Bankruptcy Problem from the Talmud

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May 22, 2013

Original paper

- Game Theoretic Analysis of a Bankruptcy Problem from the Talmud
- Robert J. Aumann and Michael Maschler
- Journal of Economic Theory **36**. 195-213 (1985)

Outline

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Bankruptcy Problem

- A man dies, leaving debts $d_1, d_2, \cdots, d_n$ totalling more than his estate E. How should the estate be divied among the creditiors?

Bankruptcy Problem from the Talmud

		Debt		
		100	200	300
	100	$\frac{100}{3}$	$\frac{100}{3}$	$\frac{100}{3}$
Estate	200	50	75	75
	300	50	100	150

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- The lesser claimant concedes half the garment to the greater one. The remaining half is divided equally.

Two person bankruptcy problem

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- We will say that this division is prescibed by the CG(contested garment) principle.

Consistency

- A bankruptcy problem is defined as a pair $(E; d)$, where $d = (d_1, d_2, \cdots, d_n)$, $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and $0 \leq E \leq D := d_1 + \cdots d_n$.

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- A solution to such a problem is an $n - tuple$ $x = (x_1, \cdots, x_n)$ of real numbers with

$$x_1 + x_2 + \cdots + x_n = E.$$

Consistency

- A solution is called CG-consistent or consistent if for all $i \neq j$, the division of $x_i + x_j$ prescribed by the CG-principle for claims d_i, d_j is (x_i, x_j).

Remark

The bankruptcy problem from the Talmud are consistent solution.

Consistent solution of bankruptcy problem

Theorem 1

Each bankruptcy problem has the unique consistent solution.

Uniqueness of consistent solution

Let x, y be two consistent solutions of given bankruptcy problem. We can find $i \neq j$ such that

$$y_i > x_i, \;\; y_j < x_j, \;\; y_i + y_j \geq x_i + x_j.$$

Since CG principle has monotonicity with respect to estate, $y_j \geq x_j$. It is contradiction.

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- This continues untill 1 has received $d_1/2$. For the time being she then stops, and each additional dollar is divided equally between the remaining $n - 1$ claimants.

Existence of consistent solution

- This, in turn, continues until 2 has received $d_2/2$, at which point she stops receving payment for the time being, and each additional dollar is divided equally between the remaining $n-2$ claimants.

Existence of consistent solution

Case2) $E > D/2$.

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- The process is the mirror image of the above. Instead of thinking i's award x_i, one thinks in terms of her loss $d_i - x_i$.
- When $D - E$ is small, the tatal loss is shared equally between all creditors.
- This continues untill 1 has lost $d_1/2$. For the time being she then stops losing and each additional loss is divided equally between the remaining $n - 1$ claimants.

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2. Let us denote $f(E;d)$ be the unique consistent solution of bankruptcy problem $(E;d)$. Then the following duality holds;

$$f(E;d) = d - f(D-E;d).$$

Outline

Definition of bankruptcy game

Definition

We define the bankruptcy game $v_{E;d}$ corresponding to the bankruptcy problem $(E; d)$ by

$$v_{E;d} := (E - d(N \setminus S))_+.$$

Remark

$v_{E;d}$ is a coalition game with superadditivity.

$v_{E;d}(S)$: Accepting either nothing, or what is left of the estate E after each menber i of the coalition $N \setminus S$ is paid his complete claim d_i.

Main theorem

Theorem 2

The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

From now on, let v be a game, S a coalition and x a payoff vector.

The reduced game

Definition

The reduced game $v^{S,x}$ is defined on the player space S as

$$v^{S,x}(T) = \begin{cases} x(T) & \text{if } T = \emptyset \text{ or } T = S \\ \max_{Q \subset N \setminus S} \{v(Q \cup T) - x(Q)\} & \text{if } \emptyset \subsetneq T \subsetneq S \end{cases}$$

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In the reduced game, the players of S consider how to divide the tatal amount assigned to them by x under the assumption that the players i outside S get exactly x_i.

The reduced game

Lemma 3

Let x be a solution of the bankruptcy problem $(E; d)$ such that $0 \leq x_i \leq d_i$ for all i. Then for any coalition S the following holds;

$$v_{E;d}^{S,x} = v_{x(S);d|S}.$$

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Let x be a solution of the bankruptcy problem $(E;d)$ such that $0 \leq x_i \leq d_i$ for all i. Then for any coalition S the following holds;

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proof. Set $v := v_{E;d}$, $v^S := v_{E;d}^{S,x}$. First let $\emptyset \subsetneq T \subsetneq S$ and let the maximum in the definition of $v^S(T)$ be attained at Q.

The reduced game

$$
\begin{aligned}
v^S(T) &= v(T \cup Q) - x(Q) \\
&= [E - d(N \setminus (Q \cup T))]_+ - x(Q)_+ \\
&\leq [x(N) - d(N \setminus (Q \cup T)) - x(Q)]_+ \\
&= [x(S) - d(S \setminus T) - (d - x)(N \setminus (S \cup Q))]_+ \\
&\leq (x(S) - d(S \setminus T))_+
\end{aligned}
$$

Setting $Q = N \setminus S$

$$
\begin{aligned}
v^S(T) &\geq v(T \cup (N \setminus S)) - x(N \setminus S) \\
&\geq [E - d(N \setminus (T \cup (N \setminus S)))]_+ - (x(N) - x(S)) \\
&\geq (E - d(S \setminus T)) - (E - x(S)) = x(S) - d(S \setminus T).
\end{aligned}
$$

The reduced game

and setting $Q = \emptyset$

$$v^S(T) \geq v(T \cup \emptyset) - x(\emptyset) = v(T) = (E - d(N \setminus T))_+ \geq 0.$$

Therefore $v^S(T) = (x(S) - d(S \setminus T))_+ = v_{x(S);d|S}(T)$.

Kernel and pre-kernel

Let v be a game. For each payoff vector x and players i, j, define

$$S_{ij}(x) = \max\{v(S) - x(S) | S \text{ contains } i \text{ but not } j\}.$$

Definition

The pre-kernel of v is the set of all payoff vectors x with $x(N) = v(N)$ and $S_{ij}(x) = S_{ji}(x)$ for all $i \neq j$.

Definition

The kernel of v is the set of all payoff vectors x with $x(N) = v(N)$, $x_i \geq v(i)$ for all i and for all $i \neq j$

$$S_{ij}(x) > S_{ji}(x) \text{ implies } x_i = v(j).$$

Standard solution of 2 person game

- For a given 2 person game v, the payoff vector

$$x_i = \frac{v(12) - v(1) - v(2)}{2} + v(i)$$

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Remark

The nucleouls and kernel, the pre-kernel and the Shapley value of 2 person game all coincide with its standard solution.

Pre-kernel and standard solution

Lemma 4

Let x be in the pre-kernel of a game v and let S be a coalition with exactly two person. Then $x|S$ is the standard solution of $v^{S,x}$.

proof) Let $S = \{i, j\}$ Then

$$S_{ij}(x) = \max_{Q \subset N \setminus S} (v(Q \cup i) - x(Q \cup i)) = \max_{Q \subset N \setminus S} (v(Q \cup i) - x(Q)) - x_i$$
$$= v^{S,x}(i) - x_i.$$

Similarly, we know that $S_{ji}(x) = v^{S,x}(j) - x_j$. Since x is in the pre-kernel $S_{ij}(x) = S_{ji}(x)$. This implies

$$x_i - x_j = v^{S,x}(i) - v^{S,x}(j)$$

also by definition of reduced game

$$x_i + x_j = x(i, j) = v^{S,x}(i, j).$$

Kernel and the consistent solution of bankrutpcy problem

Lemma 5

The contested garment solution of 2 person bankruptcy problem is the standard solution of the corresponding game.

Proof.

Directly from Lemma 4 and definition of bankruptcy game and the standard solution. □

Kernel and the consistent solution of bankrutpcy problem

Proposition 6

The kernel of a bankruptcy game $v_{E;d}$ consists of a single point, namely the consistent solution of the problem $(E;d)$.

Proof) Set $v := v_{E;d}$. v is superadditive, hence 0-monotonic, i.e. $S \subset T$ implies $v(S) + \sum_{i \in T \setminus S} v(i) \leq v(T)$. In 0-monotonic game, the kernel coincides with the pre-kernel. Hence x is in the pre-kernel of v. Let S be an arbitrary 2 person coalition. By lemma 4, $x|S$ is the standard solution of $v^{S,x}$ and hence, of $v_{x(S);d|S}$. Therefore $x|S$ is the CG-solution of $(x(S);d|S)$, but that means x is the consistent solution of $(E;d)$.

Proof of main theorem

Theorem

The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

Proof.

Since the nucleolus is always in the kernel, by proposition 6, it coincides with the consistent solution of the bankruptcy problem. $\square$