CS234r: Markets for Networks and Crowds
Instructors: Nicole Immorlica and Brendan Lucier
Problem Set #2
Due: **Oct. 6, 2017**

Answer 5 of the following 6 questions.

1. (20 points) This question tests your basic comprehension of the lectures.

   (a) (10 points) **Auction Design**. Recall the example of a matching market from the second lecture: there are 3 items, $\{a, b, c\}$ and 3 agents, $\{1, 2, 3\}$, with valuations given by

   $$v_1(a) = 2, v_1(b) = 3, v_1(c) = 0$$
   $$v_2(a) = 0, v_2(b) = 2, v_2(c) = 4$$
   $$v_3(a) = 0, v_3(b) = 4, v_3(c) = 5.$$

   - What are the VCG outcomes and payments in this market, when all agents report their valuations truthfully? Compare this with the Walrasian Equilibria of the market — what do you notice?
   - We know that truth-telling is one equilibrium of the VCG mechanism, but there can be others as well. Construct a pure Nash equilibrium of the VCG mechanism for this example, in which bidder 1 wins item b and pays 0. Does this bidding behavior seem reasonable? Explain.

   (b) (10 points) **Discrete Allocation**. Consider the following market. There are n schools. School j has 2^{j-1} seats. There are n types of students with 2^n students of each type (so $n2^n$ students in total). Students of type i have value $v_{ij} = 1 + 2^{-100j}$ for schools $j, 1 \leq j \leq i$, and $v_{ij} = 2^{-100j}$ for schools $j, i < j \leq n$.[1]

   - Calculate the expected value of a student of type i in the random serial dictatorship mechanism (RSD).
   - Describe a different lottery that improves each student's expected value by a factor $n/2$.

2. (20 points) In this problem we will study a mechanism for a computationally hard allocation problem. The setting is a combinatorial auction with n bidders and m different items for sale, where each bidder can win at most d items. Bidder valuations are non-decreasing and normalized so that $v_i(\emptyset) = 0$ for all i.

   Our mechanism will use the following allocation rule. Each bidder i specifies a single set S_i of at most d items, as well as a bid $b_i \geq 0$. The bidders are then sorted from highest bid to lowest, breaking ties randomly. For each bidder i in this order, the bidder "wins" and is given set S_i if all items in S_i are still available (i.e., haven't been allocated to another bidder earlier in the order), otherwise they "lose" and are given nothing.

[1] The point of the 2^{-100j} term in the values is to give all students the same total ordering over the options; you can ignore this term in the welfare calculations.

(a) (6 points) Consider the special case where each valuation function v_i is defined by a set T_i and a value a_i, as follows:

$$v_i(S) = \begin{cases} a_i \text{ if } T_i \subseteq S, \\ 0 \text{ otherwise.} \end{cases}$$

In other words, each bidder just has a single set they are interested in, and their value is a_i if they receive that set. For this special case, prove that our auction is "truthful" when paired with an appropriate payment rule, in the sense that each bidder maximizes utility by bidding $S_i = T_i$ and $b_i = a_i$. Describe the payment rule.

(b) (2 points) Give an example showing that this mechanism might generate only a $1/d$ fraction of the optimal welfare, even if the bidders bid truthfully.

(c) (10 points) Now consider general valuation functions, and suppose the auction uses the "pay-your-bid" payment rule: a winning bidder i pays b_i and a losing bidder pays nothing. Prove that this mechanism has price of anarchy (in pure strategies) at most $2d$. (It's possible to get a better bound, but $2d$ suffices for full credit.) HINT: consider the welfare-optimal allocation, and recall the analysis of the GSP auction from lecture.

(d) (2 points) Give an example showing that the price of anarchy of this auction is at least d.

3. (20 points) In this question we will explore the allocative efficiency of a raffle mechanism for school choice. There are m schools. School j has c_j seats. There are n students. Student i has an arbitrary value v_{ij} for school j. The raffle works as follows: there is a bucket B_j for each school j. Each student has a single ticket which he/she can put in a bucket of his/her choice. The mechanism selects c_j tickets at random from bucket B_j and assigns those students to school j. Fix an equilibrium and let n_j be the number of tickets in bucket B_j. We will bound the approximate efficiency of this equilibrium lottery with respect to an arbitrary other lottery, ALT.

(a) (5 points) Suppose that $n_j \geq c_j$ for all j (i.e., the buckets are *congested*). Define a *unit* of school j to be the probability $q_j = \frac{c_j}{n_j}$ of assignment for a ticket in bucket B_j. Note each student is allocated one unit in equilibrium by this accounting. Argue that some student i must be allocated *at most* one unit in ALT.

(b) (10 points) Use the equilibrium condition to argue that student i's expected value in ALT cannot be more than twice her expected value in the equilibrium. HINT: compare her value for a unit of her equilibrium choice school to her value for a unit of any other school.

(c) (5 points) Finally, allow for uncongested buckets by removing the assumption that $n_j \geq c_j$ for all j. Use the equilibrium condition to argue that student i's expected value from ALT, restricted to the uncongested buckets, cannot be more then her expected value in the equilibrium. Combine this with the previous parts to conclude that this raffle mechanism is approximately efficient, in the sense that at least one student is at most 4 times better off in lottery ALT. (It's possible to get a better constant, but 4 suffices for full credit.)

4. (20 points) For this problem, you will participate in a sequence of simulated school-choice markets. For each market, you will choose how to rank the schools on your school application form given your preferences, information about others' preferences, and the mecanism being used. You will also **provide a written description** (1-2 paragraphs each) explaining why you ranked schools the way you did. It is more important to explain your reasoning than to try to find a precisely optimal ranking – in many cases, there is not a single "best" ranking.

   In each example, there are five schools: A, B, C, D, and E. Each school has 20 openings. There are 120 students in the market, including yourself. School preferences are determined by a uniformly random ranking over students, drawn independently for each school. Each student has a cardinal value for being matched to each school, and all values are drawn independently from the uniform distributions below. Your realized values are also listed below. The value for not getting matched to a school is 0.

School	Value Distribution	Your Value
A	$U[10, 20]$	12
B	$U[6, 12]$	11
C	$U[4, 10]$	5
D	$U[3, 7]$	6
E	$U[1, 4]$	3
$\emptyset$	0	0

   (a) (4 points) The deferred acceptance mechanism is being used. You can rank all 5 schools.

   (b) (4 points) Deferred acceptance is being used, but you can rank at most 3 schools.

   In the Boston mechanism, each student first proposes to their favorite school, and each school accepts their favorite students from among those that proposed, up to the school's capacity. These acceptances are then fixed, and accepted students leave the market. Each remaining student then points to their second-favorite school, and schools accept their favorite students from among these new proposals, up to their remaining capacity. (In particular, a school that has already reached capacity will not accept any further students.) This repeats until all remaining students have already proposed to every school on their list.

   (c) (4 points) The Boston Mechanism is being used. You can rank all 5 schools.

   (d) (4 points) The Boston Mechanism is being used, and you can rank at most 2 schools. But now every school's preferences are are determined by a single standardized test. Scores are distributed uniformly in $[0, 100]$. You scored 62.

(e) (4 points) In this case you do not know your preferences for the schools; you only know that your values will be drawn from the distributions listed above. You can choose to interview at one or more schools, which will reveal the value you have for them. Each interview you take costs 2 utility, and you must choose all your interviews up front. After all interviews are complete, a Gale-Shapley mechanism will be used. There is no limit on the number of schools listed, but students can only list schools they interviewed at. Which schools do you choose to interview at? (There is no standardized test — school preferences are uniformly random.)

5. (20 points) Prepare a discussion of Vickrey's *Counterspeculation, Auctions, and Competitive Sealed Tenders*. Summarize the contribution and any models described in this chapter, discuss the key new ideas, and/or suggest some next steps that build upon the paper. Do you like the paper? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations of the paper?

6. (20 points) Prepare a discussion of Roth's Nobel prize lecture on *The Theory and Practice of Market Design*. Summarize the models described in this lecture, discuss the key new ideas, and/or suggest some next steps that build upon the lecture. Do you like the lecture? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations?