MSO3120 Coursework 1

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October 2021

Submission deadline: **Friday** 19^{th} **November at 23:59**.

Learning outcomes:

Knowledge 3	Demonstrate an enhanced knowledge of analysis, specifically relating to lines and surfaces in $\mathbb{R}^3$.
Skills 1	Apply your cumulative learning to solve unfamiliar problems using familiar techniques from analysis.
Skills 3	Formulate abstract problems in analysis in order to solve them.

Marking

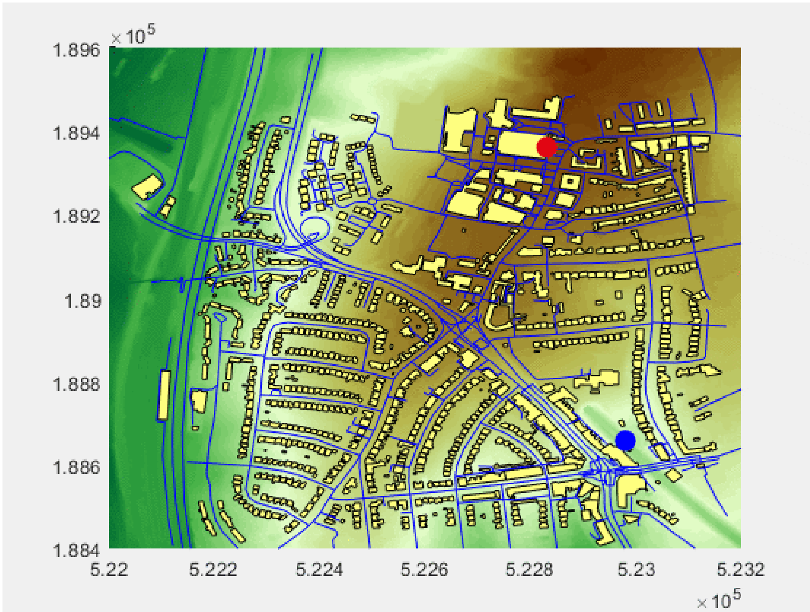
The **Assessment Criteria** can be found on page 15 of the Module Handbook.

Answers to questions	40 marks
Presentation	5 marks
Total	**45 marks**

In this coursework we will look at an applied problem in multivariable calculus. First, an introduction:

Ordnance Survey: Mapping Great Britain

We will be looking at the topography (the altitude of the terrain) near Middlesex University.



The above image shows a map of the area near Middlesex University with roads, building and landmarks highlighted.

- Middlesex University College Building main entrance.
- Hendon Central tube station.

The coordinate system is in **meters**, and has origin $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ corresponding to a point in the sea off the coast of Cornwall. The origin can be thought of as the most South-West point of Great Britain.

We see, for example, that the College Building main entrance $= \left(\begin{smallmatrix}522831\\189361\end{smallmatrix}\right)$ is about 520 km East and 190 km North of the origin.

This co-ordinate system was developed by Ordnance Survey: the national mapping agency of Great Britain.[1]

[1] Choosing good coordinate systems is **hard** for deep mathematical reasons. This would make a great third year project and would be very applicable in the real world as spatial data comes in a staggering variety of formats.

LIDAR: Measuring altitude

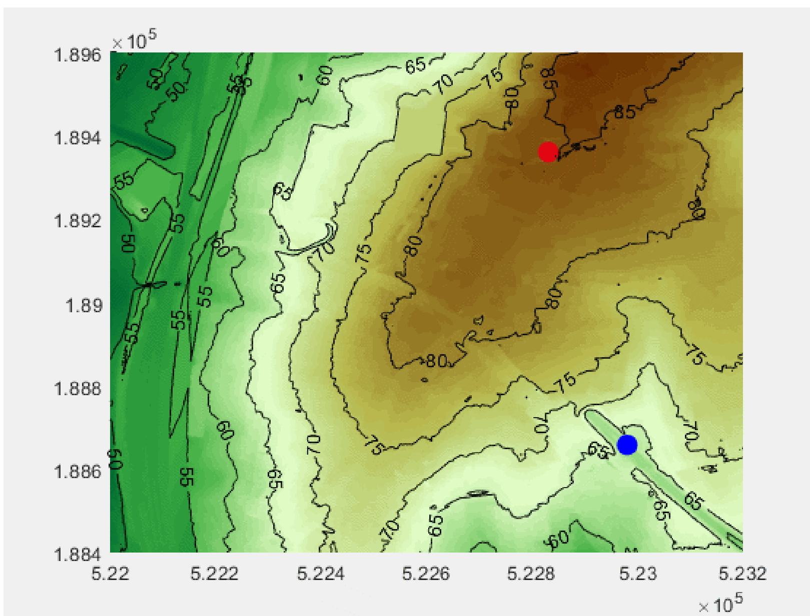
We're interested in the altitude of the terrain, which is indicated by the colours on the above map - green is low altitude and brown is high altitude.

This data didn't come from Ordnance Survey but from DEFRA - the government Department for Environment Farming and Rural Affairs. Data can be found here.

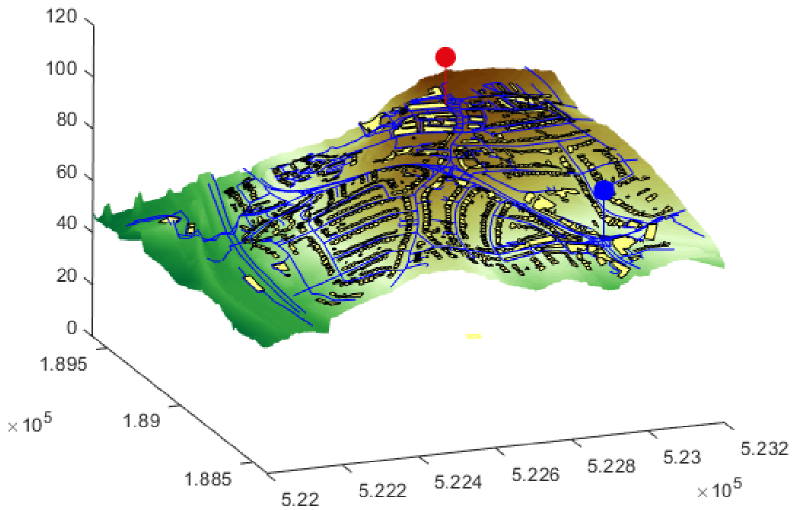
These altitudes are measured by shining lasers from planes and precisely measuring the time it takes to see a reflection. This technique is called LIDAR (LIght Detection and Ranging) - like RADAR but with LIght instead of RAdio.

This is important work that will inform future flood defence plans. Previous measurements found the altitude for every square meter on the map. Currently, a large amount of money is being spent to improve the resolution of the data and get the altitude for every square 0.25 meters. Consequently, the data from DEFRA is incomplete. Here we use the 1 meter resolution data.

We can also look at the altitude using contours (i.e. the level sets)



or by projecting into 3D - see also the animated version here



Combining these two datasets was a bit tricky because of the two different co-ordinate systems in use. The LIDAR data uses Latitude and Longitude, which have to be converted to Ordnance Survey co-ordinates before use. You can look at my MatLab code here.

Polynomial approximation

We want to apply the tools and techniques of multivariable calculus to this altitude data. But this data is just a big array of numbers (a height for each square meter), not a function that we can do calculus with.

So, we must first fit a function to this data. I've chosen to fit a quartic (degree 4) polynomial and have asked Matlab to find the quartic polynomial that best fits this data centred on the College Building co-ordinates $\binom{522831}{189361}$ (again, you can check out the Matlab code here.

Matlab returns the polynomial

$$
\begin{aligned}
&83.46 && \text{constant}\\
&+0.008222(x-522831) - 0.003587(y-189361) && \text{linear}\\
&-7.749\times10^{-5}(x-522831)^2 + 7.779\times10^{-5}(x-522831)(y-189361) - 2.777\times10^{-5}(y-189361)^2 && \text{quadratic}\\
&\left.\begin{aligned}&+4.624\times10^{-8}(x-522831)^3 - 3.901\times10^{-8}(x-522831)^2(y-189361)\\ &+4.444\times10^{-8}(x-522831)(y-189361)^2 + 2.094\times10^{-8}(y-189361)^3\end{aligned}\right\} && \text{cubic}\\
&\left.\begin{aligned}&+1.161\times10^{-10}(x-522831)^4 - 1.808\times10^{-10}(x-522831)^3(y-189361)\\ &+1.078\times10^{-10}(x-522831)^2(y-189361)^2\\ &-3.287\times10^{-12}(x-522831)(y-189361)^3 + 1.928\times10^{-11}(y-189361)^4\end{aligned}\right\} && \text{quartic}
\end{aligned}
$$

which gives the altitude in meters of the terrain at position $\binom{x}{y}$ in the Ordnance Survey co-ordinates.

This is a bit of a pain, as there are lots of factors of $(x-522831)$ and $(y-189361)$. More problematic is the use of some very small numbers, such as 10^{-10}, multiplied by very large numbers, such as 522831^3. We will lose precision if we use these values in software such as Geogebra.

We will fix both of these issues:

- First, we will translate the co-ordinate system so the origin is now at the College Building co-ordinates.
- Next, we will rescale the Ordnance Survey co-ordinates so that $\binom{x}{y}$ is measured in **kilometers** from the origin, rather than meters.

This gives the friendlier polynomial

Our polynomial model of altitude

$$\begin{aligned} f(x,y) = {} & 83.46 + 8.222x - 3.587y - 77.49x^2 + 77.79xy - 27.77y^2 \\ & + 46.24x^3 - 39.01x^2y + 44.44xy^2 + 20.94y^3 \\ & + 116.1x^4 - 180.8x^3y + 107.8x^2y^2 - 3.287xy^3 + 19.28y^4 \end{aligned}$$

Defined on the region

$$\begin{aligned} -0.831 \leq x \leq 0.369 \\ -0.961 \leq y \leq 0.239 \end{aligned}$$

We can write this is a format that's less easy to read but easier to copy and paste:

f(x,y) =
83.46 + 8.222*x -3.587*y -77.49*x^2 + 77.79*x*y -27.77*y^2 + 46.24*x^3 -39.01*x^2*y + 44.44* x*y^2 + 20.94*y^3 + 116.1*x^4 -180.8*x^3*y + 107.8*x^2*y^2 -3.287*x*y^3 + 19.28 y^4

In this new coordinate system let's specify our points of interest:

●	College Building main entrance.	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
●	Hendon Central tube station.	$\begin{pmatrix} 0.149 \\ -0.703 \end{pmatrix}$
	the point **p**	$\begin{pmatrix} -0.2 \\ -0.7 \end{pmatrix}$

Assessment questions

In the questions below we will assume that the polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ accurately describes the altitudes in the region around Middlesex University.

1. What is the altitude $0.1\ \mathrm{km}$ South and $0.1\ \mathrm{km}$ East of the College Building main entrance ? 1 mark

2. Plot the polynomial f in Geogebra (perhaps using the Augmented Reality feature) and include an image of the surface from an interesting angle. 2 marks

   > Pro-tip:
   >
   > You should limit your plotting region to $-0.831 \leq x \leq 0.369$ and $-0.961 \leq y \leq 0.239$ to cover the geographic region we have data for. The polynomial is a bad approximation outside of this range and grows rapidly which will distort your plot.

3. Write down the derivative of f at the point $\mathbf{p}$. 2 mark

4. Consider the cubic term

   $$g(x, y) = 44.44xy^2$$

   From first principles (i.e. working directly from Definition 3.3 and without using other theorems) prove that g is differentiable at all points $\binom{x}{y}$. 5 marks

5. In fact **all** multivariable polynomials are differentiable at every point in the domain. Sketch an argument to show this. Either

   - write a formal argument,
   - draw an illustration,
   - record an audio explanation or
   - record a video explanation.

   Imagine you're trying to convince a mathematician of this fact.

   > Pro-tip:
   >
   > You may recall last year we demonstrated all single variable polynomials $f : \mathbb{R} \to \mathbb{R}$ were differentiable.

   3 marks

6. We are using a polynomial to approximate the altitude of terrain near Middlesex University. In general would you expect the true altitude of terrain to be differentiable? If not, provide a mathematical description of a non-differentiable geographic feature. Either

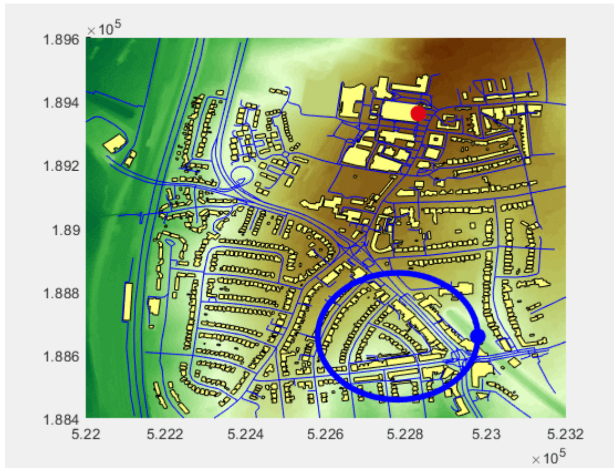
   - write an explanation,
   - draw an illustration,
   - record an audio explanation or
   - record a video explanation

   to justify your answer. 4 marks

7. Suppose you set off from the point $\mathbf{p}$ in a straight line towards the College Building main entrance.
   What is the rate-of-change of altitude in this direction? 3 marks

8. Suppose you are at point **p** and want to **reduce** your altitude as quickly as possible.
   What direction should you travel in? 2 marks

9. A mountain-biker sets off due North from Hendon Central. Her route projected on the xy plane is an anti-clockwise circle of radius $200\ \mathrm{m}$ which she traverses at a constant speed of $2\ \mathrm{m/s}$.



The cyclist's route.

Write down a function that describes the mountain-biker's projection onto the xy plane as a function of time. 4 marks

10. Hence find the rate-of-change of altitude of the mountain-biker as a function of time. It isn't necessary to simplify your answer but you should write down an explicit function. Either

    - write a formal argument,
    - draw an illustration,
    - record an audio explanation or
    - record a video explanation

    to justify your answer.

    Pro-tip:

    Don't find the altitude of the mountain-biker as a function of time!

    3 marks

11. It's a windy day in Hendon! The wind at point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is described by the vector field

$$\mathbf{w}(x, y, z) = \begin{pmatrix} -zy\left(x^2 + y^2\right) \\ zx\left(x^2 + y^2\right) \\ 0 \end{pmatrix}$$

Sketch this vector field in the $z = 90$ horizontal plane. Either

- sketch on paper and include a photo in your work
- sketch on your iPad
- use appropriate graphing software.

The purpose of this sketch is to get a justified idea of the shape of the wind flow. You can plot some vectors and use the definition of **w** to provide reasons for the vector field you sketch. To explain your reasons either

- write a formal argument,
- draw an illustration,
- record an audio explanation or
- record a video explanation.

3 marks

12. A student sets off South from the College Building main entrance at a speed of $1\ \mathrm{m/s}$. Find the derivative of the wind for this student as a function of time. Write your answer in terms of f and its partial derivatives. 4 marks

> Pro-tip:
>
> It may be helpful to find the derivative of
>
> $$\mathbf{h}(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}$$

13. It suddenly begins to rain in Hendon! To escape the rain students use a map of the area to decide whether to walk in a straight line to Hendon Central tube station or walk in a straight line to the College Building main entrance. They will choose to walk to whichever destination is closer according to the map.

    As the student are using a two-dimensional map, these decisions and distance measurements only use the x and y coordinates.

    Let $d\colon \mathbb{R}^2 \to \mathbb{R}$ be the distance a student starting at $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ has to walk to reach their destination (which will be either Hendon Central tube station $\begin{pmatrix} 0.149 \\ -0.703 \end{pmatrix}$ or the College Building main entrance $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$).

    By evaluating directional derivatives at an appropriate point, show that d is **not** differentiable everywhere. 4 marks

> Pro-tip:
>
> To find the distance they need to walk the students will calculate the distance from $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the distance from $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} 0.149 \\ -0.703 \end{pmatrix}$. They will then take the minimum of both of these distances. Can you write this as a function?

