CS 234r: Markets for Networks and Crowds **Lecture 3**
Two-Sided Matching, Stability, Deferred Acceptance

Two-Sided Matching

Last time: Matching markets with money

- buyers
- sellers

Prices coordinate market, get

- economic efficiency
- no envy (demand correspondence)

These Walrasian prices are

- computable via tatonnement
- strategyproof for one side (pset)

This time: Matching markets without money

- Marriage markets like OKCupid
- Job markets like National Residency Matching Program (NRMP)

Note: No unit of comparison $\rightarrow$ agents have preference orderings instead of values.

Goal: *What do you think of these?*

- Pareto efficiency: no one can improve without harming others
- No justified envy: if a envies b's match, then b's match prefers b to a
- Algorithmic: polytime alg to find matching
- Strategyproof (SP): reporting true pref's maximizes rank of match

Model

Def: A *two-sided one-to-one matching market* has

- set M of m *men*
- set W of n *women*

Def: *Preferences* $\succ_x$ of agent x are strict total orders over

- $W \cup \{m\}$ for man m
- $M \cup \{w\}$ for woman w

where $a \succ_x b \rightarrow$ agent x prefers a to b and $x \succ_x a \rightarrow$ agent x prefers being unmatched to a.

Def: A *matching* $\mu : M \rightarrow W$ is a one-to-one mapping. Overloading notation, if $\mu(m) = w$, we say $(m, w) \in \mu$ and $\mu(w) = m$.

Def: A matching μ is *Pareto efficient* (PE) if there is no matching ν s.t.

- $\nu(x) \succeq_x \mu(x)$ for all agents x

- and $\nu(x) \succ_x \mu(x)$ for some agent x.

Def: A matching μ is *stable* (aka has *no justified envy*) if it is

- individually rational (IR): x prefers $\mu(x)$ to being single,
- and there is no blocking pair (m, w) s.t. $m \succ_w \mu(m)$ and $w \succ_m \mu(m)$.

Goal: Find a PE and stable μ if it exists.

Example: *Can you find a stable matching here?*

$m_1 : w_1 \succ_{m_1} w_2 \succ_{m_1} w_3$ $\quad$ $w_1 : m_3 \succ_{w_1} m_2 \succ_{w_1} m_1$

$m_2 : w_2 \succ_{m_2} w_3 \succ_{m_2} w_1$ $\quad$ $w_2 : m_1 \succ_{w_2} m_3 \succ_{w_2} m_2$

$m_3 : w_2 \succ_{m_3} w_3 \succ_{m_3} w_1$ $\quad$ $w_3 : m_1 \succ_{w_3} m_3 \succ_{w_3} m_2$

Matching $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ is PE but not stable because (m_3, w_2) are a blocking pair.

Note: When preferences are strict, a stable matching is always PE.

Deferred Acceptance

Algorithm: Tatonnement (Sketch)

1. Buyers increasing prices
2. Sellers tentatively accept highest offer, rejecting others

Buyers' options get worse, sellers' get better.

Algorithm: Men-proposing Deferred Acceptance (Sketch)

1. Men successively offer to marry favorite woman who hasn't rejected them
2. Women tentatively accept best man, rejecting others

Men's options get worse, women's get better.

Note: For convenience, assume complete lists, i.e., $a \succ_x x$ for all agents a.

Algorithm: Men-Proposing Deferred Acceptance (m-DA)

1. Let $\mu(m) = m$ for all $m \in M$.
2. Let S be the set of unmatched men, i.e., $S = \{m : \mu(m) = m\}$.
3. While there's an unmatched man $m \in S$,
   (a) Man m applies to favorite woman w who has not yet rejected him.
   (b) Let $m' = \mu(w)$ be w's current match. If $m \succ_w m'$, w rejects m' ($\mu(m') = m'$) and tentatively accepts m ($\mu(m) = w$).
4. Return matching μ.

Example: For pref's in previous example,

1. $m_1 \to w_1$, $\mu = \{(m_1, w_1)\}$
2. $m_2 \to w_2$, $\mu = \{(m_1, w_1), (m_2, w_2)\}$
3. $m_3 \to w_2$, $\mu = \{(m_1, w_1), (m_3, w_2)\}$
4. $m_2 \to w_3$,
   $\mu = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$

Note: Can run on more complex markets, might not converge to stable outcome.

Theorem 1 *DA computes a stable matching.*

Proof:

- Terminates: each man proposes to each woman at most once.

- Stable: IR since pref's complete. No blocking pairs since,
  - if $w \succ_m \mu(m)$, m proposed to w
  - if w rejected m, it was for m' where $m' \succ_w m$
  - w's options only improve, so $\mu(w) \succ_w m' \succ_w m$

  so (m, w) don't block.

$\square$

Properties

Example: w-DA for running example

1. $w_3 \to m_1$, $\mu = \{(m_1, w_3)\}$
2. $w_2 \to m_1$, $\mu = \{(m_1, w_2)\}$
3. $w_3 \to m_3$, $\mu = \{(m_1, w_2), (m_3, w_3)\}$
4. $w_1 \to m_3$, $\mu = \{(m_1, w_2), (m_3, w_3)\}$
5. $w_1 \to m_2$
   $\mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$

Note: Every person prefers outcome when they propose!

Claim: There's a unique man-optimal stable matching and m-DA finds it. In fact, stable matchings form a lattice (as do WE).

Claim: m-DA is group SP for men.

Example: women's incentives for m-DA, suppose w_1 claims m_1 unacceptable.

1. ..., $\mu = \{(m_2, w_3), (m_3, w_2)\}$
2. $m_1 \to w_1$, $\mu = \{(m_2, w_3), (m_3, w_2)\}$
3. $m_1 \to w_2$, $\mu = \{(m_1, w_2), (m_2, w_3)\}$
4. $m_3 \to w_3$, $\mu = \{(m_1, w_2), (m_3, w_3)\}$
5. $m_2 \to w_1$
   $\mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$

This is a *rejection chain.*

Note: No mech that always outputs stable μ is SP: if x doesn't always get favorite stable partner y, can report y as only acceptable unique.

Claim: Best partner agent can receive is favorite stable partner.

Unique stable partners

Can you identify a restriction on prefs in which stable partners are unique?

Theorem 2 *(Rural hospital or lone wolf theorem.) Set of unmatched agents same at every stable matching.*

Proof:

- for matching μ, let $\mu(M)$ = matched women, $\mu(W)$ = matched men.
- consider μ^M, man-opt SM, and μ, another SM: μ^M is
  - worst for women:
    $$\mu(M) \supseteq \mu^M(M)$$
    so
    $$|\mu(M)| \geq |\mu^M(M)|$$
  - best for men:
    $$\mu(W) \subseteq \mu^M(W)$$
    so
    $$|\mu(W)| \leq |\mu^M(W)|$$

but then, as the number of matched men and women are equal for all matchings, $|\mu(M)| \geq |\mu^M(M)| = |\mu^M(W)| \geq |\mu(W)|$. Hence all cardinalities equal and so set containment relation implies sets are equal as well.

□

Note: Can count stable partners of x by

- have x start rejection chain (i.e., truncate list just above current stable partner)
- stop if single agent receives proposal
- stop if married man runs through his list

Example: 2 men, 3 women, $w_1 \succ_m w_2 \succ_m w_3$, $m_1 \succ_w m_2$

To show $\{(m_1, w_1), (m_2, w_2)\}$ unique, have w_2 truncate at m_2. Then m_2 applies to w_1 and is rejected, then applies to w_3 violating rural hospital theorem.

Claim: If men's lists are random lists of length $k << n$, almost all agents have unique stable partner.

From Immorlica-Mahdian, *Marriage, Honesty, and Stability.*

Reasoning: Balls and bins

First compute $m - DA$:

- women are bins $\rightarrow n$ bins
- men's proposals are balls $\rightarrow k$ balls per man
  - m throws ball into random bin
  - if m rejected, try again up to k times

To see if w has > 1 stable partner, have w reject partner, continue alg., halt if single woman gets proposal $\rightarrow$ probability $1/(\#singles + 1) \approx e^k/n$.

Claim: If lists are random and $|M| = |W|$, then ave. rank of men in w-DA is $O(n/\log n)$ whereas ave. rank of men in m-DA is $O(\log n)$.

Proposing side does much better than receiving side.

Reasoning: Above balls and bins setup, use coupon collector to count load # balls and load in bins once every bin has been hit.

Claim: If lists are random and $|M| = |W| + 1$, then ave. rank of men in w-DA is $\approx$ ave. rank of men in m-DA!

From Ashlagi-Kanoria-Leshno, *Unbalanced Random Matching Markets.*

Discussion

What do you think about applying this to our examples from the beginning of class?

- Marriage markets like OKCupid
- Job markets like National Residency Matching Program (NRMP)