Affine planes:

(a) $(V, \mathcal{L})$ is a linear space containing a 3-arc and "unique parallel lines" condition

(b) $(V, \mathcal{L})$ is a $(n^2, n, 1)$-BIBD

(c) $(V, \mathcal{L})$ is the residual of a projective plane

Proof:

(b) $\Rightarrow$ (a) Done.

(c) $\Rightarrow$ (b) Done.

Today: (a) $\Rightarrow$ (c)

Let $(V, \mathcal{L})$ be a linear space satisfying conditions of (a).

First we note that parallelism is an equivalence relation. Symmetry + Reflexive are obvious. For transitivity, suppose $l_1 \| l_2$ and $l_2 \| l_3$ but $l_1 \nparallel l_3$, then let $x = l_1 \wedge l_3$. But now l_1 and l_3 are two different parallel lines going through x, violating the uniqueness condition.

Call the equivalence class "parallel classes" and write $[l]$ to denote the parallel class of $l \in \mathcal{L}$.

We need to construct a projective plane $(\bar{V}, \bar{\mathcal{L}})$ whose residual is $(V, \mathcal{L})$

Proof: (cont)

$\hat{\mathcal{L}} = \mathcal{L} \cup \{l_\infty\}$ ← New line

$\hat{V} = V \cup \{$parallel classes$\}$

$\{\Pi_1, \ldots, \Pi_{n+1}\}$, where $\Pi_1, \ldots, \Pi_{n+1}$ are the parallel classes of $(V, \mathcal{L})$

with the following incidence relations:

- If $x \in V$, $l \in \mathcal{L}$, incidence is as in $(V, \mathcal{L})$
- For $x \in V$, l_∞, x is not incident with l_∞
- For Π_i, $l \in \mathcal{L}$ Π_i is incident with l iff $l \in \Pi_i$.
- For Π_i, l_∞ Π_i is incident with l_∞

Check that $(\hat{V}, \hat{\mathcal{L}})$ is linear, dual-linear and has a 4-arc

$\boxtimes$

Ovals and Hyperovals:

Def'n Let $\mathcal{C}$ be an s-arc in a projective plane. A

Secant

- A secant is a line that contains exactly 2 points of $\mathcal{C}$.
- A tangent ———— " ———— 1 point of $\mathcal{C}$.
- A passant ———— " ———— 0 points of $\mathcal{C}$.

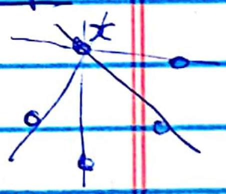
Lemma:

Let $\mathcal{C}$ be an s-arc in a projective plane of order n. Then, every point of $\mathcal{C}$ has $n+2-s$ tangents. In particular, $s \leq n+2$. Moreover if $s = n+2$, then n is even.

Proof:

Let $x \in \mathcal{C}$. There are $n+1$ lines through x. $s-1$ of them are secants and none are passants (since $x \in \mathcal{C}$)

Example:



All secants.

So, there are $(n+1)-(s-1)$ tangents

To show the moreover, Suppose $s=n+2$, then there are no tangents (since $n+2-s=0$). So, every line is either a secant or a passant. Let $y \notin \mathcal{C}$, every line through y meets $\mathcal{C}$ at an even # of points (0 or 2), so $s = |\mathcal{C}|$ is even. $\Rightarrow$ $n = s-2$ is even. ∎

Def'n

An oval is an $(n+1)$-arc in a projective plane of order n.

A hyperoval is an $(n+2)$-arc — " ——————.

Example:

Let q be a prime power. Let $(V, \mathcal{L})$ be the standard projective plane of order q:

points = 1-dim'l linear subspaces of $GF(q)^3$

lines = 2-dim'l ——— " ———

Write $[x:y:z] = \mathrm{Span}\{(x,y,z)\} \in V$ $((x,y,z) \neq (0,0,0))$.

Claim: $\Theta = \{[x:y:z] \mid xy + yz + xz = 0\}$ is an oval

Proof:

First, note that the intersection b/w any line l and Θ is given by a quadratic equation. ∴ l has at most two points of Θ. ∴ no three points of Θ are collinear.

Next: Count points $[x:y:z] \in \mathcal{C}$.

Case I: $x=0$

Then, $[x:y:z] \in \Theta$ iff $yz=0$, so either $y=0$ or $z=0$ (but not both)

This gives 2 points $[0:0:1]$, $[0:1:0]$

Case 2: $x \neq 0$.

Rescaling, we can assume $x=1$, ~~then~~ Now, $[1:y:z] \in \Theta$ iff

$$y + yz + z = 0 \Rightarrow z = -\frac{y}{y+1}$$

This gives $q-1$ points since $y=-1$ is no good.

In total, this is $q+1$ points, ∴ Θ is an oval. ∎

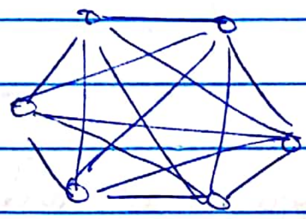
Exercise: If $q=2^d$, then check that $\Theta \cup \{[1:1:1]\}$ is a hyperoval.

Feb 26th

Theorem: All projective planes of order 4 are isomorphic

Proof:

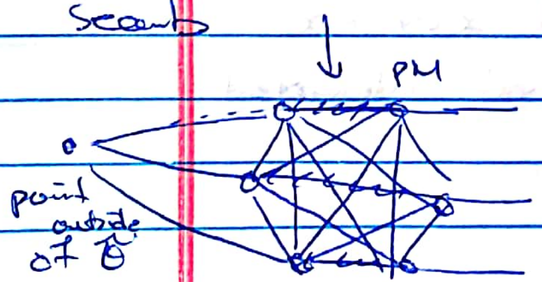
Let $\hat{\mathcal{O}}$ be a hyperoval in a projective plane $(V, \mathcal{L})$ of order 4. Regard points $\hat{\mathcal{O}}$ as vertices of K_6:



Every point/line can be identified with some feature of K_6

Points	Incidence Relation	Lines
6 points on $\hat{\mathcal{O}}$		15 secants of $\hat{\mathcal{O}}$
$\equiv$	$\longleftrightarrow$	$\equiv$
6 vertices of K_6	incidence	15 edges of K_6
15 perfect matchings		6 persents
$\equiv$	$\longleftrightarrow$	$\equiv$
15 points not on $\hat{\mathcal{O}}$		1-factorization of K_6

Note: Each PM corresponds to a unique point since w/ we ~~have~~ ~~points~~ draw overlapping secants



□

Latin Squares:

Def'n A latin square of order n is an $n \times n$ array in which every row and every column is a permutation of $[n]$

Example:

$$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$ is a latin square of order 3

Example: Multiplication table for any group

~~Notation: Use matrix notation A_{ij} entry in row i, column j~~

Notation: We will either use:

- Matrix Notation:

  A_{xy} = Entry in row x, col. y

-or-

- Write $x \circ y$ for entry in row x, col y.

  "Think of $\circ$ as a binary operation"

Def'n A latin square is idempotent if $x \circ x = x$

Example: The latin square above is not idempotent.

The following is:

$$\begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}$$

Defin: A latin square is symmetric if $x \circ y = y \circ x$.
Example: Our previous example is also symmetric

Lemma: A symmetric, idempotent latin square of order n iff n is odd

Proof:
($\Leftarrow$) If n is odd, define $x \circ y = \frac{x+y}{2}$ (mod n)
($\Rightarrow$) Conversely, consider the set of pairs $\{\{x,y\} \mid x \circ y = 1\}$

Note: Can have 1 or 2 elements (since they are sets)
1-element $\Rightarrow$ $x \circ x = 1$

If a latin square is symmetric and idempotent, then this set contains 1 singleton and $\frac{n-1}{2}$ pairs. So, n is odd

$\square$

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