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741 - ~~[illegible]~~ Random

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Feb 26th

Martingales and Azuma-Höffding Inequality:

Suppose X, Y are r.v.s in a probability space. let $Z = \mathbb{E}[Y|X]$ (the expectation of Y conditioned on X) is a random variable defined by:

$$\int_A Z dP = \int_A Y dP$$

for every X-measurable set A.

Alternative definition for "nice" X and Y:

$$\mathbb{E}(Y|X)(\omega) = \mathbb{E}(Y \mid X = X(\omega)) \text{ for every } \omega \in \Omega. \quad (*)$$

Whenever the above $\mathbb{E}(Y \mid X = X(\omega))$ is well defined.

Recall: $\mathbb{E}(X|A) = \frac{\mathbb{E}(X\, I\{A\})}{P(A)} = \frac{1}{P(A)} \int_A X(\omega)\, dP(\omega)$

where A is an event and $I\{A\}$ is the indicator variable for A.

When is $(*)$ well-defined?

For instance,

① If X is a discrete random variable, then

$$\mathbb{E}(Y|X = X(\omega)) = \frac{\mathbb{E}(Y\, I\{X = X(\omega)\})}{P(X = X(\omega))}$$

② If the conditional distribution of Y given X is a continuous distribution and the conditional density function is denoted $f_{Y|X}(y|x)$, then $\mathbb{E}(Y|X=x) = \int y f_{Y|X}(y|x) dy$.

Example:

Rolling 2 dices independently with X_1, X_2 denoting the points shown. Let $X = X_1 + X_2$. What is $\mathbb{E}(X_2 | X_1)$?

Conditioning on $X_1 = x$, we have

$$\mathbb{E}(X | X_1 = x) = \mathbb{E}(X_2 + x | X_1 = x)$$

$$= x + \mathbb{E}X_2$$

$$= x + 7/2$$

⟲ Since X_1, X_2 are independent

This holds for every x so $\mathbb{E}(X|X_1) = X_1 + 7/2$

Example:

Let X be a real number uniformly distributed in $[0,1]$ and Y be a r.v. uniformly chosen from $[X,1]$. What is $\mathbb{E}(Y|X)$?

Conditional Density function $f_{Y|X}(y|x) = \frac{1}{1-x}$

Hence,

$$\mathbb{E}(Y | X = x) = \int_x^1 \frac{y}{1-x} dy = \frac{1+x}{2}$$

This holds for all $x \in [0,1]$ so $\mathbb{E}(Y|X) = \frac{1+x}{2}$

Defn Assume $Y_0, Y_1, Y_2, \dots$ is a random process. We say $X_0, X_1, X_2, \dots$ is a martingale with respect to $(Y_i)_{i \geq 0}$ if for every $t \geq 0$:

$$\mathbb{E}(X_{t+1} \mid Y_1, \dots, Y_t) = X_t$$

Decreasing sequence

$(X_t)_{t \geq 0}$ is called a super-martingale if the above "=" is replaced with "$\leq$", and a sub-martingale if instead we have "$\geq$"

Increasing sequence

Example (Doob's Martingale)

Consider a set of random variables $Y_1, \dots, Y_n$ each taking values in a domain A. Let $f: A^n \to \mathbb{R}$. Define $(X_t)_{0 \leq t \leq n}$ as follows:

$$X_0 = \mathbb{E}(f(Y_1, \dots, Y_n))$$

$$X_t = \mathbb{E}(f(Y_1, \dots, Y_n) \mid Y_1, \dots, Y_t) \quad \text{for all } 1 \leq t \leq n$$

(Aside: An example of the Y's is $Y_1, \dots, Y_{\binom{n}{2}}$ the edges of a random graph, (this is a Bernoulli r.v.) and we can have f be any function, such as the chromatic number. Then X_0 is the expected value of the ~~fixed~~ chromatic number, and X_t is the chromatic # given the values first t edges + ~~[illegible]~~. Finally, X_n is the actual chromatic number).

Let's verify that this is martingale

Aside: Tower Property

Suppose $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ (sigma al

Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(X|\mathcal{G}_1), \text{ and}$$

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1)$$

i.e. "Smaller" sigma field wins.

Example (con't)

For every $0 \leq t \leq n-1$

$$\mathbb{E}(X_{t+1} | Y_1, \ldots, Y_t)$$
$$= \mathbb{E}(\mathbb{E}(f | Y_1, \ldots, Y_{t+1}) | Y_1, \ldots, Y_t) \quad \text{By Tower Property}$$
$$= \mathbb{E}(f | Y_1, \ldots, Y_t)$$
$$= X_t.$$

Example:

$\Omega = \{\text{Canadian Citizens}\}$

P: Uniform Distribution

$\mathcal{F}: 2^{\Omega}$.

$f: \Omega \to \mathbb{R}$ (height of citizen)

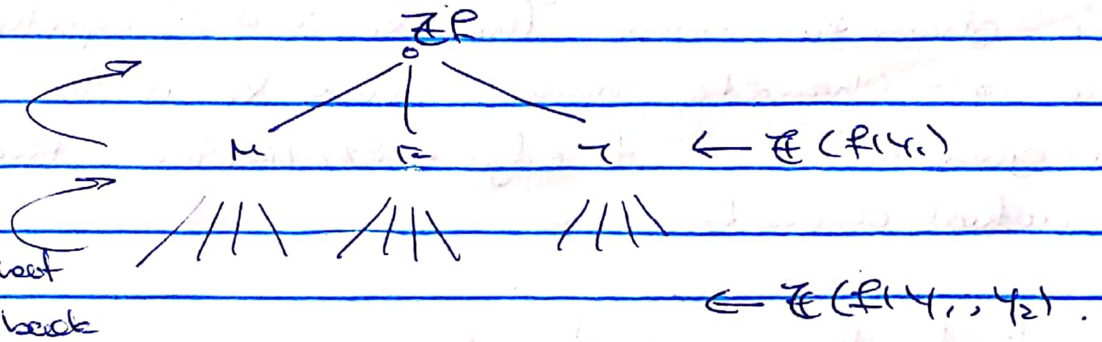
$Y_1: \Omega \to \{M, F, T\}$ (Gender)

$Y_2: \Omega \to \mathbb{Z}$ (Age)

$X_0 = \mathbb{E} f$ → Average height of Canadian citizens

$X_1 = \mathbb{E}(f | Y_1)$ → Average height ~~of~~ segregated by Gender

$X_2 = \mathbb{E}(f | Y_1, Y_2)$ → — " ——— gender AND age



Note that going back upwards gives us exactly Doob's martingale

Example (Edge Exposure Martingale)

Consider ~~G~~ $G(n,p)$ and an ~~ord~~ arbitrary order of the $\binom{n}{2}$ edges of $K_{[n]}$. Let Y_i be the indicator variable that the i-th edge in $K_{[n]}$ is in $G(n,p)$. Let f be an graph function (E.g. The chromatic #).

Define a martingale $X_0, \dots, X_{\binom{n}{2}}$ by

$$X_0 = \mathbb{E}(f(G(n,p))$$

$$X_t = \mathbb{E}(f(G(n,p)) \mid Y_1, \dots, Y_t) \text{ for every } 1 \le t \le \binom{n}{2}$$

Note that this is a special case of the Doob's martingale

We can expose the graph quicker though:

Example (Vertex Exposure Martingale)

Consider $G(n,p)$ and any graph function f. Define $(X_t)_{0 \le t \le \binom{n}{2}}$ by:

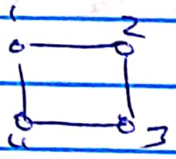
$$X_0 = \mathbb{E}(f(G(n,p))$$

$$X_t(H) = \mathbb{E}(f(G(n,p)) \mid G_{[t]} = H_{[t]}) \text{ for all } 1 \le t \le n.$$

where $G_{[t]}$ is the subgraph induced by the vertex set $[t] = \{1, \dots, t\}$

Example:

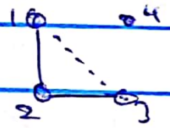
$G(4, 1/4)$. Consider the vertex exposure Martingale $(X_i)_{i=0}^{4}$ with respect to $f(g)$ as the chromatic number of g (g is a graph).

Let $g =$ 

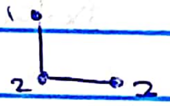
. What is $X_3(g)$ and $X_4(g)$?

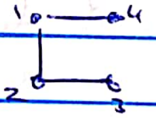
$X_4(g) = \chi(g) = 2 \rightarrow$ This is just the chromatic number since we've exposed all the information ~~&~~ on g.

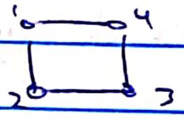
$$X_3(g) = \mathbb{E}\left(f(g') \mid \{1,2\}, \{2,3\} \in g', \{1,3\} \notin g'\right)$$



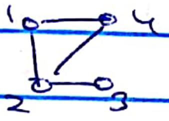
$$= 2\left(\left(\tfrac{3}{4}\right)^3 + 3\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)\right)$$

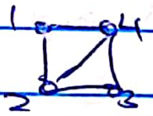






$$+ 3\left(2\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4}\right)^3\right) = \frac{135}{64}$$





Example:

Let $Y_1, \dots, Y_n$ be ind. r.v.s Let $X_t = \sum_{j=1}^{t} (Y_j - \mathbb{E}Y_j)$

Claim: $(X_t)_{1 \le t \le n}$ is a martingale w.r.t. $(Y_t)_{1 \le t \le n}$.

Proof:

~~$\mathbb{E}(X$~~ We want to verify $\mathbb{E}(X_{t+1} \mid Y_1 \dots Y_t) = $ ~~$\mathbb{E}$~~ X_t.

$$\mathbb{E}(X_{t+1} \mid Y_1 \dots Y_t) = \mathbb{E}\left(\sum_{j=1}^{t+1} Y_j - \mathbb{E}Y_j \mid Y_1, \dots Y_t\right)$$

$$= \sum_{j=1}^{t} (Y_j - \mathbb{E}Y_j) + \underbrace{\mathbb{E}(Y_{t+1} - \mathbb{E}Y_{t+1}}_{} \mid Y_1, \dots Y_t)$$

$$= X_t$$

Independent from $Y_1, \dots Y_t$, so ~~$\mathbb{E}$~~ this cancels out.

Theorem (Azuma's Inequality)

Let $(X_i)_{i \ge 0}$ be a martingale w.r.t $(Y_i)_{i \ge 0}$. Assume for all $k \ge 1$

$$|X_k - X_{k-1}| \le C_k$$

Then,

$$\Pr(|X_n - X_0| \ge t) \le 2 \exp\left(-\frac{t^2}{2\sum_{k=1}^{n} C_k^2}\right).$$