749 - Coloring

Mon 5th

Theorem (Bollobás, Thomason '96)
If G is $22k$-connected, then G is k-linked.

(State-of-the-art: Thomas-Wollan '07: $10k$-connected $\Rightarrow$ k-linked)

Theorem (Kostochka, Thomason '80's) indep. by
If G has no K_t-minor, then $ad(G) \leq O(t\sqrt{\log t})$
$\quad$ average degree

Idea: If G has large avg. degree, then G contains a minor of large avg. degree and a small # of vertices.

Lemma:
If $e(G) \geq k \cdot v(G)$, then G contains a minor H s.t. $v(H) \leq 2k+1$ and $\delta(H) \geq k+1$

Proof:
We may assume that $\delta(G) \geq k+1$; o/w we can delete a min. degree vertex and apply induction.
Similarly, we may assume that $e(G) = k \cdot v(G)$, o/w we can delete edges and apply induction.

Thus, $\exists$ a vertex $v \in V(G)$ with $d(v) \leq 2k$ (and $d(v) \geq k+1$).
Let $H = N[v]$. If $\exists u \in N(v)$ s.t. $|N(u) \cap N(v)| \leq k-1$, then induct on G/uv. So $\forall u \in N(v)$, $|N(u) \cap N(v)| \geq k$ and hence $\delta(H) \geq k+1$ and $v(H) \leq 2k+1$, as desired.

In fact, something stronger is true!

Lemma (Thomason, '84)

Let $0<\beta<1$ be the root of the equation $1=\beta(1+\ln(\frac{2}{\beta}))$ (Note: $\beta\approx.37$) and let $k\geq 3$ be an integer. If G is a graph s.t. $e(G)\geq k v(G)$, then G contains a minor H s.t. $v(H)\leq k+2$ and $2\delta(H)\geq v(H)+\lfloor\beta k\rfloor-1$.

Why is this nice? The guarantee $2\delta(H)\geq v(H)+\lfloor\beta k\rfloor-1$ means we get min. degree at least ~~...~~ ~60% of the original degree. For min. degrees higher than 50%, we get ~~a~~ a linear amount of common neighbors!

Proof (of Kostochka-Thomason)

Idea: We will prove the contrapositive, if we have high average degree, then we can use the previous lemma to find small minors, and then we finish with probabilistic method.

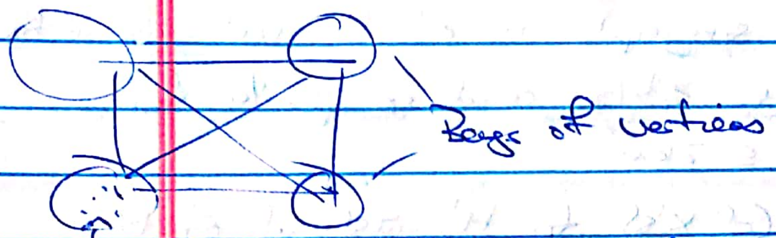
We'll prove that if c is large enough and $ad(G)\geq \underbrace{ct\sqrt{\log t}}_{:=2k}$, then G contains a K_t-minor.

Using the easier lemma, $\exists$ minor H of G s.t. $v(H)\leq 2k+1$ and $\delta(H)\geq k+1$.

(Note: We will try to bag vertices up to use for k-linked theorems, without worrying about the connectivity of the bags)

Proof (con't)

Example: We want smth like:



And we'll worry about individual connectivity in a bit

Let $p = \frac{e(H)}{v(H)}$, the density.
Let $X_1, \dots, X_t$ be vertex-disjoint subsets of H of size $\frac{c}{200}\sqrt{\log t}$ chosen u.a.r.

$$\Pr[e(X_i, X_j) = 0] = (1-p)^{|X_i||X_j|}$$
$$= (1-p)^{\frac{c^2}{40000}\log t}.$$

and we claim this is $< 1/t^2$, and so:

$$\mathbb{E}[(i,j) \in \tbinom{[t]}{2} : e(X_i, X_j) = 0] < t^2 \cdot \frac{1}{t^2} < 1.$$

So, there exist a choice of X's s.t. $e(X_i, X_j) > 0$ $\forall i \neq j \in [t]$.

Now, we're almost done, but each X_i may not be connected! (So, we may not be able to actually contract them)

i.e. (Thomason's lemma)

Finish #1: Use better lemma to guarantee that $|N(u) \cap N(v)| \geq .05k$ $\forall u,v \in V(H)$, then greedily pick common neighbors for each X_i outside of $\bigcup_{i=1}^{t} X_i$ to connect each X_i, avoiding previously picked common neighbors.

Proof (Con't)

Finish #2: Using Mader, $\exists$ subgraph H' of H with connectivity $\geq k/2$ and hence min. deg. $\geq k/2$ and so $p \geq 1/4$ for each H' and $v(H') \leq v(H) \leq 2k_{(1)}$.
So, apply random partition (of X's) to H' instead of H.
Now, by Bollobás, Thomason, H' is $k/4$-linked. Take a matching from $\bigcup_{i \neq j} X_i$ to a set of vertices in $G - \bigcup X_i$. (Such exists since $\delta(H') \geq k/2$ and $\bigcup X_i \leq k/4$). Now, we the matched vertices and stk. vertices of X as terminals to find vertex disjoint paths connecting each X_i

$\square$.

Proof (Thomason's Lemma)
By the other lemma, there exists a minor H' of G s.t. $v(H') \leq 2k$ and $\delta(H) \geq k$. (We can drop the '+1's)
We define a $\mathbb{R}$-valued function $f(G)$ on graphs as

$$f(G) = \frac{\beta k\, v(G)}{2}\left(\log\left(\frac{v(G)}{\beta k}\right) + 1\right).$$

and let:

$$D := \{ G : v(G) \geq \beta k \text{ and } e(G) \geq f(G) \}$$

Claim: $H' \in D$
Proof: $v(H') \geq k \geq \beta k$ and $e(H') \geq \frac{k\,v(H')}{2} \cdot 1$

$$\geq \frac{k\,v(H')}{2}\left(\beta\left(1 + \log\left(\frac{2}{\beta}\right)\right)\right)$$

Proof (cont'd).

$$\boxtimes = \frac{\beta k v(H)}{2}\left(\ln\left(\underbrace{\tfrac{2}{\beta}}_{\geq \frac{v(H')}{\beta k}}\right)+1\right) \quad \text{since } v(H') \leq 2k.$$

$$\geq f(H')$$

$\square$ (Claim)

Let H'' be a minor of H' minimal w.r.t. containment in D (and such exists since $H' \in D$, by claim).

Since the complete graph of order $\lceil \beta k \rceil$ is not in D (b/c $f(G_{\lceil\beta k\rceil}) = \frac{\beta k \lceil \beta k \rceil}{2}\left(\log\left(\frac{\lceil\beta k\rceil}{\beta k}\right)+1\right) \geq \lceil \beta k\rceil^2 \geq \binom{\lceil \beta k \rceil}{2} \geq e(G_{\lceil\beta k\rceil})$),

then it follows that $v(H'') \geq \lceil \beta k \rceil + 1$.

Moreover, $e(H'') \geq \lceil f(H'') \rceil$, o/w deleting an edge stays in D, contradicting the minimality of H''.

By minimality of H'', $\forall e = uv \in E(H'')$, we have that

$$e(H''/uv) \underset{\max}{\leq} f(H''/uv)$$

(since $v(H''/uv) \geq v(H'') - 1 \geq \beta k$)

Let u be a vertex in H'' of min. degree. Let $H = H''[N_{H''}(u)]$

$$v(H) \leq d_{H''}(u) \underset{\substack{\uparrow \\ \text{b/c } u \text{ has} \\ \text{min. degree in } H''}}{\leq} \frac{2e(H'')}{v(H'')} = \frac{2\lceil f(H'')\rceil}{v(H'')} = \frac{2\,\frac{\beta k v(H'')}{2}\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right)}{v(H'')}$$

$$= \beta k\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right)$$

Proof: (cont.)

And since $v(H'') \le v(H') \le 2k$:

$$\log\left(\frac{v(H'')}{\beta k}\right) \le \log\left(\frac{2}{\beta}\right) \text{ and hence}$$

$v(H) \le \beta k\left(\log\left(\frac{2}{\beta}\right)+c\right) = k$ (since $\beta\left(\log\left(\frac{2}{\beta}\right)+c\right) = 1$).

It remains to show that $2\delta(H) \ge v(H) + \lfloor \beta k \rfloor - 1$.

Note:

$$\delta(H) \ge \min_{v \in N_{H''}(u)} |N_{H''}(u) \cap N_{H''}(v)|.$$

(Now, we have to show that the # of common neighbors is actually high)

By previous obs., we have that

$$e(H'') - e(H''/uv) \ge f(H'') - f(H''/uv)$$

So,

$$|N_{H''}(u) \cap N_{H''}(v)| \ge f(H'') - f(H''/uv) \underbrace{-\,1}_{\text{For } uv \text{ itself.}}$$

Then

$$2\delta(H) - v(H)$$

$$\ge 2\left(f(H'') - f(H''/uv) - 1\right) - v(H)$$

$$\ge 2\left(\frac{\beta k\, v(H'')}{2}\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right) - \frac{\beta k\,(\cancel{v(H'')-1})\, v(H''/uv)}{2}\left(\log\left(\frac{v(H''/uv)}{\beta k}\right)+1\right) - 1\right) - v(H).$$

$$= \beta k\left(v(H'')\left(\log\left(\frac{v(H'')}{\beta k}\right)+1\right) - (v(H'')-1)\left(\log\left(\frac{v(H'')-1}{\beta k}\right)+1\right)\right) - 2 - v(H)$$

Proof (cont)

$$= \beta k \left(v(H'') \left(\log\left(\frac{v(H'')}{\beta k}\right) - \log\left(\frac{v(H'')-1}{\beta k}\right) + 1 + \log\left(\frac{v(H'')-1}{\beta k}\right)\right)\right.$$

$$\left. - 2 - \underbrace{v(H)}_{\substack{\text{Recall that} \\ v(H) \leq \beta k \left(\log\left(\frac{v(H'')}{\beta k}\right) + 1\right).}} \right)$$