Feb 26th

Linear Spaces:

Let $(V, \mathcal{L})$ be an incidence structure. Call the elements of V points and the elements of $\mathcal{L}$ lines.

For $x, y \in V$, if there is $l \in \mathcal{L}$ incident with both, we say that l joins x and y. If l is unique, we write $l = x \vee y$ (x "join" y)

For $l, m \in \mathcal{L}$, if there is $x \in V$ incident with both, we say that l and m meet at x. If x is unique, we write $x = l \wedge m$ (l "meet" m)

Defn

$(V, \mathcal{L})$ is a linear space if for any 2 distinct $x, y \in V$, there is a unique $l \in \mathcal{L}$ joining x and y. (i.e. $x \vee y$ is always defined)

And, $(V, \mathcal{L})$ is a dual linear space if for any 2 distinct $l, m \in \mathcal{L}$, there is a unique point $x \in V$ at which they meet (i.e. $l \wedge m$ is always defined)

Example: Every $(v, k, 1)$-BIBD is a linear space

Notes: (1) If $(V, \mathcal{L})$ is a linear space, then any 2 distinct lines meet at 1 or 0 points

Proof:

If l, m meet at both x and y, then l joins x and y and so does m, so $(V, \mathcal{L})$ is not a linear space.

If l, m do not meet ~~at~~ at any point or $l=m$, then we say l and m are parallel.

(2) $(V, \mathcal{L})$ is a dual linear space $\Leftrightarrow$ its dual is a linear space. So, any statement about linear spaces also apply to dual linear spaces, with the roles of lines/points swapped.

Examples: A projective plane is both a linear space and a dual linear space

since it is a $(v,k,1)$-BIBD

because it is symmetric

(i.e. Its dual is also a linear space)

In this case, $v=n^2+n+1$, $k=n+1$, $\lambda=1$.

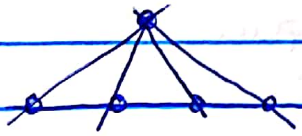
Exercise (In-class)

Find an example of an incidence structure that is linear and dual linear, but not a projective plane

Example:

(a) 

(b) 

(c) 

Turns out that apart from these silly examples, ~~the~~ the 2 properties: linear and dual linear $\Rightarrow$ projective plane

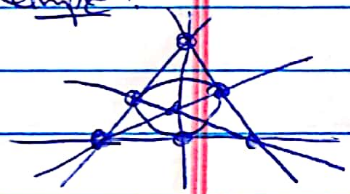
Def'n

A set of points is <u>collinear</u> if they are all incident with a common line.

An s-<u>arc</u> is a set of s points such that no 3 are collinear ((b) doesn't have 3-arcs)

An incidence structure is 'thick' if every point lies on at least 3 lines and every line has at 3 points ((c) doesn't have a 4-arc and is <u>not</u> thick)

<u>Example</u>:



The Fano plane has 4-arcs and is thick

These characterize the "trivial"/silly examples. If we throw these out:

<u>Theorem</u> (Characterization of Projective Planes)

TFAE:

(a) $(V, \mathcal{L})$ is linear, dual linear and contains a 4-arc

(b) $(V, \mathcal{L})$ is —— " —— thick

(c) $(V, \mathcal{L})$ is a symmetric $(n^2+n+1, n+1, 1)$-BIBD.

Proof:

(a) $\Leftrightarrow$ (b) Exercise (Fussy)

(c) $\Rightarrow$ (b) Linear + dual linear we've already shown.
It remains to show thick (Exercise)
(Use properties of symmetric designs)

(b) $\Rightarrow$ (c)

Assume (b). Let $l \in \mathcal{L}$ and let $n+1$ be the # of points on l. Let x be any point not on l. (Why does x exist? - Exercise to think about)

There is a bijection b/w points on l and ~~lies~~ lines through x:

$$\left.\begin{array}{l} y \mapsto x \vee y \\ m \wedge l \leftarrow m \end{array}\right\} \text{Mutual Inverses.}$$

"point on l" "line through x"

$\Rightarrow$ x is ~~on~~ on $n+1$ lines

By the same reasoning, any line not through x has $n+1$ points.

Since $(V, \mathcal{L})$ is thick, for any 2 lines l, m, we can find a point x not on either $\therefore$ All lines have $n+1$ points

Similarly, all points are on $n+1$ lines.

$\therefore$ $(V, \mathcal{L})$ is a BIBD, with $k = r = n+1$

$\square$

(Exercise: $v = n^2 + n + 1$)

Recall the construction of affine planes:

$V = \mathbb{F}^2$

$\mathcal{B} = \{\text{affine lines in } \mathbb{F}^2\}$

i.e. $\{x + W \mid \dim W = 1\}$

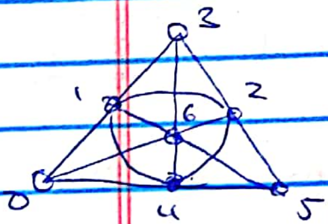
Recall: Derived and Residual designs of a symmetric BIBD

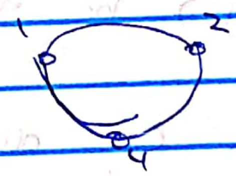
Derived: $(\alpha, \{\beta \cap \alpha \mid \beta \in \mathcal{B} \setminus \{\alpha\}\})$

Residual: $(V \setminus \alpha, \{\beta \setminus \alpha \mid \beta \in \mathcal{B}\})$

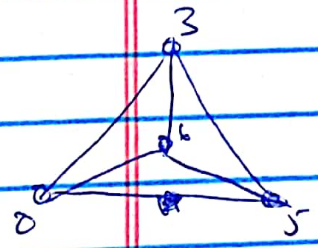
In the case of the projective plane:
Fix a line $l_\infty \in \mathcal{L}$. The residual with respect to l_∞ is the incidence structure obtained by deleting all points on l_∞ from each line and deleting l_∞ itself.

Example:
For Fano Plane:



$l_\infty =$ 

Residual:



(which is just ~~a~~ K_4.)

Theorem (Characterization of Affine Planes)

TFAE:

(a) $(V, \mathcal{L})$ is a linear space containing a 3-arc and for every line l and every point x, there is a unique line through x parallel to l.

(b) $(V, \mathcal{L})$ is an $(n^2, n, 1)$-BIBD

(c) $(V, \mathcal{L})$ is the residual of a projective plane.

Proof:

(c) $\Rightarrow$ (b) ~~Exercise~~ (Figure out params of the residual design)

(b) $\Rightarrow$ (a) Assume (b).

If $(V, \mathcal{L})$ is an $(n^2, n, 1)$-BIBD Then,

$$r = \frac{\lambda(v-1)}{k-1} = \frac{1(n^2-1)}{(n-1)} = n+1$$

Let $x \in V$, $l \in \mathcal{L}$. If x is on l, then l is the unique line through x parallel to l.

O/w, let $(y_1, \ldots, y_n)$ be the point on l.

Then,

$xy_1, \ldots, xy_n$ are all lines through x that meet l. Since $r = n+1$, there is therefore one more line through x and it doesn't meet l, which is what we want.

Check that $(V, \mathcal{L})$ has a 3-arc.

Next time: (a) $\Rightarrow$ (c)

$\square$