Molloy-Reed-Book, Proof - Kahn.

Finishing Kahn's proof:

Retain $:= \left(1-\frac{1}{|L|}\right)^{2D-2} \approx e^{-2}$.

Vertex-keep $:= 1 -$ Retain $\approx 1-\frac{1}{e^2}$.

Edge-keep $\approx$ (Vertex-keep)^2 $\approx \left(1-\frac{1}{e^2}\right)^2$ (mostly independent) (Can argue that the dependence is small).

$E[|L'(e)|] \approx |L|\cdot\left(1-\frac{1}{e^2}\right)^2$

$E[D_{L,G}(v,c)] \approx |\Delta|\left(1-\frac{1}{e^2}\right)^2$

Approximates since there are small error terms.

Even assuming the errors from ~~exp~~ expectations won't hurt us too badly & that we concentrate these variables so as to have little error from concentration, unfortunately, at best these decrease at roughly the same rate - and so we won't make progress.

Need a new finish!

Reserve Colours:

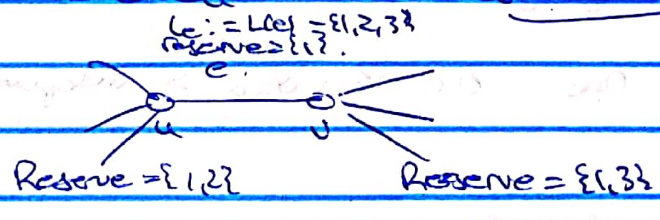
Idea: Before using Nibble and naive colouring procedure, we reserve colours around vertices to be used by edges in the "final step".

Choose Reserve$_v \subseteq L(v) := \bigcup_{e \ni v} L(e)$ (uniformly at random w/ prob p)

For each edge $e = uv$, define

$L_e' := L(e) - (\text{Reserve}_u \cup \text{Reserve}_v)$ → Lose colours from list (reserved)

Reserve$_e$:= Reserve$_u \cap$ Reserve$_v$ → Now e can use the overlapping colours safely



Lemma:

(Setting $p = \frac{\log^4 \Delta}{\sqrt{\Delta}}$) $\exists$ a choice of $Reserve_v$ for $V(G)$ such that $\forall u \in V(G)$, $\forall e \in E(G)$, $\forall c \in Reserve_v$:

(a) $|L(e) \cap (Reserve_u \cup Reserve_v)| \leq 3\sqrt{\Delta} \log^4 \Delta$ $\rightarrow L(e) - L_e$

(b) $|\underbrace{L(e)}_{L_e} \cap (Reserve_u \cap Reserve_v)| \geq \frac{1}{2} \log^8 \Delta$ $\rightarrow$ Reserve

(c) $|\{e = uv \in N_{L_G}(v, c) : c \in Reserve_w\}| \leq 2\sqrt{\Delta} \log^4 \Delta$ $(\ast)$

Intuition:

(a) You don't lose too many colors

(b) There ~~are~~ is ~~many~~ at least one color to use near the end

Proof:

$\mathbb{E}[|L(e) \cap (Reserve_u \cup Reserve_v)|]$

$\leq |L(e)| \cdot 2p$ $\leftarrow$ Since ~~must~~ may appear in either $Reserve_u$ OR $Reserve_v$.

$\leq \Delta \cdot 2 \frac{\log^4 \Delta}{\sqrt{\Delta}}$

$= 2\sqrt{\Delta} \log^4 \Delta$.

$\mathbb{E}[|L(e) \cap (Reserve_u \cap Reserve_v)|]$

$\leq |L(e)| \cdot p^2$ $\leftarrow$ Needs to appear in both $Reserve_u$ and $Reserve_v$.

$\geq \Delta \frac{\log^8 \Delta}{\Delta} = \log^8 \Delta$.

$\mathbb{E}[(\ast)] = \Delta \cdot p \leq \sqrt{\Delta} \log^4 \Delta$

Since each color $c \in Reserve_w$ is independent

You'll notice these variables are all sums of $\{0,1\}$-random variables (even more, all Bernoulli), so we can apply Chernoff Bounds:

$$\Pr[\,|L(e) \cap (\text{Reserve}_u \cup \text{Reserve}_v)| \geq 3/2\, \mathbb{E}[\ ]\,] \leq e^{-\frac{\Delta \log^3 \Delta}{3/2}}$$

$$\Pr[\qquad (b) \qquad] \leq e^{-\frac{(\log^8 \Delta)^2}{3/2}}$$

$$\Pr[\qquad (c). \qquad] \leq e^{-\frac{\Delta \log^3 \Delta}{3}}$$

So all at most $e^{-\log^8 \Delta}$ for large enough Δ which is $\leq \frac{1}{\Delta^c}$ for any c and large enough Δ.

So, we apply LLL with following bad events:

A_e = (a) $\geq 3\sqrt{\Delta} \log^4 \Delta$

B_e = (b) $\leq 1/2 \log^8 \Delta$

$C_{v,c}$ = (c) $\geq 2\sqrt{\Delta} \log^4 \Delta$.

Probability for any bad event $\leq \frac{1}{\Delta^c}$ for any c, while each event is mutually independent of the set of events depending on edges/vertices of distance >4.

So: using $c = 5$ (or 6) suffices for LLL $\square$

What is (c) saying?: If $c \notin \text{Reserve}_v$, this counts $e \in N_{u,c}(v,c)$ with $c \in \text{Reserve}_e$

We define now the reserve degree of a vertex:

$$d_{res}(v,c) = |\{ e \in N_{u,c}(v,c) : c \in \text{Reserve}_e \}|.$$

Idea for finishing w/ Reserved Colours:

Reserved degrees decreases rather quickly because edges in "reserved neighborhood" will be deleted from G, however Reserve never changes during Nibble.

So, if we can get $\max d_{res}(v,c) \leq \frac{1}{2}|Reserve| = \frac{1}{4}\log^{8}\Delta$, we can finish by applying ~~Havell~~ Haxell to reserve colour assignment

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Expectation of new reserved degree in one step of Nibble:

$$\mathbb{E}[d_{res,L',G'}(v,c)]$$

$$= d_{res,L,G}(v,c)\cdot(\underbrace{1-Retain}_{\text{Probability an edge not deleted}})$$

$$\approx d_{res,L,G}(v,c)\cdot\left(1-\frac{1}{e^2}\right)$$

Decreases at half the rate multiplicatively compared to $|L'|$ and Δ'

We can run Nibble for T iterations, where T is about the solution to the following:

$$1 = |L|\cdot((1-Retain)^2)^T = |L|(\underbrace{(1-\tfrac{1}{e^2})^2}_{=k})^T$$

$$\Rightarrow T = \frac{\ln|L|}{-\ln k}$$

(So a roughly logarithmic amount of steps)

Yet,

$$
\begin{aligned}
d_{res,L_T^*,G_T}(u,c) &\approx d_{res,L,G}(u,c)\cdot\left(\left(1-\tfrac{1}{e}\right)\right)^T \\
&= d_{res,L,G}(u,c)\cdot K^{T/2} \\
&\le 2\sqrt{\Delta}\log^4\Delta\cdot\frac{1}{\sqrt{|L|}} \\
&\le 2\log^4\Delta \\
&\le \tfrac{1}{4}\log^8\Delta \quad \text{for large enough } \Delta
\end{aligned}
$$

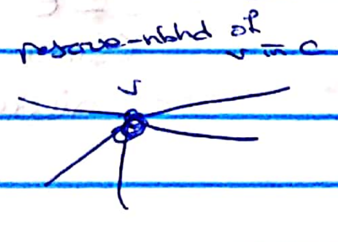
(So we don't have to run too many iterations for the reserved degree to be small enough)

<u>Remark</u>: We actually want to use $T = \frac{\ln|L'|}{-\ln|K|} - \text{poly}\log\log\Delta$ since we need to concentrate d_{res} during nibble enough to apply LLL, so need $d_{res} \ge \text{polylog}\,\Delta$

So, this gives an L-coloring if $|L| \ge \Delta + 4(\log^4\Delta)\sqrt{\Delta}$.
(Since we need $3\sqrt{\Delta}\log^4\Delta$ for reserves and an extra $\sqrt{\Delta}\,\text{poly}\log\Delta$ for concentration errors)

<u>Concentrations</u> for $|L'|$, Δ', d'_{res}:

We'll do <u>d'_{res}</u> first!
d'_{res} counts # edges from d_{res} <u>not</u> deleted.



c-Lipshitz for $c =$ ______ :

- If we change colouring on an edge not incident with v, this changes ≤ 2 of these edges
- If we change colouring incident with v, this can change at most itself and 1 other edge (they both lose the colour). Otherwise, ~~there~~ if there are ≥ 3 edges coloured the same colour, then changing v won't matter.

Note: We can't use Chernoff b/c not sum of independent — " — Simple conc. bound b/c may depend on Δ^2 trials ($\Rightarrow$ Bad Bound)

So: We'll use Talagrand's and we need to verify this variable

Note: Verifying ~~$\neq$~~ retaining a colour requires 2Δ trials.

But we can verify not retaining using $r=2$ by showing its colour $\phi(e)$ and some edge from w w/ $\phi(f)=\phi(e)$.

So, we can apply Talagrand to show this is within about $\sqrt{E[\deg]}$ with probability $\underbrace{e^{-cE[\deg]}}$ (c constant)

$\leq e^{-c\log^8\Delta}$.

(since $\deg \geq \log^8\Delta$ during nibbles)

Feb 4th

Concentrating Δ':

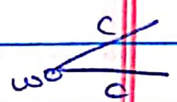
Assume c is not retained around v, this counts # of edges in $X_{k,G}(v,c)$ which are not deleted and the other end of the edge keeps c.

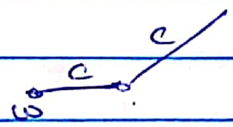
Again, we are 2-Lipschitz, since changing a single color can affect at most 1 other vertex. (If we have ≥ 3, then changing the color won't matter) (Same ~~argu~~ argument as before)

As before, not deleted is 2-verifiable, so we'd be happy if "other end keeps c" or its complement were r-verifiable for constant r since the expectation of intersections of these are on the same order as Δ.
(Note: As before, verifying edge around w retains c takes around 2Δ trials)

What about verifying the complement:
i.e. # of edges e where other end w does not have an edge retaining c?

Cases:





if 2 colored c, this is easy, show 2

To verify this one, we need to check all neighbors!

Trick: Change Variables

Define

$X_{j,k}$ to be # of ~~vertices~~ edges e where e and

other end w

$\geq j$ are assigned color c

$\geq k$ have color c subsequently removed.

So we can write the variable above (the component count) as: $X_{1,0} - \underbrace{(X_{1,1} - X_{2,1})}_{\text{counts receive exactly 1 and remove} \geq 1.}$ ~~...~~

Easy to check that $X_{j,k}$ concentrates b/c 2-Lipschitz and $(j+k)$-verifiable and we can use Talagrand's on each. This works since they have roughly same expectation and j, k constant.

For $\mathbb{L}'$, we use the same trick, but ~~...~~ counting both ends of e.