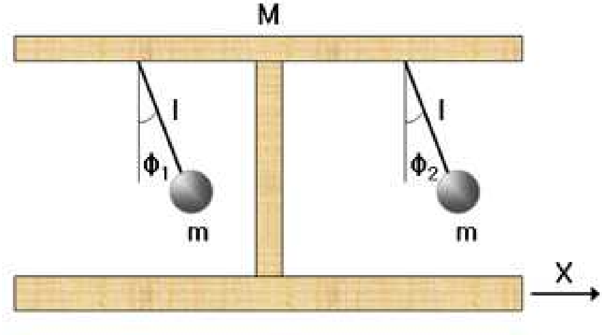
Chapter 12.
Coupled Oscillators and Normal Modes



In chapter 6, we studied the oscillations of a single body subject to a Hooke's law force. Now, we wish to examine the oscillations of several bodies, each of which is connected by springs to one or more of the others. An example of such a system is a molecule, which we can model as a system of masses (the atoms) connected by springs (the force between atoms, which takes a Hooke's law form around the atoms' equilibrium separation.) If each of the masses were attached to a separate, fixed spring, with no connections between the masses, then each would oscillate independently. However, the forces that connect the masses to each other turn this into a *coupled system*. We shall find that - unlike the case of a single oscillator which exhibits a single natural oscillation frequency - two or more coupled oscillators have a corresponding number of natural (or "normal") frequencies, and that the general motion is a combination of vibrations at all the different normal frequencies.

Table of Contents

Two Masses and Three Springs JRT §11.1

As a first example, consider the two carts shown in Fig. 1. The carts move without friction on a horizontal track, between two fixed walls. Each is attached to its adjacent wall by a spring (force constants k_1 and k_3), and to each other by a spring with force constant k_2. Clearly, in the absence of spring 2, the carts would oscillate independently of one another (assuming that they don't collide, of course.) However, the presence of the coupling spring ensures that - to at least a small extent - the motion of one cart affects the motion of the other.

In the following discussion, it is assumed that when the two carts are at their equilibrium positions the three springs are neither stretched nor compressed. However, depending on the distance between the walls, it could be that all three springs are compressed or all three are stretched. It is a relatively simple task to show that the results of this discussion are not affected by these possibilities.

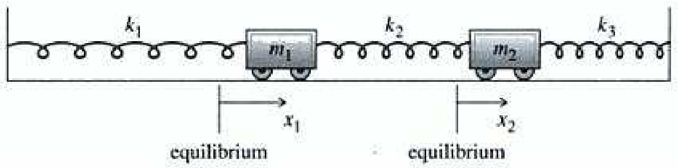


Fig. 1: *Two carts attached by springs to fixed walls and to each other - see JRT Fig. 11.1*

Two Masses and Three Springs *cont'* JRT §11.1

For this system, the equations of motion can be found using either Newton's second law or Lagrange's equations. The latter are easier to write down, but the former is a bit more instructive, so that is the path we will take.

Suppose the carts have moved distances x_1 and x_2, measured to the right, relative to their equilibrium positions. Spring 1 is stretched by an amount x_1, so it exerts a force k_1x_1 to the left on cart 1. Spring 2 is more complicated since it is affected by the position of both carts; it exerts a force $k_2(x_2 - x_1)$ to the right on cart 1. Of course, these forces can be negative if the situation warrants. The net force on cart 1 is

$$-k_1x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2x_2. \tag{12.1}$$

The net force on cart 2 is found in an identical fashion. By Newton's second law, the two equations of motion are

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2, \quad \text{and} \quad m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2. \tag{12.2}$$

Before we try to solve these coupled equations, we note that they can be written in a compact matrix form

$$\boxed{\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}.} \tag{12.3}$$

Here, we have introduced the vector and matrices

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}. \tag{12.4}$$

Two Masses and Three Springs *cont'* JRT §11.1

The "mass matrix" **M** for this simple case is a diagonal matrix, with the masses m_1 and m_2 down the diagonal. The "spring-constant matrix" **K** has nonzero off-diagonal elements, reflecting that the elements of **x** are coupled to each other. Notice that the matrix equation (12.3) has the form of Newton's second law, and indeed, if there was just one cart then **M**, **K**, and **x** would reduce to scalars. Notice as well that both **M** and **K** are symmetric, as will be true of all the corresponding matrices in this chapter.

To solve the equations of motion, we might reasonably guess that there could be solutions in which both carts oscillate sinusoidally with the same angular frequency ω; that is,

$$x_1(t) = \alpha_1 \cos(\omega t - \delta_1), \quad \text{and} \quad x_2(t) = \alpha_2 \cos(\omega t - \delta_2). \tag{12.5}$$

Furthermore, if such a solution exists, then there must also be a solution of the same form but with the cosines replaced by sines (this is just a shift in the time axis):

$$y_1(t) = \alpha_1 \sin(\omega t - \delta_1), \quad \text{and} \quad y_2(t) = \alpha_2 \sin(\omega t - \delta_2). \tag{12.6}$$

And then, we can combine these two solutions into a single complex solution,

$$z_1(t) = x_1(t) + iy_1(t) = \alpha_1 e^{i(\omega t - \delta_1)} = a_1 e^{i\omega t}, \tag{12.7}$$

where $a_1 = \alpha_1 e^{-i\delta_1}$, and likewise,

$$z_2(t) = x_2(t) + iy_2(t) = \alpha_2 e^{i(\omega t - \delta_2)} = a_2 e^{i\omega t}, \quad \text{where } a_2 = \alpha_2 e^{-i\delta_2}. \tag{12.8}$$

Two Masses and Three Springs *cont'* JRT §11.1

Next, we combine the two complex solutions into a single vector solution of the form

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \mathbf{a}e^{i\omega t} \tag{12.9}$$

where the vector **a** is a constant, made up of two complex numbers,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{bmatrix}. \tag{12.10}$$

In seeking solutions of the equation of motion, we must remember that when we find such solutions, the actual motion $\mathbf{x}(t)$ is equal to the *real part* of $\mathbf{z}(t)$.

When we substitute eq. (12.9) into eq.(12.3), we obtain the equation

$$-\omega^2 \mathbf{M}\mathbf{a}e^{i\omega t} = -\mathbf{K}\mathbf{a}e^{i\omega t}, \tag{12.11}$$

or, cancelling the common terms,

$$\boxed{(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = 0.} \tag{12.12}$$

This is known as a *generalized eigenvalue equation*.

Two Masses and Three Springs *cont'* JRT §11.1

If the matrix $(\mathbf{K} - \omega^2\mathbf{M})$ has a nonzero determinant, then the only solution to eq. (12.12) is the trivial solution $\mathbf{a} = 0$, which corresponds to no motion at all. On the other hand, if

$$\boxed{\det(\mathbf{K} - \omega^2\mathbf{M}) = 0,} \tag{12.13}$$

then there is a nontrivial solution of eq. (12.12) and hence a solution of the equations of motion with our assumed sinusoidal form. This is the solution that we seek.

In the present case, since **M** and **K** are 2-by-2 matrices, there will be two (possibly identical) solutions to eq. (12.13). This implies that there are two possible frequencies ω at which the carts can oscillate sinusoidally.

These frequencies are called the normal frequencies of the system. They depend on the values of the two masses and the three spring constants.

If we had N coupled masses, we would have an N-by-N eigenproblem to solve, which would provide us with N normal frequencies. In PC235, we will restrict ourselves to the case where $N = 2$ or 3. In many branches of physics (condensed matter physics and optics in particular), we often have N so large that it is essentially infinite! Thankfully, mathematicians have provided us with closed-form expressions for the properties of symmetric infinite matrices.

Identical Springs and Equal Masses JRT §11.2

We will continue to examine the two carts of Fig. 1, but now **we suppose that the two masses are equal**, $m_1 = m_2 = m$, and **that the three springs constants are also equal**, $k_1 = k_2 = k_3 = k$. In that case, the matrices defined in the previous section become

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}. \tag{12.14}$$

The matrix of the eigenvalue equation is then

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \tag{12.15}$$

and its determinant is

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = (2k - m\omega^2)^2 - k^2 = (k - m\omega^2)(3k - m\omega^2). \tag{12.16}$$

The two normal frequencies are determined by the condition that this determinant be zero, and are therefore

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{3k}{m}}. \tag{12.17}$$

These are the frequencies at which the two carts can oscillate in purely sinusoidal motion. The first one, ω_1, is precisely the frequency of a single mass m on a single spring k. We will soon see why this is the case.

Identical Springs and Equal Masses - The First Normal Mode

JRT §11.2

The preceding equation tells us the two possible frequencies of the system, but we have not yet described the corresponding motions. Recall that the actual motion is given by the vector $\mathbf{x}(t) = \mathbb{R}\mathrm{e}\{\mathbf{z}(t)\}$, where the complex vector $\mathbf{z}(t) = \mathbf{a}e^{i\omega t}$, and $\mathbf{a}$ is the vector

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{12.18}$$

which must satisfy the eigenvalue equation

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0. \tag{12.19}$$

Now that we know the possible normal frequencies, we must solve this equation for the vector $\mathbf{a}$ for each normal frequency in turn. The sinusoidal motion with any one of the normal frequencies is called a normal mode of the coupled oscillators.

If we choose ω equal to the first normal frequency, $\omega_1 = \sqrt{k/m}$, then the matrix $(\mathbf{K} - \omega^2\mathbf{M})$ becomes

$$(\mathbf{K} - \omega_1^2\mathbf{M}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}. \tag{12.20}$$

Identical Springs and Equal Masses - The First Normal Mode *cont'* JRT §11.2

For this case, the eigenvalue equation, eq. (12.19), reads

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0, \tag{12.21}$$

which is equivalent to the two equations

$$\begin{aligned} a_1 - a_2 &= 0 \\ -a_1 + a_2 &= 0. \end{aligned} \tag{12.22}$$

These equations imply that $a_1 = a_2 = Ae^{-i\delta}$. The complex vector **z**(t) is therefore

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)}, \tag{12.23}$$

and the corresponding actual motion is given by the vector $\mathbf{x}(t) = \mathbb{R}\mathrm{e}\,\{\mathbf{z}(t)\}$, or

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta). \tag{12.24}$$

That is,

$$x_1(t) = A\cos(\omega_1 t - \delta), \quad \text{and} \quad x_2(t) = A\cos(\omega_1 t - \delta). \tag{12.25}$$

Identical Springs and Equal Masses - The First Normal Mode *cont'* JRT §11.2

We see that **in the first normal mode the two carts oscillate in phase and with the same amplitude** A, as shown in Figs. 2-3. A striking feature of these figures is that, because $x_1(t) = x_2(t)$, **the middle spring is at no time either stretched or compressed**. This spring is in fact irrelevant, and each cart oscillates simply as if it were attached to the nearest wall by a single spring. **This explains why the first normal frequency is the same as for a single cart on a single spring**.

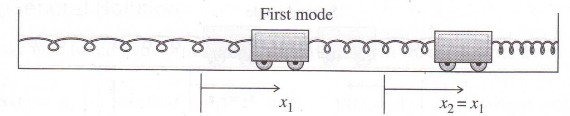


Fig. 2: *The first normal mode for two equal-mass carts with three identical springs - see JRT Fig. 11.2*

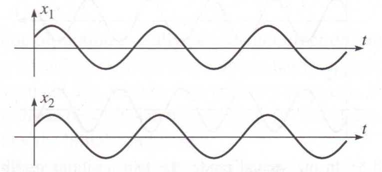


Fig. 3: *In the first normal mode, the two positions oscillate with equal amplitudes and in phase - see JRT Fig. 11.3*

Identical Springs and Equal Masses - The Second Normal Mode JRT §11.2

The second normal frequency at which our system can oscillate is $\omega_2 = \sqrt{3k/m}$, which, when substituted into eq. (12.15), gives

$$(\mathbf{K} - \omega_2^2\mathbf{M}) = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}. \tag{12.26}$$

For this case, the eigenvalue equation, eq. (12.19), reads

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0, \tag{12.27}$$

which implies that $a_1 = -a_2 = Ae^{-i\delta}$. That is,

$$x_1(t) = A\cos(\omega_2 t - \delta), \quad \text{and} \quad x_2(t) = -A\cos(\omega_2 t - \delta). \tag{12.28}$$

In this second normal mode, **the two carts oscillate with the same amplitude** A, but **exactly out of phase**, as shown in Figs. 4-5.

Note that in the second normal mode, when cart 1 is displaced to the right, cart 2 is displaced to the left by an equal distance. This means that when the outer two springs are stretched, the middle spring is compressed by twice as much. Thus, for example, when the left spring pulls cart 1 to the left, the middle spring is pushing cart 1, also to the left, with a force that is twice as large. This means that *each cart moves as if it were attached to a single spring with force constant* $3k$.

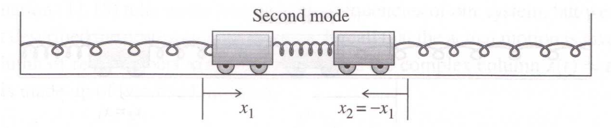


Fig. 4: *The second normal mode for two equal-mass carts with three identical springs - see JRT Fig. 11.4*

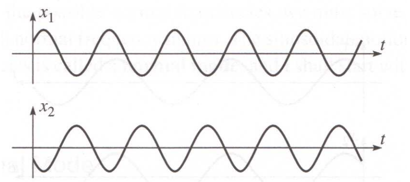


Fig. 5: *In the second normal mode, the two positions oscillate sinusoidally with equal amplitudes but exactly out of phase - see JRT Fig. 11.5*

Identical Springs and Equal Masses - The General Solution JRT §11.2

We have now found two normal-mode solutions, which we can rewrite as

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \mathbf{x}(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2), \tag{12.29}$$

where ω_1 and ω_2 are the normal frequencies. Both of these solutions satisfy the equation of motion for any values of the four real constants A_1, A_2, δ_1, and δ_2. Because the equation of motion is linear and homogeneous, the sum of these two solutions is also a solution:

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2). \tag{12.30}$$

Because the equation of motion is really two second-order differential equations for the two variables $x_1(t)$ and $x_2(t)$, its general solution has four constants of integration. Thus, eq. (12.30), with its four arbitrary constants, is the general solution.

When we allow A_1, A_2, δ_1, and δ_2 to be completely arbitrary, the general solution can be difficult to visualize. The motion of each cart is a mixture of the two frequencies ω_1 and ω_2. Since $\omega_2 = \sqrt{3}\omega_1$, the motion never repeats itself except in the special case that one of the constants A_1 or A_2 is zero. Figure 6 shows graphs of the two positions in a typical non-normal mode.

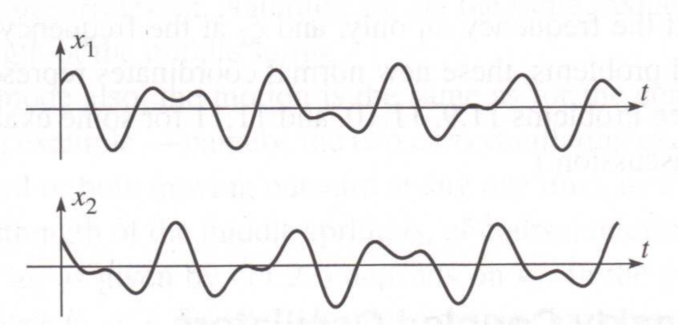


Fig. 6: *General solution for* $x_1(t)$ *and* $x_2(t)$*, for* $A_1 = 1, A_2 = 0.7, \delta_1 = 0, \delta_2 = \pi/2$ *- see JRT Fig. 11.6*

Identical Springs & Equal Masses - Normal Coordinates JRT §11.2

We have just seen that in any possible motion of the two-cart system, both of the coordinates $x_1(t)$ and $x_2(t)$ vary with time. In the normal modes, the time dependence is sinusoidal and coupled; one cart cannot move without the other.

It is also possible to introduce an alternative set of normal coordinates, which have the convenient property that **each can vary independently of the other**. This statement is true for any system of coupled oscillators, but is especially easy to see in the present case of two equal masses joined by three identical springs. For this case, the normal coordinates are

$$\xi_1 = \frac{1}{2}(x_1 + x_2), \quad \text{and} \quad \xi_2 = \frac{1}{2}(x_1 - x_2). \tag{12.31}$$

It is easy to show that for the first normal mode, the new variables are given by

$$\xi_1(t) = A\cos(\omega_1 t - \delta), \quad \text{and} \quad \xi_2(t) = 0, \tag{12.32}$$

whereas for the second normal mode, we have

$$\xi_1(t) = 0, \quad \text{and} \quad \xi_2(t) = A\cos(\omega_2 t - \delta). \tag{12.33}$$

In this sense, the new coordinates are *independent*; oscillation of one does not result in oscillation of the other. Of course, the *general* motion of the system is a superposition of both modes, and thus ξ_1 and ξ_2 both oscillate, but ξ_1 oscillates with frequency ω_1 only, while ξ_2 oscillates with frequency ω_2 only. In problems that are much more complicated than the simple system of this section, the use of normal coordinates represents a considerable simplification.

Two Weakly Coupled Oscillators JRT §11.3

In the last section we discussed the oscillations of two equal masses joined by three equal springs. For this system, the two normal modes were easy to understand and to visualize, but the non-normal oscillations were somewhat complicated (Fig. 6). A system where some of the non-normal oscillations are readily visualized is a pair of oscillators which have the same natural frequency and which are weakly coupled. For example, the system in Fig. 7 consists of two identical carts, attached to their respective walls by identical springs (force constant k) and to each other by a much weaker spring (force constant $k_2 \ll k$).

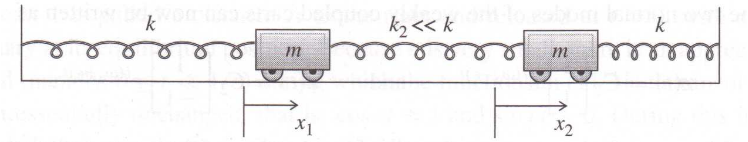


Fig. 7: *Two weakly coupled carts. The middle spring which couples the two carts is much weaker than the outer two springs - see JRT Fig. 11.7*

Two Weakly Coupled Oscillators *cont'* JRT §11.3

We can solve for the normal modes of this system in exactly the same fashion as before. The spring matrix **K** takes the form

$$\mathbf{K} = \begin{bmatrix} k + k_2 & -k_2 \\ -k_2 & k + k_2 \end{bmatrix}, \tag{12.34}$$

and thus

$$(\mathbf{K} - \omega^2 \mathbf{M}) = \begin{bmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{bmatrix}. \tag{12.35}$$

The determinant is $(k - m\omega^2)(k + 2k_2 - m\omega^2)$, and we conclude that the two normal frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k + 2k_2}{m}}. \tag{12.36}$$

The first frequency is the same as in the previous example, and for the same reason (the carts move in unison, so the middle spring is irrelevant.) As for the second mode, the carts oscillate exactly out of phase; this will be proven shortly. However, in this mode, the strength of the middle spring becomes relevant since it does experience tension and compression.

Two Weakly Coupled Oscillators *cont'* JRT §11.3

In the present case, ω_2 is very close to ω_1, since $k_2 \ll k$. To take advantage of this closeness, we will define ω_0 to be the average of the two normal frequencies,

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}. \tag{12.37}$$

Since ω_1 and ω_2 are very close to each other, ω_0 is very close to either, and for most purposes, we can think of ω_0 as being the same as $\omega_1 = \sqrt{k/m}$. To show the difference between ω_1 and ω_2, we can write

$$\omega_1 = \omega_0 - \epsilon \quad \text{and} \quad \omega_2 = \omega_0 + \epsilon. \tag{12.38}$$

That is, the small number ϵ is half the difference between the two normal frequencies.

The two normal modes of the weakly coupled carts can now be written as

$$\mathbf{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)t} \quad \text{and} \quad \mathbf{z}(t) = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \epsilon)t}, \tag{12.39}$$

and, as before, their sum is also a solution,

$$\mathbf{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \epsilon)t}. \tag{12.40}$$

The constants C_1 and C_2 are determined by the initial conditions - the positions and velocities of the two carts at $t = 0$.

Two Weakly Coupled Oscillators *cont'* JRT §11.3

To see some general features of the solution, we can factor out a term $e^{i\omega_0 t}$, leaving

$$\mathbf{z}(t) = \left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right\} e^{i\omega_0 t}. \tag{12.41}$$

The term in braces, $\{\cdots\}$, is a vector which depends on t. But since ϵ is very small, this column varies very slowly compared to the second factor $e^{i\omega_0 t}$. Over a reasonably short time interval, the first factor is essentially constant, and the solution behaves like $\mathbf{z}(t) = \mathbf{a}e^{i\omega_0 t}$, with $\mathbf{a}$ constant. That is, over any short time interval, the two carts oscillate sinusoidally with angular frequency ω_0. However, if we wait long enough, the details of the motion will change.

Let us now examine the behaviour of eq. (12.41) for some simple values of the constants C_1 and C_2. First, if either of the constants is zero, then the solution reverts to one of the normal modes. A more interesting case is that C_1 and C_2 are equal in magnitude. For instance, let $C_1 = C_2 = A/2$. In this case,

$$\mathbf{z}(t) = \frac{A}{2} \begin{bmatrix} e^{-i\epsilon t} + e^{i\epsilon t} \\ e^{-i\epsilon t} - e^{i\epsilon t} \end{bmatrix} e^{i\omega_0 t} = A \begin{bmatrix} \cos \epsilon t \\ -i \sin \epsilon t \end{bmatrix} e^{i\omega_0 t}. \tag{12.42}$$

Two Weakly Coupled Oscillators *cont'* JRT §11.3

Of course, the *actual* motion of the carts is given by the real part of $\mathbf{z}(t)$, resulting in

$$x_1(t) = A\cos\epsilon t\cos\omega_0 t, \quad \text{and} \quad x_2(t) = A\sin\epsilon t\sin\omega_0 t. \tag{12.43}$$

This solution has a simple and elegant interpretation. At time $t = 0$, we have $x_1 = A$, whereas $x_2 = \dot{x}_1 = \dot{x}_2 = 0$. That is, the solution describes the motion when cart 1 is pulled a distance A to the right and released at $t = 0$. Because ϵ is very small, there is an appreciable interval ($0 \leq t \ll 1/\epsilon$) during which the functions in eq. (12.43) that involve ϵt remain essentially unchanged. During this interval, we have

$$x_1(t) \approx A\cos\omega_0 t, \quad \text{and} \quad x_2(t) \approx 0. \tag{12.44}$$

That is, initially cart 1 oscillates with amplitude A and frequency ω_0, while cart 2 remains stationary.

Eventually, of course, the factor ϵt starts to become appreciable, and cart 2 starts to oscillate, also at frequency ω_0. The terms involving $\sin\epsilon t$ grow, while the terms involving $\cos\omega t$ shrink toward zero. Eventually, when $t = \pi/2\epsilon$, the motion is

$$x_1(t) \approx 0, \quad \text{and} \quad x_2(t) \approx A\sin\omega_0 t. \tag{12.45}$$

The complete motion is shown in Fig. 8.

Those of you in photonics will return to the concept of weakly coupled oscillators when you study such devices as directional couplers and gratings.

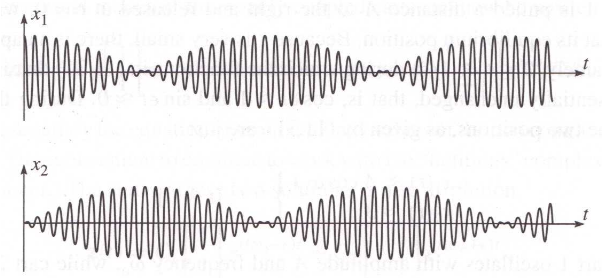


Fig. 8: *The positions $x_1(t)$ and $x_2(t)$ of two weakly coupled oscillators - see JRT Fig. 11.8*

Lagrangian Approach: The Double Pendulum JRT §11.4

The derivations in the previous sections were based on Newton's second law. For such relatively simple systems, there is no real advantage to using the Lagrangian approach. However, for more complicated systems, the Lagrangian approach is considerably simpler.

Here, we will re-derive the equations for the system of two masses and three springs using the Lagrangian. We shall then do the same for another simple system with two degrees of freedom, the double pendulum.

Consider the two carts as in Fig. 1. The kinetic energy (assuming massless springs) is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2. \tag{12.46}$$

To write down the potential energy, we must identify the extensions of the three springs as $x_1, x_2 - x_1$, and $-x_2$, from which it follows that the potential energy is

$$\begin{aligned} U &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2 \\ &= \frac{1}{2}(k_1 + k_2)x_1^2 - k_2x_1x_2 + \frac{1}{2}(k_2 + k_3)x_2^2. \end{aligned} \tag{12.47}$$

Lagrangian Approach: The Double Pendulum *cont'* JRT §11.4

These results immediately give us the Lagrangian $\mathcal{L} = T - U$, from which we find the Lagrange equations of motion for the two variables x_1 and x_2:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \quad \text{or} \quad m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2 \tag{12.48}$$

and

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \quad \text{or} \quad m_2\ddot{x}_2 = k_2 x_1 - (k_2 + k_3)x_2. \tag{12.49}$$

These are precisely the two equation of motion that we found by the Newtonian method, which we rewrote in the form $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$.

Lagrangian Approach: The Double Pendulum *cont'* JRT §11.4

Now, consider a *double pendulum*, comprising a mass m_1 suspended by a massless rod of length L_1 from a fixed pivot, and a second mass m_2 suspended by a massless rod of length L_2 from m_1, as shown in Fig. 9. We choose as our generalized coordinates the two angles ϕ_1 and ϕ_2.

It is a straightforward matter to write down the Lagrangian $\mathcal{L}$ as a function of the two generalized coordinates. When the angle ϕ_1 increases from 0, the mass m_1 rises by an amount $L_1(1 - \cos\phi_1)$ and gains a potential energy

$$U_1 = m_1 g L_1 (1 - \cos\phi_1). \tag{12.50}$$

Similarly, as ϕ_2 increases from 0, the second mass rises by $L_2(1 - \cos\phi_2)$ but, in addition, its point of support (m_1) has risen by $L_1(1 - \cos\phi_1)$. Thus,

$$U_2 = m_2 g \Big[L_1(1 - \cos\phi_1) + L_2(1 - \cos\phi_2) \Big]; \tag{12.51}$$

that is, U_1 depends only on ϕ_1, while U_2 depends on both ϕ_1 and ϕ_2. The potential energy is therefore

$$U(\phi_1, \phi_2) = (m_1 + m_2) g L_1 (1 - \cos\phi_1) + m_2 g L_2 (1 - \cos\phi_2). \tag{12.52}$$

Lagrangian Approach: The Double Pendulum *cont'* JRT §11.4

The velocity of m_1 is just $L_1\dot{\phi}_1$ in the tangential direction (remember, for circular motion, $v = r\omega$), as shown in Fig. 9, so its kinetic energy is

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2. \tag{12.53}$$

The velocity of m_2 is the vector sum of two velocities - the velocity $L_2\dot{\phi}_2$ of m_2 relative to its support m_1 plus the velocity $L_1\dot{\phi}_1$ of its support. The angle between these two velocities is $(\phi_2 - \phi_1)$, so the kinetic energy of m_2 is

$$T_2 = \frac{1}{2} m_2 \left[L_1^2 \dot{\phi}_1^2 + 2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)\right] + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2, \tag{12.54}$$

and the total KE is

$$T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2. \tag{12.55}$$

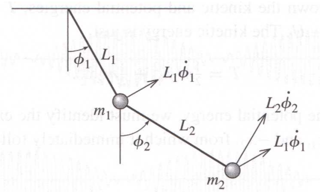


Fig. 9: *A double pendulum. The velocity of m_2 is the vector sum of the two velocities shown, separated by an angle $\phi_2 - \phi_1$ - see JRT Fig. 11.9*

Lagrangian Approach: The Double Pendulum *cont'* JRT §11.4

At this point, we could write down the Lagrangian $T - U$ and then find the two Lagrange equations for ϕ_1 and ϕ_2. However, the resulting equations are complicated and not terribly illuminating (and they can't be solved analytically.) This situation is reminiscent of the simple pendulum, for which the equation of motion ($L\ddot{\phi} = -g\sin\phi$) can be solved analytically only by resorting to "special functions". As in that case, we can make a small-angle approximation for the double pendulum. We assume that ϕ_1 and ϕ_2 are small, as are their derivatives $\dot{\phi}_1$ and $\dot{\phi}_2$. Then, we Taylor-expand the expressions for T and U and drop all terms that are of third power or higher in these four small quantities. This gives

$$T = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 \tag{12.56}$$

and

$$U = \frac{1}{2}(m_1 + m_2)gL_1\phi_1^2 + \frac{1}{2}m_2gL_2\phi_2^2. \tag{12.57}$$

Now, the Lagrangian $\mathcal{L} = T - U$ gives us the two Lagrange's equations,

$$\frac{\partial\mathcal{L}}{\partial\phi_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}, \quad \text{or} \quad (m_1 + m_2)L_1^2\ddot{\phi}_1 + m_2L_1L_2\ddot{\phi}_2 = -(m_1 + m_2)gL_1\phi_1 \tag{12.58}$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2}, \quad \text{or} \quad m_2L_1L_2\ddot{\phi}_1 + m_2L_2^2\ddot{\phi}_2 = -m_2gL_2\phi_2. \tag{12.59}$$

Lagrangian Approach: The Double Pendulum *cont'* JRT §11.4

These two equations for ϕ_1 and ϕ_2 can be rewritten as a single matrix equation

$$\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi}, \tag{12.60}$$

where we have introduced the following vector and matrices:

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} (m_1 + m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}. \tag{12.61}$$

The matrix equation is completely analogous to the case for the two carts on springs. In the present case, the "mass" matrix **M** is not actually made up of masses, but it still plays the role of inertia (note that the elements of **M** have the same units as the moment of inertia **I**). Similarly, the "spring-constant" matrix **K** is not actually made up of spring constants; its elements have units of energy.

Nevertheless, the problem is solved by the same methods. We seek to find solutions - normal modes - in which the two coordinates ϕ_1 and ϕ_2 vary sinusoidally with the same angular frequency ω. As before, any such solution $\boldsymbol{\phi}(t)$ can be written as the real part of a complex solution $\mathbf{z}(t)$ whose time dependence is just $e^{i\omega t}$. That is,

$$\boldsymbol{\phi}(t) = \mathbb{R}\mathrm{e}\{\mathbf{z}(t)\} \quad \text{where} \quad \mathbf{z}(t) = \mathbf{a}e^{i\omega t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}. \tag{12.62}$$

Lagrangian Approach: The Double Pendulum - Equal Lengths and Masses JRT §11.4

To simplify the discussion, let us now restrict our attention to the case that our double pendulum has equal masses, $m_1 = m_2 = m$, and equal lengths, $L_1 = L_2 = L$. The equations become much tidier if we recognize that $\sqrt{g/L}$ is the frequency of a single pendulum of the same length L. If we call this frequency ω_0, then we can replace g with $L\omega_0^2$ and the matrices **M** and **K** become

$$\mathbf{M} = mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = mL^2 \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}. \tag{12.63}$$

The matrix $(\mathbf{K} - \omega^2\mathbf{M})$ of the eigenvalue equation is therefore

$$(\mathbf{K} - \omega^2\mathbf{M}) = mL^2 \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{bmatrix}. \tag{12.64}$$

The normal frequencies are determined by setting the determinant of the above matrix to zero, which gives

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0, \tag{12.65}$$

with the two solutions $\omega^2 = (2 \pm \sqrt{2})\omega_0^2$. That is, the two normal frequencies are

$$\omega_1 = \sqrt{2 - \sqrt{2}}\omega_0 \approx 0.77\omega_0 \quad \text{and} \quad \omega_2 = \sqrt{2 + \sqrt{2}}\omega_0 \approx 1.85\omega_0. \tag{12.66}$$

Lagrangian Approach: The Double Pendulum - Equal Lengths and Masses *cont'* JRT §11.4

Knowing the two normal frequencies, we can now find the motion of the double pendulum in the corresponding normal modes, by solving the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ with $\omega = \omega_1$ and $\omega = \omega_2$ in turn. If we use ω_1, we get

$$(\mathbf{K} - \omega_1^2\mathbf{M}) = mL^2\omega_0^2(\sqrt{2} - 1)\begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}. \tag{12.67}$$

Therefore, the equation $(\mathbf{K} - \omega_1^2\mathbf{M})\mathbf{a} = 0$ implies that $a_2 = \sqrt{2}a_1$, and if we write $a_1 = A_1 e^{-i\delta_1}$, the two coordinates are

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \mathbb{R}\mathrm{e}\left\{\mathbf{a}e^{i\omega_1 t}\right\} = A_1\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\cos(\omega_1 t - \delta_1). \tag{12.68}$$

In this first normal mode, the two pendulums oscillate exactly in phase, with the amplitude of the lower pendulum $\sqrt{2}$ times that of the upper pendulum, as shown in Fig. 10.

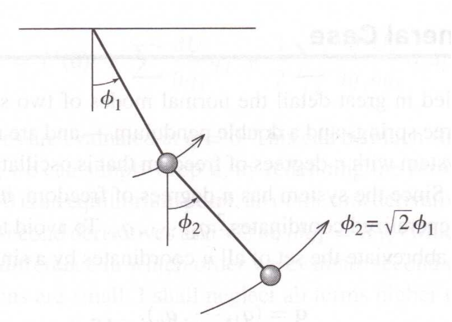


Fig. 10: *The first normal mode for a double pendulum with equal masses and equal lengths. The two angles ϕ_1 and ϕ_2 oscillate in phase, with the amplitude for ω_2 larger by a factor of $\sqrt{2}$ - see JRT Fig. 11.10*

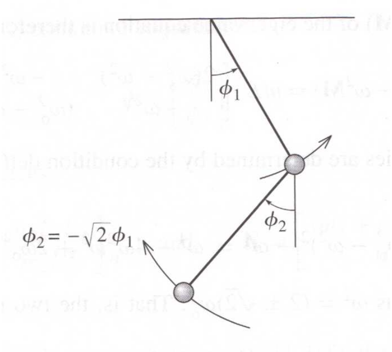


Fig. 11: *The second normal mode for a double pendulum with equal masses and equal lengths. The two angles ϕ_1 and ϕ_2 oscillate exactly out of phase, with the amplitude for ω_2 larger by a factor of $\sqrt{2}$ - see JRT Fig. 11.11*

Lagrangian Approach: The Double Pendulum - Equal Lengths and Masses *cont'* JRT §11.4

Turning to the second mode, we find that

$$(\mathbf{K}-\omega_2^2\mathbf{M}) = -mL^2\omega_0^2(\sqrt{2}+1)\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}. \tag{12.69}$$

The equation $(\mathbf{K}-\omega_2^2\mathbf{M})\mathbf{a} = 0$ implies that $a_2 = -\sqrt{2}a_1$, and if we write $a_1 = A_2 e^{-i\delta_2}$, the two coordinates are

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \mathbb{R}\mathrm{e}\left\{\mathbf{a}e^{i\omega_2 t}\right\} = A_2\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\cos(\omega_2 t - \delta_2). \tag{12.70}$$

In this second normal mode, the two pendulums oscillate exactly out of phase, with the amplitude of the lower pendulum $\sqrt{2}$ times that of the upper pendulum, as shown in Fig. 11.

The general solution, of course, is a linear combination of these normal modes, with relative amplitudes and phases that are determined by the initial conditions.

Example #1 - Cart and Pendulum JRT Prob. 11.19

Problem: A simple pendulum (mass M and length L is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in the figure.

(a) **Assuming that the angle ϕ remains small, write down the system's Lagrangian and the equations of motion for x and ϕ.**

(b) **Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units), find the normal frequencies, and for each normal frequency find and describe the motion of the corresponding normal mode.**

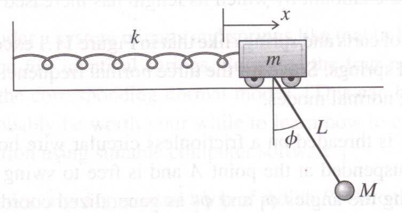


Fig. 12: *Geometry of the cart and pendulum problem*

Example #1 - Cart and Pendulum *cont'* JRT Prob. 11.19

Solution:

(a) Using the equilibrium position ($x = \phi = 0$) as the reference point for potential energy, we have

$$U = U_{\text{spr}} + U_{\text{pen}} = \frac{1}{2}kx^2 + MgL(1 - \cos\phi) \approx \frac{1}{2}kx^2 + \frac{1}{2}MgL\phi^2, \tag{12.71}$$

where the small-angle approximation has been used. The kinetic energy has two components due to the two moving masses. That of the cart is simply $T = \frac{1}{2}m\dot{x}^2$. As for the pendulum, the velocity of the bob is a bit trickier to describe - it's equal to the velocity of the bob relative to the pivot plus the velocity of the pivot relative to the origin. Therefore,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left(\dot{x} + L\dot{\phi}\right)^2 = \frac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \frac{1}{2}ML^2\dot{\phi}^2. \tag{12.72}$$

All together,

$$\mathcal{L} = T - U = \frac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \frac{1}{2}ML^2\dot{\phi}^2 - \frac{1}{2}kx^2 - \frac{1}{2}MgL\phi^2. \tag{12.73}$$

The Lagrange equations read

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \longrightarrow -kx = (m+M)\ddot{x} + ML\ddot{\phi} \\ \frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \longrightarrow -MgL\phi = ML\ddot{x} + ML^2\ddot{\phi}. \end{aligned} \tag{12.74}$$

(b) We can combine these equations in the form $\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}$, where

$$\mathbf{q} = \begin{bmatrix} x \\ \phi \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & MgL \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{12.75}$$

(note that our choice of numerical values has made this problem mathematically equivalent to that of the double pendulum with equal lengths and equal masses).

Example #1 - Cart and Pendulum *cont'* JRT Prob. 11.19

Continuing, we have

$$\mathbf{K} - \omega^2 \mathbf{M} = \begin{bmatrix} 2 - 2\omega^2 & \omega^2 \\ -\omega^2 & 1 - \omega^2 \end{bmatrix}. \tag{12.76}$$

The requirement that the determinant of this matrix equals zero leads to the characteristic equation $\omega^4 - 4\omega^2 + 2 = 0$, with resulting normal frequencies

$$\omega_1 = \sqrt{2 - \sqrt{2}} = 0.77, \quad \omega_2 = \sqrt{2 + \sqrt{2}} = 1.85 \tag{12.77}$$

(with proper units implied). As for the normal modes, as for the double pendulum problem, we have

$$\omega_1 \longrightarrow a_2 = \sqrt{2} a_1, \quad \omega_2 \longrightarrow a_2 = -\sqrt{2} a_1, \tag{12.78}$$

In the first mode, ϕ and x are in phase, with ϕ (in radians) having $\sqrt{2}$ times the magnitude of x (in meters). In the second mode, they are out of phase with the same ratio of magnitudes.

The General Case JRT §11.5

We have now studied in detail the normal modes of two systems - a pair of carts attached to three springs and a double pendulum. Now, we are ready to discuss the general case of a system with n degrees of freedom that is oscillating about a point of stable equilibrium. Since the system has n degrees of freedom, its configuration can be specified by n generalized coordinates, $q_1, \cdots, q_n$. We abbreviate this set as $\mathbf{q} = (q_1, \cdots, q_n)$. For example, for the two carts that we studied earlier, $\mathbf{q} = (x_1, x_2)$ and for the double pendulum, $\mathbf{q} = (\phi_1, \phi_2)$. Note carefully that $\mathbf{q}$ is not, in general, a three-dimensional vector; it is a vector in the n-dimensional space of the generalized coordinates.

We shall assume that the system is conservative, so that it has a potential energy

$$U(q_1, \cdots, q_n) = U(\mathbf{q}) \tag{12.79}$$

and Lagrangian $\mathcal{L} = T - U$. The kinetic energy, as usual, is

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2, \tag{12.80}$$

where the sum runs over all of the particles that make up the system.

The General Case *cont'* JRT §11.5

We must rewrite the KE in terms of the generalized coordinates, using the relation between the Cartesian coordinates $\mathbf{r}_\alpha$ and the generalized coordinates,

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \cdots, q_n), \tag{12.81}$$

where we take for granted that this relation does not involve the time t explicitly (that is, the coordinates are "natural.") We saw back in our initial study of Hamiltonian mechanics that if we differentiate eq. (12.81) with respect to t and substitute into the kinetic energy, we find that

$$T = T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{j,k} A_{jk}(\mathbf{q})\dot{q}_j\dot{q}_k \tag{12.82}$$

where the coefficients $A_{jk}(\mathbf{q})$ may depend only on the coordinates $\mathbf{q}$. Under our present assumptions, the Lagrangian has the general form $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q})$, where $T(\mathbf{q}, \dot{\mathbf{q}})$ is given by the previous equation and $U(\mathbf{q})$ is an as-yet-unspecified function of the coordinates $\mathbf{q}$.

The General Case *cont'* JRT §11.5

As a final assumption, we presume that the system is making small oscillations about a stable equilibrium configuration. By redefining the coordinates if necessary, we can arrange that the equilibrium position is $\mathbf{q} = 0$. Then, since we are interested only in small oscillations, we have only to concern ourselves with small values of the coordinates $\mathbf{q}$, and we can use Taylor expansions of T and U about the equilibrium point $\mathbf{q} = 0$. For U, this gives

$$U(\mathbf{q}) = U(0) + \sum_j \frac{\partial U}{\partial q_j} q_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \cdots \tag{12.83}$$

where all derivatives are evaluated at $\mathbf{q} = 0$. Luckily, this equation can be simplified. Since $U(0)$ is a constant, we can simply redefine the zero-level of energy so that this term equals zero. Next, since $\mathbf{q} = 0$ is an equilibrium point, all of the first derivatives $\partial U / \partial q_j$ are zero.

As a further simplification, we shall rename the second derivatives as $\partial^2 U / \partial q_j \partial q_k = K_{jk}$, where $K_{jk} = K_{kj}$ by inspection. And finally, since the oscillations are small, we shall neglect all terms higher than second order in the small quantities $\mathbf{q}$ or $\dot{\mathbf{q}}$. This reduces U to

$$U = U(\mathbf{q}) = \frac{1}{2} \sum_{j,k} K_{jk} q_j q_k . \tag{12.84}$$

The General Case *cont'* JRT §11.5

The kinetic energy is a bit simpler to rearrange. Every term in eq. (12.82) contains a factor $\dot{q}_j\dot{q}_k$ which is already second order in small quantities. Therefore, we can ignore everything but the constant term in the expansion of $A_{jk}(\mathbf{q})$. If we call this constant term $A_{jk}(0) = M_{jk}$, this reduces the KE to

$$T = T(\dot{\mathbf{q}}) = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k, \tag{12.85}$$

and the Lagrangian to

$$\mathcal{L}(\mathbf{q},\dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) - U(\mathbf{q}). \tag{12.86}$$

Now we can easily write down the equations of motion. Since there are n generalized coordinates, there are n corresponding Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} \qquad [i = 1,\cdots,n]. \tag{12.87}$$

To write these equations explicitly, we need to differentiate our expressions for T and U. Differentiating sums can be a bit tricky. For example, for a system with $n = 2$, the equation for U reads

$$\begin{aligned} U = \frac{1}{2}\sum_{j,k} K_{jk}q_jq_k &= \frac{1}{2}\left(K_{11}q_1^2 + K_{12}q_1q_2 + K_{21}q_2q_1 + K_{22}q_2^2\right) \\ &= \frac{1}{2}\left(K_{11}q_1^2 + 2K_{12}q_1q_2 + K_{22}q_2^2\right). \end{aligned} \tag{12.88}$$

The General Case *cont'* JRT §11.5

In this form we can easily differentiate with respect to either q_1 or q_2. For example,

$$\frac{\partial U}{\partial q_1} = K_{11}q_1 + K_{12}q_2 \tag{12.89}$$

with a corresponding expression for $\partial U/\partial q_2$. For systems with more than 2 degrees of freedom, we have a similar expression,

$$\frac{\partial U}{\partial q_i} = \sum_j K_{ij}q_j \quad [i = 1, \cdots, n]\,. \tag{12.90}$$

Since differentiation of the kinetic energy works in exactly the same way, we can write down the n Lagrange equations

$$\sum_j M_{ij}\ddot{q}_j = -\sum_j K_{ij}q_j \quad [i = 1, \cdots, n]\,. \tag{12.91}$$

These n equations can be grouped into a single matrix equation

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}} \tag{12.92}$$

where **M** and **K** are the n-by-n "mass" and "spring constant" matrices with elements M_{ij} and K_{ij} respectively.

Note that we only need to find the elements of the matrices **M** and **K** to solve the problem at hand; it is not necessary to write down the Lagrangian or the Lagrange equations.

The General Case *cont'* JRT §11.5

The preceding matrix equation is just the n-dimensional equivalent of the two-dimensional equations that we derived earlier for the two carts and for the double pendulum. It is solved in exactly the same way; the eigenvalues determine the normal frequencies, and the eigenvectors determine the normal modes. Of course, finding these values for an n-by-n matrix is often no easy task.

Three Coupled Pendulums JRT §11.6

Consider three identical pendulums coupled by two identical springs as shown in Fig. 13. We use the generalized coordinates ϕ_1, ϕ_2, and ϕ_3, with the system at equilibrium when all three angles are zero. Our first task is to write down the Lagrangian for small displacements.

From the results of the previous sections the KE of the three pendulums is seen to be

$$T = \frac{1}{2}mL^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2). \tag{12.93}$$

The gravitational potential energy of each pendulum has the form $mgL(1 - \cos\phi) \approx \frac{1}{2}mgL\phi^2$. Thus, the total gravitational potential energy is

$$U_{\text{grav}} = \frac{1}{2}mgL(\phi_1^2 + \phi_2^2 + \phi_3^2). \tag{12.94}$$

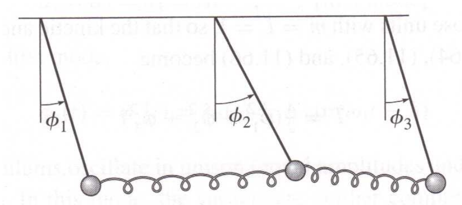


Fig. 13: *Three identical pendulums of lengths L and masses m are coupled by two identical springs with spring constants k - see JRT Fig. 11.13*

Three Coupled Pendulums *cont'* JRT §11.6

We also need to account for the potential energy due to the springs. This is a function of how much each is stretched or compressed. For arbitrary values of the angles ϕ, this is rather messy. However, for small angles, the only appreciable stretching and compression comes from the horizontal displacement of the masses, each of which moves a distance of approximately $L\phi$ (to the right, for positive ϕ.) Thus, for example, the left spring is stretched by about $L(\phi_2 - \phi_1)$. The total spring potential energy is

$$\begin{aligned} U_{\text{spr}} &= \frac{1}{2}kL^2\left[(\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2\right] \qquad (12.95) \\ &= \frac{1}{2}kL^2\left(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3\right). \end{aligned}$$

At this point, the textbook introduces a new notation system called "natural units" in which certain parameters (in this case, mass *m* and length *L*) are given the dimensionless value of unity. While there are good reasons for this approach, I disagree with its use at this level, as it obscures some of the physics. Thus, the remainder of this section of notes will differ slightly from the way it is presented in the textbook.

Three Coupled Pendulums *cont'* JRT §11.6

At this point, we could write down the Lagrangian and then the Lagrange equations. But we've already done this. We already know that the result will be the familiar matrix equation

$$\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi} \tag{12.96}$$

where in this case, $\boldsymbol{\phi}$ is the 3-element vector containing the three angles ϕ_1, ϕ_2, and ϕ_3, and the matrices are

$$\mathbf{M} = mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} mgL + kL^2 & -kL^2 & 0 \\ -kL^2 & mgL + 2kL^2 & -kL^2 \\ 0 & -kL^2 & mgL + kL^2 \end{bmatrix}. \tag{12.97}$$

The normal modes of our system have the familiar form $\boldsymbol{\phi}(t) = \mathbb{R}\mathrm{e}\,\{\mathbf{z}(t)\} = \mathbb{R}\mathrm{e}\left\{\mathbf{a}e^{i\omega t}\right\}$, where $\mathbf{a}$ and ω are determined by the eigenvalue equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. Written out in full (and dividing through by L^2,) the matrix is

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} \frac{mg}{L} + k - m\omega^2 & -k & 0 \\ -k & \frac{mg}{L} + 2k - m\omega^2 & -k \\ 0 & -k & \frac{mg}{L} + k - m\omega^2 \end{bmatrix}. \tag{12.98}$$

The determinant is (somewhat) easily evaluated as

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = \left(\frac{g}{L} - \omega^2\right)\left(\frac{g}{L} + \frac{k}{m} - \omega^2\right)\left(\frac{g}{L} + \frac{3k}{m} - \omega^2\right). \tag{12.99}$$

Three Coupled Pendulums *cont'* JRT §11.6

Thus, the three normal frequencies are

$$\omega_1 = \sqrt{\frac{g}{L}}, \quad \omega_2 = \sqrt{\frac{g}{L} + \frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{g}{L} + \frac{3k}{m}}. \tag{12.100}$$

For the first mode, $a_1 = a_2 = a_3 = Ae^{-i\delta}$. That is,

$$\phi_1(t) = \phi_2(t) = \phi_3(t) = A\cos(\omega_1 t - \delta). \tag{12.101}$$

In this case, the three pendulums oscillate in unison (with equal amplitudes and phases,) as shown in Fig. 14(a). In this mode, the springs are neither stretched nor compressed, and their presence is irrelevant; each pendulum oscillates just like a single pendulum with frequency $\omega_1 = \sqrt{g/L}$.

Similarly, the second mode gives us $a_1 = -a_3 = Ae^{-i\delta}, a_2 = 0$. In this case, the outer two pendulums oscillate exactly out of phase, while the middle one sits at rest, as shown in Fig. 14(b). Finally, the third mode gives us $a_1 = -\frac{1}{2}a_2 = a_3 = Ae^{-i\delta}$. In this mode, the outer two pendulums oscillate in unison while the middle one oscillates with twice the amplitude and exactly out of phase, as shown in Fig. 14(c).

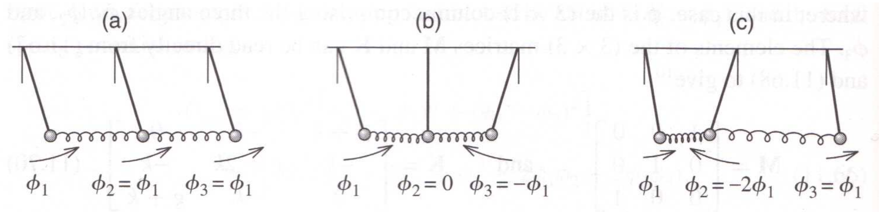


Fig. 14: *The three normal modes for three coupled pendulums (equal masses, equal lengths, equal spring constants - see JRT Fig. 11.14*

Example #2 - Linear Triatomic Molecule

Problem: A simplistic model of a *linear triatomic molecule* such as carbon dioxide is shown in the figure below. For this molecule, two identical atoms of mass m are connected by two identical springs (spring constant k) to a single atom of mass M. We assume that the system is confined to move in one dimension (that is, the three atoms are always colinear). The equilibrium state of the molecule occurs when both springs are at their natural length.

(a) **Write down the kinetic energy, the potential energy, and the Lagrangian for this system, using the displacement of each atom from its equilibrium position as the three generalized coordinates.**

(b) **Write down the Lagrange equations for the three generalized coordinates, and then express these equations as a single matrix equation.**

(c) **Solve for the normal frequencies of the system, as a function of $m, M,$ and k.**

(d) **Find and describe the motion in each of the three normal modes.**

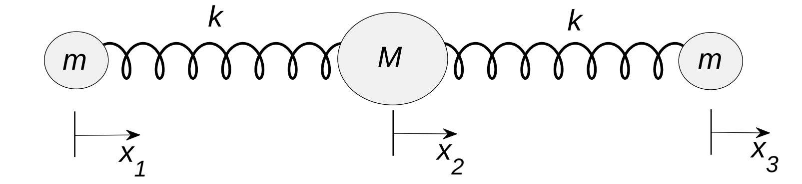


Fig. 15: *Geometry of the linear triatomic molecule*

Example #2 - Linear Triatomic Molecule *cont'*

Solution: We will introduce the shorthand notation $\lambda = M/m$.

(a) The total KE is

$$\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 = \frac{1}{2}m(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2). \tag{12.102}$$

The total PE is

$$\frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 = \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^3 - 2x_1x_2 - 2x_2x_3). \tag{12.103}$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^3 - 2x_1x_2 - 2x_2x_3). \tag{12.104}$$

(b) The Lagrange equations read

$$\frac{\partial\mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_1} \implies m\ddot{x}_1 = -kx_1 + kx_2 \tag{12.105}$$

$$\frac{\partial\mathcal{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_2} \implies m\lambda\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 \tag{12.106}$$

$$\frac{\partial\mathcal{L}}{\partial x_3} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_3} \implies m\ddot{x}_3 = kx_2 - kx_3. \tag{12.107}$$

This can be expressed as the matrix equation $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, where

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & \lambda m & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}. \tag{12.108}$$

Example #2 - Linear Triatomic Molecule *cont'*

(c) To find the normal frequencies, we must set $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$, where

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - \lambda m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix}. \qquad (12.109)$$

This determinant is a bit easier to deal with if we divide through by m and recall that $k/m = \omega_0^2$:

$$\frac{1}{m}\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \begin{bmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \lambda\omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{bmatrix}. \qquad (12.110)$$

The determinant is $\omega^2(\omega^2 - \omega_0^2)\left(\omega^2 - \left(\frac{2+\lambda}{\lambda}\right)\omega_0^2\right)$, and therefore the normal frequencies are

$$\omega_1 = 0, \quad \omega_2 = \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{2+\lambda}{\lambda}}\omega_0 = \sqrt{\frac{k}{m} + \frac{2k}{M}}. \qquad (12.111)$$

(d) For the case of ω_1, the zero frequency just means that all three masses have identical motion; that is, the molecule is drifting through space. This type of motion is possible because there are no forces anchoring the molecule to a particular position in space (in contrast, the other systems we have studied, such as two carts with three springs and the double pendulum, always have at least one fixed reference point...motion away from these points produces a restoring force.) Since neither spring expands or contracts in this mode, there is zero oscillation frequency.
For the case of ω_2, substituting $\omega = \sqrt{k/m}$ into $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = \mathbf{0}$ results in the solution $a_2 = 0, a_3 = -a_1$. In this mode, the center mass M is stationary, while the two outer masses oscillate with equal amplitudes and out of phase.

For the case of ω_3, substituting $\omega = \sqrt{\frac{k}{m} + \frac{2k}{M}}$ into $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = \mathbf{0}$ results in the solution $a_1 = a_3 = -(\lambda/2)a_2$. In this mode, the two outer masses move with equal amplitude and in phase, while the middle mass moves out of phase with the outer masses, and with an amplitude that is $\lambda/2$ as large.