C749 - Colorings

Jan 18th

Review: (Color Degree)

G graph, L list assignment

$|L| := \min_{v \in V(G)} |L(v)|$

$d_L(v,c) := |\{ v \in N(v) : c \in L(v) \}|$

$d_L(v) := \max_{c \in L(v)} d_L(v,c)$

$\Delta_L(G) := \max_{v \in V(G)} d_L(v)$

Theorem (Alon '92)

If $|L| \geq 2e(\Delta_L(G)+1)$, then G has an L-coloring.

Proof:

(We may assume WLOG $|L(v)| = L \ \forall v$)

- Color every vertex, $\phi(v)$ uniformly at ~~random~~ random from its list

- Bad events: $A_{\underset{=uv}{e},c} : \phi(u) = \phi(v) = c \quad \forall c \in L(u) \cap L(v)$

- Then $Pr[A_{e,c}] = \frac{1}{L^2} = p$.

- $A_{e,c}$ is mutually independent of $A_{e',c'}$ whose $e' \nsim e$ (and $\neq$)

$B_{\underset{=uv}{e,c}} = \{ A_{\underset{=u'v'}{e',c'}} : e' \sim e \text{ or } e' = e \text{ and } c \in L(u') \cap L(v') \}$

$= 2 \cdot |L| \cdot \Delta_L(G) = d+1$ ← Since we counted ourselves

↓ Pick an end (u or v) Call this x

↓ Pick a color c' in $L(x)$

→ Pick a neighbor w of x where $c \in L(w)$.

Proof (cont'd)

Then, by LLL:

Since $ep(d+1) = e\frac{1}{|L|^2}(2|L| \cdot \Delta_L(G))$

$$= \frac{2e\Delta_L(G)}{|L|} \leq 1.$$

Then $\exists$ a coloring ϕ avoiding all $A_{e,c}$, i.e. an L-coloring of G, as desired. $\square$

Note: We can actually get rid of the "+1" in the statement of the theorem!

Theorem (Haxell)

If $|L| \geq 2\Delta_L(G)$, then G has an L-coloring.

Actually follows from the following more general theorem:

Theorem (Haxell)

Let $k \geq 1$ be an integer. If $V_1, V_2, \ldots, V_r$ is a partition of $V(G)$ into independent sets for a graph G such that:

- $|V_i| \geq 2k$ $\forall i \in [r]$, and
- $\Delta(G) \leq k$, ~~and~~

Then $\exists$ an independent set I of G s.t. $\forall i \in [r]$, $I \cap V_i \neq \emptyset$.

(Called an independent ~~set~~ transversal)

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How does it imply previous thm?
Let H be such that
$V(H) = \{(v,c) : v \in V(G),\ c \in L(v)\}$.
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c = c'\}$
Then H satisfies independent transversal theorem by assumption on G. So, by that theorem, $\exists$ an independent transversal, i.e. An L-coloring

Remark: Clearly also this implies correspondence version by letting
$E(H) = \{(v,c)(v',c') : vv' \in E(G),\ c$ matched to c' in $M_{vv'}\}$.

~~Remark~~ Remark: This is tight! (i.e. The ratio of 2). For general independent transversals.

Conjecture (Reed, ~'90s)
If $|L| \geq \Delta_L(G) + 1$, then G has an L-coloring

But Bohman and Holzman disproved this conjecture!
Still open if $|L| \geq \Delta_L(G) + 2$ works! (has been open for about 20 years)

Theorem (Reed-Sudakov, '02)
If $|L| \geq (1 + o(1))\Delta_L(G)$, then G has an L-coloring

(Note: Equivalent to:
$\forall \varepsilon > 0, \exists \Delta_\varepsilon$ s.t. $\forall \Delta \geq \Delta_\varepsilon$, if $\Delta(G) = \Delta$ and $|L| \geq (1+\varepsilon)\Delta$, then G has an L-coloring).

Remark:
~~Rbddr~~ Loh and Sudakov proved that the ratio for independent transversals is $(1+o(1))$ assuming that every vertex has at most $o(k)$ neighbours in any other partitions..

Proof (Reed-Sudakov)

Note: Uses Rödl Nibble (aka semirandom) method of iteratively constructing a solution a little bit at a time. (In particular, uses the "Wasteful Colouring Procedure" to implement one such iterative procedure)

Note #2: Uses LLL and concentration inequalities to prove that there is a decent enough outcome of the WCP so as to continue iterating, until the Big Finish.

What is our finish? Alon's Thm (or Haxell's), i.e. can L-colour if $|L| \geq 2e\Delta_L(G)$..

How will our iteration step be improving?: Progress will be in the ratio, $|L|/\Delta_L(G)$. In fact, in our proof, it will take some constant, depending on ε, # of steps.

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Proof details:

Big assumption for now:

Let's assume $\forall c \in L(v)$ that $d_L(v,c) = d(v) = \Delta(G)$

(of course, we may assume wlog $|L(v)| = |L|$ $\forall v$)

Wasteful Colouring Procedure:

- Independently "activate" each vertex of G for some probability p to be fixed later.

(Remark (if the calculations are correct):

$$p = \frac{\varepsilon(1+\varepsilon)}{2}$$

works and any smaller $p \geq \text{polylog}(\Delta)$).

- Now colour every activated vertex v with a colour $\phi(v)$ selected uniformly at random from $L(v)$

This is the "wasteful" step!

- Remove $\phi(v)$ from the list of all v's neighbours
Uncolour any vertex which has the same colour as neighbour (or equiv. $\phi(v)$ is no longer in its list)

- Let ϕ' be the resulting colour, and
L' be the resulting lists

Remark: In the "naive colouring procedure" we only remove colours from neighbour's list if they keep the colour.

Here, we use WCP over NCP b/c

1) NCP is harder to analyze for concentrations, and
2) The probability that an active keeps its color will be close to 1, so in practice, much difference here

We let U ~~Ξ~~ be the set of vertices that do not receive a color from Φ'

let $G' = G[U]$

Clotus $L'(u) = L(u) \setminus \{\Phi'(v) : v \in N(u) \cap A\}$ · ← where A is activated Lovasfasl procedures

$\to \setminus U$ would be for naive.

We'll prove that there is an outcome with

$$\frac{|L'|}{\Delta_{L'}(G')} > 1+\varepsilon \quad \leftarrow \underline{\text{GOAL!}}$$

(In fact, $(1+\varepsilon)(1+\varepsilon/4)$.)

~~uuuuuuuuuu~~

Expectations:

- Lists: Want to calculate ~~uuuu~~ $E[|L'(u)|]$ for a given u

$$E[|L'(u)|] = \sum_{c \in L(u)} \Pr[c \in L'(u)]$$

By linearity of expectation

$$= \sum_{c \in L(u)} \underbrace{\Pr[\nexists u \in N(u) \cap A \text{ with } \Phi(u) = c]}_{\text{X4 ~~[illegible]~~}}$$

$$\prod_{u \in N(u)} \Pr[u \notin A \text{ ~~and~~ or } \Phi(u) \neq c]$$

See note on next pg. $u \in N_c(u)$ ←

$$\prod_{u \in N(u)} ~~\text{Pr}~~(1 - ~~\Pr[u \in A \text{ and } \Phi(u) = c]~~)$$

But activation is independent of colouring:

$$\prod_{u \in N(v)} 1 - \Pr[u \in A] \cdot \Pr[\phi(u) = c]$$

$$= \prod_{u \in N_L(v,c)} \left(1 - p \cdot \frac{1}{|L|}\right).$$

In fact, since $\phi(u) = c$ only happens if $c \in L(u)$, we can write $u \in N_L(v, c)$.

$$= \left(1 - \frac{p}{|L|}\right)^{|N_L(v,c)|}$$

$$= \left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)} \leftarrow \text{by color regularity}$$

$$\approx e^{-\frac{p \Delta_L(G)}{|L|}}$$

So, the sum is:

$$\mathbb{E}[|L'(v)|] = e^{-\frac{p\Delta_L(G)}{|L|}} \cdot |L|.$$

So, note that when $p \approx 1$, the we cut the list by around a $1/3$ each time. But, if p small, then we retain a decent chunk.

Color-degrees:

$$\mathbb{E}[d_{L'}(v, c)] = \sum_{u \in N_L(v,c)} \Pr[u \in V(G') \text{ and } c \in L'(u)]$$

$c \in L'(v)$ (on G')

(Remark: Do this whether or not $c \in L'(v)$)

$$\mathbb{E}[d_{L'}(v,c)] = \sum_{u \in N_L(v,c)} \Pr[u \in N(G') \text{ and } c \in L'(u)]$$

$$= \sum_u \Pr\left[\begin{matrix}(u \in A \text{ and } c \in L'(u)) \\ \text{or } (u \in A \cap \mathcal{U} \text{ and } c \in L'(u))\end{matrix}\right]$$

$$= \sum_u \Pr[u \in A] \Pr[c \in L'(u)]$$

$$+ \Pr[\qquad]$$

$c = \phi(v)$ $\qquad$ $c \neq \phi(v)$

$\Pr[u \in A \cap \mathcal{U}]$ $\qquad$ $\leq \Pr[u \in A \cap \mathcal{U}]$ $\Pr[c \in L'(v)]$

Let's pretend that $c = \phi(v)$ does not matter:

$$\leq \sum_u \left(1 - p\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right.$$

$\Pr[u \in A] \Pr[u \in \mathcal{U}]$

$$+ \Pr[u \in A \cap \mathcal{U}] \cdot \frac{1}{|L|}$$

$$+ \left(p\left(1 - \underbrace{\Pr[\phi(v) \in L'(v)]}_{\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}}\right)\left(1 - \frac{p}{|L|}\right)^{\Delta_L(G)}\right)$$

$= \Delta_L(G)$ times

Asymptotically the dist among cardinal is 1