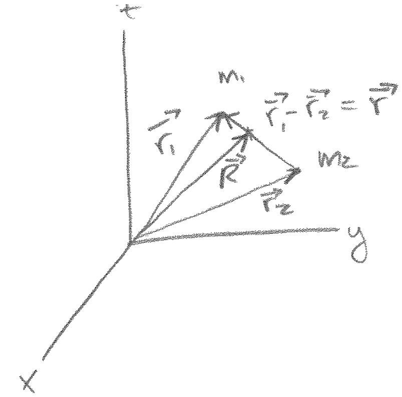
8.2 a)



$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$U = U(r) + m_1 g z_1 + m_2 g z_2 \qquad \vec{g} = g\hat{z}$$

$$L = T - U$$

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - (U(r) + m_1 g z_1 + m_2 g z_2)$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \quad , \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Express $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$.

r_1: $$\vec{R} = \frac{m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r})}{M}$$

$$M\vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_1 - m_2 \vec{r}$$

$$= \vec{r}_1 (m_1 + m_2) - m_2 \vec{r}$$

$$\frac{M\vec{R} + m_2 \vec{r}}{m_1 + m_2} = \vec{r}_1$$

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r}$$

$\vec{r}_2$: $$M\vec{R} = m_1(\vec{r} + \vec{r}_2) + m_2 \vec{r}_2$$

$$= m_1 \vec{r} + m_1 \vec{r}_2 + m_2 \vec{r}_2$$

$$\frac{M\vec{R} - m_1 \vec{r}}{m_1 + m_2} = \vec{r}_2$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2} \vec{r}$$

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1 + m_2} \dot{\vec{r}}$$

$$\dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1 + m_2} \dot{\vec{r}}$$

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r) - m_1 g z_1 - m_2 g z_2$$

$$\dot{\vec{r}}_1^{\,2} = \left[\dot{\vec{R}} + \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right]^2$$

$$= \dot{\vec{R}}^2 + 2\frac{m_2}{m_1+m_2}\dot{\vec{R}}\dot{\vec{r}} + \left(\frac{m_2}{m_1+m_2}\right)^2\dot{\vec{r}}^2$$

$$\dot{\vec{r}}_1^{\,2} = \dot{\vec{R}}^2 - 2\frac{m_1}{m_1+m_2}\dot{\vec{R}}\dot{\vec{r}} + \left(\frac{m_1}{m_1+m_2}\right)^2\dot{\vec{r}}^2$$

$$\frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}\left[(m_1+m_2)\dot{\vec{R}}^2 + \left(\frac{2m_1m_2}{m_1+m_2} - \frac{2m_1m_2}{m_1+m_2}\right)\dot{\vec{R}}\dot{\vec{r}} + \left(\frac{m_1m_2^2}{(m_1+m_2)^2} + \frac{m_2m_1^2}{(m_1+m_2)^2}\right)\dot{\vec{r}}^2\right]$$

$$= \frac{1}{2}\left[M\dot{\vec{R}}^2 + \frac{m_1m_2}{m_1+m_2}\left(\frac{m_2+m_1}{m_1+m_2}\right)\dot{\vec{r}}^2\right] \quad \{M = m_1+m_2\}$$

$$= \frac{1}{2}\left[M\dot{\vec{R}}^2 + \frac{m_1m_2}{M}\dot{\vec{r}}^2\right]$$

$$= \frac{1}{2}\left[M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right] \quad \left\{\mu = \frac{m_1m_2}{M}\right\}$$

$$Z = \frac{m_1z_1 + m_2z_2}{M}$$

$$MZ = m_1z_1 + m_2z_2$$

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) - g(m_1z_1 + m_2z_2)$$

$$L = \frac{1}{2}M\dot{\vec{R}}^2 - MgZ + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

$$L_{cm} = \frac{1}{2}M\dot{\vec{R}}^2 - MgZ$$

$$L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

b) $\vec{R} = X\hat{x} + Y\hat{y} + Z\hat{z}$

$$\dot{\vec{R}}^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \quad \{\dot{\vec{R}}^2 = \dot{\vec{R}}\cdot\dot{\vec{R}}\}$$

$$L_{cm} = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ$$

$$\frac{\partial L_{cm}}{\partial X} = \frac{\partial L_{cm}}{\partial Y} = 0 \ , \ \frac{\partial L_{cm}}{\partial Z} = -Mg$$

$$\frac{\partial L_{cm}}{\partial \dot{X}} = M\dot{X} \ , \ \frac{\partial L_{cm}}{\partial \dot{Y}} = M\dot{Y} \ , \ \frac{\partial L_{cm}}{\partial \dot{Z}} = M\dot{Z}$$

$$\frac{\partial L_{cm}}{\partial q_i} - \frac{d}{dt}\frac{\partial L_{cm}}{\partial \dot{q}_i} = 0$$

$X: \ M\ddot{X} = 0$

$Y: \ M\ddot{Y} = 0$

$Z: \ M\ddot{Z} = -Mg$

The X and Y components have zero acceleration, and the Z componet accelerates at $-g$.

$$L_{rel} = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - U(r)$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$= (x_1 - x_2)\hat{x} + (y_1 - y_2)\hat{y} + (z_1 - z_2)\hat{z}$$

$$\vec{r} = x_r\hat{x} + y_r\hat{y} + z_r\hat{z} \quad \{r = \text{relative}\}$$

$$\dot{\vec{r}}^{\,2} = \dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2$$

$$L_{rel} = \frac{1}{2}\mu[\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2] - U(|\vec{r}_1 - \vec{r}_2|) \qquad \{r = |\vec{r}_1 - \vec{r}_2|\}$$

$$= \frac{1}{2}\mu[\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2 - U(x_r) - U(y_r) - U(z_r)$$

$$\frac{\partial L_{rel}}{\partial x_r} = \frac{-\partial U(x_r)}{\partial x_r}, \quad \frac{\partial L_{rel}}{\partial y_r} = \frac{-\partial U(y_r)}{\partial y_r}, \quad \frac{\partial L_{rel}}{\partial z_r} = \frac{-\partial U(z_r)}{\partial z_r}$$

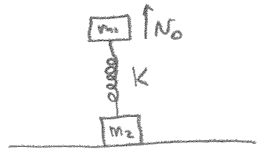
$$\frac{\partial L_{rel}}{\partial \dot{x}_r} = \mu\dot{x}_r, \quad \frac{\partial L_{rel}}{\partial \dot{y}_r} = \mu\dot{y}_r, \quad \frac{\partial L_{rel}}{\partial \dot{z}_r} = \mu\dot{z}_r$$

$$\mu\ddot{x}_r = \frac{-\partial U(x_r)}{\partial x_r}, \quad \mu\ddot{y}_r = \frac{-\partial U(y_r)}{\partial y_r}, \quad \mu\ddot{z}_r = -\frac{\partial U(z_r)}{\partial z_r}$$

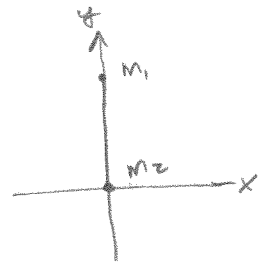
This is equivalent to

$$\mu\ddot{\vec{r}} = -\frac{\partial U(r)}{\partial r},$$

the components of $\mu(\ddot{x}_r + \ddot{y}_r + \ddot{z}_r) = -\left(\frac{\partial}{\partial x_r}U(x_r) + \frac{\partial}{\partial y_r}U(y_r) + \frac{\partial}{\partial z_r}U(z_r)\right)$.



L = natural length of spring



$$\vec{Y} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2}$$

$$\vec{Y} = \frac{m_1\vec{y_1} + m_2\vec{y_2}}{m_1+m_2}$$

$$\dot{\vec{Y}} = \frac{m_1\dot{\vec{y_1}} + m_2\dot{\vec{y_2}}}{M} \quad \{M = m_1+m_2\}$$

$$\vec{r} = \vec{y_1} - \vec{y_2}$$

$$\dot{\vec{r}} = \dot{\vec{y_1}} - \dot{\vec{y_2}}$$

Express $\vec{y_1}$ and $\vec{y_2}$ in terms of $\vec{r}$ and $\vec{Y}$

$$\vec{Y} = \frac{m_1\vec{y_1} + m_2(\vec{y_1} - \vec{r})}{M}$$

$$= \frac{\vec{y_1}(m_1+m_2) - m_2\vec{r}}{M}$$

$$\frac{M\vec{Y} + m_2\vec{r}}{m_1+m_2} = \vec{y_1}$$

$$\vec{y_1} = \vec{Y} + \frac{m_2}{m_1+m_2}\vec{r}$$

$$\boxed{\vec{y_1} = \vec{Y} + \frac{\mu}{m_1}\vec{r}} \quad \left\{\mu = \frac{m_1 m_2}{m_1+m_2}\right\}$$

$$\vec{Y} = \frac{m_1(\vec{y_2} + \vec{r}) + m_2\vec{y_2}}{M}$$

$$\frac{M\vec{Y} - m_1\vec{r}}{m_1+m_2} = \vec{y_2}$$

$$\vec{y_2} = \vec{Y} - \frac{m_1}{m_1+m_2}\vec{r}$$

$$\boxed{\vec{y_2} = \vec{Y} - \frac{\mu}{m_2}\vec{r}}$$

$$\dot{\vec{y}}_1 = \dot{\vec{Y}} + \frac{\mu}{m_1}\dot{\vec{r}}$$

$$\dot{\vec{y}}_2 = \dot{\vec{Y}} - \frac{\mu}{m_2}\dot{\vec{r}}$$

$$T = \frac{1}{2}m_1\dot{\vec{y}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{y}}_2^{\,2}$$

$$= \frac{1}{2}\left[m_1\left(\dot{\vec{Y}} + \frac{\mu}{m_1}\dot{\vec{r}}\right)^2 + m_2\left(\dot{\vec{Y}} - \frac{\mu}{m_2}\dot{\vec{r}}\right)^2\right]$$

$$= \frac{1}{2}\left[m_1\left(\dot{\vec{Y}}^2 + \frac{2\mu}{m_1}\dot{\vec{Y}}\dot{\vec{r}} + \left(\frac{\mu}{m_1}\right)^2\dot{\vec{r}}^2\right) + m_2\left(\dot{\vec{Y}}^2 - \frac{2\mu}{m_2}\dot{\vec{Y}}\dot{\vec{r}} + \left(\frac{\mu}{m_2}\right)^2\dot{\vec{r}}^2\right)\right]$$

$$= \frac{1}{2}\left[m_1\dot{\vec{Y}}^2 + 2\mu\dot{\vec{Y}}\dot{\vec{r}} + \frac{\mu^2}{m_1}\dot{\vec{r}}^2 + m_2\dot{\vec{Y}}^2 - 2\mu\dot{\vec{Y}}\dot{\vec{r}} + \frac{\mu^2}{m_2}\dot{\vec{r}}^2\right]$$

$$= \frac{1}{2}\left[\dot{\vec{Y}}^2(m_1+m_2) + \dot{\vec{r}}^2\left(\frac{\mu^2}{m_1} + \frac{\mu^2}{m_2}\right)\right]$$

$$= \frac{1}{2}\left[M\dot{\vec{Y}}^2 + \mu^2\left[\frac{m_1+m_2}{m_1m_2}\right]\dot{\vec{r}}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{\vec{Y}}^2 + \mu\dot{\vec{r}}^2\right]$$

$$U = m_1gy_1 + m_2gy_2 + \frac{1}{2}k(y_1-y_2-L)^2$$

$$= g(m_1y_1 + m_2y_2) + \frac{1}{2}k(r-L)^2$$

$$U = MgY + \frac{1}{2}k(r-L)^2 \quad \left\{Y = \frac{m_1y_1+m_2y_2}{M}\right\}$$

$$L = T - U$$

$$L = \frac{1}{2}M\dot{\vec{Y}}^2 - MgY + \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{1}{2}k(r-L)^2$$

$$\frac{\partial L}{\partial Y} = -Mg$$

$$\frac{\partial L}{\partial \dot{Y}} = M\dot{Y} \rightarrow \frac{d}{dt}M\dot{Y} = M\ddot{Y}$$

$$\ddot{Y} = -g$$

$$\dot{Y} = -gt + v_{Y_0}$$

$$Y = -\frac{1}{2}gt^2 + v_{Y_0}t + Y_0$$

$$\frac{\partial L}{\partial r} = -k(r-L)$$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \rightarrow \frac{d}{dt}\mu\dot{r} = \mu\ddot{r}$$

$\mu \ddot{r} + k(r-L) = 0$ ③

$\mu \ddot{r} + kr = kL$

$\ddot{r} + \frac{k}{\mu} r = \frac{k}{\mu} L$

$r_h: \; r = A_1 e^{\lambda t}$

$\lambda^2 + \frac{k}{\mu} = 0$

$\lambda = \pm i\omega \;, \; \omega = \sqrt{\frac{k}{\mu}}$

$r_h = A_1 e^{i\omega t} + A_1 e^{-i\omega t}$

$r_h = A\cos(\omega t - \delta)$

$r_p: \; r = C_1$

$\frac{k}{\mu} C_1 = \frac{k}{\mu} L$

$C_1 = L$

$r_p = L$

$r = r_h + r_p$

$r = A\cos(\omega t - \delta) + L$

$y_1 - y_2 = A[\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)] + L$

$y_1(0) - y_2(0) = L = A[\cos(\delta)] + L$

$A\cos(\delta) = 0$

$y_1 - y_2 = A\sin(\omega t)\sin(\delta) + L$

$\dot{y}_1 - \dot{y}_2 = A\omega\cos(\omega t)\sin(\delta)$

$\dot{y}_1(0) - \dot{y}_2(0) = v_0 = A\omega\sin(\delta)$

$\frac{v_0}{\omega} = A\sin(\delta)$

$y_1 - y_2 = \frac{v_0}{\omega}\sin(\omega t) + L$

$Y = Y_0 + v_{Y_0} t - \frac{1}{2} g t^2$

$\frac{m_1 y_1 + m_2 y_2}{M} = Y_0 + v_{Y_0} t - \frac{1}{2} g t^2$

$$\frac{m_1 y_1(0) + m_2 y_2(0)}{M} = Y_0$$

$$\frac{m_1 L}{M} = Y_0 \qquad \{ y_1(0) = L, y_2(0) = 0 \}$$

$$\frac{m_1 y_1 + m_2 y_2}{M} = \frac{m_1 L}{M} + v_{Y_0} t - \frac{1}{2} g t^2$$

$$\frac{m_1 \dot{y}_1 + m_2 \dot{y}_2}{M} = v_{Y_0} - g t$$

$$\frac{m_1 \dot{y}_1(0) + m_2 \dot{y}_2(0)}{M} = v_{Y_0}$$

$$\frac{m_1 v_0}{M} = v_{Y_0} \qquad \{ \dot{y}_1(0) = v_0, \dot{y}_2(0) = 0 \}$$

$$\frac{m_1 y_1 + m_2 y_2}{M} = \frac{m_1 L}{M} + \frac{m_1 v_0}{M} t - \frac{1}{2} g t^2$$

Now, solve for y_1 and y_2.

$$m_1 y_1 + m_2 \left(y_1 - \frac{v_0}{\omega} \sin(\omega t) - L \right) = m_1 L + m_1 v_0 t - \frac{M}{2} g t^2$$

$$m_1 y_1 + m_2 y_1 - \frac{v_0 m_2}{\omega} \sin(\omega t) - m_2 L = m_1 L + m_1 v_0 t - \frac{M}{2} g t^2$$

$$y_1 M = m_1 L + m_2 L + \frac{v_0 m_2}{\omega} \sin(\omega t) + m_1 v_0 t - \frac{M}{2} g t^2$$

$$\boxed{y_1 = L + \frac{m_1 v_0}{M} t + \frac{v_0 m_2}{M\omega} \sin(\omega t) - \frac{1}{2} g t^2}$$

$$m_1 \left(y_2 + \frac{v_0}{\omega} \sin(\omega t) + L \right) + m_2 y_2 = m_1 L + m_1 v_0 t - \frac{M}{2} g t^2$$

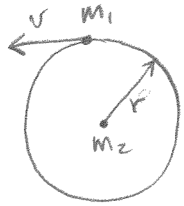
$$m_1 y_2 + m_2 y_2 + \frac{m_1 v_0}{\omega} \sin(\omega t) + m_1 L = m_1 L + m_1 v_0 t - \frac{M}{2} g t^2$$

$$y_2 M = m_1 v_0 t - \frac{m_1 v_0}{\omega} \sin(\omega t) - \frac{M}{2} g t^2$$

$$\boxed{y_2 = \frac{m_1 v_0}{M} t - \frac{m_1 v_0}{M\omega} \sin(\omega t) - \frac{1}{2} g t^2}$$

8.1

a)



$$F_1 = -\frac{G m_1 m_2}{r^2}$$

$$m_1 a = -\frac{G m_1 m_2}{r^2}$$

$$a = -\frac{G m_2}{r^2}$$

$$-\frac{v^2}{r} = -\frac{G m_2}{r^2}$$

$$v^2 = \frac{G m_2}{r}$$

$$v = \left(\frac{G m_2}{r}\right)^{1/2}$$

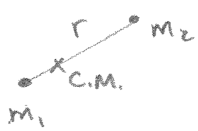
$$C = 2\pi r$$

$$T = t_p = \frac{C}{v}$$

$$= \frac{2\pi r}{\left(\frac{G m_2}{r}\right)^{1/2}}$$

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{G m_2}}}$$

b)



$$\vec{R} = \frac{m_1 r_1 + m_2 r_2}{M}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$T = \frac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right), \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad M = m_1 + m_2$$

$$U = U(r)$$

$$U(r) = -\int_{r_0}^{r} \frac{G m_1 m_2}{r'^2}\, dr'$$

$$= \left.\frac{G m_1 m_2}{r'}\right|_{r_0 = \infty}^{r}$$

$$U(r) = \frac{G m_1 m_2}{r}$$

$$L = T - U$$

Since the C.M. is not accelerating $L_{cm} = 0$

$$L = L_{cm} + L_{rel}$$

$$L = L_{rel}$$

$$L = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$$

$$\frac{\partial L}{\partial r} = \frac{-\partial U(r)}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{Gm_1m_2}{r}\right)$$

$$\frac{\partial L}{\partial r} = \frac{Gm_1m_2}{r^2}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \rightarrow \frac{d}{dt}\mu\dot{r} = \mu\ddot{r}$$

$$\mu\ddot{r} = \frac{Gm_1m_2}{r^2} \quad \left\{\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}\right\}$$

$$\frac{m_1m_2}{M}\ddot{r} = \frac{Gm_1m_2}{r^2}$$

$$\ddot{r} = \frac{GM}{r^2}$$

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

$$v^2 = \frac{GM}{r}$$

$$C = 2\pi r$$

$$T_{rel} = t_p = \frac{C}{v}$$

$$= \frac{2\pi r}{\left(\frac{GM}{r}\right)^{1/2}}$$

$$\boxed{T_{rel} = 2\pi\sqrt{\frac{r^3}{GM}}}$$

As $m_2 \rightarrow \infty$, $M = m_1 + m_2 \rightarrow m_2$ $\{m_1 << m_2\}$, So $T_{rel} \rightarrow T$.

c) $m_1 = m_2 = M_\odot$

$r = 1.5\times10^8\,km$ {earth orbital radius}

$r = 1.5\times10^{11}\,m$

$M = 2M_\odot$

$M_\odot = 1.99\times10^{30}\,kg$

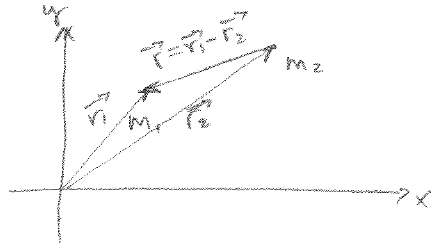
$G = 6.67\times10^{-11}\,\frac{m^3}{kg\,s^2}$

$$T_{rel} = 2\pi\sqrt{\frac{(1.5\times10^{11}\,m)^3}{6.67\times10^{-11}\frac{m^3}{kg\,s^2}\cdot(1.99\times10^{30}\,kg)\cdot 2}}$$

$$= 2.24\times10^7\,s$$

$$\boxed{T_{rel} = 0.71 \text{ years}}$$

0.8



$T=\frac{1}{2}M\dot{\vec{R}}^2+\frac{1}{2}\mu\dot{\vec{r}}^2$

$\vec{R}=\frac{m_1\vec{r_1}+m_2\vec{r_2}}{M}$

$U=\frac{1}{2}kr^2$

$\vec{r}=\vec{r_1}-\vec{r_2}$, $\vec{r_1}=x_1\hat{x}+y_1\hat{y}$, $\vec{r_2}=x_2\hat{x}+y_2\hat{y}$

$\mu=\frac{m_1m_2}{m_1+m_2}$

$L=T-U$

$L=\frac{1}{2}M\dot{\vec{R}}^2+\frac{1}{2}\mu\dot{\vec{r}}^2-\frac{1}{2}kr^2$

$\frac{\partial L}{\partial R}=0$

$\frac{\partial L}{\partial \dot{R}}=M\dot{R}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}}=M\ddot{R}$

$M\ddot{R}=0$

$\dot{R}=v_{0_R}$

$\boxed{R=v_{0_R}t+R_0}$

$\frac{\partial L}{\partial r}=-kr$

$\frac{\partial L}{\partial \dot{r}}=\mu\dot{r}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}=\mu\ddot{r}$

$\mu\ddot{r}+kr=0 \quad \left\{\frac{\partial L}{\partial r}-\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}=0\right\}$

$\ddot{r}+\frac{k}{\mu}r=0$

$r=A_1e^{\lambda t}$

$\lambda^2=-\frac{k}{\mu}$

$\lambda=\pm i\omega$, $\omega=\sqrt{\frac{k}{\mu}}$

$r=A_1e^{i\omega t}+A_1e^{-i\omega t}$

$r=A\cos(\omega t-\delta)$

Initial conditions:

$r(0) = r_0$

$\dot{r}(0) = v_{0_r}$

$r = A\cos(\omega t)\cos(\delta) + A\sin(\omega t)\sin(\delta)$

$r_0 = A\cos(\delta)$

$r = r_0\cos(\omega t) + A\sin(\omega t)\sin(\delta)$

$\dot{r} = -r_0\omega\sin(\omega t) + A\omega\cos(\omega t)\sin(\delta)$

$v_{0r} = A\omega\sin(\delta)$

$\frac{v_{0r}}{\omega} = A\sin(\delta)$

$$\boxed{r = r_0\cos(\omega t) + \frac{v_{0r}}{\omega}\sin(\omega t)}$$

$R = v_{0R}(t) + R_0$

$\frac{m_1 r_1 + m_2 r_2}{M} = v_{0R}(t) + R_0$

$m_1 r_{10} + m_2 r_{20} = MR_0 \quad \{ r_{10} = r_1(0),\ r_{20} = r_2(0) \}$

$m_1 \dot{r}_{10} + m_2 \dot{r}_{20} = M v_{0R} \quad \{ \dot{r}_{10} = \dot{r}_1(0),\ \dot{r}_{20} = \dot{r}_2(0) \}$

$m_1(\dot{x}_1\hat{x} + \dot{y}_1\hat{y}) + m_2(\dot{x}_2\hat{x} + \dot{y}_2\hat{y}) = M v_{0R}$

$$X\hat{x} + Y\hat{y} = \frac{m_1\dot{\vec{r}}_{10} + m_2\dot{\vec{r}}_{20}}{M}t + \frac{m_1\vec{r}_{10} + m_2\vec{r}_{20}}{M}$$

$\vec{r}_{10} = x_{10}\hat{x} + y_{10}\hat{y} \quad , \quad \vec{r}_{20} = x_{20}\hat{x} + y_{20}\hat{y}$

$\dot{\vec{r}}_{10} = \dot{x}_{10}\hat{x} + \dot{y}_{10}\hat{y} \quad , \quad \dot{\vec{r}}_{20} = \dot{x}_{20}\hat{x} + \dot{y}_{20}\hat{y}$

$$X\hat{x} + Y\hat{y} = \frac{m_1(\dot{x}_{10}\hat{x} + \dot{y}_{10}\hat{y}) + m_2(\dot{x}_{20}\hat{x} + \dot{y}_{20}\hat{y})}{M}t + \frac{m_1(x_{10}\hat{x} + y_{10}\hat{y}) + m_2(x_{20}\hat{x} + y_{20}\hat{y})}{M}$$

Equating components:

$$\boxed{X = \frac{m_1\dot{x}_{10} + m_2\dot{x}_{20}}{M}t + \frac{m_1 x_{10} + m_2 x_{20}}{M}}$$

$$\boxed{Y = \frac{m_1\dot{y}_{10} + m_2\dot{y}_{20}}{M}t + \frac{m_1 y_{10} + m_2 y_{20}}{M}}$$

$$r = r_0\cos(\omega t) + \frac{v_{0r}}{\omega}\sin(\omega t)$$

$$r_1 - r_2 = r_0\cos(\omega t) + \frac{v_{0r}}{\omega}\sin(\omega t)$$

$$(x_1-x_2)\hat{x} + (y_1-y_2)\hat{y} = \left[(x_{10}-x_{20})\hat{x} + (y_{10}-y_{20})\hat{y}\right]\cos(\omega t) + \frac{(\dot{x}_{10}-\dot{x}_{20})\hat{x} + (\dot{y}_{10}-\dot{y}_{20})\hat{y}}{\omega}\sin(\omega t)$$

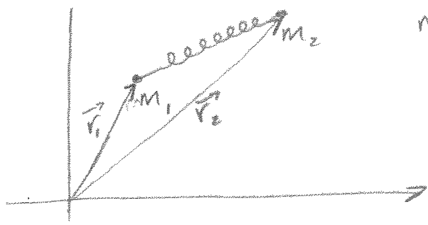
Equating components:

$$x_1 - x_2 = (x_{10}-x_{20})\cos(\omega t) + \frac{\dot{x}_{10}-\dot{x}_{20}}{\omega}\sin(\omega t)$$

$$y_1 - y_2 = (y_{10}-y_{20})\cos(\omega t) + \frac{\dot{y}_{10}-\dot{y}_{20}}{\omega}\sin(\omega t)$$

The X and Y motions are not accelerated. The $x_1 - x_2$ and $y_1 - y_2$ motion is periodic with angular frequency $\sqrt{\frac{k}{\mu}}$.

8.9



$m_1 = m_2 = m$

a)

$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$

$T = \frac{1}{2} m (\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2})$

$U = \frac{1}{2} k (r - L)^2 \quad , \quad r = |\vec{r}_1 - \vec{r}_2|$

$L = T - U$

$\boxed{L = \frac{1}{2} m (\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}) - \frac{1}{2} k (|\vec{r}_1 - \vec{r}_2| - L)^2}$

In terms of CM:

$L_{cm} = \frac{1}{2} M \dot{\vec{R}}^2$

$L_{rel} = \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - \frac{1}{2} k (r - L)^2$

$\dot{\vec{r}}$ in polar coordinates

$\dot{\vec{r}} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi}$

$\dot{\vec{r}}^{\,2} = \dot{\vec{r}} \cdot \dot{\vec{r}}$

$\dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2 \dot{\phi}^2$

$L = L_{cm} + L_{rel}$

$\boxed{L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{1}{2} k (r - L)^2}$

b) $\frac{\partial L}{\partial R} = 0$

$\frac{\partial L}{\partial \dot{R}} = M \dot{R}$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{R}} = M \ddot{R}$

$$M\ddot{R} = 0$$

$$\dot{R} = \dot{R}_0$$

$$R = \dot{R}_0 t + R_0$$

$$\vec{R} = X\hat{x} + Y\hat{y}$$

$$\dot{\vec{R}} = \dot{X}\hat{x} + \dot{Y}\hat{y}$$

$$\boxed{\begin{aligned} X &= \dot{X}_0 t + X_0 \\ Y &= \dot{Y}_0 t + Y_0 \end{aligned}}$$

c) $\dfrac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - k(r-L)$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \mu\ddot{r}$$

$$\boxed{\mu\ddot{r} - \mu r\dot{\phi}^2 + k(r-L) = 0} \qquad \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi}$$

$$\frac{d}{dt}(\mu r^2\dot{\phi}) = 0 \qquad \left\{\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0\right\}$$

$$\boxed{\mu r^2\dot{\phi} = \text{const.}}$$

i) r is constant

$$\ddot{r} = 0$$

$$\mu r^2\dot{\phi} = \text{const.}$$

$$\dot{\phi} = \frac{\text{const.}}{\mu r^2} = \text{const.}$$

$$\mu r\dot{\phi}^2 - k(r-L) = 0$$

$$\mu r\dot{\phi}^2 - kr + kL = 0$$

$$r(\mu\dot{\phi}^2 - k) = -kL$$

$$\boxed{r = \frac{kL}{(k - \mu\dot{\phi}^2)}}$$

relative uniform circular motion.

ii) ϕ is constant

$$\dot{\phi} = 0$$

$$\mu\ddot{r} + kr = kL$$

$$\ddot{r} + \frac{k}{\mu}r = \frac{k}{\mu}L$$

$$r_h = A\cos(\omega t - \delta), \quad \omega^2 = \frac{k}{\mu}$$

r_p: $r = C$

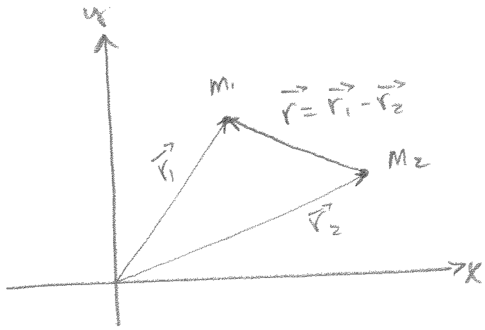
$C = L$

$r_p = L$

$$\boxed{r = A\cos(\omega t - \delta) + L, \quad \omega^2 = \frac{k}{\mu} = \frac{k}{\frac{m_2}{2m_1}} = \frac{2k}{m_1}}$$

Oscillation relative to each other,

8.10



$m_1 = m_2$

$U_1 = \frac{1}{2}kr_1^2$

$U_2 = \frac{1}{2}kr_2^2$

$U_{12} = \frac{1}{2}\alpha k r^2$

a) $T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$

$M = 2m_1$

$\mu = \frac{m_1^2}{m_1+m_1} = \frac{m_1}{2}$

$T = m_1\dot{\vec{R}}^2 + \frac{1}{4}m_1\dot{\vec{r}}^2$

$\frac{m_1\vec{r}_1 + m_1\vec{r}_2}{2m_1} = \vec{R}$

$\vec{r}_1 + \vec{r}_2 = 2\vec{R}$

$\vec{r}_1 - \vec{r}_2 = \vec{r}$

$2\vec{r}_1 = 2\vec{R} + \vec{r}$

$\vec{r}_1 = \vec{R} + \frac{\vec{r}}{2}$

$2\vec{r}_2 = 2\vec{R} - \vec{r}$

$\vec{r}_2 = \vec{R} - \frac{\vec{r}}{2}$

$$U_1 + U_2 = \frac{1}{2}k(r_1^2 + r_2^2)$$
$$= \frac{1}{2}k\left[\vec{R}^2 + \vec{R}\vec{r} + \frac{\vec{r}^2}{4} + \vec{R}^2 - \vec{R}\vec{r} + \frac{\vec{r}^2}{4}\right]$$
$$= \frac{1}{2}k\left[2\vec{R}^2 + \frac{\vec{r}^2}{2}\right] = \frac{1}{2}k\left[2R^2 + \frac{r^2}{2}\right] \quad \left\{\vec{R}^2 = \vec{R}\cdot\vec{R} = R^2,\ \vec{r}^2 = \vec{r}\cdot\vec{r} = r^2\right\}$$

$$U = U_1 + U_2 + U_{12}$$
$$= \frac{1}{2}k\left[2R^2 + \frac{r^2}{2}\right] + \frac{1}{2}\alpha k r^2$$
$$= \frac{1}{2}k\left[2R^2 + \frac{r^2}{2} + \alpha r^2\right]$$

$L = T - U$

$$\boxed{L = m_1\dot{\vec{R}}^2 + \frac{1}{4}m_1\dot{\vec{r}}^2 - \frac{1}{2}k\left[2R^2 + \frac{r^2}{2} + \alpha r^2\right]}$$

b) $L = L_{cm} + L_{rel}$

$$L_{cm} = m_1 \dot{\vec{R}}^2 - \tfrac{1}{2} k\, 2R^2$$

$$L_{rel} = \tfrac{1}{4} m_1 \dot{\vec{r}}^2 - \tfrac{1}{2} k \left[\tfrac{r^2}{2} + \alpha r^2\right]$$

CM:

$$\frac{\partial L_{cm}}{\partial R} = -2kR$$

$$\frac{\partial L_{cm}}{\partial \dot{R}} = 2m_1 \dot{R}$$

$$\frac{d}{dt}\frac{\partial L_{cm}}{\partial \dot{R}} = 2m_1 \ddot{R}$$

$$\frac{\partial L_{cm}}{\partial R} - \frac{d}{dt}\frac{\partial L_{cm}}{\partial \dot{R}} = 0$$

$$2m_1\ddot{R} + 2kR = 0$$

$$\ddot{R} + \frac{k}{m_1}R = 0$$

$$R = A_1 e^{\lambda t}$$

$$\lambda^2 + \frac{k}{m_1} = 0$$

$$\lambda = \pm i\omega, \quad \omega_1 = \sqrt{\frac{k}{m_1}}$$

$$R = A_1 e^{i\omega t} + A_1 e^{-i\omega t}$$

$$\boxed{R = A\cos(\omega_1 t - \delta_1)}$$ → C.M. oscillates

Rel:

$$\frac{\partial L_{rel}}{\partial r} = -k\left[\frac{r}{2} + \alpha r\right]$$

$$\frac{\partial L_{rel}}{\partial \dot{r}} = \tfrac{1}{2} m_1 \dot{r}$$

$$\frac{d}{dt}\frac{\partial L_{rel}}{\partial \dot{r}} = \tfrac{1}{2} m \ddot{r}$$

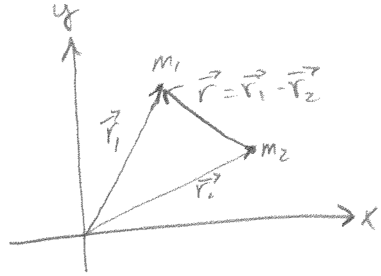
$$\frac{\partial L_{rel}}{\partial r} - \frac{d}{dt}\frac{\partial L_{rel}}{\partial \dot{r}} = 0$$

$$\tfrac{1}{2} m\ddot{r} + k\left[\tfrac{1}{2} + \alpha\right] r = 0$$

$$\ddot{r} + \frac{k}{m}\left[1 + 2\alpha\right] r = 0$$

$$\boxed{r = A\cos(\omega_2 t - \delta_2), \quad \omega_2 = \sqrt{\frac{k[1+2\alpha]}{m}}}$$ → masses oscillate relative to each other

8.11



$U = \frac{1}{2}kr^2$

$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$

$U = \frac{1}{2}kr^2$

$L_{cm} = \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{1}{2}kr^2$

$\frac{\partial L_{cm}}{\partial r} = -kr$

$\frac{\partial L_{cm}}{\partial \dot{r}} = \mu\dot{r}$

$\frac{d}{dt}\frac{\partial L_{cm}}{\partial \dot{r}} = \mu\ddot{r}$

$\mu\ddot{r} + kr = 0$

$\ddot{r} + \frac{k}{\mu}r = 0$

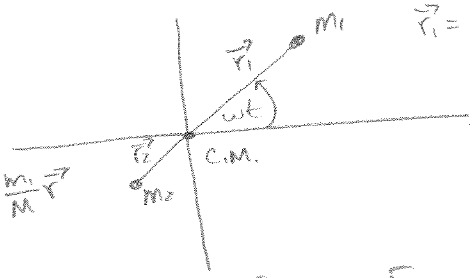
$r = A\cos(\omega t - \delta)$, $\omega = \sqrt{\frac{k}{\mu}}$

$r = A\cos(\omega t)\cos(\delta) + A\sin(\omega t)\sin(\delta)$

$A\cos(\delta) = C_1$

$A\sin(\delta) = C_2$

$r = C_1\cos(\omega t) + C_2\sin(\omega t)$

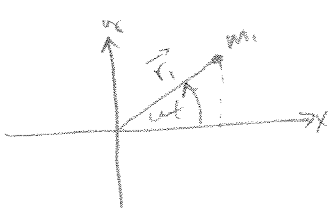


$\vec{r}_1 = \frac{m_2}{M}\vec{r}$

$\vec{r}_2 = \frac{-m_1}{M}\vec{r}$

Center of Mass Frame

Motion of m_1 in CM frame



$\vec{r}_1 = \frac{m_2}{M}\left(C_1\cos(\omega t)\hat{x} + C_2\sin(\omega t)\hat{y}\right)$

$x = \frac{C_1 m_2}{M}\cos(\omega t)$

$y = \frac{C_2 m_2}{M}\sin(\omega t)$

$$\frac{xM}{C_1 m_2} = \cos(\omega t)$$

$$\frac{yM}{C_2 m_2} = \sin(\omega t)$$

$$\cos^2(\omega t) + \sin^2(\omega t) = 1$$

$$\left(\frac{M}{C_1 m_2}\right)^2 x^2 + \frac{M}{C_2 m_2} y^2 = 1$$ → *Equation of an ellipse with CM center.*

Motion of m_2 in CM frame

$$\vec{r}_2 = \frac{-m_1}{M}\left(C_1 \cos(\omega t)\hat{x} + C_2 \sin(\omega t)\hat{y}\right)$$

$$x = \frac{-m_1 C_1}{M}\cos(\omega t)$$

$$y = \frac{-m_1 C_2}{M}\sin(\omega t)$$

$$-\frac{xM}{C_1 m_1} = \cos(\omega t)$$

$$-\frac{yM}{C_2 m_1} = \sin(\omega t)$$

$$\cos^2(\omega t) + \sin^2(\omega t) = 1$$

$$\left(\frac{M}{C_1 m_1}\right)^2 x^2 + \left(\frac{M}{C_2 m_1}\right) y^2 = 1$$ → *Equation of an ellipse with CM center*

8.12 $U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2}$

a) $\frac{dU}{dr} = \frac{Gm_1m_2}{r^2} - \frac{2l^2}{2\mu r^3}$

$-\frac{dU}{dr} = \mu\ddot{r}$

$\mu\ddot{r} = -\frac{Gm_1m_2}{r^2} + \frac{l^2}{\mu r^3}$

For a fixed radius $\ddot{r} = 0$

Let $m_1 = m_{planet} = m_p$

$m_2 = m_{sun} = m_s$

$\frac{Gm_pm_s}{r^2} = \frac{l^2}{\mu r^3}$

$r_0 = \frac{l^2}{Gm_pm_s\mu}$, $\mu = \frac{m_pm_s}{m_p+m_s}$

$\boxed{r_0 = \frac{l^2(m_p+m_s)}{G(m_pm_s)^2}}$

b) Taylor expansion of U_{eff}

$U_{eff} = U_{eff_0} + \frac{dU_{eff}}{dr}\Big|_{r_0} r + \frac{1}{2}\frac{d^2U_{eff}}{dr^2}\Big|_{r_0}(r-r_0)^2 + \cdots$

The constant U_{eff_0} can be dropped. At $r=r_0$ $\frac{dU_{eff}}{dr} = 0$. It is recognized that the quadratic term is of the form $U = \frac{1}{2}kx^2$. Therefore,

$\omega_r^2 = \frac{k}{\mu} = \frac{1}{\mu}\frac{d^2U_{eff}}{dr^2}\Big|_{r_0}$

$\frac{d}{dr}\frac{dU_{eff}}{dr} = \frac{d}{dr}\left(\frac{Gm_1m_2}{r^2} - \frac{l^2}{\mu r^3}\right)$

$= -\frac{2Gm_1m_2}{r^3} + \frac{3l^2}{\mu r^4}$

$k = \frac{d^2U_{eff}}{dr^2}\Big|_{r_0} = -\frac{2Gm_1m_2}{r_0^3} + \frac{3l^2}{\mu r_0^4}$

$= -\frac{2Gm_pm_sG^3(m_pm_s)^6}{l^6(m_p+m_s)^3} + \frac{3l^2G^4(m_pm_s)^8}{\mu l^8(m_p+m_s)^4}$

$= \frac{-2G^4(m_pm_s)^7}{l^6(m_p+m_s)^3} + \frac{3G^4(m_pm_s)^7}{l^6(m_p+m_s)^3}$

$= \frac{G^4(m_pm_s)^7}{l^6(m_p+m_s)^3}$

$$\omega_r^2 = \frac{k}{\mu}$$

$$\omega_r^2 = \frac{G^4 (m_p m_s)^6}{\ell^6 (m_p + m_s)^2}$$

The radial equation is

$$\mu \ddot{r} = -\frac{G m_p m_s}{r^2} + \frac{\ell^2}{\mu r^3}$$

and with $r = r_o + \varepsilon$, $\varepsilon \ll 1$

$$\mu \ddot{\varepsilon} = -G m_p m_s (r_o + \varepsilon)^{-2} + \frac{\ell^2}{\mu}(r_o + \varepsilon)^{-3} \qquad \{ (a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2$$

$$\cong -G m_p m_s (r_o^{-2} - 2 r_o^{-3} \varepsilon + \cdots) + \frac{\ell^2}{\mu}(r_o^{-3} - 3 r_o^{-4} \varepsilon + \cdots) \quad \{\text{terms of } \varepsilon^2 \text{ and higher are ignored}\}$$

$$\cong \frac{-G m_p m_s}{r_o^2} + \frac{2 G m_p m_s}{r_o^3}\varepsilon + \frac{\ell^2}{\mu r_o^3} - \frac{3\ell^2}{r_o^4}\varepsilon$$

$$\cong \left(\frac{2 G m_p m_s}{r_o^3} - \frac{3\ell^2}{r_o^4} \right)\varepsilon + \frac{\ell^2}{\mu r_o^3} - \frac{G m_p m_s}{r_o^2}$$

$$\cong \frac{-G^4 (m_p m_s)^7}{\ell^6 (m_p + m_s)^3}\varepsilon + \frac{\ell^2}{\mu r_o^3} - \frac{G m_p m_s}{r_o^2}$$

The homogeneous equation is of the form $F = -kx$, a restoring force, and is therefore stable. Another view is that the Taylor expansion about r_o gives an effective potential of $\frac{1}{2}k(r - r_o)^2$ which is the potential of a harmonic oscillator.

Period of oscillation:

$$T_r = \frac{2\pi}{\omega_r}$$

$$= \frac{2\pi (m_p + m_s) \ell^3}{G^2 (m_p m_s)^3}$$

Period of orbit

$$\dot{\phi} = \frac{\ell}{\mu r_o^2}$$

$$T_o = \frac{2\pi}{\dot{\phi}}$$

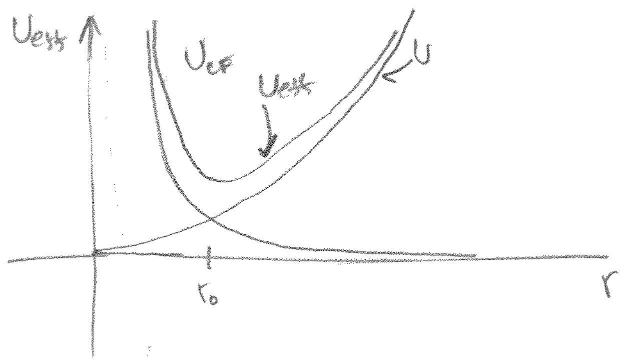
$$= \frac{2\pi \mu r_o^2}{\ell}$$

$$= \frac{2\pi}{\ell} \frac{m_p m_s}{m_p + m_s} \frac{(m_p + m_s)^2 \ell^4}{G^2 (m_p m_s)^4}$$

$$= \frac{2\pi (m_p + m_s) \ell^3}{G^2 (m_p m_s)^3} \longrightarrow T_r = T_o$$

8.13

a) $U = \frac{1}{2}Kr^2$, $U_{cf} = \frac{\ell^2}{2\mu r^2}$, $U_{eff} = U + U_{cf}$



b) $U_{eff} = \frac{1}{2}Kr^2 + \frac{\ell^2}{2\mu r^2}$

$$\frac{\partial U_{eff}}{\partial r} = Kr - \frac{\ell^2}{\mu r^3}$$

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r_0} = 0$$

$$K r_0 = \frac{\ell^2}{\mu r_0^3}$$

$$r_0^4 = \frac{\ell^2}{\mu K}$$

$$\boxed{r_0 = \left(\frac{\ell^2}{\mu K}\right)^{1/4}}$$

c) $U_{eff} = U_{eff_0} + \left.\frac{\partial U_{eff}}{\partial r}\right|_{r_0} + \frac{1}{2}\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_0}(r-r_0)^2 + \ldots$

$$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{\partial}{\partial r}\left(Kr - \frac{\ell^2}{\mu r^3}\right)$$

$$= K + \frac{3\ell^2}{\mu r^4}$$

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_0} = K + \frac{3\ell^2}{\mu r_0^4}$$

$$= K + \frac{3\ell^2}{\mu}\frac{\mu K}{\ell^2}$$

$$= K + 3K$$

$$= 4K$$

$$\omega_r^2 = \frac{4K}{\mu} \rightarrow \boxed{\omega_r = \sqrt{\frac{4K}{\mu}}}$$

0.17

$$U = kr^n \quad , \quad kn > 0$$

$$U_{eff} = U + U_{CF}$$

$$= kr^n + \frac{l^2}{2\mu r^2}$$

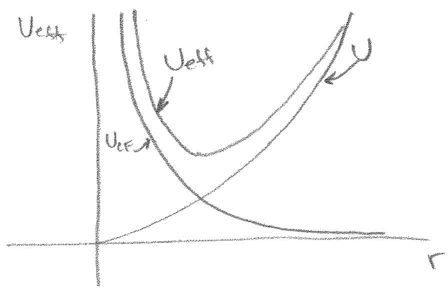
$$F = -\frac{\partial U_{eff}}{\partial r}$$

$$= -\left(knr^{n-1} - \frac{l^2}{\mu r^3}\right)$$

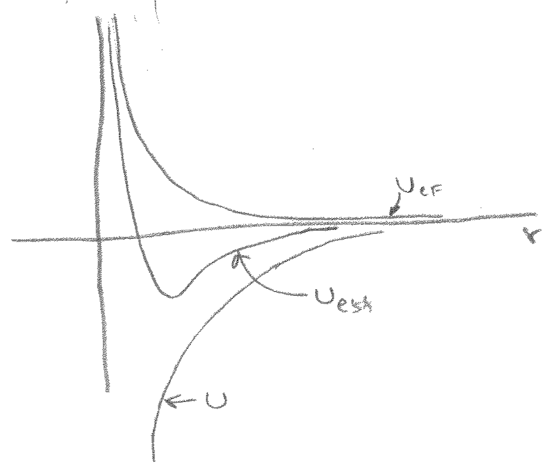
$$F = -knr^{n-1} + \frac{l^2}{\mu r^3}$$

The condition kn>0 indicates the force is a central force and a centrifugal force.

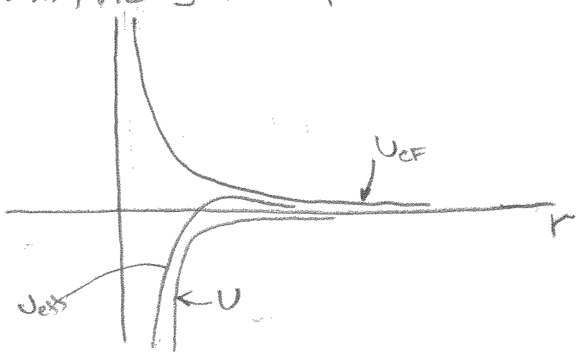
i) $n=2 \to U = kr^2$



ii) $n=-1 \to U = \frac{-k}{r}$



iii) $n=-3 \to U = \frac{-k}{r^3}$



b) Fixed radius $\to \ddot{r} = 0 \to F = 0$

$$kn r_0^{n-1} = \frac{l^2}{\mu r_0^3}$$

$$r_0^{n+2} = \frac{l^2}{kn\mu}$$

$$r_0 = \left(\frac{l^2}{kn\mu}\right)^{\frac{1}{n+2}}$$

For stability $\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_0} > 0$ {concave up}

$$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{\partial}{\partial r}\left(Knr^{n-1} - \frac{\ell^2}{\mu r^3}\right)$$

$$= Kn(n-1)r^{n-2} + \frac{3\ell^2}{\mu r_4}$$

$$Kn(n-1)r_0^{n-2} + \frac{3\ell^2}{\mu r_0^4} > 0$$

$$Kn(n-1)r_0^{n+2} + \frac{3\ell^2}{\mu} > 0$$

$$r_0 = \left(\frac{\ell^2}{Kn\mu}\right)^{\frac{1}{n+2}}$$

$$Kn(n-1)\left(\frac{\ell^2}{Kn\mu}\right)^{\frac{n+2}{n+2}} + \frac{3\ell^2}{\mu} > 0$$

$$(n-1)\frac{\ell^2}{\mu} + \frac{3\ell^2}{\mu} > 0$$

$$n+2 > 0$$

$$n > -2$$

The sketches for $n=2$ and $n=-1$ show stability. The $n=-3$ sketch shows no stable equilibrium points. So

$n > -2$ is in agreement with the sketches.

c) Small radial oscillations:

$$K = \left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r_0} = Kn(n-1)r_0^{n-2} + \frac{3\ell^2}{\mu r_0^4}$$

$$Kr_0^4 = Kn(n-1)r_0^{n+2} + \frac{3\ell^2}{\mu}$$

$$Kr_0^4 = Kn(n-1)\left(\frac{\ell^2}{Kn\mu}\right)^{\frac{n+2}{n+2}} + \frac{3\ell^2}{\mu}$$

$$= (n-1)\frac{\ell^2}{\mu} + 3\frac{\ell^2}{\mu}$$

$$= \frac{\ell^2}{\mu}(n+2)$$

$$K = \frac{\ell^2}{\mu r_0^4}(n+2)$$

$$\omega_r^2 = \frac{K}{\mu} = \frac{\ell^2}{\mu^2 r_0^4}(n+2)$$

$$\omega_r = \frac{l}{\mu r_0^2}(n+2)^{1/2}$$

$$T_{osc} = \frac{2\pi}{\omega_r}$$

$$T_{osc} = \frac{2\pi \mu r_0^2}{l\sqrt{n+2}}$$

$$\dot{\phi} = \frac{l}{\mu r_0^2}$$

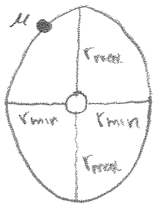
$$T_{orb} = \frac{2\pi}{\dot{\phi}}$$

$$T_{orb} = \frac{2\pi \mu r_0^2}{l}$$

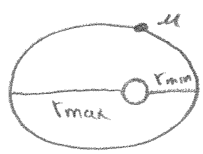
$$\boxed{T_{osc} = \frac{T_{orb}}{\sqrt{n+2}}}$$

If $\sqrt{n+2}$ is a rational number then $p\omega_{osc} = q\omega_{orb}$ and produces a Lissajous figure which is a closed orbit.

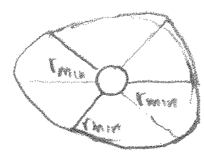
i) $n=2 \rightarrow T_{osc} = \frac{T_{orb}}{2}$



ii) $n=-1 \rightarrow T_{osc} = T_{orb}$

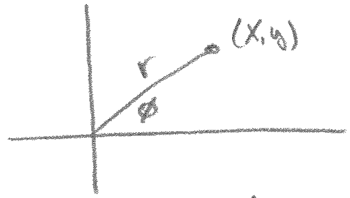


iii) $n=7 \rightarrow T_{osc} = \frac{T_{orb}}{3}$



8.16

$$r = \frac{c}{1+\varepsilon\cos\phi}$$



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$r = (x^2+y^2)^{1/2}$$

$$r + \varepsilon r\cos\phi = c$$

$$(x^2+y^2)^{1/2} + \varepsilon x = c$$

$$x^2+y^2 = (c-\varepsilon x)^2$$

$$= c^2 - 2c\varepsilon x + \varepsilon^2 x^2$$

$$x^2 - \varepsilon^2 x^2 + 2c\varepsilon x + y^2 = c^2$$

$$(1-\varepsilon^2)x^2 + 2c\varepsilon x + y^2 = c^2$$

$$x^2 + \frac{2c\varepsilon x}{1-\varepsilon^2} + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2}$$

$$x^2 + \frac{2c\varepsilon x}{1-\varepsilon^2} + \left(\frac{c\varepsilon}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2} + \left(\frac{c\varepsilon}{1-\varepsilon^2}\right)^2$$

$$\left(x + \frac{c\varepsilon}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{c^2 - c^2\varepsilon^2 + c^2\varepsilon^2}{(1-\varepsilon^2)^2} = \frac{c^2}{(1-\varepsilon^2)^2}$$

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a^2 = \frac{c^2}{(1-\varepsilon^2)^2} \longrightarrow a = \frac{c}{1-\varepsilon^2}$$

$$b^2 = \frac{c^2}{1-\varepsilon^2} \longrightarrow b = \frac{c}{\sqrt{1-\varepsilon^2}}$$

$$d = \frac{c\varepsilon}{1-\varepsilon^2} \longrightarrow d = a\varepsilon$$

8.17

a) $G = \vec{r} \cdot \vec{p}$

$$\frac{dG}{dt} = \frac{d}{dt}(\vec{r} \cdot \vec{p})$$

$$= \frac{d\vec{r}}{dt} \cdot \vec{p} + \frac{d\vec{p}}{dt} \cdot \vec{r}$$

$$\frac{dG}{dt} = \vec{v} \cdot \vec{p} + \vec{F} \cdot \vec{r}$$

$$dG = (\vec{v} \cdot \vec{p} + \vec{F} \cdot \vec{r})dt$$

$$dG = (\vec{v} \cdot m\vec{v} + \vec{F} \cdot \vec{r})dt$$

$$\int_0^t dG = \int_0^t mv^2 dt' + \int_0^t \vec{F} \cdot \vec{r}\, dt' \quad \{\vec{v} \cdot \vec{v} = v^2\}$$

$$\frac{\int dG}{t} = \frac{1}{t}\int_0^t mv^2 dt + \frac{1}{t}\int_0^t \vec{F} \cdot r\, dt$$

$$\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle, \quad \langle f \rangle = \frac{1}{t}\int_0^t f\, dt'$$

b) If the orbit is periodic then t can be made as large as desired because the righthand side of the equation is the time average. Therefore, the left side of the equation approaches zero as $t \to \infty$.

c) Let $t \to \infty$

$$0 = 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle$$

$$\vec{F} = -\frac{d}{dr}(U)\hat{r}$$

$$= -\frac{d}{dr}(kr^n)\hat{r}$$

$$\vec{F} = -nkr^{n-1}\hat{r}$$

$$\vec{F} \cdot \vec{r} = -nkr^{n-1}\hat{r} \cdot r\hat{r}$$

$$= -nkr^n$$

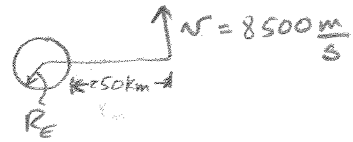
$$\vec{F} \cdot \vec{r} = -nU$$

$$2\langle T \rangle = -\langle \vec{F} \cdot \vec{r} \rangle$$

$$= n\langle U \rangle$$

$$\boxed{\langle T \rangle = \frac{n}{2}\langle U \rangle}$$

8.18



$$r_{min} = \frac{c}{1+\varepsilon}$$

$$c = \frac{l^2}{\gamma\mu}, \quad \gamma = Gm_1m_2 = GM_em_s, \quad \mu = \frac{M_E m_s}{M_e + M_s}$$

$$l = m_s r_{min}^2 \dot{\phi}$$

$$v = r_{min}\dot{\phi}$$

$$l = m_s r_{min} v, \quad r_{min} = R_E + 250\,km = 6.4\times10^6 m + 2.5\times10^5 m = 6.65\times10^6 m$$

$$c = \frac{m_s^2 r_{min}^2 v^2}{GM_em_s\left(\frac{M_em_s}{M_e+m_s}\right)}$$

$$c = \frac{r_{min}^2 v^2}{G\frac{M_e^2}{M_e+m_s}}$$

$$r_{min} + r_{min}\varepsilon = c$$

$$\varepsilon = \frac{c - r_{min}}{r_{min}}$$

$$= \frac{c}{r_{min}} - 1$$

$$= \frac{r_{min}v^2}{\frac{GM_e^2}{M_e+m_s}} - 1$$

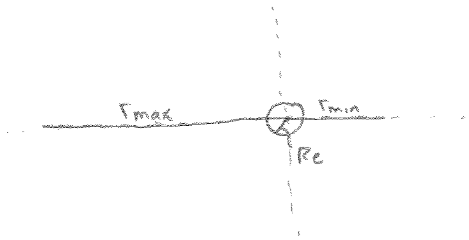
$$\varepsilon \cong \frac{r_{min}v^2}{GM_e} - 1 \quad \{M_e + m_s \cong M_e\}$$

$$\frac{GM_e}{R_e^2} = g$$

$$GM_e = R_e^2 g$$

$$\varepsilon = \frac{r_{min}v^2}{R_e^2 g} - 1 = \frac{(6.65\times10^6)(8500)^2}{(6.4\times10^6)^2(9.81)} - 1 = .1957 = \boxed{0.20}$$

8.19



$r_{min} = R_E + 300km$

$r_{max} = R_E + 3000km$

$$r_{min} = \frac{c}{1+\varepsilon}$$

$$r_{max} = \frac{c}{1-\varepsilon}$$

$$\frac{r_{min}}{r_{max}} = \frac{1-\varepsilon}{1+\varepsilon}$$

$$r_{min} + r_{min}\varepsilon = r_{max} - r_{max}\varepsilon$$

$$\varepsilon(r_{min} + r_{max}) = r_{max} - r_{min}$$

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

$$= \frac{R_E + 3000km - R_E - 300km}{R_E + 3000km + R_E + 300km}$$

$$= \frac{2700km}{2R_E + 3300km}$$

$$= \frac{2700km}{2(6.4 \times 10^3 km) + 3300km}$$

$$= .1677$$

$$\boxed{\varepsilon = 0.7}$$

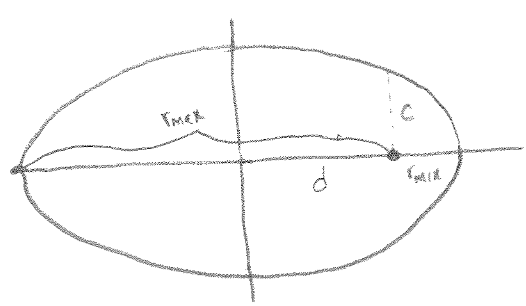
$$r = \frac{c}{1 + \varepsilon\cos\phi} \qquad \phi = 90^\circ$$

$$r = c$$

$h = c - R_e$

$= 7824 - 6400$

$\boxed{h = 1424 km}$



a = semi major axis

$$a = \frac{r_{min} + r_{max}}{2}$$

$$a = \frac{c}{1-\varepsilon^2}$$

$$c = a(1-\varepsilon^2)$$

$$= \frac{r_{min} + r_{max}}{2}(1-\varepsilon^2)$$

$$= \frac{[2(6.4 \times 10^3) + 3300]}{2}(1-(0.1677)^2)$$

$$= 7824 km$$

8.(c)

$$r_{max} = \frac{c}{1-\varepsilon}$$

$$c = \frac{l^2}{\gamma\mu}$$

$$r_{max} = \frac{l^2}{\gamma\mu(1-\varepsilon)}$$

$$r_{max}\gamma\mu(1-\varepsilon) = l^2$$

$$\lim_{l\to 0} l^2 = \lim_{\varepsilon\to x} C_1(1-\varepsilon) \quad \{C_1 = r_{max}\gamma\mu\}$$

$$0 = \lim_{\varepsilon\to x}(1-\varepsilon)$$

$$\boxed{\varepsilon\to 1}$$

$$r_{min} = \frac{c}{1+\varepsilon}$$

$$r_{min} = \frac{l^2}{\gamma\mu(1+\varepsilon)}$$

$$\lim_{l\to 0} r_{min} = \lim_{l\to 0}\frac{l^2}{\gamma\mu(1+\varepsilon)}$$

$$= \frac{0}{\gamma\mu(2)}$$

$$\lim_{l\to 0} r_{min} \to 0$$

When $l\to 0$ with fixed r_{max}, $c\to 0$, $r_{min}\to 0$, and $\varepsilon\to 1$ {parabolic orbit}. So, the elliptical orbit becomes very thin and long. Since $r_{min}\to 0$ then $a\to \frac{r_{max}}{2}$.

8.21

a) See Solution 8.20

b) $\tau^2 = 4\pi^2 \frac{a^3\mu}{\gamma}$

$a = \frac{r_{max}}{2}$

$\mu = \frac{M_s m_c}{M_s + m_c} \cong m_c$

$\gamma = G M_s m_c$

$\tau^2 = 4\pi^2 \frac{(r_{max}/2)^3 m_c}{G M_s m_c}$

$\tau^2 = \frac{\pi^2 (r_{max})^3}{2 G M_s}$

$\boxed{\tau = \frac{\pi (r_{max})^{3/2}}{\sqrt{2GM_s}}}$

c) $t = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x)}}$

$U = -\int \frac{-G M_s m_c}{r^2} dr$

$U = -\frac{G M_s m_c}{r}$

$E = T_0 + U_0$

$E = -\frac{G M_s m_c}{r_{max}}$

$U(r) = -\frac{G M_s m_c}{r}$

$m = m_c$

$r_0 = x_0 = r_{max}$

$r = x = 0$

$t = \sqrt{\frac{m_c}{2}} \int_{r_{max}}^{0} \frac{dr}{\sqrt{E + \frac{G M_s m_c}{r}}}$

let $\frac{G M_s m_c}{r} = -E \sec^2\theta \quad \{E < 0\}$

$\frac{G M_s m_c \cos^2\theta}{-E} = r$

$\frac{2 G M_s m_c \cos\theta \sin\theta \, d\theta}{E} = dr$

$$t=\sqrt{\frac{m_c}{2}}\int_{\theta_0}^{\theta}\frac{\frac{2GM_sm_c}{E}\cos\theta\sin\theta\,d\theta}{\sqrt{E-E\sec^2\theta}}$$

$$=2GM_sm_c\sqrt{\frac{m_c}{2}}\frac{1}{(-E)^{3/2}}\int_{\theta_0}^{\theta}\frac{\cos\theta\sin\theta\,d\theta}{\sqrt{-1+\sec^2\theta}}$$

$$=2GM_sm_c\sqrt{\frac{m_c}{2}}\frac{1}{\left(\frac{GM_sm_c}{r_{max}}\right)^{3/2}}\int_{\theta_0}^{\theta}\frac{\cos\theta\sin\theta\,d\theta}{\tan\theta}$$

$$=\frac{2(r_{max})^{3/2}}{\sqrt{2GM_s}}\int_{\theta_0}^{\theta}\cos^2\theta\,d\theta$$

$$\frac{Gm_sm_c\cos^2\theta}{-E}=r \qquad r=0\to\theta=\frac{\pi}{2}$$

$$r_{max}\cos^2\theta=r \qquad r_0=r_{max}\to\theta_0=0$$

$$=\frac{2(r_{max})^{3/2}}{\sqrt{2GM_s}}\frac{1}{2}\int_0^{\pi/2}(1+\cos2\theta)d\theta$$

$$=\frac{(r_{max})^{3/2}}{\sqrt{2GM_s}}\left[\theta\Big|_0^{\pi/2}+\frac{1}{2}\sin2\theta\Big|_0^{\pi/2}\right]$$

$$=\frac{(r_{max})^{3/2}}{\sqrt{2GM_s}}\left[\frac{\pi}{2}\right]$$

$$\boxed{t=\frac{\pi(r_{max})^{3/2}}{2\sqrt{2GM_s}}}$$

d) The motion would be that of a one-dimensional oscillator.

$T_{osc}=4t$ {time to complete one cycle}

$$T_{osc}=\frac{2\pi(r_{max})^{3/2}}{\sqrt{2GM_s}}$$

e) Comparing with b)

$$T_{osc}=2\tau$$

8.22

a) $F(r) = \frac{K}{r^3}$

$U_{eff} = U + U_{cF}$

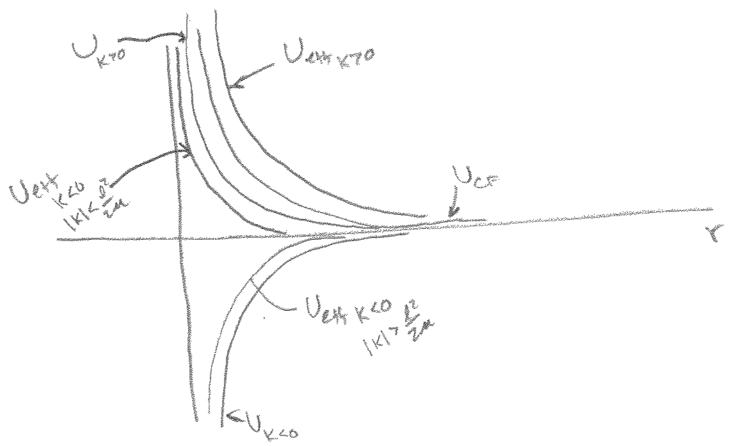
$U = -\int \vec{F} \cdot dr \qquad \{ \vec{F} = \frac{K}{r^3}\hat{r} \}$

$= -K \int r^{-3} dr$

$U = \frac{K}{2} r^{-2} = \frac{K}{2r^2}$

$U_{cF} = \frac{\ell^2}{2\mu r^2}$

$U_{eff} = \frac{K}{2r^2} + \frac{\ell^2}{2\mu r^2}$



$E>0$: $U_{eff\ K>0}$ and $U_{eff\ K<0,\ |K|<\frac{\ell^2}{2\mu}}$ produce open orbits that approach infinity.

$E<0$: $U_{eff\ K<0,\ |K|>\frac{\ell^2}{2\mu}}$ produces a well such that the object is attracted to the force center.

b) $u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F$

$= -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} \frac{K}{r^3}$

$= -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} K u(\phi)^3 \quad \{ u = \frac{1}{r} \}$

$= -u(\phi) - \frac{\mu K}{\ell^2} u(\phi)$

$= -u(\phi)\left[1 + \frac{\mu K}{\ell^2}\right]$

$u''(\phi) + u(\phi)\left[1 + \frac{\mu K}{\ell^2}\right] = 0$

$\lambda^2 + \left[1 + \frac{\mu K}{\ell^2}\right] = 0 \qquad \{ u(\phi) = A_1 e^{\lambda \phi} \}$

$\lambda^2 = -\left[1 + \frac{\mu K}{\ell^2}\right]$

$\lambda = \pm i\omega$

$\omega = \sqrt{1 + \frac{\mu k}{\ell^2}}$

i) $k > 0 \rightarrow \omega$ is real

$u = A\cos(\omega\phi - \delta)$

$\frac{1}{r} = A\cos(\omega\phi - \delta)$

$r = \frac{r_0}{\cos(\omega t - \delta)}$

This is the equation for a straight line in agreement with $U_{\underset{k>0}{eff}}$; unbounded orbit.

ii) $-1 < \frac{\mu k}{\ell^2} \leq 0$

$\omega = \sqrt{1 + \frac{\mu k}{\ell^2}} > 0 \rightarrow$ same condition as $k > 0$ and agrees with the sketch; unbounded orbit.

iii) $\frac{\mu k}{\ell^2} = -1$

$\omega = 0$

$r = r_0 \rightarrow$ no acceleration

iv) $\frac{\mu k}{\ell^2} < -1$

$\omega < 0 \rightarrow \lambda^2 = \omega^2$

$\lambda = \pm\omega$

$u = A_1 e^{\omega\phi} + A_1 e^{-\omega\phi}$

$= A_1\left[e^{\omega\phi} + e^{-\omega\phi}\right]$

$\frac{1}{r} = A_2\cosh(\phi) \quad \{A_2 = \frac{A_1}{2}\}$

$r = \frac{1}{A_2\cosh(\phi)}$

$r = A_3\,\text{sech}(\phi) \quad \{A_3 = \frac{1}{A_2}\}$

This is a hyperbolic trajectory and agrees with the sketch; unbounded orbit.

Therefore, an inverse cube central force does not produce bounded orbits.

8.23

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3} \quad , \; k>0, \lambda>0$$

a) $u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F$

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}\left[-\frac{k}{r^2} + \frac{\lambda}{r^3}\right] \quad , \quad r = \frac{1}{u}$$

$$r^2 = \frac{1}{u^2} \; , \; r^3 = \frac{1}{u^2}$$

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}\left[-k u(\phi)^2 + \lambda u(\phi)^3\right]$$

$$u''(\phi) = -u(\phi) - \frac{\lambda\mu}{\ell^2} u(\phi) + \frac{\mu k}{\ell^2}$$

$$= -u(\phi)\left[1 + \frac{\lambda\mu}{\ell^2}\right] + \frac{\mu k}{\ell^2}$$

$$u''(\phi) + u(\phi)\left[1 + \frac{\lambda\mu}{\ell^2}\right] = \frac{\mu k}{\ell^2}$$

u_h: $u(\phi) = A_1 e^{\lambda\phi}$

$$\lambda^2 + \left[1 + \frac{\lambda\mu}{\ell^2}\right] = 0$$

$$\lambda^2 = -\left[1 + \frac{\lambda\mu}{\ell^2}\right]$$

$$\lambda = \pm i\omega \; , \; \omega = \sqrt{1 + \frac{\lambda\mu}{\ell^2}}$$

$$u_h(\phi) = A_2 \cos(\omega\phi - \delta)$$

u_p: $u = C_1$

$$C_1\left[1 + \frac{\lambda\mu}{\ell^2}\right] = \frac{\mu k}{\ell^2}$$

$$C_1\left[\frac{\ell^2 + \lambda\mu}{\ell^2}\right] = \frac{\mu k}{\ell^2}$$

$$C_1 = \frac{\mu k}{\ell^2}\left[\frac{\ell^2}{\ell^2 + \lambda\mu}\right]$$

$$= \frac{\mu k}{\ell^2 + \lambda\mu}$$

$$u_p = \frac{\mu k}{\ell^2 + \lambda\mu}$$

$$u = A_2 \cos(\omega\phi - \delta) + \frac{\mu k}{\ell^2 + \lambda\mu}$$

$$\frac{1}{r} = A_2 \cos(\omega\phi - \delta) + \frac{\mu k}{\ell^2 + \lambda\mu}$$

$$= A_2 \cos(\omega\phi) + A_3 \quad \left\{ \delta = 0 \text{ for convenience}, \; A_3 = \frac{\mu k}{\ell^2 + \lambda\mu} \right\}$$

$$= A_3\left[\frac{A_2}{A_3}\cos(\omega\phi) + 1\right]$$

$$= A_3\left[A_4 \cos(\omega\phi) + 1\right] \quad \left\{ A_4 = \frac{A_2}{A_3} \right\}$$

$$r = \frac{\frac{1}{A_3}}{1 + A_4\cos(\omega\phi)}$$

$$\boxed{r = \frac{c}{1+\varepsilon\cos(\beta\phi)}}, \quad c = \frac{1}{A_3}, \; A_4 = \varepsilon, \; \omega = \beta$$

b) $c = \frac{1}{A_3}$

$$\boxed{c = \frac{l^2 + \lambda\mu}{\mu K}}$$

$\beta = \omega$

$$\boxed{\beta = \sqrt{1 + \frac{\lambda\mu}{l^2}}}$$

$$r = \frac{\frac{l^2+\lambda\mu}{\mu K}}{1+\epsilon\cos([1+\frac{\lambda\mu}{l^2}]\phi)}$$

For $0 < \epsilon < 1$, r is never ∞ $(1+\epsilon\cos(\beta\phi) > 0)$ and is therefore a bounded orbit.

c) For the orbit to be closed it has to return to its initial conditions.

$$\beta = \sqrt{1 + \frac{\lambda\mu}{l^2}}$$

$$= \sqrt{\frac{l^2 + \lambda\mu}{l^2}} = \omega$$

$$T_{osc} = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\beta}$$

$$T_{orb} = \frac{2\pi}{\dot\phi}$$

For closed orbits

$n T_{orb} = m T_{osc}$, n and m are rational numbers

$$n\frac{2\pi}{\dot\phi} = m\frac{2\pi}{\beta}$$

$$\boxed{\beta = \frac{m}{n}\dot\phi}$$

When $\lambda \to 0$: $r = \dfrac{\frac{l^2}{\mu K}}{1+\epsilon\cos(\phi)}$ which is a Kepler orbit.

8.24

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3}$$

$$u''(\phi) = -u(\phi) - \frac{\mu}{l^2 u(\phi)^2} F$$

$$u''(\phi) = -u(\phi) - \frac{\mu}{l^2 u(\phi)^2}\left[\frac{-k}{r^2} - \frac{\lambda}{r^3}\right]$$

$$u''(\phi) = -u(\phi) - \frac{\mu}{l^2 u(\phi)^2}\left[-k u(\phi)^2 - \lambda u(\phi)^3\right] \quad , \quad r = \frac{1}{u}$$

$$u''(\phi) = -u(\phi) + \frac{\mu\lambda}{l^2} u(\phi) + \frac{\mu k}{l^2}$$

$$u''(\phi) = -u(\phi)\left[1 - \frac{\mu\lambda}{l^2}\right] + \frac{\mu k}{l^2}$$

$$u''(\phi) + u(\phi)\left[1 - \frac{\mu\lambda}{l^2}\right] = \frac{\mu k}{l^2}$$

u_h: $u(\phi) = A_1 e^{\lambda_1 \phi}$

$$\lambda_1^2 + \left[1 - \frac{\mu\lambda}{l^2}\right] = 0$$

$$\lambda_1^2 = -\left[1 - \frac{\mu\lambda}{l^2}\right]$$

$$\lambda_1 = \pm i\omega \quad , \quad \omega = \sqrt{1 - \frac{\mu\lambda}{l^2}}$$

$$u_h(\phi) = A_2 \cos(\omega\phi - \delta)$$

u_p: $u = C_1$

$$C_1\left[1 - \frac{\mu\lambda}{l^2}\right] = \frac{\mu k}{l^2}$$

$$C_1\left[\frac{l^2 - \mu\lambda}{l^2}\right] = \frac{\mu k}{l^2}$$

$$C_1 = \frac{\mu k}{l^2 - \mu\lambda}$$

$$u_p = \frac{\mu k}{l^2 - \mu\lambda}$$

$$u = A_2 \cos(\omega\phi) + \frac{\mu k}{l^2 - \mu\lambda} \quad , \quad \delta = 0 \text{ for convenience}$$

$$\frac{1}{r} = A_3\left[\frac{A_2}{A_3}\cos(\omega\phi) + 1\right] \quad , \quad A_3 = \frac{\mu k}{l^2 - \mu\lambda}$$

$$r = \frac{\frac{1}{A_3}}{1 + \varepsilon\cos(\omega\phi)}$$

$$\boxed{r = \frac{\frac{l^2 - \mu\lambda}{\mu k}}{1 + \varepsilon\cos(\omega\phi)}}$$

For $l^2 > \mu\lambda$

$$\omega = \sqrt{\frac{l^2 - \mu\lambda}{l^2}} > 0$$

This is a bounded orbit since

$$\lambda_1^2 = \pm i\omega$$

resulting in a cosine solution.

For $l^2 < -\mu\lambda$

$$\omega = \sqrt{\frac{l^2 - \mu\lambda}{l^2}}$$

$$l^2 - \mu\lambda < 0$$

Then

$$\lambda_1^2 - \omega_1^2 = 0 \quad , \quad \omega_1^2 = \frac{l^2 - \mu\lambda}{l^2}$$

$$\lambda_1^2 = \omega_1^2$$

$$\lambda_1 = \pm\omega_1$$

$$u_h(\phi) = A_1 e^{\pm\omega\phi}$$

$$= A_1 e^{\omega\phi} + A e^{-\omega\phi}$$

$$= \frac{A_2}{2}\left[e^{\omega\phi} + e^{-\omega\phi}\right]$$

$$u_h(\phi) = A_2 \cosh(\omega\phi)$$

This is an unbounded orbit.

Y.26

$$r = \frac{c}{1+\varepsilon\cos\phi}, \quad 0<\varepsilon<1 \quad \{1^{st}\text{ Law}\}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \text{constant} \quad \{2^{nd}\text{ Law}\}$$

$$\frac{1}{2}r^2\dot{\phi} = C_1$$

$$\vec{r} = r\,\hat{r}(\phi)$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\frac{d\phi}{dt}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \quad \left\{\frac{d\hat{r}}{d\phi} = \hat{\phi}\right\}$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\phi}\frac{d\phi}{dt} + \dot{r}\dot{\phi}\hat{\phi} + r\left(\ddot{\phi}\hat{\phi} + \dot{\phi}\frac{d\hat{\phi}}{d\phi}\frac{d\phi}{dt}\right)$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r} \quad \left\{\frac{d\hat{\phi}}{d\phi} = -\hat{r}\right\}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$$

2^{nd} Law

$$\dot{\phi} = \frac{2C_1}{r^2}$$

$$\ddot{\phi} = \frac{-4C_1}{r^3}\dot{r}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + \left(2\dot{r}\left(\frac{2C_1}{r^2}\right) + r\left(\frac{-4C_1}{r^3}\dot{r}\right)\right)\hat{\phi}$$

$$= (\ddot{r} - r\dot{\phi}^2)\hat{r} + \left(\frac{4C_1}{r^2}\dot{r} - \frac{4C_1}{r^2}\dot{r}\right)\hat{\phi}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r}$$ {establishes that a force directed through the center exists}

$$\dot{r} = -c(1+\varepsilon\cos\phi)^{-2}(-\varepsilon\sin\phi)\dot{\phi}$$

$$= \frac{c}{1+\varepsilon\cos\phi}\,\frac{\varepsilon\sin\phi}{1+\varepsilon\cos\phi}\,\frac{2C_1}{r^2}$$

$$= \frac{r^2}{c}\varepsilon\sin\phi\,\frac{2C_1}{r^2}$$

$$\dot{r} = \frac{2\varepsilon C_1}{c}\sin\phi$$

$$\ddot{r} = \frac{2\varepsilon C_1}{c}\cos\phi\,\dot{\phi}$$

$$= \frac{2\varepsilon C_1}{c}\cos\phi\,\frac{2C_1}{r^2}$$

$$\ddot{r} = 4C_1^2\frac{\varepsilon}{c}\cos\phi\frac{1}{r^2}$$

$$\frac{c}{r} = 1+\varepsilon\cos\phi$$

$$\varepsilon\cos\phi = -1 + \frac{c}{r}$$

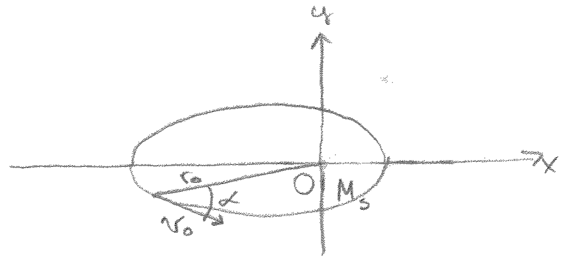
$$\ddot{r} = \frac{-4C_1^2}{cr^2} + \frac{4C_1^2}{r^3}$$

$$\ddot{\vec{r}} = \left(-\frac{4C_1^2}{cr^2} + \frac{4C_1^2}{r^3} - r\left(\frac{2C_1}{r^2}\right)^2\right)\hat{r}$$

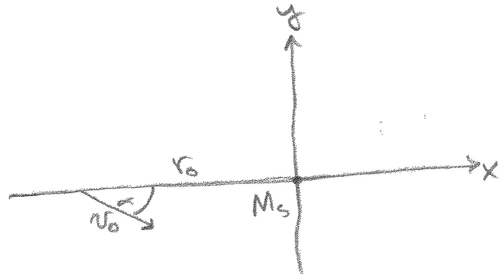
$$= \left(\frac{-4C_1^2}{cr^2} + \frac{4C_1^2}{r^3} - \frac{4C_1^2}{r^3}\right)\hat{r}$$

$$\boxed{\ddot{\vec{r}} = -\frac{4C_1^2}{c}\frac{1}{r^2}\hat{r}}$$

8.27



Orient coordinate system such that the comet is on the x-axis at t_o.



Angular momentum is conserved

$$\ell = mr^2\dot{\phi}$$

$$= m r_o v_{o\,Tan} \text{ at } t_o$$

$$\ell = m r_o v_o \sin\alpha$$

Now,

$$F(r) = \frac{-\gamma}{r^2} \rightarrow \gamma = G m_c M_s$$

$$c = \frac{\ell^2}{\gamma\mu}$$

$$= \frac{(m_c r_o v_o \sin\alpha)^2}{G m_c M_s\left(\frac{M_s m_c}{M_s + m_c}\right)}$$

$$\boxed{c \cong \frac{r_o^2 v_o^2 \sin^2\alpha}{G M_s}} \quad \left\{ \frac{M_s m_c}{M_s + m_c} \cong m_c,\ M_s >> m_c \right\}$$

Energy - eccentricity relationship

$$E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$$

Energy of comet

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{G M_s m_c}{r}$$

$\dot{r}$ = radial velocity

$$\dot{r} = v_0 \cos\alpha$$

Equating energy expressions

$$\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{GM_s m_c}{r} = \frac{r^2\mu}{2\ell^2}(\epsilon^2 - 1)$$

$$\epsilon^2 = \frac{2\ell^2}{r^2\mu}\left[\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{GM_s m_c}{r}\right] + 1$$

$$= \frac{2(m_c r_0 v_0 \sin\alpha)^2}{(GM_s m_c)^2 m_c}\left[\frac{1}{2}m_c(v_0\cos\alpha)^2 + \frac{(m_c r_0 v_0 \sin\alpha)^2}{2m_c r_0^2} - \frac{GM_s m_c}{r_0}\right] + 1$$

$$= \frac{2(r_0 v_0 \sin\alpha)^2}{(GM_s)^2}\left[\frac{(v_0\cos\alpha)^2}{2} + \frac{(v_0\sin\alpha)^2}{2} - \frac{GM_s}{r_0}\right] + 1$$

$$\boxed{\epsilon^2 = \frac{2(r_0 v_0 \sin\alpha)^2}{(GM_s)^2}\left[\frac{v_0^2}{2} - \frac{GM_s}{r_0}\right] + 1}$$

$$r_0 = \frac{c}{1 + \epsilon\cos(\phi_0)}$$

$$\frac{c}{r_0} = 1 + \epsilon\cos(\phi_0)$$

$$\frac{c}{r_0} - 1 = \epsilon\cos(\phi_0)$$

$$\boxed{\frac{c - r_0}{\epsilon r_0} = \cos(\phi_0)}$$

Given: $r_0 = 1 \times 10^{11}$ m

$v_0 = 45 \times 10^3 \ \frac{m}{s}$

$\alpha = 50^\circ$

$M_s = 2.0 \times 10^{30}$ Kg

$G = 6.7 \times 10^{-11} \ \frac{N \cdot m^2}{s^2}$

$$\epsilon^2 = -.433 + 1$$

$$= .566$$

$$\boxed{\epsilon = .753}$$

$$\boxed{c = 8.868 \times 10^{10}\,m}$$

$$\phi_0 = \cos^{-1}\left[\frac{c - r_0}{\varepsilon r_0}\right]$$

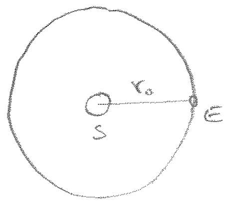
$$\delta = \cos^{-1}[-.150] \quad \{\phi_0 = \delta\}$$

$$\delta = 98.646^\circ$$

$$\boxed{\delta = 1.72\ \text{radians}}$$

8.29

Inital conditions:



$$U = -\frac{GM_sM_E}{r_0}$$

Virial Theorem: $U = Kr^n \rightarrow T = n\frac{U}{2}$

$$U_0 = \frac{-GM_sM_E}{r_0} \rightarrow T_0 = \frac{GM_sM_E}{2r_0} \quad \{n=-1\}$$

For a circular orbit velocity is alway tangential to orbit path; no radial velocity. Also, $\varepsilon = 0$.

At the instant of sun mass change $\{M_s = \frac{M_{s0}}{2}\}$

$$U = -\frac{GM_sM_e}{2r_0} = \frac{U_0}{2}$$

$$T = T_0 = \frac{GM_sM_e}{2r_0} = \frac{1}{2}M_E v_0^2$$

$$v_0^2 = \frac{GM_s}{r_0}$$

$$E = U_{eff}(r_{min})$$

$$= U_{eff}(r_0)$$

$$= -\frac{\gamma}{r_0} + \frac{\ell^2}{2\mu r_0^2} \quad \left\{ \mu = \frac{M_sM_E}{M_s+M_s} \doteq M_E \right\}$$

$$E = -\frac{GM_sM_E}{2r_0} + \frac{\ell^2}{2M_Er_0^2} \quad \left\{ \gamma = \frac{GM_sM_E}{2} \right\}$$

$$\ell = M_E r_0 v_0$$

$$\ell^2 = M_E^2 r_0^2 v_0^2$$

$$= M_E^2 r_0^2 \frac{GM_s}{r_0}$$

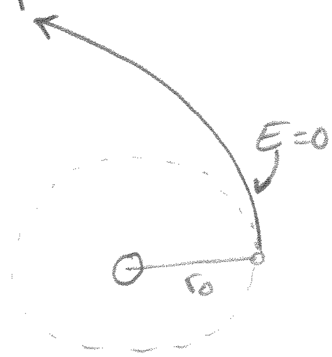
$$\ell^2 = GM_sM_E^2 r_0$$

$$E = \frac{-GM_sM_E}{2r_0} + \frac{GM_sM_E^2 r_0}{2M_E r_0^2}$$

$$= -\frac{GM_sM_E}{2r_0} + \frac{GM_sM_E}{2r_0}$$

$$E = 0$$

when $E=0$ the path is a parabola.



8.30

$$r(\phi) = \frac{c}{1+\epsilon\cos\phi}$$

i) $\epsilon = 1$

$$r = \frac{c}{1+\cos\phi}$$

$x = r\cos\phi$

$r^2 = x^2+y^2$

$$r = \frac{c}{1+\frac{x}{r}}$$

$$= \frac{c}{\frac{r+x}{r}}$$

$$r = \frac{rc}{r+x}$$

$$\frac{c}{r+x} = 1$$

$$c = r+x$$

$$c = (x^2+y^2)^{1/2}+x$$

$$x^2+y^2 = (c-x)^2$$

$$= c^2 - 2cx + x^2$$

$$y^2 = c(c-2x)$$

$$\boxed{y = [c(c-2x)]^{1/2}}$$

ii) $\epsilon > 1$

$$r = \frac{c}{1+\epsilon\cos\phi}$$

$$r = \frac{c}{1+\frac{\epsilon x}{r}}$$

$$= \frac{c}{\frac{r+\epsilon x}{r}}$$

$$r = \frac{rc}{r+\epsilon x}$$

$$1 = \frac{c}{r+\epsilon x}$$

$$r + \epsilon x = c$$

$$r = c - \epsilon x$$

$$x^2 + y^2 = (c - \epsilon x)^2$$

$$= c^2 - 2\epsilon c x + \epsilon^2 x^2$$

$$x^2(1-\epsilon^2) + 2\epsilon c x + y^2 = c^2$$

$$x^2 + \frac{2\epsilon c}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$x^2 + \frac{2\epsilon c}{1-\epsilon^2} + \left(\frac{\epsilon c}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + \left(\frac{\epsilon c}{1-\epsilon^2}\right)^2$$

$$\left(x + \frac{\epsilon c}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + \frac{\epsilon^2 c^2}{(1-\epsilon^2)^2}$$

$$= \frac{c^2(1-\epsilon^2) + \epsilon^2 c^2}{(1-\epsilon^2)^2}$$

$$= \frac{c^2 - c^2\epsilon^2 + \epsilon^2 c^2}{(1-\epsilon^2)^2}$$

$$= \frac{c^2}{(1-\epsilon^2)^2}$$

$$\frac{\left(x + \frac{\epsilon c}{1-\epsilon^2}\right)^2}{\frac{c^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{(1-\epsilon^2)c^2}{(1-\epsilon^2)^2}} = 1$$

$$\frac{\left(x - \frac{\epsilon c}{\epsilon^2-1}\right)^2}{\frac{c^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{c^2}{1-\epsilon^2}} = 1$$

$$\boxed{\frac{\left(x - \frac{\epsilon c}{\epsilon^2-1}\right)^2}{\frac{c^2}{(1-\epsilon^2)^2}} - \frac{y^2}{\frac{c^2}{\epsilon^2-1}} = 1, \quad \delta = \frac{\epsilon c}{\epsilon^2-1}, \quad \alpha^2 = \frac{c^2}{(1-\epsilon^2)^2}, \quad \beta^2 = \frac{c^2}{\epsilon^2-1}}$$

8.31

$$F(r) = \frac{\gamma}{r^2} = \gamma u^2$$

$$u''(\phi) = -u(\phi) - \frac{\mu}{l^2 u(\phi)^2} F$$

$$= -u(\phi) - \frac{\mu}{l^2 u(\phi)^2} \gamma u^2$$

$$u''(\phi) = -u(\phi) - \frac{\mu \gamma}{l^2}$$

u_h:

$$u''(\phi) + u(\phi) = 0$$

let $u(\phi) = A_1 e^{\lambda \phi}$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$u_h(\phi) = A_1 e^{\pm i\phi}$$

$$= A_1 e^{i\phi} + A_1 e^{-i\phi}$$

$$= \frac{A}{2} e^{i(\phi-\delta)} + \frac{A}{2} e^{-i(\phi-\delta)} \quad \{ A_1 = \frac{A}{2} e^{-i\delta} \quad A_1^* = \frac{A}{2} e^{i\delta} \}$$

$$u_h(\phi) = A\cos(\phi - \delta)$$

u_p:

$$u_p(\phi) = C_1$$

$$C_1 = -\frac{\mu \gamma}{l^2}$$

$$u_p(\phi) = -\frac{\mu \gamma}{l^2}$$

$$u(\phi) = u_h(\phi) + u_p(\phi)$$

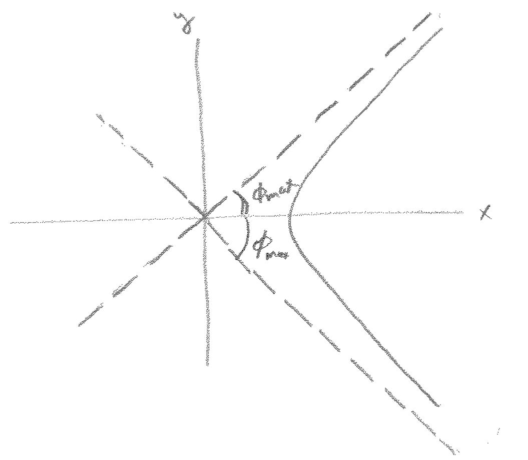
$$= A\cos(\phi - \delta) - \frac{\mu \gamma}{l^2}$$

$$u(\phi) = \frac{\gamma \mu}{l^2}(-1 + \varepsilon \cos\phi), \quad \delta = 0, \quad \varepsilon = \frac{A l^2}{\gamma \mu}$$

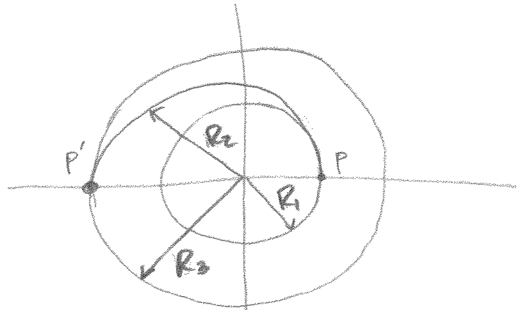
$$\frac{1}{r} = \frac{\gamma \mu}{l^2}(-1 + \varepsilon \cos\phi)$$

$$r = \frac{\frac{l^2}{\gamma \mu}}{-1 + \varepsilon \cos\phi} = \frac{c}{-1 + \varepsilon \cos\phi}$$

If $\epsilon < 1$ then $r < 0$ which is not a possible condition. Therefore, $\epsilon \geq 1$ is only allowed. this corresponds to $E \geq 0$. when $E = 0$ $(\epsilon = 1)$ the only value for ϕ is $\phi = 0$ giving $r = \infty$. For $E > 0$ $(\epsilon > 1)$ then when $\epsilon \cos\phi_{max} = 1$ $r \to \infty$. therefore, $-\phi_{max} < \phi < \phi_{max}$ is the allowable range and defines the asymptotes of the hyperbola.



8.33



$$R_2 = \frac{c_2}{1+\epsilon_2\cos\phi}$$

$$R_3 = R_{2\,apogee}$$

$$R_{2\,apogee} = \frac{c_2}{1-\epsilon_2} \quad \{\phi = 180^\circ\}$$

$$\frac{c_2}{1-\epsilon_2} = \frac{c_3}{1+\epsilon_3}$$

$$c_3 = \lambda'^2 c_2$$

$$\epsilon_3 = 0$$

$$\frac{c_2}{1-\epsilon_2} = \lambda'^2 c_2$$

$$\frac{1}{1-\epsilon_2} = \lambda'^2$$

$$\epsilon_2 = \lambda^2\epsilon_1 + (\lambda^2 - 1)$$

$$= \lambda^2 - 1 \quad \{\epsilon_1 = 0\}$$

$$\lambda'^2 = \frac{1}{1-(\lambda^2-1)}$$

$$= \frac{1}{2-\lambda^2} \qquad \lambda^2 = \frac{2R_3}{R_1+R_3} \quad \{\text{Eq. } 8.72\}$$

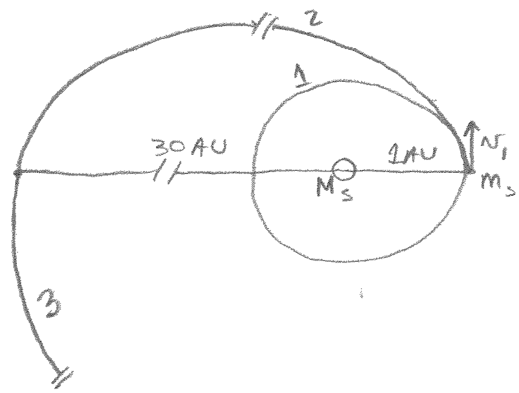
$$= \frac{1}{2 - \frac{2R_3}{R_1+R_3}}$$

$$= \frac{1}{\frac{2R_1+2R_3-2R_3}{R_1+R_3}}$$

$$= \frac{R_1+R_3}{2R_1}$$

$$\boxed{\lambda' = \sqrt{\frac{R_1+R_3}{2R_1}}}$$

8.34



Path 2 is an ellipse

$$a = \frac{30+1}{2} \text{ AU}$$

$$a = 15.5 \text{ AU}$$

$$\tau_2^2 = 4\pi^2 \frac{a^3 \mu}{\gamma} \qquad \mu = \frac{M_s m_s}{M_s + m_s} = m_s$$

$$\gamma = G M_s m_s$$

$$\tau_2^2 = \frac{4\pi^2 (15.5)^3 \text{AU}^3}{G M_s} \qquad G = 6.7 \times 10^{-11} \frac{\text{N}\cdot\text{m}^2}{\text{s}^2}$$

$$M_s = 2.0 \times 10^{30} \text{ kg}$$

$$1\text{AU} = 1.5 \times 10^{11} \text{ m}$$

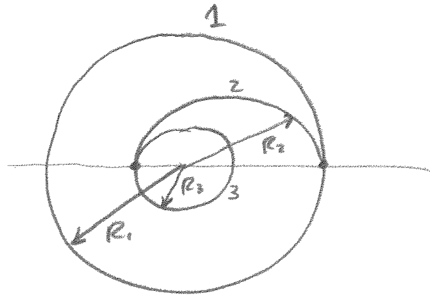
$$\tau_2^2 = \frac{4\pi^2 (23.25 \times 10^{11})^3}{(6.7\times10^{-11})(2.0\times10^{30})}$$

$$\tau_2 = 1.92 \times 10^9 \text{ s}$$

$$\tau_2 = 61.02 \text{ years}$$

$$\text{Time of transfer} = \frac{\tau_2}{2} = 30.5 = 31 \text{ years}$$

8.35



$R_3 = \frac{R_1}{4}$

1st stage:

$$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1-\varepsilon_2} \leftarrow \text{apogee}$$

$$\varepsilon_1 = 0$$

$$c_2 = \lambda^2 c_1$$

$$c_1 = \frac{\lambda^2 c_1}{1-\varepsilon_2}$$

$$1-\varepsilon_2 = \lambda^2$$

$$\varepsilon_2 = 1-\lambda^2$$

$$R_3 = \frac{c_2}{1+\varepsilon_2 \leftarrow \text{perigee}}$$

$$= \frac{\lambda^2 c_1}{1+(1-\lambda^2)}$$

$$= \frac{\lambda^2 R_1}{2-\lambda^2}$$

$$2R_3 - \lambda^2 R_3 = \lambda^2 R_1$$

$$2R_3 = \lambda^2 (R_1+R_3)$$

$$\lambda^2 = \frac{2R_3}{R_1+R_3}$$

$$= \frac{\frac{2R_1}{4}}{R_1+\frac{R_1}{4}}$$

$$= \frac{\frac{R_1}{2}}{\frac{5R_1}{4}}$$

$$\boxed{\lambda = \sqrt{\frac{2}{5}}}$$

2nd stage:

$$\frac{c_2}{1+\varepsilon_2} = \frac{c_3}{1+\varepsilon_3}$$

$$\varepsilon_3 = 0$$

$$c_3 = \lambda'^2 c_2$$

$$\frac{c_2}{1+(1-\lambda^2)} = \lambda'^2 c_2$$

$$\frac{1}{2-\lambda^2} = \lambda'^2 \quad \left\{ \lambda'^2 = \frac{1}{2-\frac{2R_3}{R_1+R_3}} = \frac{1}{\frac{2R_1+2R_3-2R_3}{R_1+R_3}} = \frac{R_1+R_3}{2R_1} \right\}$$

$$\lambda'^2 = \frac{1}{2-\frac{2}{5}}$$

$$= \frac{1}{\frac{10-2}{5}}$$

$$\boxed{\lambda' = \sqrt{\frac{5}{8}}}$$

$$v_3 = \lambda' \cdot \frac{v_{2apo}}{v_{2per}} \cdot \lambda \cdot v_1$$

$$= \sqrt{\frac{R_1+R_3}{2R_1}} \cdot \frac{R_1}{R_3} \cdot \sqrt{\frac{2R_3}{R_1+R_3}} \cdot v_1$$

$$= \sqrt{\frac{R_1}{R_3}} \cdot v_1$$

$$= \sqrt{\frac{R_1}{\frac{R_1}{4}}} \cdot v_1$$

$$= \sqrt{4} \cdot v_1$$

$$\boxed{v_3 = 2v_1}$$