~~F24 - Hopf Algebra~~
749 - Graph Colouring

2th
Jan 7th

Def'n 1: A k-colouring of a graph G is a partition of $V(G)$ into at most k independent sets.

Def'n 2: A k-colouring of a graph G is a map $f: V(G) \to [k]$ such that $\forall e = vw \in E(G)$, $f(v) \neq f(w)$

Defn 3: A k-colouring of a graph G is a graph homomorphism to K_k.

Takeaway: There are multiple ways to view what a colouring is.

Weakenings, Generalizations and Variants of Colouring:
Variants: ("Changing what you color")
- Edge Colouring: A k-edge-colouring of a graph G is
  Def 1: A partition of $V(G)$ into at most k matchings
  2: A k-colouring of $L(G)$ (the line graph of G)

~~"Changing what you color"~~:

- Total Colouring: A k-total-colouring of a graph G is a map $f: V(G) \cup E(G) \to [k]$ s.t.
  - $f(v) \neq f(w)$ $\forall vw \in E(G)$
  - $f(v) \neq f(e)$ $\forall v \sim e$, $v \in V(G)$, $e \in E(G)$
  - $f(e) \neq f(e')$ $\forall e \sim e' \in E(G)$

Generalizations: ("Changing what you are allowed to color")
- List Colouring: (idea: Lists of available colours to vertices)
  Def: A ~~k-list-colouring~~ k-list-assignment of a graph G is an assignment of lists $(L(v): v \in V(G))$ such that

$|L(v)| \geq k \quad \forall v \in V(G)$

An <u>L-colouring</u> of a graph G is a colouring ϕ of G such that $\phi(v) \in L(v) \quad \forall v \in V(G)$

~~"Changing what you are allowed to colour"~~

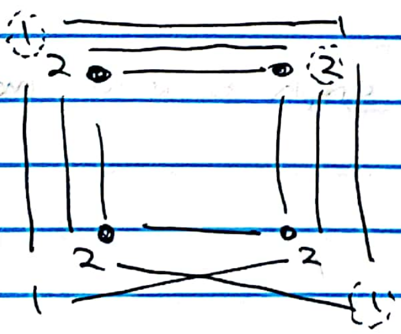
- Correspondence Colouring:

<u>Def:</u> A k-<u>correspondence</u>-assignment ~~colouring~~ is a pair $((L(v) : v \in V(G)), (M_{uv} : uv \in E(G)))$, where M_{uv} is a matching from $L(u)$ to $L(v)$

An <u>(L,M)-colouring</u> of G is a colouring ϕ of G such that:

- $\phi(v) \in L(v) \quad \forall v \in V(G)$
- $\phi(u)$ is not matched to $\phi(v)$ in M_{uv} $\forall uv \in E(G)$

<u>Ex:</u> $G = C_4$



$L(v) = \{1, 2\} \quad \forall v$

<u>Q:</u> Is there an (L,M)-colouring of G for this L, M?

No!

And we run into trouble here

Remarks:

- We may as well assume $L(v) = \{1, \dots, |L(v)|\}$ $\forall v \in V(G)$
- Correspondence has a "local notion of colour", while list colouring / ordinary colouring have a "global notion of colour"

Weakenings: ("Change what a colour is")

- No Restrictions: Improper colourings, i.e. mappings of graphs
- d-defective colouring: Each colour has maximum degree d
- c-clustered Colouring: Every monochromatic component has size (# of vertices) $\leq c$
- No mono. path of length $> l$.
- Every colour is d-degenerate
- Every colour is triangle-free (or more generally, bounded clique #)

Fractional Colouring: ("Changing how you colour")

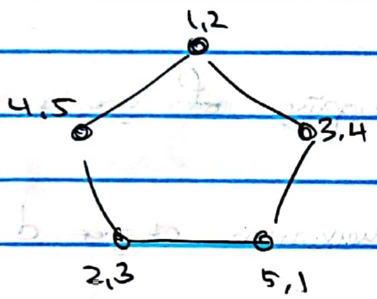
Def: An (a,b)-colouring of a graph G is a map ϕ such that $\phi(v)$ is a subset of $[a]$ of size b and $\forall uv \in E(G)$, $\phi(u) \cap \phi(v) = \phi$

Remark: If G has a k-colouring, then G has a (kb, b)-colouring $\forall b$

(i.e. Graph homomorphism to Kneser graph on a, b)

The fractional chromatic #: $\chi_f(G) = \inf\{\frac{a}{b} : G$ has an (a,b)-colouring$\}$.

Ex. $\chi_f(C_5) = 5/2$



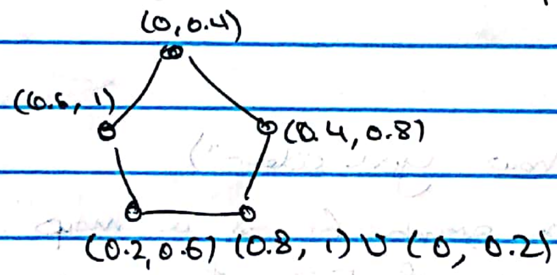
Remark:

- There is also an LP-formulation and its dual. (its dual asks about weighted independent set)
- An assignment of measurable subsets $\Phi(v)$ of $[0,1]$ $\forall v \in V(G)$ such that:

  $\mu(\Phi(u) \cap \Phi(v)) = \phi$

  $\chi_f(G) = 1 / \sup\{\varepsilon : G$ has a colouring as above, with $\mu(\Phi(v)) = \varepsilon \; \forall v\}$

Ex:



Let's call the above an f-colouring if $\mu(\phi(v)) = f(v)$ $\forall v$.

Proposition: $\chi_f(G) = k$ iff G has a $(1/k)$-colouring

Proposition: G has an f-colouring iff the vector $(f(v) : v \in V(G))$ is in the independent set polytope (i.e. $\exists$ probability distribution on independent sets such that $\Pr[v \in I] = f(v)$)

Combinations:
- Defective Clustering ×
- List Defective ✓
- Frac. Defective ✓
- List Edge ✓
- Frac. Edge ✓
- Edge Total ×
- List Correspondence × (Correspondence is already list).
- Fractional List ✓
  - Find b-colouring from a-list-assignment. (multicolourings)

$$\chi_{f,list}(G) = \inf\left\{\frac{a}{b} : \forall \text{ a-list-assign } G \text{ has a b-colouring}\right\}$$

Theorem: $\chi_f(G) = \chi_{f,list}(G)$
- Fractional Correspondence.