EULER MATRICES OF QUIVERS AND MAHLER MEASURE

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ABSTRACT. For a given quiver Q, we can define the matrix B such that $B = -(E^T)E^{-1}$, where E is the Euler matrix of Q. We investigate the roots of the characteristic polynomial of B for small quivers, including the simply laced Dynkin diagrams. We show that the characteristic polynomial of B is reciprocal and that its roots must lie on the unit circle in the complex plane when $E + E^T$ is positive definite. For this project, we consulted with Professor Harm Derksen.

1. INTRODUCTION

We investigate the properties of the characteristic polynomial for a matrix B defined on a quiver. We are interested in the roots of this characteristic polynomial and where they lie in the complex plane.

In particular, we ask these questions:

Question 1.1. *Which polynomials P can be written as the characteristic polynomial of a matrix B for some Euler Matrix E of a quiver?*

Question 1.2. *Given a triangular matrix E of some acyclic quiver, is it the case that if $E + E^T$ is positive semidefinite, then the characteristic polynomial of B has roots on the unit circle in the complex plane?*

Question 1.3. *Given two quivers Q and Q', if Q is a subgraph of Q', is*

$$M(\det(xI - B_Q)) \leq M(\det(xI - B_{Q'}))$$

We now introduce basic terminology that motivates our approach to the problem, before providing some background into the project motivations. The main objects under study are quivers, and the related matrices are defined for use in the theory of quiver representations.

1.1. Quivers.

Definition 1. [3] A *quiver* Q is a 4-tuple $Q = (Q_0, Q_1, h, t)$ where Q_0 is a finite set of vertices, Q_1 is the set of edges, and $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ are functions that give the head and tail of each edge, respectively.

Definition 2. [3] A *path* on a quiver $Q = (Q_0, Q_1, h, t)$ from u to v for $u, v \in Q_0$ is some sequence of edges $e_0, \dots e_n$ such that for $0 \leq i \leq n$, $h(e_{i-1}) = t(e_i)$ and $t(e_0) = u$, $h(e_n) = v$.

Example 1.4. Consider the quiver illustrated by the following diagram:

$$\textcircled{1} \to \textcircled{2} \to \textcircled{3} \to \textcircled{4}$$

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Then $Q_0 = \{1, 2, 3, 4\}$, $Q_1 = \{(1, 2), (2, 3), (3, 4)\}$, and $h((1, 2)) = 2$ while $t((1, 2)) = 1$.

Example 1.5. Consider

$$\begin{tikzcd} \circled{1} \arrow[loop above] \arrow[r, bend left] & \circled{2} \arrow[l, bend left] \end{tikzcd}$$

Then $Q_0 = \{1, 2\}$, $Q_1 = \{(1, 2), (2, 1), (1, 1)\}$, and $h((1, 1)) = t((1, 1)) = 1$.

Notice that self-loops and cycles are permitted, and that the functions h and t are not necessarily injective, i.e. edges with the same head and tail may be repeated. However, we will be considering quivers that do not contain *oriented cycles*:

Definition 3. [3] An *oriented cycle* of a quiver $Q = (Q_0, Q_1, h, t)$ is a path from u to u. Notice that we may equivalently say that the cycle is a path from v to v for any v contained in the cycle.

Example 1.6. Consider again the quivers in Examples 1.1 and 1.2. Observe that the quiver in Example 1.1 does not contain an oriented cycle, as there is no path from any node back to itself. However, the quiver in Example 1.2 contains multiple oriented cycles; one consists only of the self-loop on 1, while the other consists of the edges $(1, 2)$ and $(2, 1)$.

1.2. **Quiver Representations.** The matrices under study for this project are used in the study of quiver representations; however, the definitions are in terms of quivers and not quiver representations. In particular, we use the *Euler matrix* of a quiver:

Definition 4. [3] Given a quiver $Q = (Q_0, Q_1, h, t)$, then the *Euler matrix* E of Q is defined as

$$E_{i,j} = \delta_{i,j} - |\{a \in Q_1 : t(a) = i, h(a) = j\}|$$

where $\delta_{i,j}$ is the Kronecker symbol, defined as

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

1.2.1. *Ausland-Reiter Transform.* This project studies the matrix product defined by $B = -E^T E^{-1}$, where E is the Euler Matrix for some quiver Q. The matrix B is used in the Ausland-Reiter Transform; see [3]. However, defining this transform is beyond the scope of this project.

An important note is that B is defined using E^{-1}, but E is singular when its defining quiver is cyclic [1]. Therefore, we will generally be considering only acyclic quivers.

1.3. **Topological Ordering.** Importantly, we know that any directed acyclic graph has a *topological ordering* of vertices; see [10], wherein any edge e has $h(e) \geq t(e)$. We can visualize these orderings by drawing the vertices in a line from least to greatest and adding edges accordingly.

Example 1.7. Consider A_3 as numbered below:

$$\circled{1} \rightarrow \circled{2} \leftarrow \circled{3}$$

We may construct a topological ordering of this graph by re-numbering vertex 3 as vertex 2. We can then draw the graph like this:

$$\begin{tikzcd} \textcircled{1} \arrow[rr, bend right] & \textcircled{2} \arrow[r] & \textcircled{3} \end{tikzcd}$$

Notice that now the edges all flow in the same direction. If we consider the example as an acyclic quiver, we see that arranging the vertices in a topological order means that E will be an upper triangular matrix; in particular,

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

And this can similarly be done with a reversed ordering of the vertices to obtain a lower triangular E. It follows from the definitions that any topological ordering of a quiver will lead to E being triangular.

Therefore, we will use this fact to assume that E is defined as a triangular matrix with diagonal values of 1 when we use it in examples and calculations. According to professor Derksen, if such an ordering can be constructed, it will not change the eigenvalues of B [1].

Proposition 1.8. *Let E be an Euler matrix of an acyclic quiver written in triangular form. Then* $\det(E) = 1$.

Proof. Since E is defined on an acyclic quiver, there are no edges (i, i) for any vertex i. Therefore $E_{i,i} = \delta_{i,i} - 0 = 1$. Since E is triangular by assumption and has diagonal entries 1, $\det(E) = 1$. $\square$

1.4. **History.** One way to look at the roots of a polynomial is to consider its *Mahler measure*, defined as

$$M(P) = |a_d|\Pi_{k=1}^{n} \max\{1, |\alpha_k|\}$$

Where

$$P(x) = a_n\Pi_{k=1}^{n}(x - \alpha_k)$$

has integer coefficients; see [4].

It is not known if there is some $P(x)$ with $M(P) = 1+\epsilon$ for $\epsilon > 0$. This question is referred to as *Lehmer's Conjecture* or *Lehmer's Question* [4]:

Question 1.9. *Does there exist some $C > 1$ such that for every polynomial $P(x) \in \mathbb{Z}[x]$, $M(P) = 1$ or $M(P) > C$?*

Lehmer conjectured that $C \approx 1.1762808$, given by $M(P)$ for $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$; see [4].

Lehmer first introduced his question in the 1933 paper *Factorization of Certain Cyclotomic Functions* [5]. The initial paper was motivated by an approach for finding large primes [6]. A lemma of Kronecker implies that if $M(P) = 1$, then P is a product of powers of x and cyclotomic polynomials; see [6].

Some partial results bounding Mahler measure have been established. Breusch proved in 1951 that a monic, irreducible, and nonreciprocal polynomial P then

$$M(P) \geq 1.324717\ldots$$

which is the real root of $x^3 - x - 1$ [6]. Later, in 1979, Dobrowolski showed that for a monic, irreducible, and non-cyclotomic P,

$$M(P) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3$$

for some constant c; see [6].

Several computational searches have been carried out in addition to the above approaches. None have yielded a counterexample to Lehmer's question [4]. Mossinghoff carried out a search for all polynomials of degree at most 24 with Mahler measure less than 1.3; while the specific algorithm used applied only to even degrees, he was able to find 48 such polynomials of degree 22 and 46 of degree 24 [4]. He was additionally able to use several other algorithms to extend the search calculations to more degrees, and was able to find a limit point for measures near 1.309; see [4].

For more information on the work around Lehmer's question, see the cited works *Mahler Measure of Polynomials* and *Polynomials with Small Mahler Measure.*

2. The Characteristic Polynomial of $B = -E^T E^{-1}$

For this section, we assume that E is the Euler matrix of some acyclic quiver, with the vertices of that quiver numbered such that E is triangular. We consider the form of the characteristic polynomial of $B = -E^T E^{-1}$.

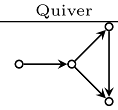
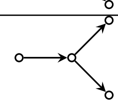
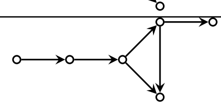
Quiver	$\det(xI - B)$
	$\lambda^4 - \lambda^3 - 3\lambda^2 - \lambda + 1$
	$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$
	$\lambda^6 - \lambda^5 - 5\lambda^4 - 7\lambda^3 - 5\lambda^2 - \lambda + 1$

Table 1: Several quivers, and the characteristic polynomial of the associated $B = -E^T E^{-1}$

Notice that the coefficients of each polynomial are palindromic; this corresponds with the notion of a *self-reciprocal* or *palindromic* polynomial, which is a polynomial

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

such that

$$P(x) = a_0 x^n + \ldots | a_{n-1} x + a_n$$

Proposition 2.1. *Let $B = -E^T E^{-1}$ where E is a triangular Euler matrix of size $n \times n$. Let $P(x) = \det xI - B$. Then $P(x)$ is self-reciprocal.*

Proof. Consider

$$xI - B = xI + E^T E^{-1}$$

so

$$(xI - B)E = xE + E^T$$

But since $\det(E) = 1$,

$$\det(xI - B) = \det(xE + E^T)$$

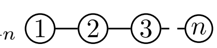
We know that since
$$\det(B) = \pm 1$$
$x = 0$ is not a root of $\det(xI - B)$, and so when
$$\det(xI - B) = \det(xE + E^T) = 0$$
we see that
$$\det(xE + E^T) = x^n \det(E + \frac{1}{x}E^T)$$
But for $x \neq 0$,
$$P(\frac{1}{x}) = \det E + \frac{1}{x}E^T$$
so
$$P(x) = x^n P(\frac{1}{x})$$
and therefore by [8] P is self-reciprocal. $\square$

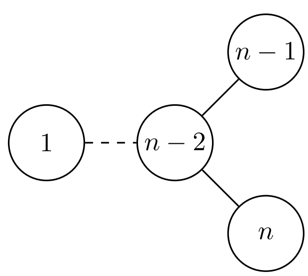
Further research could find more specific restrictions on $\det(xI - B)$.

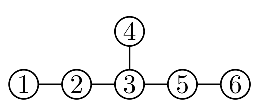
3. Roots of $\det(xI - B)$ when $E + E^T$ is positive definite

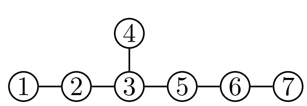
Let $B = -E^T E^{-1}$ where E is the Euler matrix of some acyclic quiver written in triangular form. For certain quivers, representation theory can tell us about the properties of B and and E; in particular, the quivers known as the simply laced Dynkin diagrams are well-studied, as they have connections to many other topics.

The *simply laced Dynkin diagrams* are A_n, D_n, E_6, E_7, and E_8; see [3]. Dynkin diagrams are typically drawn as undirected graphs; if we speak of a quiver as being related to a Dynkin diagram, we mean the quiver drawn also as an undirected graph. We present the simply laced Dynkin diagrams below:

A_n 

D_n 

E_6 

E_7 

$$\begin{tikzcd}[row sep=small, column sep=small] & & & ④ \arrow[d, dash] & & & & \\ E_8 & ① \arrow[r, dash] & ② \arrow[r, dash] & ③ \arrow[r, dash] & ⑤ \arrow[r, dash] & ⑥ \arrow[r, dash] & ⑦ \arrow[r, dash] & ⑧ \end{tikzcd}$$

We know that the roots of $\det(xI - B)$ for these diagrams are all on the unit circle. We also know that $E + E^T$ is positive definite when the quiver used to define E is a simply laced Dynkin diagram. These two facts are not obviously related, but we show below that the roots of $\det(xI - B)$ must lie on the unit circle if $E + E^T$ is positive definite.

Proposition 3.1. *Suppose an acyclic quiver has a triangular Euler matrix E and that $E + E^T$ is positive definite. Let $B = -E^T E^{-1}$. Then the solutions to* $\det(xI - B)$ *lie on the unit circle in the complex plane.*

Proof. By assumption, $E+E^T$ is positive definite. Consider some $\mathbf{v} \in \ker(xE+E^T)$ where x is a solution of $\det(xE + E^T) = 0$.

Then $(xE + E^T)\mathbf{v} = 0$ so

$$xE\mathbf{v} + E^T\mathbf{v} = 0$$

and therefore $xE\mathbf{v} = -E^T\mathbf{v}$.

Furthermore, by taking the conjugate transpose of the above equation we obtain

$$\mathbf{v}^*(xE + E^T)^* = 0$$

but since E is triangular with integer entries,

$$\mathbf{v}^*(E + xE^T) = 0$$

so

$$\mathbf{v}^*E\mathbf{v} = -\bar{x}\mathbf{v}^*E^T\mathbf{v}$$

But since $xE\mathbf{v} = -E^T\mathbf{v}$, we can see that

$$\mathbf{v}^*E\mathbf{v} = (-\bar{x})(-x\mathbf{v}^*E\mathbf{v})$$

and therefore

$$\mathbf{v}^*E\mathbf{v} = |x|^2\mathbf{v}^*E\mathbf{v}$$

Suppose $\mathbf{v}E\mathbf{v} = 0$. Then

$$(\mathbf{v}E\mathbf{v})^* = 0^*$$
$$\mathbf{v}^*(\mathbf{v}^*E)^* = 0$$
$$\mathbf{v}^*E^T\mathbf{v} = 0$$

But $E + E^T$ is positive definite, so

$$\mathbf{v}^*(E + E^T)\mathbf{v} > 0$$
$$\mathbf{v}^*E\mathbf{v} + \mathbf{v}^*E^T\mathbf{v} > 0$$
$$0 > 0$$

which is a contradiction. So we know that $\mathbf{v}^*E\mathbf{v} \neq 0$.

Therefore, since $\mathbf{v}^*E\mathbf{v} \neq 0$ and

$$\mathbf{v}^*E\mathbf{v} = |x|^2\mathbf{v}^*E\mathbf{v}$$

we see that

$$|x|^2 = 1$$

so x must lie on the unit circle in the complex plane; and since

$$\det((xI - B)(E)) = \det((xE + E^T) = \det(xI - B)\det(E) = \det(xI - B)$$

we can see that the characteristic polynomial of $B = -E^T E^{-1}$ must have roots on the unit circle in the complex plane if $E + E^T$ is positive definite. $\square$

This is not necessarily the case in reverse. Further research could investigate $E + E^T$ when the roots of $\det(xI - B)$ are known to be on the unit circle.

4. Roots of Subquivers

In this section, we again consider $B = -E^T E^{-1}$ for some triangular Euler matrix E for an acyclic quiver. Consider the following example, created by repeateddly adding a single vertex and edge to a small quiver:

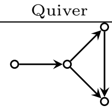
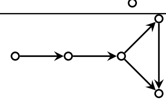
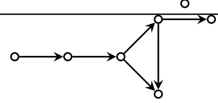
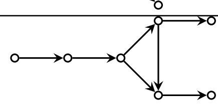
Quiver	$\det(xI - B)$	$M(\det(xI - B))$
	$\lambda^4 - \lambda^3 - 3\lambda^2 - \lambda + 1$	2.3692054071
	$\lambda^5 - \lambda^4 - 3\lambda^3 - 3\lambda^2 - \lambda + 1$	2.618033988770564
	$\lambda^6 - \lambda^5 - 5\lambda^4 - 7\lambda^3 - 5\lambda^2 - \lambda + 1$	3.3014904324145147
	$\lambda^7 - \lambda^6 - 7\lambda^5 - 13\lambda^4 - 13\lambda^3 - 7\lambda^2 - \lambda + 1$	3.899908930943441

Table 2: "Pendulum" quiver sequence and characteristic polynomials of B

Here, $M(\det(xI - B))$ refers to the *Mahler measure* [4] of the characteristic polynomial of B. The Mahler measure of a polynomial $P(x) \in \mathbb{Z}[x]$ with integer coefficients where

$$P(x) = a_n \Pi_{k=1}^n (x - \alpha_k)$$

is defined as

$$M(P) = |a_d| \Pi_{k=1}^n \max\{1, |\alpha_k|\}$$

Notice that the Mahler measure of each quiver is increasing as more edges and vertices are added. Future work could show that the addition of any edges or vertices to a quiver cannot lead to a smaller Mahler measure for $\det(xI - B)$ than the original quiver.

Appendix A. Source Code

The source code used to generate examples, graphs, and calculations can be found at https://github.com/nichols-t/f20-math4020-quiver-matrices.

References

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