7404 - Random

Feb 10

* All p's are $\frac{c}{n}$.

Recall:

$k_0 = C\log n$, $k_1 = n^{2/3}$. Prove a.a.s

(i) No components of order (k_0, k_1)

(ii) Only one component of order $\geq k_1$

(iii) # vertices in small component $\sim \rho n$, where $\rho e^c = e^{\rho c}$ $(0<\rho<1)$

We will prove (iii) this class:

Let Y denote the # of vertices lying in small components (i.e. of order $< k_0$). We will prove that the probability for a ~~random~~ uniformly random vertex to be in a small component is $\rho + o(1)$

Lemma: Fix $c>1$ and let $Z_n \sim Bin(n, p)$ where $p \sim c/n$. Then $P_{Z_n} \to \rho$ as $n \to \infty$ where $\rho e^c = e^{\rho c}$, $(0<\rho<1)$

Proof:

Let $G(x) = E(x^{Z_n}) = (1-p+px)^n = e^{\overbrace{-c+cx+O(\frac{x^2}{n})+o(1)}^{}}$,

which converges to e^{-c+cx} pointwise as $n\to\infty$.

By the proof of the theorem for the branching process, part (iii):

P_{Z_n} is the unique solution of $G(x) = x$, where $0<x<1$.

As $G(x) \to \underbrace{e^{-c+cx}}_{F(x)}$ pointwise and ρ is the unique solution of $x = F(x)$, $0<x<1$, it follows immediately $P_{Z_n} \to \rho$ as $n\to\infty$.

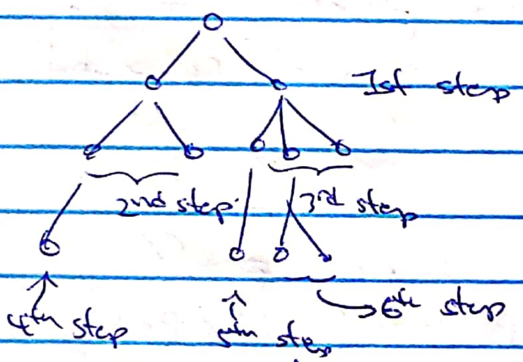
$\square$

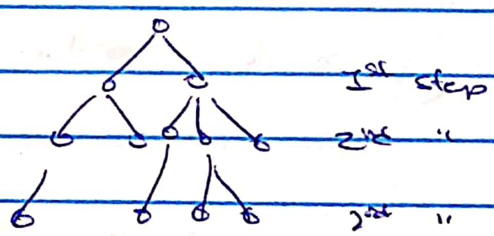
Lemma $\mathbb{E}Y = (1+o(1))\rho n$.

Proof:

Consider the graph exploration process starting from v, $\hat{\varphi}_v$.
Let E_v denote the event that $\hat{\varphi}_v$ lasts $\le k_0$ steps.
Consider the slowed down version of the branching process where $Z_n \sim Bin(n, (1+\xi)\frac{c}{n})$, where $0 < \xi = o(1)$.

(Slowed-down Branching process:





i.e. Reveal grandchildren one child at a time)

$Bin(n, (1+\xi)\frac{c}{n})$ stochastically dominates $Bin(n-(\sum_t X_t + t), \frac{c}{n})$, so we can couple $(\hat{\varphi}_v, \varphi)$ such that $\hat{\varphi}_v$ terminates no later than φ. (Exercise!)

Let P_t denote the probability that φ terminates during the first t steps. We know $P_t \to P_{Z_n}$.
That is $\forall \varepsilon > 0, \exists n_0 > n \ \forall n > n_0$: $P_{k_0} \ge P_{Z_n} - \varepsilon$
And also, $P_{Z_n} \to \rho$ as $n \to \infty \Rightarrow P_{Z_n} \ge \rho - \varepsilon$
$\Rightarrow P_{k_0} \ge \rho - 2\varepsilon$

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By the coupling $(\tilde{\varphi}_v, \varphi)$, we have $P(E_v) \geq P(\varphi$ terminates before k_0 steps$) = P_{k_0} > \rho - \varepsilon \Rightarrow P(E_v) \geq \rho - o(1)$

For the upper bound, consider the stand-down branching process with $Z_n \sim Bin(n-k, (1-\varepsilon)\frac{c}{n})$. Z_n is stochastically dominated by $Bin((X_t + t), \frac{c}{n})$ for $t < k_0$

Again, we can couple these two processes such that during the first k_0 steps the graph exploration process will terminate no earlier than the branching process

$\Rightarrow P(E_v) \leq P($Branching process lasts $\leq k_0$ steps$)$

$= \rho + o(1)$, using similar argument as above

$\Rightarrow P(E_v) = \rho + o(1)$

$\Rightarrow \mathbb{E}Y = (1+o(1))\rho n$

$\square$

$\mathbb{E}Y(Y-1)$. Fix a pair of vertices u and v.

$Pr(u$ is in a small component$) = \rho + o(1)$

Condition on that. The graph exploration process only "exposes" $\leq k_0$ vertices.

(Exercise: Easy to show that a.a.s. v is not exposed) → i.e. In the same component that u is in.

Run the graph exploration process starting from v, condition on C_u, the component containing u. This is equivalent to running the graph exploration process in $G(n', p)$ where $n' = n - O(k)$.

Repeat the proof as for $\mathbb{E}Y$. We have... →

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We have $\Pr(v \text{ is in a small component} \mid u \text{ is in a small component}) = \rho + o(1)$.

$\mathbb{E}Y(Y-1) = n(n-1)(\rho^2 + o(1)) \sim (\mathbb{E}Y)^2$.

By Chebyshev's Inequality: $\forall \varepsilon > 0$

$\Pr(|Y - \mathbb{E}Y| > \varepsilon \mathbb{E}Y) \le \frac{\operatorname{Var} Y}{\varepsilon^2 (\mathbb{E}Y)^2} = \frac{o(1) \cdot (\mathbb{E}Y)^2}{\varepsilon^2 (\mathbb{E}Y)^2} = o(1)$

So a.a.s $Y \sim \mathbb{E}Y = (\rho + o(1))n$. $\square$

Hamiltonian Cycles in $G(n,p)$:

p's are new p's

Theorem:

Let $p = \frac{\log n + \log\log n + x(n)}{n}$.

(a) If $x(n) \to \infty$, then a.a.s $G(n,p)$ is Hamiltonian

(b) If $x(n) \to -\infty$, then a.a.s $G(n,p)$ Not Hamiltonian

(Note: when degree 1 vertices disappear, then our graph becomes Hamiltonian (w.h.p)).

Lemma:

Let D_2 denote the property that min. degree is ≥ 2

HAM the property that graph is Hamiltonian

$G \in HAM \Rightarrow G \in D_2$

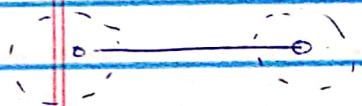
Lemma (Exercise)

(a) If $pn - (\log n + \log\log n) \to -\infty$ then a.a.s. $G(n,p) \notin D2$

(b) If $pn - (\log n + \log\log n) \to \infty$ then a.a.s. $G(n,p) \in D2$

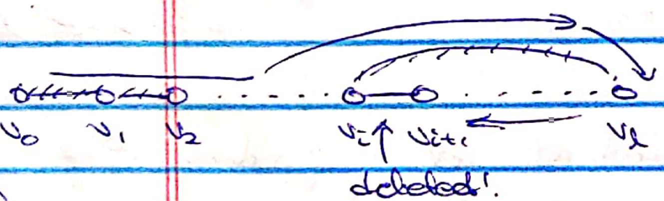
Lemma $\Rightarrow$ Part (b) of the theorem.

Idea for proof:



We try to build a longer and longer path until we build a Hamiltonian cycle.
What is the probability of us creating a longer path? $\to$ We can only build from the 2 ends

Posá Rotation:



$v_0 v_1 v_2 \dots v_i v_l v_{l-1} \dots v_{i+1}$ is a Posá Rotation

Def'n:
Let $P = v_0 v_1 \dots v_l$ be a longest path in G. Assume $v_i v_l$ is an edge. Then, the operation of deleting edge $v_i v_{i+1}$ from P and adding edge $v_i v_l$ to P is called a Posá Rotation.

The Posá Rotation converts P to another longest path

Given a longest path $P = v_0 v_1 \dots v_\ell$. Let $\mathcal{P}$ be the set of longest paths obtained by fixing end v_0 and repeatedly performing Pósa Rotations.

Let $\text{End}(v_0)$ be the set of ends of paths in $\mathcal{P}$ other than v_0.

Given a subset S of vertices, let

$$N(S) = \cup\{u \notin S : \exists v \in S \text{ s.t. } v \sim u\}$$

Lemma: $|N(\text{End}(v_0))| < 2|\text{End}(v_0)|$

Proof:

It is sufficient to show that if $v_i \in N(\text{End}(v_0))$, then one of v_{i-1}, v_{i+1} must be in $\text{End}(v_0)$

(Extremal Case:

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Consider a sequence of Pósa rotations which produces P' with $v_i \sim x$, where $x \neq v_0$ is an end of P'. Let y and z be the left and right neighbors of v_i on P'. If $\{y, z\} = \{v_{i-1}, v_{i+1}\}$, the one of them can be added to $\text{End}(v_0)$ by performing a Pósa Rotation on P'. Else, one of the edges $v_i v_{i-1}$, $v_i v_{i+1}$ has been deleted in a previous Pósa Rotation $\Rightarrow$ one of v_{i-1}, v_{i+1} have been added to $\text{End}(v_0)$.

$\square$