CO434 - Combinatorial Designs

University of Waterloo
Nicholas Pun
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Contents

Lecture 1: Introduction

This course is concerned with combinatorial objects satisfying certain regularity/balanced conditions. We will go over the following topics:

- Classical results in design theory
- Constructions of designs
- Structural theorems
- Non-existence results
- Lots of algebra (abstract and linear)
- Some number theory!

1.1 The 36 Officers Problem (Euler, 1782)

Problem statement: We have 36 pieces that are

- 1 of 6 chess pieces: King, Queen, Bishop, Rook, Knight, Pawn, and
- 1 of 6 colors (one piece of each colour)

Can we arrange these on a 6-by-6 square such that each row and each column uses exactly one piece of each type and one piece of each colour? Euler thought this was impossible.

Note that this is easy for 5 chess pieces and 5 colours. Consider (using numbers 1 through 5 in place of the different chess pieces):

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Which was constructed by taking the sequence 12345 and shifting it left each row.

Further, this is impossible for 2 pieces and 2 colours. In fact, Euler conjectured that this is impossible for $n \equiv 2 \pmod 4$

However, this was wrong! (In fact, 2 ad 6 are the only numbers for which this is impossible, and we'll gain a better understanding of why through this course!)

1.2 Statistical Experimental Design

Here is a take on design theory. Consider the following scenario where:

- We have v kinds of wine we want to compare,

- But, we can only accurately compare k kinds per day.
- And, we only want to compare each pair exactly once.

Can we decide on what to drink each day in order to accomplish these tasks?

Example 1.1 (Fano Plane)**.** Suppose $v = 7, k = 3$.

We will call the objects we want to compare (in this case, the wines) points and the experiment for the day a block.

Here our points be labelled with numbers: $\{0, 1, 2, \ldots, 6\}$

And our blocks for each day will be:

Day 1: $\{0, 1, 3\}$

Day 2: $\{1, 2, 4\}$

Day 3: $\{2, 3, 5\}$

Day 4: $\{3, 4, 6\}$

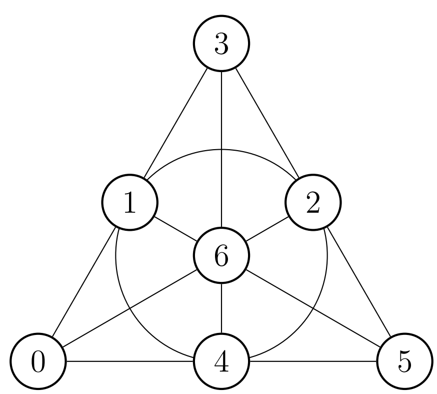
Day 5: $\{4, 5, 0\}$

Day 6: $\{5, 6, 1\}$

Day 7: $\{6, 0, 2\}$

Note that we can obtain the next day's experiment by adding 1 modulo 7. And, finally we can check that every pair appears in exactly one block. $\triangle$

This example actually has a nice visualization:



Note that each collinear line gives us a block (We count the line through 1, 2, and 4 as a "straight" line) How did we come up with this? We'll answer this through the course! For now, let's look at another example:

Example 1.2 (Game of Set)**.** In this game, we have 81 cards, each with 4 attributes:

- Colour: Red, Green, or Purple

- Number: 1, 2, or 3
- Shading: Open, Shaded, or Solid
- Shape: Oval, Diamond, or Squiggle

A Set is a triple of cards with the property that in each attribute, they are all the same or all different.

Note. Each pair of cards is in a unique Set!

This is since on each attribute, the pair is either matching or different, and there is a unique third card that can complete the pattern. △

How do the above examples compare?

Balancing Properties of the Two Examples

Fano Plane	Set
Each block has 3 points	Each Set has 3 cards
Each point is in 3 blocks	Each card is in 40 different Sets*
Each pair of points is in exactly one block	Each pair of cards is in a unique set

*Why is this true? For every card, we can choose any other card in the deck (There are 80 such choices) But, we observed earlier that for every pair, there is a unique 3rd card that completes a Set. So, in fact, we'll treat the remaining 80 cards as pairs.

1.3 Basic Definitions

Definition 1.1. A design is a pair $(V, \mathcal{B})$ where

- V is a finite set of elements called points
- $\mathcal{B}$ is a collection (multiset) of (usually non-empty) subsets of V, called blocks

We call a design simple if $\mathcal{B}$ is a set (there are no repeated blocks)

Definition 1.2. A t-design ($t \in \mathbb{N}$) is a design $(V, \mathcal{B})$ in which all blocks have the same size k, and there is a constant λ_t such that every t-tuple of points lie in exactly λ_t blocks.

The two examples we just saw were 2-designs with $\lambda_2 = 1$ and also 1-designs with $\lambda_1 = 3$ for the Fano plane and $\lambda_1 = 40$ for the game of Set.

Definition 1.3. A Balanced Incomplete Block Design (BIBD) is a design that is a 2-design and a 1-design.

Exercise. If $\lambda_2 \neq 0$, then show that every 2-design is automatically a 1-design

Lecture 2: BIBDs and Examples

Recall Definition 1.3, the definition for Balanced Incomplete Block Designs (BIBD).

The parameters (v, b, r, k, λ) of a BIBD are defined as follows:

- $v = |V|$ - The number of points
- $b = |\mathcal{B}|$ - The number of blocks (also λ_0)
- $r = \lambda_1$ - Each point lies in r blocks
- $k = |\alpha|, \forall \alpha \in \mathcal{B}$ - The size of the blocks
- $\lambda = \lambda_2$ - Each pair of points are in λ blocks

We call v, k, λ the primary parameters and b, r the secondary parameters.

We'll call a BIBD with these parameters a:

- (v, k, λ)-BIBD, or
- 2-(v, k, λ)-BIBD, or
- (v, b, r, k, λ)-BIBD

Example 2.1. The Fano Plane (Example 1.1) is a $(7, 3, 1)$-BIBD (and also a $(7, 7, 3, 3, 1)$-BIBD) $\triangle$

A BIBD is trivial if $k \in \{0, 1, v-1, v\}$

Convention: We will assume that our BIBDs are non-trivial designs (Since otherwise some statements *may* be false)

Definition 2.1. Let $(V, \mathcal{B})$ be a design, we define the complement design $(V, \overline{\mathcal{B}})$ where $\overline{\mathcal{B}} = \{V \setminus \alpha \mid \alpha \in \mathcal{B}\}$

Example 2.2 (Complement of the Fano Plane)**.**
$V = \{0, 1, 2, \ldots, 6\}$ - The points stay the same.

$\mathcal{B} = \{013, 124, 235, 346, 450, 561, 602\}$ - These were the blocks we had before.

$\overline{\mathcal{B}} = \{2456, 0356, 0146, 0125, 1236, 0234, 1345\}$ - These make a $(7, 7, 4, 4, 2)$-BIBD $\triangle$

Exercise. The complement of a (v, b, r, k, λ)-BIBD is a $(v, b, b-r, v-k, b-2r+\lambda)$-BIBD

2.1 Difference Sets

Let $(G, +, 0)$ be an abelian group. A difference set S is a subset of G with the property that there exists a constant λ such that $\forall g \neq 0, |\{(a, b) \in S \times S \mid a - b = g\}| = \lambda$. If $|G| = v, |S| = k$, we call this a (v, k, λ)-difference set

Example 2.3. Let $G = \mathbb{Z}_7, S = \{0, 1, 3\}$. We can check that this is a difference set by taking the difference between every pair of elements $(a, b) \in S \times S$

$$\begin{array}{cc|c|c|c} & & & a & \\ & & 0 & 1 & 3 \\ \hline & 0 & 0 & 1 & 3 \\ \hline b & 1 & 6 & 0 & 2 \\ \hline & 3 & 4 & 5 & 0 \end{array}$$

Each number from 1 to 6 appear exactly once, so this is a $(7, 3, 1)$-difference set. $\triangle$

Example 2.4. Let $G = \mathbb{Z}_{11}, S = \{0, 2, 3, 4, 8\}$ creates an (11,5,2)-difference set. $\triangle$

Theorem 2.1. If $S \subseteq G$ is a (v, k, λ)-difference set, then we construct a (v, k, λ)-BIBD as follows:

- $V = G$
- $\mathcal{B} = \{g + S \mid s \in S\}$

Note. $v = b$ and $k = r$

Proof. Exercise! $\square$

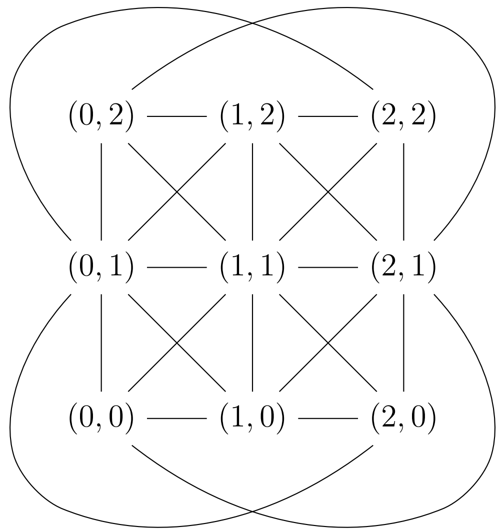
<u>Problem</u>: How to we find difference sets? We'll see later in the course!

2.2 Affine Space

Let V be a vector space over a finite field $\mathbb{F}$. An affine line in V is a set of points of the form $\{xt + y \mid t \in \mathbb{F}\}$, where $x, y \in V$, $x \neq 0$. Let $\mathcal{B}$ be the set of all affine lines in V, then $(V, \mathcal{B})$ is a BIBD with $\lambda = 1, k = |\mathbb{F}|$

When $\dim_{\mathbb{F}} V = 2$, we call this an <u>affine plane</u>.

Example 2.5. Let $\mathbb{F} = \mathbb{Z}_3, V = \mathbb{Z}_3^2$. This creates a $(9, 12, 4, 3, 1)$-BIBD.



For example, with $y = (1, 0), x = (2, 1)$, we have:

- $(1, 0) = 0x + y$
- $(0, 1) = 1x + y$
- $(2, 2) = 2x + y$

all of which are located on the same line. △

Example 2.6. The game of Set is a 4-dimensional affine space over Z_3 △

2.3 Relations between Parameters

Theorem 2.2. The parameters (v, b, r, k, λ) of a BIBD satisfy:

$$\frac{v}{k} = \frac{b}{r} \tag{2.1}$$

$$\frac{v(v-1)}{k(k-1)} = \frac{b}{\lambda} \tag{2.2}$$

$$\frac{v-1}{k-1} = \frac{r}{\lambda} \tag{2.3}$$

Lecture 3:

References

[1] Douglas R. Stinson. *Combinatorial designs: constructions and analysis.* Springer, 2004.