CO749 - Graph Colourings

(Notes Scans)

University of Waterloo

Nicholas Pun

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749 - Graph Colouring

Jan 7th

Lecture 1:

Def'n 1: A k-colouring of a graph G is a partition of $V(G)$ into at most k independent sets.

Def'n 2: A k-colouring of a graph G is a map $f: V(G) \to [k]$ such that $\forall e = vw \in E(G)$, $f(v) \neq f(w)$

Def'n 3: A k-colouring of a graph G is a graph homomorphism to K_k.

Takeaway: There are multiple ways to view what a colouring is.

Weakenings, Generalizations, and Variants of Colouring:
Variants: ("Changing what you color")
- Edge Colouring: A k-edge-colouring of a graph G is
Def 1: A partition of $V(G)$ into at most k matchings
2: A k-colouring of $L(G)$ (the line graph of G)
~~"Changing what you color"~~:

- Total Colouring: A k-total-colouring of a graph G is a map $f: V(G) \cup E(G) \to [k]$ s.t.
  - $f(v) \neq f(w)$ $\forall vw \in E(G)$
  - $f(v) \neq f(e)$ $\forall v \sim e$ $\in E(G)$, $\in V(G)$
  - $f(e) \neq f(e')$ $\forall e \sim e' \in E(G)$

Generalizations: ("Changing what you are allowed to color")
- List Colouring: (idea: Lists of available colours to vertices)

Def: ~~A k-list-colouring~~ k-list-assignment of a graph G is an assignment of lists $(L(v): v \in V(G))$ such that

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$|L(v)| \geq k \quad \forall v \in V(G)$

An L-<u>colouring</u> of a graph G is a colouring ϕ of G such that $\phi(v) \in L(v) \quad \forall v \in V(G)$

~~"Changing what you are allowed to colour"~~

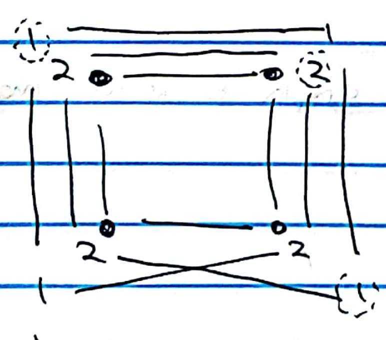
- Correspondence Colouring:

  <u>Def</u>: A k-<u>correspondence</u>-<u>assignment</u> ~~colouring~~ is a pair $((L(v): v \in V(G)), (M_{uv}: uv \in E(G)))$, where M_{uv} is a matching from $L(u)$ to $L(v)$

  An (L,M)-<u>colouring</u> of G is a colouring ϕ of G such that:

  - $\phi(v) \in L(v) \quad \forall v \in V(G)$
  - $\phi(u)$ is not matched to $\phi(v)$ in M_{uv} $\forall uv \in E(G)$

  Ex: $G = C_4$



$L(v) = \{1,2\} \;\forall v$

Q: Is there an (L,M)-colouring of G for this L,M?

No!

And we run into trouble here

Remarks:

- We may as well assume $L(v) = \{1, \dots, |L(v)|\}$ $\forall v \in V(G)$
- Correspondence has a "local notion of colour", while list colouring / ordinary colouring have a "global notion of colour"

Weakenings: ("Change what a colour is")

- No Restrictions: Improper colourings, i.e. mappings of graphs
- d-defective colouring: Each colour has maximum degree d
- c-clustered Colouring: Every monochromatic component has size (# of vertices) $\leq c$
- No mono. path of length $> l$.
- Every colour is d-degenerate
- Every colour is triangle-free (or more generally, bounded clique #)

Fractional Colouring: ("Changing how you colour")

Def: An (a,b)-colouring of a graph G is a map ϕ such that $\phi(v)$ is a subset of $[a]$ of size b and $\forall uv \in E(G)$, $\phi(u) \cap \phi(v) = \emptyset$

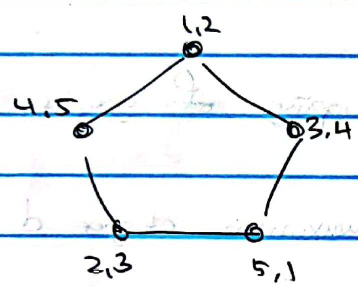
Remark: If G has a k-colouring, then G has a (kb, b)-colouring $\forall b$

(1)

(i.e. Graph homomorphism to Kneser graph on a, b)

The fractional chromatic #: $\chi_f(G) = \inf\{\frac{a}{b} : G \text{ has an } (a,b)\text{-colouring}\}$.

Ex. $\chi_f(C_5) = 5/2$



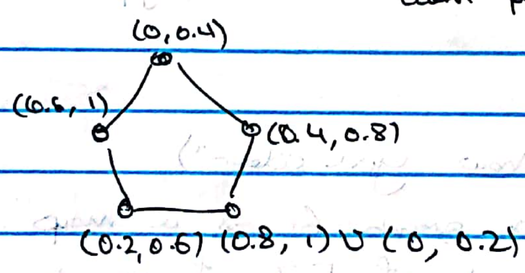
Remark:

- There is also an LP-formulation and its dual. (its dual asks about weighted independent set)
- An assignment of measurable subsets $\Phi(v)$ of $[0,1]$ $\forall v \in V(G)$ such that:

  $\mu(\Phi(u) \cap \Phi(v)) = \emptyset$

  $\chi_f(G) = 1 / \sup\{\varepsilon : G$ has a colouring as above, with $\mu(\Phi(v)) = \varepsilon \ \forall v\}$

Ex:



Let's call the above an *f-colouring* if $\mu(\phi(v)) = f(v)$ $\forall v$.

Proposition: $\chi_f(G) = k$ iff G has a $(\frac{1}{k})$-colouring

Proposition: G has an f-colouring iff the vector $(f(v) : v \in V(G))$ is in the independent set polytope (i.e. $\exists$ probability distribution on independent sets such that $Pr[v \in I] = f(v)$)

Combinatorics:

- Defective Clustering ×
- List Defective ✓
- Frac. Defective ✓
- List Edge ✓
- Frac. Edge ✓
- Edge Total ×
- List Correspondence × (Correspondence is already list).
- Fractional List ✓
  - Find b-colouring from a-list-assignment. (multicolourings)

$$\chi_{f,list}(G) = \inf\left\{\frac{a}{b} : \forall \text{ a-list-assign } G \text{ has a b-colouring}\right\}$$

Theorem: $\chi_f(G) = \chi_{f,list}(G)$

- Fractional Correspondence.

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Lecture 2:

Questions / Results: (on Graph Colourings)

Def'n (k-colourable)
A graph G is k-colourable if G has a k-colouring

Def'n (Chromatic Number)
The chromatic number, denoted $\chi(G)$, is the minimum k s.t. G is k-colourable

Question: Why is this a ~~natural~~ good ~~question~~ definition?
If a graph is k-colourable, then it is $(k+1)$-colourable, so this is a natural def'n.
(Prop: If $H \subseteq G$, then $\chi(H) \leq \chi(G)$)

Remark:
A graph is 1-colourable iff no edges
— " — 2-colourable iff no odd cycles
— " — 3-colourable iff no good answer!
↳ Since NP-hard to decide if a graph is 3-col.

A graph is critical for k-colouring if it is not k-colourable, but every proper subgraph is. (also formerly known as $(k+1)$-critical)

①

List Colouring: → (Introduced by Erdős, Rubin, Taylor in 1979 and independently by Vizing 1976)
The list chromatic number (aka choice number or choosability) denoted $\chi_\ell(G)$ is the minimum k such that G has a L-coloring $\forall$ k-list-assignments L.

Proposition: $\chi(G) \leq \chi_\ell(G)$

Proposition: If $H \subseteq G$, $\chi_\ell(H) \leq \chi_\ell(G)$
(i.e. The list chromatic number remains monotone)

Def'n G is k-list-colorable if $\chi_\ell(G) \leq k$
(aka k-choosable)

Def'n G is critical for k-list-colouring if G is not k-list-col. but every proper subgraph is

L-critical w.r.t a list assignment L if G is not L-col, but $\forall$ proper subgraph is.

Q: How is list colouring different from colourings?

Theorem: $\chi_\ell(K_{d,d}) = \Theta(\log d)$
(But note: $\chi(K_{d,d}) = 2$)

Theorem (Alon 2000)
If G is a graph of min. degree d, then $\chi_\ell(G) = \Omega(\log d)$.

Conjecture: $\forall k$, if $\chi(G) \le k$, (and ~~min. degree~~ triangle free) then $\chi_\ell(G) = O(\log d)$

Correspondence Colouring (aka DP-colouring)

Def: Corr. Chromatic # (aka DP-chr. #), denoted $\chi_c(G)$ (aka $\chi_{DP}(G)$) is min k s.t. $\forall (L,M)$ k-corr.-assign, G has a (L,M)-coloring

critical for "

(L,M)-critical for "

Theorem (Bernshteyn, 2018)
If G is d-regular, then $\chi_c(G) = \Omega\left(\frac{d}{\log d}\right)$

Back to questions:

Types of questions:
- Chromatic # as related to other graph parameters (E.g. degree, clique #, etc.)
- Chromatic # of certain graph classes (E.g. Planar, surfaces, etc.)
- Algorithmic questions, e.g. Deciding if colouring exists, finding a coloring, can you sample a colouring uniformly at random
- How many colourings?
- Re-configuration: Can we get from one colour to another?

Relations to Other Parameters:-

Colouring Ones: $\max_{H \subseteq G} \frac{v(H)}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \chi_\ell(G) \le \chi_{DP}(G)$

Hall ratio

(Recent result of 2019): $\exists f$ s.t. $\chi_\ell(G) \le f(\text{Hall ratio})$

Degree, Clique #, Girth:

$\chi_{DP}(G) \le \underline{\Delta(G)} + 1$ (Greedy Bound)

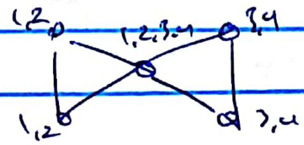
max degree

Brook's Thm (1941): $\chi(G) \le \Delta(G)$ unless G contains $K_{\Delta+1}$ or an odd cycle if $\Delta = 2$

(Other version: If connected, then $\chi \le \Delta$ unless G is iso to clique or odd cycle if $\Delta = 2$)

ERT, $\chi_\ell(G) \le$ ——— " ———
independently Vizing

ERT (Borodin) If L is a list assignment of connected G such that $|L(v)| \le d(v)$, then G is L-colorable unless every block of G is a clique or odd cycle



Jem atn

Girth:

Kim '95: If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\log\Delta}$

Johannsen '99: If G is triangle-free, then: $\chi(G) = O\left(\frac{\Delta}{\log\Delta}\right)$

Molloy '17: ——— " ——————— then: $\chi(G) \leq (1+o(1))\left(\frac{\Delta}{\log\Delta}\right)$

If G is K_r-free for fixed r, then $\chi \leq O\left(\frac{\Delta}{\log\Delta}\log\log\Delta\right)$

Conj: $\log\log\Delta$ is not neccessary.

Question: Can we do better than $\frac{\Delta}{\log\Delta}$?

Erdős '59: $\exists$ graphs of arbitrary girth and chromatic #

Best known result in Ramsey Theory
$\Rightarrow \exists$ graphs of arbitrary girth and $\chi \geq \frac{1}{2}\frac{\Delta}{\log\Delta}$

$\hookrightarrow$ So, no, we can't!

Question: Is the answer 1 or 1/2 or in-between?

Reed's Conjecture (1998) $\chi(G) \leq \left\lceil \frac{\Delta+1+\omega}{2} \right\rceil$.
$\Rightarrow$ True for $\chi_f(G)$ (Reed)
for if $\omega \geq .9999998\,\Delta$ (i.e. large ~~omega~~).

Thm (Delcourt, P.)
If Δ large enough, then $\chi \leq \left\lceil \frac{12}{13}(\Delta+1) + \frac{1}{13}\omega \right\rceil$.

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Chromatic # of Graph Classes:

Four Color Theorem (Appel & Haken, 1977/79)
(Conjectured 1852)
If G is planar, $\chi(G) \leq 4$.

→ Later proof by Robertson, Sanders, Seymour, and Thomas (1994/6)
(Formally verified by proof systems in 2000s)

Grötzsch's Theorem (1959)
If G is planar, triangle-free, then $\chi(G) \leq 3$.

Surfaces:
Euler genus of a surface =
$2 \times$ # of handles + # of crosscaps

Heawood's Bound: $\forall g \geq 1$, if G is a graph embedded in a surface of genus g, then

$$\chi(G) \leq \frac{7 + \sqrt{24g + 1}}{2}$$

Ringel-Youngs Thm (1960's)
Heawood's Bound is tight for every surface, except the Klein bottle, where $\chi \leq 6$.

Hadwiger's Conjecture:
(1943) If G has no K_t-minor, then $\chi(G) \le t-1$
→ Easy for $t \le 3$ and proved by Had for $t=4$

Wagner '37 - Showed $t=5$ is equiv to 4CT
Robertson, Seymour, Thomas '94 : $t=6$ equiv to 4CT

Open for all $t \ge 7$.

Weakenings:
Thm (Reed-Seymour, '99):
If G has no K_t-minor, then $\chi_f(G) \le 2t$

Thm (Edwards, Kang, Kim, Oum, Seymour, '15)
$\forall t, \exists d$ s.t. if G has no K_t-minor, then G is a d-defective, t-colorable

Thm (Dvorak - Norin, '18+)
— " —, G is $\exists c$-clustered, t-col.

Thm (Kostochka, Thomason '80s)
If G has no K_t-minor, then G is $O(t\sqrt{\log t})$-degenerate

Thm (Norin, Song, '19+)
$\forall \beta > 1/4$, G is $O(t(\log t)^{\beta})$-col.

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List version of Hadwiger's is false
i.e. $\exists K_t$-minor free G w/ $\chi_\ell(G) \geq 4t/3$

Strong perfect graph thm: (Chudnovsky, Robertson, Seymour, Thomas, '06)

G is perfect iff G has no induced C_{2k+1} or $\overline{C_{2k+1}}$ $\forall k \geq 2$

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Lecture 3:

The Probabalistic Method:

3 Pillars of prob. method:

- Linearity of Expectation (and basic probability)
- Lovász Local Lemma (introduced in 1975)
- Concentration Inequalities - (Markov, Chebyshev, Chernoff, Talagrand's) 1990s

Linearity of Expectation:

Expectation: Let X be a discrete $\mathbb{R}$ r.v., then: $\mathbb{E}[X] = \sum_{i \in \mathbb{R}} \mathbb{P}[X=i] \cdot i$

Linearity of Expectation: If $X = \sum_{i \in [n]} X_i$ (n finite), then $\mathbb{E}[X] = \sum_{i \in [n]} \mathbb{E}[X_i]$

Independent variables: 2 ~~random variables~~ events in a probability space A and B are independent if $Pr[A|B] = P[A]$ or $Pr[A \cap B] = P[A]P[B]$.

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be ~~events~~ a collection of events in a probability space. Then:

- We say the events in $\mathcal{A}$ are pairwise independent if $\forall i \neq j$ in $\mathcal{A}$ A_i, A_j are independent
- We say the events in $\mathcal{A}$ are mutually independent if $\forall i$, A_i is mutually independent of $\mathcal{A} \setminus \{A_i\}$, i.e. $\forall \mathcal{A}' \subseteq \mathcal{A} \setminus \{A_i\}$ $Pr[A_i \mid \bigcap_{A_j \in \mathcal{A}'} A_j] = Pr[A_i]$

Question: If $\mathcal{A}$ is pairwise independent then is it mutually independent?

A: No! Example: $n \geq 3$ coins and ~~the event~~ space of flipping where # of heads is even and A_i is coin i is Heads. and $\mathcal{A} = \{A_1, \ldots, A_n\}$.

1

Lovász Local Lemma: → often referred to as "bad events"

If $A = \{A_1, \dots, A_n\}$ is a set of events in a probability space, and for every i, there is $B_i \subseteq [n] \setminus \{i\}$ such that A_i is mutually independent of $A \setminus (\{A_i\} \cup \{A_j : j \in B_i\})$.*) i.e. Bad events are not too dependent

AND $|B_i| \leq d$, $\forall i \in [n]$

AND $Pr(A_i) \leq p$, $\forall i \in [n]$ i.e Bad events are unlikely

AND $ep(d+1) \leq 1$ (also $4pd \leq 1$ works too!)

Then,

$$Pr[\bigcap_{i \in [n]} \bar{A}_i] > 0.$$

(i.e. with positive probability, none of the bad events occur)

*Wrong Version: If A_i is independent of at most ~~d~~ other events.

(Example: 1 is mutually independent of any $n-2$ sets of coins but not $n-1$ ⇒ No unique maximal set).

Remarks:

- Use the local lemma, say, if union bound fails (Recall that the union bound is: If $\sum_{i \in [n]} Pr[A_i] < 1$, then $Pr[\cap \bar{A}_i] > 0$)
- The "positive probability" is not very large, i.e. Usually $O(2^{-|A|})$, so could be exponentially small in the # of events.
- We use it to construct/force a good outcome.

Algorithmic Q: Can we find such a good outcome efficiently?

{Sampling is bad idea since this may take exponentially long} (b/c of low probability)

Jan 14th

Moser-Tardos (2000): There exists an algorithm to find a good outcome of the LLL in "the variable model" that runs in time $O(|A|)$.

→ Independent Trials

Every event depends on a bounded # of variables w/ bounded # of states and $B_i = \{A_j \in A \setminus \{A_i\} : A_j$ and A_i depend on at least one common variable$\}$.

Moser-Tardos Algorithm:
- Sample all the independent trials in the probability space
- WHILE $\exists$ a bad event A_i:
  - Resample, at random, all the trials that A_i depends on
- RETURN good outcome

(Some) Applications of LLL to colouring:

Hypergraph Colouring:
Recall: Hypergraph $H=(V,E)$: V - Set of vertices, E - Set of hyperedges (i.e. subsets of vertices)

H is k-uniform if $|e|=k$ $\forall e \in E(H)$.

(Note: $k=2$ gives us simple graphs)

Def'n A k-colouring of a hypergraph H is a partition of $V(H)$ into at most k independent sets of H.

A set $I \subseteq V(H)$ is independent if $\nexists e \in E(H)$ s.t. $e \subseteq I$

or equivalently $\exists\, \phi : V(H) \to [k]$ such that $\forall e \in E(H)$, $\exists v_1 \neq v_2 \sim e$ such that $\phi(v_1) \neq \phi(v_2)$ (i.e. No monochromatic edges)

The chromatic number $\chi(H)$ is the min k s.t. H is k-colorable

Q: How is $\chi(H)$ related to $\Delta(H) := \max_{v \in V(H)} d_H(v)$, (where $d_H(v) := |\{e \in E(H) : e \sim v\}|$). ?

Trivial Bound: $\chi(H) \leq \Delta(H) + 1$ (Greedy!)

Theorem: If H is a k-uniform hypergraph ($k \geq 2$), then

$$\chi(H) \leq \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$$

Proof:

We will use LLL.

Assign every vertex x of H a colour $\phi(x)$ from $[L]$, where $L = \lceil (ek\Delta(H))^{\frac{1}{k-1}} \rceil$.

Define bad event:

A_e = Edge e is monochromatic in ϕ.

$Pr(A_e) = 1/L^{k-1}$ (because color of 1st vertex can be anything, but after that, $\forall v \sim e$ needs a different color). $(= p)$

Let $B_e = \{f \neq e \in E(H) : f \sim e\}$, then by variable model, it follows that A_e is mut. ind. of $\mathcal{A} \setminus (A_e \cup B_e)$

$|B_e| \leq \underset{\text{at most } k \text{ vertices}}{k} (\underbrace{\Delta(H)}_{\text{max degree.}} - 1) (= d.) \leq k\Delta(H) - 1$

So: $ep(d+1) = e \cdot \frac{1}{L^{k-1}} k\Delta(H) = \frac{ek\Delta(H)}{((ek\Delta(H))^{\frac{1}{k-1}})^{k-1}} = 1.$

So, by LLL, $\exists$ a ϕ avoiding all A_e, i.e. a k-colouring of H ☒

Jan 14th

Color Degree:

Def'n: Let G be a graph and L list assignment of G. Let $|L|$ denote the min. size of a list, i.e. $|L| := \min_{v \in V(G)} |L(v)|$

We define:

The color-degree of a vertex $v \in V(G)$ ~~is~~ in color $c \in L(v)$ as:

$d_L(v,c) := |\{u \in N(v) : c \in L(u)\}|$

The color-degree of v, denoted $d_L(v) = \max_{c \in L(v)} d_L(v,c)$

The maximum color-degree of G w.r.t. L, ~~denoted~~ is:

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

Question: Does $\exists$ function f s.t. if $|L| \geq f(\Delta_L(G))$, then G has an L-colouring? (analogous to $|L| \geq \Delta(G)+1$, then G has an L-colouring)

Theorem (Alon '88 (with constant $\geq .5$), '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring

Theorem (Haxell '00 - follows from a more general thm)

If $|L| > 2\Delta_L(G)$, then G has an L-colouring

Theorem (Reed-Sudakov, 2002)

If $|L| \geq (1 + o(1))\Delta_L(G)$, then G has an L-colouring

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Lecture 4:

Review: (Colour Degree)

G graph, & L list assignment

$|L| := \min_{v \in V(G)} |L(v)|$

$d_L(v,c) := |\{v \in N(v) : c \in L(v)\}|$

$d_L(v) := \max_{c \in L(v)} d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

Theorem (Alon '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-colouring.

Proof:

(We may assume WLOG $|L(v)| = L \ \forall v$)

- Color every vertex v, $\phi(v)$ uniformly at ~~random~~ random from its list
- Bad events: $\underbrace{A_{e,c}}_{e=uv}$: $\phi(u) = \phi(v) = c$ $\forall c \in L(u) \cap L(v)$
- Then $Pr[A_{e,c}] = \frac{1}{L^2} = p$.
- $A_{e,c}$ is mutually independent of $A_{e',c'}$ whose $e' \nsim e$ (and $\neq$)
- $\underbrace{B_{e,c}}_{=uv} = \{\underbrace{A_{e',c'}}_{=u'v'} : e' \sim e \text{ or } e' = e \text{ and } c \in L(u') \cap L(v')\}$

$= 2 \cdot |L| \cdot \Delta_L(G) = d \cdot e$ ← Since we counted ourselves

↓ Pick an end (u or v) call this x

↓ Pick a colour c' in $L(x)$

→ Pick a neighbor w of x where $c \in L(w)$.

Proof (con't)

Then, by LLL:

Since $ep(d+1) = e\frac{1}{|L|^2}(2|L|\cdot\Delta_L(G))$

$$= \frac{2e\Delta_L(G)}{|L|} \leq 1.$$

Then $\exists$ a coloring ϕ avoiding all $A_{e,c}$, i.e. an L-coloring of G, as desired. $\blacksquare$

(Note: We can actually get rid of the "+1" in the statement of the theorem)

Theorem (Haxell)

If $|L| \geq 2\Delta_L(G)$, then G has an L-coloring

Actually follows from the following more general theorem:

Theorem (Haxell)

Let $k \geq 1$ be an integer. If $V_1, V_2, \ldots, V_r$ is a partition of $V(G)$ into independent sets for a graph G such that:

- $|V_i| \geq 2k$ $\forall i \in [r]$, and
- $\Delta(G) \leq k$, and

Then $\exists$ an independent set I of G s.t. $\forall i \in [r]$, $I \cap V_i \neq \emptyset$.
(Called an independent ~~trans set~~ transversal)

Go

How does it imply previous thm?
Let H be such that
$V(H) = \{(v,c) : v \in V(G),\ c \in L(v)\}$.
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c = c'\}$
Then H satisfies independent transversal theorem by assumption on G. So, by that theorem, $\exists$ an independent transversal, i.e. An L-coloring

Remark: Clearly also this implies correspondence version by letting
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c$ matched to c' in $M_{vv'}\}$.

~~Rest~~ Remark: This is tight! (i.e. The ratio of 2). For general independent transversals.

Conjecture (Reed, ~'90s)
If $|L| \geq \Delta_L(G) + 1$, then G has an L-coloring

But, Bohman and Holzman disproved this conjecture!
Still open if $|L| \geq \Delta_L(G) + 2$ works! (Has been open for about 20 years)

Theorem (Reed - Sudakov, '02)
If $|L| \geq (1 + o(1))\Delta_L(G)$, then G has an L-coloring
(Note: Equivalent to:
$\forall \varepsilon > 0, \exists \Delta_\varepsilon$ s.t. $\forall \Delta \geq \Delta_\varepsilon$, if $\Delta(G) = \Delta$ and $|L| \geq (1+\varepsilon)\Delta$, then G has an L-coloring).

Remark:

~~Reed~~ Loh and Sudakov proved that the ratio for independent transversals is $(1+o(1))$ assuming that every vertex has at most $o(k)$ neighbors in any other partitions..

Proof (Reed-Sudakov)

Note: Uses Rödl ~~Nibble~~ Nibble (aka semirandom) method of iteratively constructing a solution a little bit at a time. (In particular, uses the "Wasteful Coloring Procedure" to implement one such iterative procedure)

Note #2: Uses LLL and concentration inequalities to prove that there is a decent enough outcome of the WCP as to continue iterating, until the Big Finish.

What is our finish? - Alon's Thm (or Haxell's), i.e. can L-color if $|L| \geq 2e\Delta_L(G)$..

How will our iteration step be improving?: Progress will be in the ratio, $|L|/\Delta_L(G)$. In fact, in our proof, it will take some constant, depending on ε, # of steps.

Proof details:

Big assumption for now:

Let's assume $\forall c \in L(v)$ that $d_L(v,c) = d(v) = \Delta(G)$

(of course, we may assume wlog $|L(v)| = |L|$ $\forall v$)

Wasteful Colouring Procedure:

- Independently "activate" each vertex of G for some probability p to be fixed later.

(Remark (if the calculations are correct):

$$p = \frac{\varepsilon(1+\varepsilon)}{2}$$

works and any smaller $p \geq \text{polylog}(\Delta)$).

- Now colour every activated vertex v with a colour $\phi(v)$ selected uniformly at random from $L(v)$

This is the "wasteful" step!

- Remove $\phi(v)$ from the list of all v's neighbours. Uncolour any vertex which has the same colour as neighbour (or equiv. $\phi(v)$ is no longer in its list)

- Let ϕ' be the resulting colour, and L' be the resulting list

Remark: In the "naive colouring procedure" we only remove colours from neighbour's list if they keep the colour.

Here, we use WCP over NCP b/c
1) NCP is harder to analyze for concentrations, and
2) The probability that an active keeps its color will be close to 1, so in practice, much difference here

We let U ~~$\mathcal{E}$~~ be the set of vertices that do not receive a color from ϕ'

let $G' = G[U]$

Claim $L'(v) = L(v) \setminus \{\phi'(v) : v \in N(v) \cap A\}$ ← where A is activated (wasteful procedure)

$\setminus U$ would be for naive.

We'll prove that there is an outcome with

$$\frac{|L'|}{\Delta_{L'}(G')} > 1+\varepsilon \qquad \leftarrow \text{GOAL!}$$

(In fact, $(1+\varepsilon)(1+\varepsilon/4)$.)

Expectations:

- Lists: Want to calculate $E[|L'(v)|]$ for a given v

$$E[|L'(v)|] = \sum_{c \in L(v)} Pr[c \in L'(v)]$$

By linearity of expectation

$$= \sum_{c \in L(v)} \underbrace{Pr[\nexists u \in N(v) \cap A \text{ with } \phi(u) = c]}$$

$$\prod_{u \in N(v)} Pr[u \notin A \text{ or } \phi(u) \neq c]$$

See note on next pg. $u \in N_c(v)$

$$\prod_{u \in N(v)} (1 - Pr[u \in A \text{ and } \phi(u) = c])$$

But activation is independent of colouring:

$$\prod_{u \in N(v)} 1 - \Pr[u \in A] \cdot \Pr[\phi(u) = c]$$

$$= \prod_{u \in N(v)} \left(1 - p \cdot \frac{1}{|L|}\right).$$

$u \in N(v, c)$

In fact, since $\phi(u) = c$ only happens if $c \in L(u)$, we can write $u \in N(v, c)$.

$$= \left(1 - \frac{p}{|L|}\right)^{|N_L(v,c)|}$$

$$= \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)} \leftarrow \text{by color regularity}$$

$$\simeq e^{-\frac{p \Delta_L(G)}{|L|}}$$

So, the sum is:

$$\mathbb{E}[|L'(v)|] = e^{-\frac{p \Delta_L(G)}{|L|}} \cdot |L|.$$

So, note that when $p \approx 1$, the we- cut the list by around a $1/3$ each time. But, if p small, then we retain a decent chunk.

Color-degrees:

$$\underset{(\text{in } G')}{\mathbb{E}[d_{L'}(v, c)]} = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

(Remark: Do this whether or not $c \in L'(v)$)

④

$$\mathbb{E}[d_{L'}(v,c)] = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

$$= \sum_{u} \Pr\left[\begin{matrix}(u \in A \text{ and } c \in L'(u)) \\ \text{or } (u \in A \cap U \text{ and } c \in L'(u))\end{matrix}\right]$$

$$= \sum_{u} \Pr[u \in A] \Pr[c \in L'(u)]$$

$$+ \Pr[\quad]$$

$c = \phi(v)$: $\Pr[u \in A \cap U]$

$c \neq \phi(v)$: $\le \Pr[u \in A \cap U] \Pr[c \in L'(u)]$

Let's pretend that $c = \phi(v)$ does not matter:

$$\le \sum_{u} \left(1 - p\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right) \Pr[u \in A] \Pr[u \in U]$$

$$+ \Pr[u \in A \cap U] \cdot \frac{1}{|L|}$$

$$+ \left(p\left(1 - \underbrace{\Pr[\phi(v) \in L'(v)]}_{\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}}\right)\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right)$$

$= \Delta_L(G)$ times

Asymptotically the dist among cardinal is ?

Lecture 5:

Recall:
We were proving Reed-Sudakov, and had the following expected values:

$$\text{Keep} := \left(1-\frac{p}{2}\right)^{\Delta_L(G)} \approx e^{-\frac{p\Delta}{L}} \quad (= \Pr(c \in L'(v) \mid c \in L(v)))$$

$$\mathbb{E}[|L'(v)|] = \text{keep} \cdot |L|$$

$$\mathbb{E}[d_{L',G'}(v,c)] = d_{L,G}(v,c)\Big(\underbrace{(1-p)}_{\text{not activated}}\cdot\underbrace{\text{keep}}_{\text{keeps } c} + p\Big(\underbrace{\text{keep}\cdot\frac{1}{|L|}}_{\text{if } \phi(v)=c \text{ and keep } c} + \underbrace{\Big(1-\frac{1}{|L|}\Big)}_{\phi(v)\neq c \text{ and keep } c.}\underbrace{\Big(1-\frac{2}{|L|}\Big)^{\Delta}}_{\text{keeping } \phi(v) \text{ and } c}\Big)\Big)$$

But it turns out that this variable doesn't concentrate well, so we'll use another.

Picking a better variable:

$$d_{L',G'}(v,c) = |N_{L',G'}(v,c)| = |\{u \in \underbrace{N_{L,G}(v,c) \cap V(G')}_{G^2 \to G'} : \underbrace{c \in L'(u)}_{L^2 \to L'}\}|$$

But, it turns out we don't need to care about the lists, so instead let's look at $|N_{L,G}(v,c) \cap V(G')|$.
Clearly

$$d_{L',G'}(v,c) \leq |N_{L,G}(v,c) \cap V(G')|.$$

$$\mathbb{E}[|N_{L,G}(v,c) \cap V(G')|]$$

$$= \sum_{u \in N_{L,G}(v,c)} \Pr[u \in V(G')] \qquad \text{(lin. of exp.)}$$

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \notin V(G')] \qquad \text{These events independent.}$$

$$= \sum_{u \in N_{L,G}(v,c)} 1 - \Pr[u \in A \text{ and } \phi(u) \text{ is kept}] = d_{L,G}(v,c)(1-p\cdot\text{keep})$$

On to some iterative calculations:

- Now, if we could show that with "high enough" probability every $|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$ are close enough to their expectations to apply the LLL, then we'll be happy via the following calculations:

$$\frac{|L'|}{\Delta_{L'}(G)} \gtrsim \frac{\mathbb{E}[|L'(v)|]}{\mathbb{E}[|N_{L',G}(v,c) \cap V(G')|]} = \frac{\text{Keep} \cdot |L|}{\underbrace{(1-p\,\text{keep}) \cdot \Delta_L(G)}_{\text{old ratio}}}$$

Same ~~ratio~~ factor $\rightarrow$ we will argue this is close to 1.

$$\geq \frac{|L|}{\Delta_L(G)} \cdot \frac{1-\frac{p\Delta}{|L|}}{1-p\left(1-\frac{p\Delta}{|L|}\right)}$$

{ using that $\left(1+\frac{x}{n}\right)^n \geq 1+x$ for $n>1$ and $|x| \leq n$.

Note: $\frac{\Delta}{|L|} = \frac{1}{1+\varepsilon}$ (some assumption we made)

$$= \frac{|L|}{\Delta_L(G)} \cdot \underbrace{\frac{1-p/1+\varepsilon}{1-p(1-p/(1+\varepsilon))}}_{}.$$

$$= 1 + \frac{p\varepsilon}{1+\varepsilon} - \frac{\varepsilon^2}{1+\varepsilon}$$

Now if we choose $p = \varepsilon/2$, then we get

$$= 1 + \frac{\varepsilon^2}{4(1+\varepsilon)}$$

{ So, we get some improvement in each iteration.

Jan 21st

How to show a variable is close to its expectation:

Concentration Inequalities:

Markov's Inequality:

If $X \geq 0$ is a random variable,

$$\Pr[X \geq k E[X]] \leq 1/k. \quad \forall k \geq 1$$

Chebyshev's Inequality:

The variance of a random variable X, denoted $Var[X]$ is

$$Var[X] = E[(X - E[X])^2]$$

The standard deviation σ of X is $\sqrt{Var[X]}$.

Let $\mu = E[X]$, then

$$\Pr[X \geq \mu + k\sigma] \leq 1/k^2.$$

Proof: Use Markov's inequality.

Chernoff Bounds:

Let $X = \sum_{i=1}^{n} X_i$ where X_i is a Bernoulli r.v. (i.e. only takes value of 0 and 1). and all independent:

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}} \quad (0 \leq \delta \leq 1)$$

$$\Pr[X \geq (1+\delta)\mu] \leq \begin{cases} e^{-\frac{\delta^2 \mu}{3}} & \text{if } 0 \leq \delta \leq 1 \\ e^{-\frac{\delta \mu}{3}} & \text{if } \delta \geq 1 \end{cases}$$

(i.e. Exponentially small if δ constant and gives sth meaningful if $\delta \gg \frac{1}{\sqrt{\mu}}$).

Example:

$\mathbb{E}[|A \cap N_{L,G}(v,c)|] = p\Delta_L(G)$

$\Pr[|A \cap N_{L,G}(v,c)| \geq p\Delta_L(G)(1+\delta)] \leq e^{-\frac{\delta^2(p\Delta_L(G))}{3}}$ (By Chernoff).

i.e. $|A \cap N_{L,G}(v,c)| = p\Delta_L(G) \pm \sqrt{\Delta_L(G)\,\mathrm{polylog}\,\Delta_L(G)}$

with prob. $1 - \frac{1}{\Delta^c}$ for any c.

So, by LLL, this holds for all v with some positive prob.

But, we need to concentrate:

$|L'(v)|$ and $|N_{L,G}(v,c) \cap V(G')|$

Depends on color $c \in L(v)$ being in $L'(v)$ which depends on activation flips and color-assignments for $N_{L,G}(v,c)$ (But not independent!)

Even worse! (Stronger interactions with neighbors + 2nd neighbors)

"Simple" Concentration Bound (Book of Molloy & Reed).

Let X be a r.v. that depends only on the outcome of a set of independent trials $T_1, \dots, T_n$. Suppose that changing the outcome of any one trial changes X by at most c (c constant) (This is called c-Lipschitz).

Then,

$$\Pr[|X - \mathbb{E}[X]| \geq t + 8c\sqrt{\mathbb{E}[X]}] \leq e^{\frac{-t^2}{3c^2 n}}$$

Remark: Note that the denominator has the # of trials. (not exponential)

Jan 21st

Concentrating $|L'(v)|$:

Trials: activating flips & colourings of $N_{L,G}(v,c)$ $\forall c \in L(v)$

- 1-lipschitz: Since changing any one trial (activ. or colouring) changes $|L'(v)|$ by at most 1.

Recall: $E[|L'(v)|] = Keep \cdot \Delta_L(G) = \Theta(\Delta_L(G))$

What is r?

- If not doing colour degree, then 2Δ (Good!)
- But, w/ colour degree it's:

$2\Delta_L(G) \cdot |L|$ (If every color the neighbours holds are disjoint)

$= 2(1+\varepsilon)\Delta_L(G)^2$

(This is Bad! The simple concentration bound is only meaningful for $t > \frac{\Delta_L(G)}{\omega}$)

Talagrand's Inequality (versions)

(A) Combinatorial version: (There is also a probability one)

Let $X \geq 0$ depend on independent trials $T_1, \ldots, T_n$. If X is C-lipschitz and r-verifiable, then for any $t > 96\sqrt{rc^2 E[X]} + 128rc^2$

then:

$$Pr[|X - E[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4E[X]+t)}}$$

Remark: If r, c constant, then get ~~expo~~ exponentially small in t, if $t = \Theta(E[X])$ and still meaningful for $t \gg \sqrt{E[X]}$.

3

r-verifiable: For every $s>0$, if $X \geq s$, then there exists a set Z of at most rs trials that "verify" that $X \geq s$, i.e. Changing any trial outside of Z still results in $X \geq s$.

So, of course, counting # of heads in coin flips is 1-verifiable.
Similarly, $|A \cap N_{L,G}(\vec{x},c)|$ is 1-verifiable
(We just exhibit the heads or the activated set).

What about $|L'(\omega)|$?
Example: How do we show $|L'(\omega)| \geq 1$ (i.e. One color kept).
For all $u \in N_{L,G}(v,c)$ show either $u \notin A$ or $\Phi(u) \neq c$.
i.e. Need $\Delta_{L,G}(v,c)$ trials to verify
i.e. Need $r \geq \Delta_{L,G}(v,c)$ (This is bad.)

Pick a better variable:

$$|L(\omega)| - |L'(\omega)| = \#\text{ of colors lost.}$$

$$\mathbb{E}[|L(\omega)| - |L'(\omega)|] = |L| - \text{keep}|L|$$

$$= |L|(1-\text{keep})$$

$$= (1+\varepsilon)\Delta_L(G)(1-\text{keep}) = \Theta(\Delta_L(G))$$

Obviously, this variable is 1-Lipschitz.
What do we need to verify a color is lost?
$\to$ Need a neighbor $v \in N_{L,G}(v,c)$ to be activated and $\Phi(v) = c$
i.e. Need 2 trials
More generally, if $|L(\omega)| - |L'(\omega)| \geq s$, need $2s$, so $r=2$ works.

7409_Coloring Jan 23rd.

Lecture 6:

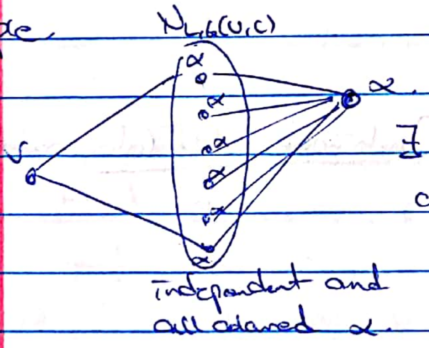
Concentrating One last variable for Reed-Sudakov:

$|N_{L,G}(v,c) \cap V(G')|$

$\mathbb{E}[\;\;] = \Delta_L(G)\cdot((1-p\cdot \text{keep})$

Can we concentrate this?
Is it r-verifiable for some r?
→ Note: $u \in N_{L,G}(v,c) \cap V(G')$ if either $u \notin A$ or $u \in A$ but $\exists$ $w \in N_{L,G}(u,\phi(w)) \cap A$ s.t. $\phi(u) = \phi(v)$
So, this is 4-verifiable! (at most 2 vertex flips and 2 colors)

But, is it c-Lipschitz for some c?
Not really: c could need to be at least $\Delta_L(G)$ as follows
For example,



$\exists$ exactly one neighbor (and it's common to all) of these colored α.
If α changes to β, then all the colors change.

Idea: However, it's unlikely for this to happen.
i.e. Unlikely that a vertex can change the outcome by $\Omega(\deg)$
(we'll prove this later)

So, if we could somehow use Talagrand's with this "likely" Lipschitz constant, we'd find.

$$\Pr(|\mathbb{E}[X]-X| > t + 60 r c^2 + 8\sqrt{rc^2\,\mathbb{E}[X]}) \le e^{-\frac{t^2}{8rc^2(t+\mathbb{E}[X])}}$$ } something like this.

$\mathbb{E}[X] \approx \Theta(\Delta)$ works for $t = \sqrt{\Delta}\,\text{polylog}\,\Delta$

1

So we'd get that Prob. $|L'(v)|$ or $|N_{L,G}(v,c) \cap V(G')|$ too far from $\mathbb{E}[X]$ is $\le \frac{1}{\Delta^c}$ for any c.

Argue each event is at most independent of all but a set of at most $(\Delta \cdot L)^2 \le \Delta^5$ events. and apply LLL to argue w/ pos. prob. that none happen

Then iterate and finish w/ Alon/Haxell when $L/\Delta \ge 2c$ or 2

□

Remark: This will only work for large enough Δ, b/c to apply the local lemma, we'd need $e^{-\frac{\log^2 \Delta}{8}} \le \frac{1}{\Delta^{10}}$. (and then extrapolate back through iterations, how large original Δ needs to be)

An Exceptional Outcome Version of Combinatorial Talagrand's Inequality.

Theorem:
Let $((\Omega_i, \Sigma_i, P_i))_{i=1}^n$ be probability spaces. Let (Ω, Σ, P) be their product space and $\Omega^* \subseteq \Omega$ be a set of "exceptional outcomes", and let $X: \Omega \to \mathbb{R}_{\ge 0}$ be a nonnegative random variable. Let $r, c \ge 0$. If X is (r,c)-certifiable w.r.t. Ω^*, then for any $t > 96c\sqrt{r\mathbb{E}[X]} + 128rc^2 + 8P[\Omega^*]\sup X$, then

$$\Pr[|X - \mathbb{E}[X]| > t] \le 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}} + 4P[\Omega^*]$$

Remark: We need to pay some costs:
- Needing $t \leq 8P[\Omega^*] \sup X$, and
- An extra $4P[\Omega^*]$ in the prob. bound.

Def'n ~~(r,c)-verifiable~~ (r,c)-certifiable)
- If $\omega = (\omega_1, \dots, \omega_n) \in \Omega$ and $s \geq 0$, an (r,c)-certificate (w.r.t. X, ω, s, Ω^*) is an index set $I \subseteq \{1, \dots, n\}$ of size at most rs s.t. $\forall k \geq 0$, we have $X(\omega') \geq s - kc$ $\forall \omega' = (\omega_1', \dots, \omega_n') \in \Omega \setminus \Omega^*$ s.t. $\omega_i \neq \omega_i'$ for at most k values of I
- If $\forall s \geq 0$ and $\omega \in \Omega \setminus \Omega^*$ s.t. $X(\omega) \geq s$, $\exists$ an (r,c)-certificate, then (r,c)-certifiable w.r.t. Ω^*

Remark:
- For $k=0$, the certificate just acts as an r-verifier for non-exceptional outcomes as in normal combinatorial Talagrand's
- What this really requires is an r-verifier for which, if you change at most k of its trials (and any # of the verifier trials), you lose at most kc.
  - $\Rightarrow$ This is kind of like changing any trial in a non-exceptional outcome changes at most c, but requires more generally changing k trials changes by at most kc for every $k \geq 0$ (Bonus: Can assume you end in a non-exceptional outcome)

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Back to concentrating $|N_{L,G}(v,c) \cap V(G')|$.

up to 2nd neighbourhood

Ω^* : $\exists w \in N^{\le 2}(v)$ (Note: At most $(\Delta L)^2$ of these if we delete edges b/w vertices w/ disjoint lists)

s.t. $\exists c' \in L(w)$ and $\ge$ polylog Δ vertices in $N_{L,G}(w,c')$ with colour c'

Actually can use $N_{L,G}(w,c) \cap N_{L,G}(v,c)$ instead)

Claim: $\Pr[\Omega^*] \le \frac{1}{\Delta^c}$ for any c and Δ large enough

(this will be enough to apply Talagrand's since $\mathrm{Exp}|N_{L,G}(v,c) \cap V(G')| \le \Delta$) ~~and only~~

This also assumes that we show it's (r,c)-certifiable for some r,c w.r.t. Ω^*.

Proof (that $|N_{L,G}(v,c) \cap V(G')|$ is (r,c)-certifiable.

So, let ω be a non-exceptional outcome. We need $\forall s > 0$ a set of at most rs trials to build certificate.

So, we use the $\le rs$ activation flip / colour assignments of vertices (and $\le$ neighbours) in $N_{L,G}(v,c) \cap V(G')$ (so, what we used before)

Need: $\forall k \ge 0$ changes to these trials, as well as any outside, that $|N_{L,G}(v,c) \cap V(G')| \le s - kc$

Can argue $k =$ polylog Δ works here b/c we start non-exceptional.

(b/c if value changes, had to change b/c of some critical change)

How to prove claim:

Note that $\Pr[\mathcal{R}^*] \underset{\text{by union bound}}{\le} \underbrace{|N^{\le 2}(v)|}_{\le \Delta^4} \Pr\left(\begin{array}{l} \exists c' \in L(w) \\ \text{and 2 polylog } \Delta \text{ in } N_{L_c}(w, c') \\ \text{Colored } c' \end{array} \;\middle|\; w \in N^{\le 2}(v) \right)$

Suffices to show $\le \frac{1}{\Delta^c}$ for any c (and large enough Δ).

Balls and Bins:

m balls and n bins. and uniformly at random assign each ball to some bin independently.

Expected # of balls in bin i: m/n.

Harder Q: What's the max # of balls in a ~~test~~ bin?

If $m = n$ with high probability the max prob is $O\left(\frac{\log n}{\log\log n}\right)$

→ Relating this back to colourings:

Vertices $\equiv$ Balls (in neighbourhood $(N_{L_c}(w, c'))$) and Colors $\equiv$ Bins (in $L(w)$)

(Note: If ball can't go to some bin, the probabilities are right, we can take this and can only get a worse bound)

We have $\Delta_L(G)$ Balls and $|L|$ Bins so $\Delta \simeq \Theta(|L|)$ and so no bin has $\ge$ polylog Δ balls with high probability

③

Balls and Bins - Bounds:

Cm $= \Delta$, $n = \lceil L \rceil$

Upper bounds: By Union bound, $\Pr[\exists \text{ bin w/} \geq k \text{ balls}]$

$\leq \Delta \cdot \Pr[\text{bin } i \text{ has} \geq k \text{ ball}]$

$$\Pr[\text{Bin } i \text{ has} \geq k \text{ balls}] \leq \binom{\Delta}{k} \cdot \frac{1}{L^k}$$

$$\approx \left(\frac{\Delta e}{k}\right)^k \cdot \frac{1}{L^k} = \left(\frac{\Delta e}{kL}\right)^k$$

So, we would need: $k \log k > c \log \Delta$, i.e. $k \geq \frac{\log \Delta}{\log\log \Delta}$

~~some constants~~.

to find $\leq \frac{1}{\Delta^c}$ for any c.

Lower Bound:

$$\Pr[\text{Bin } i = k] = \binom{\Delta}{k} \frac{1}{L^k} \left(1 - \frac{1}{L}\right)^{\Delta - k}$$

$$\mathbb{E}\left[\begin{matrix}\#\text{ of bins w/} \\ \text{exactly } k\end{matrix}\right] = \Delta \cdot \Pr[\text{Bin } i = k]$$

$$\approx \Delta \cdot \left(\frac{\Delta e}{kL}\right)^k \cdot e^{-\frac{\Delta k}{L}}$$

By fiddling w/ constant for k

Can get $\geq \frac{1}{\Delta}$, and so $\mathbb{E} \geq 1$,

(Use Tolegrand's w/ $c = 2$, $r = k$)

Same for upper bound (can use $r = 1$, $c = 1$).

Lecture 7:

Back to our assumption that all color degrees are the same:

For Reed - Sudakov:

- Keeping a color is more likely if smaller color degree
- So $E[|L'(v)|] \geq$ Expectation when regular
- $E[d_{L',G'}(v,c)] \leq d_{L,G}(v,c)(1-p\,keep) \leq$ ____ $(1-p\,keep)$

So keep changes with the list. → ~~color~~ ~~assigned~~ $keep(u,\phi(uw)) = (1-\frac{p}{|L|})^{d_{L,G}(u,\phi(w))}$ min. keep, i.e. old #. (but we can always extend every list)

Even if it wasn't the case that expectations were only better for us in the non-regular case:

(1) Regularization: Embed our graph into a regularized version where ~~the~~ coloring the resulting graph yields a coloring of the original.

(2) Equalizing Coin Flips: Here, for every $u \in V(G)$ and $c \in L(u)$, we add a coin flip $F_{u,c}$ which keeps c for u with probability $keep/keep(u,c)$ and hence every color is kept with probability ~~keep/F~~ $keep(u,c) \cdot F_{u,c} = keep$.

Note: (2) only works if ordered ~~coin~~ coin flips have probability ≤ 1 AND we would have to redo all the concentrations, adding the coin flips into verifications / Lipschitz.

Regularization:

Lemma: Every graph G is an induced subgraph of a $\Delta(G)$-regular graph.

→ Proof Sketch

①

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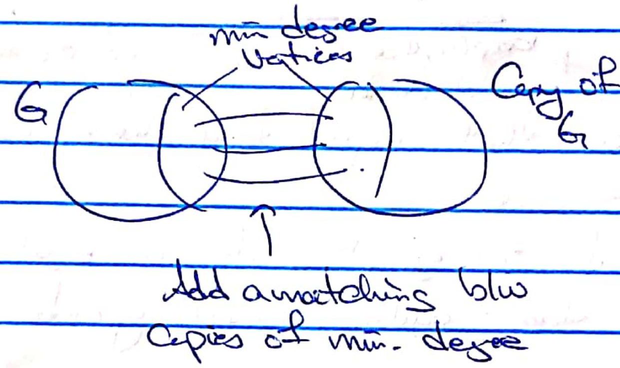
Proof Sketch:

By induction on $\Delta(G) - \delta(G)$ (min degree) (max degree)

If $\Delta(G) = \delta(G)$, then G is $\Delta(G)$-regular, as desired.
So, we may assume $\delta(G) < \Delta(G)$.

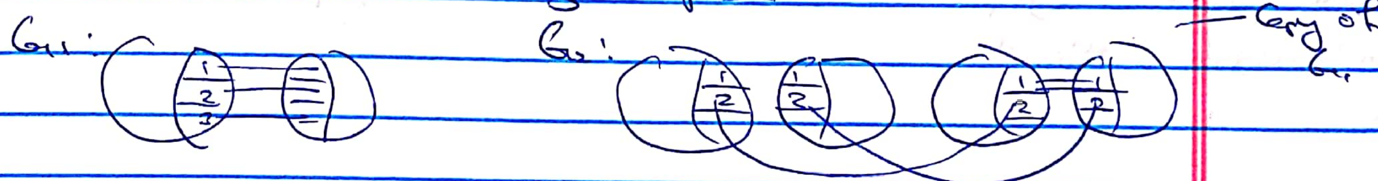
Define:

$G' =$



Now $\delta(G') > \delta(G)$ and hence by IH, $\exists\, G$ $\Delta(G)$-regular containing G' as an induced subgraph. □

Note: Construction actually gives if G is triangle-free, then G' is triangle-free.
However, if G has girth ≥ 5, G' may have 4-cycles.
We may tweak the construction to preserve girth 5 (and girth 6, but not ≥ 7)
First, find a $(\Delta+1)$-coloring of G. Only add a matching b/w color i of min-deg. copies:



So, do $\Delta+1$ doublings to up min. deg. by 1. Start over...

For Reed-Sudakov, we need to regularize color-degree, not degree:

(1) If we pass to correspondence coloring, we can use the construction above, but we only add a conflict b/w min-deg. colors b/w copies of vertices

(2) If we're more careful with making lists for the copies, this should work for list-coloring as well.

Kim's Theorem for Girth Five Graphs:

Theorem (Kim, 1995)
If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\ln \Delta}$.

Recall: This is tight up to constant factor, in particular, in random d-regular graphs have $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$ with high probability, and so $\exists$ d-regular graphs of arbitrary girths and $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$.

(Some naive) Intuition:
- Coupon Collector Problem: Suppose there are n types of coupons and when you receive a coupon you receive a type u.a.r.
- Q: How many coupons do you need to get to collect all the types? $\Theta(n \log n)$

Proof Sketch:

In the first n coupons,

$\Pr[\text{obtained} \geq 1 \text{ of coupon } i]$

$= 1 - \Pr[0 \text{ of coupon } i]$

$= 1 - (1 - \frac{1}{n})^n$

$\approx 1 - 1/e.$

$E[\text{types collected after } n] \approx n(1 - \frac{1}{e})$

$E[\text{still to collect}] \approx n/e.$

$E[\text{types still to collect after } n \cdot t \text{ coupons}] = n/e^t.$

Use concentration inequalities to make these concentration around expectation w.h.p. $\square$

So, we could think that:

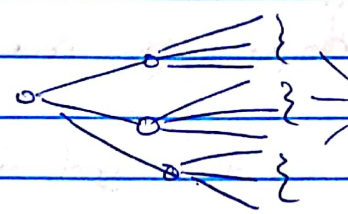
types of coupons = colors in my list $(n) = L$

coupon samples = colors neighbors receive $(n \ln n) = \Delta.$

(Solve to find $L \approx \frac{\Delta}{\ln \Delta}$).

But, coloring (randomly) is not uniformly random,

$\rightarrow$ there's where the girth 5-ness can help since for every vertex we see:



All disjoint.

This is "somewhat uniformly random"

Proof (of Kim's)
We actually prove a stronger theorem (as Kim did):
If G has girth ≥ 5 and L is a list assignment of G such that $|L| \geq (1+o(1)) \frac{\Delta_L(G)}{\ln \Delta_L(G)}$, then G

has an L-colouring.
i.e. He proved this theorem for colour-degree.

We'll use Nibble and the Wasteful Colouring Procedure.
We may assume all colour degrees and list sizes are the same by Regularization / Equalizing Coin Flips.

We will be interested in tracking the ratio: $\frac{\Delta_L(G)}{|L|}$.
(Reciprocal of what we used for Reed-Sudakov)
(Note: This starts as $\approx \ln \Delta_L(G)$).
We will show that we can (after many steps) reduce to the graph with ratio $\leq \frac{1}{20}$ (so $|L| \geq 20\Delta_L(G)$).
and then we apply Alon/Haxell to finish.

Expectations for one step of WCP:
$\Pr[c \in L'(v) \mid c \in L(v)] = \left(1-\frac{p}{|L|}\right)^{\Delta(G)} =: \text{keep} \approx e^{-\frac{p\Delta}{|L|}}$.

$\mathbb{E}[|L'(v)|] = |L| \cdot \text{keep}$.

$$\mathbb{E}[d_{L',G'}(v,c)] \leq d_{L,G}(v,c) \cdot \Big(\underbrace{\text{keep}}_{\text{keeping } c} \cdot \underbrace{\Big(\tfrac{1}{|L|}(1-p)}_{\text{if } c = \phi(u), \text{ where } v \in N_{L,G}(u,c)} + \Big(1-\tfrac{1}{|L|}\Big)\underbrace{(1-p\,\text{keep})}_{\text{keeping } \phi(u) \neq c}\Big)\Big)$$

$$\leq d_{L,G}(v,c)\,\text{keep}\,(1-p\,\text{keep})$$
$$= \mathbb{E}\,\Delta_L(G).$$

where the underbraced term $= (1-p\,\text{keep}) + \frac{1}{|L|}\underbrace{p(\text{keep}-1)}_{\leq 0}$.

Scanned with CamScanner

Remark:

In Reed-Sudakov, we threw out the keep in $\mathbb{E}[d_{L,G}(v,c)]$ for analysis purposes to get the L/Δ ratio improving by keep/1-keep which was >1, since $|L|>\Delta$.

But, for Kim's, we'll need to keep the keep in $\mathbb{E}[d_{L,G}(v,c)]$ which complicates concentration analysis, but everything will still concentrate b/c girth ≥ 5.

Concentrations:

- Show $\Pr\left[|L'(v)| - \underbrace{\mathbb{E}[|L'(v)|]}_{=\text{keep}\cdot|L|}\right] \leq \frac{1}{\Delta^{10}}$

on $|L|-|L'(v)|$

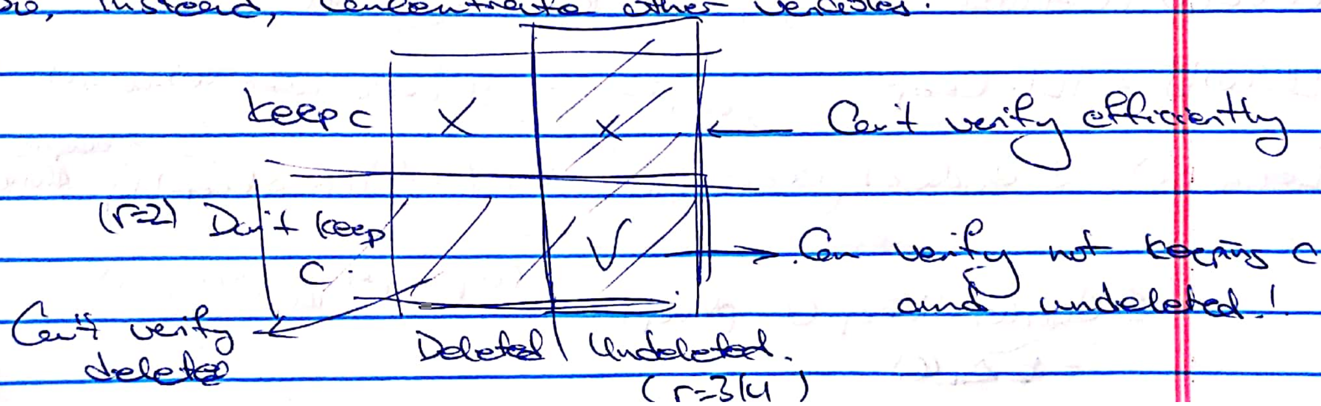
Proof same as for Reed-Sudakov, i.e. Use Talagrand's w/ $c=1$, $r=2$ (really 3 w/ activation flips)

- Show $\Pr\left[d_{L,G}(v,c) - \mathbb{E}[d_{L,G}(v,c)]\right] \leq \frac{1}{\Delta^{10}}$.

Here, we can use Talagrand's (Exc. Talagrand) with $c=1$ (except for v itself, use exceptional Talagrand for v or other trick), and $r=?$ (Is it possible to verify # of vertices undeleted and keeping c in dist?)

↳ No, can't efficiently verify.

So, instead, concentrate other variables:



Jan 28th

Can verify all boxes indirectly here assuming all expectation of size of boxes is roughly same. ~~...~~ (i.e. The intersection shouldn't be small compared to the size of the set)

Iterating the ratio:
Apply LLL to find:
$|L'| \approx |L| \cdot \text{keep}$
$\Delta' \approx \Delta \cdot \text{keep}(1 - p\,\text{keep})$
So:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta \cdot \text{keep}(1-p\,\text{keep})}{|L| \cdot \text{keep}}$$

$$= \frac{\Delta}{|L|} - \frac{p\Delta}{|L|}\text{keep} \quad (\text{Recall: } \text{keep} \approx e^{-\frac{p\Delta}{|L|}}.)$$

So, let $k := \frac{p\Delta}{|L|}$, we get:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} - ke^{-k}$$, so for $k=1$, this gives $-1/e$.

Now run this forever to get the ratio ??
↓
Can't do this since we'd run out of colors.
(Since $|L'| \approx |L| \cdot \text{keep}$ and $\text{keep} \approx \frac{1}{e}$, so we'd have to stop at about ~~...~~ $\ln \Delta$ steps)

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Lecture 8:

Finishing the iteration calculations for Kim's

$|L'| = |L| \cdot \text{keep}$, where $\text{keep} \approx e^{-\frac{p\Delta}{|L|}}$

$\Delta' \approx \Delta \cdot \text{keep}(1-p\,\text{keep})$

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} \cdot \frac{\text{keep}(1-p\,\text{keep})}{\text{keep}} \qquad k = p\frac{\Delta}{|L|}$$

$$= \frac{\Delta}{|L|} - \frac{\Delta p}{|L|}\text{keep} = \frac{\Delta}{|L|} - ke^{-k}$$

How many iterations can we do (until we run out of colours?)

Let $|L_\tau|$ be the size of L after τ iterations:

$$|L_\tau| = |L_0| \cdot \text{keep}^\tau \approx |L_0| \cdot e^{-k\tau}$$

We need $|L_\tau| \geq 1 \Rightarrow \ln|L_0| - k\tau \geq \ln 1 = 0$

$$\Rightarrow \tau \leq \frac{\ln|L_0|}{k}$$

Remark.

We actually stop when $|L_\tau| \geq$ some fixed constant to ensure that the necessary inequalities hold for the LLL in every iterative step. This only means that $\tau \leq \frac{\ln|L_0|}{k} - c$ for some constant c (so this is fine)

To finish the colouring, we need $\frac{\Delta_\tau}{|L_\tau|} \leq \frac{1}{2e}$ to apply Alon/Haxell.

Note: $\frac{\Delta_0}{|L_0|} \approx \frac{\Delta_\tau}{|L_\tau|} + ke^{-k}\tau$. So this works if

$$\frac{\Delta_0}{|L_0|} \lessapprox \frac{1}{2e} + ke^{-k}\left(\frac{\ln|L_0|}{k} - c\right)$$

$$= \text{constant} + e^{-k}\ln(L_0)$$

$\Leftrightarrow$

1

i.e. $\Delta_0 \le e^{-k}|L_0|\ln|L_0|$

$\Rightarrow |L_0| \approx \frac{e^k\Delta_0}{\ln(e^k\Delta_0)} \le \frac{e^k\Delta_0}{\ln\Delta_0}$

Recall that $k = p\frac{\Delta}{|L|}$, so when $k=1$, $p=\frac{|L|}{\Delta}$ and $k\to 0$ means $p\to 0$.

So, if $k=1$, we get $|L_0| \approx e\frac{\Delta_0}{\ln\Delta_0}$,

and if instead, we let $k\to 0$, we get $(1+o(1))\frac{\Delta_0}{\ln\Delta_0}$, as desired □

Remark: Why were colour degree necessary for the proof?
Note that in $\Delta' \approx \Delta\,\text{keep}(1-p\,\text{keep})$, the first keep only comes up in colour degree. And this keep helps cancel out some values and drive concentrations down.

Edge-Colouring:

Interested in properly colouring edges of graphs, i.e. ~~$\chi(G)$~~ $\chi(L(G))$ (the line graph)

Greedy Colouring: $\Delta(L(G)) \le 2\Delta(G)-2$

$\Rightarrow \chi(L(G)) \le 2\Delta(G)-1$

(of course $\omega(L(G)) = \Delta(G)$, if $\Delta(G)\ge 3$)

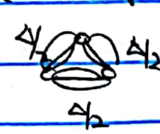
Theorem (Vizing, indep. Gupta, 1960s)

$\chi(L(G)) \le \Delta(G)+1$.

NP-hard to decide if $\chi(L(G)) = \Delta$ or $\Delta+1$

Also, if G is a multigraph

$\chi(L(G)) \le \Delta+\mu$ (where μ is the max. multiplicity of an edge)

$\le \lceil \frac{3}{2}\Delta \rceil$ (Shannon) $\rightarrow$ tight for 

Goldberg - Seymour Conjecture: (Independently in late 70's)

$\chi(L(G)) \leq \max\{\Delta(G)+1, \chi_f(L(G))\}$

$\uparrow$ fractional chromatic #.

Actually, via Edmond's matching polytope:

$$\chi_f(L(G)) = \max_{\substack{H \subseteq G \\ v(H) \text{ odd}}} \frac{e(H)}{\left(\frac{v(H)-1}{2}\right)} \Rightarrow \text{ie Defn of Hall ratio for line graphs.}$$

Known Results:

- Kahn (90's) proved ~~#~~ ~~holds~~ $\chi(L(G)) \leq \max\{\Delta+1, (1+o(1))\chi_f(L(G))\}$.
- Two groups in 2008: If $\chi_f(L(G)) \geq \Delta + \sqrt{\Delta}$, then $\chi(L(G)) = \chi_f(L(G))$.
- Another group in 2017+: —— " —— $\geq \Delta + \sqrt[3]{\Delta}$, —— " ——.
- Plantholdt, '11: $\chi(L(G)) \leq \chi_f(L(G)) + \log \chi_f(L(G))$

List - Colouring Conjecture:

If G is a simple graph, then

$$\chi_\ell(L(G)) = \chi(L(G)).$$

(i.e. There is no such thing as list-edge-colouring)

Two big results on LCC from 90's:

Theorem (Galvin, 1995)

If G is bipartite, then $\chi_\ell(L(G)) = \chi(L(G))$.

→ Proved Dinitz's conjecture: If for every square in an $n \times n$ matrix, I give you a list of integers, can you complete the matrix so that all entries in a row (or col) are distinct

(Equivalently, $\chi_\ell(K_{n,n}) = n$).

→ Related to Latin Squares, since the completion of a Latin Square is simply an n-edge-colouring of $K_{n,n}$.

②

And these exist by König's theorem, if G is bip, $\chi_\ell(L(G)) = \Delta(G)$.

Theorem (Kahn, 1996)
If G is simple, $\chi_\ell(L(G)) = (1+o(1))\chi(L(G))$
i.e. $= (1+o(1))\Delta(G)$.

Molloy + Reid (Early 2000s) improved error to $\Delta + 4\sqrt{\Delta}\log^4\Delta$.
Kahn extended this to k-uniform linear hypergraphs
(Hyperedges ~~ex~~ intersect in ≤ 1 vertex)

Proof Sketch (of Kahn's)
We'll use the Naive Colouring Procedure (i.e. Only remove colours from neighbours if retained by a vertex) and Nibble.
Also, since $|L|$ is on the order of Δ, we won't need activation probabilities (i.e. Set $p=1$)

We in fact prove a colour-degree version of Kahn's theorem as follows:
If L is a list assignment for $E(G)$ (equiv. $V(L(G))$), we define for $v \in V(G)$, a colour $c \in L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$

$d_L(v,c) := |\underbrace{\{e \in E(G) : e \ni v,\ c \in L(e)\}}_{N_{L,G}(v,c)}|$

$d_L(v) := \max_c d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

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So, restating the thm:

Stronger Kahn's Theorem (Kahn, 1996)
If L is a list assignment for $E(G)$ s.t.
$$|L| \geq (1+o(1))\Delta_L(G)$$
then G has an (edge) L-coloring.

~~(Note: For notation, let $L(v) := \bigcup_{e \ni v} L(e)$ for $v \in V(G)$).~~

Clearly, still interested in $|L|/\Delta_L(G)$

Expectations:
(Note: Assume all color degrees and list sizes are regular via equalizing coin flips (since regularization of line graphs seems odd)).

i.e. It was deleted since no inc. edge received $\phi(e)$

Retain := Pr[an edge e is not in G']
$$= \left(1-\frac{1}{|L|}\right)^{d_L(e)} = \left(1-\frac{1}{|L|}\right)^{2\Delta-2} \approx e^{-2\frac{\Delta}{|L|}} \approx e^{-2}$$

Vertex-keep := Pr[$c \in L(v)$ is not retained by any edge around vertex v]
$$\approx 1-\frac{1}{e^2}$$

Edge-keep := Pr[$c \in L(e^{uv})$ retained]
$\approx$ Pr[c not retained around u] $\times$ Pr[c — " — v]

b/c mostly independent $= (\text{Vertex-keep})^2$

③

Then,

$$\mathbb{E}[|L'(e)|] \approx |L|\left(1-\frac{1}{e^2}\right)^2$$

Colour-Degree:

$$\mathbb{E}[|N^{edge}_{LG}(v,c) \cap E(G')|] = |N^{edge}_{LG}(v,c)| \cdot (1-\text{Retain})$$
$$= \Delta \cdot \left(1-\frac{1}{e^2}\right)$$

But, how many of these e keep $c \in L'(e)$?

It's not Edge-keep, ~~since~~ if we assume $c \in L'(v)$: b/c then by assumption ~~we~~ no edge around v retains c

Under this:

$$\Pr[e \text{ keeps } c] = \Pr[\text{no edge} \longrightarrow] \times.$$

Lecture 9:

Molloy, Reed - Book
Proof - Kahn.

Finishing Kahn's proof:

Retain: $= \left(1-\frac{1}{|L|}\right)^{2\Delta-2} \approx e^{-2}$.

Vertex-keep: $= 1 -$ Retain $\approx 1 - \frac{1}{e^2}$.

Edge-keep $\approx$ (Vertex-keep)^2 $\approx \left(1-\frac{1}{e^2}\right)^2$ (Can argue that the dependence is small)
mostly independent

$\mathbb{E}[|L'(e)|] \approx |L| \cdot \left(1-\frac{1}{e^2}\right)^2$

$\mathbb{E}[\Delta_{L',G'}(v,c)] \approx |\Delta| \left(1-\frac{1}{e^2}\right)^2$

Approximates since there are small error terms.

Even assuming the errors from ~~expt~~ expectations won't hurt us too badly & that we concentrate these variables so as to have little error from concentration, unfortunately, at best these decrease at roughly the same rate - and some won't make progress.

Need a new finish!

Reserve Colours:

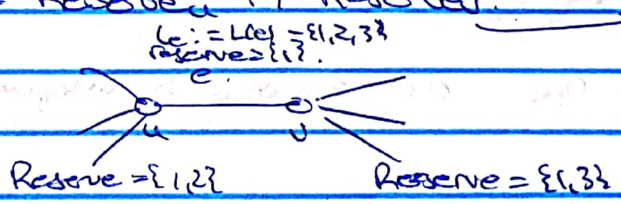
Idea: Before using nibble and naive colouring procedure, we reserve colours around ~~and~~ vertices to be used by edges in ~~during the~~ a "final step".

Choose Reserve$_v \subseteq L(v)$: $\bigcup_{e \ni v} L(e)$ (uniformly at random w/ prob p)

For each edge $e = uv$, define

$L_e' := L(e) - (\text{Reserve}_u \cup \text{Reserve}_v)$ → Lose colours from list (reserved)

$\text{Reserve}_e := \text{Reserve}_u \cap \text{Reserve}_v$ → Now e can use the overlapping colours safely



Lemma:

(Setting $p = \frac{\log^4 \Delta}{\sqrt{\Delta}}$) $\exists$ a choice of $Reserve_v$ for $V(G)$ such that $\forall u \in V(G)$, $\forall e \in E(G)$, $\forall c \in Reserve_v$:

(a) $|L(e) \cap (Reserve_u \cup Reserve_v)| \leq 3\sqrt{\Delta}\log^4\Delta \to L(e) - L_e$

(b) $|\underbrace{L(e)}_{L_e} \cap (Reserve_u \cap Reserve_v)| \geq \frac{1}{2}\log^8\Delta \to Reserve$

(c) $|\{e = uw \in N_{L_G}(v, c) : c \in Reserve_w\}| \leq 2\sqrt{\Delta}\log^4\Delta$ (*)

Intuition:

(a) You don't lose too many colors

(b) There ~~one~~ is ~~many~~ at least one color to use near the end

Proof:

$\mathbb{E}[|L(e) \cap (Reserve_u \cup Reserve_v)|]$

$\leq |L(e)| \cdot 2p \leftarrow$ Since ~~...~~ may appear in either $Reserve_u$ or $Reserve_v$.

$\leq \Delta \cdot 2\frac{\log^4\Delta}{\sqrt{\Delta}}$

$= 2\sqrt{\Delta}\log^4\Delta$.

$\mathbb{E}[|L(e) \cap (Reserve_u \cap Reserve_v)|]$

$\leq |L(e)| \cdot p^2 \leftarrow$ Needs to appear in both $Reserve_u$ and $Reserve_v$.

$\geq \Delta \frac{\log^8\Delta}{\Delta} = \log^8\Delta$.

$\mathbb{E}[(*)] = \Delta \cdot p \leq \sqrt{\Delta}\log^4\Delta$

Since each color $c \in Reserve_w$ is independent

Feb 4th

You'll notice these variables are all sums of $\{0,1\}$-random variables (even more, all Bernoulli), so we can apply Chernoff Bounds:

$$\Pr[\,|L(e) \cap (\text{Reserve}_u \cup \text{Reserve}_v)| \geq \tfrac{3}{2}\mathbb{E}[\;]\,] \leq e^{-\frac{\Delta \log^3\Delta}{3/2}}$$

$$\Pr[\qquad \cancel{\text{and}}\ (b) \qquad] \leq e^{-\frac{(\log^8\Delta)^2}{3/2}}$$

$$\Pr[\qquad (c). \qquad] \leq e^{-\frac{\Delta \log^3\Delta}{3}}$$

So all at most $e^{-\log^8\Delta}$ for large enough Δ which is $\leq \frac{1}{\Delta^c}$ for any c and large enough Δ.

So, we apply LLL with following bad events:

$A_e = (a) \geq 3\sqrt{\Delta}\log^4\Delta$

$B_e = (b) \leq \frac{1}{2}\log^8\Delta$

$C_{v,c} = (c) \geq 2\sqrt{\Delta}\log^4\Delta$.

Probability for any bad event $\leq \frac{1}{\Delta^c}$ for any c, while each event is mutually independent of the set of events depending on edges/vertices at distance ≥ 4.

So: using $c = 5$ (or 6) suffices for LLL ☒

What is (c) saying?: If $c \in \text{Reserve}_v$, this counts $e \in N_{v,c}(v,c)$ with $c \in \text{Reserve}_e$

We define now the reserve degree of a vertex:

$$d_{res}(v,c) = |\{e \in N_{v,c}(v,c) : c \in \text{Reserve}_e\}|.$$

Idea for finishing w/ Reserved Colours:
Reserved degrees decreases rather quickly because edges in "reserved neighborhood" will be deleted from G, however Reserve never changes during Nibble.
So, if we can get $\max d_{res}(v,c) \le \frac{1}{2}|\text{Reserve}| = \frac{1}{4}\log^{8}\Delta$, we can ~~st~~ finish by applying ~~Haxell~~ Haxell to reserve colour assignment

---

Expectation of new reserved degree in one step of Nibble:

$\mathbb{E}[d_{res,L',G'}(v,c)]$

$= d_{res,L,G}(v,c) \cdot (\underbrace{1 - \text{Retain}}_{\text{Probability an edge not deleted}})$

$\approx d_{res,L,G}(v,c) \cdot (1 - \frac{1}{e^2})$

Decreases at half the rate multiplicatively compared to $|L'|$ and Δ'

We can run Nibble for ~~t~~ T iterations, where T is about the solution to the following:

$1 = |L| \cdot ((1-\text{Retain})^2)^T = |L|(\underbrace{(1-\frac{1}{e^2})^2}_{=k})^T$

$\Rightarrow T = \frac{\ln|L|}{-\ln k}$

(So a roughly logarithmic amount of steps)

Yet,

$$d_{res,L_i^*,G_i}(v,c) \approx d_{res,L,G}(v,c)\cdot\left((1-\tfrac{1}{e^2})\right)^{7}$$
$$= d_{res,L,G}(v,c)\cdot K^{7/2}$$
$$\le 2\sqrt{\Delta}\log^4\Delta\cdot\frac{1}{\sqrt{|L|}}$$
$$\le 2\log^4\Delta$$
$$\le 1/4\log^8\Delta \quad \text{for large enough } \Delta'$$

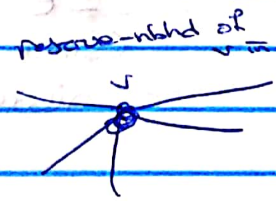
(So we don't have to run too many iterations for the reserved degree to be small enough)

Remark: We actually want to use $7 = \frac{\ln|L'|}{-\ln K} - \text{poly}\log\log\Delta$ since we need to concentrate d_{res} during while enough to apply LLL, so need $d_{res} \ge \text{poly}\log\Delta$

So, this gives an L-coloring if $|L| \ge \Delta + 4(\log^4\Delta)\sqrt{\Delta}$.
(Since we need $3\sqrt{\Delta}\log^4\Delta$ for reserves and an extra $\sqrt{\Delta}\,\text{poly}\log\Delta$ for concentration errors)

Concentration for $|L'|$, Δ', d'_{res}:

We'll do d'_{res} first!
d'_{res} counts # edges from d_{res} not deleted.



(3)

C-Lipschitz for c = ______:

- If we change colouring on an edge not incident with v, this changes ≤ 2 of these edges
- If we change colouring incident with v, this can change at most itself and 1 other edge (they both lose the color). Otherwise, ~~there~~ if there are ≥3 edges ~~x~~ coloured the same color, then changing v won't matter.

Note: We can't use Chernoff b/c not sum of independent
—— " —— Simple conc. bound b/c may depend on Δ^2 trials (⇒ Bad Bound)

So: we'll use Talagrand's and we need to verify this variable

Note: Verifying ~~x~~ retaining a colour requires ~~2Δ~~ 2Δ trials.

But we can verify not retaining using r=2 by showing its color $\phi(e)$ and some edge f near w/ $\gamma(f)=\phi(e)$.
So, we can apply Talagrand to show this is within about $\sqrt{\mathbb{E}[\deg]}$ with probability $\underbrace{e^{-C\mathbb{E}[\deg]}}_{\le e^{-c\log^8\Delta}}$ (C constant)

(since $\deg \geq \log^8\Delta$ during nibble)

Concentrating Δ':

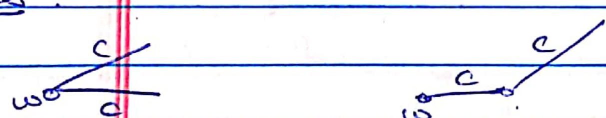
Assume c is not retained around v, this counts # of edges in $X_{L,G}(v,c)$ which are not deleted and the other end of the edge keeps c.

Again, we are 2-Lipschitz, since changing a single color can affect at most 1 other vertex. (If we have ≥ 3, then changing the color won't matter) (Same ~~argu~~ argument as before)

As before, not deleted is 2-verifiable, so we'd be happy if "other end keeps c" or its complement were r-verifiable for constant r since the expectation of intersection of these are on the same order as Δ.
(Note: As before, verifying edge around w retains c takes around 2Δ trials)

What about verifying the complement:
i.e. # of edges e where other end w does not have an edge retaining c?

Cases:



if 2 colored c, this is easy, show 2

To verify this one, we need to check all neighbors!

Trick: Change Variables

Define

~~X_a~~ $X_{j,k}$ to be # of ~~vertices~~ edges e where around other end w

$\geq j$ are assigned color c

$\geq k$ have color c subsequently removed.

So we can write the variable above (the component count) as: $X_{1,0} - \underbrace{(X_{1,1} - X_{2,1})}_{\text{counts receive exactly 1 and remove } \geq 1}$

Easy to check that $X_{j,k}$ concentrates w/ 2-Lipschitz and $(j+k)$-verifiable and we can use Talagrand's on each. This works since they have roughly same expectation and j, k constant.

For 2.1', we use the same trick, but ~~counting~~ counting both ends of e.

References