ is independent

You'll notice these variables are all sums of $\{0,1\}$-random variables (even more, all Bernoulli), so we can apply Chernoff Bounds:

$$\Pr[|L(e) \cap (\text{Reserve}_u \cup \text{Reserve}_v)| \geq \tfrac{3}{2}\mathbb{E}[\ ]] \leq e^{-\frac{\Delta \log^8 \Delta}{3/2}}$$

$$\Pr[\qquad \cancel{\ } \ (b) \qquad] \leq e^{-\frac{(\log^8 \Delta)^2}{3/2}}$$

$$\Pr[\qquad (c). \qquad] \leq e^{-\frac{\Delta \log^8 \Delta}{3}}$$

So all at most $e^{-\log^8 \Delta}$ for large enough Δ which is $\leq \frac{1}{\Delta^c}$ for any c and large enough Δ.

So, we apply LLL with following bad events:

$A_e = (a) \geq 3\sqrt{\Delta} \log^4 \Delta$

$B_e = (b) \leq \frac{1}{2} \log^8 \Delta$

$C_{v,c} = (c) \geq 2\sqrt{\Delta} \log^4 \Delta$.

Probability for any bad event $\leq \frac{1}{\Delta^c}$ for any c, while each event is mutually independent of the set of events depending on edges/vertices at distance >4.
So: using $c = 5$ (or 6) suffices for LLL. ☒

What is (c) saying?: If $c \in \text{Reserve}_v$, this counts $e \in N_{u,G}(v,c)$ with $c \in \text{Reserve}_e$

We define now the reserve degree of a vertex:

$$d_{res}(v,c) = |\{e \in N_{u,G}(v,c) : c \in \text{Reserve}_e\}|.$$

②

Idea for finishing w/ Reserved Colours:

Reserved degrees decreases rather quickly because edges in "reserved neighborhood" will be deleted from G, however Reserve never changes during Nibble.

So, if we can get $\max d_{res}(v,c) \le \frac{1}{2}|\text{Reserve}| = \frac{1}{4}\log^{s}\Delta$, we can finish by applying ~~Haxell~~ Haxell to reserve colour assignment

---

Expectation of new reserved degree in one step of Nibble:

$$\mathbb{E}[d_{res,L',G'}(v,c)]$$

$$= d_{res,L,G}(v,c)\cdot(\underbrace{1-\text{Retain}}_{\text{Probability an edge not deleted}})$$

$$\approx d_{res,L,G}(v,c)\cdot\left(1-\frac{1}{e^2}\right)$$

Decreases at half the rate multiplicatively compared to $|L'|$ and Δ'

We can run Nibble for T iterations, where T is about the solution to the following:

$$1 = |L|\cdot((1-\text{Retain})^2)^T = |L|\big(\underbrace{(1-\tfrac{1}{e^2})^2}_{=k}\big)^T$$

$$\Rightarrow T = \frac{\ln|L|}{-\ln k}$$

(So a roughly logarithmic amount of steps)

Yet,

$$d_{res,L_i^*,G_i}(v,c) \approx d_{res,L,G}(v,c)\cdot\left(\left(1-\tfrac{1}{e}\right)\right)^7$$
$$= d_{res,L,G}(v,c)\cdot K^{7/2}$$
$$\leq 2\sqrt{\Delta}\log^4\Delta \cdot \frac{1}{\sqrt{|L|}}$$
$$\leq 2\log^4\Delta$$
$$\leq \tfrac{1}{4}\log^8\Delta \quad \text{for large enough } \Delta$$

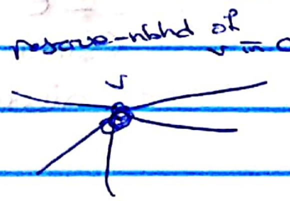
(So we don't have to run too many iterations for the reserved degree to be small enough)

Remark: We actually want to use $T = \frac{\log|L'|}{-\log|K|} - \text{poly}\log\log\Delta$ since we need to concentrate d_{res} during wave enough to apply LLL, so need $d_{res} \geq \text{poly}\log\Delta$

So, this gives an L-coloring if $|L| \geq \Delta + 4(\log^4\Delta)\sqrt{\Delta}$.
(Since we need $3\sqrt{\Delta}\log^4\Delta$ for reserves and an extra $\sqrt{\Delta}\,\text{poly}\log\Delta$ for concentration errors)

Concentration for $|L'|$, Δ', d'_{res}:

We'll do d'_{res} first!
d'_{res} counts # edges from d_{res} not deleted.



c-Lipschitz for $c =$ ______:

- If we change colouring on an edge not incident with v, this changes ≤ 2 of these edges
- If we change colouring incident with v, this can change at most itself and 1 other edge (they both lose the color). Otherwise, ~~there~~ if there are ≥ 3 edges ~~a~~ coloured the same color, then changing v won't matter.

Note: We can't use Chernoff b/c not sum of independent
—— " —— Simple conc. bound b/c may depend on Δ^2 trials ($\Rightarrow$ Bad Bound)

So: we'll use Talagrand's and we need to verify this variable.

Note: Verifying ~~x~~ retaining a colour requires $\cancel{\geq\Delta}$ 2Δ trials.

But we can verify not retaining using $r=2$ by showing its color $\phi(e)$ and some edge f near w/ $\gamma(f)=\phi(e)$.
So, we can apply Talagrand to show this is within about $\sqrt{\mathbb{E}[\deg]}$ with probability $\underbrace{e^{-c\mathbb{E}[\deg]}}$ (c constant)

$\leq e^{-c\log^8\Delta}$.
(since $\deg \geq \log^8\Delta$ during nibble)

Concentrating Δ':

~~[illegible]~~

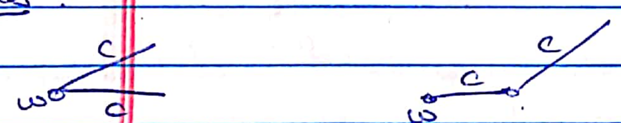
Assume c is not retained around v, this counts # of edges in $X_{L,G}(v,c)$ which are not deleted and the other end of the edge keeps c.

Again, we are 2-Lipschitz, since changing a single color can affect at most 1 other vertex. (If we have ≥ 3, then changing the color won't matter) (Same ~~argu~~ argument as before)

As before, not deleted is 2-verifiable, so we'd be happy if "other end keeps c" or its complement were r-verifiable for constant r since the expectation of intersection of these are on the same order as Δ.
(Note: As before, verifying edge around w retains c takes around 2Δ trials)

What about verifying the complement:
i.e. # of edges e where other end w does not have an edge retaining c?

Cases:



if 2 colored c, this is easy, show 2

To verify this one, we need to check all neighbors!

Trick: Change Variables
Define
~~X_{i}~~ $X_{j,k}$ to be # of ~~vertices~~ edges e where a and other end w
$\geq j$ are assigned color c
$\geq k$ have color c subsequently removed.

So we can write the variable above (the component count) as: $X_{1,0} - \underbrace{(X_{1,1} - X_{2,1})}_{\text{counts receive exactly 1 and remove} \geq 1}$ ~~[illegible]~~.

Easy to check that $X_{j,k}$ concentrates b/c 2-Lipschitz and $(j+k)$-verifiable and we can use Talagrand's on each. This works since they have roughly same expectation and j,k constant.

For IL'1, we use the same trick, but ~~[illegible]~~ counting both ends of e.

7100 - Colouring Feb 6th

Lecture 10:

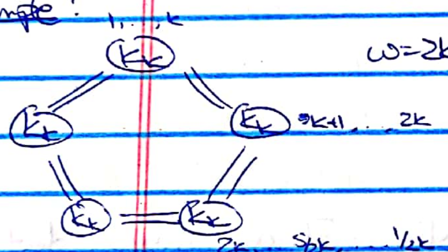
Reed's Conjecture:

Recall: Conjecture (Reed, 1998): If G is a graph, then
$$\chi(G) \geq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil.$$

Remark: If true, this is tight for some ω.

↓ Example of this.

Example:



$\omega = 2k$, $\Delta = 3k-1$ $\Rightarrow$ $\chi = \lceil \frac{5}{2}k \rceil$.

Why is this true?!
- $\exists$ a $\lceil 5/2 k \rceil$-colouring $\Rightarrow$ See labels
- We can also see this since $\alpha \leq 2$, and so the 5k colours we need can be cut in half

Reed's Conjecture has ~~[illegible]~~, been proved for various special cases of graphs, e.g: line graphs, claw-free graphs, etc.

Theorem (Reed, 1998)
If $\omega(G) \geq (1 - \frac{1}{10^{?}})(\Delta(G)+1)$ then the conjecture holds
→ So if clique # is large, then the conjecture is true.

Corollary:
$\exists \varepsilon > 0$ s.t. $\chi(G) \leq \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.
→ So if we're not satisfied w/ the half we could try some convex combination.

Scanned with CamScanner

What is ε?

- $\varepsilon = \frac{1}{2\cdot 10^8}$ (Reed '98)
- $\varepsilon = \frac{1}{320e^6}$ (King + Reed '12)
- $\varepsilon = \frac{1}{26}$ (Bonamy, Perrett, P. '16+)
- $\varepsilon = \frac{1}{13}$ (Delcourt, P. '17+)

How do we prove these bounds?:

Idea: Combine Randomness & Structure

3-Part-Plan (King & Reed)

(1) Do something different for large ω:

(a) $\omega \geq (1-\frac{1}{10^8})(\Delta(G)+1)$, use Reed.

(b) King showed that $\omega \geq \frac{2}{3}(\Delta+1)$, then $\exists$ an independent set I s.t. $\omega(G-I) < \omega(G)$, and

$$\Delta(G-I) < \Delta(G)$$

Now $G-I$ has a smaller ratio of ω/Δ, repeat until $\omega' < \frac{2}{3}(\Delta'+1)$, and now we use the I's to colour

(c) Use "harder" randomness + structure (Delcourt, P.)

(2) Now, assuming a small ω. Prove that a critical graph has neighbourhoods that are "somewhat" sparse

Structure

(3) Now, use random colouring + Nibble.

Randomness

Feb 6th

Formalizing (2) and (3):

Def'n

A graph G is σ-sparse if $\forall v \in V(G)$ we have that
$$e(G[N(v)]) \leq (1-\sigma)\binom{\Delta}{2}.$$

Remark: Note the use of Δ instead of $d(v)$, i.e. small degree vertices trivially satisfy this condition.

For (2):

Def'n

$\text{Save}(G) := \Delta(G) + 1 - \chi(G)$ → For list colourings: $\text{Save}_\ell(G) := \Delta(G) + 1 - |L|$

$\text{Gap}(G) := \Delta(G) + 1 - \omega(G)$

Reed's conjecture: $\text{Save}(G) \geq \left\lfloor \frac{\text{Gap}(G)}{2} \right\rfloor$

Our Result: $\text{Save}(G) \geq \varepsilon\, \text{Gap}(G)$ ($\varepsilon = 1/3$ say).

Theorem (Delcourt, P.)

If G is L-critical, then G is σ-sparse where ~~$\sigma = \frac{\text{Gap}(G)}{2\Delta}$~~ ~~$\frac{2\text{Save}(G)}{\Delta}$~~.

$$\sigma = \frac{\text{Gap}}{2\Delta} - \frac{2\text{Save}_\ell}{\Delta}$$

What good is sparsity?

Theorem

If G is σ-sparse and Δ is large enough, then

$\chi(G) \leq (1 - f(\sigma))(\Delta+1)$, where $f(\sigma) = \frac{1}{2e^6}\sigma$ ← Molloy & Reed '02

$\approx .1827\sigma - 0.778\sigma^{3/2}$ (Bruhn & Joos '18)

$\approx .3\sigma - .25\sigma^{3/2}$ (Bonamy, Perrett, P., '16+)

2

How to get $\varepsilon = 1/13$?

We know:

$$\mathrm{Save}(G) \geq (.3\sigma - .125\sigma^{3/2})\Delta$$

$$\gtrsim \frac{2}{9}\sigma\Delta \quad (\text{for } \Delta \geq 1/4)$$

$$\geq \frac{2}{9}\Delta\left(\frac{\mathrm{Gap}(G)}{2\delta} - \frac{2\,\mathrm{Save}(G)}{\Delta}\right) \quad (\text{By thm})$$

$$= \frac{\mathrm{Gap}}{9} - \frac{4}{9}\mathrm{Save}$$

$\Rightarrow \mathrm{Save} \geq \frac{1}{13}\mathrm{Gap}.$

Theorem (Erdős, Rubin, Taylor '79) $\longrightarrow$ Equivalently, show $\chi(K_n - M) = n - e(M)$

If M is a matching in G, then $\chi_\ell(G) \leq v(G) - e(M)$

Note:

- Obviously[?], $\chi(G) \leq v(G) - e(M)$ (color each edge of matching with same color)

Interesting part is it works for lists!

Proof:

By Induction on n. Let L be a list assign for G with $|L| = n - e(M)$

Case 1: $\exists e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \phi$. Now let $c \in L(u) \cap L(v)$

Colour u, v w/ c. Remove c from the other lists; delete u, v

Let $G' = G \setminus \{u, v\}$, $G' = K_{n-2} - \underbrace{(M - e)}_{M'}$, while $|L'| \geq |L| - 1 = v(G') - e(M')$

By induction, $\exists L'$ colouring, done

G

Proof (con't)

Case 2:

$\nexists\, e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \emptyset$.

We consider an auxiliary bipartite graph H where:

$$V(H) = \Big(\underbrace{V(G)}_{\text{vertices}},\ \underbrace{\bigcup_{u \in V(G)} L(u)}_{\text{colors}}\Big), \text{ and}$$

$$E(H) = \{(v, c) : c \in L(v)\}$$

Claim: $\exists$ a matching of H saturating $V(G)$, equivalently an L-coloring of G where every vertex receives a unique color

Let $S \subseteq V(G)$

- If $\exists\, e = uv \in M$ s.t. $u, v \in S$, then $N_H(S) \geq |L(u) \cup L(v)|$ $\overset{\text{By case 2}}{=} |L(u)| + |L(v)| \geq 2(n - e(M)) \geq n \geq |S|$
- If $\nexists$ such $e \Rightarrow |S| \leq n - e(M)$, so if $S \neq \emptyset$, then $\exists\, v \in S$ and $N_H(S) \geq |L(v)| \geq n - e(M) \geq |S|$ if $S = \emptyset$ $\square$

Theorem (Delcourt, P.)

If M is a matching in G and L is a list assignment for G s.t.:

(a) $|L(v)| \geq e(M)$ $\quad \forall v \in V(M)$

(b) $|L(a)| + |L(b)| \geq v(G)$ $\quad \forall ab \in E(M)$

(c) $|L(v)| \geq v(G) - e(M)$ $\quad \forall v \in V(G) \setminus V(M)$

③

Proof:

Same as ~~the~~ proof before, except cases for Hall's.

Cases:

(1) $\exists e \in E(H)$ s.t. $u, v \in S$ (by (b))

(2) $\exists v \in V(G) \setminus V(H)$ s.t. $v \in S$ (by (c))

(3) $w \Rightarrow |S| \le e(H)$ (by (a))

wait?? ??

$\square$

Theorem:

Let G be a graph, H be an induced subgraph of G, $s \ge 0$. Suppose that L is a list-assignment s.t. $|L(v)| \ge d(v) - s$ $\forall v \in V(H)$. If G is L-critical, then

$$e(H) \ge (m-s)(v(H) - m - s)$$

where

$m = \nu(\overline{H})$ (i.e. size of largest antimatching in H)

Obs: If G is a graph,

$$\nu(\overline{G}) \ge \frac{v(G) - \omega(G)}{2}$$ (follows from greedily removing antiedges).

Proof:

By induction on $v(H)$. Let M be a matching of $\bar{H}$ of size m.

If $m \leq s$: trivial

So, assume $v(H) > m > s \geq 0$

Since G is L-crit., $\exists$ L-colouring ϕ of $G - V(H)$.

Let $L'(u) = L(u) \setminus \{\phi(w) : w \in N(u) \setminus V(H)\}$ $\forall u \in V(H)$

Note $\forall u \in V(H)$,

$$|L'(u)| \geq |L(u)| - |N(u) \setminus V(H)|$$
$$\geq d(u) - s - |N(u) \setminus V(H)|$$
$$= d_H(u) - s$$

Since G is not L-col, H is not L'-col. So, by thm (Delcourt), $\exists$ at least one of (a)-(c) does not hold when applied to H, L', M:

Say (a) does not hold:

$\exists v \in V(M)$ s.t. $\underbrace{|L'(v)|}_{d_H(v) - s} \leq \overbrace{e(M)}^{=m} - 1$

So: $d_H(v) \leq m - 1 + s$

Let $H' = H - \{v\}$, $v(\bar{H}')$. So, by induction on H',

$$e(\bar{H}') \geq (m-1-s)(v(H) - 1 - (m-1) - s)$$
$$e(\bar{H}) \geq e(\bar{H}') + (v(H) - (d_H(v) + 1))$$
$$\geq e(\bar{H}') + v(H) - m - s$$
$$= (m-s)(v(H) - m - s)$$

4

Proof: (con't)

(b) does not hold:

$\exists\, ab \in E(H)$ s.t. $|L'(a)| + |L'(b)| \leq v(H) - 1$

$\Rightarrow d_H(a) + d_H(b) \leq v(H) - 1 + 2s$

$H' = H \setminus \{a,b\}$, $v(H') \geq m-1$

By induction:

$$e(\bar{H}') \geq (m-1-s)(v(H)-2-(m-1)-s)$$

$$= v(H) - m - s - 1$$

$$= (m-s)(v(H)-m-s) + (-v(H) + 2)$$

$$e(\bar{H}) \geq e(\bar{H}') + 1 + (v(H) - 2 - d_H(a)) + (v(H) - 2 - d_H(b))$$

... (work it out... magic happens)

(c) does not hold:

$\exists\, v \in V(H) \setminus V(M)$ s.t. $|L'(v)| \leq v(H) - m - 1$

$\Rightarrow d_H(v) \leq v(H) - m - 1 + s.$

$H' = H - v$, $v(\bar{H}') \geq m.$

By induction:

$$e(\bar{H}') \geq (m-s)(v(H) - 1 - m - s)$$

$$= (m-s)(v(H)-m-s) - (m-s)$$

$$e(\bar{H}) \geq e(\bar{H}') + (v(H) - (d_H(v) + 1))$$

$$= v(H) - m + s.$$

... can't work it out

QED.

Proving Density theorem.

Apply the previous thm to $H = N[v]$ (closed neighborhood)

$$\Rightarrow e(G[\overline{N(v)}]) \geq (m-s)(v(H)-m-s)$$

where

$S = \text{save}(G) - 1$, and

$m = v(\overline{N[v]}) \geq \frac{\Delta+1-\omega(G)}{2} = \frac{\text{Gap}(G)}{2}$, and

$v(H) = \Delta+1$

$$\approx \left(\frac{\text{Gap}}{2} - \text{Save}\right)\Big(\underbrace{\Delta+1 - \frac{\text{Gap}}{2}}_{\geq \frac{\Delta+1}{2}} - \text{Save}\Big)$$

$$\geq \frac{\text{Gap}\cdot\Delta}{4} - \text{Save}\,\Delta.$$

and since

$$e(\overline{H}) = \sigma\binom{\Delta}{2} \approx \sigma\frac{\Delta^2}{2},$$

So:

$$\sigma = \frac{2}{\Delta^2}\left(\frac{\text{Gap}\cdot\Delta}{4} - \text{Save} - \Delta\right)$$

$$= \frac{\text{Gap}}{2\Delta} - \frac{2\,\text{Save}}{\Delta}.$$

Feb 11th

Lecture 11:

Sparsity Lemma:

If G is a σ-sparse graph for $\sigma \geq \frac{\text{polylog}\,\Delta}{\Delta}$ and $\Delta(G)$ sufficiently large, then:

$$\chi(G) \leq (1 - f(\sigma))(\Delta + 1)$$

where:

$f(\sigma) \approx .023\sigma$ Molloy & Reed $\to$ Naive Coloring Procedure

$.18\sigma - .07\sigma^{3/2}$ Bruhn & Joos

$.3\sigma - .12\sigma^{3/2}$ Bonamy, Perrett, P.

New Ideas in Bruhn & Joos:

1) Naive is a bit "too naive", i.e. It's a bit wasteful to uncolor both vertices in a conflict

Idea: For every edge, we ~~independently~~ ~~flip no coin~~ direct the edge and independently flip a coin before the random coloring. We only uncolor the "head" of the edge in a conflict if coin flip is heads and similarly for tails.

2) Analyze the # of repeated colors in $N(v)$

(i.e. $\underbrace{|N(v) \setminus V(G')|}_{\substack{\text{"\# of neighbors (of v)} \\ \text{deleted"}}} - \underbrace{(|L(v)| - |L'(v)|)}_{\text{"colors lost"}}$)

i.e. ~~retained color~~ and not in G'

Via the # of pairs, triples, etc. of repeated colors.

Molloy + Reed observed that if this value is $\geq \Delta + 1 - |L|$, then we can greedily color G'.

Need to figure out $\mathbb{E}[$"$\underbrace{\text{this value}}_{\text{let's call it the "savings of } v\text{"}}$"$]$, argue its concentrated and apply LLL.

Pairs + Triples:

$$\text{Pairs}_x := \{ (u,v) \in \binom{N(x)}{2} : uv \notin E(G),\ \phi(u) = \phi(v) \text{ and } u,v \in V(G') \}$$

$$\text{Triples}_x := \{ (u,v,w) \in \binom{N(x)}{3} : uv, vw, uw \notin E(G),\ \phi(u) = \phi(v) = \phi(w), \text{ and } u,v,w \notin V(G') \}$$

Claim: $\text{Savings}_v \geq \text{Pairs}_v - \text{Triples}_v$.

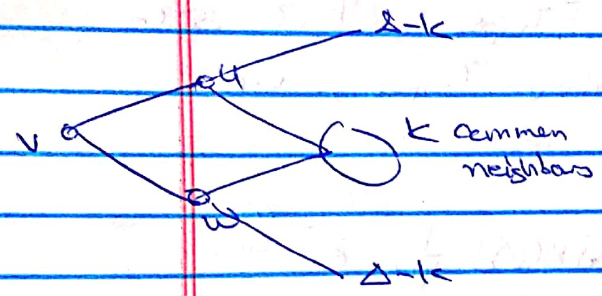
Expectation of Pairs, Triples:

Let's assume G is Δ-regular:

$$\underbrace{\Pr[v \notin V(G')]}_{\text{i.e. Retains color}} = \Big(1 - \underbrace{\frac{1}{2\cdot|L|}}_{\text{Need to lose coin flip to lose color}}\Big)^{\Delta} \approx e^{-\frac{\Delta}{2|L|}}$$

$$\mathbb{E}[\text{Pairs}_v] = \#\text{ of non-edges in } v \cdot \Pr[\text{Pairs}_v]$$

$\Pr[\phi(u) = \phi(v)] = \frac{1}{|L|}$ $\quad$ $\Pr[u,w \in V(G')]$ $\quad$ $(\cdot\, C_\Delta)$



$\Pr[u,v \in V(G')]$

$$= \left(1-\frac{1}{2|L|}\right)^{2(\Delta-k)} \left(1-\frac{3}{4|L|}\right)^{k}$$

Neighbors not in common (Bad event is losing coin flip)

Neighbors in common (Bad event is either u or w lose coin flip).

$$= \left(1-\frac{1}{2|L|}\right)^{2\Delta} \cdot \frac{\left(1-\frac{3}{4|L|}\right)^{k}}{\left(1-\frac{1}{2|L|}\right)^{2k}}$$

Can argue that $\left(1-\frac{3}{4|L|}\right) \geq \left(1-\frac{1}{2|L|}\right)^2$ for $|L|$ large enough, and so:

$$\mathbb{E}[\text{Pairs}_v] \geq \underbrace{\#\text{ of non-edges of } v}_{\geq \bar{\sigma}\binom{\Delta}{2}} \cdot \left(1-\frac{1}{2|L|}\right)^{2\Delta}$$

$$\approx e^{-\Delta/|L|}.$$

2

The calculation is similar for $\mathbb{E}[\text{Trips}_v]$, BUT since we want to argue $\text{Sewings}_v \geq \text{Pairs}_v - \text{Trips}_v$, we want an upper bound.

$$\mathbb{E}[\text{Trips}_v] \geq \underbrace{\#\text{ of triangles in } G[N(v)]}_{\text{But what is this?}} \cdot \frac{1}{|L|^2} \cdot e^{-\frac{7\Delta}{8|L|}}.$$

Theorem (Rivin, 2002)
If G is a graph w/ $\sigma\binom{V(G)}{2}$ edges, then G has at most $\sigma^{3/2}\binom{V(G)}{3}$ triangles.

So: $\mathbb{E}[\text{Pairs}_v - \text{Trips}_v]$

$$\geq \sigma\binom{\Delta}{2}\cdot\frac{1}{|L|}\cdot e^{-\frac{\Delta}{|L|}} - \sigma^{3/2}\binom{\Delta}{3}\frac{1}{|L|^2}e^{-\frac{7\Delta}{8|L|}}.$$

(where $\sigma\binom{\Delta}{2} = \#$ of nonedges in $N(v)$).

$$\approx \frac{\sigma}{2e} - \sigma^{3/2}\cdot\frac{1}{6e^{7/8}}$$

, ~~which~~ ~~is~~ ~~the #'s~~ we ~~wanted~~ which, if worked out, will match the #'s ~~Bruhn~~ we wanted (from Bruhn & Joos)

Then, use Exceptional Talagrand's and difference of variables to concentrate.

Feb 11th

New Idea in Bonamy, Perrett, P.:
- Why stop after one iteration?
  Keep using nice procedure, i.e. Use Nibble!
  Issue: Only works if uncolored subgraph is still "somewhat sparse"
  Idea: Uncolored subgraph will be "pseudorandom" subgraph of G

Defn: If G is a graph, we say a subgraph H of G is a μ-pseudorandom subgraph of G if $\forall u,v \in V(G)$,
$||N_G(u) \cap N_G(v) \cap V(H)| - \mu|N(u) \cap N(v)|| \le 10\sqrt{\Delta}\log^4\Delta$.

expected probability of going in $V(H)$

So, if we can control this #, then:

Lemma: $\forall \sigma > \sigma'$, and $\mu > 0$, if $\Delta(G)$ is large enough, and H is a μ-pseudorandom subgraph of G, and G is σ-sparse, then H is σ'-sparse

So, how does this give a savings of $(.3\sigma^{?} - .2\sigma^{3/2})\Delta$?
Savings for max deg Δ graph (from Bruhn & Joos) $\approx (.18\sigma - .07\sigma^{3/2})\Delta$
Savings in 2nd step: $\approx (.18\sigma' - .07\sigma'^{3/2})\Delta'$
$\vdots$
—"— tth " : $\approx (.18\sigma_t - .07\sigma_t)\Delta_t$.
What is Δ' (and Δ_t)?

3

What is $\Delta(G')$?

$E[|N(u) \cap V(G')|]$

$= \underbrace{|N(u)|}_{=\Delta} \cdot (1 - \underbrace{Pr[\exists v \in N(u) ; u \notin V(G')]}_{\approx e^{-\Delta/2\Delta}})$ (we've done this b/4)

$= \Delta\left(1 - e^{-\Delta/2\Delta}\right)$.

We can concentrate ~~this~~ this (using LLL) to get:

$$\Delta_t \approx \Delta_0\left(1 - e^{-\Delta/2\Delta}\right)^t$$

So:

Total Savings = $\sum$ Savings at each step

$= (.18\sigma - .17\sigma^{3/2}) \cdot \sum_{t \geq 0} \Delta_t$

$=$ " $\cdot \underbrace{\sum \Delta_0 \left(1 - e^{-\Delta/2\Delta}\right)^t}$.

$= \Delta_0 \cdot e^{\frac{\Delta}{2\Delta}} \approx \sqrt{e} \approx 1.6$

Feb 11th

Johansson / Molloy / Bernshteyn Theorems:

Theorem (Johansson)
If G is triangle-free then
$$\chi_\ell(G) \leq O\left(\frac{\Delta}{\ln\Delta}\right)$$

Molloy ('12) $(1+o(1))\frac{\Delta}{\ln\Delta}$

Bernshteyn ('18) True for DP-coloring

Bernshteyn: uses LLL

Molloy: uses entropy compression

Theorem (Johansson)
If G is K_r-free for fixed r
$$\chi_\ell(G) \leq O\left(\frac{\Delta \ln\ln\Delta}{\ln\Delta}\right).$$

Molloy: $\chi_\ell(G) \leq 200r\frac{\Delta\ln\ln\Delta}{\ln\Delta}$ (r no longer fixed)

Bernshteyn: True for DP-coloring

In fact, Molloy/Bernshteyn gives a more general result framework:
(Bonamy, Kelly, Nelson, Postle '18+)
$$\chi_\ell(G) \leq 200\Delta\sqrt{\frac{\ln w(G)}{\ln\Delta}}$$
(even $\chi_{DP}(G)$)

In spirit this answers the following question:

What assumption do we need on $\omega(G)$ to guarantee that:

$$\chi(G) \leq \frac{\Delta}{c} \qquad (c \geq 2)$$

Using BKNP: $\omega \leq \Delta^{\frac{1}{200c^2}}$

This is almost tight:

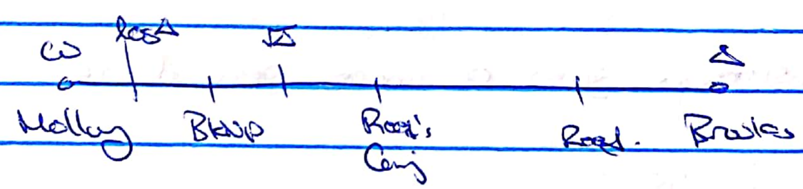
Ramsey Theory $\Rightarrow$ Need $\omega \leq {}^{\frac{1}{c-1}}$.

Conjecture: linear in c is correct

implied by the following conjecture:

$$\chi_f(G) \leq 200\Delta \cdot \frac{\ln \omega}{\ln \Delta}$$

This would give the complete picture:



Lecture 12:

Localized Coloring Theorems: → Interpolates standard thms in coloring

1) Local List Coloring → localize how many colors are available (i.e. $|L(v)|$)
2) Local Fractional Coloring
   → localize how much color a vertex recieves. (its demand $f(v)$)
   → Has connections with ~~independent set bounds.~~

Survey of ~~[illegible]~~ results.

- Can localize # of colors → $|L(v)|$
- Can localize maximum degree (Δ) → $d(v)$
- — " — clique # → $\omega(v) := \omega(G[N(v)])$ (size of largest clique containing v)

Ranging over $\omega(v)$ vs. $d(v)$

Greedy: If $|L(v)| \geq d(v)+1$ $\forall v$, then G has an L-coloring

Local List Brooks: If $|L(v)| \geq d(v)$, then G has an L-col
(ERT '79, Borodin '78~~+~~) unless G ~~has~~ contains ~~on~~ a component where every block is a clique or odd cycle

Local List Reed's Conj:
If $|L(v)| \geq \left\lceil \frac{d(v)+1+\omega(v)}{2} \right\rceil$ $\forall v \in V(G)$,
then G has an L-coloring

Theorem (Kelly, Postle)
$\exists \varepsilon > 0$ ($\varepsilon \approx \frac{1}{1000}$), if $|L(v)| \geq \lceil (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v) \rceil$ $\forall v$ and
~~δ~~ $\delta(G) \geq$ polylog $\Delta(G)$, then G has an L-coloring

70

On "edge-colorings" locations

Recall: Thm (Galvin) If G is bipartite, $\chi_e(G) = \chi(G)$.

Local List Galvin: (Borodin - Kostochka - Woodall '99)

If G is bipartite and $|L(e)| \geq \max\{d(u), d(v)\}$, then G has an L-coloring.
$e = uv$

(Note there is no restriction on degrees)

Local list Kahn's (Bonamy, Delcourt, Lang, Postle, 20+)

If $|L(e)| \geq (1+o(1)) \max\{d(u), d(v)\}$ and $\delta(G) \geq \text{polylog}\,\Delta$ then G has an L-coloring.

Local Reed-Sudakov

Theorem (Abbasi, Kim, Postle)

$\exists\, c>0$ such that if $|L(v)| \geq c \cdot d(v)\ \forall v$ and $\delta(G) > \text{polylog}\,\Delta$, then G has an L-col.

Feb 13th

Local Fractional Coloring!

Recall:

Demand function $f(v)$ $f: V(G) \to [0,1] \cap \mathbb{Q}$.

"How much color they demand"

Proposition: (Dvořák, Sereni, Volec)

Let G be a graph with demand function f. TFAE:

(a) G has an f-coloring ($\phi: V(G) \to$ measurable subset of $[0,1]$ s.t $\mu(\phi(v)) = f(v)$ $\forall v$ and $\phi(u) \cap \phi(v) = \emptyset$ $\forall e = uv \in E(G)$)

(b) There exist a common denominator N for f s.t G has a (f,N)-coloring (i.e. $\phi(v) \subseteq [N]$ where $|\phi(v)| = f(v) \cdot N$ and $\phi(v) \cap \phi(u) = \emptyset$ $\forall e$).

(c) $\exists$ probability distribution on independent sets of G s.t $\Pr[v \in I] \geq f(v)$ $\forall v$

(d) The vector of demands $(f(v) : v \in V(G))$ is in the stable set polytope of G

(e) $\forall$ nonnegative weight function $w: V(G) \to \mathbb{R}_+$, the graph G contains an independent set I s.t $\sum_{v \in I} w(v) \geq \sum_{v \in V(G)} w(v) f(v)$.

Remark! $\chi_f(G) := \min\{k : \exists f\text{-col of } G \text{ w/ } f(v) = 1/k\}$.

Also,

(e) $\Rightarrow$ $\alpha(G) \geq \sum_{v \in V(G)} f(v)$. $\left(\geq \frac{V(G)}{\chi_f(G)}\right)$.

②

Results for local frac. colouring:

Local frac. greedy? $\rightarrow f(v) = \frac{1}{d(v)+1}$?

$\rightarrow$ Does there exist such an analogue $\leftarrow$ would be the most natural analogue

Theorem (Caro-Wei; '90) — independently by

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$$

In fact, the proof gives an f-col where $f(v) = \frac{1}{d(v)+1}$.

Corollary: (Turán)

$$\alpha(G) \geq \frac{v(G)}{\underbrace{\text{avg}(d(v))}+1} \rightarrow \text{By convexity of } \frac{1}{x+1}$$

$\hookrightarrow$ average degree of v.

---

3 proofs of Caro-Wei:

Proof I: (Min-deg proof)

Let G be a minimum counterexample. Let $v \in V(G)$ be of min. deg.; since G is a min. counterexample, $G - v$ has an f-col ϕ.

Note that $\forall u \in N(v)$, $\mu(\phi(u)) = f(u) = \frac{1}{d(u)+1} \leq \frac{1}{d(v)+1}$.

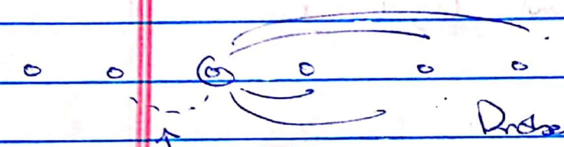
So, v "sees" at most $\frac{d(v)}{d(v)+1}$ colours and so $\mu(L_\phi(v)) \geq 1 - \frac{d(v)}{d(v)+1}$

$= \frac{1}{d(v)+1} = f(v)$

Proof #2: (Probabalistic Proof)

Choose a total ordering $<$ of $V(G)$ u.a.r. Let $v \in I$ if $v < u$ $\forall u \in N(v)$.

$$\Pr[v \in I] = \frac{1}{d(v)+1} \Rightarrow$$



Probability v only ~~sees~~ has forward edges is $\frac{1}{d(v)+1}$.

No back edges.

Proof #3: (Max degree proof)

Let G be a min. counterexample. Let $v \in V(G)$ s.t. $f(v)$ is minimum. Let $\Phi(v) \subseteq [0,1]$ be a set of measure at least $f(v)$. Let f' be a demand function for $G - v$.

s.t.

$$f'(u) = \frac{1}{d_{G-v}(u)+1}$$

(Idea: Neighbors of v are overcolored)

Since, $G-v$ has an f'-coloring

G is min counterexample,

Note $\forall u \in N(v)$,

$$f'(u)\,\mu(L_\phi(u)) = \frac{1-f(v)}{d_{G-v}(u)+1} \geq \frac{1 - \frac{1}{d_G(u)+1}}{d_G(u)} = \frac{1}{d_G(u)+1} \geq f(u).$$

By lemma, G has a f-col

↳ See back.

Lemma
Let G be a graph with demand function f, fractional list assignment L and $g: V(G) \to \mathbb{R}$. If $g(v) \leq f(v)\mu(L(v))$ $\forall v$ and $\forall S \subseteq V(G)$ s.t. $\mu(\bigcap_{v \in S} L(v)) > 0$ the graph $G[S]$ has an f-col, then G has a fractional (g, L)-col.

Frac list-assign: $L(v) \subseteq \mathbb{R}$
Frac (g, L)-col: $\phi(v) \subseteq L(v)$, $\mu(\phi(v)) \geq g(v)$

Feb 25^{th}

Lecture 13:

Recall:

Thm (Caro-Wei)

If G is a graph, then G has an f-coloring where $f(v) = \frac{1}{d(v)+1}$ $\forall v$

3 proofs:

1) Min degree - Delete & Extend

2) Probabilistic

3) Max degree - Delete, "Selectively Reduce", Extend

Q: Are there local fractional versions of other coloring theorems?

A local fractional Brooks's Theorem:

Multiple Viewpoints:

- Under what conditions can you get an f-col w/ $\frac{1}{d(v)}$?
  (Or, weaker version $\frac{1}{d(v)+\varepsilon}$ for some $\varepsilon \in (0,1)$)
- Caro-Wei is tight for disjoint union of cliques. What if we "forbid this"? (What can we save?)

  Idea #1: Control local clique #'s, i.e. $\omega(v) = \omega(G[N(v)])$
  - We could assume $\omega(v) \le d(v)$ $\forall v$. ~~Recall~~ (Instead of possible $d(v)+1$ in clique) $\rightarrow$ Recall that if $\omega(v) = d(v)+1$, v is called simplicial. So, we restrict simplicial vertices

  -OR-

  - We could try a larger f for non-simplicial and $f = \frac{1}{d(v)+1}$ for simplicial.

-CR-

Strongest Version:

Show that the necessary condition:

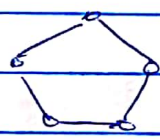
$$\sum_{v \in V(K)} f(v) \le 1 \quad \forall \text{cliques } K$$

is sufficient

(Since if cliques demand too much color, then we're screwed)

Q: For what ε is this nec. clique condition sufficient?

If $\varepsilon < 1/2$, then no!



then, $f(v) = \frac{1}{d(v)+\varepsilon} > 2/5$,

so we get a better than $2/5$-coloring.

Theorem (Kelly, Postle)

If G is a graph and f a demand function for G such that

- $f(v) \le \frac{1}{d(v)+1/2} \quad \forall v \in V(G)$, and
- $\sum_{v \in K} f(v) \le 1 \quad \forall$ clique K in G,

then G has an f-coloring.

Proof Ideas:

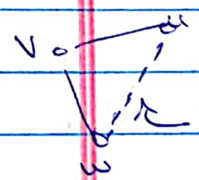
- Min. degree type proof. (Probabilistic, max-degree didn't work well)

Proof Idea (Cont.)

1st Obs) Max degree vertex has many min. degree neighbors

2nd Obs) Prove that min. deg vertices decompose into cliques

i.e.



← will always exist.

⋮

□

Corollary (Brouwer)

If G is a K_{r+1}-free graph on n vertices with:

$$e(G) \ge \left(1-\frac{1}{r}\right)\frac{n^2}{2} - \frac{n}{r}+1$$

then $\chi(G) \le r$ (r-partite)

~~Proof~~

Local Proof Vizings:

If G is a graph, $E(G)$ has an f-colouring where

$$\underset{=uv}{f(e)} = \frac{1}{\max\{d(u),d(v)\}+1} \quad \forall e\in E(G)$$

Proof:

Have to prove $(f(e): e\in E(G))$ is in the Edmond's Matching Polytope:

- $\sum_{e\ni v} f(e) \le 1$ → $\sum_{e\ni v}\frac{1}{d(v)+1} \le \frac{d(v)}{d(v)+1}\le 1$
- $\sum_{e\in G[S]} f(e) \le \left\lfloor\frac{|S|}{2}\right\rfloor$ → Show that $f'(e) \le \frac{v(G)-1}{2}$, where $f(v) = \frac{1}{2}\left(\frac{1}{d(u)+1}+\frac{1}{d(v)+1}\right)$ ← Average of max.
- $f(e)\ge 0$

$$\sum_e \frac{1}{2}\left(\frac{1}{d(u)+1}+\frac{1}{d(v)+1}\right)$$

$$=\sum_v \frac{d(v)}{d(v)+1} \le \sum_v \frac{\Delta(G)}{\Delta(G)+1} \le v(G)-1.$$

(2)

Perfect Graph:

Recall:

G is perfect if $\forall$ induced subgraphs H, $\omega(H) = \chi(H)$

Strong perfect graph thm (Chudnovsky, Robertson, Seymour, Thomas, 2006)

G perfect iff neither G nor $\bar{G}$ contain an induced odd cycle of length ≥ 5

Weak Version: (Lovasz '70s)

G perfect iff $\bar{G}$ perfect

Theorem (Kelly, Postle)

If G is perfect, then G has a $\frac{1}{\omega(G)}$-coloring

Lemma:

If you copy a vertex in a perfect graph, you remain perfect.

~~Proof (SA):~~

Blow up ~~[illegible]~~ $G \to G$.

References

[1] Tom Kelly and Luke Postle. A local epsilon version of reed's conjecture. 2019.

[2] Marthe Bonamy, Thomas Perrett, and Luke Postle. Colouring graphs with sparse neighbourhoods: Bounds and applications, 2018.

[3] Marthe Bonamy, Tom Kelly, Peter Nelson, and Luke Postle. Bounding χ by a fraction of Δ for graphs without large cliques, 2018.

[4] Tom Kelly and Luke Postle. Fractional coloring with local demands, 2018.