CS 234r: Markets for Networks and Crowds **Lecture 5**
Discrete Allocation, Pareto Efficiency, RSD, Eating Mechanism, Hylland-Zeckhauser

Discrete Allocation

Lectures 1 & 2: Matching markets

Match *agents to agents*, both sides have preferences.

- Uber: drivers to riders
- Kudu: farmers to traders
- OKCupid: men to women
- NRMP: med students to hospitals
- School Choice: schools to children

Solution properties

- Pareto efficiency, welfare
- No justified envy
- Computable
- Strategyproof for one side

Lectures 3 & today: Allocation

Match *agents to items*, one side has preferences.

- LECTURE EXAMPLE
- FCC Incentive Auction: spectrum from TV stations
- Public Housing: apartments to low-income families
- Organ Transplant: cadaver/donor organs to patients
- Food Banks: food to banks
- School Choice: schools to children

Last time

- money $\rightarrow$ (approximate) welfare

This time

- ??? $\rightarrow$ (approximate) Pareto efficiency

Model

Def: A *discrete allocation problem* has

- set N of n *agents*
- set I of m *items*

Def: Assume unit-demand. *Value* v_i of agent i is

- $v_i : I \rightarrow [0, \infty]$

Def: *Preferences* $\succ_i$ of agent i are

- strict total orders over I

where $a \succ_i b \rightarrow$ iff $v_i(a) > v_i(b)$.

Example: 4 agents, 4 items

values	a	b	c	d
agent 1	100	3	2	1
agent 2	100	99	2	1
agent 3	100	99	98	1
agent 4	100	99	98	97

For every agent i, $a \succ_i b \succ_i c \succ_i d$.

Def: An *allocation* x is a mapping,

- $x_{ia} = 1$, if i gets a
- $x_{ia} = 0$, if i doesn't get a

feasible if $x_{ia} = x_{ja} = 1$ only if $i = j$.

Def: An allocation x is *ex post Pareto efficient* (ex post PE) if there is no allocation y s.t.

- $\sum_{a \in I} v_{ia} y_{ia} \geq \sum_{a \in I} v_{ia} x_{ia}$ for all i
- strictly greater for some i.

Example: For running example, any full assignment ex post PE, e.g.,

allocation	a	b	c	d
agent 1	1	0	0	0
agent 2	0	1	0	0
agent 3	0	0	1	0
agent 4	0	0	0	1

Dictatorships

Algorithm: Serial Dictatorship

Let $I' = I$ and $x_{ia} = 0$ for all i, a.

For $i = 1$ to n,

1. let a be i's favorite item in I'
2. set $x_{ia} = 1$ and $I' \leftarrow I' \setminus a$

Example: computes allocation above for running example.

Claim: ex-post PE and strategyproof.

Question: good solution?

Def: A mechanism satisfies *equal-treatment-of-equals* if identical agents (in terms of input to mech) get same allocation in expectation.

Algorithm: Random Serial Dictatorship (RSD)

Run serial dictatorship with *uniformly random order* of agents.

Note: RSD = DA if schools have no priorities and there's a global random tie-breaking list

Claim: ex post PE, strategyproof, and equal-treatment-of-equals.

Def: a *randomized allocation* is a set of allocations $\{x^1, \ldots, x^k\}$ and a convex combination of them $\{q^1, \ldots, q^k\}$ s.t. $\sum_{l=1}^{k} q^l = 1$

Def: a *lottery* is a probability p_{ia} for each agent i and item a s.t.

- $\sum_{i=1}^{n} p_{ia} = 1$ for all a
- $\sum_{a \in I} p_{ia} = 1$ for all i

(called *bistochastic*).

Claim: (*Birkhoff-von Neumann*) Any lottery can be represented as a randomized allocation, and there's an efficient algorithm to do so.

Example: each agent gets equal chance at each item

- $p_{ia} = 1/4$ for all i, a.
- $E[v_i] = \sum_{a \in I} v_{ia} p_{ia} \approx 25i$

no way to weakly improve everyone.

Def: A lottery is ex ante Pareto efficient (ex ante PE) if there is no lottery q s.t.

- $\sum_{a\in I} v_{ia}q_{ia} \geq \sum_{a\in I} v_{ia}p_{ia}$ for all i
- strictly greater for some i.

Claim: ex ante PE $\rightarrow$ ex post PE

Question: Is RSD ex ante PE?

Example: 8 agents, 8 items

values	a	2 copies of b
4 copies of agent 1	100	1
4 copies of agent 2	100	99

and a remaining 7 items that no one likes.

Agents of type 1: only value a, and to get a, must be first to select

$$E[v_1^{RSD}] = (1/8) \times 100 = 12.5$$

Agents of type 2: value a and b, and to get a or b, must be among first 3 to select

$$E[v_1^{RSD}] \approx (3/8) \times 100 = 37.5$$

Question: Alternative? randomly allocate a to agents of type 1, 2 copies of b to agents of type 2

$$E[v_1^{ALT}] = 100, E[v_2^{ALT}] = 49.5$$

Note: Interpretation is that type 2 agents want to trade probability share of item a for shares of item b

Claim: (*Zhou*) No ex ante PE, equal-treatment-of-equals, and strategyproof mech.

Question: How much worse is RSD?

- for type 1: 8 times worse,

$$E[v_1^{ALT}]/E[v_1^{RSD}] = 8$$

- for type 2: 4/3 times worse,

$$E[v_1^{ALT}]/E[v_2^{RSD}] \approx (1/2)/(3/8) = 4/3$$

Def: A lottery p is *α-approximately ex ante PE* (α-PE) if there is no other lottery q s.t.

$$\sum_{a\in I} v_{ia}q_{ia} \geq \alpha \sum_{a\in I} v_{ia}p_{ia}$$

for all i (where $\alpha \geq 1$).

Question: Good definition?

Claim: RSD not α-PE for any constant (pset).

Hylland-Zeckhauser

Idea: competitive equilibrium from equal incomes (CEEI), each person has equal budget of artificial currency to spend on allocation

Question: allocation is

- items? need unequal budget to have a CE, but then becomes RSD.
- probability share? captures interpretation above.

Algorithm: HZ

1. Input v_{ia}, define artificial budgets B_i
2. Find prices z_a for probability shares of a (probability p of receiving item a costs $z_a p$) and allocation $\{p_{ia}\}$ s.t.
    - demand correspondence

$$\max_{p_{ia}} \sum_{a\in I} v_{ia}p_{ia} \tag{1}$$

subject to

$$\sum_{a\in I} p_{ia}z_a \leq B_i$$
$$0 \leq p_{ia} \leq 1 \tag{2}$$

where indifferent agents buy cheapest bundle

- clear market, i.e., for all items a

$$\sum_{i=1}^{n} p_{ia} = 1$$

- assignment, i.e., for all agents i

$$\sum_{a \in I} p_{ia} = 1$$

Note: Picking cheapest bundle necessary for PE (helps others as those are less valued).

Claim: This is ex ante PE.

Proof: Let $\{p_{ia}\}, \{z_a\}$ be HZ assignment and prices, and $\{p^*_{ia}\}$ be ALT, a Pareto improving lottery.

- let i be agent that strictly prefers ALT:

$$\sum_{a \in I} v_{ia} p^*_{ia} > \sum_{a \in I} v_{ia} p_{ia}$$

- by demand correspondence

$$\sum_{a \in I} p^*_{ia} z_a > B_i \geq \sum_{a \in I} p_{ia} z_a$$

- by market clearing, for all a

$$\sum_{i=1}^{n} p_{ia} = 1 \geq \sum_{i=1}^{n} p^*_{ia}$$

so total money spent in HZ

$$\begin{aligned} \sum_{i=1}^{n} \sum_{a \in I} p_{ia} z_a &= \sum_{a \in I} z_a \sum_{i=1}^{n} p_{ia} \\ &\geq \sum_{a \in I} z_a \sum_{i=1}^{n} p^*_{ia} \end{aligned}$$

is more than total money spent in ALT

- as i spends more in ALT, someone, say i', spends less

$$\sum_{a \in I} p^*_{i'a} z_a < \sum_{a \in I} p_{i'a} z_a$$

but gets as much value since $\{p^*_{ia}\}$ is Pareto improving

- contradicts i' demand correspondence, since i' picks *cheapest* value-maximizing bundle.

□

Question: Compare to RSD?

- computational issues
- incentives
- simplicity

Eating Mechanism

Question: Efficiency with ordinal reports?

Note: Previous example used cardinal values to claim inefficiency

Example: RSD inefficient for all values consistent with ordinal prefs.

- agents $1, 2$: $a \succ b \succ c \succ d$
- agents $3, 4$: $b \succ a \succ d \succ c$

RSD allocation	a	b	c	d
agents 1,2	5/12	1/12	5/12	1/12
agent 3,4	1/12	5/12	1/12	5/12

Question: improvement for *any* values?

RSD allocation	a	b	c	d
agents 1,2	6/12	0	6/12	0
agent 3,4	0	6/12	0	6/12

Def: Lottery $\{p_{ia}\}$ is *ordinally efficiency* if there is no lottery $\{q_{ia}\}$ s.t.

- $\sum_{b:b\succ_i a} q_{ib} \geq \sum_{b:b\succ_i a} p_{ib}$ for all i, a
- strictly greater for some i, a.

Idea: Each item is a pie of unit size

Algorithm: Eating mechanism.

While there's uneaten pie,

- each agent eats at constant speed from favorite remaining pie

Allocation of agent is fraction of pies eaten.

Example: Setting from before

- from time 0 to 1/2,
  - agents 1, 2 eat from pie a
  - agents 3, 4 eat from pie b
- from time 1/2 to 1,
  - agents 1, 2 eat from pie c
  - agents 3, 4 eat from pie d

Claim: Eating mechanism is ordinally efficient

Proof: (*Sketch*)

1. Define graph
   - nodes are items
   - directed edge (a, b) if for some agent i,
     - $a \succ_i b$
     - $p_{ib} > 0$
2. Lottery $\{p_{ia}\}$ is ordinally efficient iff no cycles.
3. HZ lottery has no cycles
   - agents eat from favorite available pie at any given time
   - can be an edge (a, b) only if pie a is finished before pie b

□

Example: But not SP for some values.

values	a	b	c
agent 1	3	2	1
agent 2	3	1	2
agent 3	99	100	0

HZ lottery	a	b	c
agent 1	1/2	1/4	1/4
agent 2	1/2	0	1/4+1/4
agent 3	0	1/2+1/4	1/4

So 3 gets expected value 75.

Suppose 3 claims $a \succ b \succ c$:

HZ lottery w/lie	a	b	c
agent 1	1/3	1/2	1/6
agent 2	1/3	0	1/2+1/6
agent 3	1/3	1/2	1/6

Now 3 gets expected value $50 + 33 = 88 > 75$.

Discussion

What do you think about applying this to our examples from the beginning of class?

- Public Housing: apartments to low-income families
- Organ Transplant: cadaver/donor organs to patients
- Food Banks: food to banks
- School Choice: schools to children