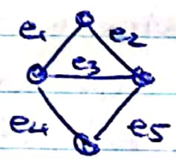
Vector Spaces for Graphs.

Def'n (Edge Space)

The edge space of G, denoted $\mathcal{E}(G)$, is the set of vectors of the form $GF(2)^{E(G)}$ (Note: $GF(2) \cong \mathbb{Z}_2$). We are indexing each entry with an edge in G. Each such vector represents a subgraph of G. (i.e. $e \in E(G)$ is in the subgraph iff the entry for e is 1 in the vector)

Ex:



$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

corresponds to:

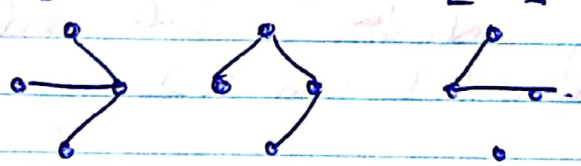
Vector Addition in $\mathcal{E}(G)$

(Note: Scalar multiplication is uninteresting)

Vector addition corresponds to the symmetric difference of the edge sets.

Ex:

$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



Note:

(i) The zero vector corresponds to the subgraph with no edges

(ii) The fundamental basis is $\{e_1, \ldots, e_n\}$, so $\dim \mathcal{E}(G) = |E(G)|$.

Cut Spaces:

Def'n (Subspace)

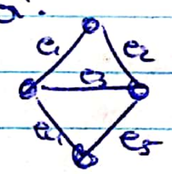
A subspace S is a subset that contains the zero vector and is closed under addition

Def'n (Cut Space)

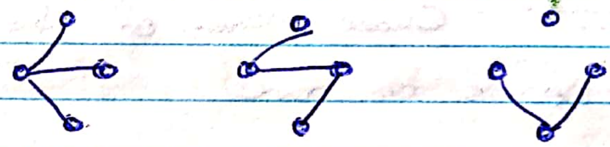
The cut space of G, denoted $C^*(G)$, is the set of all edge cuts in G (which includes the empty set)

Ex:

G:



Things in $C^*(G)$ include:



$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Proposition 12.1:

$C^*(G)$ is a subspace of $\mathcal{E}(G)$

[Proof]

This follows from the fact that the symmetric difference of a cut is still a cut □

Proposition 12.2:

Let B be the set of all bonds of G. Then B spans $C^*(G)$.

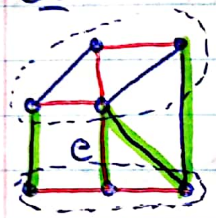
[Proof]

We proved that every cut is a disjoint union of bonds □.

Note: However the set of all bonds may not necessarily give a basis

Def'n (Fundamental Cuts)
Let G be a connected graph, and let T be a spanning tree of G. For each $e \in E(T)$, $T-e$ has 2 components. Let S be vertices of one such component. Then, $\delta_G(S)$ is the fundamental cut in G with respect to e, T. We write D_e.

Ex!



■ - Spanning Tree (T)
■ - Fundamental Cut wrt e, T (D_e)

- Choose either of these to be S.

Note: D_e is a bond

Theorem 13.1
Let G be connected and T a spanning tree of G. The set of all fundamental cuts in G with respect to T form a basis for the cut space $C^*(G)$.

[Proof]
For each edge $e = xy \in E(T)$, e is in D_e, but not in any $D_{e'}$ for $e' \in E(T) \setminus \{e\}$, since x and y are adjacent in $T - e'$, and hence must be in the same component.
If a linear combination of D_e's give 0, then the coefficient of the D_e's must be 0, since only one D_e contains e (out of all the D_e's). So, the set of fundamental cuts is independent

Con't

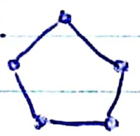
[Proof] (cont)
Let F be a cut. Consider the sum $S = F + \sum_{e \in F \cap E(T)} D_e$.
Every edge $e' \in F \cap E(T)$ appears twice, once in F, once in $D_{e'}$. So $e' \notin S$, since the sum will be 0.
Any other edges in T do not appear in S. So S does not contain any edge in T. So, $S = \emptyset$.
Then,

$$F = -\sum_{e \in F \cap E(T)} D_e = \sum_{e \in F \cap E(T)} D_e$$

So, the set of fundamental cuts span $C^*(G)$ □

Corollary: B.1
If G is connected, then $\dim C^*(G) = |V(G)| - 1$.
[Proof]
This is exactly the number of edges in a spanning tree □

Corollary B.2:
If G has k components, then $\dim C^*(G) = |V(G)| - k$.

Cycle Spaces:
Def'n
The cycle space of G, denoted $C(G)$ is the set of all cycles in G. We write:
$$C(G) = \operatorname{Span}\{1_C\}_{C \subseteq E(G),\ C \text{ cycle}}$$
formally (where 1_C is the characteristic vector for a cycle C). Commonly we write $C(G) = \operatorname{Span}\{C\}$.
Ex:
G:

Then, $C(G) = \operatorname{Span}\left\{\begin{bmatrix} \vdots \end{bmatrix}\right\}$.

(there is only one cycle).

Recall: Every connected even graph (all vertices have even degree) has an Eulerian circuit.

Lemma 14.1:
Suppose we have 2 even edge sets E_1, E_2, then $E_1 + E_2$ is even.
[Proof]
Let $v \in V(G)$. Let D_1, D_2 be the edges incident to v in E_1, E_2 respectively.
Let D be the edges incident to v in $E_1 + E_2$.
Then,
$$|D| = \underbrace{|D_1|}_{\substack{\text{even,} \\ \text{since } E_1 \text{ even.}}} + \underbrace{|D_2|}_{\text{even}} - \underbrace{2|D_1 \cap D_2|}_{\text{even.}}$$
So, $|D|$ is even, and hence $E_1 + E_2$ is even ☐

Proposition 14.1:
Elements of $C(G)$ are precisely even subgraphs of G.
[Proof]
($\Rightarrow$) Let $F \in C(G)$
Consider:
$$F = C_1 + \dots + C_i + \dots + C_k$$
We proceed by induction on the # of cycles.

Base Case: C_1 is even (it is a cycle).

IH: Assume $C_1 + \dots + C_i$ is even

Con't

[Proof] (Con't)

($\Rightarrow$) (Con't)

[redacted] IC: We know that $C_1 + \dots + C_i$ is even and C_{i+1} is even. Hence, by lemma 14.1, $(C_1 + \dots + C_i) + C_{i+1}$ is also even. So the claim follows by induction.

($\Leftarrow$) Let F be an even subgraph, we want to show that F is a sum of cycles. We will proceed by induction on the # of edges in F.

Base Case: F is empty, so $F \in C(G)$ trivially.

IH: Assume that every non-trivial component has min. degree 2. Thus it has a cycle C

IC: Now,

$$\underbrace{F - C}_{\text{By IH, sum of cycles.}} = \underbrace{F + C}_{\text{By lemma, this is even}} \qquad \text{(Since we are in } GF(2))$$

$\Rightarrow (F - C) + C = F$

So, we've shown that F is a sum of cycles $\blacksquare$

Def'n (Fundamental Cycle)

Let T be a spanning tree of G. For any $e \notin E(T)$, the unique cycle of $T + e$ is a fundamental cycle, denoted C_e

$\rightarrow$ Theorem

Hilroy

Theorem 14.1:
Let T be a spanning tree of G, then, the fundamental cycles, with respect to T, form a basis for $C(G)$.

[Proof]

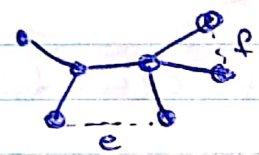
(Independence)

We claim that:

$$\{C_e\}_{e \in E(G) \setminus E(T)}$$

is linearly independent

Notice that:



Each $e, f \in E(G) \setminus E(T)$ are only part of one fundamental cycle.

In other words, each C_e has a unique non-zero entry that is 0 for all other fundamental cycles. So $\{C_e\}_{e \in E(G) \setminus E(T)}$ must be linearly independent.

(Spanning)

We want to show that any even $F \in C(G)$ is a sum of C_e's. Let $F' = F \setminus E(T)$

Consider:

$$S = \underbrace{F}_{\text{even}} + \underbrace{\sum_{e \in F'} C_e}_{\text{sum of even cycles, so even by lemma}}$$

So, by lemma, S is even, so $S \in C(G)$. And S can't contain any tree edges, so $S = 0$. Then,

$$F = -\sum_{e \in F'} C_e = \sum_{e \in F'} C_e$$

$\square$

Corollary 15.1:
For a connected graph G:

$$\dim C(G) = |E(G)| - |V(G)| + 1$$

[Proof]
There are $|V(G)| + 1$ edges in the spanning tree $\square$

Corollary 15.2:
If G has k components, then

$$\dim C(G) = |E(G)| - |V(G)| + k$$

Orthogonal Complements:
We will show that the cut space is the orthogonal complement of the cycle space.

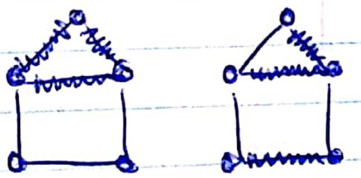
Recall: In $\mathbb{R}^N$, $\vec{v}, \vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

Def'n (Dot product in $\mathcal{E}(G)$)
The dot product in $\mathcal{E}(G)$ gives the # of edges in common. In particular, given H_1, H_2:

$$H_1 \cdot H_2 = \begin{cases} 0, & \text{when \# of edges is even} \\ 1, & \text{—— '' —— odd} \end{cases}$$

Note: This is NOT an inner product space since $\langle \vec{x}, \vec{x} \rangle = 0 \not\Leftrightarrow \vec{x} = 0$. However, this is a bilinear form!

Ex:



H_1 H_2

$$H_1 \cdot H_2 = 0 + 1 + 1 + 0 = 0.$$

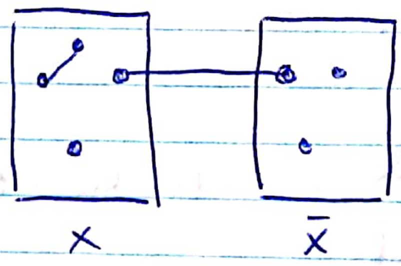
Def'n (Orthogonal Complement)
For a vector space V with subspace S, the orthogonal complement of S is the set of all vectors in V that are orthogonal to all vectors in S, denoted $S^\perp$.

$$S^\perp = \{v \in V \mid v \cdot w = 0, \ \forall w \in S\}$$

Remark:
1) $S^\perp$ is also a subspace
2) $(S^\perp)^\perp = S$
3) $\dim S + \dim S^\perp = \dim V$

Theorem 15.1
$C^*(G)^\perp = C(G)$
[Proof]
($\Rightarrow$) Let $F \in C^*(G)^\perp$, so F is orthogonal to every cut in G. In particular, F is orthogonal to cuts of the form $\delta(\{v\})$ for each vertex $v \in V(G)$. So, F has even degree at every vertex, hence $F \in C(G)$.

($\Leftarrow$) Now, suppose $F \in C(G)$. Let $D = \delta(X)$ be any cut. We want to show that $|F \cap D|$ is even.



(Two types of edges)

Con't

[Proof] (Con't)

Consider the sum $S = \sum_{v \in X} \deg_F(v)$

Let F' be the set of edges with both endpoints in X. Each edge contribute 2 to S. (one for each end). Edges in $F \cap D$ contribute 1 to the sum.

So,

$$S = 2|F'| + |F \cap D|$$

Since $F \in C(G)$, $\deg_F(v)$ is even for all v. So, S is even, hence

$$|F \cap D| = S - 2|F'|$$

is even. So $F \in C^*(G)^\perp$. □

Note:

1) $C(G)^\perp = C^*(G)$

2) $\dim C^*(G) + \dim C(G)$

$= |V(G)| - k + |E(G)| - |V(G)| + k$

$= |E(G)|$

$= \dim \mathcal{E}(G)$

Hilroy