CS 234r: Markets for Networks and Crowds **Lecture 7**
Approximately Optimal Auctions, Pricing, Price Discrimination

Recap

Last time, we argued that it's without loss of generality to limit our attention to auctions that are *truthful* (i.e., have a truth-telling BNE). We also learned two surprising things about truthful auctions in single-parameter settings:

1. A mechanism is truthful iff the expected allocation functions $x_i(v_i)$ are monotone non-decreasing, and its expected payments follow the payment identity.

2. The expected revenue of a truthful mechanism is equal to its expected virtual welfare, $E[\sum_i \phi_i(v_i)x_i(\mathbf{v})]$.

The first fact implies that when we describe a truthful mechanism, it's enough to focus on the allocation rule. The payments are essentially determined (in expectation) by the payment identity! This is convenient for designing mechanisms, but potentially inconvenient if we want to argue about the payments directly. The second fact helps address this inconvenience: the expected revenue of a mechanism can be expressed in terms of its allocation rule. If we adjust each agent's value (in a way that depends only on that agent's value distribution), the expected revenue is the total value generated under these "adjusted" (i.e., virtual) values.

With these facts in hand, we are ready to describe revenue-optimal auctions.

Myerson's Auction

Recall from last time: for a single item and a single agent with value $v \sim F$, what is the revenue-optimal mechanism?

Def: Distribution F is *regular* if $\phi(\cdot)$ is monotone non-decreasing.

Claim: For regular F, the optimal mechanism for selling a single item to a single bidder is posted price $\phi^{-1}(0)$.

Note: The regularity condition is to make sure the allocation rule "allocate iff virtual value is positive" is monotone, and that $\phi^{-1}(0)$ is uniquely defined. (A process called "ironing" allows us to extend to non-regular distributions, but that's beyond the scope of this lecture.)

Question: what about n agents, $v_i \sim F_i$?

From Myerson's characterization, we want to find an allocation rule x that solves:

$$\max_x \sum_i E_{v_i}\left[x_i(v_i)\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right)\right]$$

s.t.

$$x_i(v_i) \text{ non}-\text{decreasing},$$

$$0 \le \sum_i x_i(v_i) \le 1.$$

In fact, this is the goal not just for allocating a single item, but for ANY allocation problem described by a set of constraints on the feasible allocations $\mathbf{x}$.

Idea: Optimize pointwise.

Given $(v_1, \ldots, v_n)$,

- choose feasible allocation $x(\cdot)$ to maximize $\sum_i \phi(v_i) x(v_i, v_{-i})$,
- check incentive constraint (i.e., monotonicity of allocation),
- choose payments according to threshold rule.

This will maximize expectation and satisfy expected payments specified by payment rule.

This mechanism is *Myerson's Optimal Auction*. It is revenue-optimal for single-parameter settings, assuming the value distributions are regular (so that the allocation is monotone).

The fact that Myerson's optimal auction maximizes the expected virtual welfare follows from linearity of expectation:

$$\begin{aligned}
&\sum_i E_{v_i}\left[x_i(v_i)\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right)\right] \\
&= \sum_i E_{v_i}\left[E_{v_{-i}}[x_i(v_i, v_{-i})]\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right)\right] \\
&= \sum_i E_{v_1,\ldots,v_n}\left[x_i(v_i, v_{-i})\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right)\right] \\
&= E_{v_1,\ldots,v_n}\left[\sum_i x_i(v_i, v_{-i})\phi_i(v_i)\right]
\end{aligned}$$

We can now apply Myerson's optimal auction to the special case of allocating a single item to n agents, with $v_i \sim F_i$.

Claim: For regular distributions $F_1, \ldots, F_n$, optimal auction gives item to agent w/highest positive virtual value $\phi_i(v_i)$,

$$\begin{aligned}
&x_i(v_i, v_{-i}) = 1 \text{ if and only if} \\
&i = \text{argmax}_j \phi_j(v_j) \text{ and } \phi_i(v_i) \geq 0.
\end{aligned}$$

If i is allocated, then i pays

$$p_i(v_i, v_{-i}) = \max\{\phi_i^{-1}(0), \phi_i^{-1}(\phi_j(v_j))\}$$

where v_j has second-highest virtual value.

Proof: Relax IC, maximize pointwise:

$$\begin{aligned}
&\max_x \sum_i x_i(v_i, v_{-i})\phi_i(v_i) \\
&\text{s.t.} \\
&0 \leq \sum_i x_i(v_i, v_{-i}) \leq 1.
\end{aligned}$$

Note allocation is monotone (for regular distributions), solve for thresholds. □

Question: What auction is this for, e.g., $F_i = U[0,1]$ for all i?

Second-price auction with reserve 1/2.

Discussion: issues with Myerson

- Requires detailed knowledge of value distributions
- Allocation rule can be non-obvious (more on this later)
- Centralized protocol, requires all agents be present at once.

We'll spend the rest of the lecture addressing some of these concerns.

Approximate Revenue Maximization

Let's explore a few ways we might design "simpler" ways of selling an item that are still approximately optimal.

Setting 1: Unknown Distribution

The setting:

- 1 item to sell
- n buyers, values $v_i \sim F$ (note identical distributions)

Suppose the seller doesn't know F. What can she do?

Prior-independent auction design: compare mechanism without prior knowledge to optimal mechanism given the prior.

Idea: Run a Vickrey auction: a 2nd-price auction with no reserve.

Example: Suppose F is actually $U[0, 1]$. In this case, the Myerson auction is a 2nd-price auction with reserve price of $\phi^{-1}(0) = 1/2$. How well does the Vickrey auction do? Write $Rev(opt)$ for optimal expected revenue, and $Rev(Vickrey)$ for expected revenue of a Vickrey auction.

- 1 bidder: $E[Rev(Vickrey)] = 0$, $E[Rev(opt)] = 1/4$
- 2 bidders: $E[Rev(Vickrey)] = 1/3$, $E[Rev(opt)] = 5/12$
- 3 bidders: $E[Rev(Vickrey)] = 1/2$, $E[Rev(opt)] = 17/32$

Note: that last calculation, 17/32, can be obtained by arguing about virtual value. The expected revenue is equal to the expected virtual value of the winner. For a given agent i with value $v > 1/2$, the probability that agent i is the winner is v^2 (since the distributions are $U[0, 1]$). Note that if $v \leq 1/2$, then agent i is never a winner. Since there are 3 agents, the expected virtual value of the winner of the auction is $3 \int_{1/2}^{1} \phi(v)v^2 dv = 3 \int_{1/2}^{1} (2v - 1)v^2 dv = 17/32$.

Note that the Vickrey revenue doesn't approximate the Myerson revenue in general (e.g., when $n = 1$). But Vickrey with one more bidder always gets more revenue than Myerson!

Claim (Bulow-Klemperer '96): Let F be a regular distribution, and $n > 0$. Then $E_{v_1,\ldots,v_{n+1}\sim F}[Rev(Vickrey)] \geq E_{v_1,\ldots,v_n\sim F}[Rev(OPT)]$.

In other words, recruiting 1 additional bidder is worth more than setting the right reserve price!

Proof: Consider the following intermediate auction A on $n + 1$ bidders:

- simulate OPT on the first n bidders
- if item not allocated by this simulation, allocate to the $(n + 1)$st bidder for free.

Then note that

(1) $E_{v_1,\ldots,v_{n+1}\sim F}[Rev(A)] = E_{v_1,\ldots,v_n\sim F}[Rev(OPT)]$

(2) A always allocates the item

What is the revenue-maximizing auction that always allocates the item? By Myerson's characterization, it's the allocation rule x that maximizes $E[\sum_i \phi_i(v_i)x_i(v)]$, subject to $\sum_i x_i(v) = 1$ for all profiles v.

But note that since ϕ is an increasing function, allocating to the highest virtual value is the same as allocating to the highest value. So this is just the Vickrey auction!

This implies that $E_{v_1,\ldots,v_{n+1}\sim F}[Rev(Vickrey)] \geq E_{v_1,\ldots,v_{n+1}\sim F}[Rev(A)]$. Putting this together with (1) from above finishes the proof.

$\square$

A simple corollary is that the second-price auction is a good approximation to the optimal auction, as long as the number of bidders is not too small.

Claim: For all $n \geq 2$, and regular F, $E_{v_1,\ldots,v_n \sim F}[Revenue(Vickrey)] \geq \frac{n-1}{n} E_{v_1,\ldots,v_n \sim F}[Revenue(OPT)]$.

Proof: (sketch) Consider an auction A for $n-1$ bidders that just samples a "fake" extra bidder, runs the Myerson optimal auction including the fake bidder, and discards the item if the fake bidder wins. By Bulow-Klemperer, $E[Rev(Vickrey)] \geq E[Rev(A)]$. But, by symmetry, auction A gets revenue at least $(1-1/n)$ times the revenue of Myerson on n bidders, since it just omits the revenue from one of the bidders in the optimal auction. $\square$

Setting 2: Multiple Distributions

The setting:

- 1 item to sell
- n buyers, values $v_i \sim F_i$

The Myerson auction is optimal, but can be unintuitive.

Suppose we are selling a used car on eBay. There are 2 bidders. We notice bidder 1 has zip code 90210, bidder 2 has zip code 02139.

- bidder 1 is probably richer
- bidder 1 probably has more value for a car
- we expect bidder 1 to pay more when winning, in an optimal auction

If F_1 is more skewed toward higher values than F_2, then there may be values $v_1 > v_2$ such that $\phi_1(v_1) < \phi_2(v_2)$. In this case, the optimal auction might allocate to the bidder with the lower value!

Example: Say $F_1 = U[0,2]$, $F_2 = U[0,3]$.

- $\phi_1(v_1) = 2v_1 - 2$
- $\phi_2(v_2) = 2v_2 - 3$

If $v_1 = 1.5$ and $v_2 = 1.9$, then $\phi_1(v_1) = 1$ and $\phi_2(v_2) = 0.8$, so OPT allocates to bidder 1.

Suppose we want to restrict attention to "natural" auctions that allocate to the bidder with the highest *value*, subject to clearing a (possibly bidder-specific) reserve. How well does this approximate the optimal auction?

Def: The second-price auction with monopoly reserves works as follows:

- eliminate any bidders i such that $\phi_i(v_i) < 0$
- allocate to the remaining highest bidder at price $\max(\phi_i^{-1}(0),\ \text{2nd-highest remaining bid })$.

Claim: The second-price auction with monopoly reserves is a 2-approx to OPT.

Proof: We'll refer to the 2nd-price auction with monopoly reserves as SMR, for convenience. Write I for the winner of OPT and J for the winner of SMR. I and J are random variables. We'll say $I = 0$ if OPT doesn't allocate to any agent, and similarly $J = 0$ if SMR doesn't allocate to any agent. Note that $I = 0$ iff $J = 0$, since in either auction the no-sale event occurs precisely when $\phi_i(v_i) < 0$ for all i. So we can assume $I > 0$ and $J > 0$. Note that

$$\begin{aligned}
&E[Rev(OPT)] \\
&= E[Rev(OPT) \mid I = J]Pr[I = J] \\
&\quad + E[Rev(OPT) \mid I \neq J]Pr[I \neq J].
\end{aligned}$$

That is, we can decompose $E[Rev(OPT)]$ into the contribution from cases where I and J are the same, and cases where I and J are different. We'll write X for the first term and Y for the second term. Let's analyze these terms separately.

$$\begin{aligned}
X &= E[\phi_I(v_I) \mid I = J]Pr[I = J] \\
&= E[\phi_J(v_J) \mid I = J]Pr[I = J] \\
&\leq E[\phi_J(v_J) \mid I = J]Pr[I = J] \\
&\quad + E[\phi_J(v_J) \mid I \neq J]Pr[I \neq J] \\
&= E[Rev(SMR)]
\end{aligned}$$

where the inequality is because J passed the monopoly reserve, so $\phi_J(v_J) \geq 0$.

For the other term, we have

$$\begin{aligned}
Y &= E[\phi_I(v_I) \mid I \neq J]Pr[I \neq J] \\
&\leq E[v_I \mid I = J]Pr[I = J] \\
&\leq E[p_J(\mathbf{v}) \mid I = J]Pr[I = J] \\
&\leq E[p_J(\mathbf{v}) \mid I = J]Pr[I = J] \\
&\quad + E[p_J(\mathbf{v}) \mid I \neq J]Pr[I \neq J] \\
&= E[Rev(SMR)]
\end{aligned}$$

where the first inequality is from the form of the function ϕ_I, and the second inequality is because J was the winner in a second-price auction, and hence paid at least the bid of each other agent, including agent I. (This is where we used the fact that I and J are different.)

Combining these two terms, we get that $E[Rev(OPT)] \leq 2E[Rev(SMR)]$. □

Note: this auction still sets bidder-specific reserve prices, based on information the seller has about each bidder's type. This is a form of *price discrimination.* (Since it's based on distributions known for each bidder, it's called *third-degree price discrimination* in the econ literature. More on that later.)

Setting 3: Posted Prices

Same setting as before:

- 1 item to sell
- n buyers, values $v_i \sim F_i$

Suppose that we are unable to run a centralized auction. Instead, we must interact with each bidder individually in sequence. Think: buyers arriving over time to decide whether or not to buy a car you have listed for sale.

We can offer each buyer a take-it-or-leave-it price when they arrive. If a buyer leaves, they don't return. If a buyer decides to buy, then we must turn away all subsequent buyers.

Claim: there is a way to set prices so that the expected revenue is at least $\frac{1}{2}E[Rev(OPT)]$, half the expected revenue of the Myerson optimal auction.

We'll prove this claim using *prophet inequalities.*

Aside: prophet inequality

Game: we are given n distributions $G_1, \ldots, G_n$

- in stage (i), we observe prize $\pi_i \sim G_i$.
- Can accept prize and end game, or reject prize and go to stage $(i + 1)$.
- Goal: maximize the accepted prize.

Example: $G_i \sim U[0,1]$

- if $n = 2$: continue iff 1st draw is $< 1/2$
- if $n = 10$: have some threshold for continuing, which decreases over time.

Claim (Prophet Inequality): For all $G_1, \ldots, G_n$, there is a threshold strategy t, which simply accepts the first prize greater than t, such that the expected reward of the t-threshold strategy is at least $\frac{1}{2} E_{\pi_i \sim G_i}[\max_i \pi_i]$.

Moreover, one threshold that achieves this guarantee is $t = \frac{1}{2} E_{\pi_i \sim G_i}[\max_i \pi_i]$.

Applying the Prophet Inequality

Recall our claim: it is possible to achieve at least half of the optimal revenue (over all possible auctions) using posted prices.

Want to approximate $E[Rev(OPT)] = E[\max_i \phi_i(v_i)^+]$. Consider virtual values as prizes in the prophet inequality!

Posted Price Mechanism:

- Choose $t = \frac{1}{2} E[\max_i \phi_i(v_i)^+]$
- Offer bidder i posted price $\phi_i^{-1}(t)$

Claim: For regular distributions, the expected revenue of this posted price mechanism is at least $\frac{1}{2} E[Rev(OPT)]$

Proof: By regularity, offering posted price $\phi_i^{-1}(t)$ is a threshold strategy of t for virtual values. By prophet inequality, this yields expected virtual value at least $\frac{1}{2} \cdot E[\max_i \phi_i(v_i)^+]$. By Myerson's characterization, this means the expected revenue is at least $\frac{1}{2} E[Rev(OPT)]$. $\square$

Note: We assume the seller knows the distribution for each individual bidder, and can offer each one a different price. For example: student discounts. This is called *3rd-degree price discrimination.* In contrast, 1st-degree price discrimination is a situation where the seller knows the exact value of each buyer, rather than a distribution. There's also 2nd-degree price discrimination, which we'll discuss next.

Discussion: downsides to price discrimination? Sequential posted pricing vs. auctions?

Applications: airline tickets, theatre tickets, supermarkets, eBay, ...

More on Price Discrimination

Sometimes it is not possible to do 3rd-degree price discrimination, either due to regulations or because we don't know the types of individual buyers.

In these cases, it can still be possible to price discriminate by modifying the goods for sale. E.g., bulk discounts, selling goods of varying quality tiers. This is known as *2nd-degree price discrimination.*

Example: We are selling airline tickets. Our plane is filled with beautiful, comfortable, first-class-style seats.

There are two types of customers: high-value (H) and low-value (L). Each customer has a value for getting a seat, as follows:

- $v_H = \$1,000$
- $v_L = \$500$

Suppose there are 10 high types and 10 low types, and our plane is large enough to let everyone fly. But we don't know which customers are which type. If we have to post a single price and offer it to all customers, then posting a price of $\$1,000$ gives us revenue

$10,000, and posting a price of $500 again gives us $10,000, so the most revenue we can obtain is $10,000.

But now suppose we take half the seats in the plane and make them less comfortable. The values of customers for these damaged seats are:

- $v_H = \$400$
- $v_L = \$300$

If we set prices $p_1 = \$900$ for the good seats and $p_2 = \$300$ for the bad seats, then the high types choose the good seats and the low types choose the bad seats. Our total revenue is $9000 from the good seats and $3000 from the bad seats, for a total of $12,000, which is higher than before we damaged the goods.

Discussion: advantages and disadvantages of this type of (indirect) price discrimination? Other types of indirect price discrimination? Policies to affect this type of practice, and their implications?

Setting 4: Multiple Items

The setting:

- n items to sell
- 1 buyer, values $v_i \sim F_i$ for the items
- additive valuation over bundles: $v(\text{bundle B}) = \sum_{i \in B} v_i$

Question: Items don't "interact." Should we monopoly price each item?

Surprisingly, the answer is NO!

Example: Item pricing can be suboptimal. n items, $v_i \sim U[0,1]$.

- item pricing: $p_i = 1/2$ for each i, revenue is $n/4$
- bundle price: sell bundle of all items for $n/2 - \sqrt{n}$, revenue is approximately $n/2 - \sqrt{n}$.

Example: Bundle pricing can be suboptimal. n items, $v_i = 2^i$ w.prob. 2^{-i}, otherwise 0.

- bundle pricing at price 2^k: probability of sale is at most $\sum_{i \geq k} 2^{-i} < 2^{-(k-1)}$, so revenue is at most 2.
- item pricing: set $p_i = 2^i$, then revenue is $\sum_i 2^i \cdot 2^{-i} = n$.

Claim (Babaioff, Immorlica, Lucier, Weinberg'14): the better of item-pricing and bundle-pricing is at least $1/6 \cdot E[Rev(OPT)]$