749 - Random

Feb 2th

Recall:

Lemma: $pn - (\log n + \log\log n) \to \infty$ and $pn < 2\log n$. Then, $\exists\ \varepsilon_0 > 0$ s.t. a.a.s $\forall S \subseteq [n]$, $|S| \le \varepsilon_0 n$ and $|N(S)| \ge 2|S|$

Proof:

Aas, no S with $n^{3/4} \le |S| \le \varepsilon_0 n$, $|N(S)| < 2|S|$ (last class)

...

And we've also previously proved the following properties:

A.a.s in $G(n,p)$:

(a) $S \subseteq [n]$ with $|S| \le \frac{n}{\log^2 n}$ induces $\le 3|S|$ edges

(b) No path of length at most 5 joins 2 light vertices

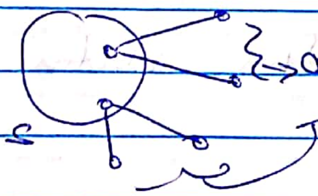
(c) No cycle of length ≤ 5 has a light vertex

We will now complete the proof of the lemma: (Note: Not probabilistic)

For $S < n^{3/4}$. Assume G satisfies (a)-(c) and $G \in D2$. Assume $S \subseteq [n]$ with $s = |S| < n^{3/4}$. and $|N(S)| < 2|S|$

Let X be the set of light vertices in S and $Y = S - X$

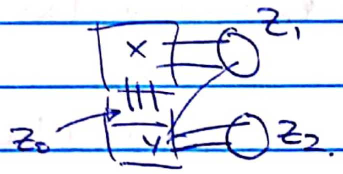
If $Y = \varnothing$, then $|N(S)| \ge 2|S|$ (by (b) and D2)



$\Rightarrow$ all vertex-disjoint, since otherwise we violate (b).

Proof: (cont)

If $Y \neq \emptyset$ let $Z_0 = N(X) \cap Y$, $Z_1 = N(X) \setminus Y$, $Z_2 = N(Y) \setminus (X \cup Z_1)$



$|N(S)| = |Z_1| + |Z_2| < 2|S| = 2(|X| + |Y|)$

By assumption

| e(A,B) : # of edges b/w A and B.

By GEDZ and (b), (c)

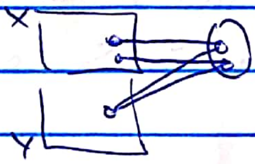
(i) $|Z_0| + |Z_1| \geq 2|X|$ (D2 and (b))

(ii) $|Z_0| = e(X,Y) \leq |Y|$

(iii) $e(Y, Z_1) \leq |Y|$

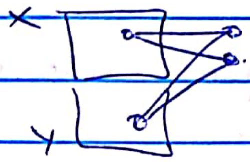
→ Every vertex in Y can only see one vertex in Z_1

Suppose o/w, then we get smth like:



But this is a path of length <5, that joins 2 vertices in X (violating (b))

OR



and this violates (c).

Hence,

$$2|X| - |Z_0| + |Z_2| \underset{\text{By (i)}}{\leq} |Z_1| + |Z_2| \underset{\text{By assumption}}{<} 2(|X| + |Y|)$$

Thus:

$$|Z_2| < |Z_0| + 2|Y| \underset{\text{By (ii)}}{\leq} 3|Y|.$$

By (a), $e(Y, Z_2) \leq e(Y \cup Z_2) \leq 3(|Y| + |Z_2|) \leq 12|Y|$.
Since every vertex in Y has degree ≥ 100, we have:

$$\underbrace{e(Y, Z_1)}_{\substack{\leq |Y| \\ \text{(By (iii))}}} + \underbrace{2e(Y)}_{\leq 6|Y|} + \underbrace{e(X, Y)}_{\substack{\leq |Y| \\ \text{(By (ii))}}} + \underbrace{e(Y, Z_2)}_{\leq 12|Y| \text{ (above)}} \geq 100|Y|$$

$\Rightarrow 100|Y| \leq 20|Y| \Rightarrow |Y| = 0$

Contradicting that $Y \neq \emptyset$ ◻

Note: The 100 was arbitrarily chosen, we could have chosen any large #.

A1Q3:

Show that if $p = o\left(\frac{\log n}{n}\right)$ a.a.s $\Delta(G(n,p))$ has a 2-point concentration.

Let $\lambda_k = n \cdot P(Bin(n-1, p) = k)$ = Expected # of vertices with degree k.

① Prove that λ_k monotonely decreasing on $k > np$

Let k^* be the integer $\gg np$ which minimizes $\max\{\lambda_k, \frac{1}{\lambda_k}\}$
... But, we have to show that this is actually well-defined

Instead we could try:

② Let $k^* \geq np$ be the max. integer s.t. $\lambda_{k^*} \geq 1$. (And let $k^* = n-1$ if $\lambda_{n-1} \geq 1$)

③ Prove that $\forall$ constant $c > 0$, $\cancel{\lambda_{np}}$ $\cancel{\lambda_{cnp}}$ $\lambda_{cnp} = \omega(1)$
$\Rightarrow k^* = \omega(np)$

④ $\frac{\lambda_k}{\lambda_{k-1}} \sim \frac{np}{k} \Rightarrow \lambda_{k^*} \geq 1$ and $\lambda_{k^*+1} < 1$

Case I: $\lambda_{k^*} = \omega(1)$

$\lambda_{k^*+2} = o(1)$ and $\sum_{j \geq k^*+2} \lambda_j = o(1)$ (By (4))

$\Rightarrow$ a.a.s $\Delta \in \{k^*, k^*+1\}$

Case 2: $\lambda_{k^*} = O(1)$, $\lambda_{k^*-1} = \omega(1)$, $\lambda_{k^*+1} = o(1)$

Then, $\sum_{j \geq k^*+1} \lambda_j = o(1)$ (By (4)).

$\Rightarrow$ a.a.s $\Delta \in \{k^*-1, k^*\}$

Feb 24th

Subsubsequence Principle:
If for every subsequence of a sequence there is a subsubsequence converging to a limit ~~a~~, then the entire sequence must converge to the same limit.
(True for any sequence in a topological space)

Suppose you want to prove a limit theorem for ~~g~~ $g(n,p)$
You just need to prove such a limit theorem for $p = p_n$
s.t. p_n has an expression that converges to a limit
(E.g. $n^a p^b \to c$ for some a, b)
(E.g:
$p = \Theta(n^{-2/3})$, then $n^2 p^3 \to c$)

In the example of the 2-point concentration:
We have $p = o(\frac{\log n}{n})$. Assume the 2-point concentration is not true. Then, there ~~is~~ is a subsequence S of $\{p_n\}$
s.t. $\exists\, \varepsilon > 0$, $\Pr(\exists \{i,j\} : \Delta \in \{i,j\}) < 1 - \varepsilon$ $\forall n$.
In the subsequence.

However we can show that there is a subsequence of S (so a subsubsequence of p_n), s.t. 2-point concentrations holds in this subsubsequence, a contradiction.