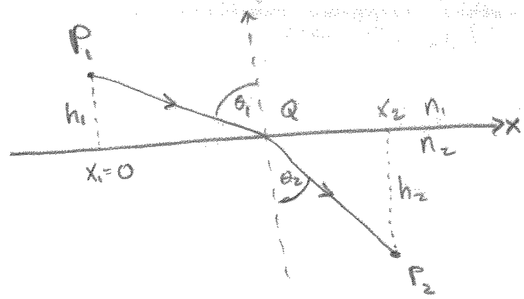


$$t = \int_{P_1}^{P_2} dt = \int_{P_1}^{P_2} \frac{ds}{v} = \frac{1}{c}\int_{P_1}^{P_2} n\,ds \quad \left\{ v = \frac{c}{n} \right\}$$

Let $P_1(x,y,z) = P_1(0,h_1,0)$ { P_1 lies on the y-axis }

Let $P_2(x,y,z) = P_2(x_2,-h_2,0)$ { P_2 lies in the x-y plane }

Let $Q(x,y,z) = Q(x,0,z)$ { Q lies in th x-z plane }

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$t = \int_{P_1}^{Q} dt + \int_{Q}^{P_2} dt$$

$$= \frac{n_1}{c}\int_{P_1}^{Q} ds + \frac{n_2}{c}\int_{Q}^{P_2} ds$$

$$t = \frac{n_1}{c}\int_{0}^{x} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx + \frac{n_2}{c}\int_{x}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Since n_1 and n_2 are constant, the shortest path between P_1 and Q, and Q and P_2 are straight lines. The distance between two points is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$d_1 = \sqrt{(x-0)^2 + (h_1-0)^2 + (0-z)^2} = \sqrt{x^2 + h_1^2 + z^2}$$

$$d_2 = \sqrt{(x_2-x)^2 + (-h_2-0)^2 + (0-z)^2} = \sqrt{(x_2-x)^2 + h_2^2 + z^2}$$

$$t = \frac{n_1}{c} d_1 + \frac{n_2}{c} d_2$$

$$= \frac{n_1}{c}\sqrt{x^2+h_1^2+z^2} + \frac{n_2}{c}\sqrt{(x_2-x)^2+h_2^2+z^2}$$

$$\frac{dt}{dz} = \frac{n_1}{c}\frac{z}{(x^2+h_1^2+z^2)^{1/2}} + \frac{n_2}{c}\frac{z}{((x_2-x)^2+h_2^2+z^2)^{1/2}}$$

$$0 = \frac{n_1}{c}\frac{z}{(x^2+h_1^2+z^2)^{1/2}} + \frac{n_2}{c}\frac{z}{((x_2-x)^2+h_2^2+z^2)^{1/2}}$$

$z=0$ is a solution.

Therefore Q lies in the same vertical plane as P_1 and P_2

Now,

$$t = \frac{n_1}{c}\sqrt{x^2+h_1^2} + \frac{n_2}{c}\sqrt{(x_2-x)^2+h_2^2}$$

$$0 = \frac{dt}{dx} = \frac{n_1}{c}\frac{x}{(x^2+h_1^2)^{1/2}} - \frac{n_2}{c}\frac{(x_2-x)}{((x_2-x)^2+h_2^2)^{1/2}}$$

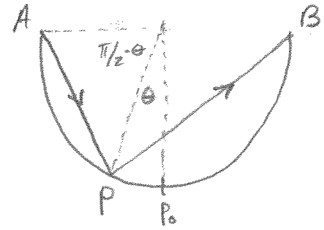
$$n_1\frac{x}{(x^2+h_1^2)^{1/2}} = n_2\frac{(x_2-x)}{((x_2-x)^2+h_2^2)^{1/2}}$$

$$\frac{x}{(x^2+h_1^2)^{1/2}} = \sin\theta_1$$

$$\frac{x_2-x_1}{((x_2-x)^2+h_2^2)^{1/2}} = \sin\theta_2$$

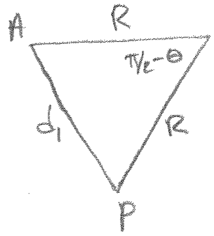
$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$$

6.3



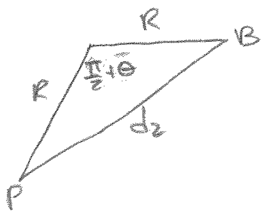
$R = \frac{AB}{2}$

i) Path AP:



$$d_1^2 = R^2 + R^2 - 2R^2\cos(\pi/2-\theta)$$
$$= 2R^2\left(1-\cos\left(\tfrac{\pi}{2}-\theta\right)\right)$$

Path PB:



$$d_2^2 = R^2 + R^2 - 2R^2\cos\left(\tfrac{\pi}{2}+\theta\right)$$
$$= 2R^2\left(1-\cos\left(\tfrac{\pi}{2}+\theta\right)\right)$$

$$t = \frac{1}{c}(d_1+d_2)$$
$$= \frac{R\sqrt{2}}{c}\left(\left(1-\cos\left(\tfrac{\pi}{2}-\theta\right)\right)^{1/2} + \left(1-\cos\left(\tfrac{\pi}{2}+\theta\right)\right)^{1/2}\right)$$
$$= \frac{R\sqrt{2}}{c}\left(\sqrt{2}\sin\frac{\beta}{2} + \sqrt{2}\sin\frac{\gamma}{2}\right)$$
$$= \frac{2R}{c}\left(\sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right), \quad \beta = \frac{\pi}{2}-\theta,\ \gamma = \frac{\pi}{2}+\theta$$
$$= \frac{2R}{c}\left[2\left(\sin\frac{\pi}{4}\cos\frac{\theta}{2}\right)\right]$$
$$= \frac{2R}{c}\left[2\frac{\sqrt{2}}{2}\cos\frac{\theta}{2}\right]$$
$$\boxed{t = \frac{2\sqrt{2}R\cos\frac{\theta}{2}}{c}}$$

$$\frac{(1-\cos\alpha)}{2} = \sin^2\left(\frac{\alpha}{2}\right)$$
$$(1-\cos\alpha)^{1/2} = \sqrt{2}\sin\left(\frac{\alpha}{2}\right)$$

$$\sin\frac{\beta}{2} = \sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \sin\frac{\pi}{4}\cos\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\pi}{4}$$
$$\sin\frac{\gamma}{2} = \sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \sin\frac{\pi}{4}\cos\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\pi}{4}$$

ii) $\frac{dt}{d\theta} = \frac{d}{d\theta}\left(\frac{2\sqrt{2}R}{C}\cos\frac{\theta}{2}\right)$

$0 = -\frac{2\sqrt{2}R}{C}\frac{1}{2}\sin\frac{\theta}{2}$

$0 = -\frac{\sqrt{2}R}{C}\sin\frac{\theta}{2}$

$\theta = 0 \rightarrow P = P_0$

$\frac{d^2t}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\sqrt{2}R}{C}\sin\frac{\theta}{2}\right)$

$= -\frac{\sqrt{2}R}{2C}\cos\frac{\theta}{2}$

At $\theta = 0$ $\frac{d^2t}{d\theta^2} = -\frac{\sqrt{2}R}{2C}(1) < 0$ $\rightarrow$ concave down indicating a maximum

6.6

a) $y = y(x)$

$ds = \sqrt{dx^2 + dy^2}$

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ {factor out dx}

b) $x = x(y)$

$ds = \sqrt{dx^2 + dy^2}$

$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ {factor out dy)

c) $r = r(\phi)$

$ds = \sqrt{dr^2 + (r\,d\phi)^2}$

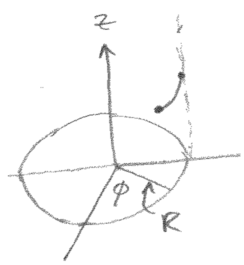
$ds = \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2}\, d\phi$ {factor out $d\phi$}

d) $\phi = \phi(r)$

$ds = \sqrt{dr^2 + (r\,d\phi)^2}$

$ds = \sqrt{1 + \left(\frac{r\,d\phi}{dr}\right)^2}\, dr$ {factor out dr}

e) $\phi = \phi(z)$



$ds = \sqrt{(R\,d\phi)^2 + dz^2}$

$ds = \sqrt{\left(\frac{R\,d\phi}{dz}\right)^2 + 1}\, dz$

f) $z = z(\phi)$

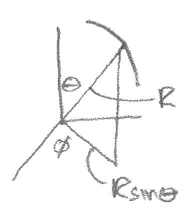
$ds = \sqrt{(R\,d\phi)^2 + dz^2}$

$ds = \sqrt{R^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$

g) $\theta = \theta(\phi)$

$ds = \sqrt{(R\sin\theta\, d\phi)^2 + (R\,d\theta)^2}$

$ds = R\sqrt{\sin^2\theta + \left(\frac{d\theta}{d\phi}\right)^2}\, d\phi$



h) $\phi = \phi(\theta)$

$ds = R\sqrt{(\sin\theta\, d\phi)^2 + (d\theta)^2}$

$ds = R\sqrt{\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2 + 1}\; d\theta$

6.11

$$\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,dx$$

Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$f(y, y', x) = \sqrt{x}\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y'} = \text{constant}$$

$$\frac{\partial f}{\partial y'} = \frac{\sqrt{x}\,y'}{(1+y'^2)^{1/2}}$$

$$\frac{\sqrt{x}\,y'}{(1+y'^2)^{1/2}} = C$$

$$\frac{x y'^2}{1+y'^2} = C^2$$

$$x y'^2 = C^2(1+y'^2)$$

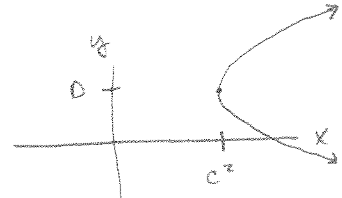
$$x y'^2 = C^2 + C^2 y'^2$$

$$x y'^2 - C^2 y'^2 = C^2$$

$$y'^2(x - C^2) = C^2$$

$$y'^2 = \frac{C^2}{(x-C^2)}$$

$$y' = \pm C(x-C^2)^{-1/2}$$



$$y = \pm\int C(x-C^2)^{-1/2}\,dx$$

$$y = \pm 2C(x-C^2)^{1/2} + D \rightarrow \frac{(y-D)^2}{4C} + C^2 = \pm x \rightarrow \text{parabolic path}$$

6.12

$$\int_{x_1}^{x_2} x\sqrt{1-y'^2}\,dx$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y'} = \text{constant}$$

$$\frac{\partial f}{\partial y'} = \frac{-xy'}{(1-y'^2)^{1/2}}$$

$$\frac{-xy'}{(1-y'^2)^{1/2}} = C$$

$$\frac{x^2y'^2}{1-y'^2} = C^2$$

$$x^2y'^2 = C^2 - C^2y'^2$$

$$x^2y'^2 + C^2y'^2 = C^2$$

$$y'^2(x^2+C^2) = C^2$$

$$y' = \frac{C}{(x^2+C^2)^{1/2}}$$

$$y = \int \frac{C}{(x^2+C^2)^{1/2}}\,dx$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$1 + \sinh^2 u = \cosh^2 u$$

$$x = C\sinh u$$

$$dx = C\cosh u\,du$$

$$y = \int \frac{C\cosh u\,du}{C(1+\sinh^2 u)^{1/2}}$$

$$= \int \frac{\cosh u\,du}{\cosh u}$$

$$= \int du$$

$$= u + D$$

$$y = \sinh^{-1}\left(\frac{x}{c}\right) + D$$

6.15

$$ds = \frac{2}{1-r^2}\sqrt{dr^2 + r^2 d\phi^2}$$

let $r = r(\phi)$

$$ds = \frac{2}{1-r^2}\sqrt{r'^2 + r^2}\, d\phi$$

$$f(r, r', \phi) = \frac{2}{1-r^2}\sqrt{r'^2+r^2}$$

$$\frac{\partial f}{\partial r} - \frac{d}{d\phi}\frac{\partial f}{\partial r'} = 0$$

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(\frac{2}{1-r^2}(r'^2+r^2)^{1/2}\right)$$

$$= \frac{2}{1-r^2}\left(\frac{1}{2}\right)(r'^2+r^2)^{-1/2}(2r) + (r'^2+r^2)^{1/2}(-2)(1-r^2)^{-2}(-2r)$$

$$\frac{\partial f}{\partial r} = \frac{2r}{1-r^2}(r'^2+r^2)^{-1/2} + \frac{4r(r'^2+r^2)^{1/2}}{(1-r^2)^2}$$

$$\frac{\partial f}{\partial r'} = \frac{2}{1-r^2}\frac{1}{2}(r'^2+r^2)^{-1/2}(2r')$$

$$= \frac{2r'}{1-r^2}(r'^2+r^2)^{-1/2}$$

$$\frac{d}{d\phi}\frac{\partial f}{\partial r'} = 0 \quad \left\{\frac{\partial f}{\partial r'} \text{ is not a function of } \phi\right\}$$

$$\frac{\partial f}{\partial r} = 0$$

$$\frac{-2r}{(1-r^2)(r'^2+r^2)^{1/2}} = \frac{4r(r'^2+r^2)^{1/2}}{(1-r^2)^2}$$

$$\frac{-1}{(1-r^2)(r'^2+r^2)^{1/2}} = \frac{2(r'^2+r^2)^{1/2}}{(1-r^2)^2}$$

$$-\frac{1}{2}(1-r^2) = r'^2 + r^2$$

$$-\frac{1}{2} + \frac{r^2}{2} = r'^2 + r^2$$

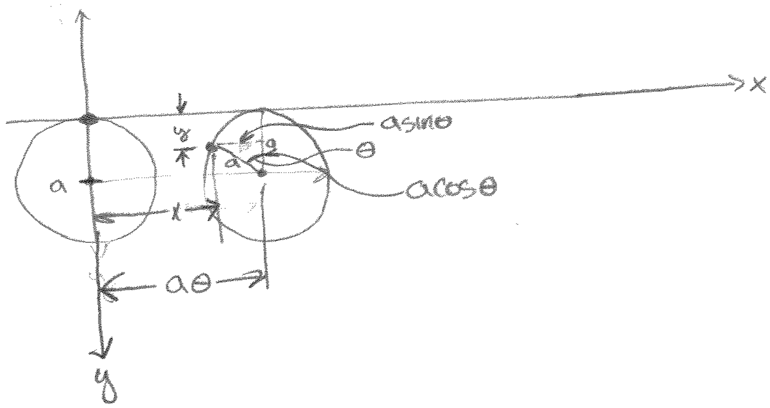
$$r'^2 = -\frac{1}{2} - \frac{r^2}{2}$$

$$r'^2 = -\left(\frac{1}{2}+\frac{r^2}{2}\right) \rightarrow r' = \sqrt{-\left(\frac{1}{2}+\frac{r^2}{2}\right)} \rightarrow \frac{dr}{d\phi} \text{ does not exist.}$$

Therefore, r can change when ϕ is constant. This is a straight line.

6.14

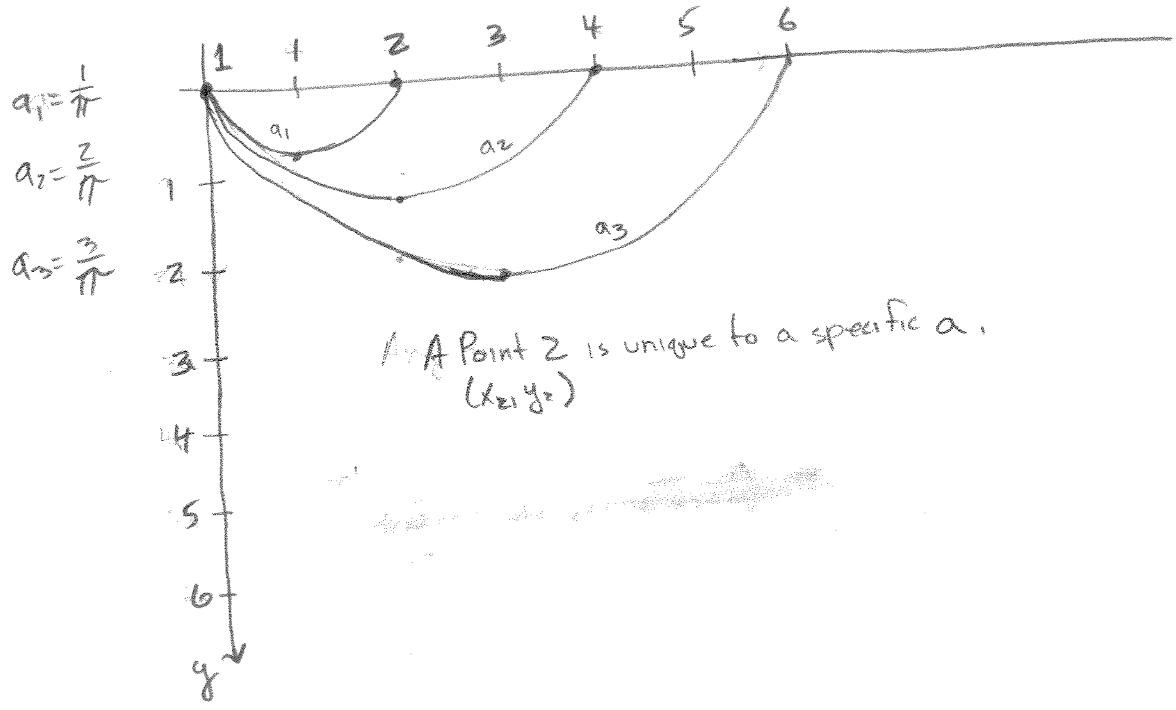
a) $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ {Eq. 6.26}



From the diagram } → $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$

$x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

b) $x = a(\theta - \sin\theta)$ $y = a(1-\cos\theta)$



∴ A Point 2 is unique to a specific a.
(x_2, y_2)

c)

i) $X_2 = \pi b$, $y_2 = 2b$

$$X = a(\theta - \sin\theta) \quad , y = a(1-\cos\theta)$$

$$\frac{X}{\theta - \sin\theta} = \frac{y}{1-\cos\theta}$$

$$\frac{\pi b}{\theta - \sin\theta} = \frac{2b}{1-\cos\theta}$$

$$\pi - \pi\cos\theta = 2\theta - 2\sin\theta$$

Try $\theta = \pi$

$$\pi - \pi(-1) = 2\pi - 2(0)$$

$$2\pi = 2\pi$$

$$\theta = \pi \checkmark$$

$$\pi b = a(\pi - \sin\pi)$$

$$\pi b = a(\pi)$$

$$\boxed{a = b}$$

ii) $X_2 = 2\pi b$, $y_2 = 0$

It's evident $\theta = 2\pi$ since $y_2 = 0$

$$2\pi b = a(\theta - \sin\theta)$$

$$2\pi b = a(2\pi - \sin(2\pi))$$

$$2\pi b = a2\pi$$

$$\boxed{a = b}$$

$$\underset{1\to 2}{\text{time}} = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2+1}}{\sqrt{y}}\, dy$$

$$x = b(\theta - \sin\theta) \qquad y = b(1-\cos\theta)$$

$$\frac{dx}{dy} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dy} \qquad dy = b\sin\theta \, d\theta$$

$$\frac{dx}{d\theta} = b - b\cos\theta$$

$$\frac{d\theta}{dy} = \frac{1}{b\sin\theta}$$

$$\frac{dx}{dy} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$x'^2 = \left(\frac{dx}{dy}\right)^2 = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\underset{1\to 2}{\text{time}} = \frac{1}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{\frac{\frac{(1-\cos\theta)^2}{\sin^2\theta}+1}{b(1-\cos\theta)}}\; b\sin\theta\, d\theta$$

$$= \frac{\sqrt{b}}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{\frac{(1-\cos\theta)^2+\sin^2\theta}{\sin^2\theta(1-\cos\theta)}}\,\sin\theta\, d\theta$$

$$= \frac{\sqrt{b}}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{\frac{(1-\cos\theta)^2+(1-\cos^2\theta)}{1-\cos\theta}}\, d\theta$$

$$= \frac{\sqrt{b}}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{\frac{(1-\cos\theta)^2+(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}}\, d\theta$$

$$= \frac{\sqrt{b}}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{(1-\cos\theta)+(1+\cos\theta)}\, d\theta$$

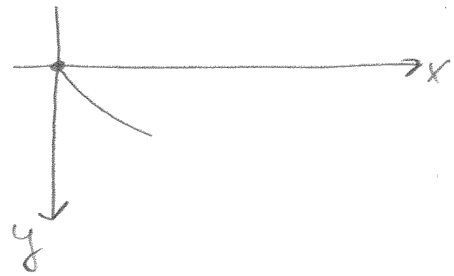
$$= \frac{\sqrt{b}}{\sqrt{2g}}\int_0^{\theta_2} \sqrt{2}\, d\theta$$

$$\underset{1\to 2}{\text{time}} = \frac{\sqrt{2b}}{\sqrt{2g}}\int_0^{\theta_2} d\theta = \sqrt{2b}\,\theta\Big|_0^{\theta_2}$$

when $\theta_2 = \pi$ $\quad t = \pi\sqrt{\frac{b}{g}}$

when $\theta_2 = 2\pi$ $\quad t = 2\pi\sqrt{\frac{b}{g}}$

6.15



$$\frac{1}{2}mv^2 = mgy + \frac{1}{2}mv_0^2$$

$$v^2 = 2gy + v_0^2$$

$$v = \sqrt{2gy + v_0^2}$$

$$t = \int_1^2 \frac{ds}{v}$$

$$t = \int_1^2 \frac{\sqrt{x'^2 + 1}}{\sqrt{2gy + v_0^2}}\, dy$$

$$f = \frac{\sqrt{x'^2+1}}{\sqrt{2gy+v_0^2}} = f(x, x', y)$$

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = C$$

$$\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'} \frac{(x'^2+1)^{1/2}}{(2gy+v_0^2)^{1/2}}$$

$$= \frac{\frac{1}{2}(x'^2+1)^{-1/2}(2x')}{(2gy+v_0^2)^{1/2}}$$

$$= \frac{x'}{(x'^2+1)^{1/2}(2gy+v_0^2)^{1/2}}$$

$$\frac{x'^2}{(x'^2+1)(2gy+v_0^2)} = C^2 = \frac{1}{2a}$$

$$x'^2 = \frac{(x'^2+1)(2gy+v_0^2)}{2a}$$

$$2ax'^2 = x'^2(2gy+v_0^2) + (2gy+v_0^2)$$

$$x'^2 = \frac{2gy+v_0^2}{2a-(2gy+v_0^2)}$$

$$x' = \sqrt{\frac{2gy + V_0^2}{2a - (2gy + V_0^2)}}$$

let $w = 2gy + V_0^2$

$dw = 2g\,dy$

$$2g\frac{dx}{dw} = \sqrt{\frac{w}{2a - w}}$$

$$2gx = \int \sqrt{\frac{w}{2a-w}}\,dw$$

$w = a(1-\cos\theta)$

$dw = a\sin\theta\,d\theta$

$$= \int \sqrt{\frac{a(1-\cos\theta)}{2a - a(1-\cos\theta)}}\,a\sin\theta\,d\theta$$

$$= a\int \sqrt{\frac{a(1-\cos\theta)}{a + a\cos\theta}}\,\sin\theta\,d\theta$$

$$= a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,\sin\theta\,d\theta$$

$$= a\int \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}}\,\sin\theta\,d\theta$$

$$= a\int \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\,\sin\theta\,d\theta$$

$$= a\int (1-\cos\theta)\,d\theta$$

$$= a(\theta - \sin\theta) + \text{const.}$$

$$\boxed{x = \frac{a}{2g}(\theta - \sin\theta)} \quad \{x_0 = y_0 = 0\}$$

$w = a(1-\cos\theta)$

$2gy + V_0^2 = a(1-\cos\theta)$

$2gy = a(1-\cos\theta) - V_0^2$

$$\boxed{y = \frac{a}{2g}(1-\cos\theta) - \frac{V_0^2}{2g}}$$

when $\theta = 2\pi$ (cusp), $y = -\frac{V_0^2}{2g}$

6.16

$$L = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\,\phi'^2}\,d\theta$$

$$f(\phi,\phi',\theta) = \sqrt{1+\sin^2\theta\,\phi'^2}$$

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = 0$$

$$\frac{\partial f}{\partial \phi} = 0$$

$$\frac{\partial f}{\partial \phi'} = \text{constant}$$

$$\frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\left(1+\sin^2\theta\,\phi'^2\right)^{1/2}$$

$$= \frac{1}{2}\left(1+\sin^2\theta\,\phi'^2\right)^{-1/2} 2\phi'\sin^2\theta$$

$$= \frac{\phi'\sin^2\theta}{\left(1+\sin^2\theta\,\phi'^2\right)^{1/2}}$$

$$\frac{\phi'\sin^2\theta}{\left(1+\sin^2\theta\,\phi'^2\right)^{1/2}} = C$$

$$\frac{\phi'^2\sin^4\theta}{1+\sin^2\theta\,\phi'^2} = C^2$$

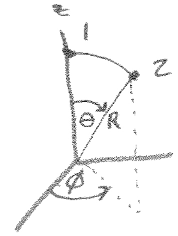
$$\phi'^2\sin^4\theta = C^2 + C^2\sin^2\theta\,\phi'^2$$

$$\phi'^2\left(\sin^4\theta - C^2\sin^2\theta\right) = C^2$$

$$\phi'^2 = \frac{C^2}{\sin^2\theta\left(\sin^2\theta - C^2\right)}$$

$$\phi' = \frac{C}{\sin\theta\left(\sin^2\theta - C^2\right)^{1/2}}$$

Let Point 1 be on the z-axis. Then to reach Point 2 only requires a change of θ and ϕ remains constant. therefore,



$\frac{d\phi}{d\theta} = 0$. So, $C = 0$

$$\phi' = 0$$

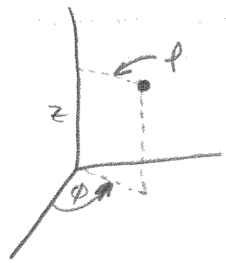
$$\phi = \text{const.}$$

This means the geodesic is a path on the sphere surface with starting point rotated to be on the z-axis, establishing a plane with the three points 1, 2, and the origin, and the intersection of the plane with the surface defines the geodesic.

6.17

$$z = \lambda \rho$$

let $\phi = \phi(\rho)$



$$ds^2 = d\rho^2 + (\rho\, d\phi)^2 + dz^2$$

$$= d\rho^2 + (\rho\, d\phi)^2 + \lambda^2 d\rho^2 \quad \{dz = \lambda d\rho\}$$

$$= (1+\lambda^2)\, d\rho^2 + \rho^2 d\phi^2$$

$$ds = \sqrt{(1+\lambda^2) + \rho^2 \phi'^2}\, d\rho$$

$$f(\phi, \phi', \rho) = \sqrt{(1+\lambda^2) + \rho^2 \phi'^2}$$

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\rho}\frac{\partial f}{\partial \phi'} = 0$$

$$\frac{\partial f}{\partial \phi} = 0$$

$$\frac{\partial f}{\partial \phi'} = \text{const.}$$

$$\frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\left((1+\lambda^2) + \rho^2 \phi'^2\right)^{1/2}$$

$$= \frac{1}{2}\left((1+\lambda^2) + \rho^2 \phi'^2\right)^{-1/2} 2\rho^2 \phi'$$

$$= \frac{\rho^2 \phi'}{\left((1+\lambda^2) + \rho^2 \phi'^2\right)^{1/2}}$$

$$\frac{\rho^4 \phi'^2}{(1+\lambda^2) + \rho^2 \phi'^2} = C^2$$

$$\rho^4 \phi'^2 = C^2(1+\lambda^2) + C^2 \rho^2 \phi'^2$$

$$\phi'^2(\rho^4 - C^2\rho^2) = C^2(1+\lambda^2)$$

$$\phi'^2 = \frac{C^2(1+\lambda^2)}{\rho^2(\rho^2 - C^2)}$$

$$\frac{d\phi}{d\rho} = \phi' = \frac{C(1+\lambda^2)^{1/2}}{\rho(\rho^2 - C^2)^{1/2}}$$

$$\phi = \int \frac{C(1+\lambda^2)^{1/2}}{\rho(\rho^2 - C^2)^{1/2}}\, d\rho$$

$$\int \frac{C(1+\lambda^2)^{1/2}}{\rho(\rho^2 - C^2)^{1/2}}\, d\rho = \int \frac{C^2(1+\lambda^2)^{1/2} \sec\theta \tan\theta\, d\theta}{C\sec\theta\,(C^2\sec^2\theta - C^2)^{1/2}}$$

$\rho = C\sec\theta$

$d\rho = C\sec\theta\tan\theta\, d\theta$

$$= (1+\lambda^2)^{1/2} \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\,(\tan^2\theta)^{1/2}}$$

$$= (1+\lambda^2)^{1/2} \int d\theta$$

$$= (1+\lambda^2)^{1/2}[\theta + D]$$

$$= (1+\lambda^2)^{1/2}\left[\sec^{-1}\left(\frac{\rho}{C}\right) + D\right]$$

$$\phi = (1+\lambda^2)^{1/2}\left[\sec^{-1}\left(\frac{\rho}{C}\right)\right] + b$$

$$\phi - \phi_0 = (1+\lambda^2)^{1/2} \sec^{-1}\left(\frac{\rho}{C}\right) \quad \{b = \phi_0\}$$

$$\sec\Theta = \frac{\rho}{c}$$

$$\cos\Theta = \frac{c}{\rho}$$

$$\sec^{-1}\left(\frac{\rho}{c}\right) = \cos^{-1}\left(\frac{c}{\rho}\right)$$

$$\phi - \phi_0 = (1+\lambda^2)^{1/2}\cos^{-1}\left(\frac{c}{\rho}\right)$$

Let $\lambda \to 0$ {x-y plane}

$$\phi - \phi_0 = \cos^{-1}\left(\frac{c}{\rho}\right)$$

$$\cos(\phi - \phi_0) = \frac{c}{\rho}$$

$$\rho\cos(\phi - \phi_0) = c$$

But,

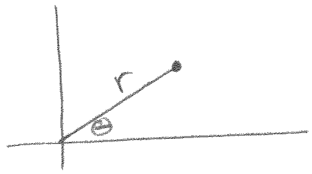
$$\rho\cos\phi = x$$

Let $\phi_0 = 0$

$$\rho\cos\phi = c$$

$$x = c$$

6.18



$$ds^2 = dr^2 + r^2 d\theta^2$$

Let $\theta = \theta(r)$

$$ds = \sqrt{1 + r^2\theta'^2}\, dr$$

$$f(\theta, \theta', r) = \sqrt{1 + r^2\theta'^2}$$

$$\frac{\partial f}{\partial \theta} = 0$$

$$\frac{d}{dr}\frac{\partial f}{\partial \theta'} = 0$$

$$\frac{\partial f}{\partial \theta'} = c$$

$$\frac{\partial f}{\partial \theta'} = \frac{\partial}{\partial \theta'}\left(1 + r^2\theta'^2\right)^{1/2}$$

$$= \frac{1}{2}\left(1 + r^2\theta'^2\right)^{-1/2} 2r^2\theta'$$

$$= \frac{r^2\theta'}{(1 + r^2\theta'^2)^{1/2}}$$

$$\frac{r^4\theta'^2}{1 + r^2\theta'^2} = c^2$$

$$r^4\theta'^2 = c^2 + c^2 r^2\theta'^2$$

$$\theta'^2(r^4 - c^2r^2) = c^2$$

$$\theta'^2 = \frac{c^2}{r^2(r^2 - c^2)}$$

$$\frac{d\theta}{dr} = \theta' = \frac{c}{r(r^2 - c^2)^{1/2}}$$

$$\theta = \int \frac{c}{r(r^2 - c^2)^{1/2}}\, dr$$

$$\int \frac{c}{r(r^2 - c^2)^{1/2}}\, dr = \int \frac{c^2 \sec\theta \tan\theta\, d\theta}{c\sec\theta (c^2\sec^2\theta - c^2)^{1/2}} = \int d\theta = \sec^{-1}\left(\frac{r}{c}\right) + a$$

Let $r = c\sec\theta$

$dr = c\sec\theta\tan\theta\, d\theta$

$$\theta = \sec^{-1}\left(\frac{r}{c}\right) + b$$

$$\theta - b = \cos^{-1}\left(\frac{c}{r}\right)$$

$$\cos(\theta - \theta_0) = \frac{c}{r} \quad \{b = \theta_0\}$$

$$r\cos(\theta - \theta_0) = C$$

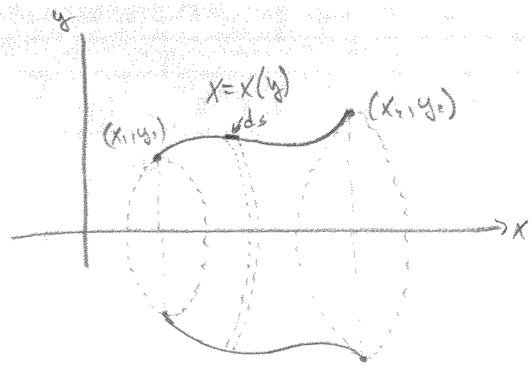
Let $\theta_0 = 0$

$$r\cos\theta = C$$

$$r\cos\theta = x$$

$$x = C$$

6.19



$$dA = 2\pi y\, ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{x'^2+1}\; dy$$

$$A = \int 2\pi y\, ds$$

$$= \int 2\pi y (x'^2+1)^{1/2} dy$$

$$f(x,x',y) = 2\pi y (x'^2+1)^{1/2}$$

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial x'} = C$$

$$\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'} 2\pi y (x'^2+1)^{1/2}$$

$$= 2\pi y x' (x'^2+1)^{-1/2}$$

$$\frac{4\pi y^2 x'^2}{x'^2+1} = C^2$$

$$4\pi y^2 x'^2 = C^2 + C^2 x'^2$$

$$x'^2 (4\pi y^2 - C^2) = C^2$$

$$x'^2 = \frac{C^2}{4\pi y^2 - C^2}$$

$$\frac{dx}{dy} = x' = \frac{C}{(4\pi y^2 - C^2)^{1/2}}$$

$$dx = \frac{C}{2\pi (y^2 - \frac{C^2}{4\pi^2})^{1/2}} dy$$

$$= \frac{a}{(y^2-a^2)^{1/2}} dy \quad \{a = \tfrac{C}{2\pi}\}$$

$$\int dx' = a\int \frac{dy'}{(y'^2-a^2)^{1/2}} = a\int \frac{a \sinh u\, du}{a(\cosh^2 u - 1)^{1/2}}$$

let $y' = a\cosh u$

$dy' = a\sinh u\, du$

$$= a\int \frac{\sinh u\, du}{\sinh u}$$

$$= a\int du$$

$$= a u + b$$

$$= a \cosh^{-1}\left(\frac{y'}{a}\right) + b$$

$$x = a\cosh^{-1}\left(\frac{y}{a}\right) + b$$

$$x = a\cosh^{-1}\left(\frac{y}{a}\right) + x_0 \quad \{b = x_0\}$$

$$x - x_0 = y_0 \cosh^{-1}\left(\frac{y}{y_0}\right) \quad \{a = y_0\}$$

$$\frac{x-x_0}{y_0} = \cosh^{-1}\left(\frac{y}{y_0}\right)$$

$$\boxed{y = y_0 \cosh\left(\frac{x-x_0}{y_0}\right)}$$

6.20

$$f = f(y, y')$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{df}{dx} = \frac{d}{dx} f(y, y')$$

$$= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \quad \{\text{chain rule}\}$$

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''}$$

$$\frac{df}{dx} = (0)y' + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$\boxed{\frac{df}{dx} = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)}$$

$$df = d\left(y'\frac{\partial f}{\partial y'}\right)$$

$$\int df = \int d\left(y'\frac{\partial f}{\partial y'}\right)$$

$$f = y'\frac{\partial f}{\partial y'} + C$$

$$\boxed{f - y'\frac{\partial f}{\partial y'} = C}$$

6.21

$$t = \int_1^2 \frac{ds}{v}$$

$$v = \sqrt{2gy}$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1+y'^2}\,dx$$

$$t = \frac{1}{\sqrt{2g}} \int_0^{x_2} \sqrt{\frac{1+y'^2}{y}}\,dx$$

$$f(y, y', x) = \sqrt{\frac{1+y'^2}{y}}$$

$$f - y' \frac{\partial f}{\partial y'} = \text{const.}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'} \left(\frac{1+y'^2}{y} \right)^{1/2}$$

$$= \frac{1}{2} \left(\frac{1+y'^2}{y} \right)^{-1/2} \frac{2y'}{y}$$

$$= \frac{y'}{y \left(\frac{1+y'^2}{y} \right)^{1/2}}$$

$$= \frac{y'}{y^{1/2}(1+y'^2)^{1/2}}$$

$$y' \frac{\partial f}{\partial y'} = \frac{y'^2}{y^{1/2}(1+y'^2)^{1/2}}$$

$$f - y' \frac{\partial f}{\partial y'} = \frac{(1+y'^2)^{1/2}}{y^{1/2}} - \frac{y'^2}{y^{1/2}(1+y'^2)^{1/2}}$$

$$= \frac{1+y'^2 - y'^2}{y^{1/2}(1+y'^2)^{1/2}}$$

$$= \frac{1}{y^{1/2}(1+y'^2)^{1/2}}$$

$$\frac{1}{y^{1/2}(1+y'^2)^{1/2}} = C$$

$$\frac{1}{y(1+y'^2)} = C^2$$

$$\frac{1}{y} = C^2 + C^2 y'^2$$

$$\frac{1}{y} - C^2 = C^2 y'^2$$

$$\frac{1-C^2y}{y} = C^2 y'^2$$

$$\frac{1-C^2y}{C^2y} = y'^2$$

$$\sqrt{\frac{1-C^2y}{C^2y}} = y'$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{1-C^2y}{C^2y}}$$

$$dy = \sqrt{\frac{1-C^2y}{C^2y}}\, dx$$

$$dx = \sqrt{\frac{C^2y}{1-C^2y}}\, dy$$

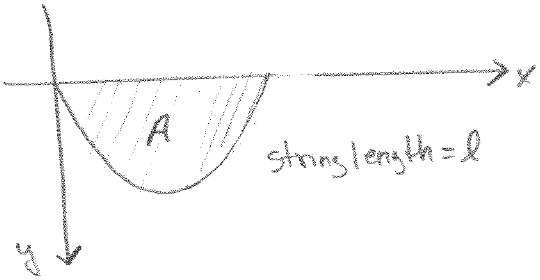
$$dx = \sqrt{\frac{y}{\frac{1}{C^2}-y}}\, dy$$

let $\frac{1}{C^2} = 2a$ {same as Eq. 6.22}

$$dx = \sqrt{\frac{y}{2a-y}}\, dy$$

$$\boxed{x = \int \sqrt{\frac{y}{2a-y}}\, dy}$$

6.22



$dA = y\,dx$

$A = \int y\,dx$

$ds^2 = dx^2 + dy^2$

$dx^2 = ds^2 - dy^2$

$dx = \sqrt{1-y'^2}\,ds$

$A = \int_0^\ell y\sqrt{1-y'^2}\,ds \qquad \{y' = \frac{dy}{ds}\}$

$f(y, y', s) = y\sqrt{1-y'^2}$

f is not a function of s, so

$f - y'\frac{\partial f}{\partial y'} = \text{const.}$

$y'\frac{\partial f}{\partial y'} = y' y (1-y'^2)^{-1/2}(-y')$

$= \frac{-y y'^2}{(1-y'^2)^{1/2}}$

$f - y'\frac{\partial f}{\partial y'} = y(1-y'^2)^{1/2} + \frac{y y'^2}{(1-y'^2)^{1/2}}$

$= \frac{y(1-y'^2) + y y'^2}{(1-y'^2)^{1/2}}$

$= \frac{y}{(1-y'^2)^{1/2}}$

$\frac{y^2}{1-y'^2} = c^2$

$y^2 = c^2 - c^2 y'^2$

$y^2 - c^2 = -c^2 y'^2$

$y'^2 = \frac{c^2 - y^2}{c^2}$

$$\frac{dy^2}{ds^2} = \frac{c^2-y^2}{c^2}$$

$$ds^2 = dx^2 + dy^2$$

$$\frac{dy^2}{dx^2+dy^2} = \frac{c^2-y^2}{c^2}$$

$$dy^2 = \frac{(dx^2+dy^2)(c^2-y^2)}{c^2}$$

$$c^2dy^2 = dx^2(c^2-y^2) + dy^2(c^2-y^2)$$

$$c^2dy^2 - dy^2(c^2-y^2) = dx^2(c^2-y^2)$$

$$c^2dy^2 - c^2dy^2 + y^2dy^2 = dx^2(c^2-y^2)$$

$$y^2dy^2 = dx^2(c^2-y^2)$$

$$\frac{y^2dy^2}{(c^2-y^2)} = dx^2$$

$$dx = \frac{y\,dy}{(c^2-y^2)^{1/2}}$$

$$x = \int \frac{y\,dy}{(c^2-y^2)^{1/2}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -u^{1/2} + a$$

$$u = c^2 - y^2$$

$$du = -2y\,dy$$

$$x = -(c^2-y^2)^{1/2} + a$$

$$0 = -(c^2)^{1/2} + a \quad \{x_0 = y_0 = 0\}$$

$$= -c + a$$

$$a = c$$

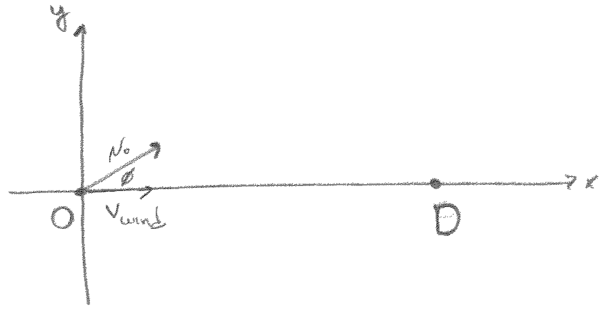
$$x = -(c^2-y^2)^{1/2} + c$$

$$x - c = -(c^2-y^2)^{1/2}$$

$$(x-c)^2 = c^2 - y^2$$

$(x-c)^2 + y^2 = c^2$ → This is the equation of a semi circle of radius c below the x-axis with center at $(c,0)$. $2\pi c = \ell \rightarrow c = \frac{\ell}{2\pi}$.

6.23



$\vec{V}_{wind} = Vy\,\hat{x}$

airspeed of aircraft $= v_0$

a) From the diagram

$\vec{v_0} = v_0\cos\phi\,\hat{x} + v_0\sin\phi\,\hat{y}$

$\vec{V} = \vec{v_0} + \vec{V}_{wind}$

$= (v_0\cos\phi + Vy)\hat{x} + v_0\sin\phi\,\hat{y}$

$V^2 = (v_0\cos\phi + Vy)^2 + (v_0\sin\phi)^2$

$V = \sqrt{(v_0\cos\phi + Vy)^2 + (v_0\sin\phi)^2}$

b) $t_{O\to D} = \int_O^D \frac{ds}{V}$ $\qquad ds = \sqrt{1+y'^2}\,dx$

$t_{O\to D} = \int_O^D \sqrt{\frac{1+y'^2}{(v_0\cos\phi + Vy)^2 + (v_0\sin\phi)^2}}\,dx$

If y' and ϕ are small then

$(1+y'^2)^{1/2} = 1 + \frac{1}{2}y'^2 + \dots$ {binomial expansion: $(a+b)^n = a^n + na^{n-1}b + \dots$}

$\left((v_0\cos\phi + Vy)^2 + (v_0\sin\phi)^2\right)^{1/2} = (v_0\cos\phi + Vy)\left[1 + \frac{(v_0\sin\phi)^2}{(v_0\cos\phi + Vy)^2}\right]^{1/2}$

$\approx (v_0 + Vy)\left[1 + \frac{1}{2}\frac{\sin^2\phi}{(v_0+Vy)^2} + \dots\right]$ {$\cos\phi \approx 1$}

$\approx \left(1 + \frac{Vy}{v_0}\right)\left[v_0 + \frac{v_0\sin^2\phi}{(v_0+Vy)^2} + \dots\right]$

$\approx \left(1 + \frac{Vy}{v_0}\right)\left[v_0 + \frac{\sin^2\phi}{v_0\left(1+\frac{Vy}{v_0}\right)^2} + \dots\right]$

$\approx \left(1 + \frac{Vy}{v_0}\right)\left[v_0 + \frac{\phi^2}{v_0\left(1+\frac{Vy}{v_0}\right)^2} + \dots\right]$ {$\sin\phi \approx \phi$}

$f = \sqrt{\frac{1+y'^2}{(v_0\cos\phi + Vy)^2 + (v_0\sin\phi)^2}} \approx \frac{1+\frac{1}{2}y'^2}{1+ky}\cdot C, \quad k = \frac{V}{v_0}$

c) $$f = \frac{1+\frac{1}{2}y'^2}{1+Ky} \cdot c$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(1+\tfrac{1}{2}y'^2\right)(1+Ky)^{-1} \cdot c$$

$$\frac{\partial f}{\partial y} = \frac{-K\left(1+\frac{1}{2}y'^2\right)}{(1+Ky)^2} \cdot c$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(1+\tfrac{1}{2}y'^2\right)(1+Ky)^{-1} \cdot c$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{1+Ky} \cdot c$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{y'}{1+Ky} \cdot c$$

$$= \frac{(1+Ky)y'' - y'(Ky')}{(1+Ky)^2} \cdot c$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{-K\left(1+\frac{1}{2}y'^2\right)}{(1+Ky)^2} - \frac{(1+Ky)y'' - Ky'^2}{(1+Ky)^2} = 0$$

$$(1+Ky)y'' - Ky'^2 + K\left(1+\tfrac{1}{2}y'^2\right) = 0$$

$$(1+Ky)y'' - Ky'^2 + K + \tfrac{K}{2}y'^2 = 0$$

$$(1+Ky)y'' - \tfrac{K}{2}y'^2 + K = 0$$

$$2(1+Ky)y'' - Ky'^2 + 2K = 0$$

Apply the guess $y = \lambda x(D-x)$ as a solution.

$$y = \lambda Dx - \lambda x^2$$

$$y' = \lambda D - 2\lambda x$$

$$y'' = -2\lambda$$

Insert into the Euler-Lagrange Equation.

$$2(1+k(\lambda x(D-x))(-2\lambda) - k(\lambda(D-2x))^2 + 2k = 0$$

Solve for λ.

$$-4\lambda(1+k(\lambda Dx - \lambda x^2) - k\lambda^2(D-2x)^2 + 2k = 0$$

$$-4\lambda(1 + kD\lambda x - k\lambda x^2 - k\lambda^2(D^2-4Dx+4x^2) + 2k = 0$$

$$-4\lambda - 4kD\lambda^2 x + 4k\lambda^2 x^2 - k\lambda^2 D^2 + 4Dk\lambda^2 x - 4k\lambda^2 x^2 + 2k = 0$$

$$-4\lambda - k\lambda^2 D^2 + 2k = 0$$

$$kD^2\lambda^2 + 4\lambda - 2k = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 + 4(2k)(kD^2)}}{2kD^2}$$

$$= \frac{-4 \pm 2\sqrt{4+2k^2D^2}}{2kD^2}$$

$$= \frac{-2 \pm \sqrt{4+2k^2D^2}}{kD^2}$$

$$\boxed{\lambda = \frac{\sqrt{4+2k^2D^2} - 2}{kD^2} \quad \{\text{positive root} \rightarrow y \geq 0\}}$$

$$y = x\,\frac{\sqrt{4+2k^2D^2} - 2}{kD^2}(D-x)$$

Given $D = 2000$ miles, $V_0 = 500$ mph, $V = .5$ mph/mi.

$$k = \frac{V}{V_0} = \frac{.5}{500} = .001$$

$$\lambda = \frac{\sqrt{4+2(.001)^2(2000)^2} - 2}{(.001)(2000)^2}$$

$$\lambda = 3.66 \times 10^{-4}$$

$$y = \lambda Dx - \lambda x^2$$

Find y max.

$$\frac{dy}{dx} = \lambda D - 2\lambda x$$

$$0 = \lambda D - 2\lambda x$$

$$2\lambda x = \lambda D$$

$$x = \frac{D}{2}$$

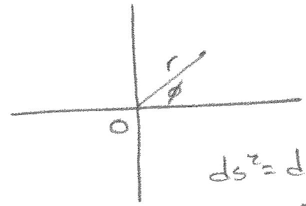
$x = 1000\,\text{mi}$ at y_{max}.

$$y_{max} = \lambda(1000)(2000-1000)$$
$$= \lambda(1000)^2$$
$$= (3.66 \times 10^{-4})(10^6)$$
$$= 3.66 \times 10^2\,\text{mi}$$

$$\boxed{y_{max} = 366\,\text{mi}}$$

6.24

$$n = \frac{a}{r^2}$$



$$ds^2 = dr^2 + (r\,d\phi)^2$$

$$ds = \sqrt{1 + r^2\phi'^2}\,dr$$

Fermat's Principle

$$\int_{x_1}^{x_2} n(x,y)\sqrt{1+y'^2}\,dx \quad \text{is a minimum.}$$

Express the path as $\phi = \phi(r) \rightarrow ds = \sqrt{1+r^2\phi'^2}\,dr$

$$f = \frac{a}{r^2}\sqrt{1+r^2\phi'^2} = f(\phi, \phi', r) \quad \left\{ n(r,\phi) = \frac{a}{r^2} \right\}$$

$$\frac{\partial f}{\partial \phi} - \frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$$

$$\frac{\partial f}{\partial \phi} = 0$$

$$\frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$$

$$\frac{\partial f}{\partial \phi'} = \text{const.}$$

$$\frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\frac{a}{r^2}(1+r^2\phi'^2)^{1/2}$$

$$= \frac{a}{r^2}\frac{1}{2}(1+r^2\phi'^2)^{-1/2} 2r^2\phi'$$

$$= \frac{a\phi'}{(1+r^2\phi'^2)^{1/2}}$$

$$\frac{a\phi'}{(1+r^2\phi'^2)^{1/2}} = C$$

$$\frac{a^2\phi'^2}{1+r^2\phi'^2} = C^2$$

$$a^2\phi'^2 = C^2 + C^2r^2\phi'^2$$

$$a^2\phi'^2 - C^2r^2\phi'^2 = C^2$$

$$\phi'^2(a^2 - C^2r^2) = C^2$$

$$\phi'^2 = \frac{C^2}{(a^2 - C^2r^2)}$$

$$\phi' = \frac{C}{(a^2 - C^2r^2)^{1/2}}$$

$$\int d\phi = \int \frac{c}{(a^2-c^2r^2)^{1/2}}\,dr$$

$$\int \frac{c\,dr}{a(1-\frac{c^2}{a^2}r^2)^{1/2}} = \int \frac{\cos\theta\,d\theta}{(1-\sin^2\theta)^{1/2}} = \int d\theta$$

$$= \theta + b$$

$$\sin\theta = \frac{c}{a}r$$

$$\cos\theta\,d\theta = \frac{c}{a}dr$$

$$c\,dr = a\cos\theta\,d\theta$$

$$= \sin^{-1}(\tfrac{c}{a}r) + b$$

$$\phi = \sin^{-1}(\tfrac{c}{a}r) + b$$

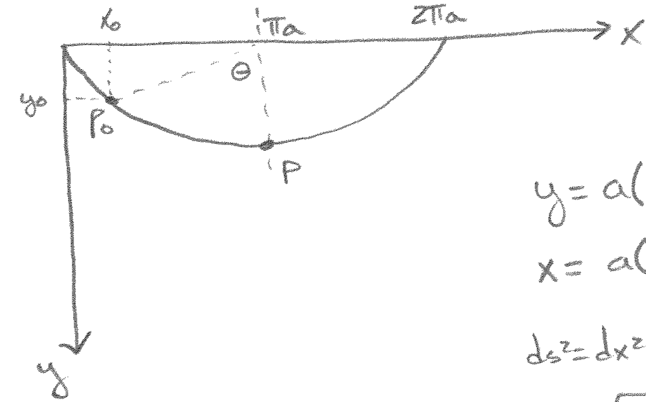
$$\phi - \phi_0 = \sin^{-1}(\tfrac{c}{a}r)$$

$$\sin(\phi-\phi_0) = \frac{c}{a}r$$

$$r = \frac{a}{c}\sin(\phi-\phi_0)$$

This is the equation of a circle that passes through the origin with a radius of $\frac{a}{2c}$. with $n = \frac{a}{r^2}$ it is seen that when $r=0$ $n=\infty$. The velocity of light through a medium is $V=\frac{c}{n}$. Therefore, the velocity of light at the origin is zero. Since Fermat's Principle states the time traveled is a minimum it follows that a path of light will always pass through the origin to minimize the travel time.

6.25



$$y = a(1-\cos\theta)$$

$$x = a(\theta - \sin\theta)$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1+y'^2}\,dx$$

$$v = \sqrt{2g(y-y_0)}$$

$$t_{P_0-P} = \int \frac{ds}{v}$$

$$= \int_{x_0}^{\pi a} \sqrt{\frac{1+y'^2}{2g(y-y_0)}}\,dx$$

$$dy = a\sin\theta\,d\theta$$

$$dx = a(d\theta - \cos\theta\,d\theta)$$

$$= a(1-\cos\theta)\,d\theta$$

$$= \int_{\theta_0}^{\pi} \sqrt{\frac{1+\frac{\sin^2\theta}{(1-\cos\theta)^2}}{2ga(\cos\theta_0-\cos\theta)}}\,a(1-\cos\theta)\,d\theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= \int_{\theta_0}^{\pi} \sqrt{\frac{a}{2g}} \sqrt{\frac{\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}}{\cos\theta_0-\cos\theta}}\,(1-\cos\theta)\,d\theta$$

$$= a\sin\theta \cdot \frac{1}{a(1-\cos\theta)}$$

$$y' = \frac{\sin\theta}{(1-\cos\theta)}$$

$$= \int_{\theta_0}^{\pi} \sqrt{\frac{a}{2g}} \sqrt{\frac{(1-\cos\theta)^2+\sin^2\theta}{\cos\theta_0-\cos\theta}}\,d\theta$$

$$y'^2 = \frac{\sin^2\theta}{(1-\cos\theta)^2}$$

$$= \int_{\theta_0}^{\pi} \sqrt{\frac{a}{2g}} \sqrt{\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\cos\theta_0-\cos\theta}}\,d\theta$$

$$y - y_0 = a(1-\cos\theta) - a(1-\cos\theta_0)$$

$$= a(\cos\theta_0 - \cos\theta)$$

$$= \int_{\theta_0}^{\pi} \sqrt{\frac{a}{2g}} \sqrt{\frac{2-2\cos\theta}{\cos\theta_0-\cos\theta}}\,d\theta$$

$$\boxed{t_{P_0-P} = \int_{\theta_0}^{\pi} \sqrt{\frac{a}{g}} \sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\,d\theta}$$

To evaluate the integral let

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta)$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{(1-\cos\theta)^{1/2}}{\sqrt{2}}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1+\cos\theta)$$

$$2\cos^2\left(\frac{\theta}{2}\right) - 1 = \cos\theta$$

$$\sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\,d\theta = \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\frac{\sqrt{2}\sin(\frac{\theta}{2})}{\sqrt{2\cos^2(\frac{\theta_0}{2})-1-(2\cos^2(\frac{\theta}{2})-1)}}\,d\theta$$

$$= \sqrt{\frac{2a}{g}}\int_{\theta_0}^{\pi}\frac{\sin(\frac{\theta}{2})}{\sqrt{2\cos^2(\frac{\theta_0}{2})-2\cos^2(\frac{\theta}{2})}}\,d\theta$$

$$= \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\frac{\sin(\frac{\theta}{2})}{\sqrt{\cos^2(\frac{\theta_0}{2})-\cos^2(\frac{\theta}{2})}}\,d\theta$$

let $\cos(\frac{\theta}{2}) = u\cos(\frac{\theta_0}{2})$

$$-\frac{1}{2}\sin(\tfrac{\theta}{2})\,d\theta = \cos(\tfrac{\theta_0}{2})\,du$$

$$d\theta = -\frac{2\cos(\frac{\theta_0}{2})}{\sin(\frac{\theta}{2})}\,du$$

$$= \sqrt{\frac{a}{g}}\int_{u_0}^{u_1}\frac{\sin(\frac{\theta}{2})}{\sqrt{\cos^2(\frac{\theta_0}{2})-u^2\cos^2(\frac{\theta_0}{2})}}\cdot\frac{-2\cos(\frac{\theta_0}{2})}{\sin(\frac{\theta}{2})}\,du$$

$$= \sqrt{\frac{a}{g}}\int_{1}^{0}\frac{-2\,du}{\sqrt{1-u^2}} \qquad u_0 = \frac{\cos(\frac{\theta_0}{2})}{\cos(\frac{\theta_0}{2})} = 1,\quad u_1 = \frac{\cos(\frac{\pi}{2})}{\cos(\frac{\theta_0}{2})} = 0$$

$$u = \sin w$$

$$du = \cos w\,dw$$

$$= \sqrt{\frac{a}{g}}\int_{w_0}^{w_1}\frac{-2\cos w\,dw}{\sqrt{1-\sin^2 w}}$$

$$= (-2)\sqrt{\frac{a}{g}}\int_{w_0}^{w_1} dw$$

$$= (-2)\sqrt{\frac{a}{g}}\,w\Big|_{w_0}^{w_1}$$

$$= (-2)\sqrt{\frac{a}{g}}\,\sin^{-1}(u)\Big|_{u_0}^{u_1}$$

$$= -2\sqrt{\frac{a}{g}}\,\sin^{-1}(u)\Big|_{1}^{0}$$

$$= -2\sqrt{\frac{a}{g}}\left[\sin^{-1}(0)-\sin^{-1}(1)\right]$$

$$= -2\sqrt{\frac{a}{g}}\left[0-\frac{\pi}{2}\right]$$

$$\boxed{t = \sqrt{\frac{a}{g}}\,\pi}$$

This is true because the greater $\theta_0^{(1)}$ becomes the velocity increases thus covering more distance in the same time as any other $\theta_0^{(2)} < \theta_0^{(1)}$.

6.26

$$S = \int_{u_1}^{u_2} f(x(u), y(u), x'(u), y'(u), u)\,du$$

In parametric form:

$$x = x(u) \quad , y = y(u)$$

The "wrong" path is

$$x = x(u) + \alpha\xi(u) \quad , \quad y = y(u) + \beta\eta(u)$$

$$x'(u) = x'(u) + \alpha\xi'(u) \quad y'(u) = y'(u) + \beta\eta'(u)$$

$$S(\alpha,\beta) = \int_{u_1}^{u_2} f(x(u) + \alpha\xi(u), y(u + \beta\eta(u)), x'(u) + \alpha\xi'(u), y'(u) + \beta\eta'(u), u)\,du$$

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \alpha}(x(u) + \alpha\xi(u), x'(u) + \alpha\xi'(u), y(u) + \beta\eta(u), y'(u) + \beta\eta'(u), u)$$

$$= \xi\frac{\partial f}{\partial x} + \xi'\frac{\partial f}{\partial x'}$$

$$\frac{dS}{d\alpha} = \int_{u_1}^{u_2} \frac{\partial f}{\partial \alpha}\,du = 0$$

$$= \int_{u_1}^{u_2} \left(\xi\frac{\partial f}{\partial x} + \xi'\frac{\partial f}{\partial x'}\right)du = 0$$

$$\int \xi'\frac{\partial f}{\partial x'}\,du = \xi(u)\frac{\partial f}{\partial x}\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} \xi(u)\frac{d}{du}\frac{\partial f}{\partial x'}\,du$$

$$= -\int_{u_1}^{u_2} \xi(u)\frac{d}{du}\frac{\partial f}{\partial x'}\,du \qquad \{\text{endpoint term is zero}\}$$

$$\frac{dS}{d\alpha} = \int_{u_1}^{u} \xi\frac{\partial f}{\partial x}\,du - \int_{u_1}^{u_2} \xi\frac{d}{du}\frac{\partial f}{\partial x'}\,du$$

$$= \int_{u}^{u_2} \left(\xi\frac{\partial f}{\partial x} - \xi\frac{d}{du}\frac{\partial f}{\partial x'}\right)du = 0$$

$$= \int_{u_1}^{u_2} \xi\left(\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'}\right)du = 0$$

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0}$$

$$\frac{\partial f}{\partial \beta} = n\frac{\partial f}{\partial y} + n'\frac{\partial f}{\partial y'}$$

$$\frac{dS}{d\beta} = \int_{u_1}^{u_2} \frac{\partial f}{\partial \beta}\,du$$

$$= \int_{u_1}^{u_2} \left(n\frac{\partial f}{\partial y} + n'\frac{\partial f}{\partial y'} \right) du = 0$$

This expression is the same as previously but with a change in variables. So,

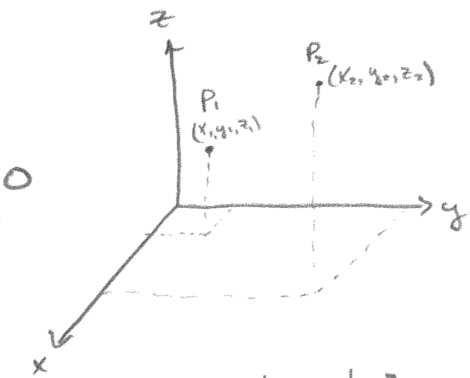
$$\frac{dS}{d\beta} = \int_{u_1}^{u_2} n\frac{\partial f}{\partial y}\,du - \int_{u_1}^{u_2} n\frac{d}{du}\frac{\partial f}{\partial y'}\,du = 0$$

$$0 = \int_{u_1}^{u_2} n\left(\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} \right) du$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0}$$

6.27

$$x = x(u) \ , \ y = y(u) \ , \ z = z(u)$$



$$\frac{\partial f}{\partial x} - \frac{d}{dx}\frac{\partial f}{\partial x'} = 0 \ , \ \frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0 \ , \ \frac{\partial f}{\partial z} - \frac{d}{du}\frac{\partial f}{\partial z'} = 0$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$dx = x'du \ , \ dy = y'du \ , \ dz = z'du$$

$$ds = \sqrt{x'^2 + y'^2 + z'^2}\, du$$

$$f(x, x', y, y', z, z', u) = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial x'} = C_1 \ , \ \frac{\partial f}{\partial y'} = C_2 \ , \ \frac{\partial f}{\partial z'} = C_3$$

$$\frac{\partial f}{\partial x'} = \frac{1}{2}(x'^2 + y'^2 + z'^2)^{-1/2} 2x' = \frac{x'}{(x'^2 + y'^2 + z'^2)^{1/2}}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2}(x'^2 + y'^2 + z'^2)^{-1/2} 2y' = \frac{y'}{(x'^2 + y'^2 + z'^2)^{1/2}}$$

$$\frac{\partial f}{\partial z'} = \frac{1}{2}(x'^2 + y'^2 + z'^2)^{-1/2} 2z' = \frac{z'}{(x'^2 + y'^2 + z'^2)^{1/2}}$$

Solving for x:

$$\frac{d}{du}\frac{\partial f}{\partial x'} = 0$$

$$\frac{d}{du}\frac{x'}{(x'^2 + y'^2 + z'^2)^{1/2}} = 0 = \frac{d}{du}\frac{\frac{dx}{du}}{(x'^2 + y'^2 + z'^2)^{1/2}}$$

let $dw = (x'^2 + y'^2 + z'^2)^{1/2} du$

$$(x'^2 + y'^2 + z'^2)^{1/2} \frac{d}{dw}\frac{dx}{dw} = 0 \longrightarrow \frac{d^2x}{dw^2} = 0$$

$$\frac{dx}{dw} = A$$

$$x = Aw + B$$

Likewise, with y and z:

$$y = Cw + D$$
$$z = Ew + F$$

and $x = Aw + B$, $y = Cw + D$, $z = Ew + F$ define a straight line in three dimensions.