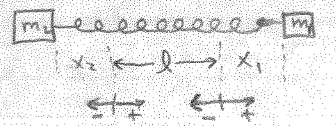
7.8

a)



$m_1 = m_2 = m$

$T_1 = \frac{1}{2} m_1 \dot{x}_1^2 = \frac{1}{2} m \dot{x}_1^2$

$T_2 = \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2} m \dot{x}_2^2$

$U = \frac{1}{2} k x^2 \quad x = (x_1 - x_2 - \ell)$

$U = \frac{1}{2} k (x_1 - x_2 - \ell)^2$

$L = T - U$

$$\boxed{L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k (x_1 - x_2 - \ell)^2}$$

b) $X = \frac{1}{2}(x_1 + x_2)$ {C.M.} , $x = x_1 - x_2 - \ell$

$$\begin{array}{rl} & 2X = x_1 + x_2 \\ + & x = x_1 - x_2 - \ell \\ \hline & 2X + x = 2x_1 - \ell \end{array} \qquad \begin{array}{rl} & 2X = x_1 + x_2 \\ - & x = x_1 - x_2 - \ell \\ \hline & 2X - x = 2x_2 + \ell \end{array}$$

$x_1 = X + \frac{1}{2} x + \frac{1}{2}\ell \qquad x_2 = X - \frac{1}{2} x - \frac{1}{2}\ell$

$\dot{x}_1 = \dot{X} + \frac{1}{2}\dot{x} \qquad \dot{x}_2 = \dot{X} - \frac{1}{2}\dot{x}$

$L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k (x_1 - x_2 - \ell)^2$

$= \frac{1}{2} m \left[(\dot{X} + \frac{1}{2}\dot{x})^2 + (\dot{X} - \frac{1}{2}\dot{x})^2 \right] - \frac{1}{2} k x^2$

$= \frac{1}{2} m \left[\dot{X}^2 + \dot{X}\dot{x} + \frac{1}{4}\dot{x}^2 + \dot{X}^2 - \dot{X}\dot{x} + \frac{1}{4}\dot{x}^2 \right] - \frac{1}{2} k x^2$

$= \frac{1}{2} m \left[2\dot{X}^2 + \frac{1}{2}\dot{x}^2 \right] - \frac{1}{2} k x^2$

$$\boxed{L = m[\dot{X}^2 + \tfrac{1}{4}\dot{x}^2] - \tfrac{1}{2} k x^2}$$

$X: \quad \frac{\partial L}{\partial X} = \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = 0 \rightarrow \frac{d}{dt}(2m\dot{X}) = 0 \rightarrow \ddot{X} = 0$

$x: \quad \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \rightarrow -kx = \frac{d}{dt}\left(\frac{1}{2} m \dot{x}\right) \rightarrow -kx = \frac{1}{2} m \ddot{x}$

c) $\ddot{X} = 0$

$\dot{X} = A = v_0$

$$\boxed{X = v_0 t + B = v_0 t + X_0}$$

$\frac{1}{2} m\ddot{x} + kx = 0 \rightarrow \ddot{x} + 2\frac{k}{m}x = 0$

$x = Ae^{\lambda t}$ $\quad\quad \lambda^2 + 2\frac{k}{m} = 0$

$\dot{x} = A\lambda e^{\lambda t}$ $\quad\quad \lambda^2 = -\frac{2k}{m}$

$\ddot{x} = A\lambda^2 e^{\lambda t}$ $\quad\quad \lambda = \pm i\sqrt{\frac{2k}{m}}$

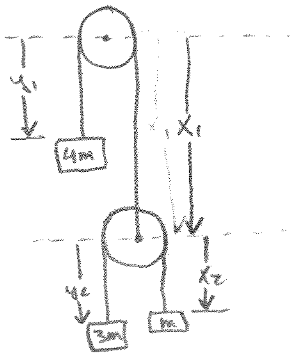
$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$, $\omega = \sqrt{\frac{2k}{m}}$

$= \frac{A}{2} e^{i(\omega t - \delta)} + \frac{A}{2} e^{-i(\omega t - \delta)}$, $A_1 = \frac{A e^{-i\delta}}{2}$ $\quad A_2 = A_1^*$

$$\boxed{x = A\cos(\omega t - \delta)}$$

The C.M. moves at constant velocity since no external forces are acting on it, and the two masses oscillate with respect to each other.

7.27



$l_1 = y_1 + x_1 + 2\pi R \qquad l_2 = y_2 + x_2 + 2\pi R$

$x_1 = -y_1 + l_1 - 2\pi R \qquad x_2 = -y_2 + l_2 - 2\pi R$

$\dot{x}_1 = -\dot{y}_1 \qquad \dot{x}_2 = -\dot{y}_2$

$$T = \tfrac{1}{2}(4m)\dot{y}_1^2 + \tfrac{1}{2}(3m)(\dot{x}_1 + \dot{y}_2)^2 + \tfrac{1}{2}(m)(\dot{x}_1 + \dot{x}_2)^2$$

$$= \tfrac{1}{2}4m\dot{y}_1^2 + \tfrac{1}{2}3m(-\dot{y}_1 + \dot{y}_2)^2 + \tfrac{1}{2}m(-\dot{y}_1 - \dot{y}_2)^2$$

$$= \tfrac{1}{2}4m\dot{y}_1^2 + \tfrac{1}{2}3m(\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2) + \tfrac{1}{2}m(\dot{y}_1^2 + 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2)$$

$$= \frac{m}{2}\left[4\dot{y}_1^2 + 3\dot{y}_1^2 - 6\dot{y}_1\dot{y}_2 + 3\dot{y}_2^2 + \dot{y}_1^2 + 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2\right]$$

$$T = \frac{m}{2}\left[8\dot{y}_1^2 - 4\dot{y}_1\dot{y}_2 + 4\dot{y}_2^2\right]$$

$$U = -4mgy_1 - 3mg(y_2 + x_1) - mg(x_2 + x_1)$$

$$= -4mgy_1 - 3mg(y_2 - y_1) - mg(-y_2 - y_1) + \text{const.}$$

$$= -4mgy_1 - 3mgy_2 + 3mgy_1 + mgy_2 + mgy_1 + \text{const.}$$

$$U = -2mgy_2 + \text{const.}$$

$$L = T - U$$

$$\boxed{L = \frac{m}{2}\left[8\dot{y}_1^2 - 4\dot{y}_1\dot{y}_2 + 4\dot{y}_2^2\right] + 2mgy_2}$$

$$\frac{\partial L}{\partial y_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} = 0$$

$$\frac{\partial L}{\partial y_1} = 0$$

$$\frac{\partial L}{\partial \dot{y}_1} = \frac{m}{2}\left[16\dot{y}_1 - 4\dot{y}_2\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} = \frac{m}{2}\left[16\ddot{y}_1 - 4\ddot{y}_2\right]$$

$$\frac{m}{2}\left[16\ddot{y}_1 - 4\ddot{y}_2\right] = 0$$

$$4\ddot{y}_1 - \ddot{y}_2 = 0 \quad (\text{Eq. } 1)$$

$$\frac{\partial L}{\partial y_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} = 0$$

$$\frac{\partial L}{\partial y_2} = 2mg$$

$$\frac{\partial L}{\partial \dot{y}_2} = \frac{m}{2}\left[-4\dot{y}_1 + 8\dot{y}_2\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} = \frac{m}{2}\left[-4\ddot{y}_1 + 8\ddot{y}_2\right]$$

$$= m\left[-2\ddot{y}_1 + 4\ddot{y}_2\right]$$

$$2mg - m\left[-2\ddot{y}_1 + 4\ddot{y}_2\right] = 0$$

$$-2\ddot{y}_1 + 4\ddot{y}_2 = 2g$$

$$2\ddot{y}_2 - \ddot{y}_1 = g \quad (\text{Eq. } 2)$$

$$4\ddot{y}_1 - \ddot{y}_2 = 0 \quad (1)$$

$$-\ddot{y}_1 + 2\ddot{y}_2 = g \quad (2)$$

$$8\ddot{y}_1 - 2\ddot{y}_2 = 0$$

$$-\ddot{y}_1 + 2\ddot{y}_2 = g$$

$$7\ddot{y}_1 = g$$

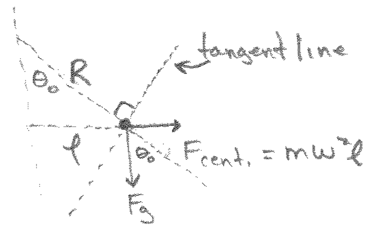
$$\boxed{\ddot{y}_1 = \frac{g}{7}}$$

The top pulley rotates when considering the extremes of mass m. When m=0, the tension in the string is zero so there is no downward force applied to the movable pulley, and 4m is in free fall. If m= 2m and 3m=2m the masses of the movable pulley are always in equilibrium and the tensions in the string supporting the two 2m masses is 2mg each, so the 4m mass will have zero acceleration.

7.28

a) Equilibrium points

$$\theta_0 = \pm \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$$



$F_{g_{Tan}} = -mg \sin\theta_0$

$F_{cent_{Tan}} = m\omega^2 \ell \cos\theta_0$

$$F_{cent_{Tan}} = m\omega^2 \ell \left(\frac{g}{\omega^2 R}\right)$$
$$= mg \frac{\ell}{R}$$
$$= mg \sin\theta_0$$

$F_{cent_{Tan}} = -F_{g_{Tan}}$

b) $\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$

At $\theta = \pi$ and the hoop rotating very slowly gives

$\cos\theta \cong -1$ $\quad (\cos(\pi+\theta) = \cos\pi \cos\theta - \sin\pi \sin\theta = -\cos\theta \cong -1,\ \theta \ll 1)$

$\sin\theta \cong -\theta$ $\quad (\sin(\pi+\theta) = \sin\pi \cos\theta + \cos\pi \sin\theta = -\sin\theta \cong -\theta,\ \theta \ll 1)$

$$\ddot{\theta} \cong \left(-\omega^2 - \frac{g}{R}\right)(-\theta)$$
$$\cong \left(\omega^2 + \frac{g}{R}\right)\theta$$
$$\ddot{\theta} \cong (\text{positive number})\,\theta$$

This is not a restoring force $\{F = -kx\}$. A small positive angular displacement results in a positive acceleration. therefore, instability at equilibrium exists.

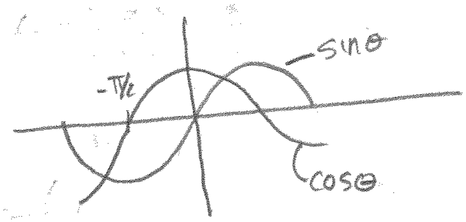
c) $\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$

$\theta_0 = \pm \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$

$\cos\theta_0 = \pm \frac{g}{\omega^2 R}$

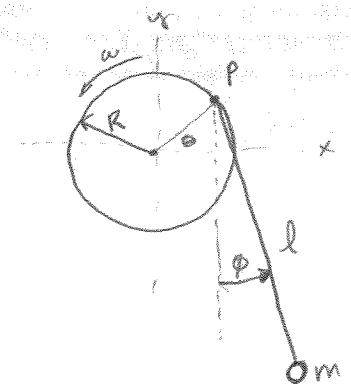
The left $\theta_0 = -\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$ $\quad \{0 \leq \theta \leq -\frac{\pi}{2}\}$

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$



At the $-\theta_0$ position $\ddot{\theta}=0$. As θ becomes more negative $\sin\theta$ is negative and $\cos\theta$ decreases ($\omega^2\cos\theta < \frac{g}{R}$) so $\ddot{\theta}$ is positive and the bead accelerates back toward the equilibrium point. If θ becomes more positive $\sin\theta$ is still negative but $\cos\theta$ increases so $\ddot{\theta}$ is negative and the bead accelerates back to equilibrium.

7.29



Position of P: $R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y}$

position of bob: $l\sin\phi\hat{x} - l\cos\phi\hat{y}$

$$x = R\cos(\omega t) + l\sin\phi$$

$$y = R\sin(\omega t) - l\cos\phi$$

$$\dot{x} = -R\omega\sin(\omega t) + l\cos\phi\dot{\phi}$$

$$\dot{y} = R\omega\cos(\omega t) + l\sin\phi\dot{\phi}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[\left(-R\omega\sin(\omega t) + l\cos\phi\dot{\phi}\right)^2 + \left(R\omega\cos(\omega t) + l\sin\phi\dot{\phi}\right)^2\right]$$

$$= \frac{1}{2}m\left\{R^2\omega^2\sin^2(\omega t) + l^2\dot{\phi}^2\cos^2\phi - 2R\omega\sin(\omega t)l\cos\phi\dot{\phi} + R^2\omega^2\cos^2(\omega t) + l^2\dot{\phi}^2\sin^2\phi + 2R\omega\cos(\omega t)l\sin\phi\dot{\phi}\right\}$$

$$= \frac{1}{2}m\left[R^2\omega^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right) + l^2\dot{\phi}^2\left(\cos^2\phi + \sin^2\phi\right) + 2R\omega l\dot{\phi}\left(\cos(\omega t)\sin\phi - \sin(\omega t)\cos\phi\right)\right]$$

$$T = \frac{1}{2}m\left[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)\right]$$

$$U = mgy$$

$$U = mg(R\sin(\omega t) - l\cos\phi)$$

$$L = T - U$$

$$L = \frac{1}{2}m\left\{R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)\right\} - mg(R\sin(\omega t) - l\cos\phi)$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\frac{\partial L}{\partial \phi} = mR\omega l\dot{\phi}\cos(\phi-\omega t) - mgl\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + mR\omega l\sin(\phi-\omega t)$$

$$mR\omega l\dot{\phi}\cos(\phi-\omega t) - mgl\sin\phi = \left(ml^2\dot{\phi} + mR\omega l\sin(\phi-\omega t)\right)\frac{d}{dt}$$

$$R\omega l\phi\cos(\phi-\omega t) - gl\sin\phi = l^2\ddot{\phi} + R\omega l\cos(\phi-\omega t)(\dot{\phi}-\omega)$$

$$= l^2\ddot{\phi} + R\omega l\cos(\phi-\omega t)\dot{\phi} - R\omega^2 l\cos(\phi-\omega t)$$

$$-gl\sin\phi = l^2\ddot{\phi} - R\omega^2 l\cos(\phi-\omega t)$$

$$\boxed{l\ddot{\phi} = R\omega^2\cos(\phi-\omega t) - g\sin\phi}$$

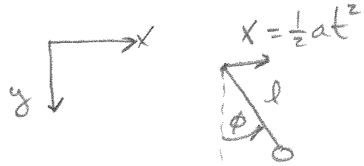
when $\omega=0$ the equation becomes that of a fixed pivot pendulum, i.e.

$$l\ddot{\phi} = -g\sin\phi$$

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0$$

1.50

a)



$$x = \tfrac{1}{2}at^2 + l\sin\phi$$

$$y = l\cos\phi$$

$$\dot{x} = at + l\cos\phi\,\dot{\phi}$$

$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \tfrac{1}{2}m\left[(at + l\cos\phi\dot{\phi})^2 + (-l\sin\phi\dot{\phi})^2\right]$$

$$= \tfrac{1}{2}m\left[a^2t^2 + 2atl\cos\phi\dot{\phi} + l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi\right]$$

$$T = \tfrac{1}{2}m\left[a^2t^2 + l^2\dot{\phi}^2 + 2atl\cos\phi\dot{\phi}\right]$$

$$U = -mgy$$

$$U = -mgl\cos\phi$$

$$L = T - U$$

$$\boxed{L = \tfrac{1}{2}m\left[a^2t^2 + l^2\dot{\phi}^2 + 2atl\cos\phi\dot{\phi}\right] + mgl\cos\phi}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\frac{\partial L}{\partial \phi} = -matl\sin\phi\dot{\phi} - mgl\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + matl\cos\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = ml^2\ddot{\phi} + mal\cos\phi - matl\sin\phi\dot{\phi}$$

$$-matl\sin\phi\dot{\phi} - mgl\sin\phi - ml^2\ddot{\phi} - mal\cos\phi + matl\sin\phi\dot{\phi} = 0$$

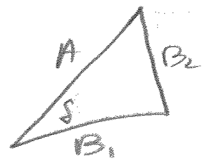
$$-gl\sin\phi - l^2\ddot{\phi} - al\cos\phi = 0$$

$$l\ddot{\phi} = -g\sin\phi - a\cos\phi$$

Using the hint of Eq. 5.11 (and related equations)

$$x = B_1 \cos(\omega t) + B_2 \sin(\omega t)$$

$$A = \sqrt{B_1^2 + B_2^2}$$



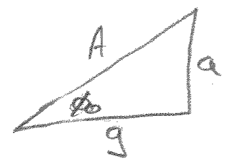
$$x = A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right]$$

$$= A\left[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right]$$

$$x = A\cos(\omega t - \delta)$$

Let $B_1 = g$, $B_2 = a$ {so result is of form $\sin(\alpha+\beta)$}

$$l\ddot{\phi} = -\sqrt{a^2+g^2}\left(\frac{g}{\sqrt{a^2+g^2}}\sin\phi + \frac{a}{\sqrt{a^2+g^2}}\cos\phi\right)$$



$$= -\sqrt{a^2+g^2}\left(\cos\phi_0\sin\phi + \sin\phi_0\cos\phi\right)$$

$$\boxed{l\ddot{\phi} = -\sqrt{a^2+g^2}\,\sin(\phi+\phi_0)}$$

b) $\ddot{\phi} = 0$

$$0 = \sin(\phi_{eq} + \phi_0) \longrightarrow \phi_{eq} = -\phi_0 = \tan^{-1}\left(\frac{a}{g}\right)$$

$$\phi_{eq} = \tan^{-1}\left(\frac{a}{g}\right)$$

Let $\phi = \phi_{eq} + \delta\phi = -\phi_0 + \delta\phi$

$$l\ddot{\phi} = -\sqrt{a^2+g^2}\,\sin(\delta\phi)$$

$$l\delta\ddot{\phi} \approx -\sqrt{a^2+g^2}\,\delta\phi$$

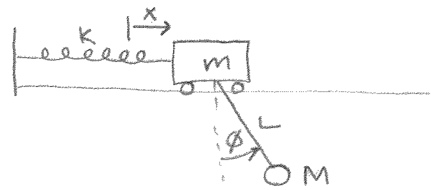
$$\delta\ddot{\phi} \approx -\frac{\sqrt{a^2+g^2}}{l}\,\delta\phi$$

Thus, for a positive displacement from ϕ_{eq} the acceleration is in the opposite direction. the same is true for a negative displacement. therefore, ϕ_{eq} is a stable equilibrium point.

$$\delta\ddot{\phi} + \frac{\sqrt{a^2+g^2}}{l}\,\delta\phi = 0$$

$$\omega^2 = \frac{\sqrt{a^2+g^2}}{l} \quad \text{\{small oscillations\}}$$

7.31



$m: \; x_1 = x \rightarrow \dot{x}_1 = \dot{x}$

$M \quad x_2 = x + L\sin\phi \rightarrow \dot{x}_2 = \dot{x} + L\cos\phi\,\dot{\phi}$

$\quad\; y_2 = L\cos\phi$

$T_1 = \frac{1}{2}m\dot{x}_1^2$

$\quad = \frac{1}{2}m\dot{x}^2$

$T_2 = \frac{1}{2}M\dot{x}_2^2$

$\quad = \frac{1}{2}M\left(\dot{x} + L\cos\phi\,\dot{\phi}\right)^2$

$T = \frac{1}{2}M\left[\dot{x}^2 + L^2\dot{\phi}^2\cos\phi + 2\dot{x}\dot{\phi}L\cos\phi\right]$

$U_1 = \frac{1}{2}Kx_1^2$

$\quad = \frac{1}{2}Kx^2$

$U_2 = -Mgy_2$

$\quad = -MgL\cos\phi$

$L = T - U$

$$\boxed{L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}M\left(L^2\dot{\phi}^2 + 2\dot{x}\dot{\phi}L\cos\phi\right) - \frac{1}{2}Kx^2 + MgL\cos\phi}$$

$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$\frac{\partial L}{\partial x} = -Kx$

$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ML\dot{\phi}\cos\phi$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (M+m)\ddot{x} + ML\left(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\right)$

$$\boxed{(M+m)\ddot{x} + ML\left(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\right) = -Kx}$$

$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$

$\frac{\partial L}{\partial \phi} = -M\dot{x}\dot{\phi}L\sin\phi - MgL\sin\phi$

$\frac{\partial L}{\partial \dot{\phi}} = ML^2\dot{\phi} + M\dot{x}L\cos\phi$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = ML^2\ddot\phi + ML(\ddot x\cos\phi - \dot x\dot\phi\sin\phi)$$

$$-M\dot x\dot\phi L\sin\phi - MgL\sin\phi - ML^2\ddot\phi - ML\ddot x\cos\phi - ML\dot x\dot\phi\sin\phi = 0$$

$$ML^2\ddot\phi + ML\ddot x\cos\phi = -MgL\sin\phi$$

$$\boxed{M(L\ddot\phi + \ddot x\cos\phi) = -Mg\sin\phi}$$

b) $x \ll 1,\ \phi \ll 1$

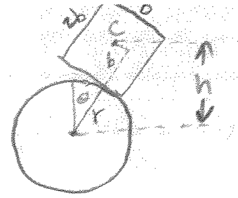
$\cos\phi \approx 1$

$\sin\phi \approx \phi$

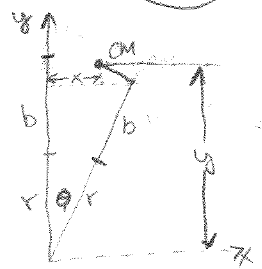
$$\boxed{(M+m)\ddot x + ML\ddot\phi = -kx}$$ $\{\dot\phi^2$ negligently small$\}$

$$\boxed{M(L\ddot\phi + \ddot x) = -Mg\phi}$$

7.32



$b < r$



$$x_{cm} = (r+b)\sin\theta - r\theta\cos\theta \qquad y_{cm} = (r+b)\cos\theta + r\theta\sin\theta$$

$$\dot{x}_{cm} = (r+b)\cos\theta\dot{\theta} - r(\dot{\theta}\cos\theta - \theta\dot{\theta}\sin\theta) \qquad \dot{y}_{cm} = -(r+b)\sin\theta\dot{\theta} + r(\dot{\theta}\sin\theta + \theta\dot{\theta}\cos\theta)$$

$$\dot{x}_{cm} = r\dot{\theta}\cos\theta + b\dot{\theta}\cos\theta - r\dot{\theta}\cos\theta + r\theta\dot{\theta}\sin\theta$$

$$\dot{x}_{cm} = b\dot{\theta}\cos\theta + r\theta\dot{\theta}\sin\theta$$

$$\dot{y}_{cm} = -r\dot{\theta}\sin\theta - b\dot{\theta}\sin\theta + r\dot{\theta}\sin\theta + r\theta\dot{\theta}\cos\theta$$

$$\dot{y}_{cm} = -b\dot{\theta}\sin\theta + r\theta\dot{\theta}\cos\theta$$

$$T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\dot{\theta}^2 \quad , \; I = \frac{2}{3}mb^2$$

$$v_{cm}^2 = \dot{x}_{cm}^2 + \dot{y}_{cm}^2 \quad \{ v^2 = \vec{v}\cdot\vec{v} \}$$

$$= (b\dot{\theta}\cos\theta + r\theta\dot{\theta}\sin\theta)^2 + (-b\dot{\theta}\sin\theta + r\theta\dot{\theta}\cos\theta)^2$$

$$= b^2\dot{\theta}^2\cos^2\theta + r^2\theta^2\dot{\theta}^2\sin^2\theta + 2rb\dot{\theta}^2\cos\theta\sin\theta +$$
$$b^2\dot{\theta}^2\sin^2\theta + r^2\theta^2\dot{\theta}^2\cos^2\theta - 2rb\dot{\theta}^2\cos\theta\sin\theta$$

$$= b^2\dot{\theta}^2 + r^2\theta^2\dot{\theta}^2$$

$$T = \frac{1}{2}m(b^2\dot{\theta}^2 + r^2\theta^2\dot{\theta}^2) + \frac{1}{3}mb^2\dot{\theta}^2$$

$$= \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mr^2\theta^2\dot{\theta}^2 + \frac{1}{3}mb^2\dot{\theta}^2$$

$$T = \frac{5}{6}mb^2\dot{\theta}^2 + \frac{1}{2}mr^2\theta^2\dot{\theta}^2$$

$$U = mgy$$

$$U = mg[(r+b)\cos\theta + r\theta\sin\theta]$$

$$L = T - U$$

$$L = \frac{5}{6} m b^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \theta^2 \dot{\theta}^2 - mg\left[(r+b)\cos\theta + r\theta\sin\theta\right]$$

For $\theta \ll 1$

$\cos\theta \cong 1$

$\sin\theta \cong \theta$

$$L = \frac{5}{6} m b^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \theta^2 \dot{\theta}^2 - mg\left[(r+b) + r\theta^2\right]$$

$$\frac{\partial L}{\partial \theta} = m r^2 \theta \dot{\theta}^2 - 2mgr\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{5}{3} m b^2 \dot{\theta} + m r^2 \theta^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{5}{3} m b^2 \ddot{\theta} + m r^2 \left(2\theta\dot{\theta}^2 + \theta^2 \ddot{\theta}\right)$$

$$= \frac{5}{3} m b^2 \ddot{\theta} + 2 m r^2 \theta \dot{\theta}^2 + m r^2 \theta^2 \ddot{\theta}$$

$$= \ddot{\theta}\left(\frac{5}{3} m b^2 + m r^2 \theta^2\right) + 2 m r^2 \theta \dot{\theta}^2$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$m r^2 \theta \dot{\theta}^2 - 2mgr\theta - \ddot{\theta}\left(\frac{5}{3} m b^2 + m r^2 \theta^2\right) - 2 m r^2 \theta \dot{\theta}^2 = 0$$

Again, since $\theta \ll 1$, this gives

$$-2mgr\theta - \ddot{\theta}\left(\frac{5}{3} m b^2\right) = 0$$

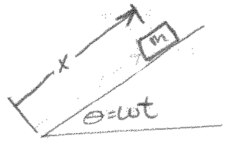
$$\ddot{\theta}\left(\frac{5}{3} m b^2\right) + 2mgr\theta = 0$$

$$\ddot{\theta} + \frac{2mgr}{\frac{5}{3} m b^2}\theta = 0$$

$$\ddot{\theta} + \frac{6gr}{5b^2}\theta = 0$$

$$\boxed{\omega^2 = \frac{6gr}{5b^2}}$$

7.33



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$I = mx^2, \quad \dot{\theta} = \omega$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2$$

$$U = mgh$$

$$h = x\sin\theta$$
$$= x\sin(\omega t)$$

$$U = mgx\sin(\omega t)$$

$$L = T - U$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2 - mgx\sin(\omega t)$$

$$\frac{\partial L}{\partial x} = mx\omega^2 - mg\sin(\omega t)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$mx\omega^2 - mg\sin(\omega t) - m\ddot{x} = 0$$

$$\boxed{\ddot{x} - \omega^2 x = -g\sin(\omega t)}$$

$$x(0) = x_0$$

Homogeneous Solution:

$$\ddot{x} - \omega^2 x = 0$$

$$\lambda^2 - \omega^2 = 0$$

$$\lambda^2 = \omega^2$$

$$\lambda = \pm\omega$$

$$x_h = Ae^{\pm\omega t}$$

$$x_h = A_1 e^{\omega t} + A_2 e^{-\omega t}$$

Particular Solution:

$$x = A\sin(\omega t)$$

$$\dot{x} = A\omega\cos(\omega t)$$

$$\ddot{x} = -A\omega^2\sin(\omega t)$$

$$\ddot{x} - \omega^2 x = -g\sin(\omega t)$$

$$-A\omega^2 - \omega^2 A = -g$$

$$2A\omega^2 = g$$

$$A = \frac{g}{2\omega^2}$$

$$x_p = \frac{g}{2\omega^2}\sin(\omega t)$$

$$x = x_h + x_p$$

$$x = A_1 e^{\omega t} + A_2 e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$$

$$x(0) = A_1 + A_2 = x_0$$

$$\dot{x} = A_1\omega e^{\omega t} - A_2\omega e^{-\omega t} + \frac{g}{2\omega}\cos(\omega t)$$

$$\dot{x}(0) = A_1\omega - A_2\omega + \frac{g}{2\omega} = 0$$

$$A_1\omega - A_2\omega = -\frac{g}{2\omega}$$

$$A_1\omega + A_2\omega = \omega x_0$$

$$2A_1\omega = \omega x_0 - \frac{g}{2\omega}$$

$$A_1 = \frac{x_0}{2} - \frac{g}{4\omega^2}$$

$$A_2 = x_0 - A_1$$

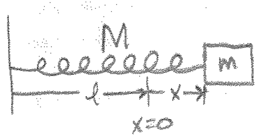
$$A_2 = \frac{x_0}{2} + \frac{g}{4\omega^2}$$

$$x = \left(\frac{x_0}{2} - \frac{g}{4\omega^2}\right)e^{\omega t} + \left(\frac{x_0}{2} + \frac{g}{4\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$$

$$= \frac{x_0}{2}\left(e^{\omega t} + e^{-\omega t}\right) + \frac{g}{4\omega^2}\left(e^{-\omega t} - e^{\omega t}\right) + \frac{g}{2\omega^2}\sin(\omega t)$$

$$\boxed{x = x_0\cosh(\omega t) - \frac{g}{2\omega^2}\sinh(\omega t) + \frac{g}{2\omega^2}\sin(\omega t)}$$

1.34



a) Length element is dx

$$dm = \frac{dx}{l} M$$

$$T = \int_0^l \frac{1}{2} v^2 \, dm$$

V is proportional to the length. So,

$$V = \frac{v x}{l}$$

$$T = \int_0^l \frac{1}{2}\left(\frac{v x}{l}\right)^2 \frac{dx}{l} M$$

$$T = \frac{M v^2}{2 l^3} \int_0^l x^2 \, dx$$

$$= \frac{M v^2}{2 l^3} \frac{x^3}{3}\Big|_0^l$$

$$T = \frac{M v^2}{6}$$

$$v = \dot{x}$$

$$\boxed{T = \frac{1}{6} M \dot{x}^2}$$

$$T = T_s + T_m$$

$$T = \frac{1}{6} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$

$$L = T - U$$

$$= \frac{1}{6} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\boxed{L = \left(\frac{1}{6} M + \frac{1}{2} m\right) \dot{x}^2 - \frac{1}{2} k x^2}$$

b) $\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$

$$\frac{\partial L}{\partial x} = -Kx$$

$$\frac{\partial L}{\partial \dot{x}} = \left(\frac{M}{3}+m\right)\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \left(\frac{M}{3}+m\right)\ddot{x}$$

$$-Kx - \left(\frac{M}{3}+m\right)\ddot{x} = 0$$

$$\ddot{x}\left(\frac{M+3m}{3}\right) + Kx = 0$$

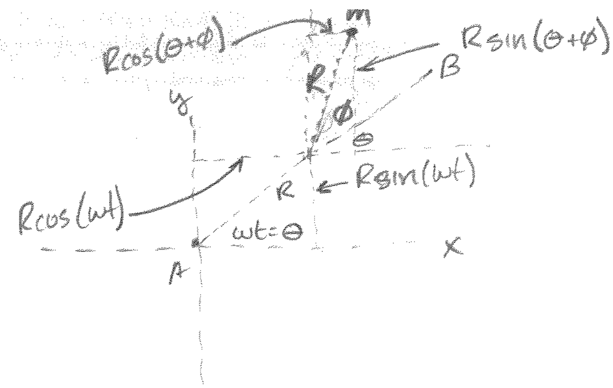
$$\ddot{x} + \frac{3K}{M+3m}x = 0$$

$$\omega^2 = \frac{3K}{M+3m}$$

$$= \frac{K}{\frac{M}{3}+m}$$

$$\boxed{\omega = \sqrt{\frac{K}{\frac{M}{3}+m}}}$$

7.35



$$x = R\cos(\omega t) + R\cos(\omega t+\phi) \qquad y = R\sin(\omega t) + R\sin(\omega t+\phi)$$

$$\dot{x} = -R\omega\sin(\omega t) - R(\omega+\dot{\phi})\sin(\omega t+\phi) \qquad \dot{y} = R\omega\cos(\omega t) + R(\omega+\dot{\phi})\cos(\omega t+\phi)$$

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$= \tfrac{1}{2}m\left[\left(-R\omega\sin(\omega t) - R(\omega+\dot{\phi})\sin(\omega t+\phi)\right)^2 + \left(R\omega\cos(\omega t) + R(\omega+\dot{\phi})\cos(\omega t+\phi)\right)^2\right]$$

$$= \tfrac{1}{2}m\left[R^2\omega^2\sin^2(\omega t) + R^2(\omega+\dot{\phi})^2\sin^2(\omega t+\phi) + 2R^2\omega(\omega+\dot{\phi})\sin(\omega t)\sin(\omega t+\phi) + R^2\omega^2\cos^2(\omega t) + R^2(\omega+\dot{\phi})^2\cos^2(\omega t+\phi) + 2R^2\omega(\omega+\dot{\phi})\cos(\omega t)\cos(\omega t+\phi)\right]$$

$$= \tfrac{1}{2}m\left[R^2\omega^2 + R^2(\omega+\dot{\phi})^2 + 2R^2\omega(\omega+\dot{\phi})\left(\cos(\omega t)\cos(\omega t+\phi) + \sin(\omega t)\sin(\omega t+\phi)\right)\right]$$

$$T = \tfrac{1}{2}m\left[R^2\omega^2 + R^2(\omega+\dot{\phi})^2 + 2R^2\omega(\omega+\dot{\phi})\cos\phi\right] \quad \left\{\cos(\omega t)\cos(\omega t+\phi) + \sin(\omega t)\sin(\omega t+\phi) = \cos(\omega t - (\omega t+\phi)) = \cos(-\phi) = \cos\phi\right\}$$

There is no potential energy.

$$L = T - U$$

$$\boxed{L = \tfrac{1}{2}m\left[R^2\omega^2 + R^2(\omega+\dot{\phi})^2 + 2R^2\omega(\omega+\dot{\phi})\cos\phi\right]}$$

$$\frac{\partial L}{\partial \phi} = -mR^2\omega(\omega+\dot{\phi})\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = mR^2(\omega+\dot{\phi}) + mR^2\omega\cos\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = mR^2\ddot{\phi} - mR^2\omega\sin\phi\,\dot{\phi}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$-mR^2\omega(\omega+\dot{\phi})\sin\theta - mR^2\ddot{\phi} + mR^2\omega\sin\phi\,\dot{\phi} = 0$$

$$-\omega(\omega+\dot{\phi})\sin\theta - \ddot{\phi} + \omega\dot{\phi}\sin\phi = 0$$

$$-\omega^2\sin\theta - \ddot{\phi} = 0$$

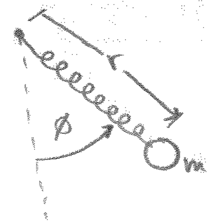
$$\ddot{\phi} + \omega^2\sin\theta = 0 \quad \text{(same form as a simple pendulum)}$$

For $\phi \ll 1$

$$\ddot{\phi} + \omega^2 \phi = 0$$

Frequency of small oscillations is ω

7.36



a) $x = r\sin\phi$ $\qquad$ $y = -r\cos\phi$

$\dot{x} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$ $\qquad$ $\dot{y} = -\dot{r}\cos\phi + r\dot{\phi}\sin\phi$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2 + (-\dot{r}\cos\phi + r\dot{\phi}\sin\phi)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi + \dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\right]$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right]$$

$$U = mgh + \frac{1}{2}kr^2$$

$$h = r - r\cos\phi$$

$$U = mgr(1-\cos\phi) + \frac{1}{2}kr^2$$

$$L = T - U$$

$$\boxed{L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right] - mgr(1-\cos\phi) - \frac{1}{2}kr^2}$$

b) $\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - mg(1-\cos\phi) - kr$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 - mg(1-\cos\phi) - kr - m\ddot{r} = 0$$

1) $$\boxed{m\ddot{r} - mr\dot{\phi}^2 = -mg(1-\cos\phi) - kr}$$

$$\frac{\partial L}{\partial \phi} = -mgr\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$-mgr\sin\phi - mr^2\ddot{\phi} - 2mr\dot{r}\dot{\phi}$$

2) $$\boxed{mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} + mgr\sin\phi = 0}$$

Eq. 1.48

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

$$\boxed{F_r = -mg(1-\cos\phi) - kr, \quad rF_\phi = -mgr\sin\phi \rightarrow F_\phi = -mg\sin\phi}$$

c) $\phi \ll 1$, $r = l + \varepsilon$ where $\varepsilon \ll 1$

$$m\ddot{r} - mr\dot{\phi}^2 = -mg(1-\cos\phi) - kr$$

$$m\ddot{\varepsilon} - m(l+\varepsilon)\dot{\phi}^2 = mg(1-1) - k(l+\varepsilon) \quad \{\cos\phi \cong 1,\ r = l+\varepsilon\}$$

$$m\ddot{\varepsilon} = -k(l+\varepsilon) \quad \{\dot{\phi}^2 \ll 1\}$$

$$\boxed{m\ddot{\varepsilon} + k\varepsilon = -kl}$$

$$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} + mgr\sin\phi = 0$$

$$m(l+\varepsilon)^2\ddot{\phi} + 2m\frac{d}{dt}(l+\varepsilon)\dot{\phi} + mg(l+\varepsilon)\phi = 0 \quad \{(l^2 + 2l\varepsilon + \varepsilon^2)\ddot{\phi} \cong \ddot{\phi}l^2;\ (l+\varepsilon)\phi \cong l\phi),\ \sin\phi \approx \phi\}$$

$$\boxed{ml^2\ddot{\phi} + mgl\phi = 0 \quad \{l+\varepsilon \cong l,\ \frac{d\varepsilon}{dt}\dot{\phi} \ll 1}$$

$$m\ddot{\varepsilon} + k\varepsilon = -kl$$

$$\ddot{\varepsilon} + \frac{k}{m}\varepsilon = \frac{-k}{m}l$$

$$\varepsilon_h: \ \ddot{\varepsilon} + \frac{k}{m}\varepsilon = 0$$

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda^2 = -\frac{k}{m}$$

$$\lambda = \pm i\sqrt{\frac{k}{m}}$$

$$\lambda = \pm i\omega_r \quad \left\{\omega_r = \sqrt{\frac{k}{m}}\right\}$$

$$\varepsilon_h = A\cos(\omega t - \delta)$$

$$\varepsilon_p: \ \varepsilon = C_1$$

$$\frac{k}{m}C_1 = \frac{-k}{m}l$$

$$C_1 = -l$$

$$\varepsilon_p = -l$$

$$\varepsilon = \varepsilon_h + \varepsilon_p$$

$$\boxed{\varepsilon = A_1\cos(\omega t - \delta) - l}$$

$$ml^2\ddot{\phi} + mgl\phi = 0$$

$$\ddot{\phi} + \frac{g}{l}\phi = 0$$

$$\lambda^2 + \frac{g}{l} = 0$$

$$\lambda^2 = -\frac{g}{l}$$

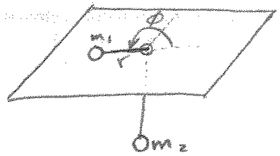
$$\lambda = \pm i\sqrt{\frac{g}{l}}$$

$$\lambda = \pm i\omega_\phi \quad \left\{\omega_\phi = \sqrt{\frac{g}{l}}\right\}$$

$$\boxed{\phi = A_2\cos(\omega_\phi t - \alpha)}$$

The motions are independent of each other; the mass oscillates along the r direction with $\omega_r = \sqrt{\frac{k}{m}}$ about the unstretched length l. The mass swings like a simple pendulum with $\omega_\phi = \sqrt{\frac{g}{l}}$.

1.31



Length of massless string = L
$M_1 = M_2 = M$

a) position of m_1: $x = r\cos\phi$ $\qquad$ $y = r\sin\phi$

$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \qquad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\sin\phi\cos\phi + \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\right]$$

$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right]$ but $\dot{r}$ has a mass of $m_1 + m_2 = 2m$

$T = \frac{1}{2}m\left[2\dot{r}^2 + r^2\dot{\phi}^2\right]$

$U = mgr$ {when $r = L$ then m_2 is at highest point}

$$L = T - U$$

$$L = \frac{1}{2}m(2\dot{r}^2 + r^2\dot{\phi}^2) - mgr$$

$$\boxed{L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr}$$

b) $\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - mg$

$$\frac{\partial L}{\partial \dot{r}} = 2m\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 2m\ddot{r}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\phi}^2 - mg - 2m\ddot{r} = 0$$

$$\boxed{2m\ddot{r} - mr\dot{\phi}^2 = -mg}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{d}{dt} mr^2\dot\phi$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = 0$$

$$\frac{d}{dt} mr^2\dot\phi = 0$$

$$\boxed{mr^2\dot\phi = \text{const.}}$$

The angular momentum of m_1

$$l_1 = I_1\dot\phi$$

$$I_1 = mr^2$$

$$l_1 = mr^2\dot\phi$$

So,

$$l_1 = l = \text{const.}$$

Angular momentum is conserved

c) $l = mr^2\dot\phi$

$$\dot\phi = \frac{l}{mr^2}$$

$$2m\ddot r - mr\dot\phi^2 = -mg$$

$$2m\ddot r - mr\left(\frac{l}{mr^2}\right)^2 = -mg$$

$$2m\ddot r - \frac{l^2}{mr^3} = -mg$$

For circular motion $\ddot r = 0$

$$\frac{l^2}{mr^3} = mg$$

$$r = r_0$$

$$r_0^3 = \frac{l^2}{m^2 g}$$

$$\boxed{r_0 = \left(\frac{l^2}{m^2 g}\right)^{1/3}}$$

d) $r(t) = r_0 + \varepsilon(t)$

$$2m\ddot{r} - \frac{\ell^2}{mr^3} = -mg$$

$$2m\ddot{\varepsilon} - \frac{\ell^2}{m(r_0+\varepsilon)^3} = -mg$$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots$$

$$(r_0+\varepsilon)^{-3} = r_0^{-3} + (-3)r_0^{-4}\varepsilon + 6r_0^{-5}\varepsilon^2 + \cdots$$

$$\approx r_0^{-3} - 3r_0^{-4}\varepsilon$$

$$2m\ddot{\varepsilon} - \frac{\ell^2}{m}\left(r_0^{-3} - 3r_0^{-4}\varepsilon\right) = -mg$$

$$2m\ddot{\varepsilon} - \frac{\ell^2}{mr_0^3} + \frac{3\ell^2\varepsilon}{mr_0^4} = -mg$$

$$2m\ddot{\varepsilon} + \frac{3\ell^2\varepsilon}{mr_0^4} = \frac{\ell^2}{mr_0^3} - mg$$

$$\ddot{\varepsilon} + \frac{3\ell^2}{2m^2r_0^4}\varepsilon = \frac{\ell^2}{2m^2r_0^3} - \frac{g}{2}$$

$$\ddot{\varepsilon} + \frac{3\ell^2}{2m^2r_0^4}\varepsilon = C \quad , \quad C = \frac{\ell^2}{2m^2r_0^3} - \frac{g}{2}$$

$$\varepsilon_h : \ddot{\varepsilon} + \omega^2\varepsilon = 0 \quad , \quad \omega^2 = \frac{3\ell^2}{2m^2r_0^4}$$

$$\varepsilon = \varepsilon_0\cos(\omega t - \delta)$$

$$\varepsilon_p : \varepsilon = C_1$$

$$\omega^2 C_1 = C$$

$$C_1 = \frac{C}{\omega^2}$$

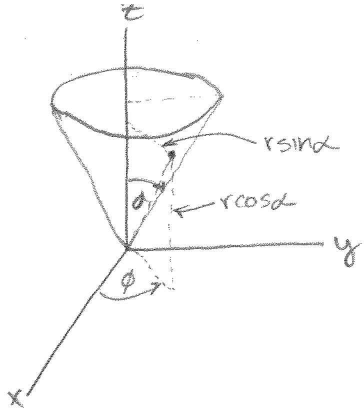
$$\varepsilon = \varepsilon_h + \varepsilon_p$$

stable about r_0

$$\varepsilon = \varepsilon_0\cos(\omega t - \delta) + \underbrace{\frac{\ell^2}{2m^2\omega^2r_0^3} - \frac{g}{2\omega^2}}_{\frac{C}{\omega^2}}$$

$$\boxed{\omega = \left(\frac{3}{2}\right)^{1/2}\frac{\ell}{mr_0^2}}$$

7.38



a) $x = r\sin\alpha\cos\phi$ $\quad y = r\sin\alpha\sin\phi$ $\quad z = r\cos\alpha$

$\dot{x} = \dot{r}\sin\alpha\cos\phi - r\dot{\phi}\sin\alpha\sin\phi$ $\quad \dot{y} = \dot{r}\sin\alpha\sin\phi + r\dot{\phi}\sin\alpha\cos\phi$ $\quad \dot{z} = \dot{r}\cos\alpha$

$\dot{x} = \sin\alpha(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)$ $\quad \dot{y} = \sin\alpha(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m\left[\sin^2\alpha(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + \sin^2\alpha(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2 + \dot{r}^2\cos^2\alpha\right]$$

$$= \frac{1}{2}m\left[\sin^2\alpha\left[\dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\sin\phi\cos\phi + \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\right] + \dot{r}^2\cos^2\alpha\right]$$

$$= \frac{1}{2}m\left[\sin^2\alpha\left[\dot{r}^2 + r^2\dot{\phi}^2\right] + \dot{r}^2\cos^2\alpha\right]$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right]$$

$$U = mgh$$

$$= mgr\cos\alpha$$

$$L = T - U$$

$$\boxed{L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right] - mgr\cos\alpha}$$

b) $$\frac{\partial L}{\partial r} = mr\dot{\phi}^2\sin^2\alpha - mg\cos\alpha$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$m r \dot{\phi}^2 \sin^2\alpha - mg\cos\alpha - m\ddot{r} = 0$$

$$\boxed{m\ddot{r} - m r \dot{\phi}^2 \sin^2\alpha = -mg\cos\alpha}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} \sin^2\alpha$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const.}$$

$$\boxed{m r^2 \dot{\phi} \sin^2\alpha = \text{const.}}$$

$$l_z = \text{const.}$$

$$m r^2 \dot{\phi} \sin^2\alpha = l_z$$

$$\dot{\phi} = \frac{l_z}{m r^2 \sin^2\alpha}$$

$$m\ddot{r} - m r \dot{\phi}^2 \sin^2\alpha = -mg\cos\alpha$$

$$m\ddot{r} - m r \left(\frac{l_z^2}{m^2 r^4 \sin^4\alpha}\right)\sin^2\alpha = -mg\cos\alpha$$

$$\boxed{m\ddot{r} - \frac{l_z^2}{m r^3 \sin^2\alpha} = -mg\cos\alpha}$$

When $l_z = 0$: $m\ddot{r} = -mg\cos\alpha$. This is a falling object constrained to the cone surface.

Set $\ddot{r} = 0$ to find r_0

$$\frac{l_z^2}{m r_0^3 \sin^2\alpha} = mg\cos\alpha$$

$$r_0^3 = \frac{l_z^2}{m^2 g \sin^2\alpha \cos\alpha}$$

$$\boxed{r_0 = \left(\frac{l_z^2}{m^2 g \sin^2\alpha \cos\alpha}\right)^{1/3}}$$

c) $r(t) = r_0 + \epsilon(t) \rightarrow \ddot{r} = \ddot{\epsilon}$

$$m\ddot{r} - \frac{l_z^2}{mr^3 \sin^2\alpha} = -mg\cos\alpha$$

$$(r_0+\epsilon)^{-3} = r_0^{-3} + (-3) r_0^{-4}\epsilon + 6 r_0^{-5}\epsilon^2 + \cdots \quad \left\{ (a+x)^n = a^n + n a^{n-1} x + \frac{n(n-1)}{2} a^{n-2} x^2 + \cdots \right\}$$

$$\cong r_0^{-3} - 3 r_0^{-4}\epsilon$$

$$m\ddot{\epsilon} - \frac{l_z^2}{m\sin^2\alpha}\left(r_0^{-3} - 3r_0^{-4}\epsilon\right) = -mg\cos\alpha$$

$$m\ddot{\epsilon} + \frac{3l_z^2}{mr_0^4\sin^2\alpha}\epsilon = \frac{l_z^2}{mr_0^3\sin^2\alpha} - mg\cos\alpha$$

$$\ddot{\epsilon} + \frac{3l_z^2}{m^2r_0^4\sin^2\alpha}\epsilon = \frac{l_z^2}{m^2r_0^3\sin^2\alpha} - g\cos\alpha$$

$$\ddot{\epsilon} + \omega^2\epsilon = C_1 \quad , \quad C_1 = \frac{l_z^2}{m^2r_0^3\sin^2\alpha} - g\cos\alpha \, , \quad \omega^2 = \frac{3l_z^2}{m^2r_0^4\sin^2\alpha}$$

ϵ_h: $\lambda^2 + \omega^2 = 0$

$$\lambda = \pm i\omega$$

$$\epsilon_h = A e^{\pm i\omega t}$$

$$\epsilon_h = \epsilon_0 \cos(\omega t - \delta)$$

ϵ_p: $\epsilon_p = C_2$

$$\omega^2 C_2 = C_1$$

$$C_2 = \frac{C_1}{\omega^2}$$

$$\epsilon_p = \frac{C_1}{\omega^2}$$

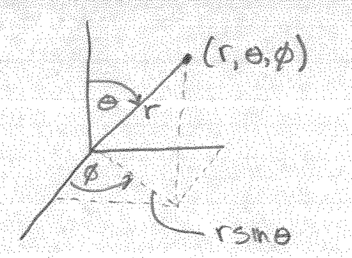
$$\epsilon = \epsilon_0\cos(\omega t - \delta) + \frac{C_1}{\omega^2}$$

The circular path is stable and oscillates about r_0.

$$\omega_r = \sqrt{3}\,\frac{l_z}{mr_0^2\sin\alpha}$$

7.39

a)



$x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

$\dot{x} = \dot{r}\sin\theta\cos\phi + r\left[-\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\cos\theta\cos\phi\right]$

$\dot{y} = \dot{r}\sin\theta\sin\phi + r\left[\dot{\phi}\sin\theta\cos\phi + \dot{\theta}\cos\theta\sin\phi\right]$

$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \tfrac{1}{2}m\Big[\left(\dot{r}\sin\theta\cos\phi + r\left[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right]\right)^2 + \left(\dot{r}\sin\theta\sin\phi + r\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right]\right)^2 + \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2\Big]$$

$\dot{x}^2 = \left(\dot{r}\sin\theta\cos\phi + r\left[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right]\right)^2 =$

$\dot{r}^2\sin^2\theta\cos^2\phi + r^2\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)^2 + 2r\dot{r}\sin\theta\cos\phi\left[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right] =$

$\dot{r}^2\sin^2\theta\cos^2\phi + r^2\left[\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi - 2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi\right] +$

$2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi - 2r\dot{r}\dot{\phi}\sin^2\theta\sin\phi\cos\phi =$

$\dot{r}^2\sin^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\cos^2\phi + r^2\dot{\phi}^2\sin^2\theta\sin^2\phi - 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi +$

$2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi - 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\sin^2\theta$

$\dot{y}^2 = \left(\dot{r}\sin\theta\sin\phi + r\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right]\right)^2 =$

$\dot{r}^2\sin^2\theta\sin^2\phi + r^2\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right]^2 + 2r\dot{r}\sin\theta\sin\phi\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right] =$

$\dot{r}^2\sin^2\theta\sin^2\phi + r^2\left[\dot{\theta}^2\cos^2\theta\sin^2\phi + \dot{\phi}^2\sin^2\theta\cos^2\phi + 2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi\right] +$

$2r\dot{r}\dot{\theta}\cos\theta\sin\theta\sin^2\phi + 2r\dot{r}\dot{\phi}\sin^2\theta\cos\phi\sin\phi =$

$\dot{r}^2\sin^2\theta\sin^2\phi + r^2\dot{\theta}^2\cos^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi + 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi +$

$2r\dot{r}\dot{\theta}\cos\theta\sin\theta\sin^2\phi + 2r\dot{r}\dot{\phi}\sin^2\theta\cos\phi\sin\phi$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\sin^2\theta\cos^2\phi + \dot{r}^2\sin^2\theta\sin^2\phi + r^2\dot{\theta}^2\cos^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\sin^2\phi +$$
$$r^2\dot{\phi}^2\sin^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi - 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi + 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi +$$
$$+ 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta\sin^2\phi - 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\sin^2\theta + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi\sin^2\theta$$

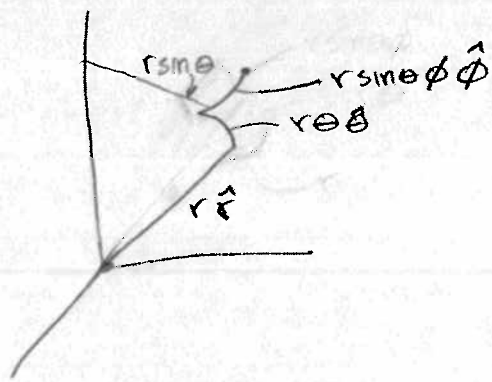
$$= \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$$

$$\dot{z}^2 = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2$$
$$= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2\sin^2\theta + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$$
$$- 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$$

The quick way is to identify the orthogonal velocities.



$\hat{r}$: $\dot{r}$

$\hat{\theta}$: $r\dot{\theta}$

$\hat{\phi}$: $r\sin\theta\dot{\phi}$

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right]$$

$$U = U(r)$$

$$\boxed{L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right] - U(r)}$$

b) $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - \frac{\partial U}{\partial r}$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$

$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$

$mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - \frac{\partial U}{\partial r} - m\ddot{r} = 0$

1) $\boxed{m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\phi}^2\sin^2\theta = -\frac{\partial U}{\partial r}}$

$\frac{\partial L}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta$

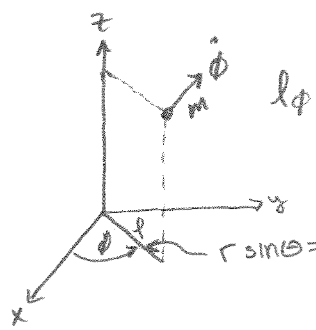
$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

2) $\boxed{\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\dot{\phi}^2\sin\theta\cos\theta}$

$\frac{\partial L}{\partial \phi} = 0$

$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta$

3) $\boxed{\frac{d}{dt}(mr^2\dot{\phi}\sin^2\theta) = 0}$



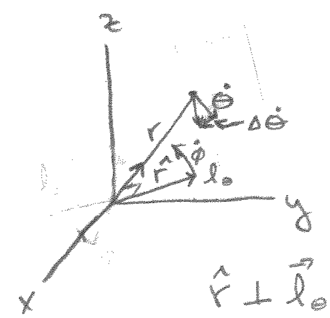
$l_\phi = m\ell\ell\dot{\phi}$
$= m\ell^2\dot{\phi}$
$= mr^2\dot{\phi}\sin^2\theta$

3) $mr^2\dot{\phi}\sin^2\theta = l_\phi$

Angular momentum in the $\hat{\phi}$ direction is conserved (about the z-axis).

2) $\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\dot{\phi}^2\sin\theta\cos\theta$

$\Delta\dot{\theta}$ points in the $-\hat{r}$ direction. So l_θ is about the origin and rotates about the z-axis at a rate $\dot{\phi}$ in the x-y plane. This makes sense because if $\dot{\phi} = 0$ then $mr^2\dot{\theta} = $ const. Therefore, l_θ is conserved; it's rotating in the x-y plane.



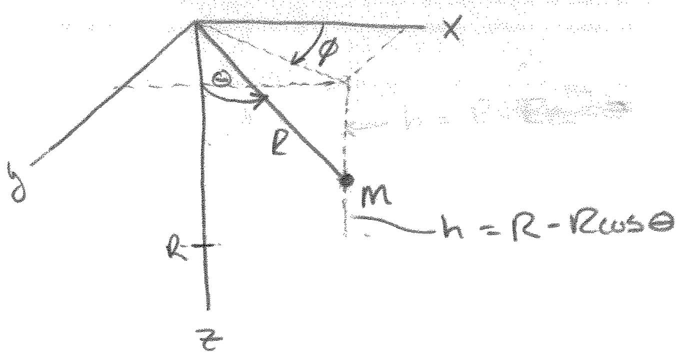
c) $\theta_0 = \frac{\pi}{2}$, $\dot{\theta}_0 = 0$

$\dot{\theta}_0 = 0$ produces no rotation in the $\hat{\theta}$ direction. Therefore, the motion remains in the equatorial plane $\theta = \frac{\pi}{2}$

d) $\dot{\phi}_0 = 0$

$\dot{\phi}_0 = 0$ produces no rotation in $\hat{\phi}$ direction. Therefore, the motion remains longitudinal at $\phi = \phi_0$.

7.40



a)

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

$$\dot{r} = 0 \quad \{r = R\}$$

$$V^2 = R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2$$

$$T = \tfrac{1}{2}m\left[R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right]$$

$$U = mgh$$

$$U = mgR(1-\cos\theta)$$

$$\boxed{L = \tfrac{1}{2}m\left[R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right] - mgR(1-\cos\theta)}$$

$$\frac{\partial L}{\partial\theta} = mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta}$$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0$$

$$\boxed{mR^2\ddot{\theta} = mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta}$$

$$\frac{\partial L}{\partial\phi} = 0$$

$$\frac{\partial L}{\partial\dot{\phi}} = mR^2\sin^2\theta\,\dot{\phi}$$

$$\frac{d}{dt}\left(mR^2\sin^2\theta\,\dot{\phi}\right) = 0$$

$$\boxed{mR^2\sin^2\theta\,\dot{\phi} = \text{const.}}$$

b) The ϕ equation shows the angular momentum about the z-axis is conserved.

c) when ϕ = constant the motion is along a line of longitude $\phi = \phi_0$.

d) $mR^2 \sin^2\theta \dot{\phi} = l_z$

$$\dot{\phi} = \frac{l_z}{mR^2 \sin^2\theta}$$

$$mR^2 \ddot{\theta} = mR^2 \dot{\phi}^2 \sin\theta\cos\theta - mgR\sin\theta$$

$$= mR^2 \frac{l_z^2}{m^2R^4\sin^4\theta} \sin\theta\cos\theta - mgR\sin\theta$$

The existance of θ_0 such that θ remains constant implies $\ddot{\theta} = \dot{\theta} = 0$.

$$0 = \frac{l_z^2}{m^2R^4\sin^3\theta_0}\cos\theta_0 - \frac{g}{R}\sin\theta_0$$

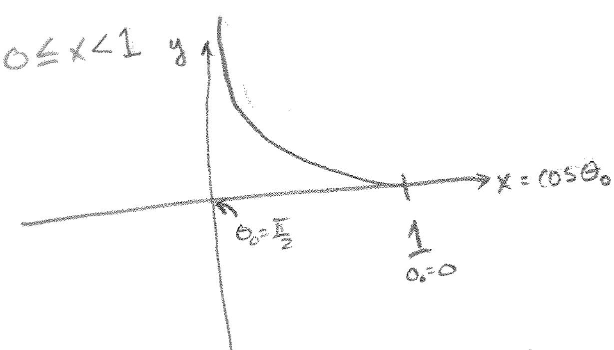
$$\frac{g}{R}\sin^4\theta_0 = \frac{l_z^2}{m^2R^4}\cos\theta_0$$

$$(\sin^2\theta_0)^2 = \frac{l_z^2}{m^2gR^3}\cos\theta$$

$$(1-\cos^2\theta_0)^2 = C_1\cos\theta_0 \qquad C_1 = \frac{l_z^2}{m^2gR^3}$$

$$\frac{(1-x^2)^2}{x} = C_1 \qquad \{x = \cos\theta_0\}$$

$0 \leq x < 1$



Therefore, a unique θ_0 exists for a given l_z and R. This motion sweeps out a cone hence the term "conical pendulum".

e) $\theta = \theta_0 + \varepsilon$

$$mR^2\ddot{\theta} = \frac{l_z^2}{mR^2\sin^3\theta}\cos\theta - mRg\sin\theta$$

$$\ddot{\theta} = \frac{l_z^2}{m^2R^4\sin^3\theta}\cos\theta - \frac{g}{R}\sin\theta$$

$$\cos(\theta_0+\varepsilon) = \cos\theta_0\cos\varepsilon - \sin\theta_0\sin\varepsilon$$
$$\cong \cos\theta_0 - \varepsilon\sin\theta_0$$

$$\sin(\theta_0+\varepsilon) = \sin\theta_0\cos\varepsilon + \cos\theta_0\sin\varepsilon$$
$$\cong \sin\theta_0 + \varepsilon\cos\theta_0$$

$$\sin^{-3}(\theta_0+\varepsilon) \cong (\sin\theta_0 + \varepsilon\cos\theta_0)^{-3}$$
$$\cong \sin^{-3}\theta_0 - 3\sin^{-4}\theta_0\,\varepsilon\cos\theta_0 + \text{terms of } \varepsilon^2 \text{ and higher} \quad \left\{(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots\right\}$$

$$\theta = \theta_0 + \varepsilon$$
$$\ddot{\theta} = \ddot{\varepsilon}$$

$$\ddot{\theta} = \frac{l_z^2\cos\theta}{m^2R^4\sin^3\theta} - \frac{g}{R}\sin\theta$$

$$\ddot{\varepsilon} = \frac{l_z^2}{m^2R^4}\cos(\theta_0+\varepsilon)\sin^{-3}(\theta_0+\varepsilon) - \frac{g}{R}\sin(\theta_0+\varepsilon)$$

$$\ddot{\varepsilon} \cong \frac{l_z^2}{m^2R^4}\left[\cos\theta_0 - \varepsilon\sin\theta_0\right]\left[\sin^{-3}\theta_0 - 3\varepsilon\cos\theta_0\sin^{-4}\theta_0\right] - \frac{g}{R}\left[\sin\theta_0 + \varepsilon\cos\theta_0\right]$$

$$\frac{l_z^2}{m^2R^4}\,\frac{\cos\theta_0}{\sin^3\theta_0} = \frac{g}{R}\sin\theta_0 \quad \{\text{condition that } \theta \text{ remains constant}\}$$

$$\frac{l_z^2}{m^2R^4} = \frac{g}{R}\,\frac{\sin^4\theta_0}{\cos\theta_0}$$

$$\ddot{\varepsilon} = \frac{g}{R}\frac{\sin^4\theta_0}{\cos\theta_0}\left[\cos\theta_0 - \varepsilon\sin\theta_0\right]\left[\sin^{-3}\theta_0 - 3\varepsilon\cos\theta_0\sin^{-4}\theta_0\right] - \frac{g}{R}\left[\sin\theta_0 + \varepsilon\cos\theta_0\right]$$

$$= \frac{g}{R}\frac{\sin^4\theta_0}{\cos\theta_0}\left[\frac{\cos\theta_0}{\sin^3\theta_0} - \frac{3\varepsilon\cos^2\theta_0}{\sin^4\theta_0} - \frac{\varepsilon}{\sin^2\theta_0} + \frac{3\varepsilon^2\cos\theta_0}{\sin^3\theta_0}\right] - \frac{g}{R}\left[\sin\theta_0 + \varepsilon\cos\theta_0\right]$$

$$= \frac{g}{R}\left[\sin\theta_0 - 3\varepsilon\cos\theta_0 - \frac{\varepsilon\sin^2\theta_0}{\cos\theta_0}\right] - \frac{g}{R}\left[\sin\theta_0 + \varepsilon\cos\theta_0\right] \quad \{\varepsilon^2 \text{ term dropped}\}$$

$$= \frac{g}{R}\left[-3\varepsilon\cos\theta_0 - \frac{\varepsilon\sin^2\theta_0}{\cos\theta_0} - \varepsilon\cos\theta_0\right]$$

$$= \frac{g}{R}\left[-4\varepsilon\cos\theta_0 - \frac{\varepsilon\sin^2\theta_0}{\cos\theta_0}\right]$$

$$= \frac{g\varepsilon}{R}\left[\frac{-4\cos^2\theta_0 - \sin^2\theta_0}{\cos\theta_0}\right]$$

$$= \frac{g\varepsilon}{R}\left[\frac{-4\cos^2\theta_0 - 1 + \cos^2\theta_0}{\cos\theta_0}\right]$$

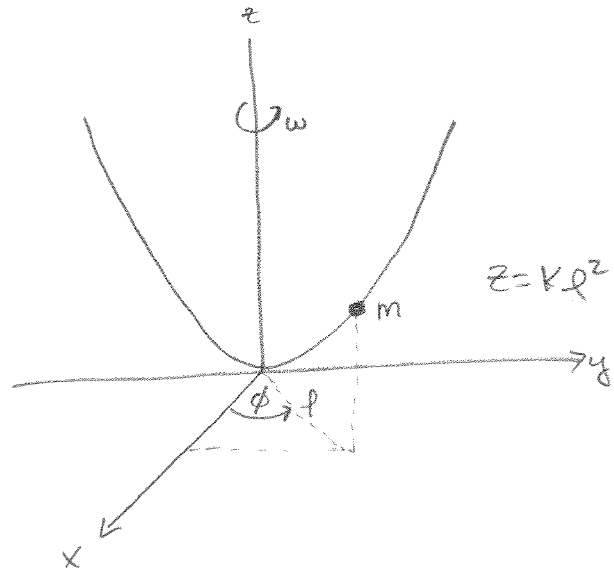
$$= \frac{g\varepsilon}{R}\left[\frac{-1 - 3\cos^2\theta_0}{\cos\theta_0}\right]$$

$$\ddot{\varepsilon} + \varepsilon \frac{g}{R}\left[\frac{1+3\cos^2\theta_0}{\cos\theta_0}\right] = 0 \quad \{\text{harmonic oscillator}\}$$

$$\boxed{\omega^2 = \frac{g}{R}\left[\frac{1+3\cos^2\theta_0}{\cos\theta_0}\right]}$$

The motion is small vibrations about θ_0 with constant $\dot{\phi}$

7.41



$x = \rho\cos\phi$ $\quad$ $y = \rho\sin\phi$ $\quad$ $z = k\rho^2$

$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$ $\quad$ $\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$ $\quad$ $\dot{z} = 2k\rho\dot{\rho}$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m\left[(\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi)^2 + (\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi)^2 + 4k^2\rho^2\dot{\rho}^2\right]$$

$$= \frac{1}{2}m\left[\dot{\rho}^2\cos^2\phi + \rho^2\dot{\phi}^2\sin^2\phi - 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + \dot{\rho}^2\sin^2\phi + \rho^2\dot{\phi}^2\cos^2\phi + 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + 4k^2\rho^2\dot{\rho}^2\right]$$

$$= \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + 4k^2\rho^2\dot{\rho}^2\right]$$

$$U = mgz$$

$$U = mgk\rho^2$$

$$L = T - U$$

$$\boxed{L = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + 4k^2\rho^2\dot{\rho}^2\right] - mgk\rho^2}$$

However, $\dot{\phi} = \omega$

$$\boxed{L = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right] - mgk\rho^2}$$

$$\frac{\partial L}{\partial \rho} = m\left[\rho\omega^2 + 4k^2\rho\dot{\rho}^2\right] - 2mgk\rho$$

$$\frac{\partial L}{\partial \dot{\rho}} = m\left[\dot{\rho} + 4k^2\rho^2\dot{\rho}\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\rho} = m[\ddot\rho + 4k^2(\rho^2\ddot\rho + 2\rho\dot\rho\dot\rho)]$$

$$= m[\ddot\rho + 4k^2(\rho^2\ddot\rho + 2\rho\dot\rho^2)]$$

$$\frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot\rho} = 0$$

$$m[\ddot\rho + 4k^2(\rho^2\ddot\rho + 2\rho\dot\rho^2)] = m[\rho\omega^2 + 4k^2\rho\dot\rho^2] - 2mgk\rho$$

$$m\ddot\rho + 4mk^2\rho^2\ddot\rho + 8mk^2\rho\dot\rho^2 = m\omega^2\rho + 4mk^2\rho\dot\rho^2 - 2mgk\rho$$

$$\boxed{(1+4k^2\rho^2)\ddot\rho + 4k^2\rho\dot\rho^2 = (\omega^2 - 2gk)\rho}$$

For determining equilibrium positions set $\ddot\rho = 0 = \dot\rho \rightarrow (\omega^2 - 2gk)\rho = 0$

$\rho = 0$ is a solution

The equation of motion is for small ρ

$$\ddot\rho = (\omega^2 - 2gk)\rho$$

For stability $\omega^2 - 2gk < 0$ {restoring force}

$\omega^2 < 2gk$

If $\omega^2 > 2gk$ then the equilibrium point $\rho = 0$ is unstable.

If $\omega^2 = 2gk$ then

$$(1+4k^2\rho^2)\ddot\rho + 4k^2\rho\dot\rho^2 = 0$$

Setting $\ddot\rho = 0 \rightarrow \dot\rho = \text{const.}$

$$4k^2\rho(\text{const.})^2 = 0$$

So, any value of ρ is an equilibrium point.

$$\ddot\rho = \frac{-4k^2\rho\dot\rho^2}{1+4k^2\rho^2}$$

For small ρ second order terms of $\rho, \dot\rho, \ddot\rho$ are ignored $(\rightarrow 0)$

$$\ddot\rho = 0$$

No restoring force exists so all points, except $\rho = 0$, are unstable when $\omega^2 = 2gk$.

7.45

a) Show that $A_{ij} = A_{ji}$

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2}$$

$$\dot{r}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i$$

$$\dot{\vec{r}}_\alpha^{\,2} = \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha = \sum_j \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j \right) \cdot \sum_k \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_k \right)$$

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{r}_\alpha^2$$

$$T = \frac{1}{2} \sum_{j,k} A_{jk} \dot{q}_j \dot{q}_k$$

$$A_{jk} = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_k} \right)$$

The dot product of two vectors is communitive:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = \text{scalar}$$

Let $\vec{A} = \frac{\partial \vec{r}_\alpha}{\partial q_i}$ and $\vec{B} = \frac{\partial \vec{r}_\alpha}{\partial q_j}$

then

$$\left(\frac{\partial \vec{r}_\alpha}{\partial q_i} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) = \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) \cdot \frac{\partial \vec{r}_\alpha}{\partial q_i}$$

Let

$$A_{ij} = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_i} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right)$$

$$A_{ji} = \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_i} \right)$$

then

$$A_{ij} = A_{ji}$$

b) Prove

$$\frac{\partial}{\partial v_i} \sum_{j,k} A_{jk} v_j v_k = 2 \sum_j A_{ij} v_j$$

for any n variables $v_1, \dots, v_n$

Let $n=2$

$i=2$

$i=1)$ $\frac{\partial}{\partial v_1} \sum_{j,k}^{2} A_{jk} v_j v_k = \frac{\partial}{\partial v_1}\left(A_{11} v_1 v_1 + A_{12} v_1 v_2 + A_{21} v_2 v_1 + A_{22} v_2 v_2\right)$

$= A_{11} v_1 + A_{11} v_1 + A_{12} v_2 + A_{21} v_2$

$= 2A_{11} v_1 + 2A_{12} v_2 \quad \{A_{12} = A_{21}\}$

$= 2(A_{11} v_1 + A_{12} v_2)$

$i=2)$ $\frac{\partial}{\partial v_2} \sum_{j,k}^{2} A_{jk} v_j v_k = \frac{\partial}{\partial v_2}\left(A_{11} v_1 v_1 + A_{12} v_1 v_2 + A_{21} v_2 v_1 + A_{22} v_2 v_2\right)$

$= A_{12} v_1 + A_{21} v_1 + A_{22} v_2 + A_{22} v_2$

$= 2A_{21} v_1 + 2A_{22} v_2 \quad \{A_{12} = A_{21}\}$

$= 2(A_{21} v_1 + A_{22} v_2)$

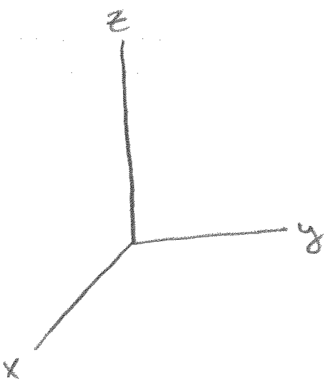
$i=1)$ $2(A_{11} v_1 + A_{12} v_2) = 2\sum_{j}^{2} A_{1j} v_j$

$i=2)$ $2(A_{21} v_1 + A_{22} v_2) = 2\sum_{j}^{2} A_{2j} v_j$

Therefore,

$$\frac{\partial}{\partial v_n} \sum_{j,k}^{n} A_{jk} v_j v_k = 2\sum_{j}^{n} A_{ij} v_j \quad \{i=n\}$$

7.46



a) Given $\delta L = 0$ about an axis of symmerty. Let the axis be z.

If $\phi_\alpha \to \phi_\alpha + \epsilon$ then

$$\delta L = \epsilon \frac{\partial L}{\partial \phi_1} + \cdots + \epsilon \frac{\partial L}{\partial \phi_N} = 0 \quad \{N \text{ particles}\}$$

$$\sum_{\alpha=1}^{N} \frac{\partial L}{\partial \phi_\alpha} = 0$$

b) Now,

$$\frac{\partial L}{\partial \phi_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_\alpha}$$

and

$$\sum_{\alpha=1}^{N} \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_\alpha} = 0$$

$$\sum_{\alpha=1}^{N} \frac{d}{dt} P_{\alpha\phi} = 0$$

Therefore, $L_z = P_\phi = \sum_\alpha P_{\alpha\phi}$ is constant and therefore conserved. And if the Lagrangian is invariant under rotations about all axes then all the components of $\vec{L}$ are conserved.

7.48

$$L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$$

$$L' = L + \frac{dF}{dt}$$

where $F = F(q_1, \cdots, q_n)$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial F}{\partial q_2}\frac{dq_2}{dt} + \cdots \quad \{\text{chain rule}\}$$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i$$

The Langrangian equation of F is

$$\frac{\partial F}{\partial q_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{q}_i} = 0$$

F is a function of q_i only. So,

$$\frac{\partial F}{\partial q_i} = 0$$

Therefore,

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i$$

$$\frac{dF}{dt} = 0$$

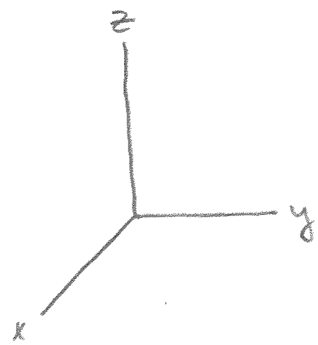
And,

$$L' = L + \frac{dF}{dt}$$

$$L' = L$$

So, L and L' give the same equations of motion.

1.44



$\vec{B} = B\hat{z}$

$\vec{E} = 0$

a) $\vec{B} = \vec{\nabla} \times \vec{A}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= i\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$B\hat{z} = \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

A solution to this equation is

$A_y = \frac{1}{2}Bx$

$A_x = -\frac{1}{2}By$

It's recognized A_y is a function of x, and A_x is a function of y. This implies a cross product relationship of $\vec{A}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\begin{vmatrix} i & j & k \\ A_x & A_y & 0 \\ x & y & 0 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{1}{2}B & \frac{1}{2}B & 0 \\ x & y & 0 \end{vmatrix} = \frac{1}{2}\vec{B} \times \vec{r}$$

So, $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$

In cylindrical coordinates

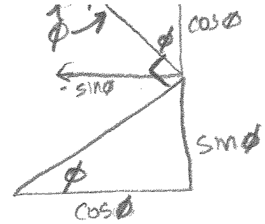
$A_y = \frac{1}{2}B\rho\cos\phi \quad \{x = \rho\cos\phi\}$

$A_x = \frac{1}{2}B\rho\sin\phi \quad \{y = \rho\sin\phi\}$

$|A| = \sqrt{A_x^2 + A_y^2} = \frac{1}{2}B\rho$

The unit vectors are

$$-\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi}$$



Therefore,

$$\vec{A} = \tfrac{1}{2}B\rho\hat{\phi}$$

b) $L = \frac{1}{2}m\dot{\vec{r}}^2 - q(V - \dot{\vec{r}}\cdot\vec{A})$

$$\vec{r} = \rho\hat{\rho} + \rho\phi\hat{\phi} + z\hat{z}$$

$$\dot{\vec{r}}^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$$

$$\vec{A} = \tfrac{1}{2}B\rho\hat{\phi}$$

$$L = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + q([\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}]\cdot\tfrac{1}{2}B\rho\hat{\phi})$$

$$\boxed{L = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + q(\tfrac{1}{2}B\rho^2\dot{\phi})}$$

$$\frac{\partial L}{\partial \rho} = qB\rho\dot{\phi} + m\rho\dot{\phi}^2$$

$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$$

$$\frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = 0$$

$$\boxed{m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} + \tfrac{1}{2}qB\rho^2$$

$$\boxed{\frac{d}{dt}\left(m\rho^2\dot{\phi} + \tfrac{1}{2}qB\rho^2\right) = 0}$$

$$\frac{\partial L}{\partial z} = 0$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\boxed{m\ddot{z} = 0}$$

c) when ρ is constant

i) $m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}$

$0 = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}$

$\dot{\phi}$ is quadratic. Solving for $\dot{\phi}$:

$\dot{\phi}^2 + \frac{qB}{m}\dot{\phi} = 0$

$\dot{\phi}(\dot{\phi} + \frac{qB}{m}) = 0$

$\boxed{\dot{\phi} = 0 \rightarrow \phi \text{ is constant}}$

$\dot{\phi} = \frac{-qB}{m}$

$\boxed{\phi = -\frac{qB}{m}t + C \rightarrow \omega = \frac{qB}{m}}$

ii) $m\rho^2\dot{\phi} + \frac{1}{2}qB\rho^2 = \text{constant}$

$\dot{\phi} = -\frac{qB}{2m} + \text{constant}$

$\boxed{\phi = (-\frac{qB}{2m} + C_1)t + C_2}$

iii) $m\ddot{z} = 0$

$\dot{z} = v_{z0}$

$\boxed{z = v_{z0}t + C}$

Angular velocity is constant and if $v_{z0} \neq 0$ then a helical path exists.