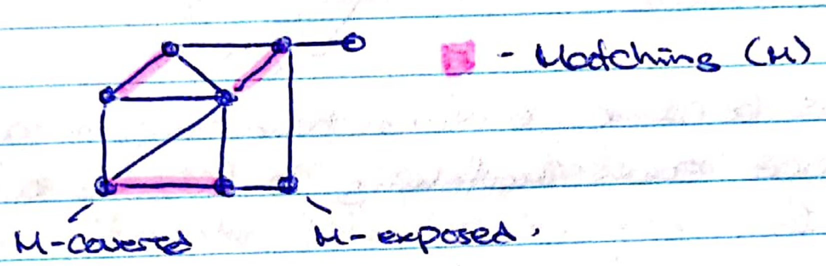
Matchings.

Def'n (Matching, Covered/M-covered, exposed/M-exposed)
A matching M of G is a set of edges that do not share any vertices. A vertex v is covered by M (or M-covered) if v is incident with an edge in M. Otherwise, v is exposed (or M-exposed)

Ex:



Def'n (Deficiency)
We denote $\nu(G)$ to be the size of a maximum matching in G. The deficiency of G is:

$$\operatorname{def}(G) = |V(G)| - 2\nu(G)$$

(i.e. The min. # of vertices "missed" by a matching)

Def'n (Maximal)
A matching is maximal if no edge has both endpoints exposed by the matching.

Def'n (Perfect Matching / 1-factor)
A perfect matching (or 1-factor) is a matching that covers all vertices.

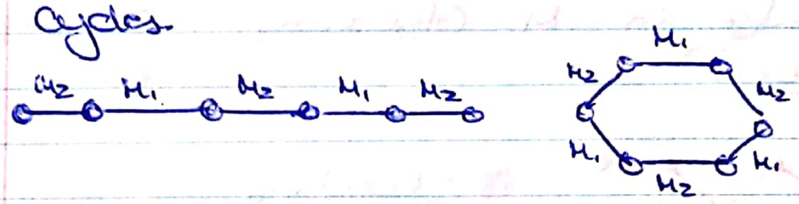
Theorem 27.1
Let M be a maximal matching and M^* be a maximum matching. Then $|M^*| \leq 2|M|$

[Proof] Follows from Prop'n 27.1 (let M^* be a maximal matching) ☐

Q: Let M_1, M_2 be 2 matchings. What does $M_1 \Delta M_2$ look like?

A:

We get components that are paths and cycles.



This is because vertices have degree at most 2, since each matching contributes at most 1.

Furthermore, notice that you can't have odd cycles, since edges alternate between M_1 and M_2.

For any component K,

$$||K \cap M_1| - |K \cap M_2|| \leq 1$$

with equality holding iff K is an odd-length path.

Proposition 27.1

If M_1, M_2 are maximal matchings, then $|M_1| \leq 2|M_2|$ and $|M_2| \leq 2|M_1|$.

[Proof]

Suppose $H_1, \ldots, H_k$ are the nontrivial components of $M_1 \Delta M_2$. Then $||H_i \cap M_1| - |H_i \cap M_2|| \leq 1$, and since the # of edges in M_1, M_2 and not in $M_1 \Delta M_2$ are the same $|M_1| - |M_2| \leq k$. So, we get that $|M_1| \leq k + |M_2|$ and claim that $k \leq |M_2|$. Suppose not, then there is H_i without an edge in M_2, so it must be a single edge $e = xy$ in M_1, but then both x and y are both M_2-exposed, contradicting maximality of M_2. So $|M_1| \leq 2|M_2|$, and by symmetry $|M_2| \leq 2|M_1|$

$\square$

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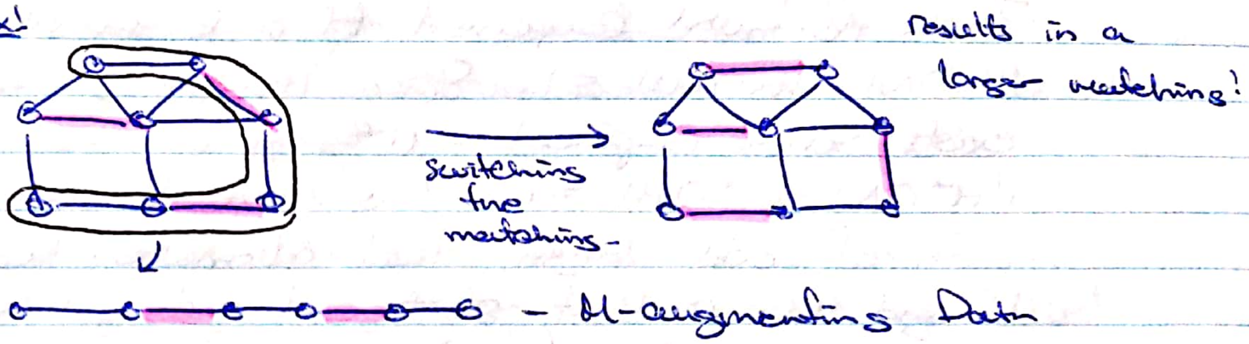
Augmenting Paths:

Ex!



Def'n (M-alternating, M-augmenting)
Given a matching M, a path is M-alternating if edges alternate b/w in M and out of M in the path. A path is M-augmenting if it is M-alternating, and starts and ends with M-exposed vertices.

Remark 28.1: There are more edges not in M than in M, so "switching" the matching increases the size of the matching. (For M-augmenting paths)

Theorem 28.1:
A matching is maximum iff there are no M-augmenting paths.

[Proof]

($\Rightarrow$) (Contrapositive)
Suppose there is an M-augmenting xy-path P. Then, $P \triangle M$ is also a matching, but it covers x, y in addition to M-covered vertices. So, M is not maximum

($\Leftarrow$) (Contrapositive)
Suppose M is not maximum. Then, there is a matching N s.t. $|N| > |M|$. Consider $K = M \triangle N$.

$\hookrightarrow$ Con't

[Proof] (Con't)
Each nontrivial component H in K satisfies $||H \cap N| - |H \cap M|| \leq 1$. Since $|N| > |M|$, there exists some component H^*, where:
$||H^* \cap N| - |H^* \cap M|| = 1$. This must be a path of odd length, that alternates between N and M, and it starts and ends with edges in N. So, the 2 ends of the path are M-exposed. Hence, it is an M-augmenting path
◻

Bipartite Matchings:

Defn (Neighbor Set)
For a subset $S \subseteq V(G)$, the neighbor set $N_G(S)$ or $N(S)$ is the set of all vertices in $V(G) \setminus S$ that is adjacent to some vertex in S.

Theorem 28.2: (Hall's Theorem)
Let G be bipartite with bipartition (A, B). Then G has a matching that covers A iff for all $S \subseteq A$, $|N_G(S)| \geq |S|$.
[Proof]
($\Rightarrow$) Any matching that covers A must cover any subset $S \subseteq A$, and their matching neighbours must be distinct in $N(S)$.
So, $|N(S)| \geq |S|$.

→ Con't

[Proof] (cont)

($\Leftarrow$) We prove by induction on $|A|$.
This is trivially true when $|A|=0$. Next, we break the proof into 2 cases.

Case 1: ($|N_G(S)| > |S|$ for all $\emptyset \neq S \subset A$)
Consider any edge xy, where $x \in A$, $y \in B$. Let $G' = G - \{x, y\}$, and $S' \subseteq A \setminus \{x\}$. By assumption, $|N_G(S')| > |S'|$, but since we removed y from B in G_2, then $|N_{G'}(S')| \geq |N_G(S')| - 1 \geq |S'|$. By induction, there is a matching in G' that covers $A \setminus \{x\}$. Together, with xy, we get a matching in G that covers A.

Case 2: ($\exists \emptyset \neq S \subset A$, where $|N_G(S)| = |S|$)
Let G_1 be the subgraph induced by $S \cup N_G(S)$ and $G_2 = G - G_1$. In G_1, for any $S' \subseteq S$, $|N_{G_1}(S')| \geq |S'|$ since all neighbours in G of S' are in $N_{G_1}(S')$ by def'n. By induction, there is a matching that covers S in G_1.
Now, consider G_2. Let $T \subseteq A \setminus S$ in G. Suppose $|N_{G_2}(T)| < |T|$. Then $N_{G_1}(S \cup T) = N_G(S) \cup N_{G_2}(T)$.
So, we get:

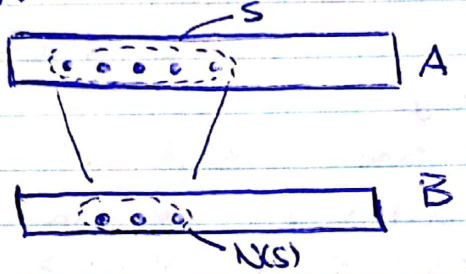
$$\begin{aligned}
|N_{G_1}(S \cup T)| &= |N_G(S)| + |N_{G_2}(T)| \\
&= |S| + |N_{G_2}(T)| \\
&< |S| + |T| = |S \cup T|
\end{aligned}$$

Contradicting that $|N_{G_1}(S \cup T)| \geq |S \cup T|$. So $|N_{G_2}(T)| \geq |T|$ for all $T \subseteq A \setminus S$, and by induction, there is a matching in G_2 that covers $A \setminus S$. Together with the matching in G_1, we get a matching that covers A in G. $\square$

Q: Suppose no M covers A, what is the size of the maximum matching?

Ex:

By Hall's Thm, there must be $S \subseteq A$ s.t. $|N(S)| < |S|$.



Notice that, in this example, we miss at least 2 vertices.

Def'n (Deficiency)

In a bipartite graph, with bipartition (A, B), a subset $S \subseteq A$ has deficiency

$$def(S) = |S| - |N(S)|.$$

Corollary 29.1

For a bipartite G, with bipartition (A, B).

$$\nu(G) = |A| - \max\{def(S) : S \subseteq A\}.$$

Note: $def(\emptyset) = 0$, so this max is nonnegative

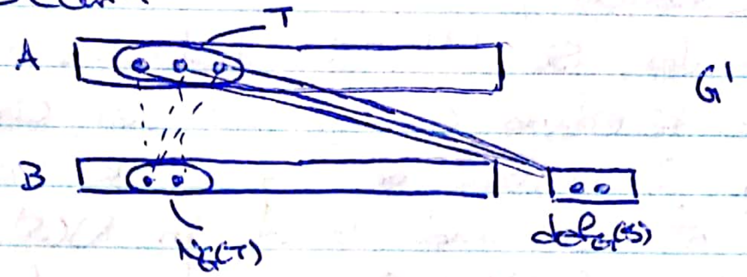
[Proof]

Let $S \subseteq A$ be a set with maximum deficiency. Any matching must miss at least $def(S)$ vertices. So, $\nu(G) \leq |A| - def(S)$.

Now, we obtain G' from G by adding $def_G(S)$ new vertices to B and join every one to all vertices in A.

To con't

[Proof] (Con't)



For any $T \subseteq A$,

$$|N_{G'}(T)| = |N_G(T)| + def_G(S)$$
$$= |T| \underbrace{- def_G(T) + def_G(S)}_{\geq 0,\text{ since } S \text{ has max. deficiency}}$$
$$\geq |T|$$

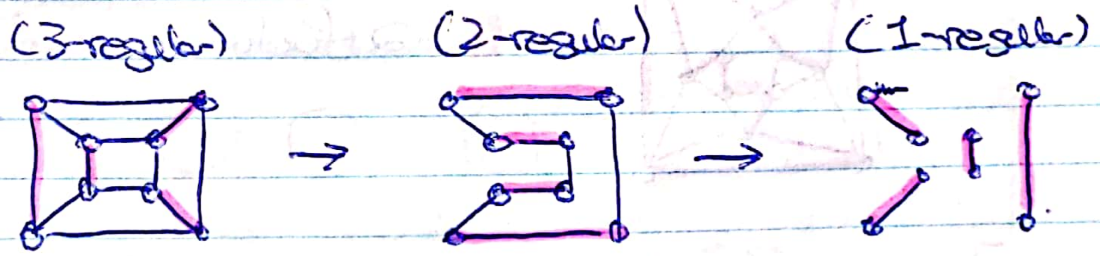
So, by Hall's Theorem, there is a matching M' that covers A in G', where at most there are $def_G(S)$ many edges in M' that use the newly added vertices.

So, $\nu(G) \geq |A| - def(S)$, and equality holds ◻

Corollary 29.2

If G is a k-regular bipartite graph ($k \geq 1$), then G has a perfect matching. Moreover, the edges of G can be partitioned into k perfect matchings.

Ex:



So, we get 3 perfect matchings.

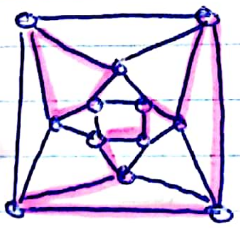
⟸ [Proof]

Hilroy

[Proof] (Sketch)
G is k-regular, so $|A| = |B|$. Let $S \subseteq A$.
There are k edges coming out of each vertex in S, so we get that $k|S|$ edges come out of S and land in N(S). Since each vertex in N(S) cover at most k vertices, $|N(S)| \geq |S|$.
Then, by Hall's Thm, there exists a matching that covers A, and is a perfect matching since $|A| = |B|$.
Removing such a matching results in a (k-1)-regular bipartite graph. And so, by induction, we get k perfect matchings □.

Note: This is sometimes called the 1-factorization of the graph

Def'n (2-factor)
A 2-factor of G is a 2-regular spanning subgraph.
Ex!



- 2-factor

Theorem 30.1:
Any 2k-regular graph with $k \geq 1$ has a 2-factor. Moreover, the edges can be partitioned into k 2-factors.

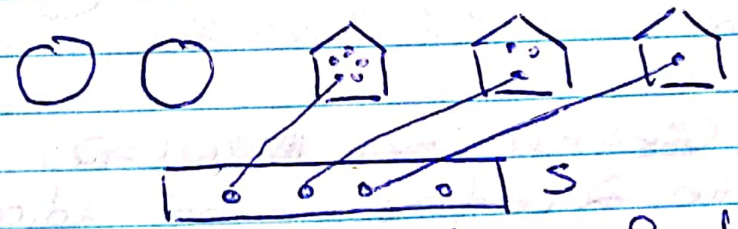
⟹ [Proof]

[Proof]
Assume G is connected. Since each vertex has even degree, it has an eulerian circuit.
Say the walk is: $v_0 e_0 v_1 e_1 \cdots v_{k-1} e_{k-1} v_k = v_0$.

We create a new graph H from G by splitting each vertex into v^- (v_{out}), v^+ (v_{in}) in H, and for each edge $e_i = v_i v_{i+1}$ (as in the walk) in G, we add the edge $v_i^- v_{i+1}^+$ in H. Then H is also bipartite with bipartition $\{v^- \mid v \in V(G)\}$ and $\{v^+ \mid v \in V(G)\}$. Since each $v \in V(G)$ has degree $2k$, the circuit will visit v k times, using k edges to go in, and k edges to go out. So, H is k-regular. Then, by previous corollary (20.2), H has a perfect matching. By merging v^-, v^+, we get a 2-factor in G.

The 2nd result is done by induction ☐

General Matchings:
Ex:
Suppose we have a perfect matching for G, and we remove a subset $S \subseteq V(G)$.



What can we say about the # of odd components?
- # odd components $\leq |S|$, since if not, we cannot have a perfect matching with an odd # of vertices.

Is the converse of this true?

Con't

Defn (Odd Component)
An odd component is a component with an odd number of vertices. Denote $o(G)$ to be the # of odd components in G.

Theorem 30.2 (Parity Lemma)
Let $S \subseteq V(G)$. Then $o(G-S) - |S| \equiv |V(G)| \pmod 2$

Theorem 30.3 (Tutte's Perfect Matching Theorem)
G has a perfect matching iff $|S| \geq o(G-S)$ for all $S \subseteq V(G)$.

[Proof]
($\Rightarrow$) If G has a perfect matching M, then any odd component of $G-S$ must have at least one vertex matched to a vertex in S, since any perfect matching will leave one vertex out. So, $|S| \geq o(G-S)$.

($\Leftarrow$) Assume $|S| \geq o(G-S)$ for all $S \subseteq V(G)$
By taking $S = \emptyset$, $|S| = 0 \geq o(G-S) = o(G)$.
So $o(G) = 0$, and G has no odd components, hence $|V(G)|$ must be even.
Let $|V(G)| = 2n$, we proceed by induction on n.

Base Case: When $n=1$, $|V(G)| = 2$.
If the 2 vertices are not adjacent, then we have 2 odd components, contradiction.
So, an edge exists and we have a perfect matching.

Now, assume $n > 1$.

(Con't)

[Proof] (a→f)
We break into 2 cases:

Case 1: ($|S| > o(G-S)$ for all $S \subseteq V(G)$, where $2 \le |S| \le 2n$)
By Parity lemma, $|S| \ge o(G-S) + 2$.
Let xy be an edge and $G' = G - \{x, y\}$.
Let $T \subseteq V(G')$, and consider $G' - T$. We can write $G' - T = G - (T \cup \{x,y\})$. Then, by assumption:

$$|T \cup \{x,y\}| \ge o(G - (T \cup \{x,y\})) + 2$$
$$\Rightarrow |T| + 2 \ge o(G' - T) + 2$$
$$\Rightarrow |T| \ge o(G' - T)$$

And, by induction, G' has a perfect matching which, together with xy, is a perfect matching for G.

Case 2: (There is $|S| = o(G-S)$ for some $S \subseteq V(G)$, where $2 \le |S| \le 2n$)
Among all such subsets, we pick a maximal one.
We first prove that $G-S$ has no even components.
Suppose there is an even component C (of $G-S$).
Let $x \in V(C)$. Then $C-x$ has an odd number of vertices, so it must contain at least one odd component. So, $o(G-(S \cup \{x\})) \ge o(G-S) + 1$.
Then:

$$|S \cup \{x\}| = |S| + 1 \underset{\text{By assumption}}{=} o(G-S) + 1 \le o(G-(S \cup \{x\}))$$

But also, $|S \cup \{x\}| \ge o(G-(S \cup \{x\}))$ by assumption, so $|S \cup \{x\}| = o(G-(S \cup \{x\}))$, which contradicts the maximality of S.

Cont

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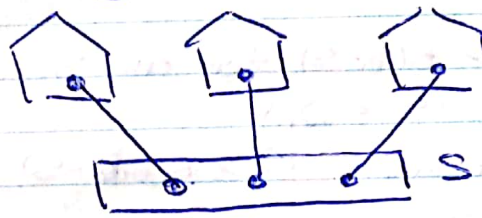
[Proof] (Con't)

So, $G-S$ only consists of odd components

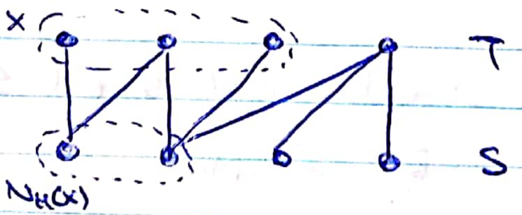


([Idea: We find a perfect matching from S to the components, using Hall's Theorem, then a P.M. within the components)

We obtain a graph H from G by contracting each odd component of $G-S$ into a vertex, and remove any edge joining 2 vertices in S. Let T be the set of all contracted vertices, then H is bipartite with bipartition (S,T). Further, since $|S| = o(G-S)$, $|S| = |T|$. We want to show that H has a perfect matching.

Let $X \subseteq T$, and suppose $|N_H(X)| < |X|$. Then in G, the odd components of $G - N_H(X)$ are those represent by X and possibly other odd components. So, $o(G-N_H(X)) > |N_H(X)|$

H:



which is a contradiction, so $|N_H(X)| \geq |X|$

So, by Hall's Theorem, H has a perfect matching, which in G, is a matching from each vertex in S to a distinct ...

Con't

[Proof] (con't)

... odd component of $G-S$. Call this matching M.

Now, consider any odd component C of $G-S$. Suppose $a \in V(C)$ is covered by M. We want to show that there is a P.M. in $C-a$.

Suppose there is $Y \subseteq V(C) \setminus \{a\}$, where $o((C-a)-Y) > |Y|$. Since $C-a$ has an even # of vertices, by Parity Lemma: $o((C-a)-Y) \geq |Y| + 2$.

Let $Z = S \cup \{a\} \cup Y$ so that the odd components of $G-Z$ are: the odd components of $C-a-Y$, and any other odd components of $G-S$.

Then:

$$o(G-Z) = (o(G-S) \underbrace{-1}_{\text{since we lose component } C}) + o(C-a-Y)$$

$$\geq (o(G-S) - 1) + |Y| + 2$$

$$= |S| - 1 + |Y| + 2$$

$$= |S| + |Y| + 1 = |Z|$$

But $o(G-Z) \leq |Z|$ by assumption, so $o(G-Z) = |Z|$, contradicting maximality of S.

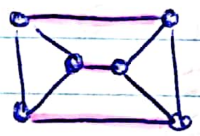
So, $o(C-a-Y) \leq |Y|$ for all $Y \subseteq V(C) \setminus \{a\}$, then, by induction $C-a$ has a matching. This, along with a P.M. for each odd component and M gives a P.M. for G. $\square$

Hilroy

Def'n (Deficiency)
For $S \subseteq V(G)$, the deficiency of S is
$$\text{def}(S) = o(G-S) - |S|$$
Note: Any matching must miss at least $\text{def}(S)$ vertices.

Corollary 32.1 (Tutte-Berge Formula)
For any graph G,
- $\text{def}(G) = \max\{\text{def}(S) \mid S \subseteq V(G)\}$
- $\nu(G) = \dfrac{|V(G)| - \text{def}(G)}{2}$

Corollary 32.2 (Petersen)
A 3-regular graph with no cut edges has a perfect matching
Ex:



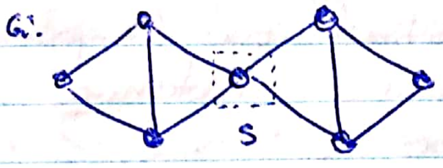
■ - Perfect matching.

[Proof]
Let G be 3-regular with no cut edges, and $S \subseteq V(G)$. Let $C_1, \dots, C_k$ be the odd components of $G-S$, so $k = o(G-S)$. For each C_i, the sum of vertex degrees in C_i is odd since $|V(C_i)|$ is odd and each has deg 3. Each edge in C_i contributes 2 to this total degree, so the number of edges joining $V(C_i)$ with S is odd. Further, this number cannot be 1, since such an edge would be a cut edge. So, there must be at least 3 edges between C_i and S. Over all k components, there are at least $3k$ edges with one edge in S. Since each vertex in S has deg 3, $|S| \geq 3k/3 = k = o(G-S)$. By Tutte's Thm, G has a perfect matching $\square$

Def'n (Tutte Set)

A tutte set is a set of vertices with max. deficiency

Ex:



S is a Tutte set, since $o(G-S) - |S| = 2 - 1 = 1$.

Note: $T = \emptyset$ is also a Tutte set.

Def'n (Essential, avoidable)

A vertex v is essential if every maximum matching G covers v (or $\nu(G-v) = \nu(G) - 1$). It is avoidable if some max. matching exposes v (or $\nu(G-v) = \nu(G)$).

Lemma 32.1:

Every vertex in a Tutte set is essential

[Proof]

Let S be a Tutte set, and suppose $x \in S$ is avoidable

Let $G' = G - x$, $S' = S \setminus \{x\}$, then $G - S = G' - S'$.

Then,

$$\begin{aligned} \operatorname{def}(G') &\geq o(G'-S') - |S'| \\ &= o(G-S) - |S| + 1 \\ &= \operatorname{def}(G) + 1 \quad (\text{since } S \text{ is a Tutte set}) \end{aligned}$$

So, $\nu(G') < \nu(G)$, and hence x must be essential

□.

↪ Theorem

Hilroy

Theorem 32.1:

Let S be a Tutte set of G.

(1) Every vertex of S is essential, and is matched to a distinct odd component of $G-S$ by any maximum matching

(2) The even components of $G-S$ have perfect matchings

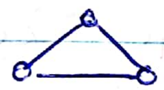
(3) The odd components of $G-S$ have near-perfect matchings (a matching that covers all but one vertex).

Defn (Factor-Critical)

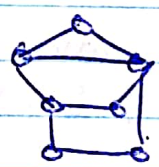
G is factor-critical if $G-v$ has a perfect matching for $v \in V(G)$.

Note: Such a graph must have an odd number of vertices

Ex:



- and -



are both factor-critical.

Remark 33.1: Factor Critical graphs are connected

Theorem 33.1:

Let G be a connected graph. Then, G is factor-critical iff every vertex of G is avoidable

$\Leftarrow$ Proof

[Proof]

($\Rightarrow$) If G is factor-critical, then $G-v$ has a perfect matching, which must also be a maximum matching for G. So, v is avoided by this max. matching.

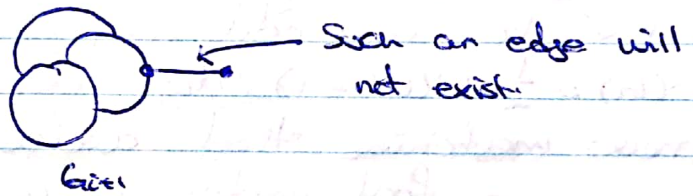
($\Leftarrow$) Suppose all vertices are avoidable. Let T be a Tutte set. Since any vertex in T is essential, $T=\emptyset$. So, $def(G) = o(G-T) - |T| = o(G)$, which is 1 or 0, since G is connected.
If G is an even component, then $o(G)=0$, and so $def(G)=0$, and G has a perfect matching, contradicting that every vertex is avoidable.
So, G must be an odd component, $def(G)=1$, and $\nu(G) = \frac{1}{2}(|V(G)|-1)$. For any $v \in V(G)$, there is a max. matching that avoids v. So, then $G-v$ has a perfect matching, as the max. matching has size $\nu(G)$. So, G is factor-critical. $\square$.

Theorem 33.2
If T is a maximal Tutte set, then each odd component of $G-T$ is factor critical.
[Proof] Similar to proof for Tutte's Thm (Thm 30.7) $\square$

Def'n (odd closed-ear decomposition)
An odd closed-ear decomposition of G is a sequence of graphs $(G_0, \ldots, G_k)$, where:
(1) G_0 is a vertex. $G_k = G$.
(2) For each $0 \leq i \leq k-1$, $G_{i+1} = G_i + P_i$, where P_i is either an ear of G_i of odd length, or an odd cycle, where $|V(G_i) \cap V(P_i)| = 1$.

Theorem 33.3 (Lovász)
A graph G is factor-critical iff G has an odd closed-ear decomposition.
[Proof]
($\Leftarrow$) Induction + Case Analysis.

($\Rightarrow$) Suppose G is factor-critical. Let G_0 consist of one vertex v in G. Let M_v be a P.M. in $G-v$. We will iteratively build G_{i+1} from G_i by adding P_i so that no edge in M_v is in the cut induced by $V(G_{i+1})$ (called the invariant) i.e.



Suppose we have built G_i. If $G_i = G$, then we're done.
If there is an edge xy in G, not in G_i, where $x, y \in V(G_i)$, then $G_{i+1} = G_i + xy$, and no edge in M_v is in $\delta(V(G_{i+1}))$.
Now, suppose $x \in V(G_i)$, $y \notin V(G_i)$. By the invariant, $xy \notin M_v$. Let M_y be a P.M. of $G-y$.
Consider $M_v \triangle M_y$. The only vertices of degree 1 are v and y, since they are covered by exactly one of M_v, M_y, and all other vertices are covered by both. So, v, y are the endpoints of a path in $M_v \triangle M_y$.

Con't

[Proof] (Con't)
Say this path P is $v_0v_1\dots v_m$, where $v_0 = y$, $v_m = v$. Since $v_0 \notin V(G_i)$ and $v_m \in V(G_i)$, there is a smallest index j, where $v_j \in V(G_i)$. Let Q be $v_0 \dots v_j$. By the invariant, $v_{j-1}v_j \in M_y$ and $v_0v_1 \in M_v$. Since the edges alternate between M_v and M_y, Q has even length. Then, $xy + Q$ is either an odd length ear or an odd cycle.
Set $G_{i+1} = G_i + xy + Q$.

Now we still need to prove the invariant! Note that every vertex $v_0, \dots, v_{j-1}$ is covered by an edge in M_v, which is also in the path Q. So, no edge in M_v is in $\delta(V(G_{i+1}))$, so the invariant holds. $\square$

Theorem 35.1 (Gallai-Edmonds Structure Theorem)
Given G, let D be the set of all avoidable vertices in G, let A be the set of all vertices in $V(G) \setminus D$, that is adjacent to some vertex in D, let $C = V(G) \setminus (D \cup A)$.
Then,
(1) The subgraph induced by D are odd components, which are factor-critical.
(2) The subgraph induced by C are even components that have perfect matchings.
(3) A is a Tutte set
(4) Each non-empty ~~sets~~ subset S of A is adjacent to at least $|S|+1$ odd components in D.
(5) If M is a maximum matching then it contains a near-perfect matching for each component of D, a perfect matching for C, and matches each vertex of A to a distinct odd component in D.

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