Planarity

Def'n (Planar graph, planar embedding)
Planar graphs are ones with a drawing where no edges cross each other. Such a drawing is called a planar embedding.

Def'n (Plane Graph)
We call a planar embedding of a graph a plane graph. We will use points and lines to refer to the representation of vertices and edges in the embeddings, respectively.

Def'n (Curve)
A curve is the continuous image of a closed unit line segment

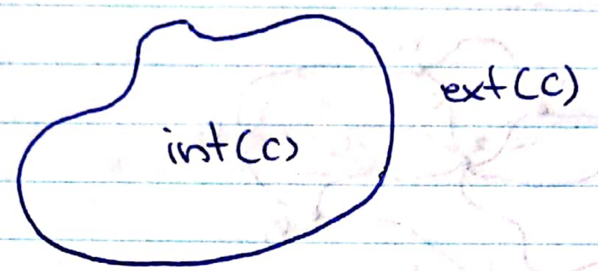
Def'n (Closed Curve)
A closed curve is the continuous image of a unit circle
Note: Such a curve is simple if it doesn't intersect itself (mapping is one-to-one)

Def'n (Arcwise-Connected)
A set of points is arcwise-connected if any 2 of its points form the endpoints of a curve that is entirely inside the set

Def'n (Region)
A region is a maximal arcwise-connected set.

Jordan Curve Theorem:

Any simple closed curve in the plane partitions the rest of the plane into 2 disjoint arcwise-connected open sets: one inside, one outside.



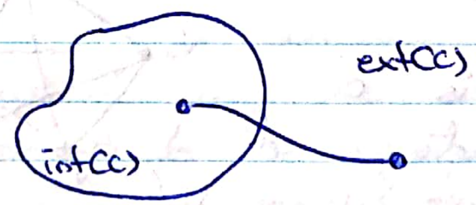
Def'n (Interior, Exterior)

The open set inside a simple closed curved C is its *interior*, denoted int(C). The open set outside of C is its *exterior*, denoted ext(C). (These do not include C).

Together, with C, we call them [redacted] Int(C), Ext(C) respectively.

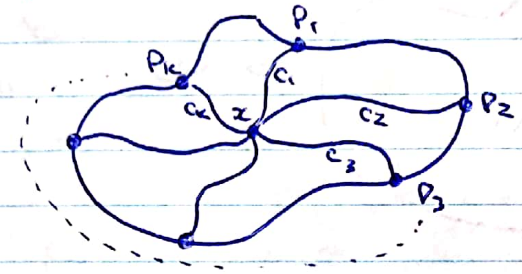
Some assumptions:

1) Any curve between a point in int(C) and a point in ext(C) must intersect C.

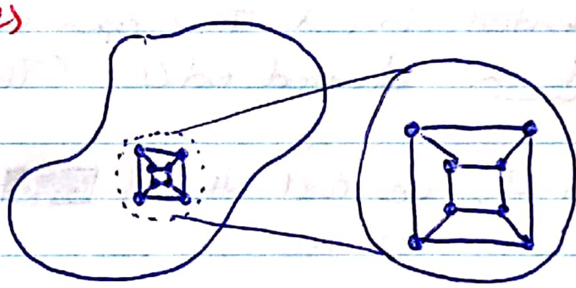


Con't

2) If $p_1, \ldots, p_k$ are distinct points on C, and $x \in \mathrm{int}(C)$, then there exist curves $c_1, \ldots, c_k$ where c_i joins x to p_i. $c_i \subseteq \mathrm{Int}(C)$, and they do not intersect except at x.



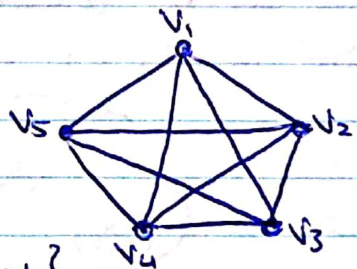
3) Any planar embedding can be redrawn in int(C)



Theorem 16.1:

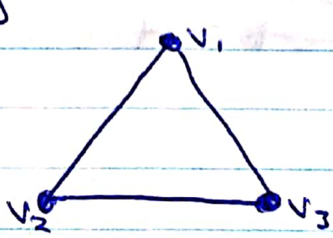
K_5 is not planar

[Proof]

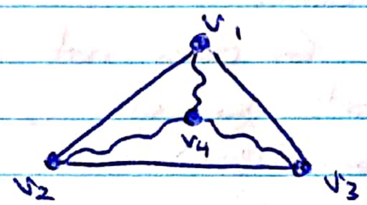


Suppose there is a planar embedding of K_5. Let $V(K_5) = \{v_1, \ldots, v_5\}$.

Let C be the simple closed curve, formed by the cycle $v_1 v_2 v_3 v_1$.



[Proof] (cont)
The point v_4 is either in $\mathrm{int}(C)$ or $\mathrm{ext}(C)$. WLOG, say $v_4 \in \mathrm{int}(C)$. Then, the lines v_4v_1, v_4v_2, v_4v_3 are entirely in $\mathrm{int}(C)$, except at v_1, v_2, v_3.



Let C_1, C_2, C_3 be the closed curves from the cycles $v_4v_2v_3v_4$, $v_4v_1v_3v_4$, $v_4v_1v_2v_4$. Note that C_1, C_2, C_3 do not include v_1, v_2, v_3 respectively. (So, $v_i \in \mathrm{ext}(C_i)$ for each i, since $\mathrm{int}(C_i) \subseteq \mathrm{int}(C)$)
Since there is a line joining v_5 to v_i for each i, v_5 is in $\mathrm{ext}(C_i)$ for each i. So $v_5 \in \mathrm{ext}(C)$. But there is a line joining $v_4 \in \mathrm{int}(C)$ and $v_5 \in \mathrm{ext}(C)$, which is a contradiction. $\square$.

Theorem 16.2:
$K_{3,3}$ is not planar.
[Proof] Similar argument, as above, can be made $\square$

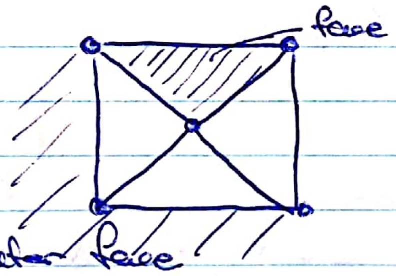
Def'n (Subdivision)
A subdivision of G is obtained by replacing each edge with a new path of length at least 1.
(i.e. Introducing new vertices of deg 2 to the edges)

Proposition 16.1:
G is planar iff every subdivision of G is planar.

Corollary 16.1:
Any graph containing a subdivision of K_5 or $K_{3,3}$ is not planar.

Kuratowski's Theorem:
G is planar iff G does not contain any subdivision of K_5 or $K_{3,3}$.
[Proof] (See Theorem 23.1)

Faces:
Def'n (face, outer face, incident, adjacent)
A face in a planar embedding is a maximal subset of points that are arcwise-connected, and do not include any part of the embedding. Every embedding has one unbounded face called the outer face.



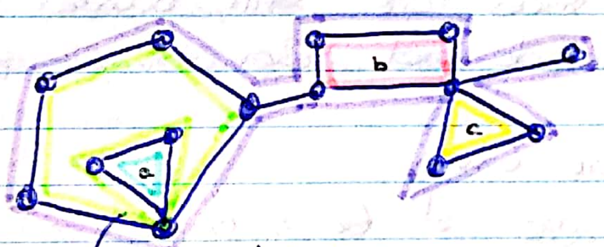
Each face is incident with the vertices and edges on its boundary. Two faces are adjacent if they share at least one edge in their boundaries.

Proposition 16.2:
A graph is embeddable on a sphere iff it is embeddable on a plane.
(We use stereographic projection from sphere to planes)

Proposition 16.3:
Let f be a face of a plane graph G. Then, there is a planar embedding of G, where the boundary of f is the boundary of the outer face.

Note: These 2 propositions together tell us that we can make any face the outer face.

We now look at the boundary walk.
Ex:

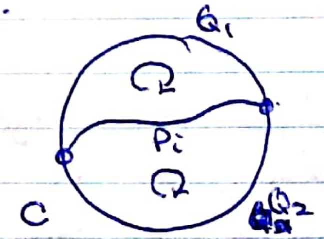


Notice that faces a, b, c are bounded by a cycle!

*Note that this is not a boundary walk, it touches one vertex twice!

Theorem 17.1
In a 2-connected plane graph G, each face is bounded by a cycle.
(i.e. The boundary walks are all cycles)
[Proof]
Let $(G_0, \ldots, G_k)$ be an ear decomposition of G, where G_{i+1} is obtained from G_i by adding the ear P_i.
Now, G_0 is a cycle, which divides the plane into 2 faces (Jordan Curve Thm), and so G_0 is the cycle that bounds both faces

[Proof] (con't)
Suppose each face of G_i is bounded by a cycle. Since G_{i+1} is a plane graph, the curve representing P_i must be entirely inside one face f, which is bounded by a cycle C.



The 2 endpoints of P_i divides C into 2 paths Q_1 and Q_2. Then f is divided into 2 faces, one bounded by $Q_1 + P_i$ and the other bounded by $Q_2 + P_i$, both are cycles.
The remaining faces are still bounded by cycles since we do not touch them ☐.

Def'n (Cycle Double Cover)
A <u>cycle double cover</u> of G is a set of cycles where every edge of G is in exactly 2 such cycles.
<u>Note</u>: 2-connected planar graphs have cycle double covers; we take the set of all facial cycles in a planar embedding

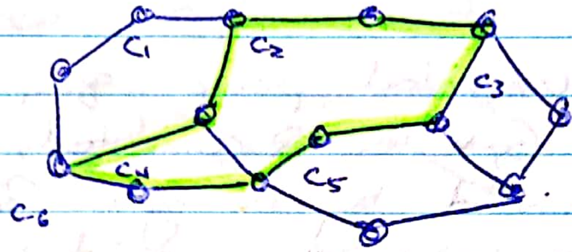
<u>Conjecture</u> 17.1:
Every 2-connected graph has a cycle double cover

⟶ Remark

Remark:

In a 2-connected plane graph, every cycle is a sum of the facial cycles in its interior.

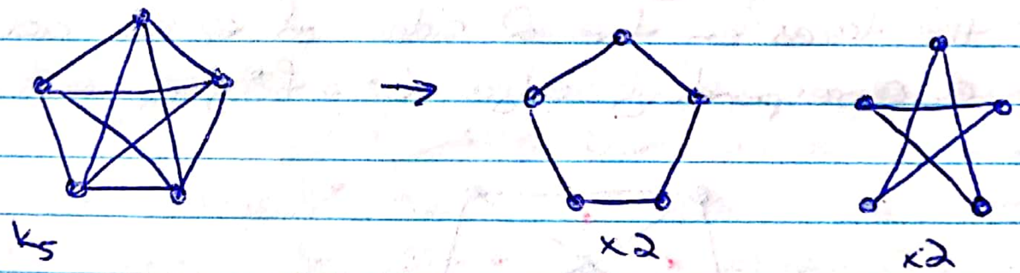
Ex!



The cycle in green is $C_2 + C_4$ (Interior) or $C_1 + C_3 + C_5 + C_6$ (but C_6 is exterior)

So, the cycle space is spanned by the facial cycles, which is a cycle double cover. However, this doesn't happen in nonplanar graphs! (i.e. No cycle double cover spans the cycle space if the graph is nonplanar). This is a consequence of Maclane's Theorem. (Theorem 26.1)

Ex!



is a cycle double cover, but does not span the cycle space.

$\hookrightarrow$ Corollary

Corollary 17.1:
In a 3-connected plane graph, all neighbours of a vertex lie on a common cycle.

[Proof]
Let G be a 3-connected plane graph. Let $v \in V(G)$. Then, $G-v$ is 2-connected, so every face is bounded by a cycle. Consider the face that contains the point v. Then, all neighbors of v must be in the face, hence they are in a cycle. ∎

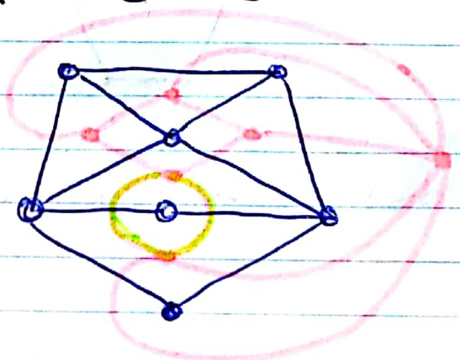
Dual Graphs:

Def'n

Given a plane graph G, define the dual G^* as follows:

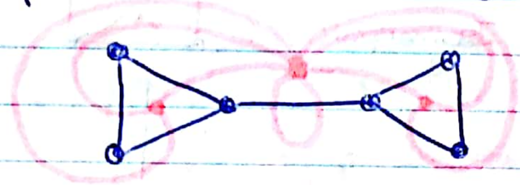
- Each face f of G corresponds to a vertex f^* in G^*
- For each edge e in G, if f, g are the faces on the 2 sides of e, we add a corresponding edge $e^* = f^* g^*$.

Ex:



Note(s):

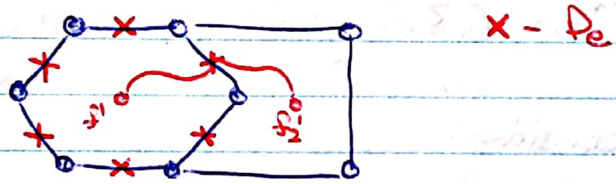
- Multiple edges may occur when 2 faces share more than one common edge (See green in previous example)
- Loops occur when there is a cut edge



Proposition 18.1:

If G is a plane graph, then G^* is a plane graph.

[Proof]

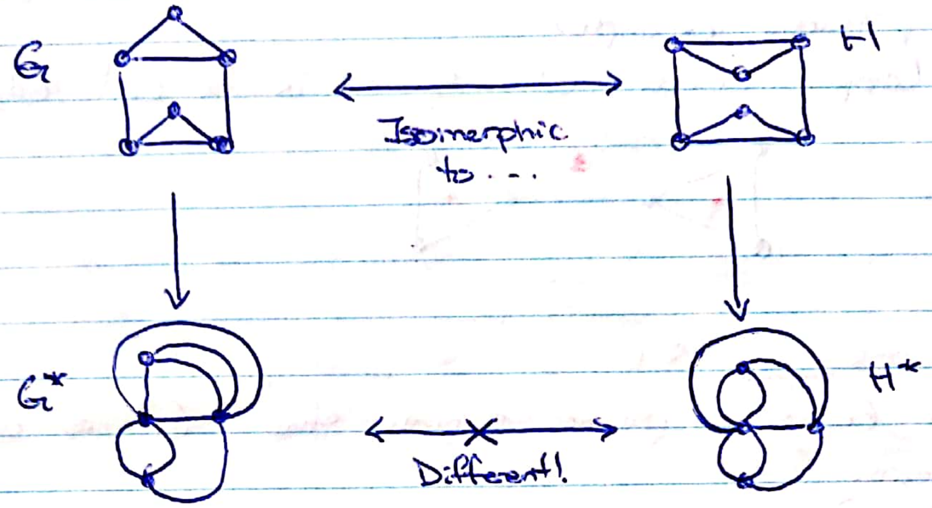


For each edge e in G, select a point p_e on the line for e. For each face f, we place f^* in the interior of the boundary of f. For each e in the boundary of f, draw a curve joining f^* to p_e that is in f. We can do so such that all curves from f^* to p_e do not cross. Joining the curves of either side of e creates a line for e^* in G. □

Con't

Dual graphs are dependent of the embedding

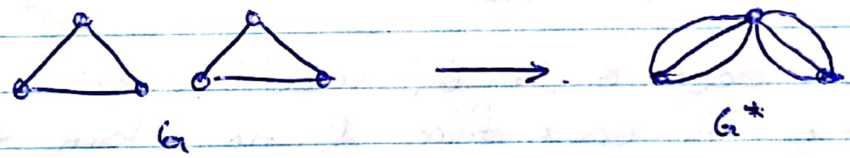
Ex:



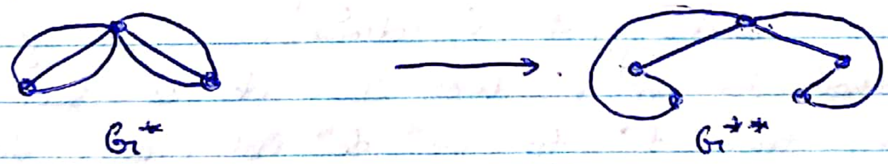
Q: Is $G^{**} = G$?

A:

We can consider:



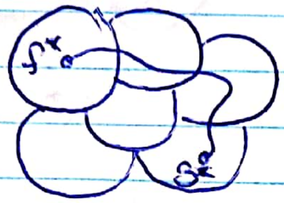
But,



So, $G^{**} \neq G$ always. Can we impose a stronger condition so that this is always true?

$\hookrightarrow$ Proposition

Proposition 18.2
The dual of any plane graph is connected
[Proof]
Let G be a plane graph, let G^* be a plane dual of G. Let f^*, g^* be 2 vertices of G^*



We alternate between faces and edges.

There is a curve in the plane connecting f^* and g^*, which avoids any vertex (in G). Consider the sequence of faces and edges of G that this curve intersects. This corresponds to a walk from f^* to g^* in G^*. So, G^* is connected □.

Corollary 18.1:
If G is a connected plane graph, $G^{**} = G$.
(i.e. Our question holds true if G connected).

More properties:
Assume G is a plane graph, G^* dual, $F(G)$ is the set of faces of G

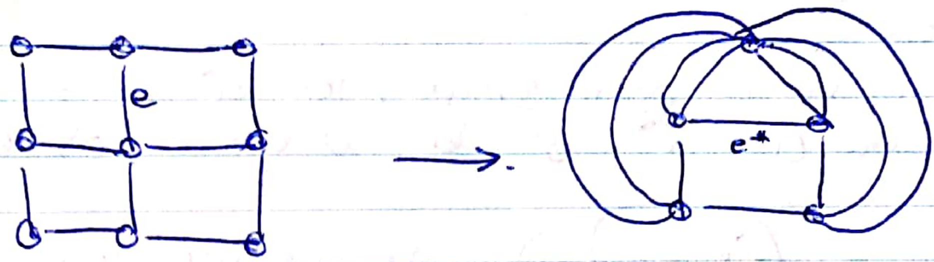
1) $|V(G)| = |F(G^*)|$ (G connected)
$|F(G)| = |V(G^*)|$
$|E(G)| = |E(G^*)|$

2) $\deg(f)$ in G is the same as $\deg(f^*)$ in G^*

Hilroy

Deletion and contraction:

Ex:

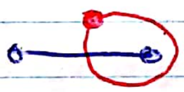


G $\qquad\qquad\qquad\qquad$ G^*

Notice that:

- Deleting e is equivalent to contracting the edge between the 2 vertices in the dual.
- Contracting e is equivalent to deleting the edge between the 2 vertices in the dual.

Exception: What if e is a cut edge?



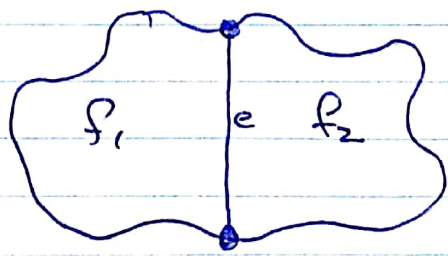
→ The loop disappears!

Proposition 18.2

Let G be a connected plane graph, let e be an edge. If e is not a cut-edge then $(G-e)^* = G^*/e^*$. And, if e is not a loop, then $(G/e)^* = G^* - e^*$.

[Proof]

Since e is not a cut-edge, the 2 sides of e are different faces, say f_1, f_2



Cont

[Proof] (Con't)

In $G-e$, f_1, f_2 merge into a new face f. The boundary of f consists of boundaries of f_1 and f_2, except e. In G^*, this corresponds to a new vertex f^* replacing f_1^*, f_2^* with f^* incident with the same set of edges as f_1^*, f_2^* except e^*. This is the same as G^*/e^*.

Since G is connected, $G^{**} = G$. We see that e^* is not a cut-edge (since e is not a loop). Then $(G^*-e^*)^* = G^{**}/e^{**} = G/e$.

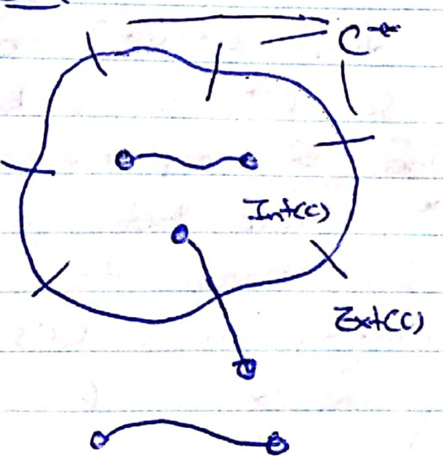
$\square$

Remark: If e is a cut-edge, then e^* is a loop in G^*, so $(G-e)^* = G^*-e^*$.

Theorem 19.1:
Let G be a connected plane graph. If C is a cycle of G, then C^* is a bond in G^*. If B is a bond in G, then B^* is a cycle in G^*.

Notation: If F is a set of edges in G, we write $F^* = \{e^* \mid e \in F\}$, which is a subset of $E(G^*)$

Idea:

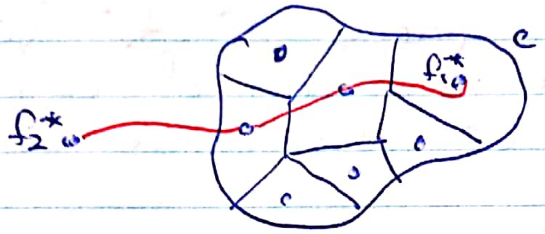


We first show that C^* is a cut (i.e. splits into interior and exterior). Then, we can show that it is a bond because there are exactly 2 components.

Proof

[Proof]

Suppose C is a cycle in G.
Let $X^* = \{f^* \mid f \text{ is a face in } \mathrm{int}(C)\}$.
(i.e. A subset of vertices in G^*)



Let $\bar{X}^* = \{f^* \mid f \text{ is a face in } \mathrm{ext}(C)\}$. Both must be nonempty since there is at least one face inside and one face outside. Pick $f_1^* \in X^*$ and $f_2^* \in \bar{X}^*$. Consider a curve from f_1^* to f_2^* that does not cross any vertices in G. The sequence of faces and edges the curve intersects in G corresponds to a f_1^*, f_2^*-walk in G^*. By Jordan Curve Theorem, the curve must intersect an edge e in C, which corresponds to using e^* in a walk. Since $e^* \in C^*$, any walk from X^* to $\bar{X}^*$ must use an edge in C^*, so C^* is a disconnecting set, and in fact $C^* = \delta(X^*)$.

Now, we want to show that $G^*[X^*]$ and $G^*[\bar{X}^*]$ are connected, so C^* is a bond since there will only be 2 components. For any f_3^*, f_4^* in X^*, since both points are in $\mathrm{int}(C)$, there is a curve from f_3^* to f_4^* contained entirely in $\mathrm{int}(C)$, without intersecting any vertex in G.

(cont)

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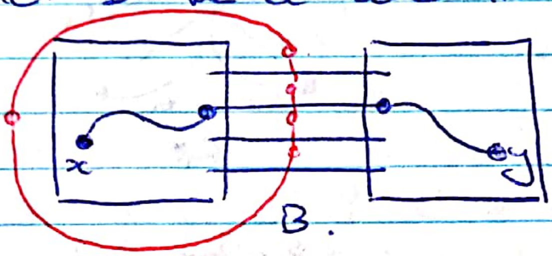
[Proof] (Con't)

The sequence of faces and edges the curve intersects gives an f_3^*, f_4^*-walk in $G^*[X^*]$. So $G^*[X^*]$ is connected. Similarly, we can show that $G^*[\bar{X}^*]$ is also connected, so C^* is a bond.

Now, let B be a bond in G.



Let x, y be vertices in different components of $G-B$. Any xy-walk uses an edge in B, so any curve from x to y must intersect an edge in B^*. So B^* must contain a cycle in order to separate x from y.
If B^* contains additional edges, then $C \subsetneq B$ (Since $C^{**} = C$ is a bond in G). But, this contradicts that B is a bond. So, B^* is a cycle. □

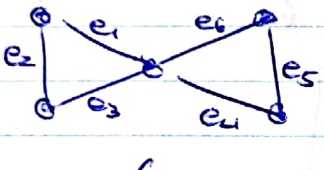
Corollary 19.1:
For a connected plane graph G, $C(G) = C^*(G^*)$. Also, $C^*(G) = C(G^*)$.

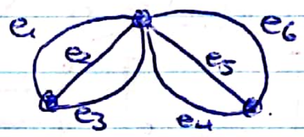
[Proof]
$C(G)$ is spanned by the cycles of G. They correspond to bonds in G^*, which span $C^*(G^*)$. Also bonds in G^* correspond to cycles in G. So $C(G) = C^*(G^*)$. And, by taking the orthogonal complement on both sides, we get $C^*(G) = C(G^*)$ □

Def'n (Abstract Dual)

Let G be a graph. Then, G^* is an abstract dual of G if $E(G) = E(G^*)$ (i.e. there is a bijection between the two), and the set of all cycles in G is the same as the set of all bonds in G^*

Ex:





Cycles:
$\{e_1, e_2, e_3\}$
$\{e_4, e_5, e_6\}$

Bonds:
$\{e_1, e_2, e_3\}$
$\{e_4, e_5, e_6\}$

Theorem 20.1:

If G^* is an abstract dual of G, then $C(G) = C^*(G^*)$ and $C^*(G) = C(G^*)$

[Proof]

See proof of Corollary 19.1, however, no mention of embeddings are relevent here

◻

Corollary 20.1:

If G^* is an abstract dual of G, then G is an abstract dual of G^*

(i.e. If cycles of G are the same as the bonds of G^*, then bonds of G are the same as the cycles of G^*)

↳ [Proof]

[Proof]
Let F be a bond of G, so $F \in C^*(G)$. Then, by Theorem 20.1, $F \in C(G^*)$.
If F is a cycle, then we're done.
Otherwise, $F \in C(G^*)$, so F is even, and hence must contain some cycle $C \subsetneq F$. But $C \in C(G)^* = C^*(G)$; then C is a cut that is a proper subset of a bond F, a contradiction. Hence, F must be a cycle.

Now, let C be a cycle in G^*, so $C \in C(G^*) = C^*(G)$ is a cut in G. If C is not a bond, then a proper subset of C is a bond that is in $C(G^*)$, which is not possible since no proper non-empty subset of a cycle is in the cycle space $\square$.

Q: What types of graphs have abstract duals?
A: We've seen that planar graphs have planar duals, in fact, we will see that ONLY planar graphs have abstract duals. This is a consequence of Whitney's Theorem (See Theorem 27.1).

Bridges:
Goal: We want to prove that 3-connected planar graphs have a unique planar embedding, and hence a unique dual. (By unique, we mean that all faces will have the same face boundaries).
This will lead up to Kuratowski's Theorem!

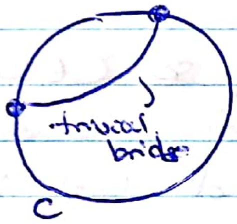
Def'n (Bridge)

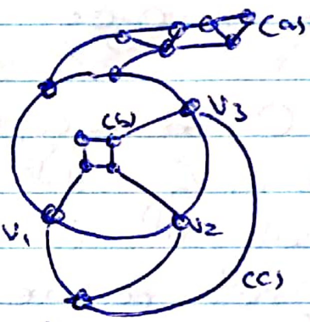
Let C be a cycle of G. A bridge of C is either:

(i) A component H of $G - V(C)$ along with the edges H to C (include the endpoints of these edges), or

(ii) An edge, not in C, joining 2 vertices of C (called trivial bridge)

Ex:





All of (a), (b), (c) are bridges.

Def'n (Vertices of attachment, Internal Vertices, k-bridge, Equivalent)

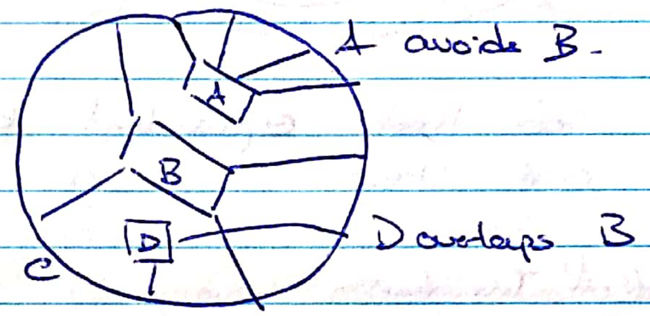
For a bridge B of C, the vertices in both B and C are vertices of attachment (VOA). The remaining vertices of B are the internal vertices. The bridge with k VOA is called a k-bridge. Two bridges with the same VOA are equivalent.

Ex:

v_1, v_2, v_3 are VOA of (b), and the remaining are internal vertices. It is also a 3-bridge, and is equivalent with (c).

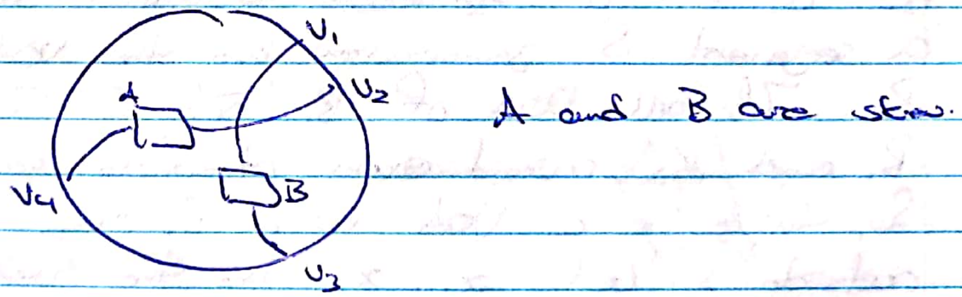
Def'n (Segment, avoid, overlap)
The k VOA of a k-bridge partitions C into k segments. Two bridges avoid each other if the VOA of one bridge is entirely within a segment of another bridge. Otherwise, they overlap. (So, the bridge spreads across more than one segment)

Ex!



Def'n (Skew)
Two bridges are skew if there is a sequence of distinct vertices v_1, v_2, v_3, v_4 in C in cyclic order such that v_1, v_3 are VOA of one bridge and v_2, v_4 are VOA of the other.

Ex!

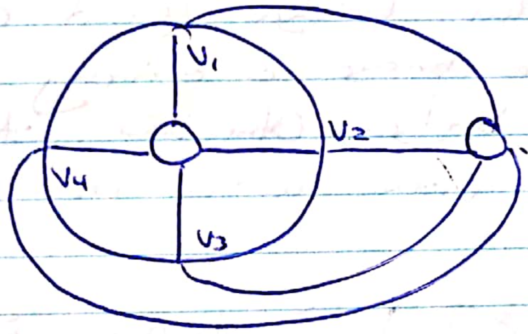


Theorem 21.1:
Overlapping bridges of a cycle C are either skew or equivalent 3-bridges.

(→ [Proof], and Not:)

Note!

What about equivalent 4 (or higher)-bridges



So, we see that equivalent k-bridges ($k \geq 4$) are skew.

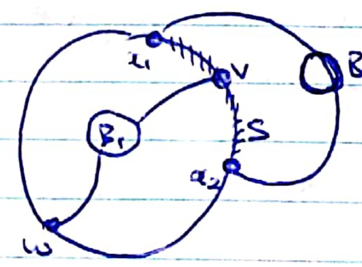
[Proof] (of Theorem)

Let B_1, B_2 be overlapping bridges, then they both need at least 2 VOA each.

We have 2 cases:

Case I: (B_1, B_2 are not equivalent)

WLOG, there is a VOA v in B_1, that is not a VOA of B_2. [redacted] v must be in a segment S generated by the VOA of B_2. If all VOA of B_1 are in S, then B_1 and B_2 avoid each other, a contradiction. So, there is a VOA w of B_1 that is outside S. Let x_1, x_2 be the vertices of B_2 that create the segment S. Then, w, x_1, v, x_2 create the skew.



Con't

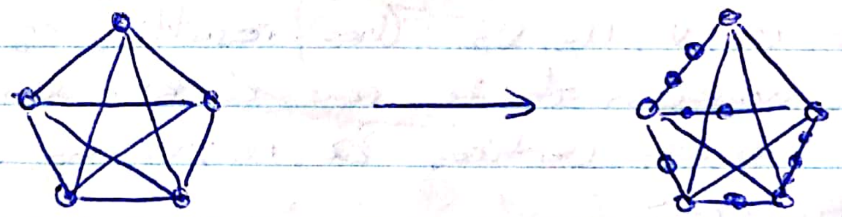
[Proof] (Con't)

Case 2: (B_1, B_2 are equivalent k-bridges)
We know k is at least 2.
If $k=2$, they do not overlap each other, a contradiction.
If $k \geq 4$, as described in the note, they are skew, and v_1, v_2, v_3, v_4 form the skew.
So $k=3$.

$\square$

Note: Given a planar embedding and a cycle C, each bridge of C is either in $Int(C)$ or $Ext(C)$. They are called inner bridges or outer bridges respectively.

We now relate bridges to planarity.
Recall:
We call an edge subdivision of G, the graph obtained by replacing edges of G by paths of length 2 or more
Ex:



Def'n (Branch Vertices)
In a subdivision, the branch vertices are the ones with degree at least 3
We will describe our subdivisions by giving the set of branch vertices.

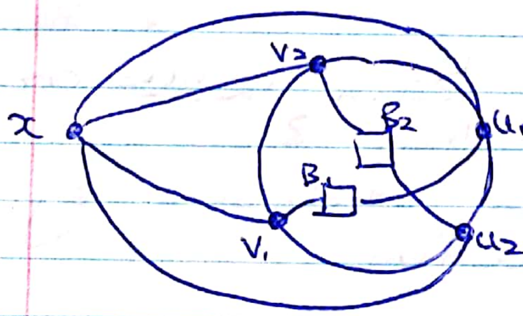
Theorem 21.2:
If G is a plane graph and C is a cycle of G, then all inner and outer bridges of C avoid each other

[Proof]

Let B_1, B_2 be 2 overlapping bridges of C. By Thm 21.1, B_1, B_2 are either skew or equivalent k-bridges.

Case I: (B_1, B_2 are skew)
Then, there exist VOA u_1, v_1 of B_1 and VOA u_2, v_2 of B_2 s.t. u_1, u_2, v_1, v_2 are in cyclic order in C.

Since B_1, B_2 are connected, there exist a u_1, v_1-path P_1 in B_1 and a u_2, v_2-path P_2 in B_2, and both paths are ~~in~~ Int(C). (We can assume B_1, B_2 are inner bridges WLOG) Let H be the subgraph that consists of $C \cup P_1 \cup P_2$. Add a vertex x to Ext(C) to H, and draw non-intersecting lines joining x to u_1, v_1, u_2, v_2. The resulting graph is plane, however it is now a K_5 subdivision with branch vertices $\{x, u_1, v_1, u_2, v_1\}$, a contradiction.

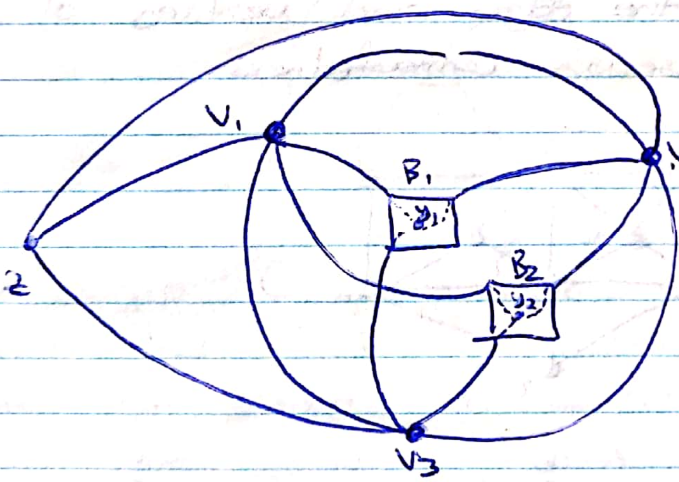


Ex. v_2 reaches u_1 by a path in the direction of the cyclic order, v_1 by a path in the reverse direction, u_2 by P_2, and x by the newly formed edge xv_2.

Cont

[Proof] (con't)
Since x was introduced so that no edge crossings would appear in ext(C), the edge crossings must occur in Int(C).

Case 2: (B_1, B_2 are equivalent 3-bridges)
Let v_1, v_2, v_3 be the v.o.a. of B_1 and B_2. There exist a 3-fan F_1 in B_1 from y_1 to $\{v_1, v_2, v_3\}$, and a 3-fan F_2 in B_2 from y_2 to $\{v_1, v_2, v_3\}$. Let H be $C \cup F_1 \cup F_2$, and we add a vertex z in ext(C) to H, and draw non-intersecting lines from z to $\{v_1, v_2, v_3\}$. The resulting graph is a $K_{3,3}$ subdivision with branch vertices $\{v_1, v_2, v_3\}$ and $\{z, y_1, y_2\}$ forming the bipartition, a contradiction.



Ex: v_1 is adjacent to z, and there is a v_1, y_1-path from F_1 and a v_1, y_2-path from F_2.

□

Unique embeddings of 3-connected planar graphs

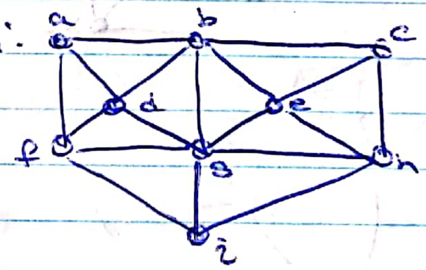
Def'n (Equivalent)
Two planar embeddings are equivalent if they have the same set of edges as face boundaries

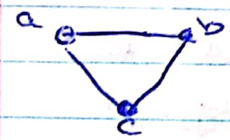
Def'n (Unique embedding)
A planar graph has a unique embedding if every embedding is equivalent.

Def'n (Induced Cycle, Non-separating)
An induced cycle is a cycle that is an induced subgraph. It is non-separating if removing the edges and vertices of the cycle preserves connectedness

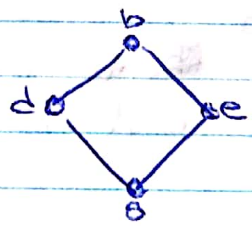
Ex:

G:

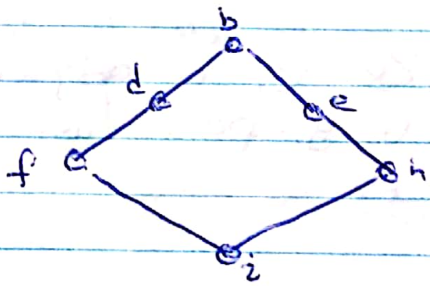




Induced,
Non-separating.



Not induced

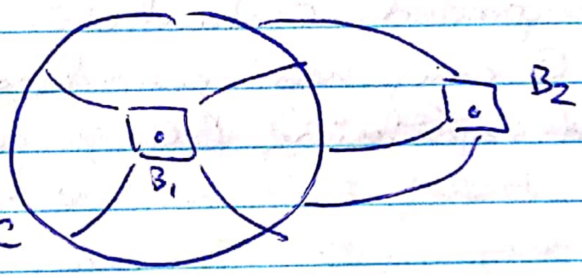


Induced,
Separating

Theorem 22.1:
A cycle C in a 3-connected plane graph G is a facial cycle iff C is an induced non-separating cycle in G

[Proof]

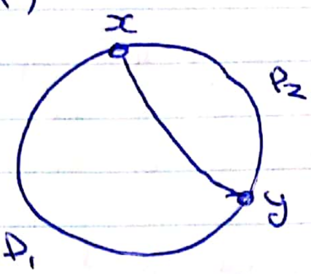
($\Leftarrow$) Suppose C is not a facial cycle, then there exist an inner bridge B_1 and an outer bridge B_2. If either bridge is trivial, then C is not an induced cycle, so this case never happens.
Otherwise, B_1, B_2 each contain a vertex not in C, which are in different components of $G - V(C)$, but then C separates these 2 vertices, contradiction.



($\Rightarrow$) Suppose C is a facial cycle. We may redraw G so that C is the outer face (Prop'n 16.2, 16.3), then all bridges of C are inner bridges.
If there exists a trivial bridge xy, then $\{x, y\}$ partitions C into two paths P_1 and P_2. Since G is 3-connected, there must be a path between the internal vertices of P_1 and P_2. But, such a path is a bridge that overlaps xy. Contradiction, since G is plane. So, C is induced

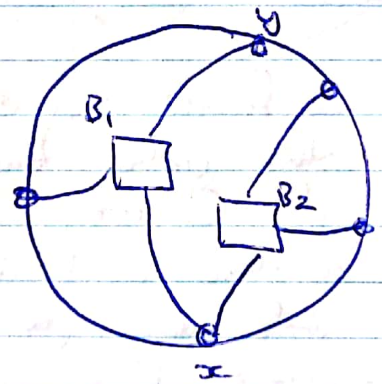
Con't

[Proof] (Con't)



Now, suppose there are 2 nontrivial bridges in C, B_1 and B_2. G is plane, so B_1, B_2 must avoid each other. Suppose x, y are the 2 ends of a segment formed by VOA of B_1 that contains all VOA of B_2. But, then $\{x, y\}$ is a separating set of size 2, as internal vertices of B_1 and B_2 are separated, contradiction.

Hence, C only has one bridge, so it is non-separating. C is also the outer cycles. ■



□

Theorem 22.2:
Every 3-connected planar graph has a unique embedding
[Proof]
The facial cycles of a 3-connected planar graph are the induced non-separating cycles, which is an abstract structure of G that is not dependent on the embedding. So, the facial cycles are the same for all embeddings of G, hence the embedding is unique □

Corollary 22.1:
Every 3-connected planar graph has a unique dual.

Kuratowski's Theorem:
We previously proved (in M239) that if a graph has a K_5 or $K_{3,3}$ subdivision, then it is not planar, we will now prove:

Theorem 23.1 (Kuratowski's)
If G is not planar, then G has a subdivision of K_5 or $K_{3,3}$.

Def'n (Kuratowski Subgraph)
We call subdivisions of K_5 or $K_{3,3}$ Kuratowski subgraphs (KS)

Outline of proof:
We will prove the theorem by induction on the # of edges.

- Base Case: We can check that the theorem holds for all graphs with at most 6 edges (They are all planar)

Then, we will have 4 cases:

(I) If G is not connected, then one component is nonplanar, we consider only this component

(II) G is 1-connected (i.e. Has a cut vertex). We consider the nonplanar blocks of G.

(III) G is 2-connected. We consider (roughly) the components of $G-\{x,y\}$. And there are 2 subcases
- i) at least One component is nonplanar
- ii) All components are planar

(IV) G is 3-connected. Then, $\exists e$ s.t. G/e is 3-connected. Again, there are 2 subcases
- i) G/e is not planar
- ii) G/e is planar, then either G is planar (Contradiction) or uncontracting e produces a K_5.

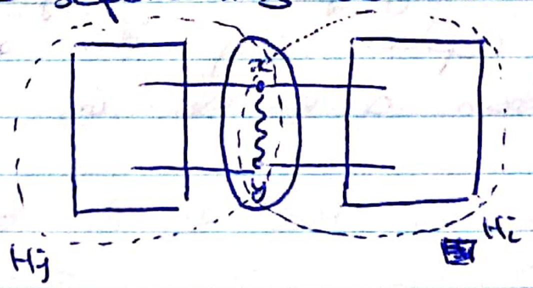
[Proof] (23.1)
Let G be any nonplanar graph

(I) If G not connected, then at least one component of G is nonplanar, and only consider that component. If the # of edges of the component is less than that of G, then by IH, that component has a KS. Otherwise, we fall into case (II), (III), or (IV).

Now, we can also assume that G is connected

(II) Suppose G has a cut vertex.
Then, G has at least 2 blocks, and at least one block B is nonplanar. Since all blocks of connected graphs are nontrivial, B has fewer edges than G. By induction, B has a KS, which is also a KS in G.

(III) Suppose G has a separating set of size 2, and let the separating set be $\{x, y\}$.



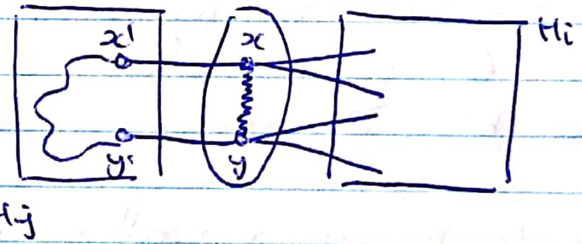
Let H_i consist of a component of $G - \{x,y\}$ together with all edges joining the component to $\{x,y\}$ and the edge xy. (We add the edge in if it is not in G).

Con't

[Proof] (cont)

Case I: (There is a subgraph H_i that is nonplanar)
There is at least one different H_j which must contain at least 2 edges from $\{x, y\}$. So H_i has fewer edges than G. (Note that even when we add xy, we still remove that and at least one other edge in H_j) So, by induction, H_i contains a KS, say J.

If J does not contain xy or xy is in G, then J is a KS in G. Otherwise, there are vertices x', y' in H_j that are adjacent to x and y, respectively.



There is an x', y'-path P in $H_j - \{x, y\}$, and so we can replace xy with $xx' + P + yy'$ in J to obtain a KS for G.

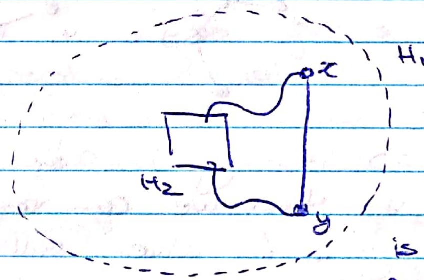
Cont

[Proof] (Con't)

Case 2: (All H_i's are planar)
We will find an embedding for G.
Start with an embedding of H_1, and now inductively create an embedding for $H_1 + H_2 + \dots + H_{i-1}$.

Consider an embedding of H_i where xy is on edge of the outer face (We can do this because H_i is planar) Now, pick a face in the embedding of $H_1 + H_2 + \dots + H_{i-1}$ that contains xy and put H_i in that face. This creates an embedding of $G + xy$, which is a contradiction, since G is non-planar.

Ex:



is a planar embedding of H_1 and H_2.

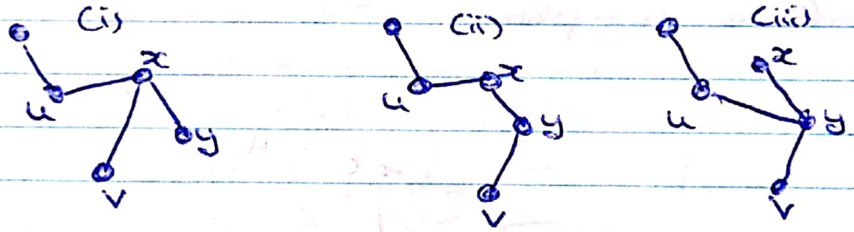
(IV) Suppose G is 3-connected.
Then, there is an edge e such that G/e is 3-connected. Let z be the contracted vertex and $e = xy$.

Con't

[Proof] (con't)

Case I: (G/e is not planar)
Since G/e has fewer edges than G, by induction, G/e has a KS H. If H does not contain z, then H is a KS of G and we are done.

Otherwise, H does contain z, and z can be a branch vertex or not.
Suppose z is not a branch vertex, then let u, v be the 2 neighbours of z in H. Uncontracting z gives one of 3 cases:

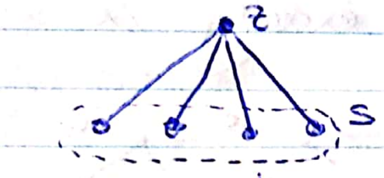


In G, if x or y is adjacent to to both u, v, then we can replace z with x or y in G, which still gives us a KS. (This is case (i) and (iii)).
Otherwise, WLOG, u is adjacent to x and v is adjacent to y. Then we can replace u, z, v in H with u, x, y, v in G to obtain a KS in G.

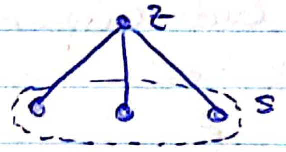
Con't

[Proof] (con't)

Now, suppose z is a branch vertex.
Let S be the set of neighbours of z in H.

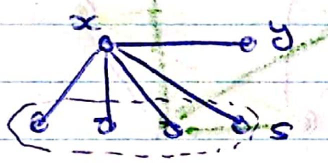


K_5 subdivision

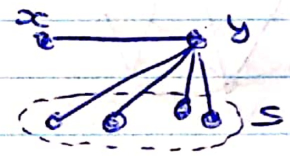


$K_{3,3}$ subdivision

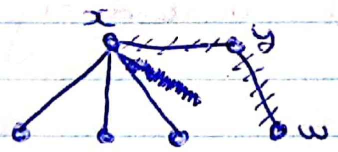
In G, if x or y is adjacent to all vertices in H, then we can replace z with such a vertex to get a KS for G:

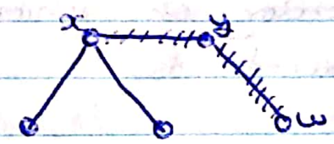


[OR]



Suppose WLOG that x is adjacent to all but one vertex in S. So, yw is an edge in G. We can replace z with x, and edge zw with the path xyw to obtain a subdivision of H, which is now a KS in G.

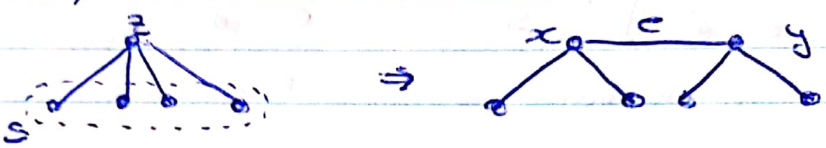




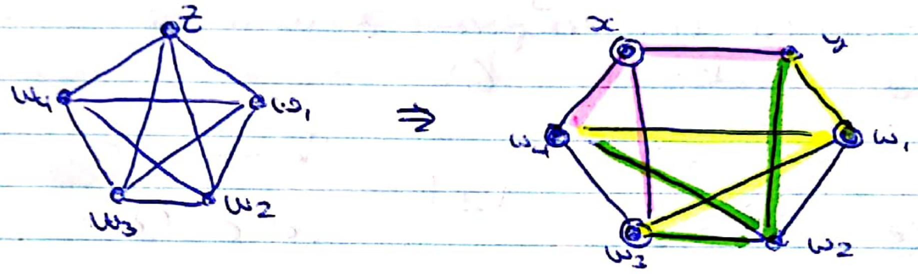
Con't

[Proof] (Con't)

The only case remaining is when H is a K_5 subdivision, and both x, y are adjacent to exactly 2 vertices in S, i.e.:



Notice that:



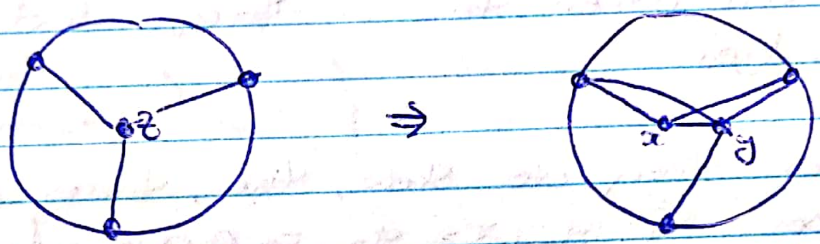
If $\{w_1, w_2, w_3, w_4\}$ are the other 4 branch vertices of H, then we can find a $K_{3,3}$ subdivision in G using $\{x, w_1, w_3\}$ and $\{y, w_2, w_4\}$ as branch vertices. (Note that we can rearrange the w_i's, depending on which vertices x and y are adjacent to)

Con't

[Proof] (cont)

Case 2: (G/e is planar)

We know that G/e is 3-connected, so $(G/e)-z$ is 2-connected. The boundary of the face in $(G/e)-z$ is a cycle C containing the point z. In G/e, all neighbours of z are in C, and so in G, all neighbours of xy are also in C.

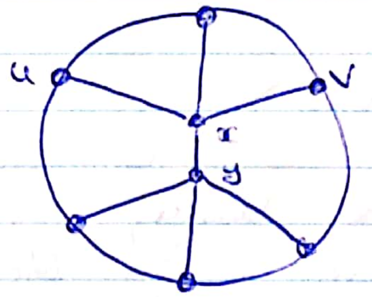


Let H_x, H_y be two subgraphs of G that consist of x, y along with their neighbours in C, respectively. So, both H_x and H_y are bridges of C in $G-e$.

Suppose H_x, H_y avoid each other. We show that we can create a planar embedding. First, embed H_x by placing x inside C, and draw a fan from x to all its VOA. Since H_x and H_y avoid each other, all VOA of H_y are inside a segment of H_x, say from u to v. This segment, along with xu, xv form a face, and we can embed H_y by placing y inside the face and joining y to all its VOA and x. This is a planar embedding, so contradiction. This case never happens

Cont

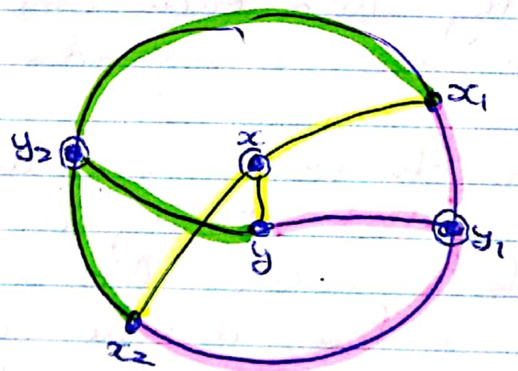
[Proof](Cont'd)



planar embedding if H_x, H_y avoid each other

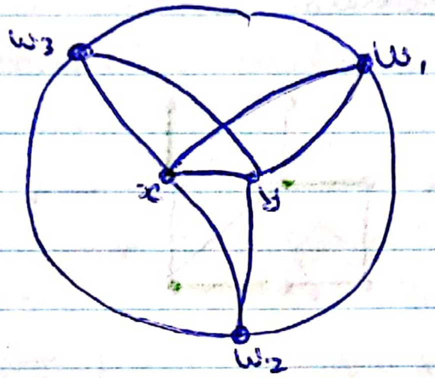
Now we can assume H_x, H_y overlap, then they are either skew or equivalent 3-bridges.

Suppose they are skew, then, there exist vertices x_1, y_1, x_2, y_2 in cyclic order where x is adjacent to x_1, x_2 and y is adjacent to y_1, y_2. Then C together with xy and the 2 paths $x_1 x x_2$, $y_1 y y_2$ form a $K_{3,3}$ subdivision with bipartition $\{x, y_1, y_2\}$ and $\{y, x_1, x_2\}$.



Cont'd

[Proof] (a⇒?)
Suppose H_x, H_y are equivalent 3-bridges with Vat $\{w_1, w_2, w_3\}$. Then $C \cup H_x \cup H_y \cup \{xy\}$ is a K_5 subdivision in G, with branch vertices $\{x, y, w_1, w_2, w_3\}$.



For graphs with connectivity 4 (or more), we are guaranteed at least a K_5 subdivision, since 4-connected implies # of vertices ≥ 5, with min. degree of 4, which is precisely a K_5 subdivision.

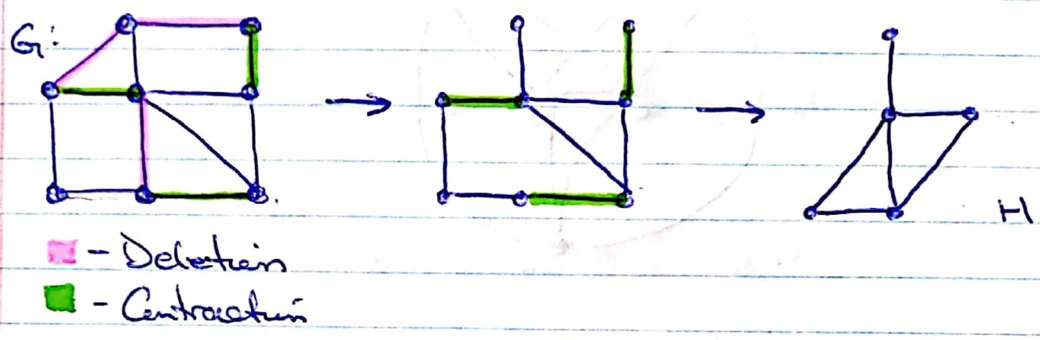
$\square$

<u>Other characterizations</u> of <u>planar graphs</u>:
We will look at 3 other characterizations of planar graphs:
1) Wagner's Theorem (25.1) - Minors
2) Maclane's Theorem (26.1) - 2-basis
3) Whitney's Theorem (27.1) - Duals

~~Co~~ Wagner's

Wagner's Theorem:

Def'n (minor)
A graph H is a minor of G, if H can be obtained from G through a series of deletions and edge contractions.
Ex:



Lemma 25.1:
If G contains a subdivision of H, then H is a minor of G.
[Proof] Contract the subdivided edges. □

Theorem 25.1: (Wagner's Thm)
A graph G is planar iff G does not contain a K_5 or $K_{3,3}$ minor.
[Proof] Use the next theorem (25.2) and Kuratowski's theorem □

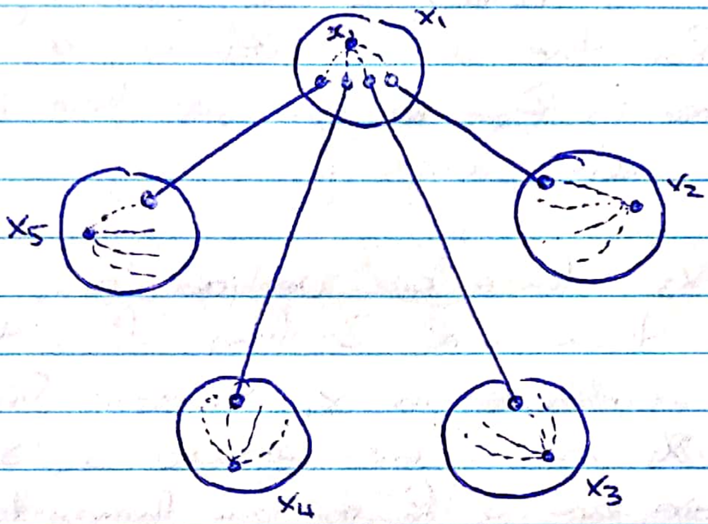
Theorem 25.2:
G contains a subdivision of K_5 or $K_{3,3}$ iff G contains a K_5 or $K_{3,3}$ minor

↪ Proof

[Proof]

($\Rightarrow$) Use previous lemma

($\Leftarrow$) G has a K_5 or $K_{3,3}$ minor.
Suppose G has a K_5 minor. Then, there exist 5 connected vertex-disjoint subgraphs $X_1, \ldots, X_5$ whose contractions result in K_5.



For each X_i, there are 4 edges joining it to the other subgraphs. Let V_i be the set of endpoints whose edges are outside of X_i.

If there exists a 4 fan for some vertex x_i to V_i in each X_i, then this forms the K_5 subdivision, and so we are done.

Otherwise, suppose such a 4-fan does not exist for one subgraph, say X_1.

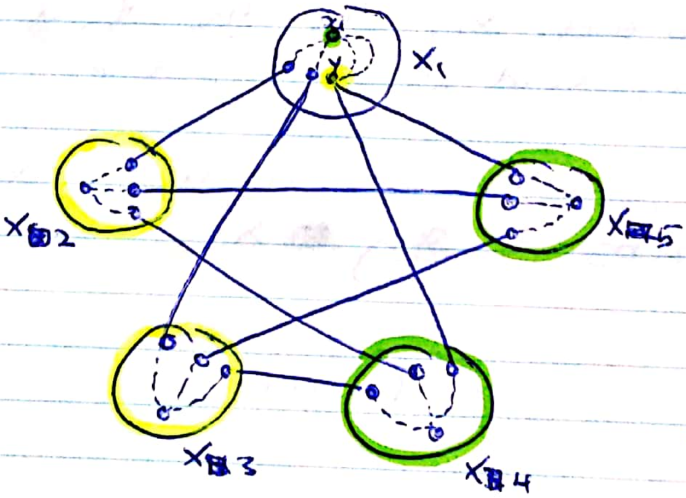
cont'

Hilroy

[Proof] (cont)

Suppose $V_1 = \{w_2, w_3, w_4, w_5\}$, where w_i is in X_i. We know that there exists a vertex x_1 in X_1 such that there is a 3-fan from x_1 to $\{w_2, w_3, w_4\}$, and P_i is an x_1, w_i-path. Since X_1 induces a connected subgraph, there is a x_1, w_5-path P_5 which is not internally disjoint from the 3-fan. Let y be the first vertex in P_5 from w_5 that is in another path, say P_4.

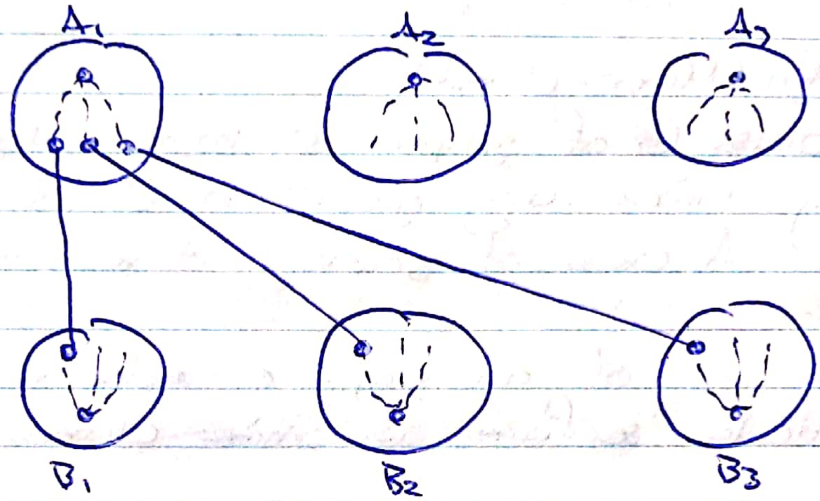
From X_2, X_3, there are vertices x_2, x_3 for which there are 3-fans from them to their neighbors in X_1, X_4, X_5. Similarly, from X_4, X_5, there are vertices x_4, x_5 for which there are 3 fans from them to their neighbors in X_1, X_2, X_3. Combine these fans, along with the paths P_2, P_3 and x_1 to y in P_4, and the paths from y to w_4, w_5 in P_4, P_5. This is a subdivision of $K_{3,3}$ with $\{x_1, x_4, x_5\}$ and $\{y, w_2, w_3\}$ as the branch vertices:



Con't

[Proof] (con't)

Now, suppose G has a $K_{3,3}$ minor. Then, there exist connected vertex-disjoint subgraphs $A_1, A_2, A_3, B_1, B_2, B_3$, whose contractions result in a $K_{3,3}$ with bipartition $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$.



In A_1, there are 3 neighbours, one in each of B_1, B_2, B_3 in the minor. There exists a vertex x_{a_1} in A_1, such that there is a 3-fan from x_1 to these 3 vertices. Do this for all 6 subgraphs to obtain a $K_{3,3}$ subdivision

$\square$

Q: Why is Wagner's Theorem useful?

A: We can use it to look at planar embeddings on non-plane surfaces. In particular, we cannot have theorems of the form:

"G can be embedded on __(surface)__ iff G does not contain ______ as a subdivision"

beyond a plane / sphere.

→ con't

Theorem 25.3 (Graph Minor Theorem)
Any infinite sequence of finite graphs: $G_1, G_2, \ldots$, includes 2 graphs G_i, G_j with $i<j$ s.t. G_i is a minor of G_j.

Def'n (Minor-closed)
A class $\mathcal{G}$ of graphs is minor-closed if any minor any graph in $\mathcal{G}$ is in $\mathcal{G}$
(Note. A class of graphs = A set of graphs)

Remark 25.1:
The set of all graphs embeddable on a particular surface is minor-closed.

Def'n (Minor-minimal)
For a minor-closed family of graphs $\mathcal{G}$, a graph G is minor-minimal if $G \in \mathcal{G}$, but any proper minor of G is in $\mathcal{G}$.

Theorem 25.4 (Wagner's Thm-Reworded)
For the class of planar graphs, K_5, $K_{3,3}$ are minor-minimal.

Ex: (Minor-minimal graph).
Let F be the set of all forests, it is a minor-closed class, since deleting anything still results in a forest. The 3-cycle is the only minor-minimal graph.

Theorem 25.5:
For any minor-closed family of graphs G, the set of all minor-minimal graphs is finite.
[Proof]
Suppose the set is infinite, by graph minor thm (Thm 25.3), one is a minor of the other, contradiction $\square$

Summary:
Wagner's Thm can be extend:
"G can be embedded on a certain surface iff G does not have a finite set of graphs as minors."
Ex. The torus has ≥ 16629 minor-minimal graphs.

Maclane's Theorem:

Def'n (2-basis)
A basis B of a subspace of $\mathcal{E}(G)$ is a 2-basis if every edge of G is in at most 2 elements of B.

Theorem 26.1 (Maclane's Thm)
A graph is planar iff the cycle space has a 2-basis.
[Proof]
($\Rightarrow$) Suppose G is planar. Notice that $\mathcal{C}(G)$ has a 2-basis iff the cycle space of each block has a 2-basis. Also, recall that G is planar iff each block of G is planar. Now, we may assume that G is 2-connected. Consider any embedding of G, each face is bounded by a cycle.

$\Rightarrow$ Con't

[Proof] (Con't)
Let B be the set of all facial cycles of the embedding. Each edge is in 2 elements of B (the two faces on either side). [redacted] We want to show that B spans $C(G)$. Let C be any cycle, and F be the set of all face boundaries of faces inside C. Consider the sum:

$$S = \sum_{C' \in F} C'$$

Each edge in C is counted once in S, since only the interior of C is counted. Each edge in the interior of C is counted twice, as both sides are in the interior. So $S = C$, and we see that B spans $C(G)$. So, a subset of B is a basis for $C(G)$, hence a 2-basis. (In particular, we could remove the outer cycle)

($\Leftarrow$) [redacted], Suppose G is not planar.
We first prove that if $C(G)$ has a 2-basis, and H is a minor of G, then $C(H)$ has a 2-basis.

Let $\{Z_1, \dots, Z_k\}$ be a 2-basis for $C(G)$, where $k = |E(G)| - |V(G)| + c$. Let e be an edge. If e is an edge, then, it is not in any Z_i. Removing e, we get $k = (|E(G)| - 1) - (|V(G)|) + (c+1)$, so $\{Z_1, \dots, Z_k\}$ is still a 2-basis for $C(G-e)$

(to con't)

[Proof] (Con't)

If e is in exactly one set, say Z_k, then $\{Z_1, \ldots, Z_{k-1}\}$ is a 2-basis for $C(G-e)$, since $k = (|E(G)| - 1) - V(G) + c$ decreases by 1.

If e is in 2 sets, say Z_{k-1} and Z_k, then $\{Z_1, \ldots, Z_{k-1} + Z_k\}$ is a 2-basis for $C(G-e)$, since $k = (|E(G)| - 1) - V(G) + c$, $Z_{k-1} + Z_k$ is still even, and any other edge will only appear at most twice, as $Z_{k-1} + Z_k$ does not "uncover" edges.

Now, suppose we want to contract e. The dimension doesn't change, but we remove e from any Z_i containing e. The resulting set is still even a linearly independent, and hence a 2-basis for $C(G/e)$

Finally, removing a vertex v is the same as removing all edges incident with v (which falls into the above cases), and then removing the isolated vertex itself, which does not change the dimension of the 2-basis.

So, we conclude that $C(H)$ has a 2-basis.

We now want to show that K_5, $K_{3,3}$ do not have 2-bases of their cycle space.

Suppose $C(K_5)$ has a 2-basis, then $\dim C(K_5) = 10 - 5 + 1 = 6$. Let $\{Z_1, \ldots, Z_6\}$ be such as 2-basis, and let $Z_7 = \sum_{i=1}^{6} Z_i$. Since $Z_1, \ldots, Z_6$ are linearly independent, $Z_7 \neq \emptyset$, and contains edges that appear exactly one in the basis.

(Con't

Hilroy

[Proof] (cont)

Then, in $\{Z_1, \dots, Z_7\}$, each edge appears at most twice (if exactly twice, this means we have no cut edges). There are 10 edges in K_5, so there are 20 times they appear in $\{Z_1, \dots, Z_7\}$, but even $|Z_i| \geq 3$ since they are/contain a cycle. So edges appear at least 21 times, contradiction.

Similarly, a cycle space of $K_{3,3}$ does not have a 2-basis. Hence, by Wagner's Thm, G must be planar, and we've shown that any non-planar graph, does not have a 2-basis for its cycle space. $\square$

Theorem 27.1 (Whitney's)
A graph G is planar iff G has an abstract dual.

[Proof]

($\Rightarrow$) If G is planar, its (geometric) dual is its abstract dual

($\Leftarrow$) Let G^* be the abstract dual. So, $C(G) = C^*(G^*)$. The set $\{\delta(v) \mid v \in V(G^*) \setminus \{w\}\}$ is a basis for $C^*(G^*)$ for some $w \in V(G^*)$. Each edge appears at most twice, once for each end, so this is a 2-basis for $C^*(G^*)$, and consequently also a 2-basis for $C(G^*)$. Then, by MacLane's Theorem, G is planar.

$\square$