The environment process $\{Z(t), t \geq 0\}$ is a CTMC with $\ell = 2$ states and $S = \{1, 2\}$ with 1 representing off and 2 representing the on-state. Therefore,

$$Q = \begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}.$$

With $r > c$, the drift matrix is

$$D = \begin{bmatrix} -c & 0 \\ 0 & r - c \end{bmatrix}.$$

The steady-state probabilities for the environment process are $p_1 = \frac{\alpha}{\alpha+\beta}$ and $p_2 = \frac{\beta}{\alpha+\beta}$. The system is stable if $0p_1 + rp_2 < c$. Thus the stability condition (notice that $B = \infty$) is

$$r\beta/(\alpha + \beta) < c.$$

State 1 has negative drift and state 2 has positive drift. Hence by solving for Equation 9.14 we would get one λ value with negative real part and one

λ value would be zero. To confirm that we solve for λ in the characteristic equation

$$det(\lambda D - Q) = 0,$$

which yields

$$(-\lambda c + \beta)(\lambda r - \lambda c + \alpha) - \alpha\beta = 0.$$

By rewriting this equation in terms of the unknown λ we get

$$c(r - c)\lambda^2 - (\beta(r - c) - \alpha c)\lambda = 0.$$

Therefore, we have the two solutions to λ as

$$\lambda_1 = \beta/c - \alpha/(r - c),$$

$$\lambda_1 = \beta/c - \alpha/(r-c),$$

$$\lambda_2 = 0.$$

From the stability condition $r\beta/(\alpha+\beta) < c$ we have $r\beta - c(\alpha+\beta) < 0$ and dividing that by the positive quantity $c(r-c)$, we get $\beta/c - \alpha/(r-c) < 0$. Hence $\lambda_1 < 0$. Thus we have verified that one λ value has negative real part and the other one is zero.

Next, using $\phi(\lambda D - Q) = [0 \;\; 0]$ for each λ, we can obtain the corresponding left eigenvectors as $\phi_1 = [(r-c)/c \;\; 1]$ and $\phi_2 = [\alpha/(\alpha+\beta) \;\; \beta/(\alpha+\beta)]$. Thereby using Equation 9.16 we can write down $F(x)$ as

$$F(x) = a_1 e^{\lambda_1 x}\phi_1 + a_2 e^{\lambda_2 x}\phi_2.$$

All we need to compute are a_1 and a_2. For that we use Equations 9.18 and 9.19. From Equation 9.18, $a_2 = 1/(\phi_2 \bar{1}) = 1$. Also, from Equation 9.19 we get $a_1\phi_1(2) + a_2\phi_2(2) = 0$ since that corresponds to state with a positive drift and that results in $a_1 = -\frac{\beta}{\alpha+\beta}$. Hence we have

All we need to compute are a_1 and a_2. For that we use Equations 9.18 and 9.19. From Equation 9.18, $a_2 = 1/(\phi_2 \bar{1}) = 1$. Also, from Equation 9.19 we get $a_1\phi_1(2) + a_2\phi_2(2) = 0$ since that corresponds to state with a positive drift and that results in $a_1 = -\frac{\beta}{\alpha+\beta}$. Hence we have

$$F(x) = [F_1(x) \;\; F_2(x)] = \left[\frac{\alpha c - (r-c)\beta e^{\lambda_1 x}}{c(\alpha+\beta)} \quad \frac{\beta}{\alpha+\beta}(1 - e^{\lambda_1 x})\right].$$

$$\frac{dF(x)}{dx} D = F(x) Q,$$