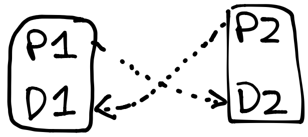
Kidney Exchange:



$\Rightarrow$ graph $G = (V, E)$ — V: patient-donor pairs; $e = (u, v) \in E$ if $\exists$ feasible exchange (pairwise compatible)

Incentives: dichotomous: $\underbrace{v_1 \sim_u v_2 \sim_u \cdots \sim_u v_i}_{\text{compatible}} \succ_u \underbrace{v_{i+1} \sim \cdots \sim v_n}_{\text{incompatible}}$

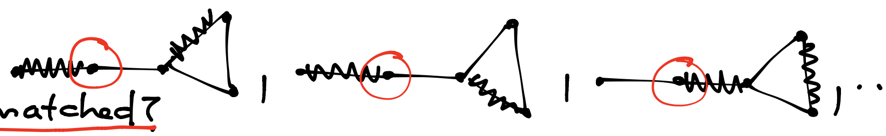
Pareto-eff.:

μ is PE iff $! \exists\, \nu$ s.t. $\nu(u) \succeq_u \mu(u)\ \forall u$ and $\exists u\ \nu(u) \succ_u \mu(u)$

$\Rightarrow$ maximal $\Rightarrow$ maximum!

Q. Max # transplants $\Rightarrow$ maximum matching in non-bipartite graphs

Example:

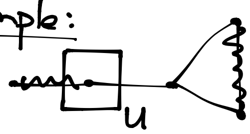


who gets matched?

Tutte-Berge Formula

$$\max_{\text{matchings } \mu} |\mu| = \frac{1}{2} \min_{U \subseteq V} \left(|U| + |V| - \underbrace{odd(G-U)}_{\#\text{ odd comp. in } G-U} \right)$$

Example:



$$\mu = 2 = \tfrac{1}{2}(1 + 5 - 2) \checkmark$$

Intuition: odd must leave someone unmatched unless it can be covered by someone in U. ($\Rightarrow$ max $\le$ min)

Proof: Suppose G connected.

Case 1: $\exists v \in V$ covered by every max matching (eg $\boxed{\cdot}$)

- max μ of $(G - \{v\}) = (\max \mu \text{ of } G) - 1$
- by induction, $\exists\, U'$ for $G - \{v\}$
- Let $U = U' \cup \{v\}$; μ, μ' max matchings in G, $G - \{v\}$

$$|\mu(G)| = |\mu'(G - \{v\})| + 1$$

$$= \tfrac{1}{2}\left(|U'| + |V'| - odd(G - \{v\} - U')\right) + 1$$

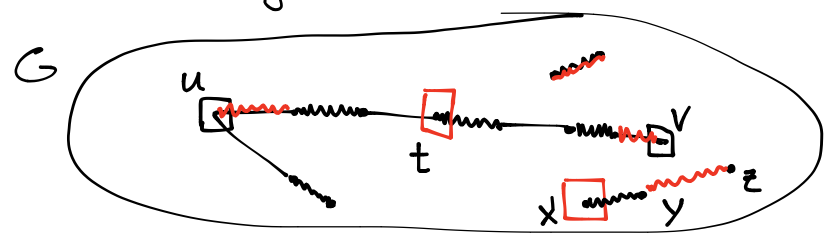
$$= \tfrac{1}{2}\left(|U| \cancel{-1} + |V| \cancel{-1} - odd(G - U)\right) \cancel{+1}$$

Case 2: $\forall v \in V$, some M leaves v uncovered (eg $\triangle$)

claim Each max matching leaves exactly 1 v uncovered.

$\Rightarrow \max|M| = \frac{1}{2}(|V|-1) \Rightarrow |V|$ is odd $\Rightarrow$ Tutte-Berge w/ $U = \emptyset$.

Proof (of claim). By contradiction.



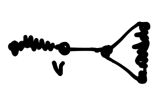
Choose M and uncovered u, v to minimize $d(u,v)$.

- $d(u,v) \geq 2$
- Let t be on shortest path from u to v
- Let ν be matching exposing t whose symmetric diff w/ M is minimized.
- By minimality of $d(u,v)$, ν must cover u, v
- Since $|M| = |\nu|$, $\exists$ x uncovered by ν but covered by M
- Let $y \neq t$ be vertex matched to x in M
- Let $z \neq t, \neq x$ ″ y in ν
- then $\nu - (y,z) + (x,y)$ leaves t exposed & has smaller symmetric difference with M.

defn. Set U satisfying Tutte-Berge w/equality is Tutte-Berge witness set.

Note: Covered in every max M!

Q. Can we find $\underbrace{U}_{\text{useful}}$? (not unique) Yes! Edmonds-Gallai.

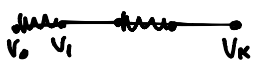


$\left.\begin{matrix} U = \emptyset \\ U = \{v\} \end{matrix}\right\}$ $|M| = 2 = \frac{1}{2}(|V| + |U| - \text{odd}(G-U))$

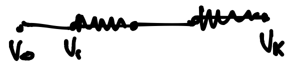
$= \frac{1}{2}(5+0-1) = \frac{1}{2}(5+1-2)$

Max matching (Edmonds-Gallai)

defn. M-alternating path is $(v_0, \dots, v_k)$ s.t. $(v_{2i}, v_{2i+1}) \in M$ or $(v_{2i-1}, v_{2i}) \in M$.



or



defn. M-augmenting if v_0, v_k uncovered by M.

Claim M maximum iff no M-augmenting paths.

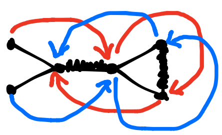
Alg. $M = \emptyset$. While $\exists$ M-augmenting path, augment.

Q. How to find M-augmenting path?

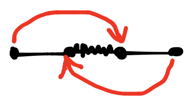
Given G, M, construct $\hat{G}$, directed graph

- vertices are V
- edge $u \to w$ if $(u,v) \notin M$, $(v,w) \in M$ (and $(u,v), (v,w) \in E$)
- X = uncovered vertices

Note: augmenting in $G \to$ path in $\hat{G}$ from X to neighbor of X.
$\leftarrow$ not true due to odd cycles



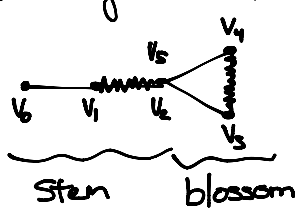
$\leftarrow$ no augmenting paths!



$\leftarrow$ augmenting path

defn. An M-flower is an M-alternating walk $v_0, \dots, v_t$ with

- $v_0 \in X$
- $v_t = v_i$ for some even i
- t is odd

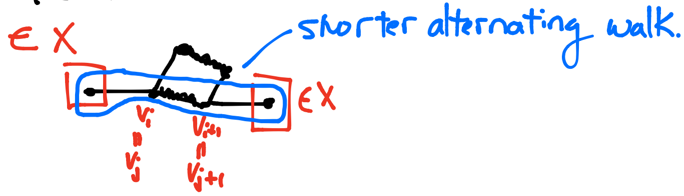


} flower

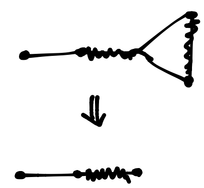
claim If $P = (v_0, \dots, v_t)$ is shortest alternating walk from X to X, then either P is augmenting or $(v_0, \dots, v_j)$ is flower for some $j < t$.

Proof. If $v_0, \dots, v_t$ all distinct, then augmenting. Else let j be smallest index st $v_i = v_j$ for some $i < j$.

- $v_0 \in X$
- j odd since otherwise $(v_{j-1}, v_j) \in M$ but $v_i = v_j$ already covered
- i even since otherwise



Shrinking Blossoms



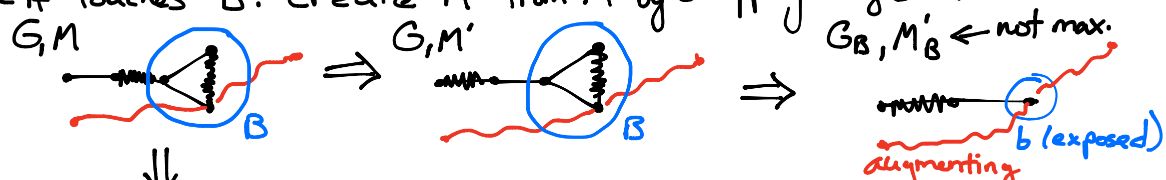
If B blossom, graph G_B w/matching M_B is:
- replace blossom w/single vertex b
- delete self loops / parallel edges
- retain all non-deleted edges from M for M_B

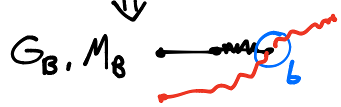
Claim. M maximum in G iff M_B in G_B

Proof.

$\rightarrow$: By contradiction. Suppose M_B not max, M max.
If larger N_B in G_B, use N_B + M's blossom edges in G.

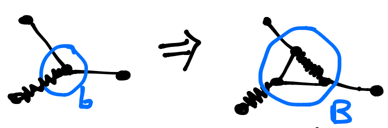
$\leftarrow$:
If M not maximum, then $\exists$ M-augmenting path in G.
- if doesn't touch B, then M_B-augmenting in G_B
- if touches B: create M' from M by swapping edges on stem





so $|M_B| = |M'_B| \Rightarrow M_B$ not max.

Note: N_B max in $G_B \not\Rightarrow$ adding blossom edges creates max N in G



why? (B not an N-blossom)

Alg. MaxMatch(G, M)

While $\hat{G}$ has directed path from X to Neighbors(X)
- find min-length such path
- if corresponding path in G is augmenting, augment.
- else, it's a flower, $N_B = \text{MaxMatch}(G_B, M_B)$
  - if $|N_B| = |M_B|$, return M (max by claim)
  - if $|N_B| > |M_B|$, set $M \leftarrow N$

Runtime: grow matching $\leq n$ times, each time shrink $\leq n$ blossoms and do $O(m)$ work $\Rightarrow O(n^2 m)$.

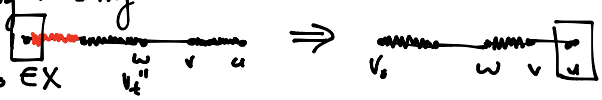
defn. Edmonds-Gallai decomposition.

- N^u = set of v uncovered by some max μ — some self-match
  $= v \in \text{odd}(G-U)$
- N^o = neighbors of N^u — Match here
  = a Tutte-Berge Witness U
- N^p = set of all remaining vertices — all self match
  $= v \in \text{even}(G-U)$

Thm.

Proof

case 1: N^u is independent set (no edges w/in N^u)

- M matching, X exposed vertices
- each $v \in N^o$ covered by M w/say, (u,v), since $v \notin N^u$
- assume $u \notin N^u$ and let w be neighbor of v that's in N^u
- if M exposes w, done:  max $\Rightarrow u \in N^u$
- let M' be matching missing w.
- $M \cup M'$ has path  exposes u + max

So, since N^u independent set,

$$\#\,\text{odd}(G-N^o) \geq |N^u| = |X| + |N^o| = |V| - 2|M| + |N^o|$$

$$\Rightarrow |M| \geq \tfrac{1}{2}\left(|V| + |N^o| - \text{odd}(G-N^o)\right), \text{ hence w/equality}$$

so N^o is Tutte-Berge witness, and N^u = odd components of $G-N^o$

case 2: N^u spans some edge (u,v)

- Let M^u, M^v expose u,v respectively.

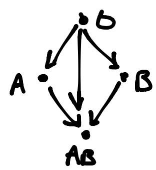
 : path of $M^u \cup M^v$ containing u must end at v or else $\exists$ augmenting path using (u,v)

- So this is an M^u-blossom
- Thus M^u_B max in G_B and $N^u(G_B) = (N^u(G) - B) \cup \{b\}$ since
  - ✓ b exposed by M^u_B as u is exposed by M^u
  - ✓ for each $v \in V - B$, even $X \rightsquigarrow v$ path in G iff even $X_B \rightsquigarrow v$ path in G_B
- Thus $N^o(G_B) = N^o(G)$, $N^p(G_B) = N^p(G)$
- By induction, $N^u(G_B)$ union odd components of $G_B - N^o(G_B)$
  $\Rightarrow N^u(G)$ union odd components ($B \in N^u(G)$ is odd)
- By induction, $|M^u_B| = \tfrac{1}{2}\left(|V_B| + |N^o_B| - \text{odd}(G_B - N^o_B)\right)$

$$\Rightarrow \text{odd}(G_{\cancel{B}} - N^o_{\cancel{B}}) = |V_{\cancel{B}}| - 2|M^u_{\cancel{B}}| + |N^o_{\cancel{B}}| \Rightarrow N^o \text{ Tutte-Burge witness for } G$$

trade b for B $\quad X = X_B \quad N^o_B = N^o$

Kidney Market: ($X-Y$: patient X, donor Y)



underdemanded N^u: $O-Y$, $X-AB$

overdemanded N^o: $AB-Y$, $X-O$

perfectly matched N^p: $A-B$, $B-A$, $X-X$

① priority mech's:

Start w/empty μ.

For agents in order $i = 1$ to n,

if $\exists\ \mu'$ matching $\{j \in \mu\} \cup \{i\}$

set $\mu = \mu'$

* incentives to fully reveal, gives max (PE) μ (pset)

② egalitarian mech's:

"max utility of poorest agent" (in expectation in lottery)

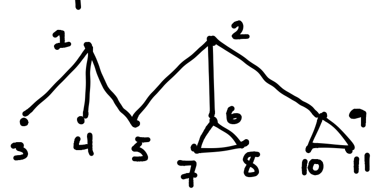
Q. Best guarantee to worst agent?

$\mathcal{J} \subseteq$ odd, $I \subseteq N^o$, neighbors$(\mathcal{J}, I) \equiv I^{\mathcal{J}}$

- if only I available, match prob. of $u \in \mathcal{J}$ is: $^{*}\ \dfrac{|\bigcup_{J \in \mathcal{J}} J| - (|\mathcal{J}| - |I^{\mathcal{J}}|)}{|\bigcup_{J \in \mathcal{J}} J|}$

Alg: Find $\mathcal{J}$ minimize *, commit available neighbors, repeat

Example:

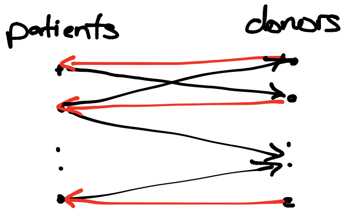


$\Rightarrow (1, 1, 1/2, 1/2, 5/7, \ldots, 5/7)$

Larger Cycles.

Claim. Easy to find max cycle cover w/unbounded length.

proof.



- Perfect matching iff cycle cover.
  (max-weight above "Perfect matching"; max above "cycle cover")
- Black weight 1, red weight 0

  $\Rightarrow$ weight = # covered

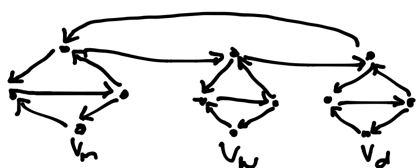
patients point to compatible donors
donors point to patients

Note: Arbitrary length cycles disallowed by doctors.

Claim. NP-hard to find max cover w/3-cycles.

Proof. Hard to find 3D-matching: given disjoint Men, Women, Dogs and compatible triples, find max-size disjoint subset of triples.

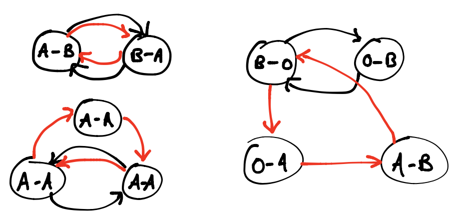
- for each $x \in$ Men $\cup$ Women $\cup$ Dogs make vertex v_x
- for each compatible triple (m, w, d) make gadget:



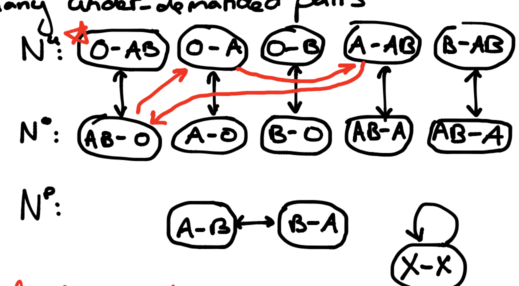
} 3D-matching iff cycle cover

How important are long cycles?

Assumption 1:
no tissue-type incompatibility w/others' donors $\Rightarrow$



Assumption 2:
many under-demanded pairs



Q. Why do 3-cycles help?
- odd # X-X
- AB-O's form 3-cycles w/O-Y, X-AB
- A-B's " w/B-O + O-A/etc.

  (sim. B-A's)

*: disadvantages these.

Q. Which gives most benefit?

Thm. Marginal benefit of 4-cycles $\leq$ # AB-O pairs

very rare

Thm. Mar inal benefit of >4-cycles $= 0$!

} hold approximately if relax assumption 1 in random graph model. [good model?]