Localized Coloring Theorems: → Interpolates standard thms in coloring

1) Local List Coloring → Localize how many colors are available (i.e. $|L(v)|$)
2) Local Fractional Coloring
   → Localize how much color a vertex recieves. (its demand $f(v)$)
   → Has connections with independent set bounds.

Survey of ~~recent~~ results.
- Can localize # of colors → $|L(v)|$
- Can localize maximum degree (Δ) → $d(v)$
- — " — clique # → $\omega(v) := \omega(G[N(v)])$ (size of largest clique containing v)

Ranging over $\omega(v)$ vs. $d(v)$
Greedy: If $|L(v)| \geq d(v)+1$ $\forall v$, then G has an L-coloring
Local List Brooks: If $|L(v)| \geq d(v)$, then G has an L-col
(ERT '79, Borodin '78 + ...) unless G ~~has~~ contains ~~an~~ a component where every block is a clique or odd cycle

Local List Reed's Conj:
If $|L(v)| \geq \left\lceil \frac{d(v)+1+\omega(v)}{2} \right\rceil$ $\forall v \in V(G)$,
then G has an L-coloring

Theorem (Kelly, Postle)
$\exists \varepsilon > 0$ ($\varepsilon \approx \frac{1}{1000}$), if $|L(v)| \geq \lceil (1-\varepsilon)(d(v)+1) + \varepsilon\omega(v) \rceil$ $\forall v$ and
$\delta(G) \geq \text{polylog}\, \Delta(G)$, then G has an L-coloring

On ^local versions^ edge-coloring:

Recall: Thm (Galvin) If G is bipartite, $\chi_\ell'(G) = \chi'(G)$.

Local List Galvin: (Borodin-Kostochka-Woodall '99)
If G is bipartite and $|L(e)| \geq \max\{d(u), d(v)\}$,
then G has an L-coloring. $\overset{e}{uv}$

(Note there is no restriction on degrees)

Local ^List^ Kahn's (Bonamy, Delcourt, Lang, Postle, 20+)
If $|L(e)| \geq (1+o(1)) \max\{d(u), d(v)\}$ and $\delta(G) \geq \text{polylog}\,\Delta$
then G has an L-coloring.

Local Reed-Sudakov
Theorem (Abbasi, Kim, Postle)
$\exists\, c>0$ such that if $|L(v)| \geq c \cdot d_L(v)$ $\forall v$ and $\delta(G) \geq \text{polylog}\,\Delta$,
then G has an L-col.

Local Fractional Coloring:

Recall:

Demand function $f(v)$ $f: V(G) \to [0,1] \cap \mathbb{Q}$.

"How much color they demand"

Proposition: (Dvořák, Sereni, Volec)

Let G be a graph with demand function f. TFAE:

(a) G has an f-coloring ($\phi: V(G) \to$ measurable subset of $[0,1]$ s.t. $\mu(\phi(v)) = f(v)$ $\forall v$ and $\phi(u) \cap \phi(v) = \emptyset$ $\forall e = uv \in E(G)$)

(b) There exist a common denominator N for f s.t. G has a (f,N)-coloring (i.e. $\phi(v) \subseteq [N]$ where $|\phi(v)| = f(v) \cdot N$ and $\phi(v) \cap \phi(u) = \emptyset$ $\forall e$).

(c) $\exists$ probability distribution on independent sets of G s.t. $\Pr[v \in I] \geq f(v)$ $\forall v$

(d) The vector of demands $(f(v) : v \in V(G))$ is in the stable set polytope of G

(e) $\forall$ nonnegative weight function $w : V(G) \to \mathbb{R}_+$, the graph G contains an independent set I s.t. $\sum_{v \in I} w(v) \geq \sum_{v \in V(G)} w(v) \cdot f(v)$.

Remark: $\chi_f(G) := \min\{k : \exists\ f\text{-col of } G \text{ w/ } f(v) = 1/k\}$.

Also,

(e) $\Rightarrow$ $\alpha(G) \geq \sum_{v \in V(G)} f(v)$. $\left(\geq \frac{V(G)}{\chi_f(G)}\right)$.

Results for local frac. colouring:

Local frac greedy? → $f(v) = \frac{1}{d(v)+1}$?

→ Does there exist such an analogue — would be the most natural analogue

Theorem (Caro–Wei, '91) — Independently by

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$$

In fact, the proof gives an f-col where $f(v) = \frac{1}{d(v)+1}$.

Corollary: (Turán)

$$\alpha(G) \geq \frac{v(G)}{\text{avg}(d(v))+1}$$ → By convexity of $\frac{1}{x+1}$.

↳ average degree of v.

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3 proofs of Caro–Wei:

Proof I: (Min-deg proof)

Let G be a minimum counterexample. Let $v \in V(G)$ be of min. deg.; since G is a min. counterexample, $G-v$ has an f-col ϕ.

Note that $\forall u \in N(v)$, $\mu(\phi(v)) = f(u) = \frac{1}{d(u)+1} \leq \frac{1}{d(v)+1}$.

So, v "sees" at most $\frac{d(v)}{d(v)+1}$ colours and so $\mu(L_\phi(v)) \geq 1 - \frac{d(v)}{d(v)+1}$

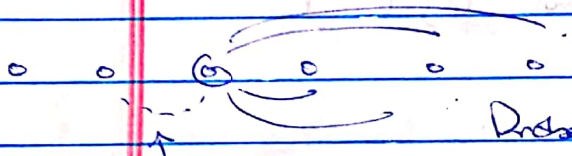
$= \frac{1}{d(v)+1} = f(v)$

$\square$

Proof #2: (Probabalistic Proof)

Choose a total ordering $<$ of $V(G)$ u.a.r. Let $v \in I$ if $v < u$ $\forall u \in N(v)$.

$$\Pr[v \in I] = \frac{1}{d(v)+1}$$



No back edges.

Probability v only has forward edges is $\frac{1}{d(v)+1}$.

Proof #3: (Max degree proof)

Let G be a min. counterexample. Let $v \in V(G)$ s.t. $f(v)$ is minimum. Let $\Phi(v) \subseteq [0,1]$ be a set of measure at least $f(v)$. Let f' be a demand function for $G-v$.

s.t.

$$f'(u) = \frac{1}{d_{G-v}(u)+1}$$

(Idea: Neighbours of v are overcoloured)

Since, $G-v$ has an f'-coloring

G is min counterexample,

Note $\forall u \in N(v)$,

$$f'(u)\,\mu(L_G(u)) = \frac{1-f(u)}{d_{G-v}(u)+1} \geq \frac{1-\frac{1}{d_G(u)+1}}{d_G(u)} = \frac{1}{d_G(u)+1} \geq f(u).$$

By lemma, G has a f-col

↳ See back.

Lemma

Let G be a graph with demand function f, fractional list assignment L and $g: V(G) \to \mathbb{R}$. If $g(v) \leq f(v)\mu(L(v))$ $\forall v$ and $\forall S \subseteq V(G)$ s.t. $\mu(\bigcap_{v \in S} L(v)) > 0$ the graph $G[S]$ has an f-col, then G has a fractional (g, L)-col.

Frac list-assign: $L(v) \subseteq \mathbb{R}$

Frac (g, L)-col: $\phi(v) \subset L(v)$, $\mu(\phi(v)) \geq g(v)$