749 - Graph Colourings Jan 9th

Questions / Results: (on Graph Colourings)

Def'n (k-colourable)
A graph G is k-colorable if G has a k-coloring

Def'n (Chromatic Number)
The chromatic number, denoted $\chi(G)$, is the minimum k s.t. G is k-colorable

Question: Why is this a ~~natural~~ good definition?
If a graph is k-colorable, then it is (k+1)-colorable, so this is a natural def'n.
(Prop: If $H \subseteq G$, then $\chi(H) \leq \chi(G)$)

Remark:
A graph is 1-colorable iff no edges
— " — 2-colorable iff no odd cycles
— " — 3-colorable iff no good answer!
↳ Since NP-hard to decide if a graph is 3-col.

A graph is critical for k-colouring if it is not k-colorable, but every proper subgraph is. (also formerly known as (k+1)-critical)

List Colouring: → (Introduced by Erdős, Rubin, Taylor in 1979 and independently by Vizing 1976)
The list chromatic number (aka choice number or choosability)
denoted $\chi_\ell(G)$ is the minimum k such that G
has a L-colouring $\forall$ k-list-assignments L.

Proposition: $\chi(G) \leq \chi_\ell(G)$

Proposition: If $H \subseteq G$, $\chi_\ell(H) \leq \chi_\ell(G)$
(i.e. The list chromatic number remains monotone)

Def'n G is k-list-colourable if $\chi_\ell(G) \leq k$
(aka k-choosable)

Def'n G is critical for k-list-colouring if G is not k-list-col.
but every proper subgraph is

L-critical w.r.t a list assignment L if G is not
L-col, but $\forall$ proper subgraph is.

Q: How is list colouring different from colourings?

Theorem: $\chi_\ell(K_{d,d}) = \Theta(\log d)$
(But note: $\chi(K_{d,d}) = 2$)

Theorem (Alon 2000)
If G is a graph of min. degree d, then $\chi_\ell(G) = \Omega(\log d)$.

Conjecture: $\forall k$, if $\chi(G) \le k$ (and ~~min. degree~~ triangle free) then $\chi_c(G) = O(\log d)$

Correspondence Colouring (aka DP-colouring)

Def: Corr Chromatic # (aka DP-chr. #), denoted $\chi_c(G)$ (aka $\chi_{DP}(G)$) is min k s.t. $\forall (L,M)$ k-corr-assign, G has a (L,M)-colouring

critical for "

(L,M)-critical for "

Theorem (Bernshteyn, 2018)

If G is d-regular, then $\chi_c(G) = \Omega\left(\frac{d}{\log d}\right)$

Back to questions:

Types of questions:

- Chromatic # are related to other graph parameters (E.g. degree, clique #, etc.)
- Chromatic # of certain graph classes (E.g. Planar, surfaces, etc.)
- Algorithmic questions, e.g. Deciding if colouring exists, finding a colouring, can you sample a colouring uniformly at random
- How many colourings?
- Re-configuration: Can we get from one colour to another?

Relations to Other Parameters:

Colourings Ones: $\underbrace{\max_{H \subseteq G} \frac{v(H)}{\alpha(H)}}_{\text{Hall ratio}} \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$

(Recent result of 2019): $\nexists f$ s.t. $\chi_f(G) \leq f(\text{Hall ratio})$

Degree, Clique #, Girth:

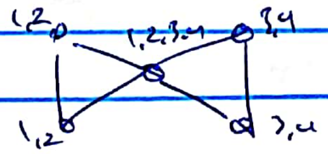
$\chi_{DP}(G) \leq \underbrace{\Delta(G)}_{\text{max degree}} + 1$ (Greedy Bound)

Brooks's Thm (1941): $\chi(G) \leq \Delta(G)$ unless G contains $K_{\Delta+1}$ or an odd cycle if $\Delta = 2$

(Other version: If connected, then $\chi \leq \Delta$ unless G is iso to clique or odd cycle if $\Delta = 2$)

ERT, independently Vizing: $\chi_\ell(G) \leq$ —— " ——

ERT (Borodin) If L is a list assignment of connected G such that $|L(v)| \leq d(v)$, then G is L-colorable unless every block of G is a clique or odd cycle



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Girth:

Kim '95: If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\log \Delta}$

Johannsen '99: If G is triangle-free, then: $\chi(G) = O\left(\frac{\Delta}{\log\Delta}\right)$

Molloy '17: — " — then: $\chi(G) \leq (1+o(1))\left(\frac{\Delta}{\log\Delta}\right)$

If G is K_r-free for fixed r, then $\chi \leq O\left(\frac{\Delta}{\log\Delta}\log\log\Delta\right)$

Conj: $\log\log\Delta$ is not neccessary.

Question: Can we do better than $\frac{\Delta}{\log\Delta}$?

Erdős '59: $\exists$ graphs of arbitrary girth and chromatic #

Best known result in Ramsey Theory
$\Rightarrow \exists$ graphs of arbitrary girth and $\chi \geq \frac{1}{2}\frac{\Delta}{\log\Delta}$

$\hookrightarrow$ So, no, we can't!

Question: Is the answer 1 or 1/2 or in-between?

Reed's Conjecture (1998) $\chi(G) \leq \left\lceil \frac{\Delta+1+\omega}{2} \right\rceil$

$\Rightarrow$ True for $\chi_f(G)$ (Reed)

for if $\omega \geq .9999948\,\Delta$ (i.e. large ~~enough~~).

Thm (Delcourt, P.)

If Δ large enough, then $\chi \leq \left\lceil \frac{12}{13}(\Delta+1) + \frac{1}{13}\omega \right\rceil$.

Chromatic # of Graph Classes:

Four Color Theorem (Appel & Hakken, 1977/79)
(Conjectured 1852)
If G is planar, $\chi(G) \le 4$.

→ Later proof by Robertson, Sanders, Seymour, and Thomas (1994/6)
(Formally verified by prover systems in 2000s)

Grötzsch's Theorem (1959)
If G is planar, triangle-free, then $\chi(G) \le 3$.

Surfaces:
Euler genus of a surface =
$2 \times$ # of handles + # of crosscaps

Heawood's Bound: $\forall g \ge 1$, if G is a graph embedded in a surface of genus g, then

$$\chi(G) \le \frac{7 + \sqrt{24g + 1}}{2}$$

Ringel-Youngs Thm (1960's)
Heawood's Bound is tight for every surface, except the Klein bottle, where $\chi \le 6$.

Hadwiger's Conjecture:

(1943) If G has no K_t-minor, then $\chi(G) \le t-1$

$\rightarrow$ Easy for $t \le 3$ and proved by Had for $t=4$

Wagner '37 - Showed $t=5$ is equiv to 4CT

Robertson, Seymour, Thomas '94 : $t=6$ equiv to 4CT

Open for all $t \ge 7$.

Weakenings:

Thm (Reed-Seymour, '98):

If G has no K_t-minor, then $\chi_f(G) \le 2t$

Thm (Edwards, Kang, Kim, Oum, Seymour, '15)

$\forall t, \exists d$ s.t. if G has no K_t-minor, then G is a d-defective, t-colorable

Thm (Dvorak - Norin, '18+)

— " —, G is $\exists c$-clustered, t-col.

Thm (Kostochka, Thomason '80s)

If G has no K_t-minor, then G is

$O(t\sqrt{\log t})$-degenerate

Thm (Norin, Song, '19+)

$\forall \beta > 1/4$, G is $O(t(\log t)^\beta)$-col.

List version of Hadwiger's is false

i.e. $\exists K_t$-minor free G w/ $\chi_\ell(G) \geq 4/3 t$

Strong perfect graph thm: (Chudnovsky, Robertson, Seymour, Thomas, '06)

G is perfect iff G has no induced C_{2k+1} or $\overline{C_{2k+1}}$ $\forall k \geq 2$