Recall:

We finished proving the key technical lemma (Bollobás - Thomason):

Let $d \geq 0$, $k \geq 2$, $l \geq d + 3k/2$ be integers. Let G be a graph with $\kappa(G)$ (connectivity) $\geq k$ containing vertex-disjoint, non-empty, connected subgraphs $C_1, \dots, C_l$ such that each of them is non-adj. to at most d others. Let $S = \{s_1, \dots, s_k\} \subseteq V(G)$. Then G contains vertex-disjoint, non-empty, connected subgraphs $D_1, \dots, D_m$ where $m = l - \lfloor k/2 \rfloor$ such that

- $\forall i \in [k]$, $s_i \in D_i$
- $\forall i \in [m]$, D_i is non-adj. to at most d of $\{D_1, \dots, D_m\}$.

(We technically proved a stronger version - that we avoid S-cut, but we'll only need this statement)

Def'n G is (k,n)-knit if $1 \leq n \leq k \leq v(G)$ and $\forall S \subseteq V(G)$, $|S| = k$ and partition $S_1, \dots, S_t$ (non-empty) of S, $t \geq n$, then $\exists$ vertex-disjoint, connected subgraphs $F_1, \dots, F_t$ s.t. $S_i \subseteq V(F_i)$ $\forall i \in [t]$

And, $(2k, k)$-knit $\Rightarrow$ k-linked.

Theorem: Let G be a graph with $\kappa(G) \geq k$ such that G has a minor H with $2\delta(H) > v(H) + \lfloor \frac{5k}{2} \rfloor - 2 - n$, then G is (k,n)-knit. In particular, if $\kappa(G) \geq 2k'$ and $2\delta(H) \geq v(H) + 4k' - 2$, then G is k-linked.

Proof:
Let the bags in the model of H in G be the C_i's. Hence let $l = v(H)$, $d := v(H) - 1 - \delta(H)$. Let $S = \{S_1, \dots, S_k\}$, then $l = 2d + 2v(H) + 2\delta(H) \geq 2d + \lfloor \frac{5k}{2} \rfloor - n$, ~~and the~~ conditions to apply key technical lemma.
So, $\exists D_1, \dots, D_m$, $\forall i \in [k]$, $s_i \in D_i$ and $\forall i \in [m]$, non-adj $\leq d$ of $D_{k+1}, \dots, D_m$.

Obs: $\forall$ pair D_i, D_j $(i, j \in [m])$ has at least $m - k - 2d \geq l - \lfloor \frac{3k}{2} \rfloor - 2d \geq k - n$ subgraphs in $D_{k+1}, \dots, D_m$ to which they are both adj. Now greedily connect S_1, then S_2, etc., to S_t by ordering each $S_i = \{S_{i,1}, S_{i,2}, \dots, S_{i,r}\}$. Find common neighbor of S_i, and $S_{i,j}$ disjoint from previous common neighbor. Let F_i be the appropriate unions of the D_i's
$\square$

Corollary: $\exists c > 0$ s.t. $\forall \frac{k}{8}$, if G is a graph w/ $\kappa(G) \geq ck$, then G is (k,n)-knit trunk.

Corollary: If $\kappa(G) \geq 22k$, then G is k-linked.

Corollary: Let H be a graph with vertices $v_1, \dots, v_m$. Let G be a graph with $\kappa(G) \geq 22e(H) + v(H)$ and let $u_1, \dots, u_m$ be distinct vertices of G. Then G contains $e(H)$ pairwise vertex-disjoint paths P_{ij} joining u_i to u_j whenever $v_iv_j \in E(H)$.

Corollary: If $\kappa(G) \geq 11p^2$ or $e(G) \geq 22p^2 v(G)$, then G contains a K_p-subdivision.

Back to Hadwiger's Conjecture:

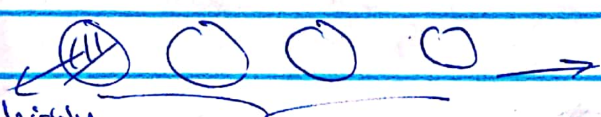
Theorem (Norin, Song, '19)
If G has no K_t-minor, then $\chi(G) \leq O(t(\log t)^{.354})$.

Theorem (Postle, '19)
— " — , then $\chi(G) \leq O(t(\log t)^{\beta})$ $\forall \beta > 1/4$.

Big Picture: no K_t-minor $\Rightarrow \alpha(G) \geq \frac{v(G)}{2t-1}$

- Small ($v(G) \leq t \operatorname{polylog} t$) graphs: Use the Duchet-Meyniel independence result iteratively to color w/ few colors.
- Large graphs:
  - We may assume large min. degree by criticality.
  - (Even further) — " — large connectivity by Kawarabayashi.

Case I:
$\exists$ many (i.e. $\geq \sqrt{\log t}$) vertex-disjoint small (as above), dense subgraphs $\Rightarrow$ we find K_t-minor.
$\geq t(\log t)^{1/4}$



Dense $\Rightarrow$ highly connected subgraphs

$\geq \sqrt{\log t}$

If there are many dense subgraphs, we can always find a K_t-minor

Case 2:

many ....

Then, we can partition G into:

- A few of these small subgraphs.
  This can be colored by our previous argument.
- And, the remainder will induce a sparse graph (density $\leq O(f(\log f)^{p})$) and hence can be colored w/ separate set of colors by greedy.

Why would this be true? ((We've removed all small "dense" subgraphs, and claim: remaining set is sparse).

~~Summary~~:

This we still need to argue:

- Small graphs
- Case I
- Sparsity in case 2.

~~Sketch~~

Small graphs:
In fact, Seymour (2016) noted that the Duchet–Meyniel implies:

Theorem: If G has no K_t-minor, then $\exists X \subseteq V(G)$ with $|X| \geq v(G)/2$ and $\chi(G[X]) \leq t-1$.

Cor: If G has no K_t-minor, then $\chi(G) \leq \left(\log_2\left(\frac{v(G)}{t}\right)+2\right)t$
Proof: By previous thm, $\forall$ integer $s \geq 0$, $\exists$ disjoint $X_1, \ldots, X_s \subseteq V(G)$ s.t. $\left|V(G) - \bigcup_{i=1}^{s} X_i\right| \leq \frac{v(G)}{2^s}$. Let $s = \left\lceil \log_2\left(\frac{v(G)}{t}\right)\right\rceil$.

then $\frac{v(G)}{2^s} \leq t$. and so $\chi(G \setminus \bigcup_{i=1}^{s} X_i) \leq v(G \setminus \bigcup_{i=1}^{s} X_i) \leq t$
So,

$$\chi(G) \leq t + \sum_{i=1}^{s} \chi(G[X_i]) \leq t + (st - 1) \leq (s+1)t.$$

$\square$

Cor: If $v(G) \leq t(\log t)^c$ and G has no K_t-minor, then

$$\chi(G) \leq t(c\log\log t) + 2.$$

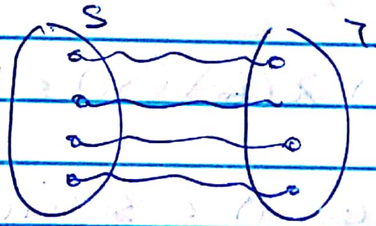
Remark: The tight examples for Kostochka–Thomason deg. bounds are random graphs on $t\sqrt{\log t}$ vertices and hence small.

Scanned with CamScanner

Onto Case I!

Lemma (Norin, Song)
$\exists\ c>0$ s.t.:

Let G be a graph and let $l \geq s \geq 2$ be positive integers.
Let $s_1, \ldots, s_l, t_1, \ldots, t_l, r_1, \ldots, r_s \in V(G)$ be distinct.
If $\kappa(G) \geq c \cdot \max\{l, s\sqrt{\log s}\}$, then $\exists$ a K_s model M rooted at $\{r_1, \ldots, r_s\}$ (i.e. each r_i is in a different bag of M) and an $\{(s_i, t_i)\}_{i \in [l]}$-linkage P in G s.t. M and P are vertex-disjoint.



Proof:
Let $d = d(G) := \frac{e(G)}{v(G)}$ be the density (Note that $d(G) \geq \frac{\kappa(G)}{2} \geq \frac{cl}{2}$). By better minor lemma, $\exists$ a model H of graph H in G, where $v(H) \leq d\varepsilon$ and every vertex in H has at $\leq \frac{v(H)}{2} - \frac{d}{10}$ non-neighbors

Now fead to key technical ~~lemma~~ lemma

$\hookrightarrow$

~~Let~~ Proof: (cont.)

Let the C_i's be the bags of H.

Let $S = \{s_1, \ldots, s_{k}, t_1, \ldots, t_k, r_1, \ldots, r_s\}$

→ Send to key technical ~~chapter~~!

~~[illegible]~~

Got lazy. Idea for rest of proof:

- Split bags not containing r_i's be used for linking and minor creation
- How? Do this randomized → Use Chernoff Bound to bound low common neighbors.

~~[illegible]~~

- Then, use these ~~sets~~ bags to link r_i's to create K_t-minor

□

Theorem (Norin, Song)

$\exists\, c > 1$ s.t. let G be a graph w/ $\chi(G) \geq ct(\log t)^{1/4}$, Let $r \geq \frac{\sqrt{\log t}}{2}$ be an integer. If $\exists$ pairwise ~~disjoint~~ p.w. vertex disjoint subgraphs $H_1, \ldots, H_r$ of G s.t. $d(H_i) \geq ct(\log t)^{1/4}$ $\forall i \in [r]$, then G has a K_t-minor.