Recall:

Last time:

$\exists$ symmetric idempotent LS of order n iff n odd

Example:

1	3	2
3	2	1
2	1	3

Order 3

Lemma: $\exists$ a symmetric LS of order 2n s.t.

$$x \circ x = \begin{cases} x & \text{if } 1 \le x \le n \\ x-n & \text{if } n+1 \le x \le 2n \end{cases}$$

Such a latin square is called "half-idempotent"

Example:

1	3	2	4	6	5
3	2	1	6	5	4
2	1	3	5	4	6
4	6	5	1	3	2
6	5	4	3	2	1
5	4	6	2	1	3

- Start with odd order
- Copy on diagonal
- Shift for every other quadrant

Gives 2,4,6.. along diagonal

Proof: Start with $x \circ y = x + y \pmod{2n}$ and then rename the entries.

so we just have to rename

Application:

Recall a steiner-triple system is a $(v,3,1)$-BIBD

Recall: Necessary conditions $\Rightarrow v \equiv 1$ or $3 \pmod 6$

We can now show that these conditions are sufficient.

Theorem:

If $v \equiv 1$ or $3 \pmod 6$, then a $(v,3,1)$-BIBD exists

Proof:

Case 1 (Bose Construction)

Let $v = 3n$ where n is odd. Take a symmetric, idempotent LS of order n, and construct a $(v,3,1)$-BIBD as follows:

Points: $[n] \times [3]$

Blocks: $\{(x,1),(x,2),(x,3)\}$ $\quad x \in [n]$

Also

$$\left.\begin{array}{l} \{(x,1),(y,1),(x \circ y,2)\} \\ \{(x,2),(y,2),(x \circ y,3)\} \\ \{(x,3),(y,3),(x \circ y,1)\} \end{array}\right\} \begin{array}{l} x,y \\ \in [n], \; x<y. \end{array}$$

Case 2 (Skolem Construction)

Let $v = 6n+1$ and take a symmetric, half-idempotent LS of order $2n$. Construct a $(v,3,1)$-BIBD as follows:

Points: $[2n] \times [3] \cup \{\infty\}$

Proof Con't)

Blocks: $\{(x,1),(x,2),(x,3)\}$ $\quad 1 \le x \le 2n$

(Copy what we had before) $\quad x,y \in [2n]$, $x<y$

$\{(x+n,1)\ (x,2),\ \infty\}$

$\{(x+n,2),(x,3),\ \infty\}$

$\{(x+n,3),(x,1),\ \infty\}$

And, check that both designs work. □

Orthogonal Arrays:

From a Latin Square of order n we can construct an orthogonal array $OA(n,3)$. This is a $n^2 \times 3$ array whose rows are $(x, y, x \circ y)$, for all $x,y \in [n]$

i.e. row index, col index, entry in LS.

Note: The order of the rows don't matter.

Def'n (Orthogonal Array)

An orthogonal array $OA(n,2)$ is an $n^2 \times 2$ array in which the rows are all pairs (x,y) with $x,y \in [n]$, in some order.

An $OA(n,k)$ is an $n^2 \times k$ array in which each pair of columns is an $OA(n,2)$.

We claim that the $n^2 \times 3$ array we write down from a latin square is equivalent to an $OA(n, 3)$.

This is since in $(x, y, x \circ y)$:

~~OA~~

(x, y) record the indices, every pair exists here (i.e. This is an $OA(n, 2)$)

$(x, x \circ y)$ is saying "~~every~~ each row has every element from 1 to n"

$(y, x \circ y)$ is saying "each col has every element from 1 to n"

Two OAs are <u>equivalent</u> if we can obtain one from the other by permuting row and columns and apply a permutation of $[n]$ to entries in any column.

From an $OA(n,k)$ A, we associate an incidence structure $([n^2], [n] \times [k])$ in which $i \to (x, j) \Leftrightarrow A_{ij} = x$.

incidence relation

$i \in [n^2]$
$j \in [k]$
$x \in [n]$

Example:

(In terms of the incidence matrix)

$$\begin{pmatrix} \{1, 1, 1\} \\ \{1, 2, 2\} \\ \{2, 1, 2\} \\ \{2, 2, 1\} \end{pmatrix} \longrightarrow \left(\begin{array}{c|c|c} 01 & 01 & 01 \\ 01 & 10 & 10 \\ 10 & 01 & 10 \\ 10 & 10 & 01 \end{array}\right)$$

Replace $1 \to (1, 0)$

$2 \to (0, 1)$

$\leftarrow$ This is the incidence matrix of the affine plane of order 2

In general, this is not the incidence structure of a BIBD.

The dual design is called a <u>transversal design</u>.
Transversal designs have the property that points are partitioned into k "groups" of size n.

Any 2 points from 2 different groups are in a unique block. Two points from the same group are not in a common block.

Application:
Suppose we have a $(v,k,1)$-BIBD $(V,\mathcal{B})$ and we want to construct a $(vk,k,1)$-BIBD.
Start by making k disjoint copies of V

$$V' = V \times [k]$$

$$\mathcal{B}' = \{ \alpha \times \{j\} \mid \alpha \in \mathcal{B},\ j \in [k] \}$$

and also add the blocks $\mathcal{B}''$ coming from a transversal design, then:

$$(V', \mathcal{B}' \cup \mathcal{B}'') \text{ is a } (vk,k,1)\text{-BIBD}$$

To better understand ~~Ods~~. O.As.
We will associate

Theorem:
If ~~OA~~ OA(n,k) exists, then $k \leq n+1$.
Equality occurs iff graph is complete.
Proof:
~~It~~ For each $j \in [k]$, the edges colored ~~j~~ j form n copies of K_n, hence there are $n\binom{n}{2}$ edges colored j.
From the defin of an OA, no edge can have more than 1 color. Hence the graph is simple and has exactly $kn\binom{n}{2}$ edges.

$$kn\binom{n}{2} \leq \binom{n^2}{2} \Rightarrow k \leq n+1$$

(and equality occurs iff the graph is complete.

☐

Thm

(i) If an $OA(n, n+1)$ exists $(n \geq 2)$, then its incidence structure is an affine plane

(ii) Conversely, for every affine plane of order n, it is the incidence structure of an $OA(n, n+1)$

(iii) If $k < n+1$, then the incidence structure of an $OA(n, k)$ is never a BIBD

Proof:

(i) Let A be an $OA(n, n+1)$. The incidence structure has points $i \in [n^2]$ and lines $(x, j) \in [n] \times [n+1]$

where $i \to (x, j) \iff A_{ij} = x$.

We must show that this is an affine plane.

Since the graph is complete, every pair of points i, i' is joined by a unique edge with some unique colour j.

$x = A_{ij} \iff (x, j) = i \vee i'$ & In particular, $i \vee i'$ is always defined, so this is a linear space

Finally, each line has exactly n points and there are n^2 points in total, so this is an $(n^2, n, 1)$-BIBD. ∎

This one is easy

(ii) and (iii) are exercises!

Corollary: If an affine plane of order n exists then there exists an $OA(n, k)$ for all $k \leq n+1$.

Product Construction:
If A is an $OA(n,k)$ and B is an $OA(m,k)$, we can construct an $OA(mn,k)$ as follows.
Let $* : [m] \times [n] \to [mn]$ be any bijection.

If $(A_{i1}, A_{i2}, \ldots, A_{ik})$ is a row of A and $(B_{i'1}, \ldots, B_{i'k})$ is a row of B, then
$$(A_{i1} * B_{i'1}, \ldots, A_{ik} * B_{i'k})$$
will be a ~~row~~ row of the new array $A*B$

Exercise: Check that $A*B$ is an $OA(mn,k)$.

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