7(a) - ~~Colouring~~ Random.

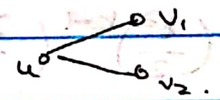
Jan 24th

$G(n,n,p)$ denotes the random bipartite graph with $V = V_1 \cup V_2$, $|V_1| = |V_2| = n$ and uv is an edge with prob p for $u \in V_1$, $v \in V_2$

Exercise: let $p = \frac{\log n + c}{n}$. Let X denote # of isolated vertices in $G(n,n,p)$. Then,

- $P(X=0) \sim e^{-2e^{-c}}$
- $P(G(n,n,p) \text{ is connected}) \sim e^{-2e^{-c}}$

A triple of vertices $\{u, v_1, v_2\}$ is called a cherry if $d(v_1) = d(v_2) = 1$ and $v_1 \sim u$ and $v_2 \sim u$.



Exercise: Let $p \sim c \log n / n$ where $c > 3/5$. Then a.a.s there are no cherries in $G(n,n,p)$

Theorem (Hall's theorem)
A bipartite graph G with vertex set $V_1 \cup V_2$ has a PM if $|V_1| = |V_2|$ and $|N(S)| \geq |S|$ $\forall S \subseteq V_1$

Exercise: Suppose G fails Hall's condition. Then, there exists a set $S \subseteq V_1$ or $S \subseteq V_2$.

(i) $|N(S)| = |S| - 1$
(ii) $|S| \leq \lceil \frac{n}{2} \rceil$
(iii) Every vertex in $N(S)$ ≥ 2 neighbours in S

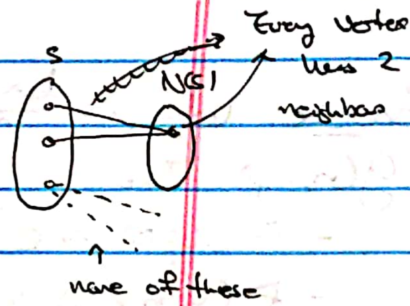
Let X_k denote a k-set S satisfying (i), (ii), (iii)

X_1 : # isolated vertices
X_2 : # cherries

Lemma:

Let $(3/4)\frac{\log n}{n} < p < \frac{2\log n}{n}$. Then, $\sum_{k=3}^{\lceil n/2 \rceil} \mathbb{E}X_k = o(1)$

Proof:



For each $3 \le k \le \lceil \frac{n}{2} \rceil$

$$\mathbb{E}X_k \le 2\cdot\binom{n}{k}\cdot\binom{n}{k-1}\left(\binom{k}{2}p^2\right)^{k-1}(1-p)^{k(n-k+1)}$$

Choose from either V_1 or V_2

neighbours

Each vertex in $N(S)$ has 2 neighbours

No edges going from S to not $N(S)$

$$\le 2\left(\frac{en}{k}\right)^k\left(\frac{en}{k-1}\right)^{k-1}\left(\frac{k^2}{2}\right)^{k-1}\left(\frac{2\log n}{n}\right)^{2k-2}\exp\left(-\frac{3\log n}{4n}\cdot k\cdot\frac{n}{2}\right)$$

since $n-k+1 \le \frac{n}{2}$

$$\le 2n\left(\frac{k}{k-1}\right)^{k-1}\left(\frac{2e^2\log^2 n}{n^{3/8}}\right)^k$$

$$= O(n\alpha^k), \text{ where } \alpha = \frac{3e^2\log^2 n}{n^{3/8}}.$$

So:

$$\sum_{k=3}^{\lceil n/2 \rceil} \mathbb{E}X_k = o(1) \qquad \square$$

Theorem

Let $p = \frac{\log n + c}{n}$, c fixed $\mathbb{R}$. Then $\Pr(\mathcal{G}(n,n,p) \text{ has a PM}) \sim e^{-2e^{-c}}$

Proof:

Let C_1 denote the event that $X_1 = 0$, i.e. no isolated vertices,

C_2 —— " —— $X_k = 0 \ \forall\, 2 \le k \le \lceil \frac{n}{2} \rceil$

Hall's theorem and Exercise $\Rightarrow C_1 \cap C_2 \Rightarrow \mathcal{G}(n,n,p)$ has a PM

So, $\Pr(\mathcal{G}(n,n,p) \text{ has PM}) \ge \Pr(C_1 \cap C_2)$

$$= \Pr(C_1) - \Pr(C_1 \cap \overline{C_2})$$

$$\ge \Pr(C_1) - \Pr(\overline{C_2}) = P(C_1) - o(1)$$

By Lemma + exercise

Proof (con't)

So, $\mathcal{G}(n,n,p)$ has a PM $\Rightarrow C_1$ ①

And,

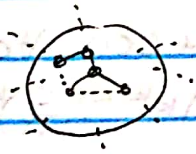
$$Pr(\mathcal{G}(n,n,p) \text{ has a PM}) \le Pr(C_1) \quad ②$$

So:

$$Pr(\mathcal{G}(n,n,p) \text{ has a PM}) = Pr(C_1) + o(1) \sim e^{-2e^{-c}}$$ □

Let T_k denote the number of components that are trees of order k.

Let $T = \sum_{k=1}^{n} k T_k$.



$$\mathbb{E}T_k = \underbrace{\binom{n}{k}}_{k \text{ vertices}} \underbrace{k^{k-2}}_{\# \text{ of trees}} \overbrace{p^{k-1}}^{\text{Edges in tree}} (1-p)^{\underbrace{\binom{k}{2} - (k-1)}_{\text{No other edges except for tree edges}} + \underbrace{k(n-k)}_{\text{edges coming out of component}}}$$

$$\mathbb{E}T = \sum_{k=1}^{n} k\, \mathbb{E}T_k$$

Theorem Let $p = \frac{c}{n}$, $c > 0$ fixed. → τ is a function of the random graph

(a) If $0 < c < 1$, then $\mathbb{E}[T(\mathcal{G}(n,p))] = n + O(1)$

(b) If $c > 1$, then $\mathbb{E}[T(\mathcal{G}(n,p))] = t(c)n + O(1)$,

where $t(c) = \frac{1}{c} \sum_{k \ge 1} \frac{k^{k-1}}{k!} (ce^{-c})^k$

Note: $t(c) \le \sum_{k \ge 1} \frac{1}{k} \left(\frac{ek}{k}\right)^k (ce^{-c})^k = \sum_{k \ge 1} \frac{1}{k} (\underbrace{ce^{1-c}}_{\text{Can show this is } <1 \text{ for } c \ne 1})^k$.

So, $t(c)$ is well-defined.

Exercise: (a) $s(c) = ct(c)$ is the unique solution of $se^{-s} = ce^{-c}$ in the range $0 < s \le 1$

(b) $t(c) = 1 \quad \forall\, 0 < c \le 1$.

Note: $\mathbb{E}[T(G(n,p))]$ is cts. in p and $t(c) \to 1$ as $c \to 1$.
So, $\mathbb{E}[T(G(n,\tfrac{1}{n}))] = n + o(n)$. (Since $t(c)n = n +$ "some error").

Proof:
Since $t(c) = 1 \;\forall\, 0 < c < 1$, it's sufficient to show that
$\mathbb{E}T = t(c)n + O(1) \quad \forall c \neq 1$.

$$\mathbb{E}T_k = \binom{n}{k} k^{k-2} \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{kn - \frac{k(k+3)}{2} + 1}.$$

Consider $1 \le k \le \sqrt{n}$. (In this range $\binom{n}{k} \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$)
... uh... actually we'll use:
$$\binom{n}{k} = \frac{n^k}{k!} \exp\left(-\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right)\right).$$

Then,
$$\mathbb{E}T_k = n \cdot \frac{k^{k-2}}{k!} c^{k-1} e^{-ck} \exp\left(-\frac{k^2}{n} - \frac{ck^2}{2n} + O\left(\frac{k}{n}\right)\right).$$

Hence,
$$\left| \sum_{k=1}^{\sqrt{n}} k\,\mathbb{E}T_k - \sum_{k=1}^{\sqrt{n}} n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck} \right|$$
$$= \sum_{k=1}^{\sqrt{n}} O\left(\frac{k^2}{n}\right) n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck}$$
$$= \sum_{k=1}^{\sqrt{n}} O(k \cdot e^{k - ck} c^k)$$
$$= \sum_{k=1}^{\sqrt{n}} O(k(\underbrace{ce^{1-c}}_{<1})^k) = O(1).$$

$$\frac{\mathbb{E}\,(k+1)T_{k+1}}{\mathbb{E}\,kT_k} = (n-k)\underbrace{\left(1+\frac{1}{k}\right)^{k-2}}_{e}\cdot\frac{c}{n}\left(1-\frac{c}{n}\right)^{n-k-2}$$

$$\le \left(1-\frac{k}{n}\right)\cdot e\cdot c\,e^{-\frac{c}{n}(n-k)}\cdot\left(1-\frac{c}{n}\right)^{-2}$$

$$= \left(1-\frac{k}{n}\right)e\,c\,e^{-c\left(1-\frac{k}{n}\right)}\left(1-\frac{c}{n}\right)^{-2}$$

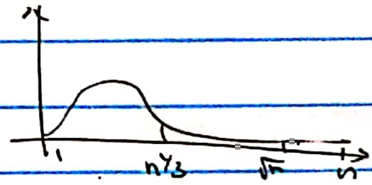
Let $\eta = \left(1-\frac{k}{n}\right)$.

$$= \underbrace{\eta\, e\, c\, e^{-c\eta}}_{c\eta e^{1-c\eta}\le 1}\left(1-\frac{c}{n}\right)^{-2}$$

$$\le \left(1-\frac{c}{n}\right)^{-2} := \lambda.$$

It is easy to verify (using formulae we've shown for $\mathbb{E}T_k$), that $\mathbb{E}T_{k_1} = o(n^{-M})$ for any fixed $M>0$, where $k_1 = n^{1/3}$

$$\sum_{k=k_1}^{n} k\,\mathbb{E}[T_k] \le k_1\,\mathbb{E}[T_{k_1}]\sum_{i=0}^{n}\lambda^i$$



$$\le O(n)\cdot k_1\,\mathbb{E}[T_{k_1}]$$

Since $\lambda^i \le \lambda^n = O(1)$

$$= o(n^{-3})$$

It is easy to check that $\sum_{k=\sqrt{n}}^{\infty}\frac{k^{k-1}}{k!}c^{k-1}e^{-ck} = \cancel{o(1)}\ o(n^{-1})$

$$|\mathbb{E}T - t(c)n| \le \left|\sum_{k=1}^{\sqrt{n}} k\,\mathbb{E}T_k - n\cdot\frac{k^{k-1}}{k!}c^{k-1}e^{-ck}\right|$$

$$+\sum_{k=\sqrt{n}}^{n} k\,\mathbb{E}T_k + \sum_{k=\sqrt{n}}^{\infty} n\cdot\frac{k^{k-1}}{k!}c^{k-1}\cdot e^{-ck}$$

$$= O(1) + o(1) + o(1) = O(1)$$

□