July 10th, 2018

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Recall:

F||C_{max}: let $T_i := \sum_{j=1}^{n} p_{ij}$ and $T_{max} := \max_{i=1,\dots,m} T_i$

Suppose we have a permutation schedule S corresponding to permutation $\sigma(1), \sigma(2), \dots, \sigma(n)$ with no unforced idle time. So

(b) $\longrightarrow$ $$C^S_{i,\sigma(j)} = \max\left(C^S_{i,\sigma(j-1)}, C^S_{i-1,\sigma(j)}\right) + p_{i,\sigma(j)}$$

$$C^S_{max} = T_m + (\text{idle time on m/c } m \text{ in } [0, C^S_{max}])$$

Let $I^S_{i,j}$ = total idle on m/c i up to completion time of $\sigma(j)$ on m/c i.

So,

(a) $\longrightarrow$ $$I^S_{i,j} = C^S_{i,\sigma(j)} - \sum_{k=1}^{j} p_{i,\sigma(k)}$$

Note that: $I^S_{1,j} = 0 \quad \forall j=1,\dots,n.$

So, $C^S_{max} = T_m + I^S_{m,n}$

We want: Relate $I^S_{i,j}$ to $I^S_{i-1,j}$ and $I^S_{i,j-1}$

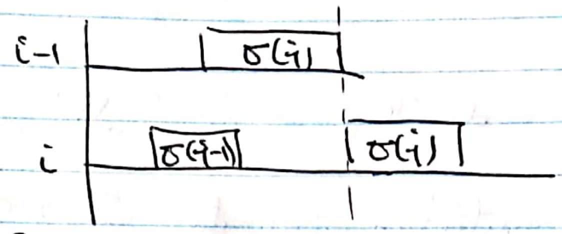
2 cases:

(1) $C^S_{i,\sigma(j)} = C^S_{i,\sigma(j-1)} + p_{i,\sigma(j)}$:

Then, $$I^S_{i,j} = C^S_{i,\sigma(j)} - \sum_{k=1}^{j} p_{i,\sigma(k)}$$
$$= \left(C^S_{i,\sigma(j-1)} - \sum_{k=1}^{j-1} p_{i,\sigma(k)}\right) + p_{i,\sigma(j)} - p_{i,\sigma(j)}$$
$$= I^S_{i,j-1}$$

Cont

(2) $C^s_{i,\sigma(j)} = C^s_{i-1,\sigma(j)} + P_{i,\sigma(j)}$



So,

$$T^s_{i,j} = C^s_{i,\sigma(j)} - \sum_{k=1}^{j} P_{i,\sigma(k)}$$

$$= C^s_{i-1,\sigma(j)} + P_{i,\sigma(j)} - \sum_{k=1}^{j} P_{i,\sigma(k)}$$

$$= \left(\sum_{k=1}^{j} P_{i-1,\sigma(k)} + T^s_{i-1,j}\right) + P_{i,\sigma(j)} - \sum_{k=1}^{j} P_{i,\sigma(k)}$$

$$= T^s_{i-1,j} + P_{i,\sigma(j)} + \sum_{k=1}^{j} \left(P_{i-1,\sigma(k)} - P_{i,\sigma(k)}\right)$$

Suppose we find a permutation σ s.t.

$$(*) \longrightarrow \sum_{k=1}^{j} \left(P_{i-1,\sigma(k)} - P_{i,\sigma(k)}\right) \leq (m-1)P_{max} \quad \forall j=1,\dots,n \quad \forall i=2,\dots,m.$$

Then,

- In case (1), $T^s_{i,j} = T^s_{i,j-1}$ is unchanged
- In case (2),

$$T^s_{i,j} = T^s_{i-1,j} + P_{i,\sigma(j)} + \sum_{k=1}^{j} \left(P_{i-1,\sigma(k)} - P_{i,\sigma(k)}\right)$$

$$\leq T^s_{i-1,j} + P_{max} + (m-1)P_{max}$$

$$= T^s_{i-1,j} + mP_{max}$$

So, $T^s_{i,j} \leq \max\{(T^s_{i,j-1}, T^s_{i-1,j} + mP_{max}) \rightarrow (1)$

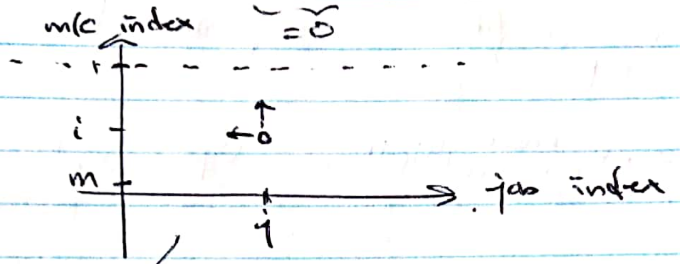
$\forall i = 2,\dots,m$, $\forall j=1,\dots,n \longrightarrow (2)$

and $T^s_{1,j} = 0 \quad \forall j=1,\dots,n$.

From (1) and (2), we infer

$$T^S_{i,j} \le T^S_{1,j} + (i-1)\, m\, p_{max}.$$ ← Can prove by induction on i

$\underbrace{T^S_{1,j}}_{=\tilde{o}}$



From (1), we know that $T^S_{i,j} \le \max(T^S_{i,j-1},\ T^S_{i-1,j} + m p_{max})$

horizontal movement (no costs)

vertical movement towards dotted line (cost of $m p_{max}$)

At most how many vertical steps can we make? ←

So, $T^S_{m,n} \le (m-1) m\, p_{max}$.

And,
$$C^S_{max} \le T_m + (m-1) m\, p_{max}.$$

$\Rightarrow$ Get an additive approximation of $(m-1) m\, p_{max}$.

Theorem 1: If we find a permutation σ satisfying $(*)$, then the corresponding permutation schedule S has makespan $C^S_{max} \le OPT + (m-1) m\, p_{max}$.

Q: How do we find a permutation σ satisfying $(*)$?

Notation:

Given a vector $v \in \mathbb{R}^d$, define

$$\|v\|_\infty = \max_{i=1,\dots,d} |v_i|$$

~~[illegible]~~

l_∞-norm of v

Lemma (Steinitz / Sevast'janov)
Given vectors $v_1, \ldots, v_n \in \mathbb{R}$ s.t. $\sum_{j=1}^{n} v_j = \vec{0}$,

we can obtain, in polytime, a permutation σ of $1, \ldots, n$ s.t.

$$\left\| \sum_{k=1}^{j} v_{\sigma(k)} \right\|_\infty \le d \cdot \max_{k=1,\ldots,n} \|v_k\|_\infty \quad \forall j = 1, \ldots, n$$

[Proof]
In a bit, let's see how we can use this first!

Recall:
We wanted to use:

$$(*) \quad \sum_{k=1}^{j} (p_{i-1,\sigma(k)} - p_{i,\sigma(k)}) \le (m-1) p_{max} \quad \begin{array}{l} \forall i = 2, \ldots, m \\ \forall j = 1, \ldots, n. \end{array}$$

To use Steinitz / ~~S~~ Sevast'janov (SS) lemma here:
We will start off by ensuring that

$\Pi_i = \Pi_{max}$ ~~for all machines~~ $\forall i = 1, \ldots, m$.

We can ensure this by repeatedly increasing p_{ij} arbitrarily up to p_{max} until $\Pi_i = \Pi_{max}$ $\forall i = 1, \ldots, m$
Ex:

Jobs

	1	2	3	4	5	Π_i
1	2	1 → 2	2	4 → 5	1	11 → 12 → 13
2	5	2	1	3	2	13 → Π_{max}, $p_{max} = 5$
3	4	1 → 4	1	1	3	10 → 13

A permutation schedule for modified p_{ij}'s give a permutation schedule for original p_{ij}'s with no greater makespan.

Define ~~for~~ ~~any~~ $\forall j = 1, \dots, n$.

$$v_j = (P_{1,j} - P_{2,j},\ P_{2,j} - P_{3,j},\ \dots,\ P_{m-1,j} - P_{m,j})$$

Now, verify that $\sum_{j=1}^{n} v_j = \vec{0}$:

The ith component of the sum is:

$$\left(\sum_{j=1}^{n} v_j\right)_i = \sum_{j=1}^{n} (P_{i,j} - P_{i+1,j})$$

$$= \pi_i - \pi_{i+1} = 0.$$

$\rightarrow$ Since $\pi_i = \pi_{max}$ $\forall i$.

So, $\sum v_j = \vec{0}$.

So, $v_j \in \mathbb{R}^{m-1}$, and $\|v_j\|_\infty \le P_{max}$. $\forall j = 1, \dots, n$

Then, SS lemma gives a permutation σ s.t.

$$\left\|\sum_{k=1}^{j} v_{\sigma(k)}\right\|_\infty \le (m-1)P_{max}.$$

$$\Rightarrow \left(\sum_{k=1}^{j} v_{\sigma(k)}\right)_{i-1} = \sum_{k=1}^{j} (P_{i-1,\sigma(k)} - P_{i,\sigma(k)}) \le (m-1)P_{max}.$$

$\forall i = 2, \dots, m$, $\forall j = 1, \dots, n$, which is exactly what we wanted.