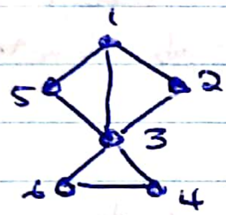
# Connectivity.

*In this course, we assume a graph $G=(V,E)$ is simple unless otherwise stated*.

Def'n (Subgraph)
Let $S$ be a set of vertices in $G$. Then, the subgraph induced by $S$, denoted $G[S]$, consists of $S$ as the vertex set, and all edges in $G$ joining 2 vertices in $S$.

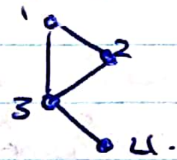
Ex:

$G$:



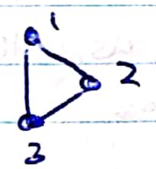
$S = \{1,2,3,4\}$

$G[S]$:



$T = \{1,2,3\}$

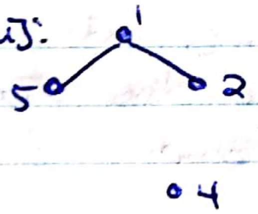
$G[T]$:



We call $G[T]$ an induced cycle.

$U = \{1,2,4,5\}$

$G[U]$:



$G[U]$ is not an induced cycle

Def'n (Min/Max degree)
We use $\delta(G)$ to denote the minimum degree among all vertices in $G$, and $\Delta(G)$ for the maximum degree

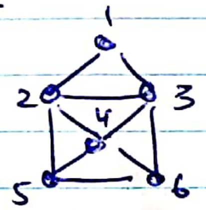
Ex: With $G$ above $\delta(G) = 2$, and $\Delta(G) = 5$.

## Edge Connectivity:

Recall:
A bridge or cut-edge in $G$ is an edge whose removal increases the # of components in $G$.

Def'n (Disconnecting Set)
A set of edges $F$ is a disconnecting set if removing all edges in $F$ from $G$ results in a disconnected graph. We use $G-F$ to denote this removal.

Ex:
G:



Some disconnecting sets are:
$\{12, 13\}$
$\{12, 23, 36, 24, 45, 46\}$
→ Set of all edges

Note:
- If a graph is already disconnected, even the empty set is a disconnecting set.
- In general, we want to minimize the size of a disconnecting set in a graph.

Def'n (k-edge-connected)
A graph is k-edge-connected if there does not exist a disconnecting set of size at most $k-1$. In other words, any disconnecting set must have size at least $k$.

Def'n (Edge Connectivity)

The edge connectivity of $G$ is the largest $k$ for which $G$ is $k$-edge-connected. (i.e. $G$ is not $(k+1)$-edge-connected). We write $\kappa'(G)$ to denote this.

Ex: For $G$ above, it is 2-edge-connected, since it does not have a cut-edge. However, it is not 3-edge-connected since it has a disconnecting set of size 2.
So, $\kappa'(G) = 2$.

Remarks:

- If $G$ is not connected, then it is 0-edge-connected
- A connected graph is 1-edge-connected. If it has a bridge, then the edge connectivity is 1
- A connected graph with no cut-edges is 2-edge-connected
- If $G$ is $k$-edge-connected, then $G$ is $(k-1)$-edge-connected
- If $\kappa'(G) = k$, then there exists a disconnecting set of size $k$, and it is a minimum disconnecting set.
- A graph with a single vertex has no disconnecting set. We set its edge connectivity to be 0.

Proposition 2.1:
$\kappa'(G) \leq \delta(G)$

[Proof]
Assume $G$ is non-trivial. So let $v$ be a vertex of degree $\delta(G)$. Let $F$ be the set of all edges incident with $v$. Then $G-F$ has no path from $v$ to any other vertices. So, $F$ is a disconnecting set of size $\delta(G)$, so $G$ is not $(\delta(G)+1)$-edge-connected, and so $\kappa'(G) \leq \delta(G)$
$\square$.

Hilroy

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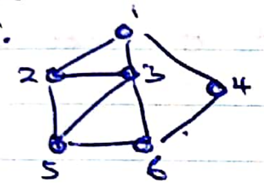
# Cuts and Bonds:

**Def'n (Cut)**

For a set $S \subseteq V(G)$, the cut induced by $S$ (or a cut), is the set of all edges with one end in $S$, and one end not in $S$, denoted $\delta(S)$. A cut is non-trivial if $S \neq \emptyset$ and $S \neq V(G)$

**Ex:**

G:




$S = \{1,2,3,5\}$

$\delta(S) = \{56, 14, 36\}$.

**Recall (from M239):**

G is connected if and only if every non-trivial cut is non-empty.

So, we get the following:

**Corollary 2.1:**

Any non-trivial cut is a disconnecting set.

**Q:** Is the converse of this true?

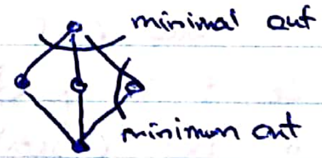
**A:** No! For example, you could take out all the edges of the graph.

**Proposition 2.2:**

For any connected graph $G$ with at least 2 vertices, any minimal disconnecting set $F$ is a cut, and $G-F$ has exactly 2 components.

**Note:**

A minimal set means there is no proper subset with the same properties.

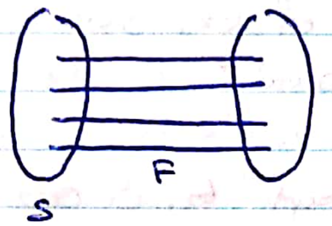



C→ Proof

[Proof]

Let $F$ be any minimal disconnecting set, we want to show a set $S \subseteq V(G)$ exists such that $\delta(S) = F$.

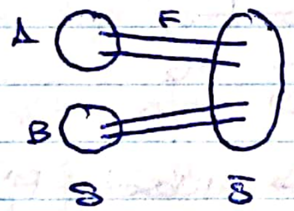
$F$ will split $G$ into at least 2 components:




Take $S$ to be the vertices of one component of $G-F$. Consider the edges of $\delta(S)$. We see that $\delta(S) \subseteq F$, since no edge is in $\delta_{G-F}(S)$ because $G-F$ is a component, and so any edge in $\delta_G(S)$ must be in $F$.

Now, since $\delta(S)$ is a disconnecting set and $F$ is minimal, $\delta(S) = F$, so $F$ is a cut.

Now, suppose $F = \delta(S)$, and $G-F$ has at least 3 components. WLOG, suppose $(G-F)[S]$ has at least 2 components




Let $A$ and $B$ be the vertex sets of 2 of the component. Now, $\delta_G(A)$ and $\delta_G(B)$ are non-empty (since $G$ connected), so using a similar argument as above $\delta_G(A) \subsetneq F$ and $\delta_G(B) \subsetneq F$, contradicting the minimality of $F$. So therefore there must be only 2 components $\square$

Def'n (Bond)

A bond is a minimal non-empty cut.

Q: Suppose you have a set of edges, that, if you remove out of $G$, gives you 2 components. Is that a bond?

Theorem 2.1:

In a connected graph $G$, a cut $\delta(S)$ is a bond if and only if $G-F$ has 2 components.

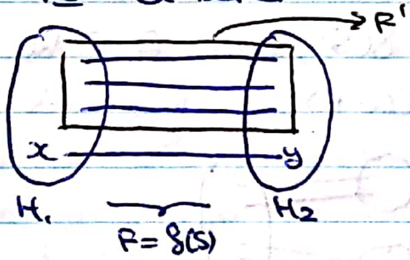
(This is a characterization of bonds in terms of cuts)

[Proof]

($\Rightarrow$) See Prop'n 2.2.

($\Leftarrow$)

Suppose $G-F$ has 2 components $H_1$, $H_2$, and suppose $F$ is not a bond.




Then, there is a proper subset $F' \subsetneq F$ that is a cut. Let $e = xy \in F \setminus F'$, where $x \in V(H_1)$ and $y \in V(H_2)$. So $e$ is in $G-F'$. Each vertex in $H_1$ has a path to $x$, and each vertex in $H_2$ has a path to $y$, and they can reach each other via $e$. So $G-F'$ is connected contradicting the fact that $F'$ is a cut.

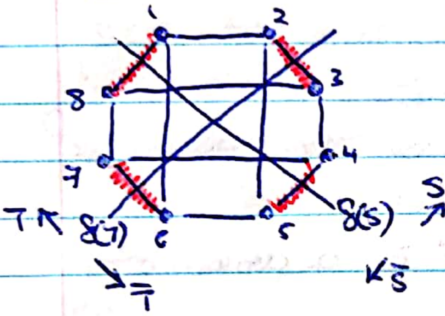
So, $F$ must be a bond. $\square$

Def'n (Symmetric Difference)
The symmetric difference of 2 sets A, B is:

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$

(i.e. The elements that appear exactly once)

Ex:



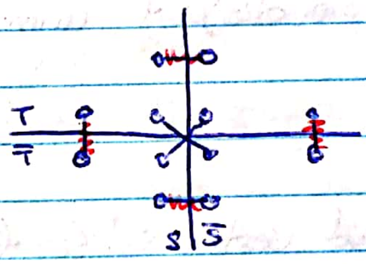

Let $S = \{1, 2, 3, 4\}$,
$T = \{1, 2, 7, 8\}$

What is $\delta(S) \triangle \delta(T)$?
$\delta(S) \triangle \delta(T) = \{18, 23, 67, 45\}$, all edges that only appear once.

Q: Is $\delta(S) \triangle \delta(T)$ a cut of some set?
A: Yes! Let $U = \{3, 4, 7, 8\}$, then $\delta(S) \triangle \delta(T) = \delta(U)$, in fact notice that $\{3, 4, 7, 8\} = S \triangle T$.

Theorem 3.1:
$\delta(S) \triangle \delta(T) = \delta(S \triangle T)$

[Sketch]
There are six types of edges

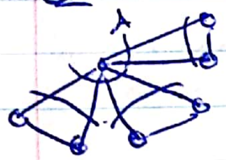



But, only the edges in red will appear in the symmetric difference. These are precisely the edges in $\delta(S \triangle T)$. $\square$

## Theorem 3.2:

Every cut is a disjoint union of bonds

Ex:



Cut A is a union of the 3 other cuts (bonds)

[Proof]

We will prove this by induction on the number of edges in the cut.

Base Case: If $|F| = 0$, then $F$ is a union of no bonds.

IH: Now, suppose $F$ is any nonempty cut

IC:

If $F$ is a bond, then we're done

So, assume $F$ is not a bond. Then, there exists a proper subset $F' \subsetneq F$ that is also a nonempty cut. Then $F'' = F \setminus F' = F \Delta F'$, and by previous result, $F''$ is also a nonempty cut. By induction, $F''$ and $F'$ are both disjoint unions of bonds. Hence $F = F' \cup F''$ is a disjoint union of bonds $\square$

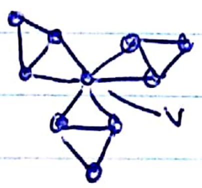
## Vertex Connectivity

Def'n (Cut-vertex)

A vertex $v$ is a cut-vertex in $G$ if removing $v$ and its incident edges from $G$ increases the number of components. (We write $G - v$)

Note: The major difference between a cut-vertex and cut-edge is that removing a cut vertex may increase the number of components by more than one

Ex:

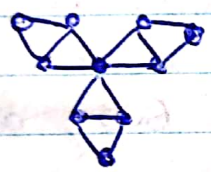



The vertex labelled $v$ is a cut-vertex.

Def'n (Seperating set)
A subset $S$ of vertices is a <u>seperating set</u> if removing all vertices in $S$ from $G$ results in a disconnected graph. (We write $G-S$)

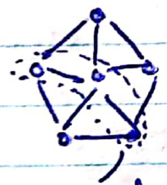
Def'n (k-connected, connectivity)
A graph is <u>k-connected</u> if does not have any seperating sets of size at most $k-1$. The <u>connectivity</u> of $G$ is the largest $k$ for which $G$ is k-connected, denoted $K(G)$.

Ex:

(i)

here $K(G)=1$

(ii)

is 2-connected, but not 3-connected
seperating set of size 2, $K(G)=2$

(iii)

is 3-connected, but not 4-connected
seperating set of size 3, $K(G)=3$

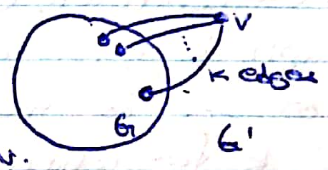
Note:
$K_n$ have no seperating set, so we define $K(K_n)=n-1$

C-Expansion
Lemma

Expansion Lemma:
Let $G$ be a k-connected graph, and let $G'$ be obtained from $G$ by adding a new vertex $v$ that is joined to at least $k$ vertices in $G$. Then, $G'$ is k-connected.

[Proof]

Let $X$ be any set of $k-1$ vertices in $G'$.




Case 1: $X$ doesn't include $v$.
Then, $G-X$ is connected since $G$ is k-connected and at least one neighbor of $v$ is not in $X$ ($v$ is incident to $k$ edges and we are removing $k-1$) So $v$ is connected to $G-X$, and so $G'-X$ is connected.

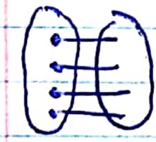
Case 2: $X$ does include $v$.
Then, $G'-X$ is the same as removing $k-2$ vertices from $G$, and since $G$ is k-connected $G'-X$ is connected.

Therefore, $G'$ is k-connected □

Theorem (Whitney)
$\kappa(G) \leq \kappa'(G) \leq \delta(G)$

vertex connectivity) edge connectivity min. degree of graph

Caution:



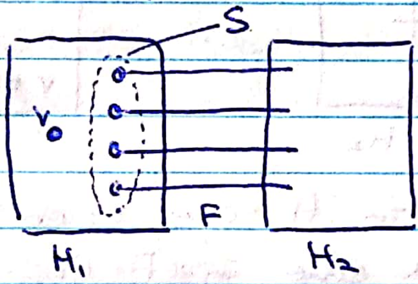
Suppose we have a min. cut, we can't just take out all the vertices, because what if they are the only vertices in the partition?
This motivates the cases in the proof.

[Proof]

[Proof]

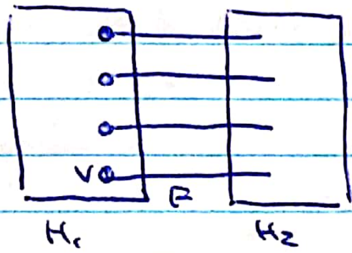
We've already proved that $\kappa'(G) \leq \delta(G)$ (Prop'n 2.1).
So, we only need to prove that $\kappa(G) \leq \kappa'(G)$.
For $G = K_n$, $\kappa(G) = \kappa'(G) = n-1$, so we're done.
Now, assume $G$ is not complete. Let $F$ be a minimum cut, so $|F| = \kappa'(G)$.
(Goal: Find a separating set of size at most $|F|$)
We have 2 cases:

Case 1:
Suppose there is a vertex $v$ not incident with any edge in $F$.




Suppose $G-F$ has 2 components $H_1, H_2$ and WLOG suppose $v \in V(H_1)$. Let $S$ be the set of vertices in $H_1$ incident with some edge in $F$. Then $|S| \leq |F|$, and it is [redacted] a separating set since $v$ is separated from $H_2$ in $G-S$. So, $\kappa(G) \leq \kappa'(G)$.

Case 2:
Now, suppose that every vertex is incident with an edge in $F$.



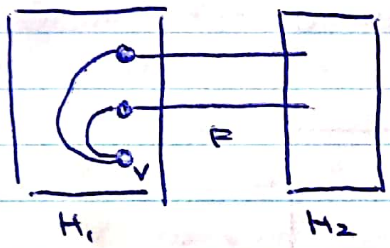

Con't

[Proof] (Con't)

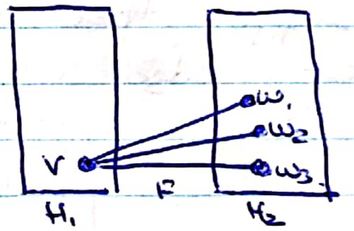
Case 2: (Con't)

Since $G$ is not complete, there exist $v, w \in V(G)$ that are not adjacent. WLOG, suppose $v \in V(H_1)$, and let $X$ be the set of all neighbours of $v$. This is a seperating set since it separates $v$ from $w$. The neighbours of $v$ can be in $H_1$ or $H_2$.

In $H_1$:




In $H_2$:

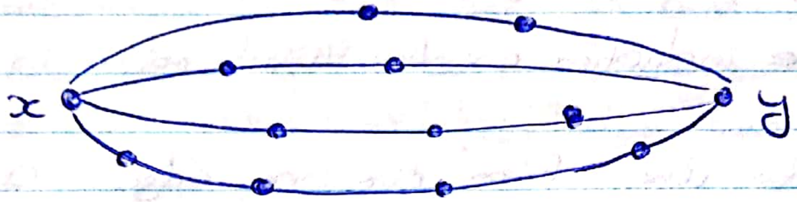



Neighbours of $v$ in $H_1$ are each [redacted] incident with an edge in $F$ (by assumption.)
Neighbours of $v$ in $H_2$ also each contribute one edge to $F$.
These must all be distinct edges, and hence $\deg(v) \leq |F|$, so $|X| \leq |F|$, and $\kappa(G) \leq \kappa'(G)$ $\square$.

Menger's Theorem:

Motivation:

Suppose we have the following graph, how many connections do we remove to destroy the connection between $x$ and $y$?




In this case, we need at least 4 vertices (one for each path)

Now. Suppose you cannot destroy the connection if you remove $v$ vertices, does that mean there are at least $v+1$ paths from $x$ to $y$?

Def'n ($x,y$-separating set)

An $x,y$-separating set $X$ is a separating set where $x,y$ are in different components of $G-X$.

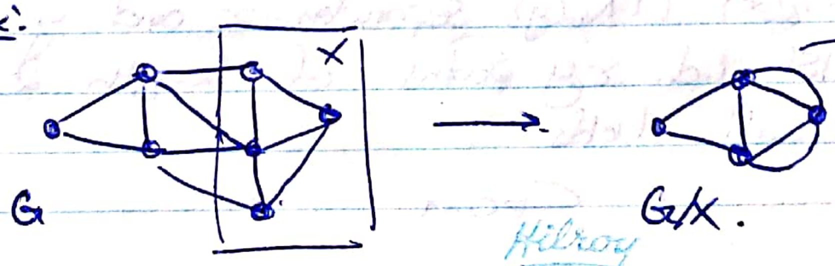
Def'n (Internally-disjoint)

A set of $x,y$-paths is internally disjoint if no two such paths share any vertices or edges except $x,y$.

Def'n (Shrinking/Contraction)

For a set $X$ of vertices where $G[X]$ is connected, we denote $G/X$ to be the graph obtained by removing all edges in $G[X]$, and identifying all vertices of $X$ into one vertex.

Ex:




→ We might have multiple edges to one vertex.

Kilroy

Theorem (Menger's Theorem)
Let $x, y$ be 2 non-adjacent distinct vertices in $G$. Then, the minimum size of an $x,y$-separating set is equal to the maximum [redacted] number of internally-disjoint $x,y$-paths.

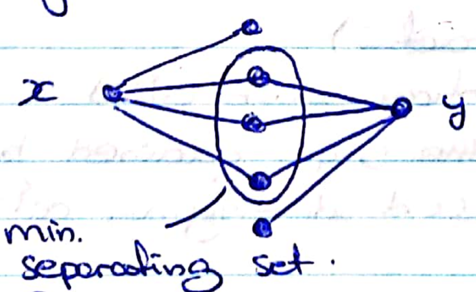
[Proof]

We use induction on the # of edges in $G$.

Base Case: When there are no edges, any minimum $x,y$-separating set has size $0$, and no $x,y$-paths exist.

IH: Let $G$ be any graph with at least one edge. Let $x, y$ be 2 non-adjacent vertices. Suppose a minimum $x,y$-separating set has size $k$.
(Goal: Find $k$ internally-disjoint $x,y$-paths)

IC:
We have 2 cases.
Case 1: (Easy)
Assume that every edge is incident with $x$ or $y$.




Then, $S = N_G(x) \cap N_G(y)$ separates $x$ and $y$, and we have $|S|$ i.d. $x,y$-paths of length 2, so the result holds

(cont.

[Proof] (cont)

Case 2:

Let $e = uv$ be an edge in $G$ not incident with $x$ or $y$. Let $H = G - e$ and let $S$ be a minimum $x,y$-seperating set in $H$.

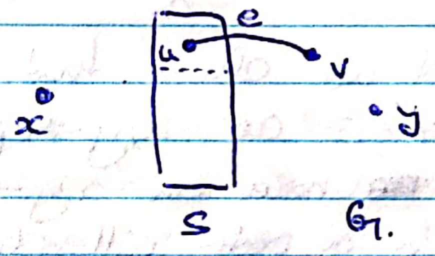
If $|S| = k$, then by induction, $H$ contains $k$ i.d $x,y$-paths which are also in $G$, and so we're done.

So, assume $|S| \le k-1$. We know that $S \cup \{u\}$ is an $x,y$-seperating set in $G$.

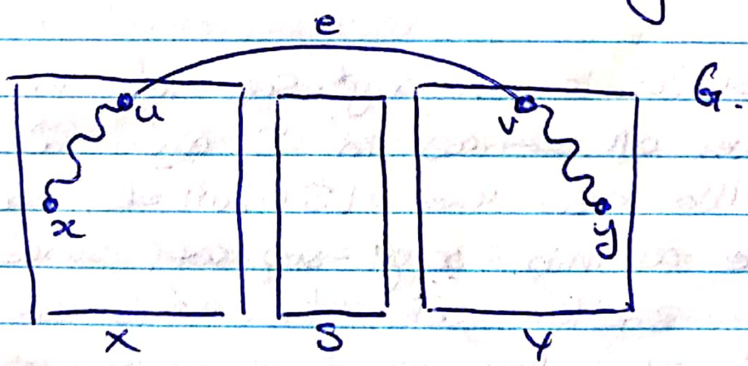
But, any $x,y$-seperating set in $G$ has size at least $k$. So,

$|S \cup \{u\}| \ge k \Rightarrow |S| \ge k-1$,

but we assumed that $|S| \le k-1$, so $\boxed{|S| = k-1.}$




Now, let $S = \{v_1, v_2, \ldots, v_{k-1}\}$, and let $X, Y$ be components containing $x$ and $y$ in $H - S$ respectively

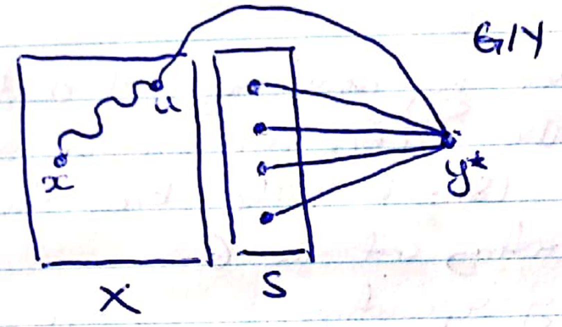



Since $|S| < k$, $S$ is not an $x,y$-seperating set in $G$, so $G - S$ has an $x,y$-path that must use $e$

(→ cont)

[Proof] (con't)

Suppose $u \in V(X)$ and $v \in V(Y)$.
Consider $G/Y$, and let $y^*$ be the contracted vertex:




First, we claim that any $x,y^*$-sep set $T$ in $G/Y$ is an $x,y$-sep set in $G$. Why? Suppose we have an $x,y$-path that avoids $T$ in $G$, such a path will still exist in $G/Y$, since we have done nothing but contract the graph. (i.e. We haven't removed paths). So, any $x,y$-separating set in $G/Y$ have size at least $k$.

Now, $S \cup \{u\}$ is an $x,y^*$-sep set in $G/Y$ since it removes all vertices in $S$ and the edge $e=uv$. We know that $|S \cup \{u\}| = k$, so it must also be a min. $x,y^*$-sep set, since if there were a smaller set, then this would imply that $G$ also has a smaller $x,y$-sep set.

We have contracted the $v,y$-path in $G/Y$, so $G/Y$ has fewer edges than $G$. Hence, by induction, there exist $k$ i.d. $x,y^*$-paths in $G/Y$.

Con't

[Proof] (Con't)
And we can describe what the paths are.
Let $P_1, \ldots, P_{k-1}$ be part of the paths that start at $x$ and end at $v_1, \ldots, v_{k-1}$ respectively, and let $P_k$ be the $x,u$-path.
Similarly, we can consider $G/X$, and find $k$ paths $Q_1, \ldots, Q_{k-1}$ from $v_i$ to $y$ and $Q_k$ from $v$ to $y$. Then $P_i + Q_i$ for $i = 1, \ldots, k-1$, and $P_k + e + Q_k$ are $k$ i.d. $x,y$-paths in $G$. $\square$.

Note:
Why do we handle the 2 cases separately?
Notice that contracting vertices in Case 1 does nothing! Hence, induction cannot be used properly, so we must deal with that case by itself.

Corollary 4.1
$G$ is $k$-connected iff there exist $k$ i.d $x,y$-paths for any distinct $x,y \in V(G)$.
[Proof]
($\Rightarrow$) We break this into 2 cases:
Case 1: ($x,y$ not adjacent)
Since $G$ is $k$-connected, any $x,y$-sep set has size at least $k$. Then, by Menger's thm, there exist $k$ i.d. $x,y$-paths.

Case 2: ($x,y$ adjacent)
$G - xy$ is $(k-1)$-connected. Then, by Menger's thm, $G-xy$ has $k-1$ i.d. $x,y$-paths and together with $xy$, they form $k$ i.d. $x$-$y$ paths.

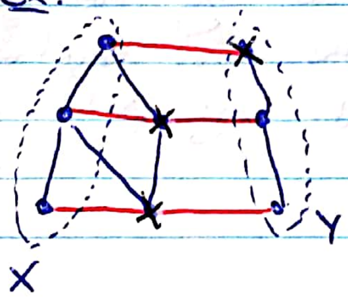
$\hookrightarrow$ Con't

[Proof] (cont)
($\Leftarrow$) Let $X$ be any set of $k-1$ vertices, and let $x$ and $y$ be any 2 vertices in $G-X$. Since there are $k$ i.d $x,y$-paths (by assumption), at least one path remains in $G-X$, (as we are only removing $k-1$ vertices) So $G-X$ is still connected, and hence $G$ is $k$-connected. □.

Variations (of Menger's Thm)
We now look at different versions of Menger's thm to be used in different contexts.

Def'n ($X,Y$-separating set)
For sets of vertices $X, Y$, an $X,Y$-separating set is a set $S$, where $G-S$ has no path from any vertex in $X$ to any vertex in $Y$
Note! $S$ may itself include vertices in $X$ or $Y$, and $S$ could also be the entire set $X$ or $Y$, i.e. $S=X$ or $S=Y$.

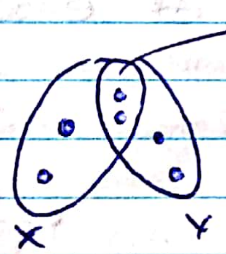
Def'n (disjoint $X,Y$-paths)
A set of disjoint $X,Y$-paths are paths that start with a vertex in $X$, ends with a vertex in $Y$, no other vertices are in $X$ or $Y$, and no paths share any vertices.
Ex.



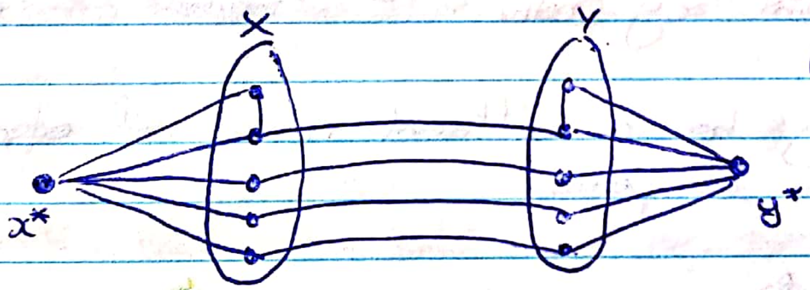

■ - 3 disjoint $x,y$-paths
X - $X,Y$-Sep set

**Theorem 6.1: (Set Version of Menger's)**
For any $X, Y \subseteq V(G)$, the minimum size of an $X,Y$-separating set is equal to the maximum number of disjoint $X,Y$-paths.

Note:
If $X, Y$ share vertices, then the vertex itself is an $X,Y$-path.




[Proof]
Add vertices $x^*, y^*$ to $G$ so that $x^*$ is adjacent to all vertices in $X$ and $y^*$ is adjacent to all vertices in $Y$. Since $x^*$ and $y^*$ are not adjacent, by Menger's thm, the max # of i.d $x^*,y^*$-paths is equal to the min size of an $x^*,y^*$-sep set. Such a sep set is an $X,Y$-sep set in $G$.

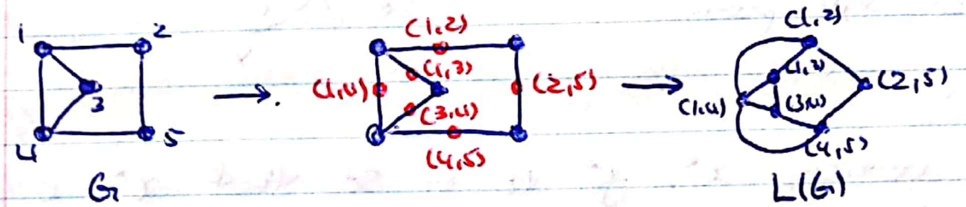



$G + x^* + y^*$

We have to be careful when choosing an disjoint $X,Y$-paths. For each $x^*,y^*$-path, we pick the part from the <u>last</u> vertex in $X$ to the <u>first</u> vertex in $Y$. This gives us the same # of disjoint $X,Y$-paths. □

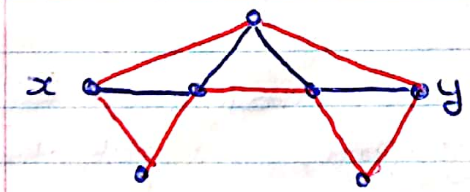
**Def'n** (Line Graph)
Let $G=(V,E)$. The line graph of $G$ is $L(G)$, where the [redacted] [redacted] vertices are $\{v_e \mid e \in E(G)\}$, and two vertices $v_e$ and $v_f$ are adjacent in $L(G)$ iff $e$ and $f$ share a common endpoint in $G$.

Ex:

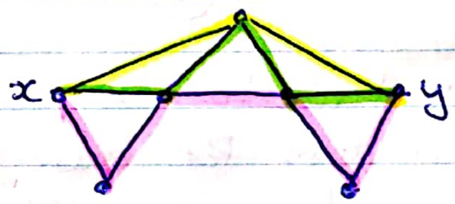



**Theorem 6.2** (Edge Version of Menger's)
Let $x,y$ be 2 distinct vertices in $G$. Then, the minimum size of an $x,y$-cut (disconnecting set) is equal to the maximum number of edge-disjoint $x,y$-paths.

Note:
There may be a different number of edge-disjoint paths than id. paths.




2 id. paths.




3 edge-disjoint paths

~~C~~Proof.

[Proof].

Consider $L(G)$. Let $X, Y$ be the set of vertices in $L(G)$ corresponding to edges incident with $x, y$ in $G$ respectively. Notice that all edge disjoint paths in $G$ are vertex disjoint in $L(G)$. (An edge only being used once in $G$ results in a vertex only being crossed once in $L(G)$). So, an edge-disjoint $x,y$-path in $G$ corresponds to disjoint $X,Y$-paths in $L(G)$, and $x,y$-cuts in $G$ correspond to $X,Y$-sep sets in $L(G)$. The result then follows from Thm 6.1. (Set Version of Menger's) □

Corollary 6.1:

$G$ is $k$-edge-connected iff there exist at least $k$ edge-disjoint $x,y$-paths for any distinct $x,y \in V(G)$

[Proof]

Similar to Corollary 4.1, but with the edge version of Menger's. □

Theorem 7.1 (Fan Lemma):

Let $G$ be $k$-connected, $v \in V(G)$ and $X \subseteq V(G) \setminus \{v\}$, where $|X| \geq k$. Then, there exist a $k$-fan from $v$ to $X$ in $G$.

[Proof]

Obtain $H$ from $G$ by adding a vertex $w$, and join $w$ to all vertices in $X$. Since $|X| \geq k$, by expansion lemma, $H$ is $k$-connected. By corollary of Menger's, there are $k$ i.d. $v,w$-paths in $H$. For each such path, we keep the part from $v$ to the first vertex in $X$ on that path. These $k$ paths form a $k$-fan from $v$ to $X$ □

Corollary 7.1':
Let $G$ be $k$-connected where $k \geq 2$, let $X$ be any set of $k$ vertices in $G$. Then, there is a cycle in $G$ containing at least all vertices in $X$.

[Proof]
We prove by induction on $k$.

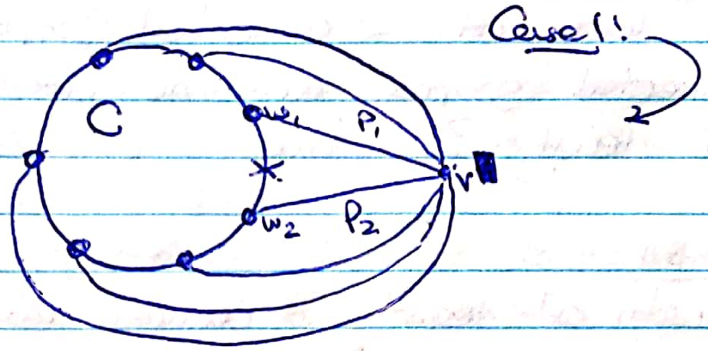
Base Case: When $k=2$, for any $X = \{u, v\}$, there exist 2 internally-disjoint $u,v$-paths. Since $G$ is 2-connected. These 2 paths form a cycle containing $u, v$

IH: Let $G$ be $k$-connected, where $k \geq 3$, let $X$ be any set of $k$ vertices in $G$.

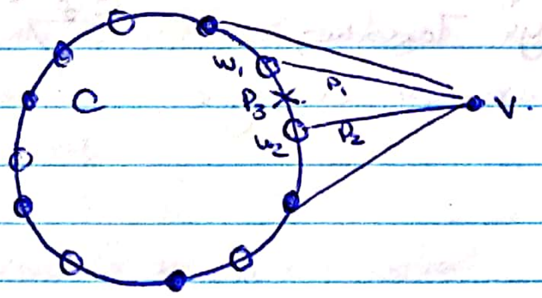
IC:
Since $G$ is $k$-connected, it is also $(k-1)$-connected. Let $v \in X$, then $X \setminus \{v\}$ have size $k-1$. By induction, there is a cycle containing all vertices in $X \setminus \{v\}$, say it is $C$.
If $C$ contains $v$, then we are done.
Otherwise, assume $v$ is not in $C$. We have 2 cases:
Case I: ($C$ has $k-1$ vertices)
So, $C$ only contains $X \setminus \{v\}$. By the fan lemma, there exist a $(k-1)$-fan from $v$ to $V(C)$. Take any consecutive vertices $w_1, w_2$ on $C$, let $P_1, P_2$ be the paths joining $w_1, w_2$ to $v$ in the fan, respectively. Then $(C - w_1 w_2) + P_1 + P_2$ is a cycle containing $X$.

Con't

[Proof] (cont.)




Case 2: (C has $\geq k$ vertices)
By fan lemma, there exists a $k$-fan from $v$ to $V(C)$. The $k-1$ vertices in $X \setminus \{v\}$ partitions C into $k-1$ i.d. paths which we call segments. By pigeonhouse principle, at least 2 of the $k$ paths in the fan must end at vertices in the same segment. Let $w_1, w_2$ be these two end vertices, let $P_3$ be the $w_1 w_2$-path in the segment. Let $P_1, P_2$ be the $w_1 v$-path and $w_2 v$-path in the fan. Now $P_3$ has no internal vertices in $X$, so $(C - P_3) + P_1 + P_2$ is a cycle containing $X$.




$\square$.

## 2-connected graphs:

We now look at 2-connected graphs, which are connected graphs with no cut vertices, and at least 3 vertices.

**Theorem 8.1:**

For $G$ with at least 3 vertices, and $\delta(G) \geq 1$, the following are equivalent:

(1) $G$ is 2-connected,

(2) For any distinct $x, y \in V(G)$, there is a cycle containing $x, y$

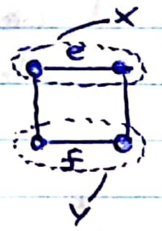
(3) For any distinct $e, f \in E(G)$, there is a cycle containing $e, f$

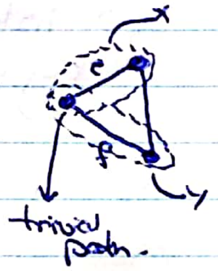
[Proof]

(1) $\Leftrightarrow$ (2): Proved by corollary of Menger's

(1), (2) $\Rightarrow$ (3):

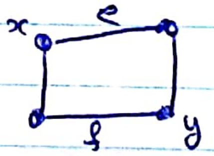
Let $e = uv$, $f = xy$. By the set version of Menger's thm, there are 2 disjoint paths between $\{u, v\}$ and $\{x, y\}$. Together, with $e$ and $f$, we get a cycle containing $e, f$.




(3) $\Rightarrow$ (2):

Let $e, f$ be edges incident with $x, y$ respectively. If $e, f$ are distinct, then, by assumption, there is a cycle which also contains $x, y$. Notice that $e, f$ cannot be the same, since if they are, then there must also be a 3rd vertex AND the edge adjacent to it cannot be in the cycle. But this is a contradiction.



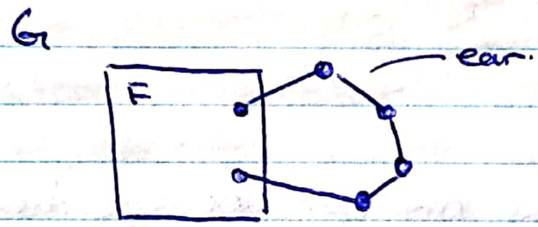

$\square$

## Ear Decomposition

Def'n (Ear / Ear Decomposition)
Let $F$ be a subgraph of $G$. An ear of $F$ in $G$ is a non-trivial path in $G$ where only the 2 endpoints of the path are in $F$.
An ear decomposition of $G$ is a sequence of graphs $(G_0, G_1, \ldots, G_k)$ where:

- $G_0$ is a cycle in $G$,
- For $0 \le i \le k-1$, $G_{i+1} = G_i + P_i$, where $P_i$ is an ear of $G_i$ in $G$,
- $G_k = G$.

Ex.




Theorem 8.2:
A graph $G$ is 2-connected iff $G$ has an ear decomposition. Furthermore, any cycle in $G$ can be the initial cycle of an ear decomposition, and all intermediate graphs in the ear decomposition are 2-connected.

[Proof]
($\Rightarrow$) Suppose $G$ is 2-connected. Let $C$ be any cycle in $G$, and let $G_0 = C$. For $i \geq 0$, we inductively construct $G_{i+1}$ from $G_i$ as follows.
If $G_i = G$, then we're done.
Otherwise, there is an edge $e \in E(G) \setminus E(G_i)$. Say $e = uv$.
By Menger's thm for sets, there are 2 disjoint paths between $\{u, v\}$ and $V(G_i)$

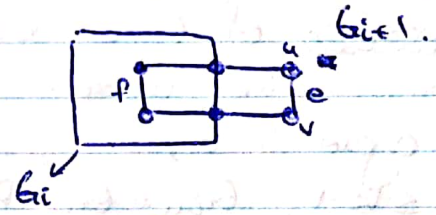
$\hookrightarrow$ Con't

Hilroy

[Proof] (Con't)

($\Rightarrow$) (Con't)

These 2 paths only meet $G_i$ at one of their endpoints, so together with $e$, we obtain an ear $P_i$ of $G_i$.




Let $G_{i+1} = G_i + P_i$. Since $G_{i+1}$ has more edges than $G_i$, this process will terminate. (Notice that we could also have used thm 8.1)

($\Leftarrow$)

Suppose that $G$ has an ear decomposition $(G_0, G_1, \ldots, G_t)$, where $G_{i+1} = G_i + P_i$.

We will prove that $G$ is 2-connected by induction on $i$ (i.e. Each $G_i$ is 2-connected).

Base Case: $G_0$ is a cycle, which is 2-connected

IH: Assume that $G_i$ is 2-connected

IC:

Let $x, y \in V(G_{i+1})$. If both $x, y \in V(G_i)$, then they are in a cycle since $G_i$ is connected. There are 2 more cases. Suppose the 2 endpoints of $P_i$ are $u, v$. If $x, y \in V(P_i)$, then $P_i$ plus a $u,v$-path in $G_i$ form a cycle containing $x, y$.

If $x \in V(P_i)$, $y \in V(G_i)$, then by fan lemma, there is a 2-fan in $G_i$ from $y$ to $\{u, v\}$. Together with $P_i$, we obtain a cycle containing $x, y$. Since there is a cycle containing $x, y$ for all cases, $G_{i+1}$ is 2-connected.

$\square$.

## Strong Orientation: (and 2-connected graphs)

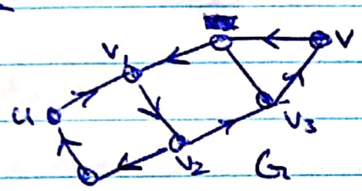
Def'n (Orientation)

An orientation of $G$ can be obtained by adding a direction to every edge $uv$, either $u \to v$ or $v \to u$.

Def'n (Directed $u,v$-path)

A directed $u,v$-path has the form $u, v_1, v_2, \ldots, v_k, v$, where the edges are directed $u \to v_1, v_1 \to v_2, \ldots, v_k \to v$.

Ex:




This orientation of $G$ gives a directed $u,v$-path $u, v_1, v_2, v_3, v$.

Def'n (Strong Orientation)

A strong orientation is one where there is a directed $u,v$-path for any ordered pairs $u,v \in V(G)$

**Theorem 9.1**

If $G$ is 2-connected, then $G$ has a strong orientation

Note:

The converse of this statement is not true. For example, we could consider the "bowtie" graph:

[Proof]

Consider an ear decomposition of $G$, $(G_0, \ldots, G_k)$.
We will inductively orient $G_i$ so that they all have a strong orientation.

Cont

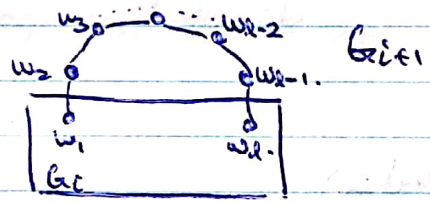
Hilroy

[Proof] (con't)

Base Case: $G_0$ is a cycle $v_1, v_2, \ldots, v_k, v_1$.
We will orient this $v_1 \to v_2$, $v_2 \to v_3$, $\ldots$, $\to v_k \to v_1$.
This is a strong orientation.

IH: Suppose $G_i$ has a strong orientation.

IC:
Let $w_1 w_2 \ldots w_\ell$ be the ear we add to $G_i$ to obtain $G_{i+1}$.




Now, orient the path $w_1 \to w_2$, $w_2 \to w_3$, $\ldots$, $w_{\ell-1} \to w_\ell$, and let $u, v \in V(G_{i+1})$. We have multiple cases:

Case 1 ($u, v \in V(G_i)$)
By IH, there is a directed $u,v$-path in $G_i$ since it is strongly oriented.

Case 2 ($u = w_j$ and $v \in V(G_i)$).
$u,v$-path: Take $w_j, \ldots, w_\ell$ followed by a directed $w_\ell, v$-path in $G_i$.
$v,u$-path: Take a $v, w_1$-path in $G_i$ followed by $w_1, w_2, \ldots, w_j$.

Case 3 ($u = w_j$, $v = w_p$, $j < p$).
$u, v$-path: Take $w_j, w_{j+1}, \ldots, w_p$.
$v, u$-path: Take $w_j, \ldots, w_\ell$, followed by $w_\ell, w_1$-path in $G_i$, and then $w_1, \ldots, w_j$.

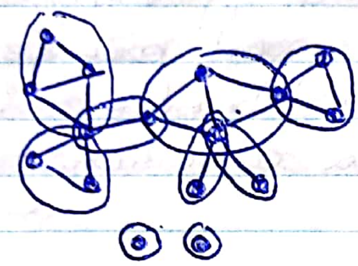
$\square$

Blocks: (and 2-connected graphs)

Def'n (Block)

A block is a maximal connected subgraph with no cut vertices.

Ex:



O-are the blocks of this graph.

Note:

1) Each block is an induced subgraph,
2) Each block is either 2-connected, a single edge, or an isolated vertex.
3) Some results may be obtained by focusing on the individual blocks (ex. A graph has a strong orientation iff each each of its blocks has a strong orientation) [Theorem 10-1].

Proposition 9.1:

Two blocks in a graph share at most one vertex. Such a vertex must be a cut-vertex.

[Proof]

Suppose blocks $B_1$, $B_2$, share at least 2 vertices, so we may also assume $B_1$, $B_2$ each have at least 3 vertices, so it is 2-connected.

Our strategy is to show that $B_1 \cup B_2$ is 2-connected, which contradicts that $B_1$ and $B_2$ are blocks

Con't

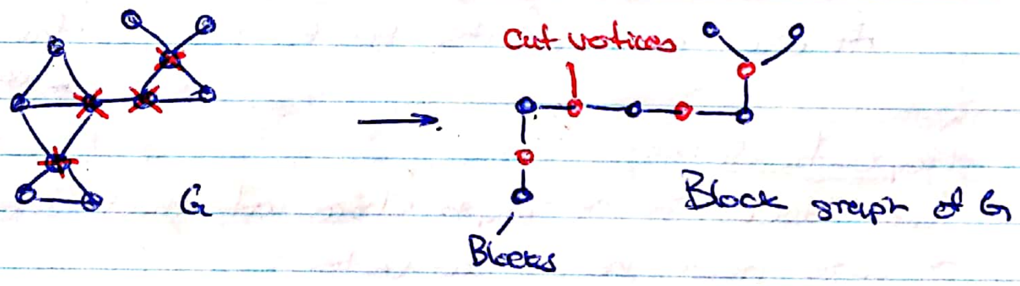
[Proof] (Con't)

Let $x \in V(B_1 \cup B_2)$, and since $|V(B_1) \cap V(B_2)| \geq 2$, there exists $y \in V(B_1 \cap B_2)$, where $y \neq x$. Since $B_1$ and $B_2$ are 2-connected, $B_1 - x$ and $B_2 - x$ are 2-connected. Since $y \in V(B_1) \cap V(B_2)$, there is a path between $y$ and each vertex in $V(B_1) \setminus \{x\}$ and $V(B_2) \setminus \{x\}$. Therefore $(B_1 \cup B_2) - x$ is connected, a contradiction. $\square$.

Def'n (Block graph)

The ~~block graph~~ of $G$ is a bipartite graph with bipartition $(B, C)$, where $B$ is the set of all blocks and $C$ is the set of all cut-vertices. $b \in B$ is adjacent to $c \in C$ iff the block $b$ contains vertex $c$.

Ex:




Proposition 10.1:

The block graph is always a forest

[Proof]

Since each component of $G$ is connected, then each component of the block graph will also be connected.

Now, we show that the block graph has no cycles. Suppose a cycle exists

Con't

[Proof] (Con't)
The cycle must alternate between cut vertices and blocks. Let $v_0, \ldots, v_k$ be the cut vertices in this cycle, and $B$ be a block. $v_0 \ldots v_k$ is a cycle in $G$, let's call $C$. Then, $B \cup C$ is a block since [redacted] $C$ is 2-connected, $B$ is 2-connected, and removing a cut vertex does not disconnect the graph. But, this is a contradiction that $B$ is a block. □.

Proposition 10.2:
If $u, v$ are in the same block $B$, then every $u,v$-path is entirely in $B$.

[Proof]
Suppose there is a $u,v$-path that includes edges not in $B$. Follow the path and consider the corresponding blocks and cut vertices visited in the block graph. This is a closed walk that visits each vertex in $C$ at most once (since you can't visit the same cut vertex more than once in a path). Since we start and end in the same block, we must have a cycle in the block graph, which contradicts Prop'n 10.1. □.

Theorem

Hilroy

Theorem 10.1:
A connected graph has a strong orientation iff each of its blocks has a strong orientation.
[Proof]
($\Rightarrow$) Suppose $G$ has a strong orientation. We keep the same orientation for each block. Let $u, v$ be 2 vertices in a block $B$. We know there is a directed $u,v$-path in $G$, and by previous prop'n (10.2), such a path must be entirely in $B$. So, $B$ has a strong orientation.

($\Leftarrow$) Suppose each of the blocks of $G$ have a strong orientation. We keep the same orientation of each block for $G$. Let $u, v \in V(G)$, we have 2 cases:
Case 1: ($u, v$ are in the same block)
The strong orientation of the block gives a directed $u,v$-path.
Case 2: ($u, v$ not in the same block)
Let $u \in B_1$, $v \in B_k$ and consider the $B_1, B_k$-path in the block graph: $B_1 v_1 \ldots v_{k-1} B_k$. (The path exists since $G$ is connected). Since each block is strongly oriented, we can find directed paths: $u$ to $v_1$ in $B_1$, $v_1$ to $v_2$ in $B_2$, etc. Together, these form a directed $u,v$-path in $G$.
$\square$.

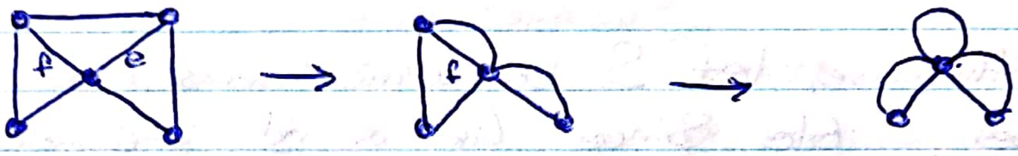
Theorem 10.2 (Robbin's Theorem)
$G$ has a strong orientation iff $G$ is 2-edge-connected.

## 3-connected graphs:

Def'n (Edge Contraction)

Suppose $e$ is an edge in $G$. The ~~contraction~~ contraction of $e$ in $G$, denoted $G/e$ is a graph obtained by removing $e$ and identifying (merging) the 2 endpoints of $e$.

Ex!



$G$ $\qquad$ $G/e$ $\qquad$ $(G/e)/f$

Note: You may encounter multiple edges or loops in contractions, but since they do not affect vertex connectivity, we will remove them (for now)

Theorem 11.1:

Let $G$ be a 3-connected graph with at least 5 vertices. Then, $G$ has an edge $e$, where $G/e$ is 3-connected.

[Proof]

We need a lemma!

Lemma 11.1:

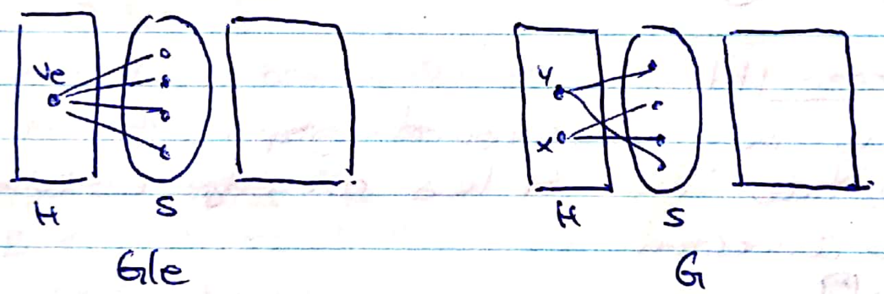
Let $k \geq 2$. If $G$ is $k$-connected with at least $k+2$ vertices and $e = xy \in E(G)$ where $G/e$ is not $k$-connected, then $G$ is a separating set of size $k$ that includes $x$ and $y$.

~~G~~ [Proof] (Lemma)

[Proof] (con't)
[Proof] (Lemma)
We first deal with the case where $G/e$ is complete, then it must have at least $k+1$ vertices. But then, it is k-connected, which contradicts our assumption. So, this case never happens.

Otherwise, let $S$ be a minimum seperating set in $G/e$. Since $G/e$ is not k-connected, $|S| \leq k-1$. Let $v_e$ be the contracted vertex in $G/e$. There are 2 cases.

Case 1: ($v_e \notin S$)




Then $G/e-S$ has at least 2 components, and $v_e$ is in one of them, say H. Neighbours of $v_e$ are in $V(H)$ or $S$. The other component of $G/e-S$ remains a component in $G-S$. Then, $S$ is a separating set in $G$ of size at most $k-1$, a contradiction. So, this case cannot occur.

Con't

[Proof] (Con't)
[Proof] (lemma) (Con't)
Case 2: ($v_e \in S$)
Then, $G/e - S$ is the same as $G - (S \setminus \{v_e\} \cup \{x, y\})$
So, $S \setminus \{v_e\} \cup \{x, y\}$ is a separating set in $G$ of size $k$ that includes $x, y$.

$\square$ lemma.

We now prove the theorem. Suppose by way of contradiction that $G/e$ is not 3-connected for all $e \in E(G)$. By Lemma 11.1, if $e = xy$, then $G$ has a separating set of the form $\{x, y, z\}$, where $z \in V(G)$. We choose $e$ and $z$ so that the # of vertices in a component $H$ of $G - \{x, y, z\}$ is minimized.

Let $S = \{x, y, z\}$. Since $S$ is a minimal separating set, we know that every vertex in $S$ must have at least one adjacent vertex in $G - S$. (Proof in A4Q3, by contradiction). Let $w \in V(G)$ be a neighbour of $z$ in $H$.

Now, $G/zw$ is also not 3-connected, so there is a separating set $S' = \{z, w, v\}$ in $G$. And since $x$ and $y$ are adjacent, there is a component $J$ in $G - S'$ not containing $x$ nor $y$. Notice that $S'$ is also a minimum separating set. Let $u \in J$ be a neighbor of $w$, and so every vertex in $J$ has a path to $w$ via $u$. Such a path will not include $x$ or $y$ (since they are in another component), and hence this path also exists in $G - S$

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[Proof] (con't)
Further, we notice that every vertex in $J$ is also in $H$. (Since we can reach $w$ from $z$, and $u$ from $w$, and all vertices in $J$ through $w$ and $u$). So $V(J) \subseteq V(H)$. However, $w \in V(H)$, but $w \notin V(J)$, and so $V(J) \subsetneq V(H)$. But we assumed that $H$ was minimal, and so we have a contradiction

$\square$

The reverse operation of contraction (i.e splitting a vertex $v$ into $v_1$ and $v_2$) from a 3-connected graph also results in a 3-connected graph. However, we must be slightly careful.

Theorem 11.2:
Let $G$ be 3-connected. Let $v$ be a vertex of degree at least 4. Obtain $H$ from $G$ by splitting $v$ into $v_1, v_2$, adding edge $e = v_1v_2$, and distributing edges incident with $v$ to $v_1, v_2$ so that $\deg_H(v_1); \deg_H(v_2) \geq 3$. Then $H$ is 3-connected

[Proof]
Suppose $H$ is not 3-connected. Let $S$ be a minimum separating set in $H$ ($|S| \leq 2$)
We have 3 cases:

Case 1: ($v_1, v_2 \in S$)
Then, $G - v = H - \{v_1, v_2\}$ and so $v$ is a cut vertex in $G$, contradicting our assumptions on $G$.

Con't

[Proof] (Con't)
Case 2 ($v_1, v_2 \notin S$)
$v_1, v_2$ must be in the same component of $H-S$, but then $S$ is still a separating set in $G$, seperating $v$ from the other component of $H-S$.

Case 3: ($v_1 \in S$, $v_2 \notin S$)
If the component containing $v_2$ has no other vertices, then all neighbors of $v_2$ are in $S$, but this contradicts that $\deg_H(v_2) \geq 3$.
So, there is at least one other vertex in the component containing $v_2$. But, then, this vertex will be seperated from the other component of $H-S$ by the set $(S \setminus \{v_1\}) \cup \{v\}$ in $G$, a contradiction
□.

Tutte's characterization of 3-connected graphs
$G$ is 3-connected iff there exists a sequence $G_0, G_1, \ldots, G_k$ of graphs such that:
1) $G_0 = K_4$, $G_K = G$, and
2) For each $0 \leq i \leq k-1$, there exist an edge $xy \in G_{i+1}$ such that $\deg_{G_{i+1}}(x) \geq 3$, $\deg_{G_{i+1}}(y) \geq 3$, and $G_i = G_{i+1} / xy$.

[Proof]
Proof follows by induction on both sides, and by using Theorem 11.1 and 11.2.
□.