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Stable matching

- men $M=\{m_1,\dots,m_n\}$
- women $W=\{w_1,\dots,w_p\}$
- preferences $a \succ_x b$ if ~~a gets~~ x prefers agent a to agent b.
- Preference lists $\succ_m$ ordered list of $W\cup\{m\}$ ($m \succ_m w \Rightarrow$ m prefers being single to marrying w)
(similarly $\succ_w$ list of $M\cup\{w\}$)
- matching $\mu: M\cup W \to M\cup W$ s.t. if $\mu(m)\neq m$ then $\mu(m)\in W$ and $\forall m\in M, w\in W$, $\underbrace{\mu(m)}_{\text{"mate" of } m}=w \Rightarrow \mu(w)=$

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defn. A matching μ is stable if

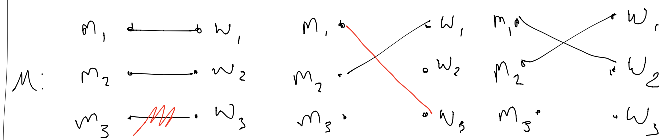
- individually rational $\forall x,\ \mu(x)\succeq_x x$
- no blocking pairs: m,w block μ if $w\succ_m \mu(m)$ and $m\succ_w \mu(w)$.

Example:

$m_1: w_1\succ w_2\succ w_3$ $\quad w_1: m_2\succ m_1\succ m_3$

$m_2: w_2\succ w_1\succ w_3$ $\quad w_2: m_1\succ m_2\succ m_3$

$m_3: w_3$ $\quad w_3: m_1$



IR	✗ ✓	✓ ✓	✓
no blocking pairs?	✓ ✓	(m_1,w_2) ~~(m_1,w_3)~~	✓
stable?	✗ ✓	✗ ✗	✓

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[[Stable matching for market thickness]]

Existence.

Algorithm Deferred Acceptance (DA)

Let $S=M$ be set of currently single men.
Initialize $\mu(w)=w\ \forall w\in W$.
While $\exists\, m\in S$ do:

- m proposes to favorite woman w that hasn't rejected him yet.
- if $m\succ_w \mu(w)$, w accepts m and rejects $\mu(w)$:
  - $S\leftarrow S\cup\{\mu(w)\}-\{m\}$
  - $\mu(w)\leftarrow m$

Example:

$m_1: \cancel{w_1}\succ w_2\succ w_3$ $\quad w_1: m_2\succ \cancel{m_1}\succ \cancel{m_3}$

$m_2: w_1\succ w_2\succ w_3$ $\quad w_2: m_1\succ \cancel{m_2}\succ \cancel{m_3}$

$m_3: \cancel{w_2}\succ \cancel{w_1}\succ w_3$ $\quad w_3: m_1\succ m_2\succ m_3$

$t=0$: $((m_1,m_1),(m_2,m_2),\dots,(w_3,w_3))$
$S=\{m_1,m_2,m_3\}$

$t=1$: $\mu(w_1)=m_1$, $S=\{m_2,m_3\}$
$t=2$: $\mu(w_1)=m_2$, $S=\{m_1,m_3\}$
$t=3$: $\mu(w_2)=m_1$, $S=\{m_3\}$

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$t=4$: $\mu(w_3)=m_3$

Claim: DA computes a stable matching.

Proof.

- Terminates: Each man proposes to each woman $\leq$ once ($\Rightarrow \leq n^2$ steps)
- Stable: IR ✓, Suppose $\exists$ blocking pair (m,w).
  - $w\succ_m \mu(m)\Rightarrow$ m proposed to w
  - $\mu(w)\neq m\Rightarrow$ w rejected m for a man she preferred
  - Women's options only improve $\Rightarrow \mu(w)\succ_w m$

so (m,w) can't block.

Properties

$m_1: w_1\succ w_2$ $\quad w_1: m_2\succ m_1$

$m_2: w_2\succ w_1$ $\quad w_2: m_1\succ m_2$

men-proposing μ: $\{(m_1,w_1),(m_2,w_2)\}$
women- " ν: $\{(m_1,w_2),(m_2,w_1)\}$

Note: all men prefer men-proposing
" women " women "

Claim. ∃ unique men-opt. stable matching and men-proposing DA produces it.

[Note: men have different favorite matchings as they compete, but stability eliminates disagreement.]]

defn. A woman w is acheivable for a man m if ∃ stable μ s.t. $\mu(m) = w$.

Proof. (of claim) We show no man is rejected by a woman who's acheivable for him.

— Let K be 1st step where a man m is rejected by an achievable woman w.

— Then w rejects m for some m' so $m' \succ_w m$.

— Furthermore, m' has not yet been rejected by an achievable w' and proposes in order $\Rightarrow$ $w \succ_{m'} w' \ \forall$ acheivable w'

— Let μ be stable matching with $\mu(m) = w$

— Then (m', w) blocking pair, contradiction

Claim. $\forall$ stable μ, ν, $\mu(m) \succsim_m \nu(m) \ \forall m \in M$ iff $\nu(w) \succsim_w \mu(w) \ \forall w \in W$.

Cor. men-opt stable matching is w-pessimal one (and vice versa).

Proof. (of claim)

If $\exists w$, $\mu(w) \succ_w \nu(w)$, then $w \succ_m \nu(m)$ where $m = \mu(w)$ by assumption, so (m, w) blocking pair.

[interpretation: pt to most preferred/least preferred acheivable mate is matching]]

Lattice Structure

defn. Lattice is partially ordered set (poset) where every two elts have a

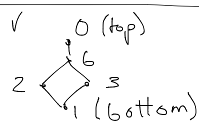
— least upper bound (join) $\wedge$

— greatest lower bound (meet) $\vee$

Complete if all subsets have meet/join

top (bottom) greatest (least) elt.

Example.

Set	order	lattice	complete
$\{1,2,3\}$	$a < b$ iff a divides b.	× 2,3 have no join	
$\{1,2,3,12,18\}$	"	× 2,3 has 2 upper bounds, neither least	
$\{0,1,2,3,6\}$	"	✓ 	✓
		(note $a \wedge b$ is least common mult. $a \vee b$ is greatest " divisor)	
$(0,1)$	usual $<$	✓	×

Claim Set of stable matchings form a distributive complete lattice.

- $\mu > \nu$ if $\forall m$, $\mu(m) \succ_m \nu(m)$
- meet: $\lambda = \mu \vee \nu$ where $\lambda(m)$ = less preferred of $\mu(m), \nu(m)$.
- join $\lambda = \mu \wedge \nu$ where $\lambda(m)$ = more preferred of $\mu(m), \nu(m)$.

Proof.

λ is a matching.

— Suppose $\lambda(m) = \lambda(m') = w$ and $m \neq m'$

— Then $\lambda(m) = \mu(m)$, $\lambda(m') = \nu(m')$

— $w \succ_m \nu(m)$, $w \succ_{m'} \mu(m')$

— if $m \succ_w m' = \nu(w)$ then ν not stable
else if $m' \succ_w m = \mu(w)$ then μ "

λ is stable

— IRV. Suppose ∃ blocking pair (m, w)

— $w \succ_m \lambda(m)$ so $w \succ_m \mu(m)$ and $\nu(m)$

— $m \succ_w \lambda(w)$ where $\lambda(w) = \mu(w)$ or $\nu(w)$
$\Rightarrow \mu$ (or ν) also blocked by (m, w)

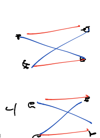
Example

$m_1: w_1 \succ w_2$ $\quad$ $w_1: m_2 \succ m_1$

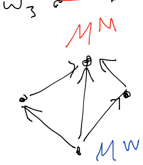
$m_2: w_2 \succ w_1$ $\quad$ $w_2: m_1 \succ m_2$

$m_3: w_3 \succ w_4$ $\quad$ $w_3: m_4 \succ m_3$

$m_4: w_4 \succ w_3$ $\quad$ $w_4: m_3 \succ m_4$



$\{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$



$\{(m_1, w_1), (m_2, w_2), (m_3, w_4), (m_4, w_3)\}$

Random Stable Matchings (9)

— list all & sample: too many

— "random walk" in lattice: takes too long to "mix" [[can get from 1 to next]]

— in fact, #P hard, but algs for some prefs.

Median Stable Matchings

defn. Median stable matching assigns every agent median acheivable mate with repetition.

Example:

$m_1: w_1 \succ w_2 \succ w_3$ $\quad$ $w_1: m_2 \succ m_3 \succ m_1$

$m_2: w_2 \succ w_3 \succ w_1$ $\quad$ $w_2: m_3 \succ m_1 \succ m_2$

$m_3: w_3 \succ w_1 \succ w_2$ $\quad$ $w_3: m_1 \succ m_2 \succ m_3$

$m_4: w_4$ $\quad$ $w_4: m_4$

Acheivable women

m_1:	w_1	w_2	w_3
m_2:	w_2	w_3	w_1
m_3:	w_3	w_1	w_2
m_4:	w_4	w_4	w_4

median

Claim: Median matchings exist and are stable. (10)

Proof. Suppose there are K stable matchings and fix $\ell \in \{1, \dots, K\}$.

— For every set of $N = \binom{K}{\ell}$ stable matchings $\{\mu_1, \dots, \mu_\ell\}$, define $\nu_i = \mu_1 \wedge \cdots \wedge \mu_\ell$.

— Define $\lambda = \nu_1 \vee \cdots \vee \nu_N$.

— Note λ is a stable matching by lattice.

— Consider a woman w.

  — $\forall i$, $\nu_i(w)$ is at best her ℓ^{th} most preferred achievable mate.

  — $\exists i$ (corresponding to w's ℓ favorite μ's) where $\nu_i(w) =$ her ℓ^{th} most preferred mate

— similarly, each man gets $(K-\ell+1)$-best choice

Set $\ell = \lceil K/2 \rceil$ or $\lfloor K/2 \rfloor$ to get median.

Note: Hard to find (can't even compute K!)

Stable Matching Polytope. (11)

Variables $x_{mw} = \begin{cases} 1 & \text{if } \mu(m) = w \\ 0 & \text{otherwise} \end{cases}$

Constraints:

$\forall m, \sum_w x_{mw} \leq 1$
$\forall w, \sum_m x_{mw} \leq 1$ } matching constraints

$\forall (m,w)$ acceptable, $\sum_{m' \succ_w m} x_{m'w} + \sum_{w' \succ_m w} x_{mw'} + x_{mw} \geq 1$

either $\mu(w) \succ_w m$ or $\mu(m) \succ_m w$ or $\mu(m) = w$

Stability constraints

Claim Integral & median exists!

Proof. Consider elt x of polytope, show can be rounded.

— define $\mu(m)$ to be most-preferred w w/ positive weight. (12)

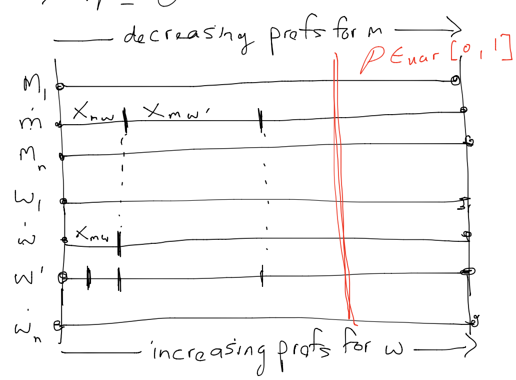
— note $\mu(w)$ is then least preferred m w/ + weight:

$1 \leq \sum_{m' \succ_w m} x_{m'w} + \sum_{w' \succ_m w} x_{mw'} + x_{mw}$

$= \sum_{m'} x_{m'w} - \sum_{m \succ_w m'} x_{m'w}$

$\leq 1 - \sum_{m \succ_w m'} x_{m'w}$

$\Rightarrow \cancel{A} = 0$



(13)

— to get median, define X^i for matching μ^i and $x_{mw} = \frac{1}{k} \sum_i X^i_{mw}$ ($k = \#$ stable μ).

— consider $p = 1/2\ (+/-\ \varepsilon) \Rightarrow$ median since all mates equal weight.

<u>Rural Hospital Thm.</u> Set of men / women unmatched is same at every stable μ.

<u>Proof.</u> Use <u>polytope</u> (V.D.I.)

— for matching μ, let $\mu(M) = \{\text{matched } w\}$
$\mu(W) = \{\text{matched } m\}$

— consider M-opt μ^M + arbitrary μ.

$|\mu^M(M)| \subseteq |\mu(M)|$ (worst for w)

$|\mu^M(W)| \supseteq |\mu(W)|$ (best for m)

— so $|\mu^M(M)| = |\mu(M)|$ + $\mu^M(M) \subseteq \mu(M)$
$\Rightarrow \mu(M) = \mu^M(M)$

— Sim for women.

(14)

<u>Stable Marriage game</u>

<u>Claim</u> DA not strategy proof.

<u>Example.</u>

$m_1 : w_1 \succ w_2$ $\qquad w_1 : m_2 \succ \cancel{m_1}$

$m_2 : w_2 \succ w_1$ $\qquad w_2 : m_1 \succ m_2$

<u>Q.</u> $\exists$ strategyproof mechs?

<u>Claim</u> No stable matching mech. is strategyproof.

<u>Prf.</u> Can change <u>set</u> of stable matchings w/report (see ex).

<u>Claim</u> In fact, always some agent improves by lying when others are truthful. (≥ 2 matchings)

<u>Prf.</u> Suppose w doesn't get $\mu^W(w)$ w/prob one. Let w claim only $\mu^W(w)$ is accept. Then μ^W still stable so by rural hospital, w matched in all stable μ (to $\mu^W(w)$).

(15)

<u>Claim</u> Any M-opt. mech (eg men-proposing DA), is group-strategyproof (GSP) for men.

<u>defn.</u> GSP if no group can alter reports s.t. everyone improves.

<u>Blocking Lemma.</u> Let μ be IR matching + $M' \neq \emptyset$ be men that prefer μ to μ^M. Then $\exists$ blocking pair (m, w) for μ s.t.

- $m \in M - M'$
- $w \in \mu(M')$

<u>Proof.</u>

① $\mu(M') \neq \mu^M(M')$

— Let $w^* \in \mu(M') - \mu^M(M')$

— Then $\mu(w^*) = m$ for $m \in M'$, so $w^* \succ_m \mu^M(m)$

— μ^M stable $\Rightarrow$ $\boxed{\mu^M(w) \succ_{w^*} m}$

— $\mu^M(w^*) = m^* \notin M'$ since $w^* \notin \mu^M(M') \Rightarrow$
$\mu^M(m^*) = \boxed{w^* \succ_{m^*} \mu(m^*)}$

(16)

② $\mu(M') = \mu^M(M') = W'$

— Let w^* be last woman in W' to receive acceptable proposal from a man in M' in M-proposing DA.

— Since men M' prefer μ to μ^M, w^* must have rejected $\mu(w^*) \in M'$

— So w^* engaged to some $m^* \notin M'$ (since w^* last woman to get proposal from M') $\Rightarrow$ $\boxed{m^* \succ_{w^*} \mu(w^*)}$

— and $m^* \notin M' \Rightarrow \mu^M(m^*) \succ_{m^*} \mu(m^*)$
plus m^* rejected by w^* in DA
$\Rightarrow w^* \succ_{m^*} \mu^M(m^*) \Rightarrow \boxed{w^* \succ_{m^*} \mu(m^*)}$

<u>Proof</u> (of GSP). Suppose $\exists$ coalition M' and prefs $\succ'$ for them st. mech outputs μ.

— μ IR + not stable wrt $\succ$

— By blocking lemma, (m, w) w/ $m \notin M'$ + $w \in \mu(M')$ block μ under $\succ$

— $m \notin M' \Rightarrow m$'s prefs didn't change

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Defn. Pref. profile $\{\succ\}$ is a Nash equilibrium (NE) of mech. A if $\forall$ agents x and reports $\succ'_x$,

$$\mu(x) \succeq_x \mu'(x)$$

where $\mu = A(\succ)$ & $\mu' = A(\succ'_x, \succ_{-x})$.

Claim. If μ stable, then $\exists$ NE $\{\succ^{NE}\}$ s.t. M-opt mech. on $\{\succ\}$ outputs μ.

Prf. Let $\succ^{NE}_w = \mu(w)$, $\succ^{NE}_m = \succ_m$

- μ is only stable matching for $\succ^{NE}$, so mech must output μ.
- M-opt mech GSP for men $\Rightarrow$ men can't profitably deviate.
- suppose $\exists w$, $\succ'_w$ s.t. w gets m' in μ'. and $m' \succ_w \mu(w)$
- then $w' = \mu(m')$ now single.
  $\Rightarrow$ $\boxed{m' \succ^{NE}_{w'} \mu'(w')}$
- but $\boxed{\underbrace{\mu(m')}_{= w'} \succ^{NE}_{m'} \underbrace{w}_{= \mu'(m')}}$ by stability of μ.

$\Rightarrow$ μ' not stable wrt $\{\succ'_w, \succ^{NE}_{-w}\}$

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Claim. If $\{\succ^{NE}\}$ is a NE for men-prop. DA where men truthful, then output μ is stable wrt $\succ$.

Prf. If $\exists$ (m,w) block μ wrt $\succ$ then profitable deviation for w is to list m only.

Note: Implies only women w/>1 stable mate can profit by reporting non-truthful prefs!