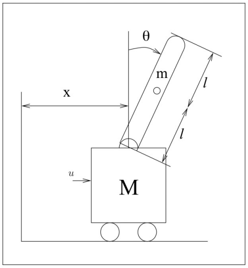
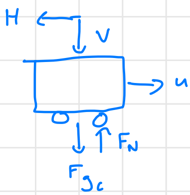
3. Write the equations of motion of the system below.

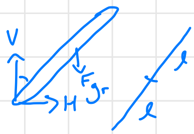
- Suppose the center of gravity of the pendulum rod is at its geometric center.
- Neglect the moment of inertia of the wheels.



FBD - Cart



FBD - Rod



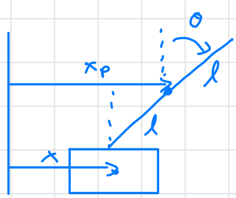
Motion of cart

$$\Sigma F_x = M\ddot{x} = u - H$$

Rotation of Rod

$$\Sigma M = I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta$$

Translation of Rod



1) Horizontal

$$x_p = x + l\sin\theta$$

$$\Sigma F_x = m\ddot{x}_p$$

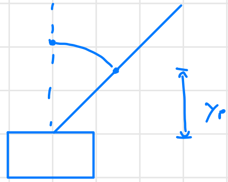
$$= \frac{d^2}{dt^2}(x + l\sin\theta)$$

$$= m\left[\ddot{x} + l\left(-\sin\theta\cdot\dot{\theta}^2 + \cos\theta\cdot\ddot{\theta}\right)\right]$$

$$= m\ddot{x} - ml\sin\theta\cdot\dot{\theta}^2 + ml\cos\theta\cdot\ddot{\theta}$$

$$\Rightarrow H = m\ddot{x} - ml\sin\theta\cdot\dot{\theta}^2 + ml\cos\theta\cdot\ddot{\theta} \quad (1)$$

Translation of Rod cont'd



2) Vertical Component

$$\Sigma F_y = m\ddot{y}_p = m\frac{d^2}{dt^2}(l\cos\theta) = ml(-\cos\theta\cdot\dot{\theta}^2 - \sin\theta\cdot\ddot{\theta})$$

$$\Rightarrow V = -ml\cos\theta\cdot\dot{\theta}^2 - m\sin\theta\cdot\ddot{\theta} + mg \quad (2)$$

Recall: $\underline{M\ddot{x} = u - H}$, $\underline{I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta}$

Sub in (1) and (2)

$$\Rightarrow \boxed{M\ddot{x} = u - m\ddot{x} + ml\sin\theta\cdot\dot{\theta}^2 - ml\cos\theta\cdot\ddot{\theta}}$$

$$\Rightarrow I\ddot{\theta} = (-ml\cos\theta\cdot\dot{\theta}^2 - m\sin\theta\cdot\ddot{\theta} + mg)\,l\sin\theta$$

$$- (m\ddot{x} - ml\sin\theta\cdot\dot{\theta}^2 + ml\cos\theta\cdot\ddot{\theta})\,l\cos\theta$$

$$I\ddot{\theta} = \cancel{-ml^2\cos\theta\sin\theta\,\dot{\theta}^2}\ \cancel{-ml\sin^2\theta\cdot\ddot{\theta}} + mgl\sin\theta$$

$$- m\ddot{x}l\cos\theta + \cancel{ml^2\sin\theta\cos\theta\,\dot{\theta}^2} - ml^2\cos^2\theta\,\ddot{\theta}$$

$$-ml^2\ddot{\theta}(1-\sin^2\theta)$$

$$-ml^2\ddot{\theta} - \cancel{ml^2\ddot{\theta}\sin^2\theta}$$

$$\boxed{I\ddot{\theta} = mgl\sin\theta - m\ddot{x}l\cos\theta - ml^2\ddot{\theta}}$$

$$\boxed{\begin{aligned} M\ddot{x} &= u - m\ddot{x} + ml\sin\theta\cdot\dot{\theta}^2 - ml\cos\theta\cdot\ddot{\theta} \\ I\ddot{\theta} &= mgl\sin\theta - m\ddot{x}l\cos\theta - ml^2\ddot{\theta} \end{aligned}}$$

$$M\ddot{x} = u - m\ddot{x} + ml\sin\theta\cdot\dot{\theta}^2 - ml\cos\theta\cdot\ddot{\theta}$$
$$I\ddot{\theta} = mgl\sin\theta - m\ddot{x}\,l\cos\theta - ml^2\ddot{\theta}$$

$$\Leftrightarrow \quad (M+m)\ddot{x} - ml\sin\theta\cdot\dot{\theta}^2 + ml\cos\theta\cdot\ddot{\theta} = u \qquad (1)$$
$$(I+ml^2)\ddot{\theta} - mgl\sin\theta + ml\cos\theta\cdot\ddot{x} = 0 \qquad (2)$$

$$\Rightarrow \quad \ddot{x} = \frac{1}{M+m}u + \frac{ml\sin\theta}{M+m}\dot{\theta}^2 - \frac{ml\cos\theta}{M+m}\ddot{\theta} \qquad (3)$$
$$\ddot{\theta} = \frac{mgl\sin\theta}{(I+ml^2)} - \frac{ml\cos\theta}{(I+ml^2)}\ddot{x} \qquad (4)$$

(3) → (2)

$$(I+ml^2)\ddot{\theta} = mgl\sin\theta - ml\cos\theta\left(\frac{1}{M+m}u + \frac{ml\sin\theta}{M+m}\dot{\theta}^2 - \frac{ml\cos\theta}{M+m}\ddot{\theta}\right)$$
$$(I+ml^2)\ddot{\theta} = mgl\sin\theta - \frac{ml\cos\theta}{M+m}u - \frac{m^2l^2\sin\theta\cos\theta}{M+m}\dot{\theta}^2 + \frac{m^2l^2\cos^2\theta}{M+m}\ddot{\theta}$$

$$I + ml^2 - \frac{m^2l^2\cos^2\theta}{(M+m)}$$

$$\frac{(I+ml^2)(M+m) - m^2l^2\cos^2\theta}{(M+m)}$$

$$\frac{\overbrace{\{IM + Im + mMl^2 + m^2l^2 - m^2l^2\cos^2\theta\}}^{a}}{(M+m)}$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{m^2l^2\sin\theta\cos\theta}{a}\dot{\theta}^2 + \frac{mgl\sin\theta\,(M+m)}{a} - \frac{ml\cos\theta}{a}u$$

$$M\ddot{x} = u - m\ddot{x} + ml\sin\theta\cdot\dot{\theta}^2 - ml\cos\theta\cdot\ddot{\theta}$$
$$I\ddot{\theta} = mgl\sin\theta - m\ddot{x}\,l\cos\theta - ml^2\ddot{\theta}$$

$$\Leftrightarrow \quad (M+m)\ddot{x} - ml\sin\theta\cdot\dot{\theta}^2 + ml\cos\theta\cdot\ddot{\theta} = u \quad (1)$$
$$(I+ml^2)\ddot{\theta} - mgl\sin\theta + ml\cos\theta\cdot\ddot{x} = 0 \quad (2)$$

$$\ddot{x} = \frac{1}{M+m}u + \frac{ml\sin\theta}{M+m}\dot{\theta}^2 - \frac{ml\cos\theta}{M+m}\ddot{\theta} \quad (3)$$

$$\ddot{\theta} = \frac{mgl\sin\theta}{(I+ml^2)} - \frac{ml\cos\theta}{(I+ml^2)}\ddot{x} \quad (4)$$

(4) → (1)

$$(M+m)\ddot{x} - ml\sin\theta\,\dot{\theta}^2 + ml\cos\theta\left(\frac{mgl\sin\theta}{I+ml^2} - \frac{ml\cos\theta}{I+ml^2}\ddot{x}\right) = u$$

$$\Bigg[\underbrace{\frac{(M+m)(I+ml^2) - m^2l^2\cos^2\theta}{I+ml^2}}_{a}\Bigg]\ddot{x} - ml\sin\theta\,\dot{\theta}^2 + \frac{m^2l^2g\sin\theta}{I+ml^2} = u$$

$$\Rightarrow \quad \ddot{x} = \frac{(ml\sin\theta)(I+ml^2)}{a}\dot{\theta}^2 - \frac{m^2l^2g\sin\theta}{a} + \frac{I+ml^2}{a}u$$

State Space Model:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{m^2 l^2 \sin x_1 \cos x_1}{a} x_2^2 + \frac{mg\, l \sin x_1 (M+m)}{a} - \frac{m\, l \cos x_1}{a} u \\ x_4 \\ \frac{(m\, l \sin\theta)(I + m l^2)}{a} x_2^2 - \frac{m^2 l^2 g \sin x_1}{a} + \frac{I + m l^2}{a} u \end{bmatrix}$$

Linearize about $\theta = 0$

$$\left.\begin{array}{l} \cdot\ \sin\theta = \theta \\ \cdot\ \cos\theta = 1 \\ \cdot\ \dot{\theta}^2 = 0 \end{array}\right\} \text{According to downloaded paper}$$

$$a = IM + Im + mMl^2 + \cancel{m^2 l^2} - \cancel{m^2 l^2}\cos^2\theta = I(M+m) + mMl^2$$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{mgl(M+m)}{I(M+m) + mMl^2} x_1 - \frac{ml}{I(M+m)+mMl^2} u \\ x_4 \\ \frac{-m^2 l^2 g}{I(M+m)+mMl^2} x_1 + \frac{I + ml^2}{I(M+m)+mMl^2} u \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{mgl(M+m)}{I(M+m)+mMl^2} x_1 - \dfrac{ml}{I(M+m)+mMl^2} u \\ x_4 \\ \dfrac{-m^2 l^2 g}{I(M+m)+mMl^2} x_1 + \dfrac{I+ml^2}{I(M+m)+mMl^2} u \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{mgl(M+m)}{I(M+m)+mMl^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-m^2 l^2 g}{I(M+m)+mMl^2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{ml}{I(M+m)+mMl^2} \\ 0 \\ \dfrac{I+ml^2}{I(M+m)+mMl^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix}$$

Only considering theta:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl(M+m)}{I(M+m)+mMl^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{ml}{I(M+m)+mMl^2} \end{bmatrix} u$$

$$y = \theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mgl(M+m)}{I(M+m)+mMl^2} & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -\frac{ml}{I(M+m)+mMl^2} \end{bmatrix} \qquad D = 0$$

$$G = C(sI-A)^{-1}B + D$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ -\frac{mgl(M+m)}{I(M+m)+mMl^2} & s \end{bmatrix}^{-1} = \frac{1}{s^2 - \frac{mgl(M+m)}{I(M+m)+mMl^2}} \begin{bmatrix} s & 1 \\ \frac{mgl(M+m)}{I(M+m)+mMl^2} & s \end{bmatrix}$$

$$\Rightarrow G = \frac{1}{s^2 - \frac{mgl(M+m)}{I(M+m)+mMl^2}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ \frac{mgl(M+m)}{I(M+m)+mMl^2} & s \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{ml}{I(M+m)+mMl^2} \end{bmatrix}$$

$$G = \frac{-\frac{ml}{I(M+m)+mMl^2}}{s^2 - \frac{mgl(M+m)}{I(M+m)+mMl^2}}$$