A discrete-time random process is a probability space which can be denoted by $(Q_0, Q_1, Q_2, \ldots)$, where each Q_i takes values in some set S. The elements of the space are sequences $(q_0, q_1, q_2, \ldots)$ where each $q_i \in S$. We use h_t to denote $(q_0, q_1, \ldots, q_t)$, which is called the history up to step t of the process. Let H_t be the random counterpart of h_t.

We care about a sequence of random processes indexed by n $(n=1,2,\ldots)$. Thus, $q_t = q_t^{(n)}$ and $S = S^{(n)}$ (We drop n from the notation if it's unambiguous) Asymptotics refers to $n \to \infty$.

$S^{(n)t}$ denotes the set of all $h_t = (q_0, \ldots, q_t)$ where each $q_i \in S^{(n)}$

We say a function $f(u_1, \ldots, u_j)$ satisfies a Lipschitz condition on $D \subseteq \mathbb{R}^j$ if there exists a constant L s.t.

$$|f(u_1, \ldots, u_j) - f(v_1, \ldots, v_j)| \le L \cdot \max_{1 \le i \le j} |u_i - v_i|$$

We are interested in the trajectories of a set of r.v. $Y_1, \ldots, Y_a$ induced by a random process.

Suppose $D \subseteq \mathbb{R}^{a+1}$, define the stopping time $T_D(Y_1, \ldots, Y_a)$ by the minimum integer $t \ge 0$ such that

$$\left(\frac{t}{n}, \frac{Y_1(t)}{n}, \ldots, \frac{Y_a(t)}{n}\right) \notin D$$

Theorem (DE Method)

For $1 \le l \le a$, where a is fixed. Let $Y_l : S^{(n)+} \to \mathbb{R}$ and ← This is an r.v.

$f_l : \mathbb{R}^{a+1} \to \mathbb{R}$ such that for some constant C_0 and for

This is the transition function

all l,

$|Y_l(h_t)| < C_0 n$.

for all $h_t \in S^{(n)+}$ for all n.

Assume the following conditions hold where in (i) and (ii) D is some bounded, connected, open set containing the closure of:

$$\{(0, z_1, \ldots, z_a) : \Pr(Y_l(0) = z_l n,\ 1 \le l \le a) \neq 0 \text{ for some } n\}$$

(i) (Boundedness Hypothesis)

For some fixed $c > 0$,

$$\max_{1 \le l \le a} |Y_l(t+1) - Y_l(t)| \le c \quad \text{for all } t < T_0$$

(ii) (Trend Hypothesis)

For all $l \le a$,

$$\left| \mathbb{E}(Y_l(t+1) - Y_l(t) \mid H_t) - f_l\left(\frac{t}{n}, \frac{Y_1(t)}{n}, \ldots, \frac{Y_a(t)}{n}\right) \right| = o(1) \quad \text{for } t < T_0$$

(iii) (Lipschitz Hypothesis)

Each function f_l is continuous and satisfies a Lipschitz condition on $D \cap \{(t, z_1, \ldots, z_a) : t \ge 0\}$

Then, the following are true:

(a) For $(0, \hat{z}_1, \ldots, \hat{z}_a) \in D$, the system of ODEs

$$\frac{dz_l}{dx} = f_l(x, z_1, \ldots, z_a) \quad l = 1, \ldots, a$$

has a unique solution in D for $z_l : \mathbb{R} \to \mathbb{R}$ passing through $z_l(0) = \hat{z}_l$, $1 \le l \le a$ which extends to points arbitrarily close to the boundary of D.

Theorem (cont'd)

(b). For every $\varepsilon > 0$, a.a.s.

$$Y_\ell(t) = n\, z_\ell(t/n) + o(n)$$

uniformly for $0 \le t \le \sigma n$ and for every $1 \le \ell \le a$, where $z_\ell(x)$ is the solution in (a) with $\hat{z}_\ell = \frac{1}{n} Y_\ell(0)$, and $\sigma = \sigma(n) = \sigma(1)$ is the supremum of x to which the solution can be extended before reaching ~~..~~ within l^∞-distance ε of the boundary of $\mathcal{D}$.

Back to the toy example!

We proved that $\mathbb{E}(X(i+1) - X(i) \mid H_i) = -\frac{X(i)}{n}$ (a)

Define $f(x, z) = -z$

We also have $0 \le X(i) \le n$ $\forall\ 0 \le t \le cn$ (b)

$X(0) = n$ (c)

$|X(t+1) - X(t)| \le 1$ for all $0 \le t \le cn$ (d)

Let $\mathcal{D} \subseteq \mathbb{R}^2$ such that $\mathcal{D} = \{(x, z) : -2c < x < 2c,\ -2 < z < 2\}$

(d) $\Rightarrow$ Boundedness Hypothesis

(a) $\Rightarrow$ Trend Hypothesis

(c) $\Rightarrow$ $\mathcal{D}$ contains all points $(0, z)$ such that $P(X(0) = zn) \neq 0$ for some n. Moreover, $\mathcal{D}$ is bounded, connected & open

(b) $\Rightarrow$ $|X(t)| \le C_0 n$ for some constant $C_0 > 0$. It is easy to see that $f(x, z) = -z$ is Lipschitz on $\mathcal{D}$.

By theorem, $\frac{dz}{dx} = -z$ has a unique solution satisfying $z(0) = 1$, which extends to points arbitrarily close to boundary of $\mathcal{D}$. Indeed, we know $z(x) = e^{-x}$

By part (ii) of the ~~DE~~ DE method, a.a.s
$X(i) = nZ(i/n) + o(n)$
$= n.e^{-i/n} + o(n)$
uniformly for all $0 \le i \le cn$.

Min-degree graph process:
Start with an empty graph on vertex set $[n]$. In each step, u.a.r choose a vertex with minimum degree, call it u, and then u.a.r choose a vertex not yet adjacent to u, call it v. Add an edge b/w u and v. Partition the graph process into phases. For $k \ge 0$, let phase k consist of the steps where $d(u) = k$.

We will consider phase 0 first. Let G_t be the graph obtained after step t and $Y_\ell(t)$ be the # of vertices of degree ℓ in G_t. again, let $H_t = (G_0, G_1, \dots, G_t)$.
In each step of phase 0: (let u_t, v_t denote the vertices u and v chosen in step t)

$$\mathbb{E}(\mathbb{1}\{d(v_{t+1}) = i\} \mid H_t)$$

$$= \frac{Y_i(t)}{n-1}$$

$Y_i(t)$ ← Total vertices with $d(v) = i$
$n-1$ ← total # of vertices

Let $X_i(t+1) = \mathbb{1}\{d(v_{t+1}) = i\}$.

$$Y_0(t+1) = Y_0(t) - 1 - X_0(t+1)$$

$$Y_1(t+1) = Y_1(t) + 1 + X_0(t+1) - X_1(t+1)$$

$$Y_i(t+1) = Y_i(t) + X_{i-1}(t+1) - X_i(t+1) \quad \text{for } i \ge 2$$

$$\Rightarrow Y_i(t+1) = Y_i(t) - \delta_{i0} + \delta_{i1} + X_{i-1}(t+1) - X_i(t+1) \quad \text{for all } i \ge 0$$

by setting $X_{-1}(t) = 0$ for all t.

$$\Rightarrow \mathbb{E}\left(Y_i(t+1) - Y_i(t) \mid H_t\right)$$

$$= -\delta_{i0} + \delta_{i1} + \underbrace{\frac{Y_{i-1}(t)}{n-1}}_{X_{i-1}(t+1)} - \underbrace{\frac{Y_i(t)}{n-1}}_{X_i(t+1)}$$