749_Colourings

Jan 23rd.

Concentrating the last variable for Reed-Sudakov:

$|N_{L,G}(v,c) \cap V(G')|$

$\mathbb{E}[\quad] = \Delta(G)\cdot((1-p\cdot \text{keep})$

Can we concentrate this?

Is it r-verifiable for some r?

→ Note: $u \in N_{L,G}(v,c) \cap V(G')$ if either $u \notin A$ or $u \in A$ but $\exists$ $w \in N_{L,G}(u,\phi(u)) \cap A$ s.t. $\phi(u) = \phi(w)$

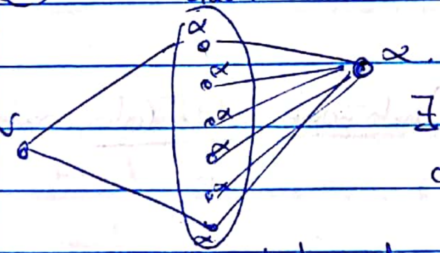
So, this is 4-verifiable! (at most 2 vertex flips and 2 colors)

But, is it ~~[illegible]~~ c-Lipschitz for some c?

Not really: c could need to be at least $\Delta(G)$ as follows

For example.

$N_{L,G}(v,c)$



Independent and all colored α.

$\exists$ exactly one neighbor (and its common to all) of these colored α.

If α changes to β, then all the colors change.

Idea: However, it's unlikely for this to happen.

i.e. Unlikely that a vertex can change the outcome by $\Omega(\deg)$

(we'll prove this later)

So, if we could somehow use Talagrand's with this "likely" Lipschitz constant, we'd find

$$\Pr(|\mathbb{E}[X]-X| > t + 60\sqrt{rc^2} + 8\sqrt{rc^2\,\mathbb{E}[X]}) \le e^{-\frac{t^2}{8rc^2(t+\mathbb{E}[X])}} \Big\} \text{ something like this.}$$

$\mathbb{E}[X] \approx \Theta(\Delta)$ works for $t = \sqrt{\Delta}\,\text{polylog}\,\Delta$

So we'd get that Prob. $|L'(u)|$ or $|N_{k,G}(v,c) \cap V(G')|$ too far from $\mathbb{E}[X]$ is $\leq \frac{1}{\Delta^c}$ for any c.

Argue each event is at most independent of all but a set of at most $(\Delta \cdot L)^2 \leq \Delta^5$ events. and apply LLL to argue w/ pos. prob. that none happen

Then iterate and finish w/ Alon/Haxell when $L/\Delta \geq 2c$ or 2.

$\square$

Remark: This will only work for large enough Δ, b/c to apply the local lemma, we'd need $e^{-\frac{\log^3 \Delta}{8}} \leq \frac{1}{\Delta^{10}}$. (and then extrapolate back through iterations, how large original Δ needs to be)

An Exceptional Outcome Version of Combinatorial Talagrand's Inequality.

Theorem:
Let $((\Omega_i, \Sigma_i, P_i))_{i=1}^n$ be probability spaces. Let (Ω, Σ, P) be their product space and $\Omega^* \subseteq \Omega$ be a set of "exceptional outcomes", and let $X: \Omega \to \mathbb{R}_{\geq 0}$ be a nonnegative random variable. Let $r, c \geq 0$. If X is (r,c)-certifiable w.r.t. Ω^*, then for any $t \geq 96c\sqrt{r\mathbb{E}[X]} + 128rc^2 + 8P[\Omega^*]\sup X$, then

$$\Pr[|X - \mathbb{E}[X]| > t] \leq 4e^{-\frac{t^2}{8rc^2(4\mathbb{E}[X]+t)}} + 4\Pr[\Omega^*]$$

Remark: We need to pay some costs:
- Needing $t > 8P[\Omega^*] \sup X$, and
- An extra $4P[\Omega^*]$ in the prob. bound.

Def'n ~~verifiable~~ (~~certifiable~~) (r,c)-certifiable)
- If $\omega = (\omega_1, \dots, \omega_n) \in \Omega$ and $s \geq 0$, an (r,c)-certificate (w.r.t. X, ω, s, Ω^*) is an index set $I \subseteq \{1, \dots, n\}$ of size at most rs s.t. $\forall k \geq 0$, we have $X(\omega') \geq s - kc$ $\forall \omega' = (\omega_1', \dots, \omega_n') \in \Omega \setminus \Omega^*$ s.t. $\omega_i \neq \omega_i'$ for at most k values of I
- If $\forall s > 0$ and $\omega \in \Omega \setminus \Omega^*$ s.t. $X(\omega) \geq s$, $\exists$ an (r,c)-certificate, then (r,c)-certifiable w.r.t. Ω^*

Remark:
- For $k=0$, the certificate just acts as an r-verifier for non-exceptional outcomes as in normal combinatorial Talagrand's
- What this really requires is an r-verifier for which, if you change at most k of its trials (and any # of the non-verifier trials), you lose at most kc.
  - ⇒ This is kind of like changing any trial in a non-exceptional outcome changes at most c, but requires more generally changing k trials changes by at most kc for any $k \geq 0$ (Bonus: Can assume you end in a non-exceptional outcome)

Back to ~~~~ concentrating $|N_{L,G}(v,c) \cap V(G')|$.

Ω^* : $\exists\, w \in N^{\le 2}(v)$ (Note: At most $(\Delta L)^2$ of these if we delete edges b/w vertices w/ disjoint lists) [up to 2nd neighbourhood]

s.t. $\exists\, c' \in L(w)$ and $\ge$ polylogΔ vertices in $N_{L,G}(w,c')$ with colour c'

(Technically could use $N_{L,G}(w,c) \cap N_{L,G}(v,c)$ instead)

Claim: $\Pr[\Omega^*] \le \frac{1}{\Delta^c}$ for any c and Δ large enough

(this will be enough to apply Talagrand's since $\mathrm{Exp}|N_{L,G}(v,c) \cap V(G')| \le \delta$) ~~and~~ ~~only~~

This also assumes that we show it's (r,c)-certifiable for some r, c w.r.t. Ω^*.

Proof (that $|N_{L,G}(v,c) \cap V(G')|$ is (r,c)-certifiable.

So, let ω be a non-exceptional outcome. We need $\forall s \ge 0$ a set of at most rs trials to build certificate.

So, we use the ≤ 4 activation flip/colour assignments of vertices (and $\le$ neighbours) in $N_G(v,c) \cap V(G')$ (so, what we used before)

Need: $\forall k \ge 0$ changes to these trials, as well as any outside, that $|N_{L,G}(v,c) \cap V(G')| \le s - kc$

Can argue $k =$ polylogΔ works here b/c we start non-exceptional. (b/c if value changes, had to change b/c of some initial change)

How to prove claim:

Note that, by union bound,
$$\Pr[R^*] \le \underbrace{|N^{\le 2}(v)|}_{\le \Delta^4} \Pr\left(\underbrace{\begin{array}{l} \exists c' \in L(w) \\ \text{and } 2\,\text{polylog}\,\Delta \text{ in } N_{L,G}(w,c') \\ \text{colored } c' \end{array}}_{\text{Suffices to show } \le \frac{1}{\Delta^c} \text{ for any } c \text{ (and large enough } \Delta)} \;\middle|\; w \in N^{\le 2}(v) \right)$$

Balls and Bins:

m balls and n bins. and uniformly at random assign each ball to some bin independently.

Expected # of balls in bin i: m/n.

Harder Q: What's the max # of balls in a ~~test~~ bin?
If $m \approx n$ with high probability the max prob is $O\left(\frac{\log n}{\log\log n}\right)$

→ Relating this back to colourings:
Vertices in neighbourhood $(N_{L,G}(w,c'))$ ≡ Balls and Colours ≡ Bins (in $L(w)$)

(Note: If ball can't go to some bin, the probabilities are right, we can fake this and can only get a worse bound)

We have $\Delta_L(G)$ Balls and $|L|$ Bins so $\Delta \approx \Theta(|L|)$ and so no bin has $\ge$ polylog Δ balls with high probability

Balls and Bins - Bounds:

$\overline{(m = \Delta, n = |L|)}$

Upper bounds: By Union bound, $Pr[\exists \text{ bin w/} \geq k \text{ balls}]$

$\leq \Delta \cdot Pr[\text{bin } i \text{ has} \geq k \text{ ball}]$

$Pr[\text{Bin } i \text{ has} \geq k \text{ balls}] \leq \binom{\Delta}{k} \cdot \frac{1}{L^k}$

$\approx \left(\frac{\Delta e}{k}\right)^k \cdot \frac{1}{L^k} = \left(\frac{\Delta e}{kL}\right)^k$

So, we would need: $k \log k > \xi \log \Delta$, i.e. $k > \frac{\log \Delta}{\log\log \Delta}$

~~some constants~~.

to find $\leq \frac{1}{\Delta^c}$ for any c.

Lower Bound:

$Pr[\text{Bin } i = k] = \binom{\Delta}{k} \frac{1}{L^k} \left(1 - \frac{1}{L}\right)^{\Delta - k}$

$E\left[\begin{matrix}\#\text{ of bins w/} \\ \text{exactly } k\end{matrix}\right] = \Delta \cdot Pr[\text{Bin } i = k]$

$\approx \Delta \cdot \left(\frac{\Delta e}{kL}\right)^k \cdot e^{-\frac{\Delta \cdot k}{L}}$

By fiddling w/ constant for k

Can get $\geq \frac{1}{\Delta}$, and so $E \geq 1$,

(Use Talagrand's w/ $c=2$, $r=k$)

Same for upper bound (can use $r=1$, $c=1$).