Math 4419 - Random

Mar. 11th

Recall the min. degree graph process:

$Y_i(t)$: # of vertices of degree i in G_t

$H_t := (G_0, \dots, G_t)$: History up to step t

We showed last class:

$$\mathbb{E}(Y_i(t+1) - Y_i(t) \mid H_t) = -\delta_{i0} + \delta_{i1} + \frac{Y_{i-1}(t) - Y_i(t)}{n-1}$$

$$= -\delta_{i0} + \delta_{i1} + \frac{Y_{i-1}(t) - Y_i(t)}{n} + O\left(\frac{1}{n}\right)$$

(By setting $Y_{-1}(t) = 0$)

Let $z_i' = -\delta_{i0} + \delta_{i1} + z_{i-1} - z_i$ for any $i \geq 0$, with $z_{-1}(x) = 0$ for all x,

with initial conditions: $z_0(0) = 1$ and $z_i(0) = 0$ $\forall i \geq 1$.

For $i=0$: $z_0' = -1 - z_0$

which gives $z_0(x) = 2e^{-x} - 1$.

For $i=1$: $z_1' = 1 + (2e^{-x} - 1) - z_1 = 2e^{-x} - z_1$

which gives $z_1(x) = 2xe^{-x}$

In general, $z_i(x) = \frac{2x^i}{i!} e^{-x}$

Theorem: Fix $a \in \mathbb{N}$ and suppose $0 < s < \ln n^2$ fixed. Then a.a.s

$$Y_\ell(t) = n z_\ell(t/n) + o(n)$$

uniformly for all $0 \leq \ell \leq a$ for all $0 \leq t \leq sn$.

Proof: By checking all hypotheses of DE Theorem (Exercise!)

Theorem

Fixed $a \in \mathbb{Z}$, $1 \le l \le a$, $Y_l : S^{(n)+} \to \mathbb{R}$ and $f_l : \mathbb{R}^{a+1} \to \mathbb{R}$

D : Bounded connected open set containing the closure of $\{(0, z_1, \dots, z_a) : P_r(Y_l(0) = z_l(n), 1 \le l \le a) \ne 0$ for some $n\}$

(i) (Boundedness) $|Y_l(t+1) - Y_l(t)| \le e$ for all $1 \le l \le a$

(ii) (Trend Hypothesis) $E(Y_l(t+1) - Y_l(t) \mid H_t) = f_l(\frac{t}{n}, \frac{Y_1(t)}{n}, \dots, \frac{Y_a(t)}{n}) + o(1)$

(iii) (Lipschitz) f_l cts and Lipschitz on $D \cap \{(t, z_1, \dots, z_a) : t \ge 0\}$

Then,

(a) Uniqueness and existence of the sol'n to the ODEs.

(b) For every $\varepsilon > 0$, a.a.s $Y_l(t) = n z_l(t/n) + o(n)$ uniformly for all $0 \le t \le \sigma n$ for all $1 \le l \le a$, where σ is the supremum of x to which the sol of the ODEs can be extended before within ε l^∞-distance of the boundary of D.

Proof:

We are only going to prove the case where $a = 1$.

Let $\lambda = o(1)$ be specified later. Let $\omega = \lambda n$.

First, we prove concentration of $Y(t+\omega) - Y(t)$.

Assume that $(t/n, Y(t)/n)$ is in l^∞-distance at least 2ε from the boundary of D

For all $k = 1, \dots, \omega$:

$$Y(t+k) = Y(t) + O(k) \quad \text{(By (i))}$$

Hence, $(\frac{t+k}{n}, \frac{Y(t+k)}{n}) \in D$ for every $k \le \omega$.

Proof (cont.)

By (ii) and (iii)

$$E\left(Y(t+k+1) - Y(t+k) \mid H_{t+k}\right)$$
$$= f\left(\frac{t+k}{n}, \frac{Y(t+k)}{n}\right) + o(1)$$
$$= \underbrace{f\left(\frac{t}{n}, \frac{Y(t)}{n}\right) + \underline{O\left(\frac{k}{n}\right)}}_{\hookrightarrow \text{ f Lipschitz.}} + o(1)$$

$\hookrightarrow$ Since $Y(t+k) = Y(t) + O(k)$.

There exists $g(n) = o(1)$ such that

$$E\left(Y(t+k+1) - Y(t+k) \mid H_{t+k}\right) \leq f\left(t/n, Y(t)/n\right) + g(n)$$

Hence,

$$\left(Y(t+k) - Y(t) - kf\left(t/n, Y(t)/n\right) - kg(n)\right)_{k=0}^{\omega}$$

is a supermartingale in k w.r.t $H_t, \ldots, H_{t+\omega}$

We want to check Lipschitz:

$$\left| Y(t+k+1) - Y(t) - (k+1)f\left(t/n, Y(t)/n\right) - (k+1)g(n) - \left(Y(t+k) - Y(t) - kf\left(t/n, Y(t)/n\right) - kg(n)\right) \right|$$
$$\leq \underbrace{\left|Y(t+k+1) - Y(t+k)\right|}_{\leq C} + \underbrace{\left|f\left(t/n, Y(t)/n\right)\right|}_{\leq C'} + \underbrace{\left|g(n)\right|}_{\in o(1)}$$

$\nearrow$ Since Lipschitz on D.

$\leq$ Some sufficiently large constant. (call it $\tilde{C}$)

So, by Azuma's Inequality,

$$\Pr\left(Y(t+\omega) - Y(t) - \omega f\left(t/n, Y(t)/n\right) \geq \omega g(n) + \tilde{C}\sqrt{2\omega\alpha} \mid H_t\right)$$
$$\leq \exp\left(-\frac{\tilde{C}^2 \cdot 2\omega\alpha}{2\omega \tilde{C}^2}\right)$$
$$= e^{-\alpha}.$$

Proof: (Cont'd)

The lower tail can be bounded using exactly the same argument, but using submartingale.

$\Rightarrow \Pr(|Y(t+\omega) - Y(t) - \omega f(t/n, Y(t)/n)| \geq \omega g(n) + c\sqrt{\omega\alpha} \mid H_t)$

$\leq 2e^{-\alpha}$.

(We will fix α later). ← Both tails.

(So, we've shown concentration of Y)

But, we also need to show that Y follows the trajectory of f.

Next, we are going to compare $Y(t)$ with $nZ(t/n)$

Define $k_i = i\omega$, $i = 0, 1, \ldots, i_0$, where $i_0 = \lceil \frac{\sigma n}{\omega} \rceil$.

(k_i is the i^{th} "~~chunk~~" ~~step~~). "Chunk" ← We chunk up the function and show at each chunk, the error is small.

We will show, by induction, that:

$$\Pr(|Y(k_i) - Z(\tfrac{k_i}{n})\cdot n| \geq B_i) = O(ie^{-\alpha})$$

where $B_i = B\omega(\tilde{\lambda} + \frac{\omega}{n})((1+\frac{B\omega}{n})^i - 1)$

and ~~$\tilde{\lambda}, B$ are~~ $\tilde{\lambda} = o(1)$ and B is a constant

Base case: $i=0$: $Z(0) = \frac{Y(0)}{n}$, so there is no error.

Induction: Assume it holds for the first i chunks

↪

Proof: (cont)

$$|Y(k_{i+1}) - z(\tfrac{k_{i+1}}{n})\cdot n| = |A_1 + A_2 + A_3 + A_4|.$$

Approximated by f.

Approximated by derivative

where

$A_1 = Y(k_i) - n\cdot z(\frac{k_i}{n})$

$A_2 = Y(k_{i+1}) - Y(k_i) - \omega f(\frac{k_i}{n}, \frac{Y(k_i)}{n})$

$A_3 = \omega z'(\frac{k_i}{n}) + nz(\frac{k_i}{n}) - nz(\frac{k_{i+1}}{n})$

$A_4 = \omega f(\frac{k_i}{n}, \frac{Y(k_i)}{n}) - \omega z'(\frac{k_i}{n})$

And, we will bound each A_i to show that the sum is bounded.

By induction $|A_1| \le B_i$ with prob $\ge 1 - O(ie^{-\alpha})$

We've shown that

$|A_2| \le \underbrace{\omega\delta(n) + O(\sqrt{2\omega\alpha})}_{\le c'\omega\tilde{\lambda}}$ with prob. $\ge 1 - 2e^{-\alpha}$

by choosing $\alpha, \tilde{\lambda}$ such that $\delta(n) = O(\tilde{\lambda})$, $\tilde{\lambda} = o(1)$ and $\alpha = o(\omega\tilde{\lambda}^2)$.

Since z is the solution of the ODE and f is Lipschitz:

$|z(\frac{k_{i+1}}{n}) - z(\frac{k_i}{n}) - \frac{\omega}{n}z'(\frac{k_i}{n})| \le C''\frac{\omega^2}{n^2}$ for some constant $C'' > 0$.

By Taylor expansion.

Thus, $|A_3| \le C''\frac{\omega^2}{n^2}$.

Proof (Cont.)

Finally, for A_4.

$$|A_4| \le \omega \underbrace{\left|f\left(\frac{k_i}{n}, \frac{Y(k_i)}{n}\right) - f\left(\frac{k_i}{n}, z\left(\frac{k_i}{n}\right)\right)\right|}_{\to \text{ Induction + Lipschitz}}$$

$$+ \omega \underbrace{\left|f\left(\frac{k_i}{n}, z\left(\frac{k_i}{n}\right)\right) - z'\left(\frac{k_i}{n}\right)\right|}_{=0, \text{ by defin.}}$$

$$\le c'' \cdot \omega \left|\frac{Y(k_i)}{n} - z\left(\frac{k_i}{n}\right)\right|$$

$$\le c''' \frac{\omega}{n} \cdot B_i \quad \text{with prob} \ge 1 - O(ie^{-\alpha})$$

So, with prob $\ge 1 - O(ie^{-\alpha}) - \underbrace{O(e^{-\alpha})}_{\leftarrow \text{ Concentration error } (A_2)}$

$$= 1 - O((i+1)e^{-\alpha}).$$

$$|Y(k_{i+1}) - nz\left(\frac{k_{i+1}}{n}\right)| \le B_{i+1}$$

by summing $\sum_{i=1}^{4} |A_i|$ and choosing sufficiently large B.

We still want to check that B_{i_0} is not too big!

$$B_i = B\omega(\tilde{\lambda} + \underbrace{\frac{\omega}{n}}_{=\lambda})((1 + \underbrace{\frac{B\omega}{n}}_{=B\lambda})^i - 1)$$

$$= B\omega(\tilde{\lambda} + \underbrace{\lambda}_{\le \tilde{\lambda}})((1 + B\lambda)^i - 1). \quad \text{since } \lambda << \delta(n) \le \tilde{\lambda} = o(1)$$

$$\le B\omega(\tilde{\lambda} + \tilde{\lambda})((1 + B\tilde{\lambda})^i - 1)$$

$$= 2B\omega\tilde{\lambda}(\underbrace{(1 + B\lambda)^i}_{\le e^{B\lambda i} = e^{B\lambda \frac{\sigma n}{\omega}} = e^{B\lambda \frac{\sigma}{\lambda}} = e^{B\sigma} = O(1)} - 1)$$

$$= O(\tilde{\lambda}\omega)$$

$$= o(n)$$

(We will finish the proof on Monday!).