

1 Graph Theory Primer

Let $G = (V, E)$ be a graph, where V is the vertex set and E is the edge set.

Definition 1.1

The **degree** of a vertex $v \in V$ (denoted $\deg(v)$) is the number of edges with one end in v . (i.e. The size of the set $\{va \mid a \in V, a \neq v\}$).

A **walk** is a sequence of vertices $v_1v_2 \dots v_k$ where v_iv_{i+1} is an edge. A **path** is a walk where all vertices are distinct. A **cycle** is a walk where $v_1 = v_k$ and v_1, \dots, v_{k-1} are distinct.

Finally, we say a graph is **connected** if there exists a path between any two vertices in G .

Definition 1.2

For $S \subset V$, the **cut** induced by S is the set of all edges with one end in S and one end not in S , denoted $\delta(S) = \{uv \in E \mid u \in S, v \notin S\}$. Given two vertices $s, t \in V$ with $s \in S, t \notin S$, we call $\delta(S)$ an s, t -**cut**

An s, t -**path** is a path with starting vertex s and ends on t .

Theorem 1.1

There exists an s, t -path if and only if every s, t -cut is nonempty

Definition 1.3

A **tree** is a connected graph with no cycles. A **spanning tree** is a subgraph that is a tree and has vertex set V

Note the following:

- A tree on n vertices contains $n - 1$ edges.
- If T is a tree, then adding an edge $uv \notin T$ creates exactly one cycle C . Moreover, if xy is an edge in C , then $T + uv - xy$

Let $D = (N, A)$ be a directed graph. N is a set of nodes and A is a set of ordered pairs of nodes (called arcs).

Definition 1.4

For an arc (u, v) , we call u the **tail** and v the **head**.

The **out-degree** of node u (denoted $d(u)$ or $d^{\text{out}}(u)$) is the number of arcs with tail u . The **in-degree** of node u (denoted $d(\bar{u})$ or $d^{\text{in}}(u)$) is the number of arcs with head u .

A **diwalk** is a sequence of nodes $v_1 v_2 \dots v_k$ where (v_i, v_{i+1}) is an arc. **Dipaths** and **Dicycle** are defined analogous to simple graphs but with arcs instead of edges.

For $S \subset N$, the **cut** induced by S is denoted $\delta(S) = \{(u, v) \in A \mid u \in S, v \notin S\}$. (sometimes written as $\delta^{\text{out}}(S)$) This is the set of arcs with tail in S . We denote the complement of S by \bar{S} , and define $\delta(\bar{S}) = \{(u, v) \in A \mid u \notin S, v \in S\}$ (sometimes written as $\delta^{\text{in}}(S)$) to be the set of arcs with head in S . Finally, if $s \in S, t \notin S$, then $\delta(S)$ is an s, t -cut.

Theorem 1.2

There is an s, t -dipath if and only if every s, t cut is non-empty

Proof. (\Rightarrow) Suppose there exists an empty s, t -cut $\delta(S)$. This partitions the graph into two sets of nodes S and $N \setminus S$, with $s \in S$ and $t \in N \setminus S$ and no outgoing edges from S to $N \setminus S$. As such, an s, t -dipath cannot exist.

(\Leftarrow) Suppose every s, t -cut is non-empty and let S be the set of nodes $v \in A$ where a s, v -dipath exists. If $t \in S$, then we're done, so suppose $t \notin S$. Then, $\delta(S)$ is an s, t -cut. By assumption, $\delta(S)$ is non-empty and so there is an arc $(x, y) \in \delta(S)$ with $x \in S$ and $y \in N \setminus S$. Since $x \in S$, an s, x -dipath P exists and since $y \notin S$, there does not exist an s, y -dipath, but $P + (x, y)$ is one. This is a contradiction \square

Definition 1.5

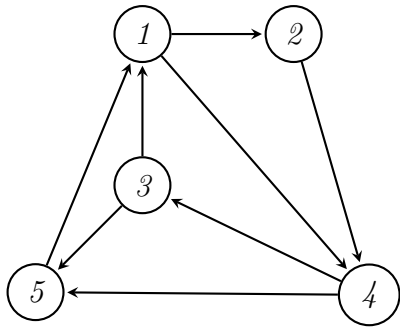
The **node-arc incidence matrix** M of a digraph $D = (N, A)$ is a matrix of $|N|$ rows and $|A|$ columns, such that:

- The rows correspond to the nodes of D ,
- The columns correspond to the arcs of D ,
- And the entry for node u and arc (i, j) , denoted $m_{u,ij}$ is given by:

$$m_{u,ij} = \begin{cases} 0 & \text{if } u \neq i \text{ and } u \neq j, \\ +1 & \text{if } u = j \text{ and,} \\ -1 & \text{if } u = i \end{cases}$$

Example 1.1

An example of a digraph $D = (N, A)$ (on the left) and its corresponding node-arc incidence matrix on the right.



	12	14	24	31	35	43	45	51
1	-1	-1	0	1	0	0	0	1
2	1	0	-1	0	0	0	0	0
3	0	0	0	-1	-1	1	0	0
4	0	1	1	0	0	-1	-1	0
5	0	0	0	0	1	0	1	-1