Sparsity Lemma:
If G is a σ-sparse graph for $\sigma \geq \frac{\text{polylog}\,\Delta}{\Delta}$ and $\Delta(G)$ sufficiently large, then:

$$\chi(G) \leq (1 - f(\sigma))(\Delta + 1)$$

where:

$f(\sigma) \approx .023\sigma$ Molloy & Reed $\rightarrow$ Naive Colouring Procedure

$.18\sigma - .07\sigma^{3/2}$ Bruhn & Joos

$.3\sigma - .12\sigma^{3/2}$ Bonamy, Perrett, P.

New Ideas in Bruhn & Joos:

1) Naive is a bit "too naive", i.e. It's a bit wasteful to uncolour both vertices in a conflict

Idea: For every edge, we ~~independently~~ ~~flip a coin~~ direct the edge and independently flip a coin before the random colouring. We only uncolour the "head" of the edge in a conflict if coin flip is heads and similarly for tails.

2) Analyze the # of repeated colours in $N(v)$

(i.e. $\underbrace{|N(v) \setminus V(G')|}_{\text{"\# of neighbours (of v) deleted: i.e. retained colour and not in } G'\text{"}} - \underbrace{(|L(v)| - |L'(v)|)}_{\text{"colours lost"}}$)

Via the # of pairs, triples, etc. of repeated colours.

Molloy + Reed observed that if this value is $\geq \Delta + 1 - |L|$, then we can greedily color G'.

Need to figure out $\mathbb{E}[$"this value"$]$, argue its concentrated and apply LLL.
let's call it the "savings of v"

Pairs + Triples:

$\text{Pairs}_{x} := \{ (u,v) \in \binom{N(x)}{2} : uv \notin E(G),\ \phi(u) = \phi(v)$ and $u,v \in V(G') \}$

$\text{Triples}_{x} := \{ (u,v,w) \in \binom{N(x)}{3} : uv, vw, uw \notin E(G),$
$\phi(u) = \phi(v) = \phi(w)$, and
$u, v, w \in V(G') \}$

Claim: $\text{Savings}_v \geq \text{Pairs}_v - \text{Triples}_v$.

Expectation of Pairs, Triples:

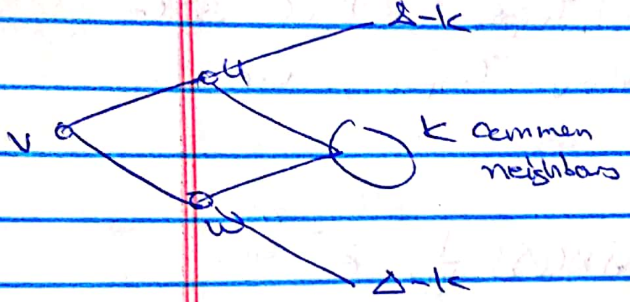
Let's assume G is Δ-regular:

$\underbrace{\Pr[v \notin V(G')]}_{\text{i.e. Retains color}} = \left(1 - \frac{1}{2 \cdot |L|}\right)^{\Delta} \approx e^{-\frac{\Delta}{2|L|}}$

Need to lose coin flip to lose color

$\mathbb{E}[\text{Pairs}_v] = \#$ of non-edges in $v \cdot \Pr[\text{Pairs}_{uv}]$

$\Pr[\phi(u) = \phi(v)] = \frac{1}{|L|}$ $\Pr[u, w \in V(G')]$ $(\cdot C_0)$



$$\Pr[u, v \in V(G')] = \left(1 - \frac{1}{2|L|}\right)^{2(\Delta-k)} \left(1 - \frac{3}{4|L|}\right)^{k}$$

Neighbors not in common (Bad event is losing coin flip)

Neighbors in common (Bad event is either u, or w lose coin flip).

$$= \left(1 - \frac{1}{2|L|}\right)^{2\Delta} \cdot \frac{\left(1 - \frac{3}{4|L|}\right)^{k}}{\left(1 - \frac{1}{2|L|}\right)^{2k}}$$

Can argue that $\left(1 - \frac{3}{4|L|}\right) \geq \left(1 - \frac{1}{2|L|}\right)^{2}$ for $|L|$ large enough, and so:

$$E[\text{Pairs}_v] \geq \underbrace{\#\text{ of non-edges of } v}_{\geq \bar{c}\binom{\Delta}{2}} \cdot \left(1 - \frac{1}{2|L|}\right)^{2\Delta}$$

$$\approx e^{-\Delta/|L|}.$$

The calculation is similar for $\mathbb{E}[\text{Trips}_v]$, BUT since we want to argue $\text{Sewings}_v \geq \text{Pairs}_v - \text{Trips}_v$, we want an upper bound.

$$\mathbb{E}[\text{Trips}_v] \geq \underbrace{\# \text{ of triangles in } G[N(v)]}_{\text{But what is this?}} \cdot \frac{1}{|L|^2} \cdot e^{-\frac{7\Delta}{8|L|}}.$$

Theorem (Rivin, 2002)
If G is a graph w/ $\sigma\binom{V(G)}{2}$ edges, then G has at most $\sigma^{3/2}\binom{V(G)}{3}$ triangles.

So: $\mathbb{E}[\text{Pairs}_v - \text{Trips}_v]$

$$\geq \sigma\binom{\Delta}{2} \cdot \frac{1}{|L|} e^{-\frac{\Delta}{|L|}} - \sigma^{3/2}\binom{\Delta}{3}\frac{1}{|L|^2} e^{-\frac{7\Delta}{8|L|}}.$$

(where $\sigma\binom{\Delta}{2} = \#$ of nonedges in $N(v)$).

$\approx \frac{\sigma}{2e} - \sigma^{3/2} \cdot \frac{1}{6e^{7/8}}$, ~~which~~ ~~is~~ ~~the~~ ~~#'s~~ we ~~wanted~~ which, if worked out, will match the #'s ~~threshold~~ we wanted (from Bruhn & Joos)

Then, use "Exceptional Zaleynemts" and difference of variables to concentrate

New Idea in Bonamy, Perrett, P.:
- Why stop after one iteration?
  Keep using nice procedure, i.e. Use Nibble!
  Issue: Only works if uncoloured subgraph is still "somewhat sparse"
  Idea: Uncoloured subgraph will be a "pseudorandom" subgraph of G

Defn: If G is a graph, we say a subgraph H of G is a μ-pseudorandom subgraph of G if $\forall u, v \in V(G)$,
$||N_G(u) \cap N_G(v) \cap V(H)| - \mu |N(u) \cap N(v)|| \leq 10\sqrt{\Delta} \log^4 \Delta$.

μ ↙ expected probability of going in $V(H)$

So, if we can control this #, then:
Lemma: $\forall \sigma > \sigma'$ and $\mu > 0$, if $\Delta(G)$ is large enough, and H is a μ-pseudorandom subgraph of G, and G is σ-sparse, then H is σ'-sparse.

So, how does this give a savings of $(.3\sigma_2 - .12\sigma^{3/2})\Delta$?
Savings for max deg Δ graph (from Bruhn & Joos) $\approx (.18\sigma - .07\sigma^{3/2})\Delta$
Savings in 2nd step: $\approx (.18\sigma' - .07\sigma'^{3/2})\Delta'$
⋮
—— " —— tth " : $\approx (.18\sigma_t - .07\sigma_t)\Delta_t$.
What is Δ' (and Δ_t)?

What is $\Delta(G')$?

$E[|N(u) \cap V(G')|]$

$= \underbrace{|N(u)|}_{=\Delta} \cdot \cancel{Pr} \, (1 - \underbrace{Pr[\cancel{v} \in N(u) ; u \notin V(G')]}_{\approx e^{-\Delta/2k}}) \quad$ (we've done this before)

$= \Delta (1 - e^{-\Delta/2k})$.

We can concentrate ~~this~~ this (using LLL) to get:

$$\Delta_t \approx \Delta_0 (1 - e^{-\Delta/2k})^t$$

So:

Total Savings $= \sum_t$ Savings at each step

$= (.18\sigma - .17\sigma^{3/2}) \cdot \sum_{t \geq 0} \Delta_t$

$= \quad " \quad \cdot \underbrace{\sum \Delta_0 (1 - e^{-\Delta/2k})^t}$

$= \Delta_0 \cdot e^{\frac{\Delta}{2k}} \approx \sqrt{e} \approx 1.6$

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Feb 11th

Johansson / Molloy / Bernshteyn Theorems:

Theorem (Johansson)
If G is triangle-free then
$$\chi_\ell(G) \leq O\left(\frac{\Delta}{\ln \Delta}\right)$$

Molloy ('12) $(1+o(1))\frac{\Delta}{\ln \Delta}$

Bernshteyn ('18) True for DP-coloring

Bernshteyn: Uses LLL

Molloy: Uses entropy compression

Theorem (Johansson)
If G is K_r-free for fixed r
$$\chi_\ell(G) \leq O\left(\frac{\Delta \ln \ln \Delta}{\ln \Delta}\right).$$

Molloy: $\chi_\ell(G) \leq 200 r \frac{\Delta \ln \ln \Delta}{\ln \Delta}$ (r no longer fixed)

Bernshteyn: True for DP-coloring

In fact, Molloy / Bernshteyn gives a more general result framework:
(Bonamy, Kelly, Nelson, Postle '18+)
$$\chi_\ell(G) \leq 200\Delta \sqrt{\frac{\ln \omega(G)}{\ln \Delta}}$$
(even $\chi_{DP}(G)$)

In spirit this answers the following question:

What assumption do we need on $\omega(G)$ to guarantee that:

$$\chi(G) \leq \frac{\Delta}{c} \quad (c \geq 2)$$

Using BKNP: $\omega \leq \Delta^{\frac{1}{200c^2}}$

This is almost tight:

Ramsey Theory $\Rightarrow$ Need $\omega \leq \frac{1}{c-1}$.

Conjecture: linear in c is correct

implied by the following conjecture:

$$\chi_f(G) \leq 200\Delta \cdot \frac{\ln \omega}{\ln \Delta}$$

This would give the complete picture: