CO749 - Random Graph Theory

(Lecture Summaries)

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Lecture 1: Introduction

Definition 1.1. The probability space we'll work in is denoted with the triple $(G, \mathbb{P}, F)$, where G is a class of graphs, $\mathbb{P}$ a probability measure and F a sigma algebra.

Normally, G is a finite set, $\mathbb{P}$ is a discrete probability measure and $F = 2^G$

Definition 1.2 (Erdős-Rényi Random Graph Model)**.**

- The $\mathcal{G}(n, p)$ model: A graph with vertex set $[n]$ is constructed randomly by including each edge in $k_{[n]}$ with probability p
- The $\mathcal{G}(n, m)$ model: A graph is chosen uniformly at random from all graphs with vertex set $[n]$ and has m edges.

(Aside: We can think of $\mathcal{G}(n, m)$ as labelling the edges)

Other models:

- $\mathcal{G}(n, d)$ is the model of random d-regular graphs
- $\mathcal{G}(n, \tilde{d})$ where $\tilde{d} = (d_1, \ldots, d_n)$ is a vector representing the degrees of vertices. (This is a generalization of $G(n, d)$)
- $\mathcal{G}(n, r)$ is the model of random geometric graphs. The construction is as follows: Pick n points uniformly in the unit square, then, add an edge if and only if the distance between two points is $\leq r$
- Random trees. A tree is chosen uniformly at random from the n^{n-2} trees on n vertices.

In this class, we will primarily focus on the Erdős-Rényi Model.

1.1 Probability Primer

Definition 1.3. A discrete probability space consists of a countable set Ω and a probability function $\mathbb{P} : \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

A subset of Ω is called an event. The probability of $A \subseteq \Omega$ is $\sum_{\omega \in A} \mathbb{P}(\omega)$, denoted $\mathbb{P}(A)$.

Proposition 1.1 (Inclusion-Exclusion)**.** For events A, B:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

and, in general:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1 < i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \ldots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Corollary 1.1. $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$

Definition 1.4. Two events are <u>independent</u> if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Definition 1.5. A <u>random variable</u> (r.v) X is a function $X : \Omega \to \mathbb{R}$. In a discrete probability space, the <u>expectation</u> of X is defined by: $\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega)\,\mathbb{P}(\omega)$

Proposition 1.2 (Linearity of Expectation)**.** $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Proof. $\mathbb{E}(X+Y) = \sum_{\omega\in\Omega}(X+Y)(\omega)\,\mathbb{P}(\omega) = \sum_{\omega\in\Omega} X(\omega)\,\mathbb{P}(\omega) + \sum_{\omega\in\Omega} Y(\omega)\,\mathbb{P}(\omega) = \mathbb{E}(X) + \mathbb{E}(Y)$ □

Lemma 1.1.

- For any $n \geq k \geq 1$
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$$

- (Stirling's Formula)
$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \mathcal{O}(n^{-2})\right)$$

- For every $t \in \mathbb{R}, e^t \geq 1 + t$

Lemma 1.2. Assume $k = o(\sqrt{n})$ Then, $\binom{n}{k} \sim \frac{n^k}{k!}$

Proof.
$$\begin{aligned}
\binom{n}{k} &= \frac{1}{k!}\prod_{i=0}^{k-1}(n-i) \\
&= \frac{n^k}{k!}\prod_{i=0}^{k-1}\left(1 - \frac{i}{n}\right) \\
&= \frac{n^k}{k!}\prod_{i=0}^{k-1} e^{\mathcal{O}(i/n)} \qquad (\log(1-x) = \mathcal{O}(x)) \\
&= \frac{n^k}{k!}\exp\left(\mathcal{O}\left(\frac{1}{n}\sum_{i=0}^{k-1} i\right)\right) \\
&= \frac{n^k}{k!}\exp\left(\mathcal{O}\left(\frac{k^2}{n}\right)\right) \\
&= (1 + o(1))\frac{n^k}{k!} \qquad \left(\text{as } k = o(\sqrt{n})\right)
\end{aligned}$$

□

Remark 1.1. $k = o\left(n^{\frac{2}{3}}\right)$, then $\binom{n}{k} \sim e^{-\frac{k^2}{n}} \cdot \frac{n^k}{k!}$

Lecture 2: Concentration Inequalities, Coupling, Connection Theorem

Definition 2.1. Given a sequence of probability spaces $(\Omega_n, P_n)_{n\geq 1}$. We say that A_n holds asympototically almost surely (a.a.s) if $P_n(A_n) \to 1$ as $n \to \infty$

Theorem 2.1 (Markov's Inequality)**.** Let X be a nonnegative random variable. Then, for any real $t > 0$, $\Pr(X \geq t) \leq \frac{\mathbb{E}\, X}{t}$

Proof. Let I_t be the indicator r.v. that $X \geq t$. Then, $X \geq t \cdot I_t$, so:

$$\mathbb{E}\, x \geq t \cdot \mathbb{E}\, I_t = t \cdot \mathbb{P}(X \geq t)$$

□

Theorem 2.2 (Chebyshev's Inequality)**.** For any $t \geq 0$

$$\mathbb{P}(|X - \mathbb{E}\, X| \geq t) \leq \frac{\operatorname{Var} X}{t^2}$$

Lecture 3: Threshold, First Order Logic of Graphs

Lecture 4: 0-1 Law, Evolution of Graphs and Theorem E

Lecture 5:

References

[1] Bollobás Béla. *Random graphs.* Academic Press, 1985.