

# Analysis of Steady-State Behavior in Server Queues using Markov Chains and Eigenvalues in the M/M/1 Model

Nicholas Wise Saragih Sumbayak - 13524037

*School of Electrical Engineering and Informatics*

*Institut Teknologi Bandung*

Jl. Ganesha 10 Bandung 40132, Indonesia

nicholasaragih@gmail.com — 13524037@std.stei.itb.ac.id

**Abstract**—This document is a model and instructions for L<sup>A</sup>T<sub>E</sub>X. This and the IEEEtran.cls file define the components of your paper [title, text, heads, etc.]. \*CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.

**Index Terms**—component, formatting, style, styling, insert.

## I. INTRODUCTION

Queueing behavior arises naturally in nearly every computing environment where resources are shared among multiple tasks. Whenever incoming work arrives faster than it can be immediately processed, the excess work must wait, forming a queue. This phenomenon appears in a broad range of systems, including CPU scheduling, packet forwarding in routers, job dispatching in cloud infrastructures, disk I/O scheduling, and networked application servers. The performance of these systems is heavily influenced by their queueing characteristics, making analytical models essential for understanding and improving real-world performance [1].

Modern computing workloads are highly variable and unpredictable. Task arrivals do not occur at fixed intervals, and service times fluctuate due to user behavior, network delays, resource contention, and software-level scheduling. These uncertainties make deterministic approaches nonoptimal. However, probabilistic theory enables engineers to determine whether a server will remain stable under a particular load, estimate average waiting times, and understand how performance degrades as traffic increases.

To formally analyze such behavior, queueing systems are commonly modeled as stochastic processes, with one of the simplest and most fundamental being the M/M/1 queue. In this model, arrivals follow a Poisson process with service times that follow an exponential distribution, all handled by a single server. Despite its simplicity, the model can capture trends and predict behaviors such as stability, queue buildup, and performance spikes. [2].

The M/M/1 queue can be naturally represented as a Markov Chain, where each state corresponds to the current number of jobs in the system. Transitions between states reflect arrivals and completions of jobs, producing a stochastic matrix representing the dynamic behavior of the queue. The steady-state

distribution corresponds to the eigenvector associated with eigenvalue 1, consistent with standard results in Markov Chain theory [3]. This connection between queueing systems and linear algebra provides a powerful framework for analyzing server load and predicting performance under varying traffic conditions.

In this paper, we examine the steady-state behavior of the M/M/1 queue using Markov Chains and eigenvalue analysis. We first derive the transition matrix of the system and compute its stationary distribution using the eigenvector method. We then compare this theoretical distribution with the known closed-form solution of the M/M/1 queue. To validate the results, a queue simulation is implemented in C++, allowing us to observe empirical queue behavior under different arrival and service rates. By comparing theoretical predictions with simulation outcomes, we illustrate how mathematical modeling can determine whether a server can withstand a particular load and how queue lengths evolve over time.

## II. THEORETICAL FOUNDATION

### A. Matrices

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. The individual items in a matrix are called its elements or entries. The size of a matrix is described in terms of the number of rows and columns it contains. Generally, a general  $m \times n$  matrix may be denoted as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

with A denoting the matrix, m the number of rows, n the number of columns, and  $a_{ij}$  the element in row  $i$  and column  $j$ . A matrix with size  $n \times n$  is called a square matrix of order n and the elements  $a_{ii}$  (where the row and column indices are equal) form the main diagonal of matrix A.

1) *Row and Column Vectors*: A matrix with only one row or one column is called a row matrix (or a row vector) or column matrix (or a column vector), respectively. A general  $m \times 1$  column matrix and a  $1 \times n$  row matrix  $\mathbf{b}$  may be denoted as:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \text{ and } \mathbf{b} = [b_1 \ b_2 \ \dots \ b_n]$$

2) *Matrix Addition and Subtraction*: If  $A$  and  $B$  are matrices of the same size, their sum ( $A + B$ ) and difference ( $A - B$ ) are obtained by adding or subtracting their corresponding entries. Matrices of different sizes cannot be added or subtracted. In matrix notation, if  $A = [a_{ij}]$  and  $B = [b_{ij}]$  have the same size, then  $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$ .

3) *Scalar Multiplication*: If  $A$  is any matrix and  $k$  is any scalar, then the scalar multiple  $kA$  is the matrix obtained by multiplying every entry of  $A$  by  $k$ . In matrix notation, if  $A = [a_{ij}]$ , then  $(kA)_{ij} = ka_{ij}$ .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow 2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

4) *Matrix Multiplication*: If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, then the product  $AB$  is defined to be the  $m \times p$  matrix  $C$  whose entries are given by:

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

In simpler terms, to find entry  $c_{ij}$  of the product matrix  $C$ , multiply the corresponding entries of the  $i^{th}$  row of matrix  $A$  with the  $j^{th}$  column of matrix  $B$  and add the results. Given the example below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 5 & 0 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 25 & 7 \\ 22 & 15 \end{bmatrix}$$

Since  $A$  is a  $2 \times 3$  matrix and  $B$  is a  $3 \times 2$  matrix, the resulting product  $AB$  is a  $2 \times 2$  matrix. For example, to determine entry  $c_{11}$  of the product matrix  $AB$ , we multiply the corresponding entries of the first row of matrix  $A$  with the first column of matrix  $B$  and add the results.

5) *Transpose of a Matrix*: Given a matrix  $A$  of size  $m \times n$ , the transpose of  $A$ , denoted by  $A^T$ , is the  $n \times m$  matrix obtained by interchanging the rows and columns of  $A$ . In matrix notation, if  $B = A^T$ , then the entries of  $B$  are defined as  $b_{ij} = a_{ji}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

## B. Eigenvalues and Eigenvectors

If  $A$  is an  $n \times n$  matrix, then a nonzero vector  $\mathbf{v}$  in  $R^n$  is called an eigenvector of  $A$  if there exists a scalar  $\lambda$  such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

The scalar  $\lambda$  is called the eigenvalue of  $A$  corresponding to the eigenvector  $\mathbf{v}$ . In other words, multiplying the matrix  $A$  by the vector  $\mathbf{v}$  results in a new vector that is a scalar multiple of the original vector  $\mathbf{v}$ .

Given a matrix  $A$  with size  $n \times n$ , the eigenvalues and eigenvectors are found by solving the following characteristic equation:

$$\begin{aligned} Ax &= \lambda x \\ IAx &= \lambda Ix \\ Ax &= \lambda Ix \\ (A - \lambda I)x &= 0 \end{aligned}$$

Since  $x = 0$  is the only trivial solution, for  $(A - \lambda I)x = 0$  to have non-trivial solutions, the matrix  $(A - \lambda I)$  must be singular, and therefore  $\det(A - \lambda I)$  must be zero. The polynomial given by  $\det(A - \lambda I) = 0$  is called the characteristic equation of  $A$ , and the solutions to such equation are the eigenvalues of  $A$ , otherwise denoted as the characteristic roots.

Geometrically, an eigenvector of a matrix  $A$  represents a direction in  $R^n$  that is preserved under the linear transformation defined by  $A$ . While most vectors are both rotated and scaled by a linear transformation, eigenvectors are only scaled by the corresponding eigenvalue. If  $|\lambda| > 1$ , the transformation stretches vectors in the direction of the eigenvector, if  $|\lambda| < 1$ , it contracts them, and if  $\lambda < 0$ , it reverses their direction.

## C. Markov Chains

A Markov Chain is a mathematical system used to model systems that transition between different states over time. The defining characteristic of a Markov Chain is that the probability of transitioning to the next state depends only on the current state, and not on the sequence of states that preceded it. This property is known as the Markov property.

A Markov Chain is commonly represented using a matrix formed by transition probabilities between states. This matrix is called the transition matrix, and the state vectors at successive time intervals are defined by

$$x(n+1) = Px(n),$$

where  $x(n)$  is a probability vector describing the state distribution at time  $n$ , and  $P_{ij}$  is the probability that the system will be in state  $i$  at time  $n+1$  given that it was in state  $j$  at time  $n$  [4].

The transition matrix  $P$  is a stochastic matrix, meaning that all of its entries are nonnegative and that the sum of each column is equal to one. These properties ensure that multiplying a probability vector by  $P$  produces another valid probability vector.

As an example, consider a system with three states. A possible transition matrix for this system is

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}.$$

Each column of this matrix sums to one, and each entry represents the probability of transitioning from one state to another in a single time step.

From a linear algebra perspective, the long-term behavior of a Markov Chain is determined by the eigenvalues and eigenvectors of its transition matrix. In particular, a steady-state distribution is a probability vector  $x$  that remains unchanged under the transition matrix, satisfying

$$Px = x.$$

This corresponds to an eigenvector associated with the eigenvalue  $\lambda = 1$ .

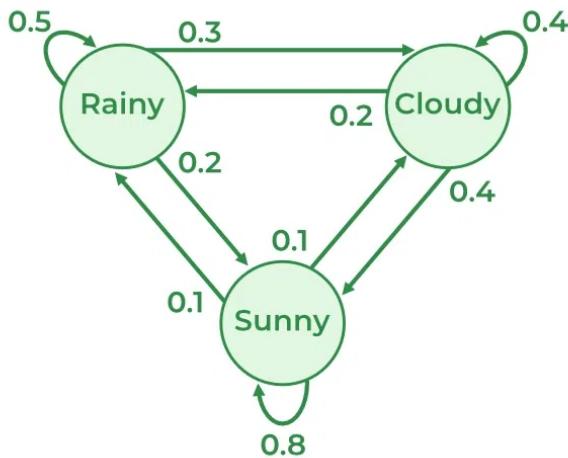


Fig. 1. Example of a Markov Chain representing weather states [6].

#### D. Equations

Number equations consecutively. To make your equations more compact, you may use the solidus (/), the exp function, or appropriate exponents. Italicize Roman symbols for quantities and variables, but not Greek symbols. Use a long dash rather than a hyphen for a minus sign. Punctuate equations with commas or periods when they are part of a sentence, as in:

$$a + b = \gamma \quad (1)$$

Be sure that the symbols in your equation have been defined before or immediately following the equation. Use “(1)”, not “Eq. (1)” or “equation (1)”, except at the beginning of a sentence: “Equation (1) is . . .”

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Please use “soft” (e.g., `\eqref{Eq}`) cross references instead of “hard” references (e.g., (1)). That will make it possible to combine sections, add equations, or change the order of figures or citations without having to go through the file line by line.

Please don’t use the `{eqnarray}` equation environment. Use `{align}` or `{IEEEeqnarray}` instead. The

`{eqnarray}` environment leaves unsightly spaces around relation symbols.

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#### F. Some Common Mistakes

- The word “data” is plural, not singular.
- The subscript for the permeability of vacuum  $\mu_0$ , and other common scientific constants, is zero with subscript formatting, not a lowercase letter “o”.
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- A graph within a graph is an “inset”, not an “insert”. The word alternatively is preferred to the word “alternately” (unless you really mean something that alternates).
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- In your paper title, if the words “that uses” can accurately replace the word “using”, capitalize the “u”; if not, keep using lower-cased.
- Be aware of the different meanings of the homophones “affect” and “effect”, “complement” and “compliment”, “discreet” and “discrete”, “principal” and “principle”.
- Do not confuse “imply” and “infer”.
- The prefix “non” is not a word; it should be joined to the word it modifies, usually without a hyphen.
- There is no period after the “et” in the Latin abbreviation “et al.”.

- The abbreviation “i.e.” means “that is”, and the abbreviation “e.g.” means “for example”.

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TABLE I  
TABLE TYPE STYLES

Table Head	Table Column Head		
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<sup>a</sup>Sample of a Table footnote.

Figure Labels: Use 8 point Times New Roman for Figure labels. Use words rather than symbols or abbreviations when writing Figure axis labels to avoid confusing the reader. As an example, write the quantity “Magnetization”, or “Magnetization, M”, not just “M”. If including units in the label, present

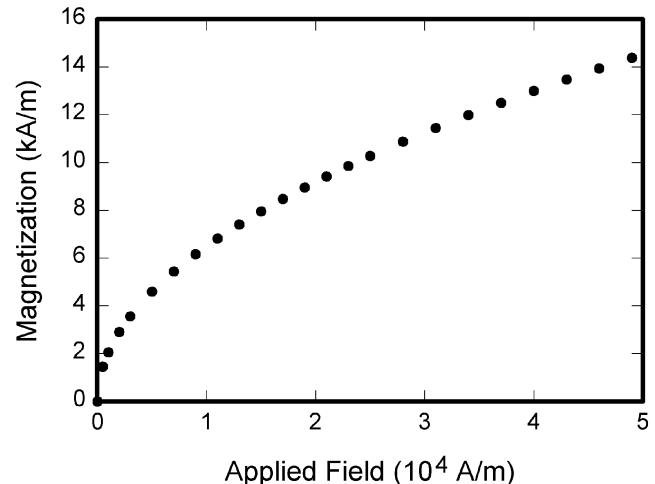


Fig. 2. Example of a figure caption.

them within parentheses. Do not label axes only with units. In the example, write “Magnetization (A/m)” or “Magnetization {A[m(1)]}”, not just “A/m”. Do not label axes with a ratio of quantities and units. For example, write “Temperature (K)”, not “Temperature/K”.

#### ACKNOWLEDGMENT

The preferred spelling of the word “acknowledgment” in America is without an “e” after the “g”. Avoid the stilted expression “one of us (R. B. G.) thanks ...”. Instead, try “R. B. G. thanks...”. Put sponsor acknowledgments in the unnumbered footnote on the first page.

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Please number citations consecutively within brackets [1]. The sentence punctuation follows the bracket [2]. Refer simply to the reference number, as in [3]—do not use “Ref. [3]” or “reference [3]” except at the beginning of a sentence: “Reference [3] was the first ...”

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For papers published in translation journals, please give the English citation first, followed by the original foreign-language citation [6].

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