7.4 - Colouring

Jan 28th

Back to our assumption that all colour degrees are the same:

For Reed - Sudakov:

- Keeping a colour is more likely if smaller colour degree
- So $E[|L'(v)|] \geq$ Expectation when regular
- $E[d_{L',G'}(v,c)] \leq d_{L,G}(v,c)(1-p\text{keep}) \leq \_\_\_\_ (1-p\text{keep})$

So keep changes with the list. → ~~colour assigned~~ $\text{keep}(u, \phi(w)) = (1-\frac{p}{|L|})^{d_{L,G}(u,\phi(w))}$

min. keep, i.e. old #. (but we can always extend every list)

Even if it wasn't the case that expectations were only better for us in the non-regular case:

(1) Regularization: Embed our graph into a regularized version where ~~the~~ colouring the resulting graph yields a colouring of the original.

(2) Equalizing Coin Flips: Here, for every $u \in V(G)$ and $c \in L(u)$, we add a coin flip $F_{u,c}$ which keeps c for u with probability $\text{keep}/\text{keep}(u,c)$ and hence every colour is kept with probability ~~keep/$F_{u,c}$~~ $\text{keep}(u,c) \cdot F_{u,c} = \text{keep}$.

Note: (2) only works if ~~desired~~ coin flips have probability ≤ 1. ~~AND~~ we would have to redo all the concentrations, adding the coin flips into verifications (Lipschitz.

Regularization:

Lemma: Every graph G is an induced subgraph of a $\Delta(G)$-regular graph.

↪ Proof Sketch

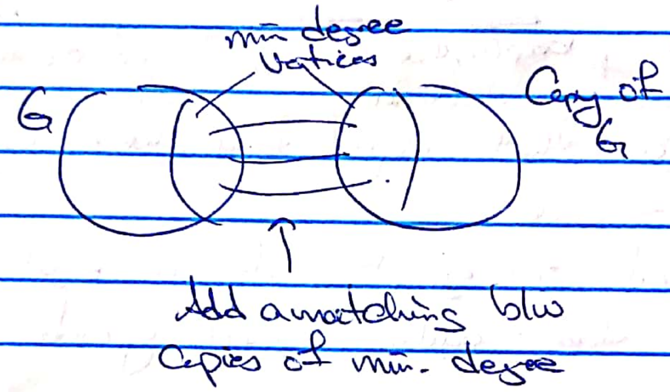
Proof Sketch:

By induction on $\Delta(G) - \delta(G)$ (min degree) (max degree)

If $\Delta(G) = \delta(G)$, then G is $\Delta(G)$-regular, as desired.
So, we may assume $\delta(G) < \Delta(G)$.

Define:

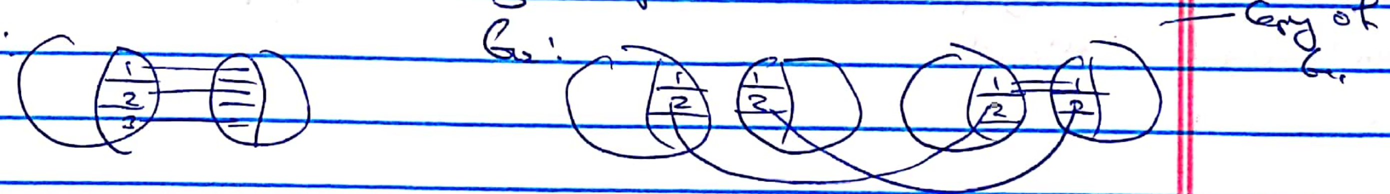
$G' =$



Now $\delta(G') > \delta(G)$ and hence by IH, $\exists$ G ~~with~~ $\Delta(G)$-regular containing G' as an induced subgraph. $\square$

Note: Construction actually gives if G is triangle-free, then G' is triangle-free.
However, if G has girth ≥ 5, G' may have 4-cycles.
We may tweak the construction to preserve girth 5 (and girth 6, but not ≥ 7)
First, find a $(\Delta+1)$-coloring of G. Only add a matching b/w color i of min. deg. copies:

G_1: G_2:



So, do $\Delta+1$ doublings to up min. deg. by 1. Start over...

For Reed-Sudakov, we need to regularize color-degree, not degree:

(1) If we pass to correspondence coloring, we can use the construction above, but we only add a conflict b/w min-deg. colors b/w copies of vertices

(2) If we're more careful with making lists for the copies, this should work for list-coloring as well.

Kim's Theorem for Girth Five Graphs:

Theorem (Kim, 1995)

If G has girth ≥ 5, then $\chi(G) \leq (1+o(1))\frac{\Delta}{\ln \Delta}$.

Recall: This is tight up to constant factor, in particular, in random d-regular graphs have $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$ with high probability, and so $\exists$ d-regular graphs of arbitrary girths and $\chi \geq (\frac{1}{2}-o(1))\frac{d}{\ln d}$.

(Some naive) Intuition:

- Coupon Collector Problem: Suppose there are n types of coupons and when you receive a coupon you receive a type u.a.r.

- Q: How many coupons do you need to get to collect all the types? $\Theta(n \log n)$

Proof Sketch:

In the first n coupons,

$\Pr[\text{Obtained} \geq 1 \text{ of coupon } i]$

$= 1 - \Pr[0 \text{ of coupon } i]$

$= 1 - (1 - \frac{1}{n})^n$

$\approx 1/e.$

$\mathbb{E}[\text{types collected after } n] \approx n(1 - \frac{1}{e})$

$\mathbb{E}[\text{still to collect}] \approx n/e.$

$\mathbb{E}[\text{types still to collect after } n \cdot t \text{ coupons}] = n/e^t.$

Use concentration inequalities to make these concentration around expectation w.h.p.

$\square$

So, we could think that:

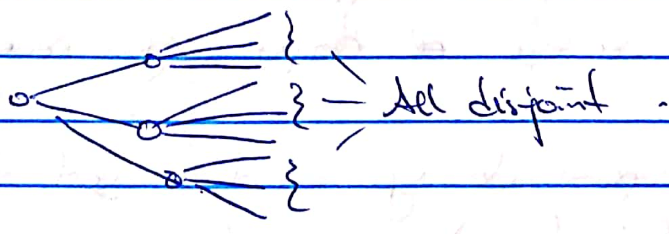
types of coupons = colors in my list $(n) = L$

coupon samples = colors neighbors receive $(n \ln n) = \Delta.$

(Solve to find $L \approx \frac{\Delta}{\ln \Delta}$).

But, coloring (randomly) is not uniformly random,

→ here's where the girth 5-ness can help, since for every vertex we see:



This is "somewhat uniformly random"

Proof (of Kim's)

We actually prove a stronger theorem (as Kim did):

If G has girth ≥ 5 and L is a list assignment of G such that $|L| \geq (1+o(1))\frac{\Delta_L(G)}{\ln \Delta_L(G)}$, then G has an L-colouring.

i.e. He proved this theorem for colour-degree.

We'll use Nibble and the Wasteful Colouring Procedure.

We may assume all colour degrees and list sizes are the same by Regularization / Equalizing Coin Flips.

We will be interested in tracking the ratio: $\frac{\Delta_L(G)}{|L|}$.

(Reciprocal of what we used for Reed-Sudakov)

(Note: This starts as $\approx \ln \Delta_L(G)$).

We will show that we can (after many steps) reduce to the graph with ratio $\leq \frac{1}{20}$ (so $|L| \geq 20\Delta_L(G)$).

and then we apply Alon/Haxell to finish.

Expectations for one step of WCP:

$$\Pr[c \in L'(v) \mid c \in L(v)] = \left(1-\frac{p}{|L|}\right)^{\Delta(G)} =: \text{keep} \approx e^{-\frac{p\Delta}{|L|}}.$$

$$\mathbb{E}[|L'(v)|] = |L| \cdot \text{keep}.$$

keeping c | If $c = \phi(u)$, where $v \in N_{L,G}(u,c)$ | keeping $\phi(u) \neq c$

$$\mathbb{E}[d_{L',G'}(v,c)] \leq d_{L,G}(v,c) \cdot \left(\text{keep} \cdot \underbrace{\left(\frac{1}{|L|}(1-p) + \left(1-\frac{1}{|L|}\right)(1-p\,\text{keep})\right)}_{= 1 - p\,\text{keep} + \frac{1}{|L|}\underbrace{p(\text{keep}-1)}_{\leq 0}}\right)$$

$$\leq \underbrace{d_{L,G}(v,c)}_{= \Delta_L(G)}\,\text{keep}\,(1-p\,\text{keep})$$

Remark:

In Reed-Sudakov, we threw out the keep in $\mathbb{E}[d_{L,G}(v,c)]$ for analysis purposes to get the L/Δ ratio improving by keep/1-keep which was >1, since $|L|>\Delta$.
But, for Kim's, we'll need to keep the keep in $\mathbb{E}[d_{L,G}(v,c)]$ which complicates concentration analysis, but everything will still concentrate b/c girth ≥ 5.

Concentrations:

- Show $\Pr[\,|L'(v)| - \underbrace{\mathbb{E}[|L'(v)|]}_{=\text{keep}\cdot|L|}\,] \leq \frac{1}{\Delta^{10}}$

  on $|L|-|L'(v)|$

  Proof same as for Reed-Sudakov, i.e. Use Talagrand's ^ with $c=1$, $r=2$ (really 3 w/ activation flips)

- Show $\Pr[d_{L,G'}(v,c) - \mathbb{E}[d'_{L,G}(v,c)]] \leq \frac{1}{\Delta^{10}}$.

  Here, we can use Talagrand's ^ (Exc. Talag) with $c=1$ (except for v itself, use exceptional Talagrand for v or other tricks), and $r=?$ (Is it possible to verify # of vertices undeleted and keeping c in dist)

  ↳ No, can't efficiently verify.

So, instead, concentrate other variables:

	Deleted	Undeleted $(r=3/4)$	
Keep c	X	X	← Can't verify efficiently
$(r=2)$ Don't keep c		✓	→ Can verify not keeping c and undeleted!

Can't verify deleted

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Can verify all boxes indirectly here assuming all expectations of size of boxes is roughly same. ~~.~~ (i.e. the intersection shouldn't be small compared to the size of the cell)

Iterating the ratio:

Apply LLL to find:

$|L'| \approx |L| \cdot \text{keep}$

$\Delta' \approx \Delta \cdot \text{keep}(1-p\,\text{keep})$

So:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta \cdot \text{keep}(1-p\,\text{keep})}{|L| \cdot \text{keep}}$$

$$= \frac{\Delta}{|L|} - \frac{p\Delta}{|L|}\text{keep} \quad (\text{Recall: } \text{keep} \approx e^{-\frac{p\Delta}{|L|}})$$

So, let $k := \frac{p\Delta}{|L|}$, we get:

$$\frac{\Delta'}{|L'|} \approx \frac{\Delta}{|L|} - ke^{-k}$$, so for $k=1$, this gives $-1/e$.

Now run this forever to get the ratio ??

Can't do this since we'd run out of colors.

(Since $|L'| \approx |L| \cdot \text{keep}$ and $\text{keep} \approx \frac{1}{e}$, so we'd have to stop at about ~~...~~ $\ln \Delta$ steps)

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