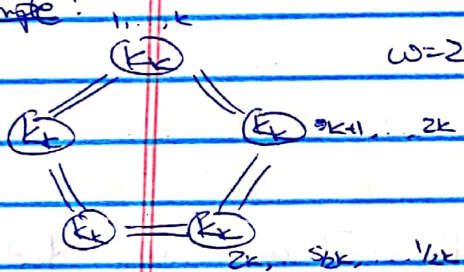
Reed's Conjecture:

Recall: Conjecture (Reed, 1998): If G is a graph, then
$$\chi(G) \geq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil.$$

Remark: If true, this is tight for some ω.

↓ Example of this:

Example:



$\omega = 2k$, $\Delta = 3k-1$ $\Rightarrow$ $\chi = \lceil \frac{5}{2}k \rceil$.

Why is this true?!
- $\exists$ a $\lceil 5/2 k \rceil$-colouring $\Rightarrow$ See labels
- We can also see this since $\alpha \leq 2$, and so the 5k colours we need can be cut in half

Reed's Conjecture has been proved for various special cases of graphs, e.g.: line graphs, claw-free graphs, etc.

Theorem (Reed, 1998)
If $\omega(G) \geq (1 - \frac{1}{10^7})(\Delta(G)+1)$ then the conjecture holds
→ So if clique # is large, then the conjecture is true.

Corollary:
$\exists \varepsilon > 0$ s.t. $\chi(G) \leq \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil$.
→ So if we're not satisfied w/ the half we could try some convex combination.

→

What is ε?

- $\varepsilon = \frac{1}{2 \cdot 10^8}$ (Reed '98).
- $\varepsilon = \frac{1}{320e^6}$ (King + Reed '12)
- $\varepsilon = \frac{1}{26}$ (Bonamy, Perrett, P. '16+)
- $\varepsilon = \frac{1}{13}$ (Delcourt, P. '17+)

How do we prove these bounds?:

Idea: Combine Randomness & Structure

3-Part-Plan (King & Reed)

(1) Do something different for large ω:

(a) $\omega \geq (1-\frac{1}{e^8})(\Delta(G)+1)$, use Reed.

(b) King showed that $\omega \geq \frac{2}{3}(\Delta+1)$, then $\exists$ an independent set I s.t. $\omega(G-I) < \omega(G)$, and

$$\Delta(G-I) < \Delta(G)$$

Now $G-I$ has a smaller ratio of ω/Δ, repeat until $\omega' < \frac{2}{3}(\Delta'+1)$, and now we use the I's to colour

(c) Use "harder" randomness + structure (Delcourt, P.)

(2) Now, assuming a small ω. Prove that a critical graph has neighborhoods that are "somewhat" sparse

(3) Now, use random colouring + Nibble.

Structure

Randomness +

Formalizing (2) and (3):

Def'n

A graph G is σ-sparse if $\forall v \in V(G)$ we have that

$$e(G[N(v)]) \leq (1-\sigma)\binom{\Delta}{2}.$$

Remark: Note the use of Δ instead of $d(v)$, i.e. small degree vertices trivially satisfy this condition

For (2):

Def'n → For list colourings: $\mathrm{Save}_L(G) := \Delta(G)+1-|L|$

$\mathrm{Save}(G) := \Delta(G)+1-\chi(G)$

$\mathrm{Gap}(G) := \Delta(G)+1-\omega(G)$

Reed's conjecture: $\mathrm{Save}(G) \geq \left\lfloor \frac{\mathrm{Gap}(G)}{2} \right\rfloor$

Our Result: $\mathrm{Save}(G) \geq \varepsilon\, \mathrm{Gap}(G)$ $(\varepsilon = 1/3$ say$)$.

Theorem (Delcourt, P.)

If G is L-critical, then G is σ-sparse where ~~$\sigma = \frac{\mathrm{Gap}(G)}{2\Delta} + \frac{2\mathrm{Save}(G)}{\Delta}$~~.

$$\sigma = \frac{\mathrm{Gap}}{2\Delta} - \frac{2\mathrm{Save}}{\Delta}$$

What good is sparsity?

Theorem

If G is σ-sparse and Δ is large enough, then ~~$\exists$~~

$\chi(G) \leq (1-f(\sigma))(\Delta+1)$, where $f(\sigma) = \frac{1}{2e^6}\sigma$ ← Molloy & Reed '02

$\approx .1827\sigma - 0.778\sigma^{3/2}$

(Bruhn & Joos '18)

$\approx .3\sigma - .25\sigma^{3/2}$

(Bonamy, Perrett, P., '16+)

How to get $\varepsilon = 1/13$?

We know:

$$\mathrm{Save}(G) \geq (.30 - .125\delta^{3/2})\Delta$$

$$\gtrsim \frac{2}{9}\delta\Delta \quad (\text{for } \Delta \geq 1/4)$$

$$\geq \frac{2}{9}\Delta\left(\frac{\mathrm{Gap}(G)}{2\Delta} - \frac{2\,\mathrm{Save}(G)}{\Delta}\right) \Leftarrow (\text{By thm})$$

$$= \frac{\mathrm{Gap}}{9} - \frac{4}{9}\mathrm{Save}$$

$$\Rightarrow \mathrm{Save} \geq \frac{1}{13}\mathrm{Gap}.$$

Theorem (Erdős, Rubin, Taylor '79) $\longrightarrow$ Equivalently, show $\chi(K_n - M) = n - e(M)$

If M is a matching in G, then $\chi_\ell(G) \leq v(G) - e(M)$

Note:

- Obviously[3], $\chi(G) \leq v(G) - e(M)$ (color each edge of matching with same color)

  Interesting part is it works for lists!

Proof:

By Induction on n. Let L be a list assign for G with $|L| = n - e(M)$

Case 1: $\exists e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \emptyset$. Now let $c \in L(u) \cap L(v)$

Colour u, v w/ c. Remove c from the other lists; delete u, v

Let $G' = G \setminus \{u, v\}$, $G' = K_{n-2} - \underbrace{(M - e)}_{M'}$, while $|L'| \geq |L| - 1$

$= v(G') - e(M')$

By induction, $\exists L'$ colouring, done

Case

Proof (con't)

Case 2:

$\nexists e = uv \in E(M)$ s.t. $L(u) \cap L(v) \neq \emptyset$.

We consider an ~~aux~~ auxiliary bipartite graph H where:

$V(H) = (V(G), \bigcup_{u \in V(G)} L(u))$, and
vertices colors

$E(H) = \{(v, c) : c \in L(v)\}$

Claim: $\exists$ a matching of H saturating $V(G)$, equivalently an L-coloring of G where every vertex receives a unique color

Let $S \subseteq V(G)$

- If $\exists e = uv \in M$ s.t. $u, v \in S$, then $N_H(S) \geq |L(u) \cup L(v)|$
By (b) $= |L(u)| + |L(v)| \geq 2(n - e(M)) \geq n \geq |S|$
- If $\nexists$ such $e \Rightarrow |S| \leq n - e(M)$, so if $S \neq \emptyset$, then $\exists v \in S$ and $N_H(S) \geq |L(v)| \geq n - e(M) \geq |S|$ if $S = \emptyset$

Theorem (Delcourt, P.)

If M is a matching in G and L is a list assignment for G s.t.:

(a) $|L(v)| \geq e(M)$ $\forall v \in V(M)$

(b) $|L(a)| + |L(b)| \geq v(G)$ $\forall a, b \in E(M)$

(c) $|L(v)| \geq v(G) - e(M)$ $\forall v \in V(G) \setminus V(M)$

Proof:

Same as ~~[illegible]~~ proof before, except cases for Hall's.

Cases:

(1) $\exists e \in uv \in H$ s.t. $u, v \in S$ (by (b))

(2) $\exists v \in V(G) \setminus V(H)$ s.t. $v \in S$ (by (c))

(3) $w \Rightarrow |S| \leq e(M)$ (by (a))

wat??   ??

$\square$

Theorem:

Let G be a graph, H be an induced subgraph of G, $s \geq 0$, Suppose that L is a list-assignment s.t. $|L(v)| \geq d(v) - s$ $\forall v \in V(H)$. If G is L-critical then

$$e(H) \geq (m-s)(v(H) - m - s)$$

where

$m = \nu(\overline{H})$ (i.e. size of largest antimatching in H)

Obs: If G is a graph.

$$\nu(\overline{G}) \geq \frac{v(G) - \omega(G)}{2}$$ (follows from greedily removing antiedges)

Proof:

By induction on $v(H)$. Let M be a matching of $\overline{H}$ of size m.

If $m \le s$: trivial

So, assume $v(H) > m > s \ge 0$

Since G is L-crit., $\exists$ L-colouring ϕ of $G - V(H)$.

Let $L'(u) = L(u) \setminus \{\phi(w) : w \in N(u) \setminus V(H)\}$ $\forall u \in V(H)$

Note $\forall u \in V(H)$,

$$
\begin{aligned}
|L'(u)| &\ge |L(u)| - |N(u) \setminus V(H)| \\
&\ge d(u) - s - |N(u) \setminus V(H)| \\
&= d_H(u) - s
\end{aligned}
$$

Since G is not L-col, H is not L'-col. So, by thm (Delcourt), $\exists$ at least one of (a)-(c) does not hold when applied to H, L', M:

Say (a) does not hold:

$\exists v \in V(M)$ s.t. $\underbrace{|L'(v)|}_{= d_H(v) - s} \le \overbrace{e(M)}^{= m} - 1$

So: $d_H(v) \le m - 1 + s$

Let $H' = H - \{v\}$, $v(\overline{H'})$. So, by induction on H',

$$
\begin{aligned}
e(\overline{H'}) &\ge (m-1-s)(v(H)-1-(m-1)-s) \\
e(\overline{H}) &\ge e(\overline{H'}) + (v(H) - (d_H(v)+1)) \\
&\ge e(\overline{H'}) + v(H) - m - s \\
&= (m-s)(v(H)-m-s)
\end{aligned}
$$

Proof: (con't)

(b) does not hold:

$\exists ab \in E(H)$ s.t. $|L'(a)| + |L'(b)| \leq v(H) - 1$

$\Rightarrow d_H(a) + d_H(b) \leq v(H) - 1 + 2s$

$H' = H \setminus \{a, b\}$, $v(H') \geq m-1$

By induction:

$$e(\bar{H}') \geq (m-1-s)(v(H)-2-(m-1)-s)$$

$$= \cancel{v(H) - m - s - 1}$$

$$= (m-s)(v(H)-m-s) + (-v(H) + 2)$$

$$e(\bar{H}) \geq e(\bar{H}') + 1 + (v(H) - 2 - d_H(a)) + (v(H) - 2 - d_H(b))$$

... (work it out... magic happens)

(c) does not hold:

$\exists v \in V(H) \setminus V(M)$ s.t. $|L'(v)| \leq v(H) - m - 1$

$\Rightarrow d_H(v) \leq v(H) - m - 1 + s$.

$H' = H - v$, $v(\bar{H}') \geq m$.

By induction:

$$e(\bar{H}') \geq (m-s)(v(H) - 1 - m - s)$$

$$= (m-s)(v(H) - m - s) - (m-s)$$

$$e(\bar{H}) \geq e(\bar{H}') + \underbrace{(v(H) - (d_H(v) + 1))}_{= v(H) - m - s}$$

... (and work it out

QED.

Proving Density theorem:

Apply the previous thm to $H = N[v]$ (closed neighborhood)

$$\Rightarrow e(G[\overline{N(v)}]) \geq (m-s)(v(H) - m - s)$$

where

$s = \text{Save}(G) - 1$, and

$m = v(\overline{N[v]}) \geq \frac{\Delta+1-\omega(G)}{2} = \frac{\text{Gap}(G)}{2}$, and

$v(H) = \Delta + 1$

$$\approx \left(\frac{\text{Gap}}{2} - \text{Save}\right)\Big(\underbrace{\Delta+1-\frac{\text{Gap}}{2}}_{\geq \frac{\Delta+1}{2}} - \text{Save}\Big)$$

$$\geq \frac{\text{Gap}\cdot\Delta}{4} - \text{Save}\,\Delta.$$

and since

$$e(\overline{H}) = \sigma\binom{\Delta}{2} \approx \sigma\frac{\Delta^2}{2},$$

~~we can substitute~~

So:

$$\sigma = \frac{2}{\Delta^2}\left(\frac{\text{Gap}\cdot\Delta}{4} - \text{Save} - \Delta\right)$$

$$= \frac{\text{Gap}}{2\Delta} - \frac{2\,\text{Save}}{\Delta}.$$