Thm E(d): Let $c>1$. a.a.s $\mathcal{G}(n, c/n)$ has a unique giant component and all other components are of order $O(\log n)$

Still need a bit more probability theory before we can prove this theorem!

Stochastic Dominance!
Given 2 real r.v.s X and Y. We say that X stochastically dominates Y if:

$$P(X \geq x) \geq P(Y \geq x) \quad \forall x$$

We can couple X and Y such that $X \geq Y$ always. Then X stochastically dominates Y always. (Proof below)

Compare: If we have 2 ~~ra~~ random graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{G}_1$ and $\mathcal{G}_2$ can be coupled so that $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then for any monotone increasing property Q:

$$Pr(\mathcal{G}_2 \in Q) \geq Pr(\mathcal{G}_1 \in Q)$$

Proof!
$\forall x$ let $\Omega^X(x) = \{\omega \in \Omega \text{ s.t. } X(\omega) \geq x\}$

$\Omega^Y(x) = \{\omega \in \Omega \text{ s.t. } Y(\omega) \geq x\}$

Since in Ω we have $X(\omega) \geq Y(\omega) \; \forall \omega \in \Omega$

$\Rightarrow \Omega^X(x) \supseteq \Omega^Y(x) \; \forall x$

So:

$$P(X \geq x) = \int_{\omega \in \Omega^X(x)} dP(\omega) \geq \int_{\omega \in \Omega^Y(x)} dP(\omega) = P(Y \geq x)$$

$\square$

* The ρ's are ρ's (rho's)

Proof (of E(d))

Let $k_0 = c\log n$ and $k_1 = n^{2/3}$ where $c>0$ is a sufficiently large constant. We will prove a.a.s.:

(i) There are no components of $G(n, c/n)$ with order between k_0 and k_1

(ii) There is only ≤ 1 component of order greater than k_1

(iii) The total # of vertices in small components (order $\leq k_0$) is asymptotic to ρn, where $\rho e^{c} = e^{\rho c}$, $0<\rho<1$

(Compare ρ with $t(c)$ in the theorem of tree component $\Rightarrow$ "$\rho = t(c)$" (Check yourself!)).

Intuition about ρ:

Take a uniformly random vertex u and explore its neighbours, its 2nd neighbours, etc. This mimics the branching process with $Z \sim Po(c)$. Recall that ρ_Z is the probability of extinction. Then, ρ_Z is the root of $\rho = G(\rho)$, $0<\rho<1$. (where $G(x)$ is the PGF for this distribution), and $G(x) = e^{-c+cx}$ (this is the PGF for Poisson)

(Aside:

$$G(x) = \sum_{j=0}^{\infty} Pr(Z=j)x^j = \sum_{j=0}^{\infty} e^{-c}\frac{c^j}{j!}x^j$$

$$= e^{-c}\sum_{j=0}^{\infty}\frac{c^j}{j!}x^j$$

$$= e^{-c+cx}$$

).

And ρ_Z is the fixed point for this PGF.

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Proof: (con't)

Parts (i)-(iii) says that $G(n,p)$ contains a unique component of linear order and all other components are small. Intuitively, the prob. that v lies in a small component coincides with P_2.

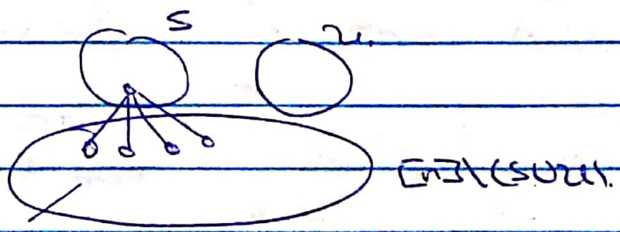
Consider the following graph exploration process. It starts with $t=0$ and $S=\{v\}$, $\mathcal{U}=\emptyset$. In every step $t\geq 1$, the process takes an arbitrary vertex $u\in S$, adds all neighbors of u in $G\setminus(S\cup\mathcal{U})$ into S, and then moves u from S to $\mathcal{U}$. The process terminates when $S=\emptyset$. and $\mathcal{U}$ contains all vertices in the same component as v and the size of $\mathcal{U}$ equals the # of steps the graph exploration process takes.

Let X_t be the size of S after step t. $X_0=1$

By the construction of the process, $|\mathcal{U}|=t$ and $|\mathcal{U}\cup S|=X_t+t$ after t steps of the process.

Let $Z_t = X_t - X_{t-1} + 1$ (# of new vertices added)

Then, $Z_{t+1} \sim Bin(n-(X_t+t), p)$



Z_t counts all the new vertices, this is a binomial distribution since an edge appears in $[n]\setminus(S\cup\mathcal{U})$ with probability p.

Proof:

To prove part (i), it is sufficient to prove that with probability $o(1/n)$ X_t reaches 0 for some $k_0 < t < k_1$, as by union bound, with probability $o(1)$, there is a component of order between k_0 and k_1.

Let $k_0 < \tau < k_1$ and E_τ be the event that $(X_t)_{t \geq 0}$ becomes 0 after step τ.

- Given E_τ, define $W_\tau := \sum_{t=1}^{\tau} Z_t = X_\tau - X_0 + \tau = \tau - 1$
- $Z_t \geq 0 \Rightarrow X_t \geq X_{t-1} - 1$ (In the worst case, we add no new ~~node~~ vertex and only remove ~~one~~ ourself).

  If for some $t' < \tau$ we have $X_{t'} > k_1$, then consequently $X_t > 0$ for all $t < k_1$, contradicting that $X_\tau = 0$.

So $E_\tau \Rightarrow X_{t'} \leq k_1$ for all $t' < \tau$.

In summary:

$$E_\tau \Rightarrow \begin{cases} W_\tau = \tau - 1 & (1) \\ X_{t'} \leq k_1 \text{ for all } t' < \tau & (2) \end{cases}$$

(2) implies that $W_\tau := \sum_{t=1}^{\tau} Z_t$ stochastically dominates $\hat{W}_\tau := \sum_{t=1}^{\tau} \hat{Z}_t$ where $\hat{Z}_t$ are i.i.d copies of $Z \sim Bin(n - 2k_1, p)$

since both $X_t \leq k_1$ and $t \leq k_1$

This r.v. will be easier to analyze since Z ~~changes~~ changes with every step'.

Exercise 2.2

Aside! $Bin(n_1,p)$ dominates $Bin(n_2,p)$ if $n_1 \geq n_2$)

Proof (cont)

since each Z_t binomial.

$\hat{\omega}_\tau \sim Bin(\tau(n-2k_1), p)$

$\mathbb{E}\,\hat{\omega}_\tau = \tau(n-2k_1)\cdot\frac{c}{n} = c\tau$ (where $\tau > 1$)

Let $\sigma = c-1 > 0$ and let $\varepsilon = \sigma/2c$. Then $(1-\varepsilon)c = 1+\frac{\sigma}{2}$

By Chernoff Bound:

$$\Pr(E_\tau) \leq \Pr(1 \text{ and } 2)$$

$$\leq \Pr(\hat{\omega}_\tau \leq 1)$$

since $\tau \leq (1-\varepsilon)\tau c\left(1-\frac{2k_1}{n}\right) = \left(1+\frac{\sigma}{2}\right)\tau\left(1-\frac{2k_1}{n}\right)$

$$= \Pr\left(\hat{\omega}_\tau < (1-\varepsilon)\tau c\left(1-\frac{2k_1}{n}\right)\right)$$

$$\leq \exp(-\kappa\varepsilon^2\tau) \quad \text{(for some constant } \kappa > 0)$$

$$= o\left(\tfrac{1}{n}\right)$$ since $\tau \geq k_0 = c\log n$ and c is sufficiently large

and part (i) follows by taking union bound over $v \in [n]$.

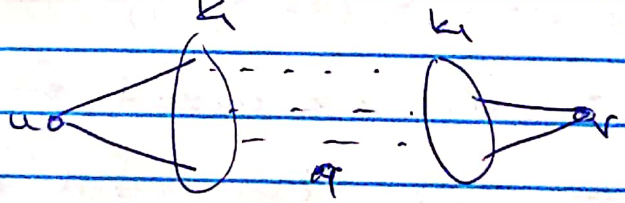
Proof (con't)

Part (ii). For any 2 vertices u and v, we prove that with prob $o(n^{-2})$, they lie in 2 different components, each of order greater than k_1.

Let $X_t(u)$ and $X_t(v)$ denote the size of $S(u)$ and $S(v)$ with respect to the graph exploration processes starting from u and v respectively.

We may assume that $X_t(u)$ and $X_t(v)$ are positive for all $t < k_1$ (Since o/w one of them lie in order k_1, i.e. the component has order $< k_1$)

Take $\varepsilon = \delta/2e$ and let $W_{k_1}(i) = \sum_{t=1}^{k_1} Z_t(i) = X_{k_1}(i) - X_0(i) + k_1$ for $i \in \{u, v\}$



We want to show w.h.p that if after k_1 steps, if we still have a large # of vertices to explore there must be an edge b/w the component containing u and the component containing v.

Proof (cont)

By Chernoff Bound, (and similar argument as (i)).

$$\Pr(W_{k_1}(i) < (1-\varepsilon)k_1 c(1-\frac{2k_1}{n})) = o(n^{-2})$$

Hence with prob $\geq 1-o(n^{-2})$, $X_{k_1}(u), X_{k_1}(v) \leq (\frac{c}{4})k_1$

If $S(u) \cap S(v) \neq \emptyset$ after k_1 steps, then we're done. Else we may assume $|S(u)|, |S(v)| \geq (\frac{c}{4})k_1$ and

$\Pr$(no edge b/w $S(u)$ and $S(v)$)

$$\leq (1-p)^{(\frac{c}{4}k_1)^2}$$

$$\leq o(n^{-2})$$

$\Rightarrow$ Part (ii) (By union Bound)