CO749 - Random Graph Theory

(Notes Scans)

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Winter 2020

Contents

Summary

Lecture 1: Introduction

Definition 1.1. The probability space we'll work in is denoted with the triple $(G, \mathbb{P}, F)$, where G is a class of graphs, $\mathbb{P}$ a probability measure and F a sigma algebra.

Normally, G is a finite set, $\mathbb{P}$ is a discrete probability measure and $F = 2^G$

Definition 1.2 (Erdős-Rényi Random Graph Model)**.**

- The $\mathcal{G}(n,p)$ model: A graph with vertex set $[n]$ is constructed randomly by including each edge in $k_{[n]}$ with probability p
- The $\mathcal{G}(n,m)$ model: A graph is chosen uniformly at random from all graphs with vertex set $[n]$ and has m edges.

(Aside: We can think of $\mathcal{G}(n,m)$ as labelling the edges)

Other models:

- $\mathcal{G}(n,d)$ is the model of random d-regular graphs
- $\mathcal{G}(n,\tilde{d})$ where $\tilde{d} = (d_1, \dots, d_n)$ is a vector representing the degrees of vertices. (This is a generalization of $G(n,d)$)
- $\mathcal{G}(n,r)$ is the model of random geometric graphs. The construction is as follows: Pick n points uniformly in the unit square, then, add an edge if and only if the distance between two points is $\leq r$
- Random trees. A tree is chosen uniformly at random from the n^{n-2} trees on n vertices.

In this class, we will primarily focus on the Erdős-Rényi Model.

1.1 Probability Primer

Definition 1.3. A discrete probability space consists of a countable set Ω and a probability function $\mathbb{P} : \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

A subset of Ω is called an event. The probability of $A \subseteq \Omega$ is $\sum_{\omega \in A} \mathbb{P}(\omega)$, denoted $\mathbb{P}(A)$.

Proposition 1.1 (Inclusion-Exclusion)**.** For events A, B:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

and, in general:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1 < i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \ldots + (-1)^{n-1}\,\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Corollary 1.1. $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$

Definition 1.4. Two events are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Definition 1.5. A random variable (r.v) X is a function $X : \Omega \to \mathbb{R}$. In a discrete probability space, the expectation of X is defined by: $\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega)\,\mathbb{P}(\omega)$

Proposition 1.2 (Linearity of Expectation)**.** $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Proof. $\mathbb{E}(X+Y) = \sum_{\omega\in\Omega}(X+Y)(\omega)\,\mathbb{P}(\omega) = \sum_{\omega\in\Omega} X(\omega)\,\mathbb{P}(\omega) + \sum_{\omega\in\Omega} Y(\omega)\,\mathbb{P}(\omega) = \mathbb{E}(X) + \mathbb{E}(Y)$ $\square$

Lemma 1.1.

- For any $n \geq k \geq 1$
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$$
- (Stirling's Formula)
$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \mathcal{O}(n^{-2})\right)$$
- For every $t \in \mathbb{R}, e^t \geq 1 + t$

Lemma 1.2. Assume $k = o(\sqrt{n})$ Then, $\binom{n}{k} \sim \frac{n^k}{k!}$

Proof.

$$\begin{aligned}
\binom{n}{k} &= \frac{1}{k!}\prod_{i=0}^{k-1}(n-i) \\
&= \frac{n^k}{k!}\prod_{i=0}^{k-1}\left(1 - \frac{i}{n}\right) \\
&= \frac{n^k}{k!}\prod_{i=0}^{k-1} e^{\mathcal{O}(i/n)} \qquad (\log(1-x) = \mathcal{O}(x)) \\
&= \frac{n^k}{k!}\exp\left(\mathcal{O}\left(\frac{1}{n}\sum_{i=0}^{k-1} i\right)\right) \\
&= \frac{n^k}{k!}\exp\left(\mathcal{O}\left(\frac{k^2}{n}\right)\right) \\
&= (1 + o(1))\frac{n^k}{k!} \qquad \left(\text{as } k = o(\sqrt{n})\right)
\end{aligned}$$

$\square$

Remark 1.1. $k = o\left(n^{\frac{2}{3}}\right)$, then $\binom{n}{k} \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$

Lecture 2: Concentration Inequalities, Coupling, Connection Theorem

Definition 2.1. Given a sequence of probability spaces $(\Omega_n, P_n)_{n\geq 1}$. We say that A_n holds asympototically almost surely (a.a.s) if $P_n(A_n) \to 1$ as $n \to \infty$

2.1 Concentration Inequalities

Theorem 2.1 (Markov's Inequality)**.** Let X be a nonnegative random variable. Then, for any real $t > 0$, $\Pr(X \geq t) \leq \frac{\mathbb{E}\,X}{t}$

Proof. Let I_t be the indicator r.v. that $X \geq t$. Then, $X \geq t \cdot I_t$, so:

$$\mathbb{E}\,x \geq t \cdot \mathbb{E}\,I_t = t \cdot \mathbb{P}(X \geq t)$$

□

Theorem 2.2 (Chebyshev's Inequality)**.** For any $t \geq 0$

$$\mathbb{P}(|X - \mathbb{E}\,X| \geq t) \leq \frac{\operatorname{Var} X}{t^2}$$

Example 2.1. Let X be the number of edges in $\mathcal{G}(n,p), N = \binom{n}{2}$. $X \sim \operatorname{Bin}(N,p)$ so $\mathbb{E}\,X = Np$ and $\operatorname{Var} X = p(1-p)N$.

Further, by Chevyshev's Inequality, for all $t > 0$:

$$\mathbb{P}(|X - \mathbb{E}\,X| \geq t) \leq \frac{p(1-p)N}{t^2}$$

△

This leads us to the following proposition:

Proposition 2.1. Let $f_n \to \infty$ as $n \to \infty$. Then,

$$\mathbb{P}(|X - Np| \geq f_n\sqrt{p(1-p)N}) \leq \frac{1}{f_n^2} = o(1)$$

So a.a.s:

$$pN - f_n\sqrt{p(1-p)N} \leq X \leq pN + f_n\sqrt{p(1-p)N}$$

2.2 Coupling

Definition 2.2. Given 2 r.v.s X, Y, a coupling of X and Y is a construction of a join distribution of $(\hat{X}, \hat{Y})$ into the probability space such that marginally $\hat{X} \sim X$ and $\hat{Y} \sim Y$

Lemma 2.1.

(a) Let $0 \le m_1 < m_2 \le N$ and $0 \le p_1 < p_2 \le 1$. There exist couplings such that:

$$\mathcal{G}(n, m_1) \subseteq \mathcal{G}(n, m_2) \quad \text{and} \quad \mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, p_2)$$

where by $\mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, p_2)$ (and respectively, m_1 and m_2), we mean that there exists a coupling (G_1, G_2) such that:

- Marginally, $G_1 \sim \mathcal{G}(n, p_1), G_2 \sim \mathcal{G}(n, p_2)$, and
- jointly, $G_1 \subseteq G_2$ always

(b) Let $m_1 = pN - f\sqrt{p(1-p)N}, m_2 = pN + f\sqrt{p(1-p)N}$ ($f = f(n)$ as before). Then, there exists a coupling (G_1, H, G_2) such that:

- $G_1 \sim \mathcal{G}(n, m_1), G_2 \sim \mathcal{G}(n, m_2), H \sim \mathcal{G}(n, p)$
- $\mathbb{P}(G_1 \subseteq H \subseteq G_2) = 1 - o(1)$

Proof.

(a) Let $G_1 \sim \mathcal{G}(n, p_1)$. For G_2, include every non-edge in G_1, include it independently with probability $q = 1 - \frac{1-p_2}{1-p_1}$. Clearly, $G_1 \subseteq G_2$. Then, check the probability that an edge is <u>not</u> included in G_2:

$$(1-p_1)(1-q) = (1-p_1)\left(1 - \left(1 - \frac{1-p_2}{1-p_1}\right)\right) = 1 - p_2$$

For $G_1 \sim \mathcal{G}(n, m_1), G_2 \sim \mathcal{G}(n, m_2)$, we choose permutation and the first m_1, m_2 edges

$\square$

2.3 Connection Theorem

Definition 2.3. Let Ω be the set of graphs on $[n]$. $Q \subseteq \Omega$ is a <u>graph property</u> if it is invariant under graph isomorphism. We say Q is <u>monotone increasing</u> if:

$$G \in Q \Rightarrow H \in Q \quad \forall H \supseteq G$$

Further, we say Q is <u>convex</u> if:

$$G_1, G_2 \in Q, G_1 \subseteq G_2 \Rightarrow H \in Q \quad \forall G_1 \subseteq H \subseteq G_2$$

Theorem 2.3. Suppose Q is monotone. Then, for any $0 \le m_1 \le m_2 \le N$, $0 \le p_1 \le p_2 \le 1$:

$$\mathbb{P}(\mathcal{G}(n, m_1) \in Q) \le \mathbb{P}(\mathcal{G}(n, m_2) \in Q)$$
$$\mathbb{P}(\mathcal{G}(n, p_1) \in Q) \le \mathbb{P}(\mathcal{G}(n, p_2) \in Q)$$

Theorem 2.4 (Connection Theorem)**.** Let Q be a graph property:

(i) Given $p = p(n)$. Suppose for all $m = pN + \mathcal{O}(\sqrt{p(1-p)N})$ we have $\mathcal{G}(n,m) \in Q$ a.a.s. Then, a.a.s $\mathcal{G}(n,p) \in Q$

(ii) Suppose Q is convex. Given $m = m(n)$ and suppose $\mathcal{G}(n, m/N) \in Q$ a.a.s. Then, a.a.s $\mathcal{G}(n,m) \in Q$

Proof Sketch.

(i) Write $\mathbb{P}(\mathcal{G}(n,p) \in G)$ in terms of the number of edges in the graph, i.e. $\mathbb{P}(\mathcal{G}(n,p) \in G) = \sum_{m=0}^{N} \mathbb{P}(X = m, \mathcal{G}(n,p) \in Q)$ (Law of total probability). Then, use Proposition 2.1.

(ii) Condition on the number of edges, and analyze the probabilities of having a graph with: less edges than 1 standard deviation (m_1), more edges than 1 standard deviation (m_2), and a number of edges within 1 standard deviation (m). Then construct graphs from $\mathcal{G}(n, m_1)$ and $\mathcal{G}(n, m_2)$ and use convexity to show $\mathbb{P}(\mathcal{G}(n,m) \in Q) = 1 - o(1)$

$\square$

Lecture 3: Threshold, First Order Logic of Graphs

3.1 Threshold

Definition 3.1. We say a property Q has a threshold p_0 if:

$$\mathbb{P}(\mathcal{G}(n,p) \in Q) \to \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}$$

Theorem 3.1 (Bollobás & Thomason, 1987)**.** Every non-trivial monotone property has a threshold

Definition 3.2. We say a property Q has a sharp threshold p_0 if $\forall \epsilon > 0$:

$$\mathbb{P}(\mathcal{G}(n,p) \in Q) \to \begin{cases} 0 & \text{if } p \leq (1-\epsilon)p_0 \\ 1 & \text{if } p \geq (1+\epsilon)p_0 \end{cases}$$

Definition 3.3. The window of a threshold is $\delta(\epsilon) = p_{1-\epsilon} - p_\epsilon$

3.2 First Order Logic of Graphs

Example 3.1.

$$\forall x \forall y \exists z \, (x = y \vee x \sim y \vee (x \sim z \wedge y \sim z))$$

is the statement characterizing the graphs of diameter ≤ 2 △

Fix $k > 0$. Let P_k be the property that for any disjoint sets W and V of order at most k, there exists a vertex $x \in V(G) \setminus (W \cup V)$ such that x is adjacent to all vertices in W and is adjacent to none of V

Lemma 3.1. Suppose $m(n), p(n)$ satisfy the following:

For every fixed $\epsilon > 0$

$$mn^{-2+\epsilon} \to \infty, \qquad (N-m)n^{-2+\epsilon} \to \infty$$
$$pn^{\epsilon} \to \infty, \qquad (1-p)n^{\epsilon} \to \infty$$

For every fixed $k > 0$, a.a.s $\mathcal{G}(n,p) \in P_k$ and $\mathcal{G}(n,m) \in P_k$

Theorem 3.2 (0-1 law of the 1st order logic of random graphs)**.** Suppose $m(n), p(n)$ satisfy the conditions of the lemma. Suppose Q is a graph property given by a 1st order sentence. Then, either Q holds a.a.s or does not hold a.a.s.

Proof Sketch. We play a k-round Ehrenfeucht-Fraïssé Game. Player 1 chooses vertices from either graph and Player 2 must choose vertices from the other graph.

After k rounds, this produces two sequences $v_1, v_2, \ldots, v_k$ in G_1 and $v'_1, v'_2, \ldots, v'_k$ in G_2. Player 2 wins if $v_i \mapsto v'_i \; \forall 1 \leq i \leq k$ is an isomorphism between $G_1[v_1, v_2, \ldots, v_k]$ and $G_2[v'_1, v'_2, \ldots, v'_k]$, and Player 1 wins otherwise.

The idea is that if G_1, G_2 are similar, then player 2 will win, but if they are not similar, then player 1 can exploit the dissimilarity.

<u>Claim</u>: Let $Th_k(G)$ be the set of graph properties of G expressible by 1st-order logic sentences with quantifier depth at most k. Player 2 has a winning strategy if and only if $Th_k(G_1) = Th_k(G_2)$ (i.e. They share the same set of properties) $\square$

Lecture 4: Evolution of Graphs and Theorem E

Theorem 4.1 (Theorem E)**.**

(a) Fix $k \geq 2$ integer. If $n^{\frac{k-2}{k-1}-2} \ll p \ll n^{\frac{k-1}{k}-2}$, then a.a.s $\mathcal{G}(n,p)$ is a forest and the largest component is of order k.

(b) If $p \ll \frac{1}{n}$, then a.a.s $\mathcal{G}(n,p)$ is a forest and the largest component is of order $o(\log n)$

(c) If $p = \frac{c}{n}$, $0 < c < 1$, then a.a.s every component of $\mathcal{G}(n,p)$ is a tree or unicyclic and the largest component has order $\Theta(\log n)$

(d) If $p = \frac{c}{n}$, $c > 1$, then a.a.s $\mathcal{G}(n,p)$ contains a unique component of linear order and all other components of order $\mathcal{O}(\log n)$

(e) When $p \geq \frac{\log n + \log\log n\omega(1)}{2n}$, a.a.s $\mathcal{G}(n,p)$ has a giant component and a few isolated vertices

(f) When $p \geq \frac{\log n + \omega(1)}{n}$, a.a.s $\mathcal{G}(n,p)$ connected and has a perfect matching if n even, or a matching of size $\frac{n-1}{2}$ if n is odd.

(g) When $p \geq \frac{\log n + \log\log n + \omega(1)}{n}$, a.a.s $\mathcal{G}(n,p)$ is Hamiltonian

4.1 Small Subgraphs

Lemma 4.1. If $p = o(1/n)$, then a.a.s $\mathcal{G}(n,p)$ has no cycles

Proof. Use Markov's Inequality □

Proof. (of Theorem 4.1 (a))

That $\mathcal{G}(n,p)$ is a forest is directly implied by above. It remains to show that every tree has order $\leq k$ and there is one tree of order k.

Let X_t be the number of trees of order t in $\mathcal{G}(n,p)$. First, we show that that $\mathbb{E}\left(\sum_{t \geq k+1} X_t\right) = o(1)$, so a.a.s $\sum_{t \geq k+1} X_t = 0$ by Markov's inequality. This tells us that we don't have trees of order $> k$

Then, we show the existence of a tree of order k by using the 2nd moment method. □

Lecture 5: Cycles, Degrees of Vertices, Critical Window Analysis

Let $X \sim \mathrm{Po}(\lambda)$ and recall that $\mathbb{P}(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$ $\forall i \geq 0$ and $\mathbb{E}(X)_r = \lambda^r$ $\forall r \geq 0$.

Theorem 5.1. Given a sequence of random variables $(X_n)_{n\geq 1}$ and $\lambda > 0$, suppose $\mathbb{E}(X_n)_r \to \lambda^r$ $\forall r \geq 0$. Then, $X_n \to \mathrm{Po}(\lambda)$ as $n \to \infty$

Let Y_k denote the number of k-cycles in $\mathcal{G}(n, p)$.

Theorem 5.2. Let $p = \frac{c}{n}$, where c fixed, then for every $k \geq 3$, $Y_k \to \mathrm{Po}\left(\frac{c^k}{2k}\right)$

Proof. $\mathbb{E}[Y_k] = \binom{n}{k}\frac{(k-1)!}{2}p^k \sim \frac{n^k}{k!}\frac{(k-1)!}{2}p^k = \frac{n^k p^k}{2k} = \frac{c^k}{2k}$

Now, fix $r \geq 2$ and consider:

$$(Y_k)_r = |\{C_1, \ldots, C_r \ : \ C_i \text{ is a } k\text{-cycle}\}|$$

So, $(Y_k)_r$ can be treated like the number of ordered lists of k-cycles. First, consider r k-cycles that are all vertex-disjoint, then consider the case where there are vertex intersections. Apply Theorem 5.1 to finish the proof. □

5.1 Degrees of Vertices

Theorem 5.3. Let $p = \frac{c \log n}{n}$ ($c > 0$ fixed).

(a) If $c < 1$, a.a.s $\exists$ isolated vertices

(b) If $c > 1$, a.a.s there are no isolated vertices

(Note: This is a sharp threshold)

Proof. Let X be the number of isolated vertices.

$$\mathbb{E}[X] = n \cdot (1-p)^{n-1} \sim n^{1-c}$$

Then, if $c > 1$, then use Markov's inequality to find (b). If $c < 1$, then use Chebyshev's inequality to find (a). □

5.2 Critical Window Analysis

Theorem 5.4. Let $p = \frac{\log n + c}{n}$, c fixed. Then

$$\mathbb{P}(X = 0) \sim e^{-e^{-c}} \tag{5.1}$$

Corollary 5.1. Let $p = \frac{\log n + c(n)}{n}$. If $c(n) \to -\infty$, then a.a.s $\mathcal{G}(n, p)$ has isolated vertices. And, if $c(n) \to +\infty$, then a.a.s $\mathcal{G}(n, p)$ has no isolated vertices

Let X_k be the number of vertices with degree k

Theorem 5.5. Let $\epsilon > 0$ be fixed, $\epsilon n^{-\frac{3}{2}} \le p \le 1 - \epsilon n^{-\frac{3}{2}}$. Let $k = k(n)$ be a nonnegative integer and $\lambda_k(n) = n \cdot \mathbb{P}(\mathrm{Bin}(n-1, p) = k)$ Then,

(i) If $\lambda_k(n) = o(1)$, then a.a.s $X_k = 0$

(ii) If $\lambda_k(n) \to o(1)$, then a.a.s $X_k \ge t$ for any fixed t.

(iii) If $0 < \lambda_k := \lim_{n\to\infty} \lambda_k(n) < \infty$, then $\mathbb{P}(X_k = t) \sim e^{\lambda_k} \cdot \frac{\lambda_k^t}{t!}$

Theorem 5.6. Let $p = \frac{\log n + c}{n}$, c fixed. Then, $\mathbb{P}(\mathcal{G}(n, p) \text{ is connected}) \to e^{-e^{-c}}$

In a feat of ultimate laziness (as the true computer scientist I am), the rest of the notes are going to be scans.

7CQ - ~~Colouring~~ Random. Jan 24th

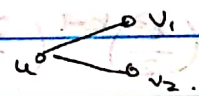
Lecture 6:

$\mathcal{G}(n,n,p)$ denotes the random bipartite graph with $V = V_1 \cup V_2$, $|V_1| = |V_2| = n$ and uv is an edge with prob p for $u \in V_1$, $v \in V_2$

Exercise: let $p = \frac{\log n + c}{n}$. Let X denote # of isolated vertices in $\mathcal{G}(n,n,p)$. Then,

- $P(X=0) \sim e^{-2e^{-c}}$
- $P(\mathcal{G}(n,n,p)$ is connected$) \sim e^{-2e^{-c}}$

A triple of vertices $\{u, v_1, v_2\}$ is called a cherry if $d(v_1) = d(v_2) = 1$ and $v_1 \sim u$ and $v_2 \sim u$.



Exercise: Let $p \sim c \log n / n$ where $c > 3/5$. Then a.a.s there are no cherries in $\mathcal{G}(n,n,p)$

Theorem (Hall's theorem)
A bipartite graph G with vertex set $V_1 \cup V_2$ has a PM if $|V_1| = |V_2|$ and $|N(S)| \geq |S|$ $\forall S \subseteq V_1$

Exercise: Suppose G fails Hall's condition. Then, there exists a set ~~S~~ $S \subseteq V_1$ or $S \subseteq V_2$.

(i) $N(S) = |S| - 1$

(ii) $|S| \leq \lceil \frac{n}{2} \rceil$

(iii) Every vertex in $N(S) \geq 2$ neighbours in S

Let X_k denote a k-set S satisfying (i), (ii), (iii)

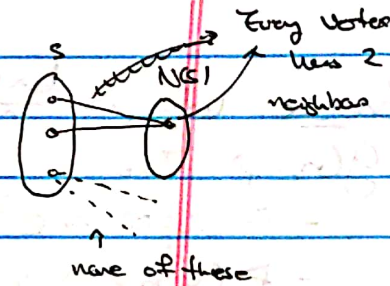
X_1 : # isolated vertices

X_2 : # cherries

Lemma:

Let $(3/4)\frac{\log n}{n} < p < \frac{2\log n}{n}$. Then, $\sum_{k=3}^{\lceil n/2 \rceil} \mathbb{E}X_k = o(1)$

Proof:



For each $3 \le k \le \lceil n/2 \rceil$

$$\mathbb{E}X_k \le \underbrace{2\cdot\binom{n}{k}}_{\text{Choose from either } V_1 \text{ or } V_2}\cdot\underbrace{\binom{n}{k-1}}_{\text{neighbors}}\underbrace{\left(\binom{k}{2}p^2\right)^{k-1}}_{\text{Each vertex in } N(S) \text{ have 2 neighbors}}\underbrace{(1-p)^{k(n-k+1)}}_{\text{No edges going from } S \text{ to not } N(S)}$$

$$\le 2\left(\frac{en}{k}\right)^k\left(\frac{en}{k-1}\right)^{k-1}\left(\frac{k^2}{2}\right)^{k-1}\left(\frac{2\log n}{n}\right)^{2k-2}\exp\Big(-\frac{3\log n}{4n}\cdot k\cdot \underbrace{\frac{n}{2}}_{\text{since } n-k+1 \le \frac{n}{2}}\Big)$$

$$\le 2n\left(\frac{k}{k-1}\right)^{k-1}\left(\frac{2e^2\log^2 n}{n^{3/8}}\right)^k$$

$$= O(n\alpha^k), \text{ where } \alpha = \frac{3e^2\log^2 n}{n^{3/8}}.$$

So: $\sum_{k=3}^{\lceil n/2 \rceil} \mathbb{E}X_k = o(1)$ □

Theorem

Let $p = \frac{\log n + c}{n}$, c fixed $\mathbb{R}$. Then $\Pr(\mathcal{G}(n,n,p)$ has a PM$) \sim e^{-e^{-c}}$

Proof:

Let C_1 denote the event that $X_1 = 0$, i.e. no isolated vertices,

C_2 — " — $X_k = 0 \ \forall\, 2 \le k \le \lceil n/2 \rceil$

Hall's theorem and Exercise $\Rightarrow C_1 \cap C_2 \Rightarrow \mathcal{G}(n,n,p)$ has a PM

So, $\Pr(\mathcal{G}(n,n,p)$ has PM$) \ge \Pr(C_1 \cap C_2)$

$= \Pr(C_1) - \Pr(C_1 \cap \overline{C_2})$

$\ge \Pr(C_1) - \Pr(\overline{C_2}) = P(C_1) - o(1)$ By lemma + exercise

Proof (con't)

So, $\mathcal{G}(n,n,p)$ has a PM $\Rightarrow C_1$ ①

And,

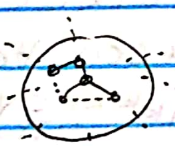
$$\Pr(\mathcal{G}(n,n,p) \text{ has a PM}) \le \Pr(C_1) \quad ②$$

So:

$$\Pr(\mathcal{G}(n,n,p) \text{ has a PM}) = \Pr(C_1) + o(1) \sim e^{-2e^{-c}} \quad \square$$

Let τ_k denote the number of components that are trees of order k.

Let $\tau = \sum_{k=1}^{n} k\tau_k$.



$$\mathbb{E}\tau_k = \underbrace{\binom{n}{k}}_{k \text{ vertices}} \underbrace{k^{k-2}}_{\# \text{ of trees}} \overbrace{p^{k-1}}^{\text{Edges in tree}} (1-p)^{\underbrace{\binom{k}{2}-(k-1)}_{\text{No other edges except for tree edges}} + \underbrace{k(n-k)}_{\text{edges coming out of component}}}$$

$$\mathbb{E}\tau = \sum_{k=1}^{n} k\, \mathbb{E}\tau_k$$

Theorem Let $p = \frac{c}{n}$, $c>0$ fixed. $\longrightarrow$ τ is a function of the random graph

(a) If $0<c<1$, then $\mathbb{E}[\tau(\mathcal{G}(n,p))] = n + O(1)$

(b) If $c>1$, then $\mathbb{E}[\tau(\mathcal{G}(n,p))] = t(c)n + O(1)$,

where $t(c) = \frac{1}{c}\sum_{k\ge 1} \frac{k^{k-1}}{k!}(ce^{-c})^k$

Note: $t(c) \le \sum_{k\ge1} \frac{1}{k}\left(\frac{ek}{k}\right)^k (ce^{-c})^k = \sum_{k\ge1}\frac{1}{k}(\underbrace{ce^{1-c}}_{\text{Can show this is } <1 \text{ for } c\neq 1})^k$.

So, $t(c)$ is well-defined.

②

Exercise: (a) $s(c) = c t(c)$ is the unique solution of $se^{-s} = ce^{-c}$ in the range $0 < s \le 1$

(b) $t(c) = 1 \;\; \forall\, 0 < c \le 1$.

Note: $\mathbb{E}[T(\mathcal{G}(n,p))]$ is cts. in p and $t(c) \to 1$ as $c \to 1$.
So, $\mathbb{E}[T(\mathcal{G}(n,\tfrac{1}{n}))] = n + o(n)$. (Since $t(c)n = n +$ "some error").

Proof:
Since $t(c) = 1 \;\forall\, 0 < c < 1$, it's sufficient to show that $\mathbb{E}T = t(c)n + O(1) \;\; \forall c \neq 1$.

$$\mathbb{E}T_k = \binom{n}{k} k^{k-2} \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{kn - \frac{k(k+3)}{2} + 1}.$$

Consider $1 \le k \le \sqrt{n}$. (In this range $\binom{n}{k} \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$)
...uh... actually we'll use:
$$\binom{n}{k} = \frac{n^k}{k!} \exp\left(-\frac{k^2}{n} + O\left(\frac{k^3}{n^2}\right)\right).$$

Then,
$$\mathbb{E}T_k = n \cdot \frac{k^{k-2}}{n!} c^{k-1} e^{-ck} \exp\left(-\frac{k^2}{n} + \frac{ck^2}{2n} + O\left(\frac{k}{n}\right)\right).$$

Hence,
$$\left| \sum_{k=1}^{\sqrt{n}} k\, \mathbb{E}T_k - \sum_{k=1}^{\sqrt{n}} n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck} \right|$$
$$= \sum_{k=1}^{\sqrt{n}} O\left(\frac{k^2}{n}\right) n \cdot \frac{k^{k-1}}{k!} c^{k-1} e^{-ck}$$
$$= \sum_{k=1}^{\sqrt{n}} O(k \cdot e^{k - ck} c^k)$$
$$= \sum_{k=1}^{\sqrt{n}} O(k(\underbrace{ce^{1-c}}_{<1})^k) = O(1).$$

$$\frac{\mathbb{E}\,(k+1)T_{k+1}}{\mathbb{E}\,kT_k} = (n-k)\underbrace{\left(1+\frac{1}{k}\right)^{k-2}}_{\le e}\cdot\frac{c}{n}\left(1-\frac{c}{n}\right)^{n-k-2}$$

$$\le \left(1-\frac{k}{n}\right)\cdot e\,c\,e^{-\frac{c}{n}(n-k)}\cdot\left(1-\frac{c}{n}\right)^{-2}$$

$$= \left(1-\frac{k}{n}\right)ece^{-c(1-\frac{k}{n})}\left(1-\frac{c}{n}\right)^{-2}$$

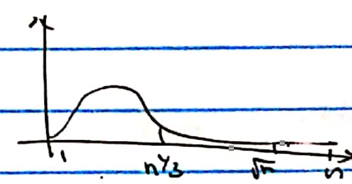
Let $\eta = (1-\frac{k}{n})$.

$$= \underbrace{\eta e c e^{-c\eta}}_{c\eta e^{1-c\eta}\le 1}\left(1-\frac{c}{n}\right)^{-2}$$

$$\le \left(1-\frac{c}{n}\right)^{-2} := \lambda.$$

It is easy to verify (using formulae we've shown for $\mathbb{E}T_k$), that $\mathbb{E}T_{k_1} = o(n^{-M})$ for any fixed $M>0$, where $k_1 = n^{1/3}$

$$\sum_{k=k_1}^{n} k\,\mathbb{E}[T_k] \le k_1\,\mathbb{E}[T_{k_1}]\sum_{i=0}^{n}\lambda^i$$



$$\le O(n)\cdot k_1\,\mathbb{E}[T_{k_1}]$$

Since $\lambda^i \le \lambda^n = O(1)$

$$= o(n^{-3})$$

It is easy to check that $\sum_{k=\sqrt{n}}^{\infty}\frac{k^{k-1}}{k!}c^{k-1}e^{-ck} = \cancel{o(1)}\ o(n^{-1})$

$$|\mathbb{E}T - t(c)n| \le \left|\sum_{k=1}^{\sqrt{n}} k\,\mathbb{E}T_k - n\cdot\frac{k^{k-1}}{k!}c^{k-1}e^{-ck}\right|$$

$$+\sum_{k=\sqrt{n}}^{n} k\,\mathbb{E}T_k + \sum_{k=\sqrt{n}}^{\infty} n\cdot\frac{k^{k-1}}{k!}c^{k-1}e^{-ck}$$

$$= O(1) + o(1) + o(1) = O(1)$$

□

749 - Random

Feb 3rd

Lecture 7:

Clarification on thresholds:

A property Q has a threshold $p_0 = p_0(n)$ if:

$$Pr(G(n,p) \in Q) \to \begin{cases} 0 & \text{if } p = o(p_0) \iff p \ll p_0 \\ 1 & \text{if } p = \omega(p_0) \iff p \gg p_0 \end{cases}$$

$\iff$

Q has a threshold $p_0 = p_0(n)$ if

$\forall \varepsilon > 0, \exists c > 0, n_0 > 0$ s.t. for all $n > n_0$

$Pr(G(n,p) \in Q) < \varepsilon$ if $p < \frac{1}{c} p_0$

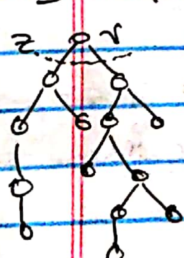
& $Pr(G(n,p) \in Q) > 1 - \varepsilon$ if $p > c p_0$

Recall: (Theorem 2(d))

If $p = c/n$ and $c > 1$, then a.a.s $G(n,p)$ contains a unique component of linear order (in n), and all other components are of order $O(\log n)$

Intuition for proof:

Branching / Exploration Process



$deg(v) \sim Bin(n-1, p) \approx Po(c)$ ← approximately (same poisson theorem)

Same parameter

$Z \sim Po(c)$

The out has poisson distribution

- Start at a vertex v, and explore the graph by doing a BFS/DFS on its children.
- Degrees (# of children) is roughly the binomial ~~distrib~~ distribution.

①

Formalizing this idea:

A Galton-Watson branching process.

Let Z be a random variable on nonnegative integers. At time $t=0$, a single particle comes into existence. At each $t \geq 1$, each existing particle gives birth to a random number of new particles and then it dies. These random numbers are i.i.d copies of Z.

Let X_t denote the number of particles born during step $t \geq 0$.

Thus, $X_0 = 1$, and X_1 has the same distribution as Z.

Let E denote the event "that $X_t = 0$ for some $t \geq 0$", i.e. the event is the process terminates after a finite # of steps.

Let $p_Z = \Pr(E)$

Exercise: For every $t \geq 1$, $\mathbb{E} X_t = (\mathbb{E} Z)^t$

Theorem

(i) If $\mathbb{E} Z < 1$, then $p_Z = 1$.

(ii) If $\mathbb{E} Z = 1$, and $\Pr(Z=0) > 0$, then $p_Z = 1$.

(iii) If $\mathbb{E} Z > 1$ and $\underbrace{\Pr(Z=0) > 0}$, then $0 < p_Z < 1$

to avoid triviality (if $\Pr(Z=0) = 0$, then we always give birth to new children, so $p_Z = 0$)

→ Proof

Proof:

(i) $\lim_{t\to\infty} \Pr(X_t > 0) \leq \lim_{t\to\infty} (\mathbb{E}Z)^t = 0$ — Since $\mathbb{E}Z < 1$.

Exercise + Markov

(ii) and (iii):

Let ρ_j be the probability that the process becomes extinct after j steps. Thus bldly occurs let $v_1, \dots, v_{X_1}$ denote the set of children born at step 1. The process terminates after j steps iff the X_1 processes starting from $v_1, \dots, v_{X_1}$ all terminate after $j-1$ steps. These X_1 processes are independent.

Then:

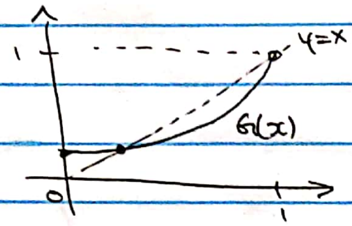
$$\rho_j = \sum_{i=0}^{\infty} \underbrace{\Pr(Z=i)}_{i \text{ children}} \underbrace{\rho_{j-1}^{i}}_{\text{each terminate in } j-1 \text{ steps}}$$

Let $G(x) = \mathbb{E}\, x^Z = \sum_{i=0}^{\infty} \Pr(Z=i) x^i$, then $\rho_j = G(\rho_{j-1})$

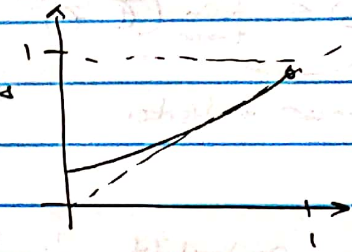
Note:

- $G(1) = 1$
- $G(0) > 0$ Since $G(0) = \Pr(Z=0) > 0$
- G convex (Since G is a probability generating function and they are always convex)
- $G'(1) = \sum_{i=1}^{\infty} i x^{i-1} \Pr(Z=i) \Big|_{x=1} = \mathbb{E}Z \geq 1$. — By assumption.

These facts can help us visualize G.



*Shape comes from that G convex

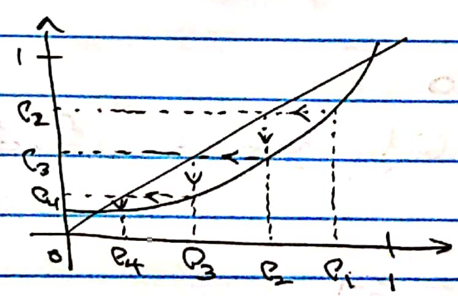


If $\mathbb{E}Z > 1$, then $G(x) = x$ has 2 roots, 1 root at $x=1$ and the other $0 < P_z < 1$

If $\mathbb{E}Z = 1$, then $G(x) = x$ has a unique root at $x=1$ (the line $y=x$ is tangent to G at $x=1$)

For (iii) (and (ii) is similar), it is sufficient to show that $P_t \to P_z$ for $t \to \infty$, $0 < P_1 < 1$, and the conclusion follows from the properties of G. $\square$

Proof by picture!



$P_2 = G(P_1)$, then find P_2 on the x-axis by following $y=x$ etc. Then $P_3 = G(G(P_2))$. And continue until we converge at the fixed point $G(P_z) = P_z$.

Theorem: (Chernoff Bound)

Let $X_1, \dots, X_n$ be independent $\{0,1\}$-value random variables. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}X$. Then the following probability bounds hold:

(a) For any $\delta > 0$:

$$\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}.$$

(a) $\Rightarrow$ (b)

(b) For any $0 < \delta \leq 1$

$$\Pr(X \geq (1+\delta)\mu) \leq \exp\left(-\frac{\mu\delta^2}{3}\right)$$

(c) For any $0 < \delta \leq 1$

$$\Pr(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

$$\Rightarrow \Pr(X \leq (1-\delta)\mu) \leq \exp\left(-\frac{\mu\delta^2}{2}\right)$$

Proof:

Let $X_i \sim \text{Bernoulli}(p_i)$

$$\Pr(X \geq (1+\delta)\mu) = \Pr(e^{tX} \geq e^{t(1+\delta)\mu}) \quad \text{holds for any } t>0,\ t \in \mathbb{R}$$

$$\leq \frac{\mathbb{E}e^{tX}}{e^{t(1+\delta)\mu}} \quad \text{Markov}$$

$$= \frac{\prod_{i=1}^{n} \mathbb{E}e^{tX_i}}{e^{t(1+\delta)\mu}} \quad \text{as } X_1, \dots, X_n \text{ are independent}$$

$$= \frac{\prod_{i=1}^{n} (1 - p_i + p_i e^{t})}{e^{t(1+\delta)\mu}}.$$

$$\leq \frac{\prod_{i=1}^{n} e^{p_i(e^t - 1)}}{e^{t(1+\delta)\mu}} \xrightarrow{\sum p_i = \mu\ (\ast)} \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}}.$$

For part (a), take $t = \log(1+\delta) > 0$

Proof: (con't)

Part (b) follows by noting that $\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \le e^{-\delta^2/3}$ for $0<\delta\le 1$

For part (c), For ~~~~ $t<0$

$$Pr(X \le (1-\delta)\mu) = Pr(e^{tX} \ge e^{t(1-\delta)\mu})$$
$$\le \frac{E(e^{tX})}{e^{t(1-\delta)\mu}}$$
Proceed as before
$$\le \frac{e^{\mu(e^t-1)}}{e^{t(1-\delta)\mu}}$$

And part (c) follows by taking $t = \log(1-\delta) < 0$.

The 2nd inequality follows by $\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \le e^{-\delta^2/2}$ for $0<\delta<1$ $\square$.

Remark:

The Chernoff bounds hold for ~~any~~ r.v. $X_1, \ldots, X_n$ that are independent $[0,1]$-valued as well.

Modifications for the proof:

At the step
$$\frac{\prod_{i=1}^{n} E e^{tX_i}}{e^{t(1+\delta)\mu}}$$
we cannot split this to $\frac{\prod_{i=1}^{n}(1-p_i+p_i e^t)}{e^{t(1+\delta)\mu}}$.

But, we will use that e^x is a convex function and $e^x \le \alpha x + \beta$ for all $x \in [0,1]$ where $\alpha = e^t - 1$ and $\beta = 1$.

Then,
$$E e^{tX_i} \le E(\alpha(tX_i) + \beta) = p_i e^t + (1-p_i)$$

and the remaining steps are the same.

Lecture 8:

Thm E(d): Let $c > 1$. a.a.s $G(n, c/n)$ has a unique giant component and all other components are of order $O(\log n)$

Still need a bit more probability theory before we can prove this theorem!

Stochastic Dominance:
Given 2 real r.v.s X and Y. We say that X stochastically dominates Y if:

$$P(X \geq x) \geq P(Y \geq x) \quad \forall x$$

We can couple X and Y such that $X \geq Y$ always. Then X stochastically dominates Y always. (Proof below)

Compare: If we have 2 ~~ra~~ random graphs G_1 and G_2 such that G_1 and G_2 can be coupled so that $G_1 \subseteq G_2$, then for any monotone increasing property Q:

$$Pr(G_2 \in Q) \geq Pr(G_1 \in Q)$$

Proof:
$\forall x$. Let $\Omega^X(x) = \{\omega \in \Omega \text{ s.t. } X(\omega) \geq x\}$

$\Omega^Y(x) = \{\omega \in \Omega \text{ s.t. } Y(\omega) \geq x\}$

Since in Ω we have $X(\omega) \geq Y(\omega)$ $\forall \omega \in \Omega$

$\Rightarrow \Omega^X(x) \supseteq \Omega^Y(x)$ $\forall x$

So:

$$P(X \geq x) = \int_{\omega \in \Omega^X(x)} dP(\omega) \geq \int_{\omega \in \Omega^Y(x)} dP(\omega) = P(Y \geq x)$$

$\square$

$\circledast$ The ρ's are ρ's (rho's)

Proof (of T(d))

Let $k_0 = C\log n$ and $k_1 = n^{2/3}$ where $C>0$ is a sufficiently large constant. We will prove a.a.s.:

(i) There are no components of $G(n, c/n)$ with order between k_0 and k_1

(ii) There is only ≤ 1 component of order greater than k_1

(iii) The total # of vertices in small components (order $\leq k_0$) is asymptotic to ρn, where $\rho e^{c} = e^{\rho c}$, $0<\rho<1$

(Compare ρ with $t(c)$ in the theorem of tree components $\Rightarrow$ "$\rho = t(c)$" (check yourself!)).

Intuition about ρ:

Take a uniformly random vertex u and explore its neighbours, its 2nd neighbour, etc. This mimics the branching process with $Z \sim Po(c)$. Recall that ρ_Z is the probability of extinction. Then, ρ_Z is the root of $\rho = G(\rho)$, $0<\rho<1$. (where $G(x)$ is the PGF for this distribution), and $G(x) = e^{-c+cx}$ (this is the PGF for Poisson)

(Aside:

$$G(x) = \sum_{j=0}^{\infty} Pr(Z=j)x^j = \sum_{j=0}^{\infty} e^{-c}\frac{c^j}{j!}x^j$$

$$= e^{-c}\sum_{j=0}^{\infty}\frac{c^j}{j!}x^j$$

$$= e^{-c+cx}$$

).

And ρ_Z is the fixed point for this PGF.

Proof: (con't)

Parts (i) - (iii) says that $G(n,p)$ contains a unique component of linear order and all other components are small. Intuitively, the prob. that v lies in a small component coincides with P_z.

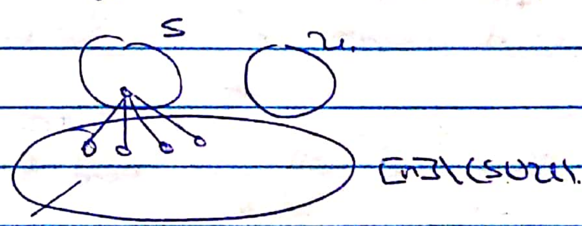
Consider the following graph exploration process. It starts with $t=0$ and $S=\{v\}$, $\mathcal{U}=\emptyset$. In every step $t\geq 1$, the process takes an arbitrary vertex $u\in S$, adds all neighbors of u in $G\setminus(S\cup\mathcal{U})$ into S, and then moves u from S to $\mathcal{U}$. The process terminates when $S=\emptyset$. and $\mathcal{U}$ contains all vertices in the same component as v and the size of $\mathcal{U}$ equals the # of steps the graph exploration process takes.

Let X_t be the size of S after step t. $X_0=1$

By the construction of the process, $|\mathcal{U}|=t$ and $|\mathcal{U}\cup S|=X_t+t$ after t steps of the process.

Let $Z_t = X_t - X_{t-1} + 1$ (# of new vertices added)

Then, $Z_{t+1} \sim \mathrm{Bin}(n-(X_t+t), p)$



Z_t counts all the new vertices,

this is a binomial distribution since an edge appears in $[n]\setminus(S\cup\mathcal{U})$ with probability p.

Proof:

To prove part (i), it is sufficient to prove that with probability $o(1/n)$ X_t reaches 0 for some $k_0 < t < k_1$, as by union bound, with probability $o(1)$, there is a component of order between k_0 and k_1.

Let $k_0 < \tau < k_1$ and E_τ be the event that $(X_t)_{t \geq 0}$ becomes 0 after step τ.

- Given E_τ, define $W_\tau := \sum_{t=1}^{\tau} Z_t = X_\tau - X_0 + \tau = \tau - 1$
- $Z_t \geq 0 \Rightarrow X_t \geq X_{t-1} - 1$ (In the worst case, we add no new vertex and only remove ourself).

  If for some $t' < \tau$ we have $X_{t'} > k_1$, then consequently

$\Rightarrow$ $X_t > 0$ for all $t < k_1$, contradicting that $X_\tau = 0$.

So $E_\tau \Rightarrow X_{t'} \leq k_1$ for all $t' < \tau$.

In summary:

$$E_\tau \Rightarrow \begin{cases} W_\tau = \tau - 1 & (1) \\ X_{t'} \leq k_1 \text{ for all } t' < \tau & (2) \end{cases}$$

(2) implies that $W_\tau := \sum_{t=1}^{\tau} Z_t$ stochastically dominates $\hat{W}_\tau := \sum_{t=1}^{\tau} \hat{Z}_t$ where $\hat{Z}_t$ are i.i.d copies of $Z \sim Bin(n - 2k_1, p)$

Since both $X_t \leq k_1$ and $t \leq k_1$

This r.v. will be easier to analyze since Z changes with every step.

Exercise ??

Aside: $Bin(n_1,p)$ dominates $Bin(n_2,p)$ if $n_1 \geq n_2$)

Proof (cont)

since each Z_t binomial.

$\tilde{W}_\tau \sim Bin(\tau(n-2k_1), p)$

$E\,\tilde{W}_\tau = \tau(n-2k_1)\cdot\frac{c}{n} = c\tau$ (where $\tau > 1$)

~~Get Event~~

~~...~~

Let $\sigma = c-1 > 0$ and let $\varepsilon = \sigma/2c$. Then $(1-\varepsilon)c = 1+\frac{\sigma}{2}$

By Chernoff Bound:

$$Pr(E_\tau) \leq Pr(① \text{ and } ②)$$

$$\leq Pr(\tilde{W}_\tau \leq 1)$$

since $\tau \leq (1-\varepsilon)\tau c(1-\frac{2k_1}{n}) = (1+\frac{\sigma}{2})\tau(1-\frac{2k_1}{n})$

$$= Pr(\tilde{W}_\tau < (1-\varepsilon)\tau c(1-\frac{2k_1}{n}))$$

$$\leq \exp(-k\varepsilon^2\tau) \quad \text{(for some constant } k>0)$$

$$= o(\tfrac{1}{n}) \quad \text{since } \tau \geq k_0 = c\log n \text{ and } c \text{ is sufficiently large}$$

And part (i) follows by taking union bound over $v \in [n]$.

36

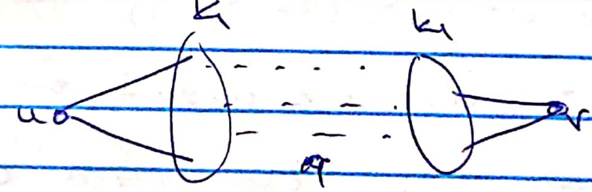
Proof (con't)

Part (ii). For any 2 vertices u and v, we prove that with prob $o(n^{-2})$, they lie in 2 different components, each of order greater than k_1

Let $X_t(u)$ and $X_t(v)$ denote the size of $S(u)$ and $S(v)$ with respect to the graph exploration processes starting from u and v respectively.

We may assume that $X_t(u)$ and $X_t(v)$ are positive for all $t < k_1$ (since o/w one of them lie in order k_1, i.e. the component has order $< k_1$)

Take $\varepsilon = \delta/2e$ and let $W_{k_1}(i) = \sum_{t=1}^{k_1} Z_t(i) = X_{k_1}(i) - X_0(i) + k_1$ for $i \in \{u, v\}$



We want to show w.h.p that if after k_1 steps, if we still have a large # of vertices to explore there must be an edge b/w the component containing u and the component containing v.

Proof (cont.)

By Chernoff Bound, (and similar argument as (i)),

$$\Pr\left(|W_{k_1}(i)| < (1-\varepsilon)k_1 c\left(1-\frac{2k_1}{n}\right)\right) = o(n^{-2})$$

Hence with prob $\geq 1-o(n^{-2})$, $X_{k_1}(u), X_{k_1}(v) \leq \left(\frac{\sigma}{4}\right)k_1$

If $S(u) \cap S(v) \neq \emptyset$ after k_1 steps, then we're done. Else we may assume $|S(u)|, |S(v)| \geq \left(\frac{\sigma}{4}\right)k_1$, and

$$\Pr(\text{no edge } b/w\ S(u) \text{ and } S(v))$$

$$\leq (1-p)^{\left(\frac{\sigma}{4}k_1\right)^2}$$

$$\leq o(n^{-2})$$

$\Rightarrow$ Part (ii) (By union Bound)

Lecture 9:

* All p's are thoes.

Recall:

$k_0 = C\log n$, $k_1 = n^{2/3}$. Prove a.a.s

(i) No components of order (k_0, k_1)

(ii) Only one component of order $\geq k_1$

(iii) # vertices in small component $\sim \rho n$, where $\rho e^c = e^{\rho c}$ $(0<\rho<1)$

We will prove (iii) this class:

Let Y denote the # of vertices lying in small components (i.e. of order $< k_0$). We will prove that the probability for a ~~random~~ uniformly random vertex to be in a small component is $\rho + o(1)$.

Lemma: Fix $c>1$ and let $Z_n \sim Bin(n, p)$ where $p \sim c/n$. Then $\rho_{Z_n} \to \rho$ as $n\to\infty$ where $\rho e^c = e^{\rho c}$, $(0<\rho<1)$

Proof:

Let $G(x) = \mathbb{E}(x^{Z_n}) = (1-p+px)^n = e^{\underbrace{-c+cx+O(\frac{x^2}{n})+o(1)}_{\text{why?}}}$,

which converges to e^{-c+cx} pointwise as $n\to\infty$.

By the proof of the theorem for the branching process, part (iii):

ρ_{Z_n} is the unique solution of $G(x)=x$, where $0<x<1$.

As $G(x) \to \underbrace{e^{-c+cx}}_{F(x)}$ pointwise and ρ is the unique solution of

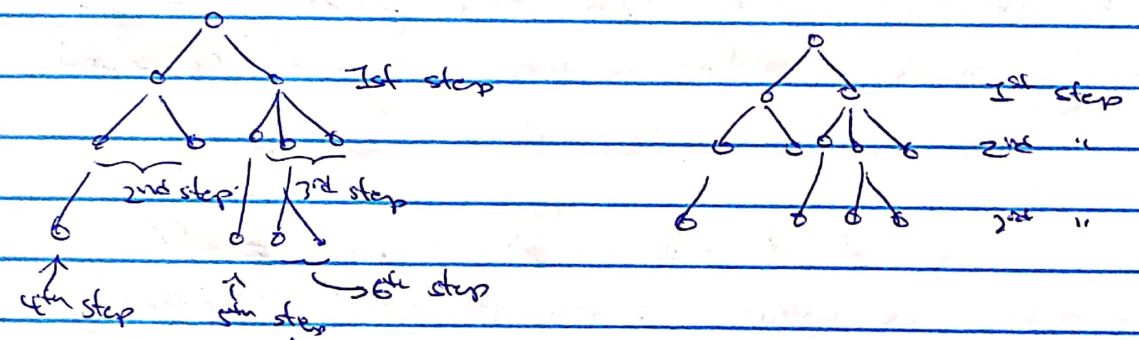
$x = F(x)$, $0<x<1$, it follows immediately $\rho_{Z_n} \to \rho$ as $n\to\infty$. ☐

Lemma $\mathbb{E}Y = (1+o(1))\rho n$.

Proof:

Consider the graph exploration process starting from v, $\hat{\varphi}_v$.
Let E_v denote the event that $\hat{\varphi}_v$ lasts $\le k_0$ steps.
Consider the slowed down version of the branching process where $Z_n \sim \text{Bin}(n, (1+\xi)\frac{c}{n})$, where $0 < \xi = o(1)$.

(Slowed-down Branching process:



i.e. reveal grandchildren one child at a time)

$\text{Bin}(n, (1+\xi)\frac{c}{n})$ stochastically dominates $\text{Bin}(n-(\sum X_t + t), \frac{c}{n})$, so we can couple $(\hat{\varphi}_v, \varphi)$ such that $\hat{\varphi}_v$ terminates no later than φ. (Exercise!)

Let P_t denote the probability that φ terminates during the first t steps. We know $P_t \to P_{Z_n}$.
That is $\forall \varepsilon > 0, \exists n_0 > n \ \forall n > n_0$: $P_{k_0} \ge P_{Z_n} - \varepsilon$
And also, $P_{Z_n} \to \rho$ as $n \to \infty \Rightarrow P_{Z_n} \ge \rho - \varepsilon$
$\Rightarrow P_{k_0} \ge \rho - 2\varepsilon$

By the coupling $(\hat{\varphi}_v, \varphi)$, we have $P(E_v) \geq P(\varphi$ terminates before k_0 steps$) = P_{k_0} > P - \varepsilon \Rightarrow P(E_v) \geq P - o(1)$

For the upper bound, consider the stand-down branching process with $Z_{t+1} \sim Bin(n-k_0, (1-\varphi)\frac{c}{n})$. Z_t is stochastically dominated by $Bin(n-(Z_t+t), \frac{c}{n})$ for $t < k_0$

Again, we can couple these two processes such that during the first k_0 steps the graph exploration process will terminate no earlier than the branching process

$\Rightarrow P(E_v) \leq P($Branching process lasts $\leq k_0$ steps$)$

$= P + o(1)$., using similar argument as above

$\Rightarrow P(E_v) = P + o(1)$

$\Rightarrow EY = (1 + o(1))Pn$

$\square$

$EY(Y-1)$. Fix a pair of vertices u and v.

$Pr(u$ is in a small component$) = P + o(1)$

Condition on that. The graph exploration process only "exposes" $\leq k_0$ vertices.

(Exercise: Easy to show that a.a.s. v is not exposed) $\to$ i.e. In the same component that u is in.

Run the graph exploration process starting from v, conditioned on C_u, the component containing u. This is equivalent to running the graph exploration process in $G(n', p)$ where $n' = n - O(k)$.

Repeat the proof as for EY. We have...

We have $\Pr(v$ is in a small component $| u$ is in a small component$)$
$= \rho + o(1)$.

$\mathbb{E}Y(Y-1) = n(n-1)(\rho^2 + o(1)) \sim (\mathbb{E}Y)^2$.

By Chebyshev's Inequality: $\forall \varepsilon > 0$

$$\Pr(|Y - \mathbb{E}Y| > \varepsilon\, \mathbb{E}Y) \le \frac{\operatorname{Var} Y}{\varepsilon^2 (\mathbb{E}Y)^2} = \frac{o(1)\cdot(\mathbb{E}Y)^2}{\varepsilon^2(\mathbb{E}Y)^2} = o(1)$$

So a.a.s. $Y \sim \mathbb{E}Y = (\rho + o(1))n$. $\square$

Hamiltonian Cycles in $\mathcal{G}(n,p)$:   p's are now p's

Theorem:
Let $p = \dfrac{\log n + \log\log n + x(n)}{n}$.

(a) If $x(n) \to \infty$, then a.a.s. $\mathcal{G}(n,p)$ is Hamiltonian
(b) If $x(n) \to -\infty$, then ——"—— not Hamiltonian

(Note: when degree 1 vertices disappear, then our graph becomes Hamiltonian (w.h.p)).

Proof:
Let D_2 denote the property that min. degree is ≥ 2
HAM ——"—— graph is Hamiltonian
$G \in HAM \Rightarrow G \in D_2$

Lemma (Exercise)

(a) If $pn-(\log n+\log\log n)\to-\infty$ then a.a.s $\mathcal{G}(n,p)\notin D2$

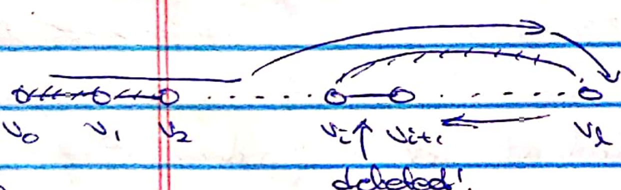
(b) If $pn-(\log n+\log\log n)\to\infty$ then a.a.s $\mathcal{G}(n,p)\in D2$

Lemma $\Rightarrow$ Part (b) of the theorem.

Idea for proof:

We try to build a longer and longer path until we build a Hamiltonian cycle.
What is the probability of us creating a longer path? $\to$ We can only build from the 2 ends

Posá Rotation:



$v_0v_1v_2\ldots v_iv_\ell v_{\ell-1}\ldots v_{i+1}$ is a Posá Rotation

Def'n:
Let $P=v_0v_1\ldots v_\ell$ be a longest path in G. Assume v_iv_ℓ is an edge. Then, the operation of deleting edge v_iv_{i+1} from P and adding edge v_iv_ℓ to P is called a Posá Rotation.

The Posá Rotation converts P to another longest path

Given a longest path $P = v_0 v_1 \ldots v_\ell$. Let $\mathcal{P}$ be the set of longest paths obtained by fixing end v_0 and repeatedly performing Pósa Rotations.

Let $\text{End}(v_0)$ be the set of ends of paths in $\mathcal{P}$ other than v_0.

Given a subset S of vertices, let

$$N(S) = \bigcup\{u \notin S : \exists v \in S \text{ s.t. } v \sim u\}$$

Lemma: $|N(\text{End}(v_0))| < 2|\text{End}(v_0)|$

Proof:

It is sufficient to show that if $v_i \in N(\text{End}(v_0))$, then one of v_{i-1}, v_{i+1} must be in $\text{End}(v_0)$

(Extremal Case:

)

Consider a sequence of Pósa rotations which produces $\mathcal{P}'$ with $v_i \sim x$, where $x \neq v_0$ is an end of $\mathcal{P}'$. Let y and z be the left and right neighbors of v_i on $\mathcal{P}'$. If $\{y, z\} = \{v_{i-1}, v_{i+1}\}$, then one of them can be added to $\text{End}(v_0)$ by performing a Pósa Rotation on $\mathcal{P}'$. O/w, one of the edges $v_i v_{i-1}$, $v_i v_{i+1}$ have been deleted in a previous Pósa Rotation $\Rightarrow$ one of v_{i-1}, v_{i+1} have been added to $\text{End}(v_0)$.

$\square$

Lecture 10:

Recall:

① Want to prove if $pn - (\log n + \log\log n) \to \infty$, then a.a.s $\mathcal{G}(n,p) \in HAM$

② Posá Rotation:

$|N(End(v_0))| < 2|End(v_0)|$.

Lemma: Then there exists a constant $\varepsilon_0 > 0$:

Assume $pn - (\log n + \log\log n) \to \infty$ and $pn \leq 2\log n$ s.t. a.a.s there is no $S \subseteq [n]$, $|S| \leq \varepsilon_0 n$ and $|N(S)| < 2|S|$.

Proof: Later! (We'll need this for the upcoming proof)

Back to the proof of hamiltonicity of $\mathcal{G}(n,p)$:

Let $\varepsilon_0 > 0$ be the constant in the lemma.

Let Exp denote the property that $\forall S \subseteq [n]$ where $|S| \leq \varepsilon_0 n$ $|N(S)| \geq 2|S|$

CNT denote the property that G is connected

Let $f = pn - (\log n + \log\log n)$

By our assumption, $f \to \infty$ as $n \to \infty$.

Let $p_1 = \frac{\log n + \log\log n + f/2}{n}$. Let $G' \sim \mathcal{G}(n, p_1)$. Let G be the random graph by adding $fn/8$ uniformly random edges to G'.

We can show (exercise!) that G and $\mathcal{G}(n,p)$ can be coupled so that a.a.s $G \subseteq \mathcal{G}(n,p)$. So, it is sufficient to show that a.a.s G is Hamiltonian

Proof (con't)

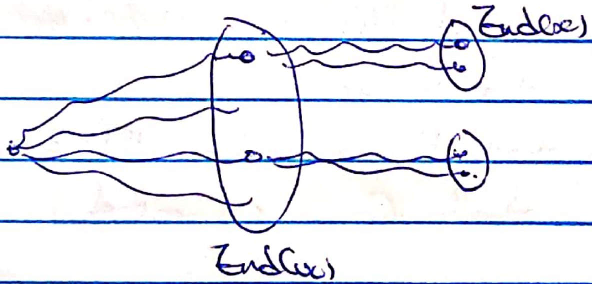
We know that a.a.s. $G' \in \mathrm{Exp} \cap \mathrm{CNT}$.

Take a longest path $P = v_0 v_1 \dots v_l$ in G'. Let $\mathcal{P}$ be the set of paths obtained from Pósa rotations with v_0 fixed.

Let $\mathrm{End}(v_0)$ denote the other ends of paths in $\mathcal{P}$.

Since $G' \in \mathrm{Exp}$ and $|N(\mathrm{End}(v_0))| < 2|\mathrm{End}(v_0)|$. $\Rightarrow$ $|\mathrm{End}(v_0)| \geq \varepsilon_0 n$.

Next for each $x \in \mathrm{End}(v_0)$ let $\mathcal{P}_x$ denote the set of paths obtained via Pósa rotations with x fixed and let $\mathrm{End}(x)$ be the set of the other ends of $p \in \mathcal{P}_x$.



Let $\xi = \{\{x,y\}, x \in \mathrm{End}(v_0), y \in \mathrm{End}(x)\}$. ξ is called the set of boosters.

As before, we have $|\mathrm{End}(x)| \geq \varepsilon_0 n$ for every $x \in \mathrm{End}(v_0)$. It follows now that $|\xi| \geq \varepsilon_0^2 n^2$. Consider G. We will sprinkle the $\delta n/8$ edges in sequence. We call each sprinkled edge a "trial". A trial is successful if it is in ξ and fails otherwise. If a trial is successful, then since G' is connected, it will result in finding a path longer than P or finding a Hamiltonian ~~pa~~ cycle if P is a Hamiltonian path $\Longrightarrow$

Feb 12th

Proof (cont)
Each trial is successful with probability $\geq c\varepsilon_0^2$ (for some constant $c>0$). Only at most n successful trials are needed to guarantee a Hamiltonian cycle in G.

$$\Pr(G \notin HAM) \leq \Pr(\mathrm{Bin}(\tfrac{n}{\varepsilon_0}, c\varepsilon_0^2) \leq n) = e^{-\Omega(\frac{n}{\varepsilon_0})} = o(1)$$

$\square$

~~Useful key lemma~~: $\to \log n \leq pn \leq 2\log n$.
Lemma: Assume ~~$pn \leq 2\log n$~~. Then:
(a) Each $S \subseteq [n]$ with $|S| \leq \frac{n}{\log^2 n}$ induces at most $3|S|$ edges (sublinear-sized subgraphs are sparse)
(b) No two vertices with degree <100 are within distance 5
(c) No vertex with degree ≤ 100 lies in a cycle of length ≤ 5
Proof: Exercise!

Back to 2-expander lemma:
Proof:
Let $n^{3/4} \leq s \leq \varepsilon_0 n$ where ε_0 is a sufficiently small constant.
Let $E(S,W)$ be the expected number of pairs of disjoint sets (S,W) s.t. $|S|=s$, $|W|=w$ and $W=N(S)$

$$E(S,W) = \underbrace{\binom{n}{s}}_{\text{Choose } S}\underbrace{\binom{n-s}{w}}_{\text{Choose } W}\left(1-\underbrace{(1-p)^{s}}_{\text{Don't see any vertex in } S}\right)^{w}\underbrace{(1-p)^{s(n-s-w)}}_{S \text{ cannot see any other vertices}}$$

every vertex in W sees a vertex in S

Complement! See a vertex in S

$\xrightarrow{}$ Since $1-(1-p)^s \le ps$

$$E(s,w) \le \left(\frac{en}{s}\right)^s \left(\frac{en}{w}\right)^w (ps)^w \exp((1-ps(n-s\epsilon w)))$$

$$= \left(\frac{en}{s}\right)^s \left(\frac{en}{w}\right)^w (ps)^w \left(\exp((1-pn\left(1-\frac{s\epsilon w}{n}\right)))\right)^s.$$

(Since $pn \ge \log n$, $ps \le p\varepsilon_0 n < \log n$, if $\varepsilon_0 < \frac{1}{2}$)

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w \exp(\underbrace{ps}_{<\log n}(s\epsilon w))$$

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w \exp(\underbrace{pn}_{\le 2\log n}.s.\frac{s\epsilon w}{n})$$

$$\le (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w n^{2s(s\epsilon w)/n}$$

Then

$$\sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} E(s,w) \le \sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} (\log n)^w \left(\frac{e}{s}\right)^s \left(\frac{en}{w}\right)^w n^{2s(s+w)/n}$$

$$\le \sum_{s=n^{3/4}}^{\varepsilon_0 n} (\log n)^{2s} \left(\frac{e}{s}\right)^s \left(\frac{en}{2s}\right)^{2s} n^{2s.3s/n}$$

$$= \sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\log^2 n \cdot \frac{e}{s} \cdot \frac{e^2n^2}{4s^2} \cdot n^{6s/n}\right)^s$$

$$\le C\sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\frac{c\log^2 n}{s^3} \cdot n^{2+\frac{6s}{n}}\right)^s$$

$$\le C\sum_{s=n^{3/4}}^{\varepsilon_0 n} \left(\frac{c\log^2 n}{s^3}\, n^{2+6\varepsilon_0}\right)^s.$$

Choose ε_0 small so that $2+6\varepsilon_0 < 9/4$. Therefore,

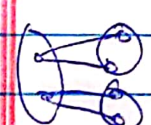
$$\sum_{s=n^{3/4}}^{\varepsilon_0 n} \sum_{w=1}^{2s} E(s,w) \le \sum_{s=n^{3/4}}^{\varepsilon_0 n} n^{-\alpha s} = o(1) \text{ for some } \alpha > 0$$

Now for $s < n^{3/4}$. Assume G is a graph satisfying properties (a)-(c) from the lemma, and $G \in D2$ (min. degree of G is ≥ 2)

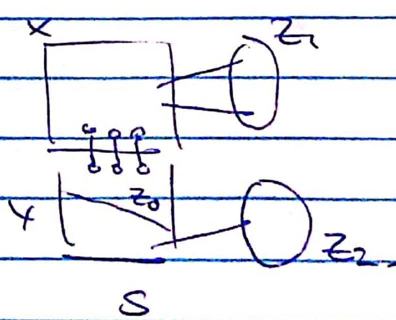
We know that a.a.s. $G(n,p)$ has these properties. It is sufficient to show that any graph G satisfying these above properties satisfies
$\forall S \subseteq [n]$ where $|S| \leq n^{3/4}$, $|N(S)| \geq 2|S|$

We call a vertex "light" if its degree is at most 100, otherwise call it "heavy". For any $S \subseteq [n]$, let $X \subseteq S$ be the set of light vertices in S and let
$Y = S \setminus X$.

Case I: $Y = \emptyset$ (i.e. All vertices in S are light)
Then $|N(S)| = |N(X)| \geq 2|X| = 2|S|$. by $G \in D2$ and property (b).

X  { All vertex disjoint

Case 2: $Y \neq \emptyset$.
Assume $|N(S)| < 2|S|$
$Z_0 = N(X) \cap Y$
$Z_1 = N(X) \setminus Y$
$Z_2 = N(Y) \setminus (X \cup Z_1)$



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Proof (con't)

$$|N(S)| = |Z_1| + |Z_2| + |Z_3| \underbrace{<}_{\text{By assumption}} 2|S| = 2(|X| + |Y|)$$

We have

Since $G \in D_2$ and by properties (b) and (c), we will have.

$|Z_0| + |Z_1| \geq 2|X|$

$|Z_0| = e(X, Y) \leq |Y|$

$e(Y, Z_1) \leq |Y|$.

(We will continue after reading week!)

749 - Random Feb 2th

Lecture 11:

<u>Recall:</u>

<u>Lemma:</u> $pn - (\log n + \log\log n) \to \infty$ and $pn < 2\log n$. Then, $\exists\ \varepsilon_0 > 0$ s.t. a.a.s $\forall S \subseteq [n]$, $|S| \leq \varepsilon_0 n$ and $|N(S)| \geq 2|S|$

<u>Proof:</u>

A.a.s, no S with $n^{3/4} \leq |S| \leq \varepsilon_0 n$, $|N(S)| < 2|S|$ (last class)

...

And we've also previously proved the following properties:

A.a.s in $\mathcal{G}(n,p)$:

(a) $S \subseteq [n]$ with $|S| \leq \frac{n}{\log^2 n}$ induces $\leq 3|S|$ edges

(b) No path of length at most 5 joining 2 light vertices

(c) No cycle of length ≤ 5 has a light vertex

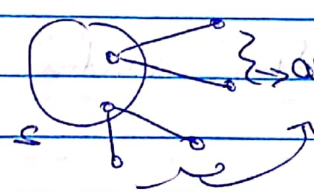
(Note: Not probabalistic)

We will now complete the proof of the lemma:

For $S < n^{3/4}$. Assume G satisfies (a)-(c) and $G \in D2$. Assume $S \subseteq [n]$ with $s = |S| < n^{3/4}$. and $|N(S)| < 2|S|$

Let X be the set of light vertices in S and $Y = S - X$.

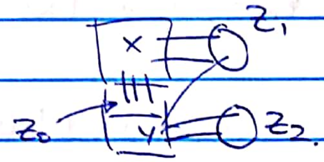
If $Y = \varphi$, then $|N(S)| \geq 2|S|$ (by (b) and D2)



all vertex-disjoint, since otherwise we violate (b).

Proof: (cont)

If $Y \neq \emptyset$. Let $Z_0 = N(X) \cap Y$, $Z_1 = N(X) \setminus Y$, $Z_2 = N(Y) \setminus (X \cup Z_1)$



$$|N(S)| = |Z_1| + |Z_2| \underbrace{<}_{\text{By assumption}} 2|S| = 2(|X| + |Y|)$$

$e(A,B)$: # of edges b/w A and B.

By GED2 and (b), (c)

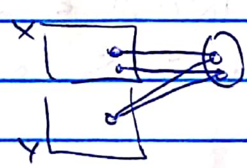
(i) $|Z_0| + |Z_1| \geq 2|X|$ (D2 and (b))

(ii) $|Z_0| = e(X,Y) \leq |Y|$

(iii) $e(Y, Z_1) \leq |Y|$

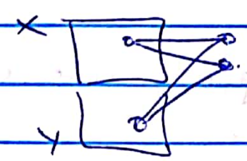
→ Every vertex in Y can only see one vertex in Z_1

Suppose o/w, then we get smth like:



But this is a path of length <5, that joins 2 vertices in X (violating (b))

OR



and this violates (c).

Hence,

$$2|X| - |Z_0| + |Z_2| \underset{\text{By (i)}}{\leq} |Z_1| + |Z_2| \underset{\text{By assumption}}{<} 2(|X| + |Y|)$$

Thus:

$$|Z_2| < |Z_0| + 2|Y| \underset{\text{By (ii)}}{\leq} 3|Y|.$$

By (a), $e(Y, Z_2) \leq e(Y \cup Z_2) \leq 3(|Y| + |Z_2|) \leq 12|Y|$.
Since every vertex in Y has degree ≥ 100, we have:

$$\underbrace{e(Y, Z_1)}_{\substack{\leq |Y| \\ \text{(By (iii))}}} + \underbrace{2e(Y)}_{\leq 6|Y|} + \underbrace{e(X, Y)}_{\substack{\leq |Y| \\ \text{(By (ii))}}} + \underbrace{e(Y, Z_2)}_{\leq 12|Y| \text{ (above)}} \geq 100|Y|$$

$\Rightarrow 100|Y| \leq 20|Y| \Rightarrow |Y| = 0$

Contradicting that $Y \neq \emptyset$ $\square$

Note: The 100 was arbitrarily chosen, we could have chosen any large #.

AIQ3:

Show that if $p = o\left(\frac{\log n}{n}\right)$ a.a.s $\Delta(G(n,p))$ has a 2-point concentration.

Let $\lambda_k = n \cdot P(Bin(n-1, p) = k)$ = Expected # of vertices with degree k.

① Prove that λ_k monotonely decreasing on $k > np$

Let k^* be the integer $\gg np$ which minimizes $\max\{\lambda_k, \frac{1}{\lambda_k}\}$

... But, we have to show that this is actually well-defined

Instead we could try:

② Let $k^* \geq np$ be the max. integer s.t. $\lambda_{k^*} \geq 1$. (And let $k^* = n-1$ if $\lambda_{n-1} \geq 1$)

③ Prove that $\forall$ constant $c > 0$, $\lambda_{cnp} = \omega(1)$

$\Rightarrow k^* = \omega(np)$

④ $\frac{\lambda_k}{\lambda_{k-1}} \sim \frac{np}{k} \Rightarrow \lambda_{k^*} \geq 1$ and $\lambda_{k^*+1} < 1$

Case I: $\lambda_{k^*} = \omega(1)$

$\lambda_{k^*+2} = o(1)$ and $\sum_{j \geq k^*+2} \lambda_j = o(1)$ (By (4))

$\Rightarrow$ a.a.s $\Delta \in \{k^*, k^*+1\}$

Case 2: $\lambda_{k^*} = O(1)$, $\lambda_{k^*-1} = \omega(1)$, $\lambda_{k^*+1} = o(1)$

Then, $\sum_{j \geq k^*+1} \lambda_j = o(1)$ (By (4))

$\Rightarrow$ a.a.s $\Delta \in \{k^*-1, k^*\}$

Feb 26th

Subsubsequence Principle:
If for every subsequence of a sequence there is a subsubsequence converging to a limit, then the entire sequence must converge to the same limit.
(True for any sequence in a topological space)

Suppose you want to prove a limit theorem for $G(n,p)$
You just need to prove such a limit theorem for $p = p_n$ s.t. p_n has an expression that converges to a limit
(E.g. $n^a p^b \to c$ for some a, b)
(E.g:
$p = \Theta(n^{-2/3})$, then $n^2 p^3 \to c$)

In the example of the 2-point concentration:
We have $p = o(\frac{\log n}{n})$. Assume the 2-point concentration is not true. Then, there is a subsequence S of $\{p_n\}$ s.t. $\exists \varepsilon > 0$, $Pr(\exists \{i,j\} : \Delta \in \{i,j\}) < 1-\varepsilon$ $\forall n$.
In the subsequence:

However we can show that there is a subsequence of S (so a subsubsequence of p_n), s.t. 2-point concentration holds in this subsubsequence, a contradiction.

Lecture 12:

Martingales and Azuma-Höffding Inequality:

Suppose X, Y are r.v.s in a probability space let $Z = \mathbb{E}[Y|X]$ (the expectation of Y conditioned on X) is a random variable defined by:

$$\int_A Z\,dP = \int_A Y\,dP$$

for every X-measurable set A.

Alternative definition for "nice" X and Y:

$\mathbb{E}(Y|X)(\omega) = \mathbb{E}(Y \mid X = X(\omega))$ for every $\omega \in \Omega$. $(*)$

whenever the above $\mathbb{E}(Y|X = X(\omega))$ is well defined.

Recall: $\mathbb{E}(X|A) = \dfrac{\mathbb{E}(X\, \mathbb{1}\{A\})}{P(A)} = \dfrac{1}{P(A)}\displaystyle\int_A X(\omega)\,dP(\omega)$

where A is an event and $\mathbb{1}\{A\}$ is the indicator variable for A.

When is $(*)$ well-defined?

For instance,

① If X is a discrete random variable, then

$$\mathbb{E}(Y|X = X(\omega)) = \frac{\mathbb{E}(Y\,\mathbb{1}\{X = X(\omega)\})}{P(X = X(\omega))}$$

② If the conditional distribution of Y given X is a continuous distribution and the conditional density function is denoted $f_{Y|X}(y|x)$, then $\mathbb{E}(Y|X=x) = \int y\, f_{Y|X}(y|x)\,dy$.

Example:

Rolling 2 dices independently with X_1, X_2 denoting the points shown. Let $X = X_1 + X_2$. What is $E(X_2 | X_1)$?

Condition on $X_1 = x$, we have

$$E(X | X_1 = x) = E(X_2 + x | X_1 = x)$$

$$= x + E X_2$$ Since X_1, X_2 are independent

$$= x + 7/2$$

This holds for every x so $E(X | X_1) = X_1 + 7/2$

Example:

Let X be a real number uniformly distributed in $[0,1]$ and Y be a r.v. uniformly chosen from $[X, 1]$. What is $E(Y|X)$?

Conditional Density function $f_{Y|X}(y | x) = \frac{1}{1-x}$

Hence,

$$E(Y | X = x) = \int_x^1 \frac{y}{1-x} dy = \frac{1+x}{2}$$

This holds for all $x \in [0,1]$ so $E(Y|X) = \frac{1+x}{2}$

Def'n Assume $Y_0, Y_1, Y_2, \ldots$ is a random process. We say $X_0, X_1, X_2, \ldots$ is a martingale with respect to $(Y_i)_{i \geq 0}$ if for every $t \geq 0$:

$$E(X_{t+1} \mid Y_1, \ldots, Y_t) = X_t$$

Decreasing sequence

$(X_t)_{t \geq 0}$ is called a super-martingale if the above "=" is replaced with "$\leq$", and a submartingale if instead we have "$\geq$"

Increasing sequence

Example (Doob's Martingale)

Consider a set of random variables $Y_1, \ldots, Y_n$ each taking values in a domain A. Let $f: A^n \to \mathbb{R}$. Define $(X_t)_{0 \leq t \leq n}$ as follows:

$$X_0 = E(f(Y_1, \ldots, Y_n))$$

$$X_t = E(f(Y_1, \ldots, Y_n) \mid Y_1, \ldots, Y_t) \quad \text{for all } 1 \leq t \leq n$$

(Aside: An example of the Y's is $Y_1, \ldots, Y_{\binom{n}{2}}$ the edges of a random graph (this is a Bernoulli r.v.), and we can have f be any function, such as the chromatic number. Then X_0 is the expected value of the ~~[illegible]~~ chromatic number, and X_t is the chromatic # given the values first t edges ~~[illegible]~~. Finally, X_n is the actual chromatic number.)

Let's verify that this is martingale

2

Aside: Tower Property

Suppose $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ (sigma alg

Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(X|\mathcal{G}_1), \text{ and}$$

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1)$$

i.e. "Smaller" sigma field wins.

<u>Example (con't)</u>

For every $0 \le t \le n-1$

$$\mathbb{E}(X_{t+1} | Y_1, \dots, Y_t)$$
$$= \mathbb{E}(\mathbb{E}(f | Y_1, \dots, Y_{t+1}) | Y_1, \dots, Y_t) \quad \text{By Tower Property}$$
$$= \mathbb{E}(f | Y_1, \dots, Y_t)$$
$$= X_t.$$

<u>Example:</u>

$\Omega = \{\text{Canadian Citizens}\}$

P: Uniform Distribution

$\mathcal{F}$: 2^{Ω}.

$f: \Omega \to \mathbb{R}$ (height of citizen)

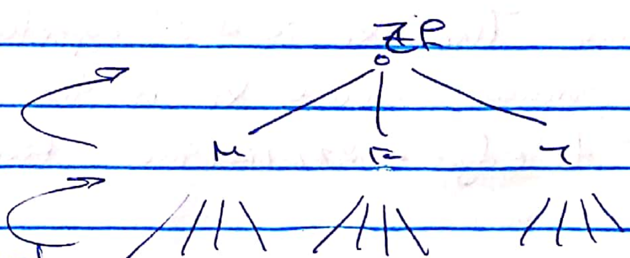
$Y_1: \Omega \to \{M, F, ?\}$ (Gender)

$Y_2: \Omega \to \mathbb{Z}$ (Age)

$X_0 = \mathbb{E} f \to$ Average height of Canadian citizens

$X_1 = \mathbb{E}(f|Y_1) \to$ Average height ~~of~~ segregated by Gender

$X_2 = \mathbb{E}(f|Y_1, Y_2) \to$ —— " —— gender <u>AND</u> age



$\leftarrow \mathbb{E}(f|Y_1)$

$\leftarrow \mathbb{E}(f|Y_1, Y_2)$.

Note that going back upwards gives us exactly Doob's martingale

Example (Edge Exposure Martingale)

Consider $G(n,p)$ and an ~~set~~ arbitrary order of the $\binom{n}{2}$ edges of $K_{[n]}$. Let Y_i be the indicator variable that the i-th edge in $K_{[n]}$ is in $G(n,p)$. Let f be an graph function (E.g. the chromatic #).

Define a martingale $X_0, \dots, X_{\binom{n}{2}}$ by

$$X_0 = \mathbb{E}(f(G(n,p)))$$

$$X_t = \mathbb{E}(f(G(n,p)) \mid Y_1, \dots, Y_t) \text{ for every } 1 \le t \le \binom{n}{2}$$

Note that this is a special case of the Doob's martingale

We can expose the graph quicker though:

Example (Vertex Exposure Martingale)

Consider $G(n,p)$ and any graph function f. Define $(X_t)_{0 \le t \le \binom{n}{2}}$ by:

$$X_0 = \mathbb{E}(f(G(n,p)))$$

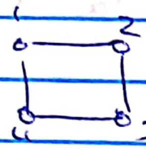
$$X_t(H) = \mathbb{E}(f(G(n,p)) \mid G_{[t]} = H_{[t]}) \text{ for all } 1 \le t \le n.$$

where $G_{[t]}$ is the subgraph induced by the vertex set $[t] = \{1, \dots, t\}$

3

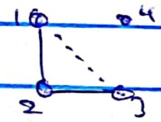
Example:

$G(4, 1/4)$. Consider the vertex exposure Martingale $(X_i)_{i=0}^{4}$ with respect to $f(g)$ as the chromatic number of g (g is a graph).

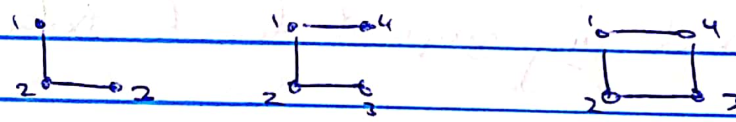
Let $g =$  . What is $X_3(g)$ and $X_4(g)$?

$X_4(g) = \chi(g) = 2$ → This is just the chromatic number since we've exposed all the information on g.

$X_3(g) = \mathbb{E}\left(f(g') \mid \{1,2\}, \{2,3\} \in g', \{1,3\} \notin g'\right)$



$$= 2\left(\left(\frac{3}{4}\right)^3 + 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)\right)$$



$$+ 3\left(2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^3\right) = \frac{135}{64}$$

Example:

Let $Y_1, \dots, Y_n$ be ind. r.v.s Let $X_t = \sum_{j=1}^{t}(Y_j - \mathbb{E}Y_j)$

Claim: $(X_t)_{1\le t\le n}$ is a martingale w.r.t. $(Y_t)_{1\le t\le n}$.

Proof:

~~$\mathbb{E}(X_{t+1}$~~ We want to verify $\mathbb{E}(X_{t+1}|Y_1 \dots Y_t) = X_t$.

$$\mathbb{E}(X_{t+1}|Y_1 \dots Y_t) = \mathbb{E}\left(\sum_{j=1}^{t+1} Y_j - \mathbb{E}Y_j \,\Big|\, Y_1, \dots Y_t\right)$$

$$= \sum_{j=1}^{t}(Y_j - \mathbb{E}Y_j) + \mathbb{E}(\underbrace{Y_{t+1} - \mathbb{E}Y_{t+1}}_{} \mid Y_1, \dots Y_t)$$

$$= X_t$$

Independent from $Y_1, \dots Y_t$, so this cancels out.

Theorem (Azuma's Inequality)

Let $(X_i)_{i\ge 0}$ be a martingale w.r.t $(Y_i)_{i\ge 0}$. Assume for all $k \ge 1$

$$|X_k - X_{k-1}| \le C_k$$

Then,

$$\Pr(|X_n - X_0| \ge t) \le 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} C_k^2}\right).$$

④

① ②

Adv-Random

Mar 2nd

Lecture 13:

Theorem (Azuma's Inequality)

Proof:

Let $Z_i = X_i - X_{i-1}$. WLOG we may assume $X_0 = 0$.

Then, $X_n = \sum_{i=1}^{n} Z_i$.

By assumption (Lipschitz condition):

$-c_i \le Z_i \le c_i \quad \forall i$

Since $\mathbb{E}(X_i \mid Y_0, \dots Y_{i-1}) = X_{i-1}$.

and $\mathbb{E}(Z_i \mid Y_0 \dots Y_{i-1}) = \mathbb{E}(X_i \mid Y_0, \dots, Y_{i-1}) - X_{i-1} = 0$

For $\alpha > 0$,

$Pr(X_n \ge t) = P(e^{\alpha X_n} \ge e^{\alpha t})$

$\le e^{-\alpha t}\, \mathbb{E}(\prod_{i=1}^{n} e^{\alpha Z_i})$ Markov.

Here, we used independence in the Chernoff Bound proof now we use the Martingale property,

$= e^{-\alpha t}\, \mathbb{E}(\mathbb{E}(\prod_{i=1}^{n} e^{\alpha Z_i} \mid Y_0, \dots Y_{n-1}))$

Tower Property

$= e^{-\alpha t}\, \mathbb{E}(\prod_{i=1}^{n-1} e^{\alpha Z_i}\, \mathbb{E}(e^{\alpha Z_n} \mid Y_0 \dots Y_{n-1}))$.

We use the following inequalities:

- $e^{x} \le \frac{\sinh(c)}{c} x + \cosh(c) \quad \forall x \in [-c, c], c > 0$
- $\cosh(x) \le e^{x^2/2} \quad \forall x > 0$

Now.

$\mathbb{E}(Z_n \mid Y_0 \dots Y_{n-1}) = 0$.

$Pr(X_n > t) \le e^{-\alpha t}\, \mathbb{E}(\prod_{i=1}^{n-1} e^{\alpha Z_i}\, \mathbb{E}(\frac{\sinh(\alpha c_n)}{\alpha c_n} \cdot \alpha Z_n + \cosh(\alpha c_n) \mid Y_0, \dots Y_{n-1}))$

$\le e^{-\alpha t}\, \mathbb{E}(\prod_{i=1}^{n-1} e^{\alpha Z_i}) \cdot e^{\alpha^2 c_n^2/2}$

$\le e^{-\alpha t + \frac{\alpha^2}{2}\sum_{i=1}^{n} c_i^2}$ (Induction)

Proof (cont)

And putting $\alpha = \frac{t}{\sum_{k=1}^{n} c_k^2}$ above gives $\Pr(X_n \geq t) \leq \exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right)$

The same argument works for $\Pr(X_n \leq -t)$ by considering $(-X_t)_{t\geq 0}$. This gives Azuma's inequality as desired $\square$

Exercise: Deduce Chernoff Bound type result using Azuma's Inequality.

Theorem: Let $(X_t)_{t\geq 0}$ be a ~~martingale~~ super-Martingale w.r.t $(Y_i)_{i\geq 0}$ s.t. $|X_t - X_{t-1}| \leq c_t \ \forall t \geq 1$.

Then
$$\Pr(X_n - X_0 \geq t) \leq \exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right)$$

(and we get the other tail with submartingale)

Proof: Same as above!

A graph function f is said to satisfy the edge Lipschitz condition with constant c if $|f(H) - f(H')| \leq c$ whenever H and H' differ by only one edge.

We say f satisfies the vertex Lipschitz condition with constant c if $|f(H) - f(H')| \leq c$ whenever $E(H) \Delta E(H')$ are incident with only one vertex.

<u>Theorem:</u>

If f satisfies the edge (vertex) Lipschitz condition with constant c, then the corresponding edge (vertex) exposure Martingale satisfies $|X_{t+1} - X_t| \le c \quad \forall t$.

<u>Proof:</u>

$X_{t+1} = \mathbb{E}(f \mid Y_1, \ldots, Y_{t+1})$

Let $\vec{y}(\omega) = (Y_1, \ldots, Y_t)(\omega)$ (ω is a specific graph)

$$X_{t+1}(\omega) = \mathbb{E}(f \mid (Y_1, \ldots, Y_{t+1}) = (\underbrace{Y_1(\omega), \ldots, y_t(\omega)}_{= \vec{y}(\omega)}, Y_{t+1}(\omega)))$$

$$X_t(\omega) = \mathbb{E}(f \mid (Y_1, \ldots, Y_t) = (\underbrace{y_1(\omega), \ldots, y_t(\omega)}_{= \vec{y}(\omega)}))$$

$$X_{t+1}(\omega) - X_t(\omega) = \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) \Pr((Y_{t+2}, \ldots, Y_{\binom{n}{2}}) = \vec{z}).$$

$$= \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} \sum_{y \in \{0,1\}} f(\vec{y}(\omega), y, \vec{z}) \Pr(\vec{z}) \Pr(Y_{t+1} = y).$$

$$= \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} P(\vec{z}) \left(f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \{0,1\}} P(y) f(\vec{y}(\omega), y, \vec{z}) \right)$$

$$|X_{t+1}(\omega) - X_t(\omega)| = \sum_{\vec{z} \in \{0,1\}^{\binom{n}{2}-(t+1)}} P(\vec{z}) \left| \underbrace{f(\vec{y}(\omega), y_{t+1}(\omega), \vec{z}) - \sum_{y \in \{0,1\}} f(\vec{y}(\omega), y, \vec{z}) P(y)}_{\le c} \right|$$

$\le c$. for every ω.

6

Theorem (Shamir & Spencer, 1987)

$Pr(|\chi(G(n,p)) - \mathbb{E}(\chi(G(n,p)))| \geq t) \leq 2e^{-t^2/2n}.$

$\Rightarrow \forall \varepsilon > 0, \exists c > 0$

$Pr(|\chi(G(n,p)) - \mathbb{E}(\chi(G(n,p)))| \leq c\sqrt{n}) \geq 1 - \varepsilon$

Proof.

Consider the vertex exposure martingale w.r.t f being the chromatic #. Obviously, f is 1-Lipschitz since modifying a set of edges incident with only one vertex can change f by ≤ 1. So,

$Pr(|\chi(G(n,p)) - \mathbb{E}\chi(G(n,p))| \geq t)$

$= Pr(|X_n - X_0| \geq t) \leq e^{-t^2/2n}.$

by Azuma's Inequality. $\square$

Theorem (Bollobás, 1988)

A.a.s. $\chi(G(n,1/2)) \sim \frac{n}{2\log_2 n}$.

Note: Since we are in $G(n,1/2)$, cliques and ind. sets are the same since G and $\bar{G}$ both in $G(n,1/2)$.

Proof.

We first establish a lower bound. Let $\omega(G)$ denote the size of a maximum clique (or ind set) of G. Obviously $\chi(G) \geq \omega(G)$.

Consider $G(n,1/2)$

The expected # of k-cliques in $G(n,1/2)$ is $f(k) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$.

Let $k_0 = \min\{k \geq 2 : f(k) \leq 1\}$ (This is well defined since $f(k)$ is decreasing on $k \geq 2$).

(Exercise: $k_0 \sim 2\log_2 n$, $f(k_0+1) = o(1)$ and $f(k_0-4) = \omega(n^3)$).

Mar 2nd.

Proof: (Con't)

C_n: Cerque #

Hence, a.a.s $\omega(C_n) \le k_0+1$ (And since $\alpha(C_n) = \omega(\bar{C_n})$ and $\bar{G} \sim \mathcal{G}(n, 1/2)$, we know a.a.s $\alpha(G) \le k_0+1$).

As $\chi(G) \cdot \alpha(C_n) \ge n$.

$\Rightarrow$ a.a.s $\chi(\mathcal{G}(n, 1/2)) \ge (1+o(1)) \frac{n}{2\log_2 n}$.

Now, we establish an upper bound:

Set $k = k_0 - 4$. We know $f(k) = \omega(n^3)$ and we want to prove that a.a.s $\omega(\mathcal{G}(n,p)) \ge k$. We want a concentration on the # of k-cliques.

(Bollobas's ~~did~~ gave the following proof:)

Consider a random variable Y that denotes the maximal size of a family of edge disjoint cliques of size k.

We will prove that $\mathbb{E}Y$ is large.

Key Lemma: $\mathbb{E}Y \ge (1+o(1)) (n^2/2k^4)$. $\leftarrow$ Will show next class.

Let $Y_0, Y_1, \ldots, Y_{\binom{n}{2}}$ be the edge exposure martingale on $\mathcal{G}(n, 1/2)$ with $Y_0 = \mathbb{E}Y$.

It is easy to see that $|Y_m - Y_{m-1}| \le 1$ because every edge is contained in at most 1 k-clique in the maximal family of edge disjoint k-cliques.

By Azuma's Inequality

Proof (cont.)

$$\Pr\left(|Y_{(2)} - \mathbb{E}Y| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\binom{n}{2}}\right)$$

Choosing $t = \frac{n^2}{\cancel{\ldots}4}$ yields

$$\begin{aligned}
\Pr\left(\omega(G(n,1/2)) < k\right) &= \Pr(Y=0) \\
&\leq \Pr\left(|Y_{(2)} - \mathbb{E}Y| \geq t\right) \\
&\leq 2\exp\left(-\Omega\left(\frac{n^2}{\log^8 n}\right)\right).
\end{aligned}$$

Use key lemma

Set $m = \lfloor n/\log_2^2 n \rfloor$. For every m-set $S \subseteq [n]$, $G[S]$ is distributed as $G(m,1/2)$. Set $k_0 = k_0(n)$ and let $k = k_0 - 4$. Then,

$$k \sim 2\log_2 m \sim 2\log_2 n$$

Same argument as above applied to $G(m,1/2)$ shows

$$\begin{aligned}
\Pr\left(\omega(G[S]) < k\right) &= \exp\left(-\Omega\left(\frac{m^2}{\log^8 m}\right)\right) \\
&= \exp\left(-\Omega\left(\frac{n^2}{\log^{12} n}\right)\right).
\end{aligned}$$

Free- Random

Mar. 4th

Lecture 14:

Recall in $\mathcal{G}(n, 1/2)$

- $\mathbb{E}(\#\text{ k-cliques}) = f(k) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$
- $k_0 = \min\{k \geq 2 : f(k) \leq 1\}$
- $k_0 \sim 2\log_2 n$, $f(k_0+1) = o(1)$, $f(k_0-4) = \omega(n^3)$
- Let $k = k_0 - 4$
- Y: Size of a maximum family of edge disjoint k-cliques

Key Lemma: $\mathbb{E}\,Y \geq (1+o(1))\frac{n^2}{2k^4}$

- $\Pr(\alpha(\mathcal{G}(n, 1/2)) < k) = \exp(-\Omega(n^2/\log^8 n))$
- Let $m = \lfloor n/\log^2 n \rfloor$
- $\Pr(\alpha(\mathcal{G}[m]) < k) = \exp(-\Omega(m^2/\log^8 m)) = \exp(-\Omega(n^2/\log^{12} n))$

There are $\binom{n}{m} < 2^n$ sets of size m.

$\Pr(\exists$ an m-set S s.t. $\alpha(\mathcal{G}[S]) < k)$

$\leq 2^n \cdot \exp(-\Omega(n^2/\log^{12} n))$

$= o(1)$

Now we color $\mathcal{G}(n, 1/2)$ as follows: Repeatedly pull out independent sets of order k and give a color to that independent set, until the # of the remaining colors is less than m. Then, give distinct colors to them.

In total, we use $\leq \frac{n}{k} + m \sim \frac{n}{2\log_2 n}$.

So, $\chi(\mathcal{G}(n, 1/2)) \leq (1+o(1))\frac{n}{2\log_2 n}$ $\square$

Proof of key lemma:
The expected # of k-cliques is $f(k)$
Let $\mathcal{C}$ be a family of k-cliques obtained by including each of the k-cliques of $G(n, 1/2)$ independently w/ prob p.
Let $\mathcal{W}$ of pairs of k-cliques (α, β) in $\mathcal{C}$ such that
$E(\alpha) \cap E(\beta) \neq \emptyset$
Let $\mathcal{C}'$ be obtained by deleting one of α, β if $\{\alpha, \beta\} \in \mathcal{W}$.
Then, $\mathcal{C}'$ is a family of edge-disjoint k-cliques.
So:

$$\mathbb{E}Y \geq \mathbb{E}|\mathcal{C}'| \geq \mathbb{E}(|\mathcal{C}| - |\mathcal{W}|).$$

$\mathbb{E}|\mathcal{C}| = p \cdot f(k)$ ← Expected # of k-cliques

← Probability of being included.

~~Exercise~~

Exercise: The expected # of pairs of k-cliques in $G(n, 1/2)$ that intersect edges is asymptotically $f(k)^2 \cdot \frac{k^4}{2n^2}$.

Then, $\mathbb{E}|\mathcal{W}| \sim p^2 f(k)^2 \frac{k^4}{2n^2}$

$\mathbb{E}Y \geq pf(k) - (1+o(1)) p^2 f(k)^2 k^4 / 2n^2$
Choosing $p = \frac{n^2}{k^4 f(k)}$ gives $\mathbb{E}(Y) \geq (1+o(1)) \frac{n^2}{2k^4}$.

$\square$.

Theorem:
Let $p = n^{-\alpha}$ where $\alpha > 5/6$ fixed. Then, there exists an integer $u = u(n,p)$ s.t. a.a.s $u \le \chi(G(n,p)) \le u+3$.

Proof:
We have shown (by assignment question) that for any fixed $c > 0$ a.a.s every subgraph induced by $\le c\sqrt{n}$ vertices can be 3-colored.

Fix $\varepsilon > 0$, and let $u = u(n,p,\varepsilon)$ be the smallest integer s.t. $Pr(\chi(G(n,p)) \le u) > \varepsilon$.

Let Y be the size of the smallest subset S s.t. $G - S$ is u-colorable. Consider the vertex exposure martingale w.r.t Y, then Y satisfies the 1-Lipschitz condition (Revealing one more vertex changes χ by at most 1).
By Azuma's inequality:

$$Pr(Y - EY > t) \le e^{-t^2/2n}, \quad Pr(Y - EY < -t) \le e^{-t^2/2n}$$

$$\Rightarrow Pr(Y > EY + \lambda\sqrt{n}) \le e^{-\lambda^2/2}, \quad Pr(Y < EY - \lambda\sqrt{n}) \le e^{-\lambda^2/2}$$

Choose λ s.t. $e^{-\lambda^2/2} = \varepsilon$, then

$$Pr(Y < EY - \lambda\sqrt{n}) \le \varepsilon. \quad (1)$$

The event $Y = 0$ is equivalent to $G(n,p)$ being u-colorable
i.e. $\{Y = 0\} = \{G(n,p) \text{ is } u\text{-colorable}\}$

$$\Rightarrow Pr(Y=0) = Pr(\chi(G(n,p)) \le u) \quad (\text{By } (1)).$$

But $Pr(\chi(G(n,p)) \le u) > \varepsilon$ and $Pr(Y < EY - \lambda\sqrt{n}) \le \varepsilon$, so $EY - \lambda\sqrt{n} < 0 \Rightarrow EY < \lambda\sqrt{n}$

$\subset$

2

Proof (con't)

Now,

$Pr(Y \geq 2\lambda\sqrt{n}) < Pr(Y > \mathbb{E}Y + \lambda\sqrt{n}) < \varepsilon.$

So with probability $1-\varepsilon$, $Y \leq 2\lambda\sqrt{n}$.

i.e. There exists a set S with $|S| \leq 2\lambda\sqrt{n}$ s.t.

$G(n,p) - S$ is u-colorable

And, we also know that a.a.s S is 3-colorable

$Pr(\chi(G(n,p)) \leq u-1$ OR $\chi(G(n,p)) \geq u+4)$

$\leq \varepsilon + \underbrace{Pr(Y \geq 2\lambda\sqrt{n})}_{\leq \varepsilon} + \underbrace{Pr(\exists S : |S| \leq 2\lambda\sqrt{n} \text{ and } S \text{ not 3-colorable})}_{o(1)}$

$\leq 3\varepsilon$

And the assertion holds by letting $\varepsilon \to 0$. $\square$

What happens when $p \sim c/n$ where $c>0$ fixed?

Theorem (Achlioptas and Friedgut, 1999)

For any $k \geq 3$ fixed there is a sharp threshold sequence $c_k = c_k(n)$ such that $\varepsilon > 0$:

$$\lim_{n\to\infty} Pr(G(n, c/n) \text{ } k\text{-colorable}) = \begin{cases} 0 & \text{if } c < (1-\varepsilon)c_k(n) \\ 1 & \text{if } c > (1+\varepsilon)c_k(n) \end{cases}$$

(~~Determining~~ Determining c_k remains open)

Differential Equation Method:

Toy Example: Let $c > 0$ fixed. Suppose cn balls are thrown into n bins sequentially uniformly at random. Let X_i be the # of empty bins after i balls are thrown.

- $X_0 = n$
- $X_{i+1} = X_i - \underbrace{\mathbb{1}\{(i+1)\text{-th ball thrown into empty bin}\}}_{:= E_{i+1}}$

  (X_i: # of empty bins after the i-th)

- $\Pr(E_{i+1}) = \dfrac{X_i}{n}$ $\xrightarrow{\text{Formally}}$ $\mathbb{E}(\mathbb{1}\{E_{i+1}\} \mid X_0, \ldots, X_i) = \dfrac{X_i}{n}$

  (Heuristically)

- $\mathbb{E}(X_{i+1} - X_i \mid H_i) = -\dfrac{X_i}{n}$

  (H_i: history of first i balls)

We want to find the trajectory of $(X_i)_{i=0}^{cn}$

↳ what does this mean? What is the trajectory of an r.v? We will formalize this later. For now...

... let's assume that all X_i are concentrated around $\mathbb{E}X_i$. So, we find the trajectory of $(\mathbb{E}X_i)_{i=0}^{cn}$.

Then,

$$\mathbb{E}(X_{i+1}) - \mathbb{E}(X_i) = -\frac{\mathbb{E}X_i}{n}$$

We write:

$$x(t) = \frac{\mathbb{E}(X_{tn})}{n}$$

↳

1

And the above becomes:

$$\mathbb{E}X_{i+1} - \mathbb{E}X_i = -\frac{\mathbb{E}X_i}{n} \Rightarrow n\underbrace{\left(x(t+\tfrac{1}{n}) - x(t)\right)}_{\xrightarrow{n\to\infty} x'(t)} = -x(t)$$

That tells us that

the solution of $x'(t) = -x$ gives (by scaling) the trajectory of $(\mathbb{E}X_i)_{i=0}^{cn}$:

$$\frac{dx}{dt} = -x, \quad \int \frac{1}{x}dx = -\int dt, \Rightarrow x(t) = ce^{-t}$$

$X_0 = n \Rightarrow x_0(0) = 1 \Rightarrow c = 1$, so $x(t) = e^{-t} \Rightarrow \mathbb{E}X_i = ne^{-i/n}$

Suppose we want to show the concentration of X_τ.

For $j \geq 0$, define $Z_j = \mathbb{E}(X_\tau \mid H_j)$

$\hookrightarrow$ The σ-field generated by the random allocation of the first j balls

We also have $|Z_j - Z_{j-1}| \leq 1$, so by Azuma's Inequality,

$$\Pr(|X_\tau - \mathbb{E}X_\tau| \geq \sqrt{\alpha\tau}) \leq 2e^{-\alpha/2}$$

$\Rightarrow$ with probability $\geq 1 - e^{-n^{1/3}}$, $X_\tau = ne^{-\tau/n} + o(n)$

Lecture 15:

A discrete-time random process is a probability space which can be denoted by $(Q_0, Q_1, Q_2, \ldots)$, where each Q_i takes values in some set S. The elements of the space are sequences $(q_0, q_1, q_2, \ldots)$ where each $q_i \in S$. We use h_t to denote $(q_0, q_1, \ldots, q_t)$, which is called the history up to step t of the process. Let H_t be the random counterpart of h_t.

We care about a sequence of random processes indexed by n $(n=1,2,\ldots)$. Thus, $q_t = q_t^{(n)}$ and $S = S^{(n)}$ (We drop n from the notation if it's unambiguous) Asymptotics refers to $n \to \infty$.

$S^{(n)t}$ denotes the set of all $h_t = (q_0, \ldots, q_t)$ where each $q_i \in S^{(n)}$

We say a function $f(u_1, \ldots, u_j)$ satisfies a Lipschitz condition on $D \subseteq \mathbb{R}^j$ if there exists a constant L s.t.

$$|f(u_1, \ldots, u_j) - f(v_1, \ldots, v_j)| \le L \cdot \max_{1 \le i \le j} |u_i - v_i|$$

We are interested in the trajectories of a set of r.v. $Y_1, \ldots, Y_a$ induced by a random process

Suppose $D \subseteq \mathbb{R}^{a+1}$, define the stopping time $T_D(Y_1, \ldots, Y_a)$ by the minimum integer $t \ge 0$ such that

$$\left(\frac{t}{n}, \frac{Y_1(t)}{n}, \ldots, \frac{Y_a(t)}{n}\right) \notin D$$

Theorem (DE Method)

For $1 \le l \le a$, where a is fixed, let $Y_l : S^{(n)+} \to \mathbb{R}$ ← This is an r.v. and $f_l : \mathbb{R}^{a+1} \to \mathbb{R}$ such that for some constant C_0 and for

↑ This is the f-sum function

all l,

$$|Y_l(h_t)| < C_0 n.$$

for all $h_t \in S^{(n)+}$ for all n.

Assume the following conditions hold where in (i) and (ii) D is some bounded, connected, open set containing the closure of:

$$\{(0, z_1, \dots, z_a) : \Pr(Y_l(0) = z_l n, 1 \le l \le a) \ne 0 \text{ for some } n\}$$

(i) (Boundedness Hypothesis)

For some fixed $c > 0$,

$$\max_{1 \le l \le a} |Y_l(t+1) - Y_l(t)| \le C \quad \text{for all } t < T_D$$

(ii) (Trend Hypothesis)

For all $l \le a$,

$$|\mathbb{E}(Y_l(t+1) - Y_l(t) \mid H_t) - f_l(\tfrac{t}{n}, \tfrac{Y_1(t)}{n}, \dots, \tfrac{Y_a(t)}{n})| = o(1) \text{ for } t < T_D$$

(iii) (~~Lipschitz~~ Lipschitz Hypothesis)

Each function f_l is continuous and satisfies a Lipschitz condition on $D \cap \{(t, z_1, \dots, z_a) : t \ge 0\}$

Then, the following are true:

(a) For $(0, \hat{z}_1, \dots, \hat{z}_a) \in D$, the system of DEs

$$\frac{dz_l}{dx} = f_l(x, z_1, \dots, z_a) \quad l = 1, \dots, a$$

has a unique solution in D for $z_l : \mathbb{R} \to \mathbb{R}$ passing through $z_l(0) = \hat{z}_l$, $1 \le l \le a$ which extends to points arbitrarily close to the boundary of D.

Theorem (con't)

(b). For every $\varepsilon > 0$, a.a.s

$$Y_\ell(t) = n Z_\ell(t/n) + o(n)$$

uniformly for $0 \le t \le \sigma n$. and for every $1 \le \ell \le a$, where $Z_\ell(x)$ is the solution in (a) with $\hat{z}_\ell = \frac{1}{n} Y_\ell(0)$, and $\sigma = \sigma(n) = O(1)$ is the supremum of x to which the solution can be extended before reaching ~~at~~ within ℓ^∞-distance ε of the boundary of $\mathcal{D}$.

Back to the toy example!

We proved that $\mathbb{E}(X_{(i+1)} - X_{(i)} \mid H_i) = -\frac{X(i)}{n}$ (a)

Define $f(x, z) = -z$

We also have $0 \le X(i) \le n$ $\forall$ $0 \le t \le \sigma n$ (b)

$X(0) = n$ (c)

$|X(t+1) - X(t)| \le 1$ for all $0 \le t \le \sigma n$ (d)

Let $\mathcal{D} \subseteq \mathbb{R}^2$ such that $\mathcal{D} = \{(x, z) : -x < x < x,\ -2 < z < 2\}$

(d) $\Rightarrow$ Boundedness Hypothesis

(a) $\Rightarrow$ Trend Hypothesis

(c) $\Rightarrow$ $\mathcal{D}$ contains all points $(0, z)$ such that $P(X(0) = zn) \ne 0$ for some n. Moreover, $\mathcal{D}$ is bounded, connected & open

(b) $\Rightarrow$ $|X(t)| \le C_0 n$ for some constant $C_0 > 0$. It is easy to see that $f(x, z) = -z$ is Lipschitz on $\mathcal{D}$.

By theorem, $\frac{dz}{dx} = -z$ has a unique solution satisfying $z(0) = 1$. which extends to points arbitrarily close to boundary of $\mathcal{D}$. Indeed, we know $z(x) = e^{-x}$

By part (a) of the ~~DE~~ DE method, a.a.s

$$x(i) = nZ(i/n) + o(n)$$
$$= n \cdot e^{-i/n} + o(n)$$

uniformly for all $0 \le i \le cn$.

Min-degree graph process:

Start with an empty graph on vertex set $[n]$. In each step, u.a.r choose a vertex with minimum degree, call it u, and then u.a.r choose a vertex not yet adjacent to u, call it v. Add an edge b/w u and v. Partition the graph process into phases. For $k \ge 0$, let phase k consist of the steps where $d(u)=k$.

We will consider phase 0 first. Let G_t be the graph obtained after step t and $Y_\ell(t)$ be the # of vertices of degree ℓ in G_t. Again, let $H_t = (G_0, G_1, \dots, G_t)$.

In each step of phase 0: (let u_t, v_t denote the vertices u and v chosen in step t)

$$\mathbb{E}(\mathbb{1}\{d(v_{t+1}) = i\} \mid H_t)$$

$$= \frac{Y_i(t)}{n-1} \begin{array}{l} \leftarrow \text{Total vertices with } d(v)=i \\ \leftarrow \text{total \# of vertices} \end{array}$$

Let $X_i(t+1) = \mathbb{1}\{d(v_{t+1}) = i\}$.

$$Y_0(t+1) = Y_0(t) - 1 - X_0(t+1)$$
$$Y_1(t+1) = Y_1(t) + 1 + X_0(t+1) - X_1(t+1)$$
$$Y_i(t+1) = Y_i(t) + X_{i-1}(t+1) - X_i(t+1) \quad \text{for } i \ge 2$$
$$\Rightarrow Y_i(t+1) = Y_i(t) - \delta_{i0} + \delta_{i1} + X_{i-1}(t+1) - X_i(t+1) \quad \text{for all } i \ge 0$$

by setting $X_{-1}(t) = 0$ for all t.

$$\Rightarrow \mathbb{E}\left(Y_i(t+1) - Y_i(t) \mid H_t\right)$$

$$= -\delta_{i0} + \delta_{i1} + \underbrace{\frac{Y_{i-1}(t)}{n-1}}_{X_{i-1}(t+1)} - \underbrace{\frac{Y_i(t)}{n-1}}_{X_i(t+1)}$$

Mar. 11th

Lecture 16:

Recall the min. degree graph process:

$Y_i(t)$: # of vertices of degree i in G_t

$H_t := (G_0, \ldots, G_t)$: History up to step t

We showed last class:

$$\mathbb{E}(Y_i(t+1) - Y_i(t) \mid H_t) = -\delta_{i0} + \delta_{i1} + \frac{Y_{i-1}(t) - Y_i(t)}{n-1}$$

$$= -\delta_{i0} + \delta_{i1} + \frac{Y_{i-1}(t) - Y_i(t)}{n} + O\left(\frac{1}{n}\right)$$

(By setting $Y_{-1}(t) = 0$)

Let $z_i' = -\delta_{i0} + \delta_{i1} + z_{i-1} - z_i$ for any $i \geq 0$, with $z_{-1}(x) = 0$ for all x,

with initial conditions: $z_0(0) = 1$ and $z_i(0) = 0$ $\forall i \geq 1$.

For $i=0$: $z_0' = -1 - z_0$

which gives $z_0(x) = 2e^{-x} - 1$.

For $i=1$: $z_1' = 1 + (2e^{-x} - 1) - z_1 = 2e^{-x} - z_1$

which gives $z_1(x) = 2xe^{-x}$

In general, $z_i(x) = \frac{2x^i}{i!} e^{-x}$

Theorem: Fix $a \in \mathbb{Z}$ and suppose $0 < s < l \cdot n^2$ fixed. Then a.a.s

$$Y_l(t) = n z_l(t/n) + o(n)$$

uniformly for all $0 \leq l \leq a$ for all $0 \leq t \leq sn$.

Proof: By checking all hypotheses of DE Theorem (Exercise!)

Theorem

Fixed $a \in \mathbb{Z}$ $1 \le l \le a$, $Y_l : S^{(n)+} \to \mathbb{R}$ and $f_l : \mathbb{R}^{a+1} \to \mathbb{R}$

D: Bounded connected open set containing the closure of $\{(0, z_1, \dots, z_a) : P_r(Y_l(0) = z_l(n), 1 \le l \le a) \ne 0$ for some $n\}$

(i) (Boundedness) $|Y_l(t+1) - Y_l(t)| \le c$ for all $1 \le l \le a$

(ii) (Trend Hypothesis) $E(Y_l(t+1) - Y_l(t) \mid H_t) = f_l(\frac{t}{n}, \frac{Y_1(t)}{n}, \dots, \frac{Y_a(t)}{n}) + o(1)$

(iii) (Lipschitz) f_l cts and Lipschitz on $D \cap \{(t, z_1, \dots, z_a) : t \ge 0\}$

Then,

(a) Uniqueness and existence of the sol'n to the ODEs.

(b) For every $\varepsilon > 0$, a.a.s $Y_l(t) = n z_l(t/n) + o(n)$ uniformly for all $0 \le t \le \sigma n$ for all $1 \le l \le a$, where σ is the supremum of x to which the sol of the ODEs can be extended before within ε l^∞-distance of the boundary of D.

Proof:

We are only going to prove the case where $a = 1$.

Let $\lambda = o(1)$ be specified later. Let $\omega = \lambda n$.

First, we prove concentration of $Y(t+\omega) - Y(t)$.

Assume that $(t/n, Y(t)/n)$ is in l^∞-distance at least 2ε from the boundary of D.

For all $k = 1, \dots, \omega$:

$$Y(t+k) = Y(t) + O(k) \quad \text{(By (i))}$$

Hence, $(\frac{t+k}{n}, \frac{Y(t+k)}{n}) \in D$ for every $k \le \omega$.

Proof (cont)

By (ii) and (iii)

$\mathbb{E}(Y(t+k+1) - Y(t+k) \mid H_{t+k})$

$= f\left(\frac{t+k}{n}, \frac{Y(t+k)}{n}\right) + o(1)$

$= \underbrace{f\left(\frac{t}{n}, \frac{Y(t)}{n}\right)}_{\text{by f Lipschitz.}} + \underbrace{O\left(\frac{k}{n}\right)}_{\text{since } Y(t+k) = Y(t) + O(k).} + o(1)$

There exists $g(n) = o(1)$ such that

$\mathbb{E}(Y(t+k+1) - Y(t+k) \mid H_{t+k}) \leq f(t/n, Y(t)/n) + g(n)$

Hence,

$\left(Y(t+k) - Y(t) - kf(t/n, Y(t)/n) - kg(n)\right)_{k=0}^{\omega}$

is a supermartingale in k w.r.t $H_t, \ldots, H_{t+\omega}$

We want to check Lipschitz:

$|Y(t+k+1) - Y(t) - (k+1) f(t/n, Y(t)/n) - (k+1)g(n)$

$\quad - (Y(t+k) - Y(t) - k f(t/n, Y(t)/n) - kg(n))|$

$\leq \underbrace{|Y(t+k+1) - Y(t+k)|}_{\leq C} + \underbrace{|f(t/n, Y(t)/n)|}_{\leq C'} + \underbrace{|g(n)|}_{\in o(1)}$

Since Lipschitz on D.

$\leq$ Some sufficiently large constant. (call it c)

So, by Azuma's Inequality,

$\Pr\left(Y(t+\omega) - Y(t) - \omega f(t/n, Y(t)/n) \geq \omega g(n) + c\sqrt{2\omega\alpha} \mid H_t\right)$

$\leq \exp\left(-\frac{c^2 \cdot 2\omega\alpha}{2\omega c^2}\right)$

$= e^{-\alpha}$.

(2)

Proof: (Con't)

The lower tail can be bounded using exactly the same argument, but using submartingale.

$\Rightarrow Pr(|Y(t+\omega) - Y(t) - \omega f(t/n, Y(t)/n)| \geq \omega g(n) + C\sqrt{\omega\alpha} \mid H_t)$
$\leq 2e^{-\alpha}$.

(We will fix α later). ← Both tails.

(So, we've shown concentration of Y)

But, we also need to show that Y follows the trajectory of f.
Next, we are going to compare $Y(t)$ with $nZ(t/n)$.
Define $k_i = i\omega$, $i = 0, 1, \ldots, i_0$, where $i_0 = \lceil \frac{\sigma n}{\omega} \rceil$.

(k_i is the i^{th} ~~chunk~~ "chunk") ← We chunk up the function and show at each chunk, the error is small.

We will show, ~~by~~ induction, that:

$$Pr(|Y(k_i) - Z(\tfrac{k_i}{n}) \cdot n| \geq B_i) = O(ie^{-\alpha})$$

where $B_i = B\omega(\lambda + \frac{\omega}{n})((1 + \frac{B\omega}{n})^i - 1)$
and ~~...~~ $\lambda = o(1)$ and B is a constant

Base case: $i = 0$. $Z(0) = \frac{Y(0)}{n}$, so there is no error.

Induction: Assume it holds for the first i chunks

↪.

Proof: (cont)

Approximated by f.

$$|Y(k_{i+1}) - z(\tfrac{k_{i+1}}{n})\cdot n| = |A_1 + A_2 + A_3 + A_4|.$$

where Approximated by derivative

$A_1 = Y(k_i) - n\cdot z(\frac{k_i}{n})$

$A_2 = Y(k_{i+1}) - Y(k_i) - \omega f(\frac{k_i}{n}, \frac{Y(k_i)}{n})$

$A_3 = \omega z'(\frac{k_i}{n}) + nz(\frac{k_i}{n}) - nz(\frac{k_{i+1}}{n})$

$A_4 = \omega f(\frac{k_i}{n}, \frac{Y(k_i)}{n}) - \omega z'(\frac{k_i}{n})$

And, we will bound each A_i to show that the sum is bounded.

By Induction $|A_1| \le B_i$ with prob $\ge 1 - O(ie^{-\alpha})$

We've shown that

$|A_2| \le \omega s(n) + O(\sqrt{2\omega\alpha})$ with prob. $\ge 1-2e^{-\alpha}$

$\le c'\omega\tilde{\lambda}$

by choosing $\alpha, \tilde{\lambda}$ such that $s(n) = o(\tilde{\lambda})$, $\tilde{\lambda} = o(1)$

and $\alpha = o(\omega\tilde{\lambda}^2)$.

Since z is the solution of the ODE and f is Lipschitz:

$|z(\frac{k_{i+1}}{n}) - z(\frac{k_i}{n}) - \frac{\omega}{n}z'(\frac{k_i}{n})| \le C''\frac{\omega^2}{n^2}$ for some constant $C'' > 0$.

↳ By Taylor expansion.

Thus, $|A_3| \le C''\frac{\omega^2}{n^2}$.

(3)

Proof (Con't)

Finally, for A_4. $\longrightarrow$ Induction + Lipschitz

$$|A_4| \le \omega \left| f\left(\tfrac{k_i}{n}, \tfrac{Y(k_i)}{n}\right) - f\left(\tfrac{k_i}{n}, z\left(\tfrac{k_i}{n}\right)\right)\right|$$

$$+ \omega \left| \underbrace{f\left(\tfrac{k_i}{n}, z\left(\tfrac{k_i}{n}\right)\right) - z'\left(\tfrac{k_i}{n}\right)}_{=0,\text{ by defin.}}\right|.$$

$$\le c'' \cdot \omega \left| \tfrac{Y(k_i)}{n} - z\left(\tfrac{k_i}{n}\right)\right|$$

$$\le c''' \tfrac{\omega}{n} \cdot B_i \quad \text{with prob} \ge 1 - O(ie^{-\alpha})$$

Concentration error (A_2)

So, with prob $\ge 1 - O(ie^{-\alpha}) - O(e^{-\alpha})$

$= 1 - O((i+1)e^{-\alpha})$.

$|Y(k_{i+1}) - nz\left(\tfrac{k_{i+1}}{n}\right)| \le B_{i+1}$

by summing $\sum_{i=1}^{4} |A_i|$ and choosing sufficiently large B.

We still want to check that B_i is not too big!

$$B_i = B\omega(\tilde{\lambda} + \underbrace{\tfrac{\omega}{n}}_{=\lambda})((1+\underbrace{\tfrac{B\omega}{n}}_{=B\lambda})^i - 1)$$

$$= B\omega(\tilde{\lambda} + \underbrace{\lambda}_{\le\tilde{\lambda}})((1+B\lambda)^i - 1). \quad \text{since } \lambda << \delta(n) << \tilde{\lambda} = o(1)$$

$$\le B\omega(\tilde{\lambda} + \tilde{\lambda})((1+B\lambda)^i - 1)$$

$$= 2B\omega\tilde{\lambda}(\underbrace{(1+B\lambda)^i}_{\le e^{B\lambda i}} - 1)$$

$$e^{B\lambda i} = e^{B\lambda \frac{\sigma n}{\omega}} = e^{B\lambda \frac{\sigma}{\lambda}} = e^{B\sigma} = O(1)$$

$= O(\tilde{\lambda}\omega)$

$= o(n)$

(We will finish the proof on Monday!).

References

[1] Bollobás Béla. *Random graphs.* Academic Press, 1985.

[2] Alan Frieze and Michał Karoński. *Introduction to Random Graphs.* Cambridge University Press, 2015.