CO342 - Introduction to Graph Theory

University of Waterloo
Nicholas Pun
Fall 2017 (Revised 2021)

Contents

Preface

Chapter 1

Connectivity

This chapter most closely follows Chapter 4 (*Connectivity and Paths*) of [1]. Coverage of similar theorems can also be found in Section 1.4 and Chapter 3 (*Connectivity*) of [2]

1.1 Introduction

Note. In this course, we will assume a graph $G = (V, E)$ is simple, unless otherwise stated.

We are interested answering two questions in this chapter:

- How many vertices/edges need to be removed in order to disconnect a graph?
- Assuming we know that number, can we characterize that class of graphs?

The first question will take us to Section 1.2 and the second will span the latter half of this chapter.

We start of with two basic definitions, the first of which the reader should already be familiar with:

> **Definition 1.1 (Induced Subgraph):** Let S be a set of vertices in G. Then, the **subgraph induced** by S, denoted $G[S]$ consists of S as the vertex set and all edges in G joining 2 vertices in S.

Example 1.1. Let $S = \{1, 2, 3, 4\}$ and $T = \{1, 2, 4, 5\}$, then $G[S]$ and $G[T]$ are as pictured in Figure 1.1

Example 1.2 (Induced Cycle)**.** Let $U = \{1, 2, 3\}$, then $G[U]$ is called an <u>induced cycle</u>.

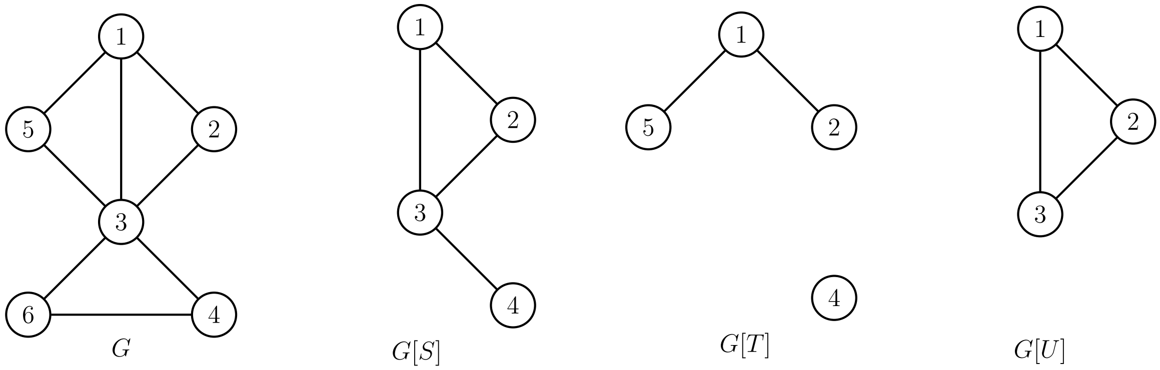


Figure 1.1

> **Definition 1.2 (Min/Max Degree):** We use $\delta(G)$ to denote the minimum degree among all vertices of G and $\Delta(G)$ for the maximum degree

Example 1.3. With G in Figure 1.1, we have $\delta(G) = 2$ and $\Delta(G) = 5$.

1.1.1 Edge Connectivity

We start our investigation into connectivity with removing edges from graphs. Recall the following definition from MATH 239:

> **Definition 1.3 (Bridge/Cut-edge):** A **bridge** or **cut-edge** in G is an edge whose removal increases the number of components in G.

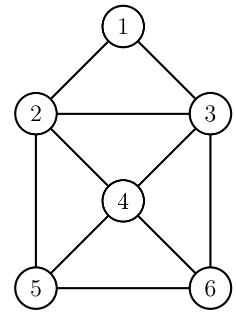
Let's extend this definition to more general sets:

> **Definition 1.4 (Disconnecting Set):** A set of edges F is called a **disconnecting set** if removing all edges in F from G results in a disconnected graph. We use $G - F$ to denote this removal

A couple remarks on this definition:

- If a graph is already disconnected, the empty set is a disconnecting set
- In general, we're interested in minimizing the size of a disconnecting set. (Since removing every edge trivially creates a disconnected graph)

Example 1.4.



Concerning the graph on the right, some disconnecting sets are $\{12, 13\}$, $\{12, 23, 34, 36\}$ and the set of all edges. Note that since this graph does not have a bridge, we must remove at least 2 edges in order to disconnect the graph.

When characterizing graphs by the size of their disconnecting sets, we'll use the following two terms:

Definition 1.5 (k-edge-connected): A graph is k**-edge-connected** if every disconnecting set has size $\geq k$.

Definition 1.6 (Edge Connectivity): The **edge connectivity** of G, written $\kappa'(G)$ is the largest k for which G is k-edge-connected, or equivalently, the minimum size of a disconnecting set.

Example 1.5.

- A connected graph is 1-edge-connected and if there is a bridge, then the edge connectivity is 1. (The bridge is the minimum disconnecting set)
- A connected graph with **no** bridges is 2-edge-connected
- A graph with a single vertex has edge connectivity 0 (The only disconnecting set is the empty set)

Example 1.6. The graph from Example 1.4 is 2-edge-connected (since it has no bridge), but **not** 3-edge-connected (since we've shown a disconnecting set of size 2). So $\kappa'(G) = 2$

Example 1.7. If a graph is k-edge-connected, then it is also $(k-1)$-edge-connected

We can always disconnect any vertex of a graph by removing all of its incident edges. Since we want to minimize the size of the disconnecting set, we can choose a vertex of minimum degree. This gives the following bound:

Proposition 1.1: $\kappa'(G) \leq \delta(G)$

Proof. Assume G is non-trivial and let v be a vertex of degree $\delta(G)$. Let F be the set of all edges incident with v, then $G - F$ has no path from v to any other vertex. So, F is a disconnecting set of size $\delta(G)$, so G is not $(\delta(G)+1)$-edge-connected and so $\kappa'(G) \leq \delta(G)$ $\square$

1.1.2 Vertex Connectivity

1.2 Menger's Theorem

1.3 2-connected Graphs

1.4 3-connected Graphs

Chapter 2

Vector Space for Graphs

Chapter 3

Planarity

Chapter 4

Matchings

Bibliography

[1] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2nd edition, 2001.

[2] Reinhard Diestel. *Graph theory*. Springer-Verlag Berlin Heidelberg, 5th edition, 2017.