7409 - Random Mar. 4th

Recall in $\mathcal{G}(n, 1/2)$

- $\mathbb{E}(\#\ k\text{-cliques}) = f(k) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$
- $k_0 = \min\{k \geq 2 : f(k) \leq 1\}$
- $k_0 \sim 2\log_2 n$, $f(k_0+1) = o(1)$, $f(k_0-4) = \omega(n^3)$
- Let $k_0 = k_0 - 4$
- Y: Size of a maximum family of edge disjoint k-cliques

Key Lemma: $\mathbb{E}Y \geq (1+o(1))\frac{n^2}{2k^4}$

- $\Pr(\alpha(\mathcal{G}(n, 1/2)) < k) = \exp(-\Omega(n^2/\log^8 n))$
- Let $m = \lfloor n/\log_2^2 n \rfloor$
- $\Pr(\alpha(\mathcal{G}[m]) < k) = \exp(-\Omega(m^2/\log^8 m)) = \exp(-\Omega(n^2/\log^{12} n))$

There are $\binom{n}{m} < 2^n$ sets of size m.

$\Pr(\exists$ an m-set S s.t. $\alpha(\mathcal{G}[S]) < k)$
$\leq 2^n \cdot \exp(-\Omega(n^2/\log^{12} n))$
$= o(1)$

Now we color $\mathcal{G}(n, 1/2)$ as follows: Repeatedly pull out independent sets of order k and give a color to that independent set, until the # of the remaining colors is less than m. Then, give distinct colors to them.

In total, we use $\leq \frac{n}{k} + m \sim \frac{n}{2\log_2 n}$.

So, $\chi(\mathcal{G}(n, 1/2)) \leq (1+o(1))\frac{n}{2\log_2 n}$

$\square$

Proof of key lemma:

The expected # of k-cliques is $f(k)$

Let $\mathcal{C}$ be a family of k-cliques obtained by including each of the k-cliques of $G(n, 1/2)$ independently w/ prob p.

Let $\mathcal{W}$ of pairs of k-cliques (α, β) in $\mathcal{C}$ such that $E(\alpha) \cap E(\beta) \neq \emptyset$

Let $\mathcal{C}'$ be obtained by deleting one of α, β if $\{\alpha, \beta\} \in \mathcal{W}$.

Then, $\mathcal{C}'$ is a family of edge-disjoint k-cliques.

So:

$$E Y \geq E|\mathcal{C}'| \geq E(|\mathcal{C}| - |\mathcal{W}|).$$

← Expected # of k-cliques

$E|\mathcal{C}| = p \cdot f(k)$

← Probability of being included.

~~Exercise~~

Exercise: The expected # of pairs of k-cliques in $G(n, 1/2)$ that intersect edges is asymptotically $f(k)^2 \cdot \frac{k^4}{2n^2}$.

Then, $E|\mathcal{W}| \sim p^2 f(k)^2 \frac{k^4}{2n^2}$

$EY \geq p f(k) - (1+o(1)) p^2 f(k)^2 k^4 / 2n^2$

Choosing $p = \frac{n^2}{k^4 f(k)}$ gives $E(Y) \geq (1+o(1)) \frac{n^2}{2k^4}$.

$\square$.

Mar 4th

Theorem:
Let $p=n^{-\alpha}$ where $\alpha>5/6$ fixed. Then, there exists an integer $u=u(n,p)$ s.t. a.a.s $u \le \chi(G(n,p)) \le u+3$.

Proof:
We have shown (by assignment question) that for any fixed $c>0$ a.a.s every subgraph induced by $\le c\sqrt{n}$ vertices can be 3-colored

Fix $\varepsilon>0$, and let $u=u(n,p,\varepsilon)$ be the smallest integer s.t. $Pr(\chi(G(n,p)) \le u) > \varepsilon$.

Let Y be the size of the smallest subset S s.t. $G-S$ is u-colorable. Consider the vertex exposure martingale w.r.t Y, then Y satisfies the 1-Lipschitz condition (Revealing one more vertex changes X by at most 1).

By Azuma's inequality:

$$Pr(Y-EY > t) \le e^{-t^2/2n}, \quad Pr(Y-EY<-t) \le e^{-t^2/2n}$$

$$\Rightarrow Pr(Y > EY + \lambda\sqrt{n}) \le e^{-\lambda^2/2}, \quad Pr(Y < EY - \lambda\sqrt{n}) \le e^{-\lambda^2/2}$$

Choose λ s.t. $e^{-\lambda^2/2} = \varepsilon$, then

$$Pr(Y < EY - \lambda\sqrt{n}) \le \varepsilon \quad (1)$$

The event $Y=0$ is equivalent to $G(n,p)$ being u-colorable

i.e. $\{Y=0\} = \{G(n,p) \text{ is } u\text{-colorable}\}$

$$\Rightarrow Pr(Y=0) = Pr(\chi(G(n,p)) \le u) \quad (\text{By } (1))$$

But $Pr(\chi(G(n,p)) \le u) > \varepsilon$ and $Pr(Y < EY - \lambda\sqrt{n}) \le \varepsilon$, so $EY - \lambda\sqrt{n} < 0 \Rightarrow EY < \lambda\sqrt{n}$

Cz

Proof (con't)

Now,

$Pr(Y \geq 2\lambda\sqrt{n}) < Pr(Y \geq EY + \lambda\sqrt{n}) < \varepsilon$.

So, with probability $\geq 1-\varepsilon$, $Y \leq 2\lambda\sqrt{n}$.

i.e. There exists a set S with $|S| \leq 2\lambda\sqrt{n}$ s.t.

$G(n,p) - S$ is u-colorable

And, we also know that a.a.s S is 3-colorable

$$Pr(\chi(G(n,p)) \leq u-1 \text{ OR } \chi(G(n,p)) \geq u+4)$$
$$\leq \varepsilon + \underbrace{Pr(Y > 2\lambda\sqrt{n})}_{\leq \varepsilon} + \underbrace{Pr(\exists S : |S| \leq 2\lambda\sqrt{n} \text{ and } S \text{ not 3-colorable})}_{o(1)}$$
$$\leq 3\varepsilon$$

And the assertion holds by letting $\varepsilon \to 0$. $\square$

What happens when $p \sim c/n$ where $c>0$ fixed?

Theorem (Achlioptas and Friedgut, 1999)

For any $k \geq 3$ fixed there is a sharp threshold sequence $c_k = c_k(n)$ such that $\varepsilon > 0$:

$$\lim_{n\to\infty} Pr(G(n, c/n) \text{ k-colorable}) = \begin{cases} 0 & \text{if } c < (1-\varepsilon)c_k(n) \\ 1 & \text{if } c > (1+\varepsilon)c_k(n) \end{cases}$$

(~~Determining~~ Determining c_k remains open)

Differential Equation Method:

Toy Example: Let $c > 0$ fixed. Suppose cn balls are thrown into n bins sequentially uniformly at random.
Let X_i be the # of empty bins after i balls are thrown.

- $X_0 = n$
- $X_{i+1} = X_i - \underbrace{\mathbb{1}\{(i+1)\text{-th ball thrown into empty bin}\}}_{:= E_{i+1}}$

  ↑ # of empty bins so far

- $\Pr(E_{i+1}) = \frac{X_i}{n}$ $\xRightarrow{\text{Formally}}$ $\mathbb{E}(\mathbb{1}\{E_{i+1}\} \mid X_0, \dots, X_i) = \frac{X_i}{n}$

  (Heuristically)

- $\mathbb{E}(X_{i+1} - X_i \mid H_i) = -\frac{X_i}{n}$

  ↑ history of first i balls.

We want to find the trajectory of $(X_i)_{i=0}^{cn}$

↳ what does this mean? What is the trajectory of an r.v?
We will formalize this later. For now...

... let's assume that all X_i are concentrated around $\mathbb{E}X_i$. So, we find the trajectory of $(\mathbb{E}X_i)_{i=0}^{cn}$.
Then,

$$\mathbb{E}(X_{i+1}) - \mathbb{E}(X_i) = -\frac{\mathbb{E}X_i}{n}.$$

We write:

$$x(t) = \frac{\mathbb{E}(X_{tn})}{n}.$$

↪

And the above becomes:

$$\mathbb{E}X_{i+1} - \mathbb{E}X_i = -\frac{\mathbb{E}X_i}{n} \Rightarrow \underbrace{n\left(x(t+\tfrac{1}{n}) - x(t)\right)}_{\xrightarrow{n\to\infty} x'(t)} = -x(t)$$

That tells us that the solution of $x'(t) = -x$ gives (by scaling) the trajectory of $(\mathbb{E}X_i)_{i=0}^{cn}$:

$$\frac{dx}{dt} = -x, \quad \int \frac{1}{x}\,dx = -\int dt \Rightarrow x(t) = Ce^{-t}$$

$X_0 = n \Rightarrow x_0(0) = 1 \Rightarrow C = 1$, so $x(t) = e^{-t} \Rightarrow \mathbb{E}X_i = ne^{-i/n}$

Suppose we want to show the concentration of X_T.
For $j \geq 0$, define $Z_j = \mathbb{E}(X_T \mid H_j)$

↳ The σ-field generated by the random allocation of the first j balls

We also have $|Z_j - Z_{j-1}| \leq 1$, so by Azuma's Inequality,

$$\Pr(|X_T - \mathbb{E}X_T| \geq \sqrt{\alpha T}) \leq 2e^{-\alpha/2}$$

$\Rightarrow$ with probability $\geq 1 - e^{-n^{1/3}}$, $X_T = ne^{-T/n} + o(n)$