749 - Random

Feb 3rd

Clarification on thresholds:

A property Q has a threshold $p_0 = p_0(n)$ if:

$$Pr(\mathcal{G}(n,p) \in Q) \to \begin{cases} 0 & \text{if } p = o(p_0) \Leftrightarrow p \ll p_0 \\ 1 & \text{if } p = \omega(p_0) \Leftrightarrow p \gg p_0 \end{cases}$$

$\Leftrightarrow$

Q has a threshold $p_0 = p_0(n)$ if

$\forall \varepsilon > 0, \exists c > 0, n_0 > 0$ s.t. for all $n > n_0$

$Pr(\mathcal{G}(n,p) \in Q) < \varepsilon$ if $p < \frac{1}{c} p_0$

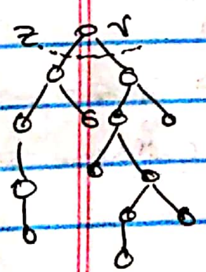
& $Pr(\mathcal{G}(n,p) \in Q) > 1 - \varepsilon$ if $p > c p_0$

Recall: (Theorem Z(d))

If $p = c/n$ and $c > 1$, then a.a.s $\mathcal{G}(n,p)$ contains a unique component of linear order (in n), and all other components are of order $O(\log n)$

Intuition for proof:

Branching (Exploration Process



$deg(v) \sim Bin(n-1, p) \approx Po(c)$ ← approximately poisson (same theorem)

$Z \sim Po(c)$ ↑ same parameter

The out has poisson distribution

- Start at a vertex v, and explore the graph by doing a BFS/DFS on its children.

- Degrees (# of children) is roughly the binomial ~~distrib~~ distribution.

Formalizing this idea:

A Galton-Watson branching process.

Let Z be a random variable on nonnegative integers. At time $t=0$, a single particle comes into existence. At each $t \geq 1$, each existing particle gives birth to a random number of new particles and then it dies. These random numbers are i.i.d copies of Z.

Let X_t denote the number of particles born during step $t \geq 0$. Thus, $X_0 = 1$, and X_1 has the same distribution as Z. Let E denote the event "that $X_t = 0$ for some $t \geq 0$", i.e. the process terminates after a finite # of steps.

Let $\rho_Z = \Pr(E)$

Exercise: For every $t \geq 1$, $\mathbb{E} X_t = (\mathbb{E} Z)^t$

Theorem

(i) If $\mathbb{E} Z < 1$, then $\rho_Z = 1$.

(ii) If $\mathbb{E} Z = 1$, and $\Pr(Z=0) > 0$, then $\rho_Z = 1$.

(iii) If $\mathbb{E} Z > 1$ and $\underbrace{\Pr(Z=0) > 0}$, then $0 < \rho_Z < 1$

to avoid triviality (if $\Pr(Z=0) = 0$, then we always give birth to new children, so $\rho_Z = 0$)

$\hookrightarrow$ Proof

Proof:

(i) $\lim_{t\to\infty} Pr(X_t > 0) \leq \lim_{t\to\infty} (\mathbb{E} Z)^t = 0$

$\longrightarrow$ Since $\mathbb{E} Z < 1$.

$\longrightarrow$ Exercise + Markov

(ii) and (iii):

Let ρ_j be the probability that the process becomes extinct after j steps. ~~Thus likely~~ ~~more~~ Let $v_1, \dots, v_{X_1}$ denote the set of children born at step 1. The process terminates after j steps iff the X_1 processes starting from $v_1, \dots, v_{X_1}$ all terminate after $j-1$ steps. These X_1 processes are independent.

Then:

$$\rho_j = \sum_{i=0}^{\infty} \underbrace{Pr(Z=i)}_{i \text{ children}} \underbrace{\rho_{j-1}^{i}}_{\text{each terminate in } j-1 \text{ steps}}$$

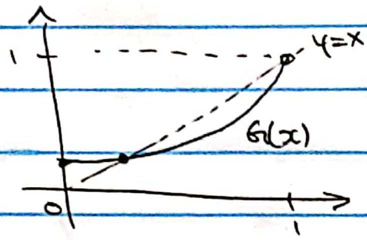
Let $G(x) = \mathbb{E}\, x^Z = \sum_{i=0}^{\infty} Pr(Z=i)\, x^i$, then $\rho_j = G(\rho_{j-1})$

Note:

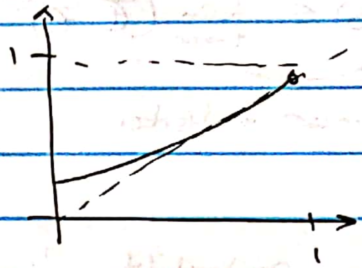
- $G(1) = 1$
- $G(0) > 0$ Since $G(0) = Pr(Z=0) > 0$
- G convex (since G is a probability generating function and they are always convex)
- $G'(1) = \sum_{i=1}^{\infty} i x^{i-1} Pr(Z=i) \Big|_{x=1} = \mathbb{E} Z \geq 1$.

$\longrightarrow$ By assumption.

These facts can help us visualize G.



*Shape comes from that G convex

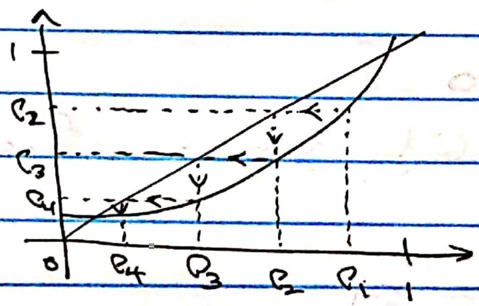


If $\mathbb{E}Z > 1$, then $G(x) = x$ has 2 roots, 1 at $x=1$ and the other $0 < P_Z < 1$

If $\mathbb{E}Z = 1$, then $G(x) = x$ has a unique root at $x=1$ (the line $y=x$ is tangent to G at $x=1$)

For (iii) (and (ii) is similar), it is sufficient to show that $P_t \to P_Z$ for $t \to \infty$, $0 < P_1 < 1$, and the conclusion follows from the properties of G. $\square$

Proof by picture!



$P_2 = G(P_1)$, then find P_2 on the x-axis by following $y=x$ ~~to~~. Then $P_3 = G(H(P_2))$. And continue until we converge at the fixed point $G(P_Z) = P_Z$.

Theorem: (Chernoff Bound)

Let $X_1, \ldots, X_n$ be independent $\{0,1\}$-value random variables. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}X$. Then the following probability bounds hold:

(a) For any $\delta > 0$:

$$\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}.$$

(a) $\Rightarrow$ (b)

(b) For any $0 < \delta \leq 1$

$$\Pr(X \geq (1+\delta)\mu) \leq \exp\left(-\frac{\mu\delta^2}{3}\right)$$

(c) For any $0 < \delta \leq 1$

$$\Pr(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

$$\Rightarrow \Pr(X \leq (1-\delta)\mu) \leq \exp\left(-\frac{\mu\delta^2}{2}\right)$$

Proof:

Let $X_i \sim \text{Bernoulli}(p_i)$

$$\Pr(X \geq (1+\delta)\mu) \underset{\text{holds for any } t>0,\ t\in\mathbb{R}}{=} \Pr(e^{tX} \geq e^{t(1+\delta)\mu})$$

$$\underset{\text{Markov}}{\leq} \frac{\mathbb{E}e^{tX}}{e^{t(1+\delta)\mu}}$$

$$= \frac{\prod_{i=1}^{n} \mathbb{E}e^{tX_i}}{e^{t(1+\delta)\mu}} \quad \text{as } X_1, \ldots, X_n \text{ are independent}$$

$$= \frac{\prod_{i=1}^{n} (1 - p_i + p_i e^{t})}{e^{t(1+\delta)\mu}}.$$

$$\leq \frac{\prod_{i=1}^{n} e^{p_i(e^t - 1)}}{e^{t(1+\delta)\mu}} \underset{\sum p_i = \mu\ (\ast)}{=} \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}}.$$

For part (a), take $t = \log(1+\delta) > 0$

Proof: (con't)

Part (b) follows by noting that $\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \le e^{-\delta^2/3}$ for $0<\delta\le 1$.

For part (c), For ~~___~~ $t<0$

$$Pr(X \le (1-\delta)\mu) = Pr(e^{tX} \ge e^{t(1-\delta)\mu})$$
$$\le \frac{\mathbb{E}(e^{tX})}{e^{t(1-\delta)\mu}} \quad \text{Proceed as before}$$
$$\le \frac{e^{\mu(e^t-1)}}{e^{t(1-\delta)\mu}}$$

And part (c) follows by taking $t=\log(1-\delta)<0$.

The 2nd inequality follows by $\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \le e^{-\delta^2/2}$ for $0<\delta<1$ $\square$.

Remark:

The Chernoff bounds hold for ~~any~~ r.v. $X_1, \dots, X_n$ that are independent $[0,1]$-valued as well.

Modifications for the proof:

At the step

$$\frac{\prod_{i=1}^{n} \mathbb{E} e^{tX_i}}{e^{t(1+\delta)\mu}} \quad \text{we cannot split this to} \quad \frac{\prod_{i=1}^{n}(1-p_i+p_ie^t)}{e^{t(1+\delta)\mu}}.$$

But, we will use that e^x is a convex function and $e^{tx} \le \alpha x+\beta$ for all $x\in[0,1]$ where $\alpha=e^t-1$ and $\beta=1$.

Then,

$$\mathbb{E}e^{tX_i} \le \mathbb{E}(\alpha(tX_i)+\beta) = p_ie^t + (1-p_i)$$

and the remaining steps are the same.