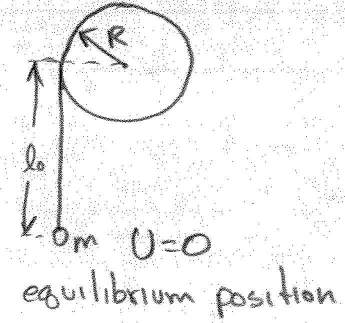
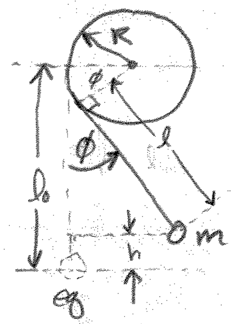
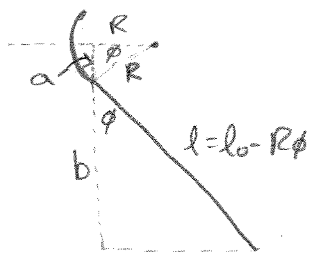
5.4





$U = mgh$



$l = l_0 - R\phi$

$\cos\phi = \frac{b}{l}$

$b = l\cos\phi$

$b = (l_0 - R\phi)\cos\phi$

$\sin\phi = \frac{a}{R}$

$a = R\sin\phi$

$h = l_0 - (a+b)$

$h = l_0 - [R\sin\phi + (l_0 - R\phi)\cos\phi]$

$$\boxed{U = mg[l_0 - (R\sin\phi + (l_0 - R\phi)\cos\phi)]}$$

For $\phi \ll 1$

$\sin\phi \approx \phi$

$\cos\phi \approx 1 - \frac{\phi^2}{2}$

$$U = mg\left[l_0 - \left(R\phi + (l_0 - R\phi)\left(1 - \frac{\phi^2}{2}\right)\right)\right]$$

$$= mg\left[l_0 - \left(R\phi + l_0 - R\phi - \frac{l_0\phi^2}{2} + \frac{R\phi^3}{2}\right)\right]$$

$$= mg\left[\frac{l_0\phi^2}{2} - \frac{R\phi^3}{2}\right]$$

$U = \frac{mg\,l_0\,\phi^2}{2}$ { terms with ϕ^3 are much smaller than ϕ^2 terms }

$K = mg\,l_0$

$U = \frac{1}{2}K\phi^2 \rightarrow$ Hooke's Law

K is the potential energy for a fixed pendulum at $\phi = \frac{\pi}{2}$.

5.11

For simple harmonic motion

$$x = A\cos(\omega t + \delta) \qquad \omega = \sqrt{\frac{K}{m}}$$

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t - \delta)$$

$$x_1 = A\cos(\omega t_1 - \delta) \qquad x_2 = A\cos(\omega t_2 - \delta)$$

$$v_1 = -A\omega \sin(\omega t_1 - \delta) \qquad v_2 = -A\omega \sin(\omega t_2 - \delta)$$

Let $\delta = 0$ for simplicity

$$x_1 = A\cos\omega t_1 \qquad x_2 = A\cos\omega t_2$$

$$v_1 = -A\omega \sin\omega t_1 \qquad v_2 = -A\omega \sin\omega t_2$$

$$v_1 + v_2 = -A\omega(\sin\omega t_1 + \sin\omega t_2)$$

$$x_1 + x_2 = A(\cos\omega t_1 + \cos\omega t_2)$$

$$\omega x_1 = A\omega \cos\omega t_1$$

$$v_1 = -A\omega \sin\omega t$$

$$\omega^2 x_1^2 = A^2\omega^2\cos^2\omega t_1$$

$$v_1^2 = A^2\omega^2 \sin^2\omega t_1$$

$$v_1^2 + \omega^2 x_1^2 = A^2\omega^2(\cos^2\omega t_1 + \sin^2\omega t_1)$$

$$v_1^2 + \omega^2 x_1^2 = A^2\omega^2$$

$$v_1^2 = A^2\omega^2 - x_1^2\omega^2$$

$$v_1^2 = \omega^2(A^2 - x_1^2)$$

$$\omega x_2 = A\omega\cos\omega t_2$$

$$v_2 = -A\omega\sin\omega t_2$$

$$\omega^2 x_2 = A^2\omega^2\cos^2\omega t_2$$

$$v_2^2 = A^2\omega^2\sin^2\omega t_2$$

$$v_2^2 + \omega^2 x_2^2 = A^2\omega^2(\cos^2\omega t_2 + \sin^2\omega t_2)$$

$$v_2^2 + \omega^2 x_2^2 = A^2\omega^2$$

$$v_2^2 = A^2\omega^2 - \omega^2 x_2^2$$

$$v_2^2 = \omega^2(A^2 - x_2^2)$$

$$\omega^2 = \frac{v_2^2}{(A^2 - x_2^2)}$$

$$v_1^2 = \frac{v_2^2}{A^2 - x_2^2}(A^2 - x_1^2)$$

$$(A^2 - x_2^2)v_1^2 = v_2^2(A^2 - x_1^2)$$

$$A^2 v_1^2 - x_2^2 v_1^2 = A^2 v_2^2 - x_1^2 v_2^2$$

$$A^2 v_1^2 - A^2 v_2^2 = x_2^2 v_1^2 - x_1^2 v_2^2$$

$$A^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2} \rightarrow \boxed{A = \pm\sqrt{\frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}}}$$

$$\omega^2 = \frac{V_2^2}{A^2 - x_2^2}$$

$$= \frac{V_2^2}{\frac{x_2^2 V_1^2 - x_1^2 V_2^2}{V_1^2 - V_2^2} - x_2^2}$$

$$= \frac{V_2^2}{\frac{x_2^2 V_1^2 - x_1^2 V_2^2 - x_2^2 (V_1^2 - V_2^2)}{V_1^2 - V_2^2}}$$

$$= \frac{V_2^2 (V_1^2 - V_2^2)}{x_2^2 V_1^2 - x_1^2 V_2^2 - x_2^2 V_1^2 + x_2^2 V_2^2}$$

$$= \frac{V_2^2 (V_1^2 - V_2^2)}{x_2^2 V_2^2 - x_1^2 V_2^2}$$

$$= \frac{V_2^2 (V_1^2 - V_2^2)}{V_2^2 (x_2^2 - x_1^2)}$$

$$= \frac{V_1^2 - V_2^2}{x_2^2 - x_1^2}$$

$$\boxed{\omega = \pm \sqrt{\frac{V_1^2 - V_2^2}{x_2^2 - x_1^2}}}$$

5.12

$$\langle f \rangle = \frac{1}{\tau}\int_0^\tau f(t)\,dt$$

$$x = A\cos(\omega t - \delta) \quad \{S.H.M.\}$$

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t-\delta)$$

$$T = \frac{1}{2}mv^2$$

$$= \frac{1}{2}mA^2\omega^2\sin^2(\omega t-\delta)$$

$$U = \frac{1}{2}kx^2$$

$$= \frac{1}{2}kA^2\cos^2(\omega t-\delta)$$

$$T = \frac{1}{2}kA^2\sin^2(\omega t-\delta) \quad \{\omega^2 = \tfrac{k}{m}\}$$

$$\langle T \rangle = \frac{1}{\tau}\int_0^\tau \frac{1}{2}kA^2\sin^2(\omega t-\delta)\,dt$$

$$= \frac{kA^2}{2\tau}\int_0^\tau \sin^2(\omega t-\delta)\,dt$$

$$\int \sin^2(\omega t-\delta)\,dt = \frac{1}{\omega}\int \sin^2 u\,du$$

$u = \omega t - \delta$

$du = \omega\,dt$

$dt = \frac{du}{\omega}$

$$= \frac{1}{\omega}\int \frac{1}{2}(1-\cos 2u)\,du$$

$$= \frac{1}{2\omega}\left[u - \frac{1}{2}\sin 2u\right] + C$$

$$= \frac{1}{2\omega}\left[\omega t - \delta - \frac{1}{2}\sin(2(\omega t-\delta))\right] + C$$

$$= \frac{kA^2}{4\tau\omega}\left[\omega t - \delta - \frac{1}{2}\sin(2(\omega t-\delta))\right]\Big|_0^\tau$$

$$= \frac{kA^2}{4\tau\omega}\left[\omega\tau - \delta - \frac{1}{2}\sin(2(\omega\tau-\delta)) - \left(-\delta - \frac{1}{2}\sin(-2\delta)\right)\right]$$

$$= \frac{kA^2}{4\tau\omega}\left[\omega\tau - \frac{1}{2}\sin(2(\omega\tau-\delta)) + \frac{1}{2}\sin(-2\delta)\right]$$

$$\sin(2\omega\tau - 2\delta) = \sin\left(2\omega\left(\tfrac{2\pi}{\omega}\right) - 2\delta\right)$$

$\tau = \frac{1}{f}$, $2\pi f = \omega$, $f = \frac{\omega}{2\pi}$, $\tau = \frac{2\pi}{\omega}$

$$= \sin(4\pi - 2\delta)$$

$$= \sin(-2\delta)$$

$$= \frac{kA^2}{4\tau\omega}\left[\omega\tau - \frac{1}{2}\sin(-2\delta) + \frac{1}{2}\sin(-2\delta)\right]$$

$$\langle T \rangle = \frac{kA^2}{4}$$

$$E = \frac{1}{2}kA^2$$

$$\langle T \rangle = \frac{1}{2}E$$

$$\langle U\rangle = \frac{1}{\tau}\int_0^{\tau}\frac{1}{2}KA^2\cos^2(\omega t-\delta)\,dt$$

$$= \frac{KA^2}{2\tau}\int_0^{\tau}\cos^2(\omega t-\delta)\,dt$$

$$= \frac{KA^2}{2\tau}\,\frac{1}{2\omega}\left[\omega t-\delta+\tfrac{1}{2}\sin(2(\omega t-\delta))\right]\Big|_0^{\tau}$$

$$= \frac{KA^2}{4\tau\omega}\left[\omega\tau-\delta+\tfrac{1}{2}\sin(2(\omega\tau-\delta))-\left(-\delta+\tfrac{1}{2}\sin(-2\delta)\right)\right]$$

$$= \frac{KA^2}{4\tau\omega}\left[\omega\tau+\tfrac{1}{2}\left(\sin(2\omega\tau-2\delta)-\sin(-2\delta)\right)\right]$$

$$= \frac{KA^2}{4\tau\omega}\left[\omega\tau\right]$$

$$\langle U\rangle = \frac{KA^2}{4}$$

$$E = \tfrac{1}{2}KA^2$$

$$\langle U\rangle = \tfrac{1}{2}E$$

$$\int\cos^2(\omega t-\delta)\,dt = \frac{1}{\omega}\int\cos^2 u\,du$$

$u=\omega t-\delta$

$du=\omega\,dt$

$dt=\frac{du}{\omega}$

$$= \frac{1}{\omega}\int\tfrac{1}{2}(1+\cos 2u)\,du$$

$$= \frac{1}{2\omega}\left[u+\tfrac{1}{2}\sin 2u\right]+C$$

$$= \frac{1}{2\omega}\left[\omega t-\delta+\tfrac{1}{2}\sin(2(\omega t-\delta))\right]+C$$

$\tau=\frac{2\pi}{\omega}$

$$\sin\left(2\omega\left(\tfrac{2\pi}{\omega}\right)-2\delta\right)=\sin(4\pi-2\delta)$$

$$=\sin(-2\delta)$$

5.13

$$U(r) = U_0\left(\frac{r}{R} + \lambda^2 \frac{R}{r}\right) \quad , \quad 0 < r < \infty$$

Equilibrium position

$$\frac{dU}{dr} = U_0\left(\frac{1}{R} - \lambda^2 \frac{R}{r^2}\right)$$

$$0 = U_0\left(\frac{1}{R} - \lambda^2 \frac{R}{r_0^2}\right)$$

$$\lambda^2 \frac{R}{r_0^2} = \frac{1}{R}$$

$$r_0^2 = \lambda^2 R^2$$

$$r_0 = \pm \lambda R$$

$$\boxed{r_0 = \lambda R \quad \{0 < r < \infty\}}$$

$$U(x) = U(r_0) + U'(r_0)x + \frac{1}{2}U''(r_0)x^2 + \cdots$$

$$= \text{constant} + 0 + \frac{1}{2}\left.\frac{d^2U}{dr^2}\right|_{r_0} x^2$$

$$\frac{d}{dr}\frac{dU}{dr} = \frac{d}{dr}\left(U_0\left(\frac{1}{R} - \lambda^2 \frac{R}{r^2}\right)\right)$$

$$= \frac{2U_0\lambda^2 R}{r^3}$$

$$\boxed{U(x) = \text{constant} + \frac{1}{2}\left(\frac{2U_0\lambda^2 R}{r_0^3}\right)x^2}$$

$$\omega^2 = \frac{K}{m}$$

$$= \frac{2U_0\lambda^2 R}{m r_0^3}$$

$$= \frac{2U_0\lambda^2 R}{m(\lambda R)^3} \quad \{r_0 = \lambda R\}$$

$$\omega^2 = \frac{2U_0}{m\lambda R^2}$$

$$\boxed{\omega = \sqrt{\frac{2U_0}{m\lambda R^2}}}$$

5.11

a) $\frac{\omega_x}{\omega_y} = \frac{p}{q}$

$\tau = \frac{1}{f}$

$\omega = 2\pi f$

$\tau = \frac{2\pi}{\omega}$

$\frac{q}{\omega_y} = \frac{p}{\omega_x}$

$\frac{2\pi q}{\omega_y} = \frac{2\pi p}{\omega_x}$

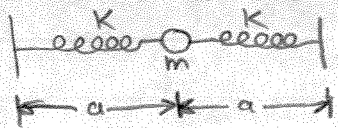
$\tau_y = \tau_x$ if q and p are integers {whole numbers of 2π revolutions}

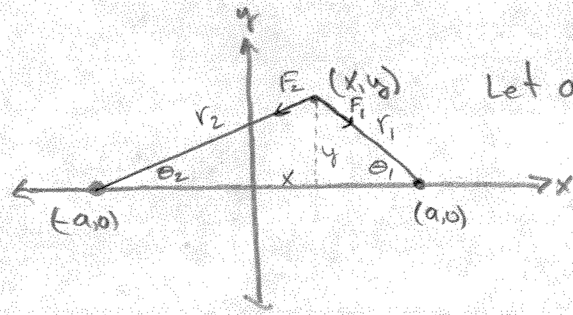
the period of the oscillator is

$\tau = \frac{2\pi p}{\omega_x}$ when $\frac{p}{q}$ is in simplest form.

b) If $\frac{p}{q}$ is irrational then p and q do not have a least common multiple that is an integer. Let q be an integer then p is irrational. Then for q revolutions of 2π there are p revolutions of 2π which is not a whole number of 2π revolutions. therefore, the two oscillations will never return to their respective starting points at the same time.

5.18





Let $a > l_0$ {springs in tension}

$$r_1 = [(a-x)^2+y^2]^{1/2}$$

$$r_2 = [(a+x)^2+y^2]^{1/2}$$

$$F_{1x} = F_1\cos\theta_1 \qquad F_{1y} = F_1\sin\theta_1$$

$$F_{2x} = F_2\cos\theta_2 \qquad F_{2y} = F_2\sin\theta_2$$

$$\cos\theta_1 = \frac{a-x}{[(a-x)^2+y^2]^{1/2}} \qquad \cos\theta_2 = \frac{a+x}{[(a+x)^2+y^2]^{1/2}}$$

$$\sin\theta_1 = \frac{y}{[(a-x)^2+y^2]^{1/2}} \qquad \sin\theta_2 = \frac{y}{[(a+x)^2+y^2]^{1/2}}$$

$$F_1 = -K(r_1 - l_0) \qquad F_2 = -k(r_2 - l_0)$$

$$F_x = F_{1x} + F_{2x}$$

$$F_x = \frac{-K\left([(a-x)^2+y^2]^{1/2} - l_0\right)(a-x)}{[(a-x)^2+y^2]^{1/2}} - \frac{K\left([(a+x)^2+y^2]^{1/2} - l_0\right)(a+x)}{[(a+x)^2+y^2]^{1/2}}$$

$$F_y = F_{1y} + F_{2y}$$

$$F_y = \frac{-K\left([(a-x)^2+y^2]^{1/2} - l_0\right)y}{[(a-x)^2+y^2]^{1/2}} - \frac{K\left([(a+x)^2+y^2]^{1/2} - l_0\right)y}{[(a+x)^2+y^2]^{1/2}}$$

For small x and y, a Taylor expansion of F about the equilibrium point (0,0) determines K_x and K_y.

Taylor expansion about a point (x_0, y_0) is

$$f(x,y) \approx f(x_0,y_0) + \vec{\nabla} f\Big|_{x_0,y_0} \cdot d\vec{x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0,y_0}(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}\Big|_{x_0,y_0}(x-x_0)(y-y_0) + \ldots$$

$$d\vec{x} = (x-x_0, y-y_0)$$

$$\vec{\nabla} f(x_0,y_0)\cdot d\vec{x} = \frac{\partial f}{\partial x}\Big|_{x_0,y_0}(x-x_0) + \frac{\partial f}{\partial y}\Big|_{x_0,y_0}(y-y_0)$$

$F(0,0) = 0$, $\frac{\partial F}{\partial x}\Big|_{x_0,y_0} = K_x$, $\frac{\partial F}{\partial y}\Big|_{x_0,y_0} = K_y$, higher order terms of x and y are ignored.

$$F_x = F_{1x} + F_{2x}$$

$$F_{1x} = \frac{-Ka\left(\left[(a-x)^2+y^2\right]^{1/2} - l_0\right)}{\left[(a-x)^2+y^2\right]^{1/2}} + \frac{Kx\left(\left[(a-x)^2+y^2\right]^{1/2} - l_0\right)}{\left[(a-x)^2+y^2\right]^{1/2}}$$

$$\frac{\partial F_{1x}}{\partial x} = \frac{-Ka\left[(a-x)^2+y^2\right]^{1/2}\left(\tfrac{1}{2}\right)\left[(a-x)^2+y^2\right]^{-1/2}(-2)(a-x) + Ka\left(\left[(a-x)^2+y^2\right]^{1/2} - l_0\right)\left(\tfrac{1}{2}\right)\left[(a-x)^2+y^2\right]^{-1/2}(-2)(a-x)}{(a-x)^2+y^2}$$

$$+ \frac{\left[(a-x)^2+y^2\right]^{1/2}\left(K\left(\left[(a-x)^2+y^2\right]^{1/2} - l_0\right) + Kx\left(\tfrac{1}{2}\right)\left[(a-x)^2+y^2\right]^{-1/2}(-2)(a-x)\right)}{(a-x)^2+y^2}$$

$$\left.\frac{\partial F_{1x}}{\partial x}\right|_{\substack{x=0\\y=0}} = \frac{-Ka[a]\tfrac{1}{2}[a]^{-1}(-2a) + Ka[a-l_0]\left(\tfrac{1}{2}\right)[a]^{-1}(-2a) + a\left(K(a-l_0)\right)}{a^2}$$

$$= \frac{Ka^2 - Ka(a-l_0) + Ka(a-l_0)}{a^2}$$

$$= K$$

By symmetry

$$\left.\frac{\partial F_{2x}}{\partial x}\right|_{\substack{x=0\\y=0}} = K$$

$$\left.\frac{\partial F_x}{\partial x}\right|_{\substack{x=0\\y=0}} = \left.\frac{\partial F_{x_1}}{\partial x}\right|_{\substack{x=0\\y=0}} + \left.\frac{\partial F_{x_2}}{\partial x}\right|_{\substack{x=0\\y=0}}$$

$$= K + K$$

$$\boxed{\left.\frac{\partial F_x}{\partial x}\right|_{x=0} = 2K = K_x}$$

$$\left.\frac{\partial F_y}{\partial y}\right|_{\substack{y=0\\x=0}} = K_y$$

$F_y(0) = 0$ {equilibrium}

$$F_y = F_{1y} + F_{2y}$$

$$F_{1y} = \frac{-K\left(\left[(a-x)^2+y^2\right]^{1/2} - l_0\right)y}{\left[(a-x)^2+y^2\right]^{1/2}}$$

③

$$\frac{\partial F_{1y}}{\partial y} = \frac{-k\left[(a-x)^2+y^2\right]^{1/2}\left[\left[\left[(a-x)^2+y^2\right]^{1/2}-l_0\right]+y\frac{1}{2}\left[(a-x)^2+y^2\right]^{-1/2}2y\right]-\left(\left[(a-x)^2+y^2\right]^{1/2}-l_0\right)y\left(\frac{1}{2}\right)\left[(a-x)^2+y^2\right]^{-1/2}2y)}{(a-x)^2+y^2}$$

$$\left.\frac{\partial F_{1y}}{\partial y}\right|_{\substack{y=0\\x=0}} = \frac{-k[a][a-l_0]}{a^2}$$

$$= \frac{-k(a-l_0)}{a}$$

$$\left.\frac{\partial F_{2y}}{\partial y}\right|_{\substack{y=0\\x=0}} = \left.\frac{\partial F_{1y}}{\partial y}\right|_{\substack{y=0\\x=0}}$$

$$= -\frac{k(a-l_0)}{a}$$

$$\boxed{\left.\frac{\partial F_y}{\partial y}\right|_{\substack{y=0\\x=0}} = -\frac{2k(a-l_0)}{a} = k_y}$$

$$U = U_x + U_y$$

$$U = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2$$

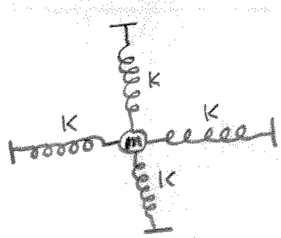
If $a < l_0$ then $k_y > 0$

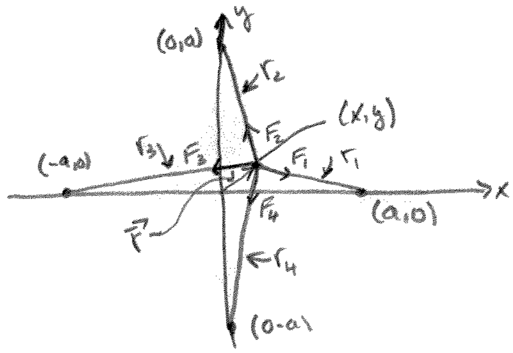
$$U_y = \frac{1}{2}k_y y^2$$

$$\frac{\partial U_y}{\partial y} = k_y y$$

$$\frac{\partial^2 U}{\partial y^2} = k_y > 0$$ → concave up → unstable

5.19





$F_x = F_{1x} + F_{2x} + F_{3x} + F_{4x}$ $\qquad$ $F_y = F_{1y} + F_{2y} + F_{3y} + F_{4y}$

$F_1 = -k(r_1 - l_0)$ $\quad F_2 = -k(r_2 - l_0)$ $\quad F_3 = -k(r_3 - l_0)$ $\quad F_4 = -k(r_4 - l_0)$

From 5.18 the F_3 and F_1 restoring forces have

$k_{x_{31}} = -2k$

$k_{y_{31}} = -\frac{2k(a-l_0)}{a}$

$U_{x_{31}} = \frac{1}{2}k_{x_{31}}x^2$

$= -kx^2$

$U_{y_{31}} = \frac{1}{2}k_{y_{31}}y^2$

$= -\frac{k(a-l_0)}{a}y^2$

Using symmetry

$k_{x_{24}} = -\frac{2k(a-l_0)}{a}$

$k_{y_{24}} = -2k$

$U_{x_{24}} = \frac{1}{2}k_{x_{24}}x^2$

$= -\frac{k(a-l_0)}{a}x^2$

$U_{y_{24}} = \frac{1}{2}k_{y_{24}}y^2$

$= -ky^2$

$U = U_{x_{31}} + U_{x_{24}} + U_{y_{31}} + U_{y_{24}}$

$= \frac{1}{2}k_{x_{31}}x^2 + \frac{1}{2}k_{x_{24}}x^2 + \frac{1}{2}k_{y_{31}}y^2 + \frac{1}{2}k_{y_{24}}y^2$

$k_x = -2k - \frac{2k(a-l_0)}{a}$ $\qquad$ $k_y = -2k - \frac{2k(a-l_0)}{a}$

$= -2k\left(1 + \frac{a-l_0}{a}\right)$ $\qquad$ $= -2k\left(1 + \frac{a-l_0}{a}\right)$

$= -2k\left(2 - \frac{l_0}{a}\right)$ $\qquad$ $= -2k\left(2 - \frac{l_0}{a}\right)$

$= -2k\left(\frac{2a-l_0}{a}\right)$ $\qquad$ $= -2k\left(\frac{2a-l_0}{a}\right)$

$U = k_x x^2 + k_y y^2$

$= k'(x^2+y^2)$

$\boxed{U = k'r^2 \quad k' = -\frac{2k(2a-l_0)}{a}}$

$F = -\nabla U$

$= -\frac{\partial U}{\partial r}$

$= -2k'r$

$\boxed{F = 4k\left(\frac{2a-l_0}{a}\right)r}$

5.25

$\beta < \omega_0$

$m\ddot{x} + b\dot{x} + kx = 0$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

$x = A e^{\lambda t}$

$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$

$\lambda = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$

$= -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$

$x = A e^{(-\beta + i\sqrt{\omega_0^2-\beta^2})t} + A e^{(-\beta - i\sqrt{\omega_0^2-\beta^2})t}$

$x = Ae^{-\beta t}(\cos(\omega_1 t - \delta))$, $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

The maxima are found by

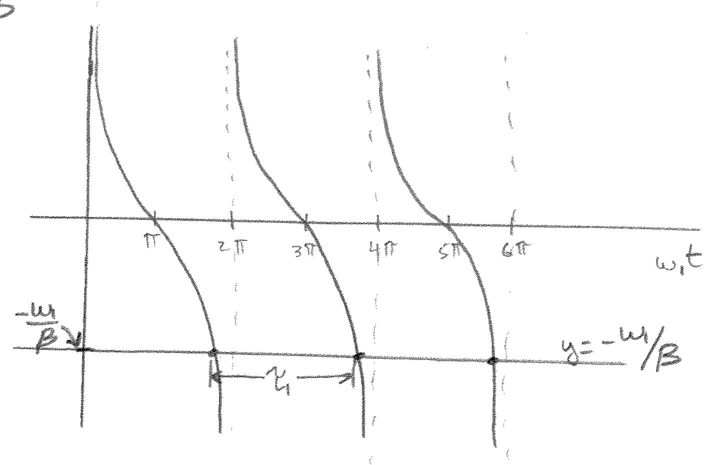
$\frac{dx}{dt} = A[e^{-\beta t}(-\omega_1)\sin(\omega_1 t - \delta) - \beta e^{-\beta t}\cos(\omega_1 t - \delta))$

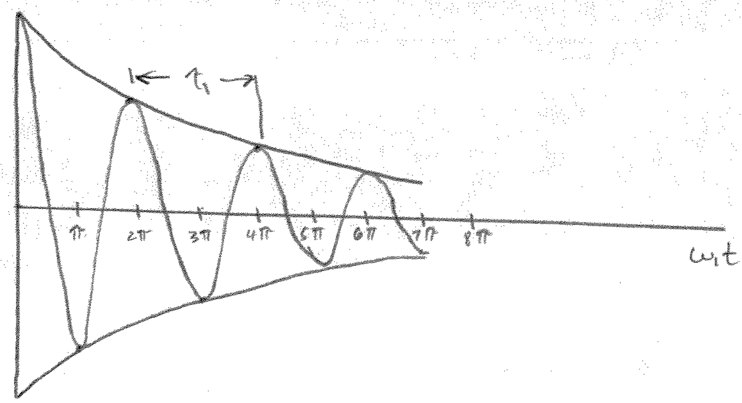
$0 = -\omega_1 \sin(\omega_1 t - \delta) - \beta\cos(\omega_1 t - \delta)$

$\frac{-\omega_1}{\beta} = \cot(\omega_1 t - \delta)$

let $\delta = 0$

$\frac{-\omega_1}{\beta} = \cot(\omega_1 t)$





$$\tau_1 = t_{2max} - t_{1max}$$
$$= \frac{4\pi}{\omega_1} - \frac{2\pi}{\omega_1}$$
$$\tau_1 = \frac{2\pi}{\omega_1}$$

b) $\Delta t_{min} = t_{2min} - t_{1min}$

$$= \frac{3\pi}{2\omega_1} - \frac{\pi}{2\omega_1}$$
$$= \frac{\pi}{\omega_1}$$
$$2\Delta t_{min} = \frac{2\pi}{\omega_1}$$
$$t_1 = 2\Delta t_{min} = \frac{2\pi}{\omega_1}$$

c) $\beta = \frac{\omega_0}{2}$

$$A = A_0 e^{-\beta t}$$
$$= A_0 e^{-\frac{\omega_0}{2}\tau_1}$$
$$= A_0 e^{-\frac{\omega_0}{2}\frac{2\pi}{\omega_1}}$$
$$= A_0 e^{-\pi\frac{\omega_0}{\omega_1}}$$
$$= A_0 e^{-\left(\frac{2\pi}{\sqrt{3}}\right)}$$
$$\boxed{A = .027 A_0}$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$
$$= \sqrt{\omega_0^2 - \frac{\omega_0^2}{4}}$$
$$= \omega_0\sqrt{\frac{3}{4}}$$
$$= \frac{\omega_0\sqrt{3}}{2}$$

5.26 $\tau_0 = 1.000\,s$

$\tau_1 = 1.001\,s$

$\tau_1 = \frac{2\pi}{\omega_1}$

$\omega_1 = \frac{2\pi}{\tau_1}$

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

$\omega_1^2 = \omega_0^2 - \beta^2$

$\beta^2 = \omega_0^2 - \omega_1^2$

$= \frac{2\pi}{1} - \frac{2\pi}{1.001}$

$= 2\pi\left(1 - \frac{1}{1.001}\right)$

$= .0063$

$\boxed{\beta = 0.0794}$

$A = A_0 e^{-\beta t}$

$A_{10} = A_0 e^{-\beta(10\tau_1)}$ {ten cycles}

$= A_0 e^{-(0.0794)(10)(1.001)}$

$\boxed{A_{10} = .452 A_0}$

$\% \Delta\tau = \frac{\tau_1 - \tau_0}{\tau_0} \times 100\%$ $\Delta A = \frac{A_0 - A_{10}}{A_0} \times 100\%$

$= \frac{1.001 - 1.000}{1.000} \times 100\%$ $= \frac{A_0 - .452 A_0}{A_0} \times 100\%$

$= .001 \times 100\%$ $= (1 - .452) \times 100\%$

$\boxed{\% \Delta\tau = 0.1\% \qquad \Delta A = 54.8\%}$

5.27

a) Critical Damping

$$\beta = \omega_0$$

$$x = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

Set $x=0$

$$0 = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

$$0 = C_1 + C_2 t$$

$$t = -\frac{C_1}{C_2}$$

If $C_1 > 0$ and $C_2 > 0$ then there is no time when the oscillator crosses $x=0$.

If $C_1 < 0$ or $C_2 < 0$ then there is one time, t, when the oscillator crosses $x=0$, namely at $t = -\frac{C_1}{C_2}$.

b) Overdamped Condition

$$\beta > \omega_0$$

$$x = e^{-\beta t}\left(C_1 e^{(\beta^2-\omega_0^2)^{1/2} t} + C_2 e^{-(\beta^2-\omega_0^2)^{1/2} t}\right)$$

Set $x=0$

$$0 = e^{-\beta t}\left(C_1 e^{(\beta^2-\omega_0^2)^{1/2} t} + C_2 e^{-(\beta^2-\omega_0^2)^{1/2} t}\right)$$

$$-C_1 e^{(\beta^2-\omega_0^2)^{1/2} t} = C_2 e^{-(\beta^2-\omega_0^2)^{1/2} t}$$

$$\frac{e^{(\beta^2-\omega_0^2)^{1/2} t}}{e^{-(\beta^2-\omega_0^2)^{1/2} t}} = \frac{-C_2}{C_1}$$

$$e^{2(\beta^2-\omega_0^2)^{1/2} t} = \frac{-C_2}{C_1}$$

$$2(\beta^2-\omega_0^2)^{1/2} t = \ln\left(\frac{-C_2}{C_1}\right)$$

If $C_1 > 0$ and $C_2 > 0$ then no time exists for the oscillator to cross $x=0$.

If $C_1 < 0$ or $C_2 < 0$ then there is one time, t, when the oscillator crossex $x=0$, namely, $t = \dfrac{\ln\left(\frac{-C_2}{C_1}\right)}{2(\beta^2-\omega_0^2)^{1/2}}$.

5.29 $\tau_0 = 1\text{ sec}$

$A = .5A_0$ at $t = \tau_1$

$A = A_0 e^{-\beta t}$

$.5A_0 = A_0 e^{-\beta \tau_1}$

$.5 = e^{-\frac{2\pi\beta}{\omega_1}}$

$\ln(.5) = -\frac{2\pi\beta}{\omega_1}$

$\ln(2) = \frac{2\pi\beta}{\omega_1}$

$\frac{\ln(2)}{2\pi} = \frac{\beta}{\omega_1}$

$= \frac{\beta}{(\omega_0^2-\beta^2)^{1/2}}$

Let $C = \frac{\ln(2)}{2\pi} = 0.1103$

$C = \frac{\beta}{(\omega_0-\beta^2)^{1/2}}$

$C^2 = \frac{\beta^2}{\omega_0^2-\beta^2}$

$C^2(\omega_0^2-\beta^2) = \beta^2$

$C^2\omega_0^2 - C^2\beta^2 = \beta^2$

$C^2\omega_0^2 = \beta^2(1+C^2)$

$\beta^2 = \frac{C^2\omega_0^2}{1+C^2}$

$\beta = \frac{C}{(1+C^2)^{1/2}}\omega_0$

$= \frac{0.1103}{(1+(0.1103)^2)^{1/2}}\omega_0$

$= 0.1096\,\omega_0$

$\boxed{\beta = 0.110\omega_0}$

$\tau_1 = \frac{2\pi}{\omega_1}$

$C = \frac{\beta}{\omega_1}$

$\omega_1 = \frac{\beta}{C}$

$\tau_1 = \frac{2\pi C}{\beta}$

$= \frac{2\pi(0.1103)}{0.1096\omega_0}$

$= \frac{0.1103}{0.1096}$

$\boxed{\tau_1 = 1.006\text{ s}}$

$\tau_0 = \frac{2\pi}{\omega_0}$

$\omega_0 = \frac{2\pi}{\tau_0} = 2\pi$

5.30 a) $x(t) = C_1 e^{-(\beta - \sqrt{\beta^2-\omega_0^2})t} + C_2 e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}$

$$x_0 = x(0) = C_1 + C_2$$

$$\dot{x} = -C_1(\beta-\sqrt{\beta^2-\omega_0^2})\, e^{-(\beta-\sqrt{\beta^2-\omega_0^2})t} - C_2(\beta+\sqrt{\beta^2-\omega_0^2})\, e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}$$

$$v_0 = \dot{x}(0) = -C_1(\beta-\sqrt{\beta^2-\omega_0^2}) - C_2(\beta+\sqrt{\beta^2-\omega_0^2})$$

$$C_1 + C_2 = x_0$$

$$C_1(\beta-\sqrt{\beta^2-\omega_0^2}) + C_2(\beta+\sqrt{\beta^2-\omega_0^2}) = -v_0$$

$$\frac{-v_0 - C_2(\beta+\sqrt{\beta^2-\omega_0^2})}{\beta-\sqrt{\beta^2-\omega_0^2}} + C_2 = x_0$$

$$\frac{-C_2(\beta+\sqrt{\beta^2-\omega_0^2})}{\beta-\sqrt{\beta^2-\omega_0^2}} + C_2 = x_0 + \frac{v_0}{\beta-\sqrt{\beta^2-\omega_0^2}}$$

$$C_2\left(1 - \frac{\beta+\sqrt{\beta^2-\omega_0^2}}{\beta-\sqrt{\beta^2-\omega_0^2}}\right) = x_0 + \frac{v_0}{\beta-\sqrt{\beta^2-\omega_0^2}}$$

$$C_2\left(\frac{\beta-\sqrt{\beta^2-\omega_0^2} - \beta-\sqrt{\beta^2-\omega_0^2}}{\beta-\sqrt{\beta^2-\omega_0^2}}\right) = \frac{x_0(\beta-\sqrt{\beta^2-\omega_0^2}) + v_0}{\beta-\sqrt{\beta^2-\omega_0^2}}$$

$$C_2(-2\sqrt{\beta^2-\omega_0^2}) = x_0(\beta-\sqrt{\beta^2-\omega_0^2}) + v_0$$

$$\boxed{C_2 = -\frac{x_0(\beta-\sqrt{\beta^2-\omega_0^2}) + v_0}{2\sqrt{\beta^2-\omega_0^2}}}$$

$$C_1 = x_0 - C_2$$

$$\boxed{C_1 = x_0 + \frac{x_0(\beta-\sqrt{\beta^2-\omega_0^2}) + v_0}{2\sqrt{\beta^2-\omega_0^2}}}$$

b)

i) Let $v_0 = 0$

$$C_1 = x_0 + \frac{x_0\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)}{2\sqrt{\beta^2-\omega_0^2}}$$

$$C_2 = -\frac{x_0\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)}{2\sqrt{\beta^2-\omega_0^2}}$$

Let $\dfrac{x_0\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)}{2\sqrt{\beta^2-\omega_0^2}} = K$

$C_1 = x_0 + K$

$C_2 = -K$

$$x(t) = (x_0+K)e^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} - Ke^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}$$

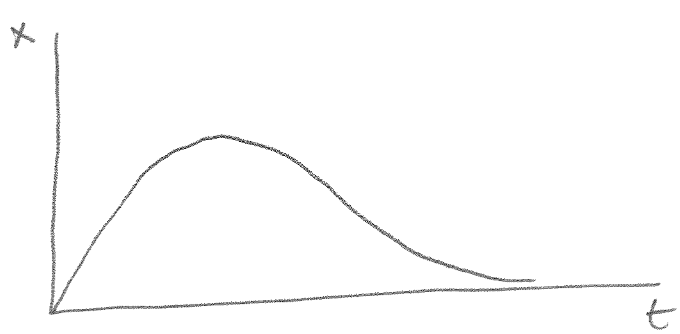
ii) $x_0 = 0$

$$C_1 = \frac{v_0}{2\sqrt{\beta^2-\omega_0^2}}$$

$$K = \frac{v_0}{2\sqrt{\beta^2-\omega_0^2}}$$

$$C_2 = -\frac{v_0}{2\sqrt{\beta^2-\omega_0^2}}$$

$$x(t) = Ke^{-\left(\beta-\sqrt{\beta^2-\omega_0^2}\right)t} - Ke^{-\left(\beta+\sqrt{\beta^2-\omega_0^2}\right)t}$$



c) $x(t) = C_1 e^{-(\beta - \sqrt{\beta^2-\omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2-\omega_0^2})t}$

$$C_1 = x_0 + \frac{x_0(\beta - \sqrt{\beta^2-\omega_0^2} + v_0}{2\sqrt{\beta^2-\omega_0^2}}$$

$$C_2 = -\frac{x_0(\beta - \sqrt{\beta^2-\omega_0^2}) + v_0}{2\sqrt{\beta^2-\omega_0^2}}$$

Let $\beta \to 0$

$$C_1 = x_0 + \frac{x_0(-\sqrt{-\omega_0^2}) + v_0}{2\sqrt{-\omega_0^2}} = x_0 - \frac{x_0}{2} + \frac{v_0}{2i\omega_0} = \frac{x_0}{2} + \frac{v_0}{2i\omega_0}$$

$$C_2 = -\frac{x_0(-\sqrt{-\omega_0^2}) + v_0}{-2\sqrt{-\omega_0^2}} = \frac{x_0}{2} - \frac{v_0}{2i\omega_0}$$

$$x(t) = \left(\frac{x_0}{2} + \frac{v_0}{2i\omega_0}\right) e^{i\omega_0 t} + \left(\frac{x_0}{2} - \frac{v_0}{2i\omega_0}\right) e^{-i\omega_0 t}$$

$$= \frac{x_0}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) + \frac{v_0}{2i\omega_0}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\boxed{x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

5.32

a) Under Damped

$$x(t) = e^{-\beta t}(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t})$$

$$x_0 = x(0)$$

$$v_0 = 0 = \dot{x}(0)$$

$$x_0 = x(0) = C_1 + C_2$$

$$\dot{x}(t) = e^{-\beta t}(C_1 i\omega_1 e^{i\omega_1 t} - C_2 i\omega_1 e^{-i\omega_1 t}) - \beta e^{\beta t}(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t})$$

$$\dot{x}(0) = C_1 i\omega_1 - C_2 i\omega_1 - \beta C_1 - \beta C_2$$

$$0 = C_1(i\omega_1 - \beta) - C_2(i\omega_1 + \beta)$$

$$C_1(i\omega_1 - \beta) = C_2(i\omega_1 + \beta)$$

$$C_1 = C_2 \frac{(i\omega_1 + \beta)}{i\omega_1 - \beta}$$

$$x_0 = C_2 \frac{(i\omega_1 + \beta)}{i\omega_1 - \beta} + C_2$$

$$= C_2\left(\frac{i\omega_1 + \beta}{i\omega_1 - \beta} + 1\right)$$

$$= C_2\left(\frac{i\omega_1 + \beta + i\omega_1 - \beta}{i\omega_1 - \beta}\right)$$

$$= C_2\left(\frac{2i\omega_1}{i\omega_1 - \beta}\right)$$

$$C_2 = x_0 \frac{(i\omega_1 - \beta)}{2i\omega_1}$$

$$C_1 = \frac{x_0(i\omega_1 - \beta)}{2i\omega_1} \frac{(i\omega_1 + \beta)}{i\omega_1 - \beta}$$

$$C_1 = \frac{x_0(i\omega_1 + \beta)}{2i\omega_1}$$

$$x(t) = e^{-\beta t}\left(\frac{x_0(i\omega_1 + \beta)}{2i\omega_1} e^{i\omega_1 t} + \frac{x_0(i\omega_1 - \beta)}{2i\omega_1} e^{-i\omega_1 t}\right)$$

$$= e^{-\beta t}\left(\frac{x_0}{2}(e^{i\omega_1 t} + e^{-i\omega_1 t}) + \frac{x_0 \beta}{2i\omega_1}(e^{i\omega_1 t} - e^{-i\omega_1 t})\right)$$

$$\boxed{x(t) = e^{-\beta t}\left(x_0 \cos\omega_1 t + \frac{x_0 \beta}{\omega_1} \sin\omega_1 t\right)}$$

b) Let $\beta \to \omega_0$

$$x(t) = e^{-\beta t}\left(x_0 \cos\omega_1 t + \frac{x_0 \beta}{\omega_1} \sin\omega_1 t\right)$$

$$\omega_1 = \sqrt{\beta^2 - \omega_0^2}$$

when $\beta \to \omega_0$, $\omega_1 \to 0$. So, $\cos\omega_1 t \to 1$ and $\sin\omega_1 t \to \omega_1 t$

$$x(t) = e^{-\omega_0 t}\left(x_0 (1) + \frac{x_0 \beta}{\omega_1} \omega_1 t\right)$$

$$\boxed{x(t) = e^{-\omega_0 t}\left(x_0 + x_0 \beta t\right)}$$

5.35

a) If $z = x+iy$ then $z = re^{i\theta}$

$x = r\cos\theta$

$y = r\sin\theta$

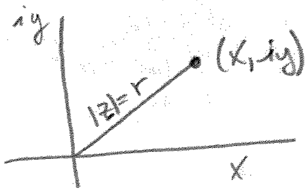
$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$z = re^{i\theta}$

b) If $|z| = r$ then $|z|^2 = zz^*$

$z = x+iy$



$r = (x^2+y^2)^{1/2}$

$r^2 = x^2+y^2$

$z = x+iy$

$z^* = x-iy$

$zz^* = (x+iy)(x-iy)$

$= x^2 - ixy + ixy - i^2y$

$zz^* = x^2+y^2$

$r^2 = zz^*$

$|z| = r$

$|z|^2 = r^2$

$|z|^2 = zz^*$

c) Prove $z^* = re^{-i\theta}$

$z^* = x - iy$

$= r\cos\theta - ir\sin\theta$

$= r(\cos\theta - i\sin\theta)$ $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

$z^* = re^{-i\theta}$

d) i) Prove $(zw)^* = z^* w^*$

$z = x + iy$

$w = u + iv$

$zw = (x+iy)(u+iv)$

$= xu + ixv + iyu + i^2 yv$

$= xu + i(xv + yu) - yv$

$= xu - yv + i(xv + yu)$

$(zw)^* = xu - yv - i(xv + yu)$

$z^* = x - iy$

$w^* = u - iv$

$z^* w^* = (x - iy)(u - iv)$

$= xu - ixv - iyu + i^2 yv$

$= xu - i(xv + yu) - yv$

$= xu - yv - i(xv + yu)$

$(zw)^* = z^* w^*$

ii) Prove $\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$

$\left(\frac{1}{z}\right) = (1 \cdot z^{-1})$

$(1 \cdot z^{-1})^* = 1^* \cdot (z^{-1})^*$ {from previous proof}

$\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$

e) If $z = \frac{a}{b+ic}$ then $|z|^2 = \frac{a^2}{b^2+c^2}$

$|z|^2 = z z^*$

$= \frac{a}{b+ic} \cdot \frac{a}{b-ic}$

$= \frac{a^2}{b^2 - ibc + ibc - i^2 c^2}$

$|z|^2 = \frac{a^2}{b^2+c^2}$

5.43 a) Determine K

$4 \cdot 80\,kg = 320\,kg$, $d = 2\,cm = .02\,m$

$F = mg$

$F = 320\,kg(9.81\frac{m}{s^2})$

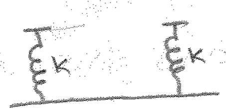
$F = 3039.2$

$K_T = \frac{F}{d}$

$K_T = \frac{3039.2\,N}{.02\,m}$

$K_T = 1.5 \times 10^5 \frac{N}{m} \rightarrow K = \frac{K_T}{4} \approx 4 \times 10^4 \frac{N}{m}$

b) $m = 50\,kg$



$k_T = 2K$

$\omega_0 = \sqrt{\frac{K_T}{m}}$

$= \sqrt{\frac{8 \times 10^4}{50}}$

$\omega_0 = 40\,s^{-1}$ $\qquad 2\pi f = \omega$

$2\pi f_0 = 40\,s^{-1}$

$f_0 = \frac{40\,s^{-1}}{2\pi}$

$f_0 = 6.4\,Hz \approx 6\,Hz$

c)

$V = \frac{6\text{ bumps}}{sec} = \frac{6 \cdot 80\,cm}{sec} = \frac{480\,cm}{sec} \times \frac{1\,in}{2.54\,cm} \times \frac{1\,ft}{12\,in} \times \frac{1\,mile}{5280\,ft} \cdot \frac{3600\,sec}{1\,hr} = 10\,\frac{mi}{hr}$

5.44

$\omega = \omega_0$

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} \quad \{\text{driven oscillator}\}$$

a) $C = \dfrac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$

$$A^2 = CC^*$$
$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$$

$$A^2(\omega_0) = \frac{f_0^2}{4\beta^2\omega_0^2}$$

$$x(t) = A\cos(\omega t - \delta) + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{\text{transient}}$$

$$x(t) = A\cos(\omega t - \delta) \rightarrow \dot{x}(t) = -A\omega\sin(\omega t - \delta)$$

$$T = \tfrac{1}{2}m\dot{x}^2$$
$$= \tfrac{1}{2}m\left(-A\omega\sin(\omega t - \delta)\right)^2$$

$$U = \tfrac{1}{2}kx^2$$
$$= \tfrac{1}{2}k\left(A\cos(\omega t - \delta)\right)^2$$

$$E = T + U$$
$$= \tfrac{1}{2}mA^2\omega^2\sin^2(\omega t - \delta) + \tfrac{1}{2}kA^2\cos^2(\omega t - \delta), \quad k = m\omega^2$$
$$= \tfrac{1}{2}m\omega^2A^2\left(\sin^2(\omega t - \delta) + \cos^2(\omega t - \delta)\right)$$

$$\boxed{E = \tfrac{1}{2}m\omega^2A^2}$$

b) The damping force is $b\dot{x}$

$$2\beta = \frac{b}{m}$$

$$b = 2m\beta$$

$$F_{dmp} = 2m\beta\dot{x}$$

$$\frac{E}{t} = Fv$$

$$\frac{E}{t} = 2m\beta \dot{x}^2$$

$$= 2m\beta(-A\omega \sin(\omega t - \delta))^2$$

$$= 2m\beta A^2\omega^2 \sin^2(\omega t - \delta)$$

$$\Delta E = 2m\beta A^2\omega^2 \sin^2(\omega t - \delta)\Delta t$$

For one cycle $t = \frac{2\pi}{\omega}$

The average value of $\sin^2(\omega t - \delta)$ over one cycle is $\frac{1}{2}$.

$$\Delta E = 2m\beta A^2\omega^2 \left(\frac{1}{2}\right)\frac{2\pi}{\omega}$$

$$\boxed{\Delta E = 2\pi m\beta\omega A^2}$$

c) $Q = \frac{\omega_0}{2\beta}$

$$E = \frac{1}{2}m\omega^2 A^2$$

$$\Delta E = 2m\beta\omega A^2$$

$$\frac{E}{\Delta E} = \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m\beta\omega A^2}$$

$$= \frac{\omega}{4\pi\beta}$$

$$\omega = \omega_0$$

$$\frac{E}{\Delta E} = \frac{\omega_0}{4\pi\beta}$$

$$2\pi\frac{E}{\Delta E} = \frac{\omega_0}{2\beta}$$

$$\boxed{2\pi\frac{E}{\Delta E} = Q}$$

3.45

a) $P = F\dot{x}$

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\dot{x}(t) = -A\omega \sin(\omega t - \delta) \quad \{\text{no transients}\}$$

$$P = F_0 \cos\omega t\left(-A\omega \sin(\omega t - \delta)\right)$$

$$P = -F_0 A\omega \cos\omega t \sin(\omega t - \delta) \qquad \sin(\omega t - \delta) = \sin\omega t\cos\delta - \cos\omega t \sin\delta$$

$$\boxed{P = -F_0 A\omega\left(\cos\omega t \sin\omega t \cos\delta - \cos^2\omega t \sin\delta\right)}$$

$$\langle P\rangle = \frac{1}{\tau}\int_0^\tau P\,dt$$

$$= \frac{1}{\tau}\int_0^\tau\left(-F_0 A\omega\left(\cos\omega t\sin\omega t\cos\delta - \cos^2\omega t \sin\delta\right)\right)dt$$

$$= -\frac{F_0 A\omega}{\tau}\left[\int_0^\tau \cos\omega t \sin\omega t\cos\delta\,dt - \int_0^\tau \cos^2\omega t \sin\delta\,dt\right]$$

$$= -\frac{F_0 A\omega}{\tau}\left[\sin^2(\omega t)\Big|_0^\tau \cos\delta - \int_0^\tau \tfrac{1}{2}(1+\cos(2\omega t))dt\,\sin\delta\right]$$

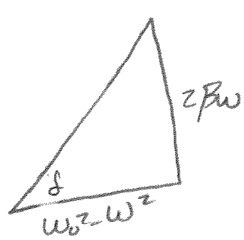
$$= -\frac{F_0 A\omega}{\tau}\left[\sin^2(\omega t)\Big|_0^{\frac{2\pi n}{\omega}} - \tfrac{1}{2}\left(t - \tfrac{1}{2}\sin(2\omega t)\right)\Big|_0^{\frac{2\pi n}{\omega}}\sin\delta\right]$$

$$= -\frac{F_0 A\omega}{\tau}\left[0 - \frac{\pi n}{\omega}\right]\sin\delta$$

$$= \frac{F_0 A\pi n}{\frac{2\pi n}{\omega}}\sin\delta$$

$$\langle P\rangle = \frac{F_0 A\omega}{2}\sin\delta$$

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$



$$\sin\delta = \frac{2\beta\omega}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}}$$

$$\langle P\rangle = \frac{F_0 A\omega}{2}\,\frac{2\beta\omega}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}}$$

$$= \frac{F_0\beta\omega^2 A}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}}$$

$F(t) = F_0 \cos\omega t$

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \quad \{Eq.\ 5.71\}$$

$$F_0 = mA\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2} \quad \left\{ f_0 = \frac{F_0}{m} \right\}$$

$$\langle P \rangle = mA\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2} \frac{\beta\omega^2 A}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}}$$

$$\boxed{\langle P \rangle = m\beta\omega^2 A^2}$$

b) Since the amplitude does not grow or decay $\langle P \rangle$ must be equal to the average power dissipation due to friction.

$$\Delta E_{dis} = 2\pi m\beta\omega A^2 \quad \{\text{from Problem } 5.44\}$$

$$P = \frac{\Delta E_{dis}}{\tau} = \frac{2\pi m\beta\omega A^2}{\tau}$$

$$= \frac{2\pi m\beta\omega A^2}{\frac{2\pi}{\omega}}$$

$$\boxed{P = m\beta\omega^2 A^2}$$

c) $$\langle P \rangle = \frac{F_0 \beta\omega^2 A}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}}$$

For $\langle P \rangle$ to be a maximum $(\omega_0^2-\omega^2)^2+4\beta^2\omega^2$ must be a minimum.

This occurs when $\omega = \omega_0$.

5.41

i) Prove

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\cos(m\omega t)\,dt = \begin{cases} \frac{\tau}{2}, & m=n\neq 0 \\ 0, & m\neq n \end{cases}$$

$$\begin{aligned} &\cos(A+B) = \cos A\cos B - \sin A\sin B \\ + \;&\cos(A-B) = \cos A\cos B + \sin A\sin B \\ \hline &\cos(A+B) - \cos(A-B) = 2\cos A\cos B \end{aligned}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$(m\neq n)$
$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\cos(m\omega t)\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{1}{2}\left[\cos(\omega t(n+m)) + \cos(\omega t(n-m))\right]dt$$

$$= \frac{1}{2}\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(\omega t(n+m))\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(\omega t(n-m))\,dt\right]$$

$$= \frac{1}{2}\left[\left.\frac{\sin(\omega t(n+m))}{\omega(n+m)}\right|_{-\frac{\tau}{2}=-\frac{\pi}{\omega}}^{\frac{\tau}{2}=\frac{\pi}{\omega}} + \left.\frac{\sin(\omega t(n-m))}{\omega(n-m)}\right|_{-\frac{\tau}{2}=-\frac{\pi}{\omega}}^{\frac{\tau}{2}=\frac{\pi}{\omega}}\right] \quad \left(\tau = \frac{2\pi}{\omega}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\omega(n+m)}\left(\sin(\pi(n+m)) + \sin(-\pi(n+m))\right) - \frac{1}{\omega(n-m)}\left(\sin(\pi(n-m)) - \sin(-\pi(n-m))\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega(n+m)}\left(2\sin(\pi(n+m))\right) + \frac{1}{\omega(n-m)}\left(2\sin(\pi(n-m))\right)\right]$$

$$= \frac{1}{\omega}\left[\frac{\sin(\pi(n+m))}{n+m} + \frac{\sin(\pi(n-m))}{n-m}\right]$$

$$= 0 \text{ when } m\neq n$$

$(m=n)$
$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\cos(m\omega t)\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos^2(n\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} (1 + \cos(2n\omega t))\,dt$$

$$= \frac{1}{2}\left[\left. t\right|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \left.\frac{\sin(2n\omega t)}{2n\omega}\right|_{-\frac{\tau}{2}=-\frac{\pi}{\omega}}^{\frac{\tau}{2}=\frac{\pi}{\omega}}\right]$$

$$= \frac{1}{2}\left[\tau + 0\right]$$

$$= \frac{\tau}{2} \text{ when } m=n$$

ii) Prove

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(n\omega t)\sin(m\omega t)\,dt = \begin{cases} \frac{\tau}{2}, & m=n\neq 0 \\ 0, & m\neq n \end{cases}$$

$$-\begin{Bmatrix} \cos(A+B) = \cos A\cos B - \sin A \sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B \end{Bmatrix}$$

$$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$(m\neq n)$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(n\omega t)\sin(m\omega t)\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{1}{2}\left[\cos(\omega t(n-m)) - \cos(\omega t(n+m))\right]dt$$

$$= \frac{1}{2}\left[\int_{-\frac{\tau}{2}=-\frac{\pi}{\omega}}^{\frac{\tau}{2}=\frac{\pi}{\omega}} \cos(\omega t(n-m))\,dt - \int_{-\frac{\tau}{2}=-\frac{\pi}{\omega}}^{\frac{\tau}{2}=\frac{\pi}{\omega}} \cos(\omega t(n+m))\,dt\right]$$

$$= \frac{1}{2}\left[\frac{\sin(\omega t(n-m))}{\omega(n-m)}\Bigg|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} - \frac{\sin(\omega t(n+m))}{\omega(n+m)}\Bigg|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\right]$$

$= 0$ when $m\neq n$ (same integrals at part i))

$(m=n)$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(n\omega t)\sin(m\omega t)\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin^2(n\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} (1-\cos(2n\omega t))\,dt$$

$$= \frac{1}{2}\left[t\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} - \frac{\sin(2n\omega t)}{2n\omega}\Bigg|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\right]$$

$$= \frac{1}{2}[\tau - 0]$$

$= \frac{\tau}{2}$ when $m=n$

iii) Prove

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\sin(m\omega t) = 0 \quad \text{for all } n \text{ and } m$$

$$-\left[\sin(A+B) = \sin A\cos B - \cos A\sin B\right]$$
$$\sin(A-B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) - \sin(A+B) = 2\cos A\sin B$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A-B) - \sin(A+B)\right]$$

$$(m\neq n)\quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t)\sin(m\omega t)\,dt = \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left[\sin(\omega t(n-m)) - \sin(\omega t(n+m))\right]dt$$

$$= \frac{1}{2}\left[-\frac{\cos(\omega t(n-m))}{\omega(n-m)} + \frac{\cos(\omega t(n+m))}{\omega(n+m)}\right]\Bigg|_{-\frac{T}{2}=-\frac{\pi}{\omega}}^{\frac{T}{2}=\frac{\pi}{\omega}}$$

$$= \frac{1}{2}\left[-\frac{1}{\omega(n-m)}\left(\cos(\pi(n-m)) - \cos(-\pi(n-m))\right) + \frac{1}{\omega(n+m)}\left(\cos(\pi(n+m)) - \cos(-\pi(n+m))\right)\right]$$

$$= \frac{1}{2}\left[\frac{-1}{\omega(n-m)}\left(\cos(\pi(n-m)) - \cos(\pi(n-m))\right) + \frac{1}{\omega(n+m)}\left(\cos(\pi(n+m)) - \cos(\pi(n+m))\right)\right] \quad \{\cos(\theta) = \cos(-\theta)\}$$

$$= \frac{1}{2}\left[\frac{-1}{\omega(n-m)}(0) + \frac{1}{\omega(n+m)}(0)\right]$$

$$= 0 \text{ when } m\neq n$$

$$(m=n)\quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t)\sin(m\omega t)\,dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t)\sin(n\omega t)\,dt$$

$$= \frac{\sin^2(n\omega t)}{n\omega}\Bigg|_{-\frac{T}{2}=-\frac{\pi}{\omega}}^{\frac{T}{2}=\frac{\pi}{\omega}}$$

$$= \frac{\sin^2(n\pi) - \sin^2(-n\pi)}{n\omega}$$

$$= \frac{0-0}{n\omega}$$

$$= 0 \text{ when } m=n$$

5.48

i) Prove

$$a_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\cos(n\omega t)\,dt \quad , n \geq 1$$

$$f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$$f(t)\cos(m\omega t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]\cos(m\omega t)$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\cos(m\omega t)\,dt = \sum_{n=0}^{\infty}\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} a_n\cos(n\omega t)\cos(m\omega t)\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} b_n\cos(m\omega t)\sin(n\omega t)\,dt\right]$$

$$= \sum_{n=1}^{\infty} a_n\left(\frac{\tau}{2}\right) \quad \{m = n \neq 0\}$$

$$= \frac{\tau}{2}\sum_{n=1}^{\infty} a_n$$

$$a_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\cos(n\omega t)\,dt, \quad n \geq 1$$

ii) Prove

$$b_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\sin(n\omega t)\,dt \quad , n \geq 1$$

$$f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$$f(t)\sin(m\omega t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]\sin(m\omega t)$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\sin(m\omega t)\,dt = \sum_{n=0}^{\infty}\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} a_n\cos(n\omega t)\sin(m\omega t)\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} b_n\sin(n\omega t)\sin(m\omega t)\,dt\right]$$

$$= \sum_{n=1}^{\infty} b_n\frac{\tau}{2} \quad \{m = n \neq 0\}$$

$$= \frac{\tau}{2}\sum_{n=1}^{\infty} b_n$$

$$b_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\sin(n\omega t)\,dt, \quad n \geq 1$$

Prove

$$a_0 = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt$$

$$f(t) = \sum_{n=0}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

$n=0$:

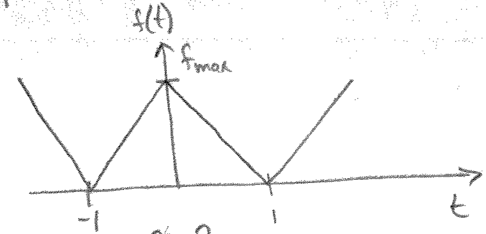
$$f(t) = a_0$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} a_0\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt$$

$$a_0\tau = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt$$

$$a_0 = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt$$

5.49



$\tau = 2$

$$f(t)_p = \begin{cases} f_{max} t + f_{max} & -1 < t < 0 \\ -f_{max} t + f_{max} & 0 < t < 1 \end{cases}$$

This is an even function $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\,dt$$

$$= \frac{1}{2} \left[\int_{-1}^{0} (f_{max} t + f_{max})\,dt + \int_{0}^{1} (-f_{max} t + f_{max})\,dt \right]$$

$$= \frac{1}{2} \left[f_{max} \left(\frac{t^2}{2} + t\right)\Big|_{-1}^{0} + f_{max}\left(-\frac{t^2}{2} + t\right)\Big|_{0}^{1} \right]$$

$$= \frac{f_{max}}{2} \left[0 - \left(\frac{1}{2} - 1\right) + -\frac{1}{2} + 1 - 0 \right]$$

$$= \frac{f_{max}}{2} \left[\frac{1}{2} + \frac{1}{2} \right]$$

$$a_0 = \frac{f_{max}}{2}$$

$$a_n = \frac{2}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \cos(n\omega t)\,dt$$

$$= \int_{-1}^{0} f_{max}(t+1) \cos(n\omega t)\,dt + \int_{0}^{1} f_{max}(-t+1) \cos(n\omega t)\,dt \quad \{\tau = 2\}$$

$$= \int_{-1}^{0} f_{max}\, t \cos(n\omega t)\,dt + \int_{-1}^{0} f_{max} \cos(n\omega t) - \int_{0}^{1} f_{max}\, t \cos(n\omega t) + \int_{0}^{1} f_{max} \cos(n\omega t)\,dt$$

$$\int_a^b t \cos(n\omega t)\,dt = \frac{t \sin(n\omega t)}{n\omega}\Big|_a^b - \int_a^b \frac{\sin(n\omega t)}{n\omega}\,dt$$

$u = t$

$du = dt$

$$= \frac{t \sin(n\omega t)}{n\omega}\Big|_a^b + \frac{\cos(n\omega t)}{(n\omega)^2}\Big|_a^b$$

$dv = \cos(n\omega t)$

$v = \dfrac{\sin(n\omega t)}{n\omega}$

$$\int_{-1}^{0} t\cos(n\omega t)\,dt = \frac{t\sin(n\pi t)}{n\pi}\Big|_{-1}^{0} + \frac{\cos(n\pi t)}{n^2\pi^2}\Big|_{-1}^{0}, \quad \omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi$$

$$= 0 + \frac{\sin(-n\pi)}{n\pi} + \frac{1}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}$$

$$= \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2}$$

$$\int_{0}^{1} t\cos(n\omega t)\,dt = \frac{t\sin(n\pi t)}{n\pi}\Big|_{0}^{1} + \frac{\cos(n\pi t)}{n^2\pi^2}\Big|_{0}^{1}$$

$$= \frac{\sin(n\pi)}{n\pi} - 0 + \frac{\cos(n\pi)}{n^2\pi^2} - \frac{1}{n^2\pi^2}$$

$$= \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}$$

$$\int_{-1}^{0} \cos(n\omega t)\,dt = \frac{\sin(n\omega t)}{n\omega}\Big|_{-1}^{0} = \frac{\sin(n\pi t)}{n\pi}\Big|_{0}^{-1} = 0$$

$$\int_{0}^{1} \cos(n\omega t)\,dt = \frac{\sin(n\omega t)}{n\omega}\Big|_{0}^{1} = \frac{\sin(n\pi t)}{n\pi}\Big|_{0}^{1} = 0$$

$$a_n = f_{max}\left[\frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2} - \left(\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right)\right]$$

$$= f_{max}\left[\frac{2}{n^2\pi^2} - \frac{2(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2f_{max}}{n^2\pi^2}\left(1 - (-1)^n\right)$$

$$= \frac{4f_{max}}{n^2\pi^2} \quad n = 1, 3, 5, \ldots$$

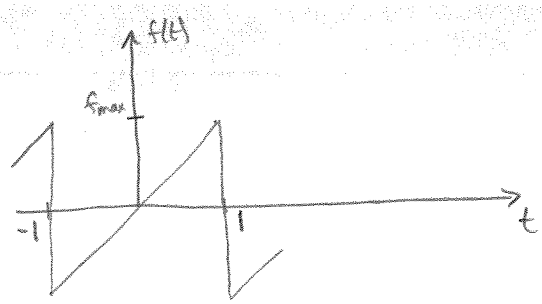
$$f(t) = a_0 + a_n\cos(n\pi t), \quad n = 1, 3, 5, \ldots$$

$$\boxed{f(t) = \frac{f_{max}}{2} + \frac{4f_{max}}{n^2\pi^2}\cos(n\pi t), \quad n = 1, 3, 5, \ldots}$$

for $f_{max} = 1$

$$\boxed{f(t) = \frac{1}{2} + \frac{4}{n^2\pi^2}\cos(n\pi t), \quad n = 1, 3, 5, \ldots}$$

5.50



$\tau = 2$

$f(t)_p = f_{max} t \quad , -1 < t < 1$

This is an odd funtion $\rightarrow a_n = 0$. $a_0 = 0$ {average value = 0}.

$$b_n = \frac{2}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\sin(n\omega t)\,dt$$

$$= \int_{-1}^{1} f_{max}\, t\sin(n\omega t)\,dt$$

$$\int_a^b t\sin(n\omega t)\,dt = -\frac{t\cos(n\omega t)}{n\omega}\Big|_a^b + \int_a^b \frac{\cos(n\omega t)}{n\omega}\,dt$$

$u = t$

$du = dt$

$dv = \sin(n\omega t)$

$v = -\frac{\cos(n\omega t)}{n\omega}$

$$= -\frac{t\cos(n\omega t)}{n\omega}\Big|_a^b + \frac{\sin(n\omega t)}{n^2\omega^2}\Big|_a^b$$

$$\int_{-1}^{1} t\sin(n\omega t)\,dt = -\frac{t\cos(n\pi t)}{n\pi}\Big|_{-1}^{1} + \frac{\sin(n\pi t)}{n^2\pi^2}\Big|_{-1}^{1} \quad , \omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi$$

$$= \frac{-\cos(n\pi) - \cos(-n\pi)}{n\pi} + \frac{\sin(n\pi) - \sin(-n\pi)}{n^2\pi^2}$$

$$= \frac{-2\cos(n\pi)}{n\pi}$$

$$= \frac{-2(-1)^n}{n\pi}$$

$$\boxed{f(t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\omega t) \quad \{f_{max} = 1\}}$$

5.51 $f = Re[g(t)]$, $g(t) = \sum_{n=0}^{\infty} f_n e^{in\omega t}$

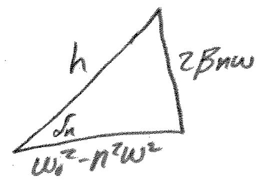
$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

$$\cos(n\omega t - \delta_n) = Re\left[e^{i(n\omega t - \delta_n)}\right]$$

$$e^{i(n\omega t - \delta_n)} = e^{-i\delta_n} e^{in\omega t}$$

$$e^{i\delta_n} = \cos\delta_n + i\sin\delta_n$$

$$\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$



$$h = \left[(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2\right]^{1/2}$$

$$e^{i\delta_n} = \frac{\omega_0^2 - n^2\omega^2 + 2i\beta n\omega}{\left[(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2\right]^{1/2}}$$

$$e^{-i\delta_n} = \frac{\left[(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2\right]^{1/2}}{\omega_0^2 - n^2\omega^2 + 2i\beta n\omega}$$

$$A_n = \frac{f_n}{\left[(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2\right]^{1/2}}$$

$$z(t) = \sum_{n=0}^{\infty} A_n e^{-i\delta_n} e^{in\omega t}$$

$$z(t) = \sum C_n e^{in\omega t}$$

$$C_n = A_n e^{-i\delta_n}$$

$$C_n = \frac{f_n}{\omega_0^2 - n^2\omega^2 + 2i\beta n\omega}$$

$$x = Re(z(t))$$

5.55

i) Prove

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)\,dt = \begin{cases} \tau, & m=n=0 \\ \frac{\tau}{2}, & m=n\neq 0 \\ 0, & m\neq n \end{cases}$$

$$\cos(A-B) = \cos A\cos B + \sin A \sin B$$

$$\cos(n\omega t - \delta_n) = \cos(n\omega t)\cos\delta_n + \sin(n\omega t)\sin\delta_n$$

$$\cos(m\omega t - \delta_m) = \cos(m\omega t)\cos\delta_m + \sin(m\omega t)\sin\delta_m$$

$$\cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m) = \left[\cos(n\omega t)\cos\delta_n + \sin(n\omega t)\sin\delta_n\right]\left[\cos(m\omega t)\cos\delta_m + \sin(m\omega t)\sin\delta_m\right]$$

$$= \cos(n\omega t)\cos(m\omega t)\cos\delta_n\cos\delta_m + \cos(n\omega t)\sin(m\omega t)\cos\delta_n\sin\delta_m + \cos(m\omega t)\sin(n\omega t)\cos\delta_m\sin\delta_n + \sin(n\omega t)\sin(m\omega t)\sin\delta_n\sin\delta_m$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m)\,dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\cos(m\omega t)\cos\delta_n\cos\delta_m\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t)\sin(m\omega t)\cos\delta_n\sin\delta_m\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(m\omega t)\sin(n\omega t)\cos\delta_m\sin\delta_n\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(n\omega t)\sin(m\omega t)\sin\delta_n\sin\delta_m\,dt$$

$$(m=n\neq 0) \rightarrow \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m)\,dt = \frac{\tau}{2}\cos\delta_n\cos\delta_m + \frac{\tau}{2}\sin\delta_n\sin\delta_m$$

$$= \frac{\tau}{2}\left[\cos\delta_n\cos\delta_m + \sin\delta_n\sin\delta_m\right]$$

$$= \frac{\tau}{2}\left[\cos\delta_n\cos\delta_n + \sin\delta_n\sin\delta_n\right]$$

$$= \frac{\tau}{2}\left[\cos^2\delta_n + \sin^2\delta_n\right]$$

$$= \frac{\tau}{2}$$

$$(m\neq n) \rightarrow \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(n\omega t-\delta_n)\cos(m\omega t-\delta_m)\,dt = 0+0+0+0$$

$$= 0$$

$$(m=n=0 \rightarrow \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(0)\cos(0)\,dt = t\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$
$$= \tau$$

ii) Prove
$$\langle x^2\rangle = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

$$\langle x^2\rangle = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x^2\,dt \quad , \quad x = \sum_{n=0}^{\infty} A_n\cos(n\omega t - \delta_n)$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left(\sum_{n=0}^{\infty} A_n\cos(n\omega t - \delta_n)\right)^2 dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_n\cos(n\omega t - \delta_n) A_m\cos(m\omega t - \delta_m)\,dt$$

Only m=n terms are considered.

$$\langle x^2\rangle = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sum_{n=m=0}^{\infty} A_n^2\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)\,dt$$

$$= \frac{1}{\tau}\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A_0^2\,dt + \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\sum_{n=m=1}^{\infty} A_n^2\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)\,dt\right]$$

$$= \frac{1}{\tau}\left[A_0^2\tau + \sum_{n=1}^{\infty} A_n^2\frac{\tau}{2}\right]$$

$$\langle x^2\rangle = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2$$

5.56

$$x(t) = \sum_{n=0}^{\infty}\left[A_n\cos(n\omega t - \delta_n) + B_n\sin(n\omega t - \delta_n)\right]$$

$$\langle x^2\rangle = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x^2\,dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\left[\sum_{n=0}^{\infty}\left[A_n\cos(n\omega t - \delta_n) + B_n\sin(n\omega t - \delta_n)\right]\right]^2 dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_n\cos(n\omega t - \delta_n) + B_n\sin(n\omega t - \delta_n)\right)\left(A_m\cos(m\omega t - \delta_m) + B_m\sin(m\omega t - \delta_m)\right)dt$$

$$\left[A_n\cos(n\omega t - \delta_n) + B_n\sin(n\omega t - \delta_n)\right]\left[A_m\cos(m\omega t - \delta_m) + B_m\sin(m\omega t - \delta_m)\right] =$$

$$A_nA_m\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m) + A_nB_m\cos(n\omega t - \delta_n)\sin(m\omega t - \delta_m) + A_mB_n\cos(m\omega t - \delta_m)\sin(n\omega t - \delta_n) +$$
$$B_nB_m\sin(n\omega t - \delta_n)\sin(m\omega t - \delta_m)$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)dt = \begin{cases}\tau, & m=n=0\\ \frac{\tau}{2}, & m=n\neq 0\\ 0, & m\neq n\end{cases} \quad \text{(Problem 5.56)}$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\cos(n\omega t - \delta_n)\sin(m\omega t - \delta_m)dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\left(\cos(n\omega t)\cos\delta_n + \sin(n\omega t)\sin\delta_n\right)\left(\sin(m\omega t)\cos\delta_m - \cos(m\omega t)\sin\delta_m\right)dt$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin(C-D) = \sin C\cos D - \cos D\sin C$$

$$\cos(A-B)\sin(C-D) = \cos A\cos B\sin C\cos D - \cos A\cos B\cos D\sin C +$$
$$\sin A\sin B\sin C\cos D - \sin A\sin B\cos D\sin C$$

$A = n\omega t$, $B = \delta_n$, $C = m\omega t$, $D = \delta_m$

$$= \cos(n\omega t)\cos\delta_n\sin(m\omega t)\cos\delta_m - \cos(n\omega t)\cos\delta_n\cos\delta_m\sin(m\omega t) +$$
$$\sin(n\omega t)\sin\delta_n\sin(m\omega t)\cos\delta_m - \sin(n\omega t)\sin\delta_n\cos\delta_m\sin(m\omega t)$$

$$= \cos(n\omega t)\sin(m\omega t)\cos\delta_n\cos\delta_m - \cos(n\omega t)\sin(m\omega t)\cos\delta_n\cos\delta_m +$$
$$\sin(n\omega t)\sin(m\omega t)\cos\delta_m\sin\delta_n - \sin(n\omega t)\sin(m\omega t)\cos\delta_m\sin\delta_n$$

$$= 0$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\cos(n\omega t - \delta_n)\sin(m\omega t - \delta_m)dt = 0 = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\cos(m\omega t - \delta_m)\sin(n\omega t - \delta_n)dt$$

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \sin(n\omega t - \delta_n)\sin(m\omega t - \delta_m)\,dt = \begin{cases} \tau, & m=n=0 \\ \frac{\tau}{2}, & m=n\neq 0 \\ 0, & m\neq n \end{cases}$$ (similarity to Problem 5.56)

m=n are only relavent

$$\langle x^2\rangle = \frac{1}{\tau}\left[A_0^2\tau + \sum_{n=1}^{\infty} A_n^2\frac{\tau}{2} + B_0^2\tau + \sum_{n=1}^{\infty} B_n^2\frac{\tau}{2}\right]$$

$$\boxed{\langle x^2\rangle = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 + B_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} B_n^2}$$