Matching w/out \$: ①

Notation:

- $a \succ_i b$ means i prefers a to b.
- $\mu : \{\text{agents}\} \to \{\text{items}\} \cup \emptyset$ is a matching if $\forall$ agents i, j
  $\mu(i) = \mu(j) \Rightarrow i = j$ or
  $\mu(i) = \mu(j) = \emptyset$ (ie i, j unmatched)

Defns:

- μ <u>Pareto efficient</u> if $\nexists\, \nu$ s.t.
  $\forall i,\ \nu(i) \succeq_i \mu(i)$ and
  $\exists i\ \ \nu(i) \succ_i \mu(i)$
- <u>Matching mechanism</u> $M : \{\succ_i\} \to \{\mu\}$ is <u>strategy proof</u> if $\forall i, \succ_i', \succ_{-i}$,
  if $\mu = M(\succ_i, \succ_{-i})$, $\mu' = M(\succ_i', \succ_{-i})$
  then $\mu(i) \succeq_i \mu'(i)$.

Serial Dictatorship (SD) ②

Algorithm:

Fix perm. π of agents (assume identity)
Set Available = {all items}
For $i = 1$ to n,
- set $\mu(i) = a \in$ Available st $\forall b \in$ Available, $a \succ_i b$
- set Available = Available $\setminus \mu(i)$

Return μ.

Claim: SD is Pareto efficient.

Proof: Suppose $\exists\, \nu, i$ s.t. $\forall j\ \ \nu(j) \succeq_j \mu(j)$; $\nu(i) \succ_i \mu(i)$.

ν:			c		a		
	1 2	$\cdots$	j	$\cdots$	i	$\cdots$	n
μ:			a		b		

$\Rightarrow \exists\, j'$ who picked $\mu(j')$ over $\nu(j')$.

Claim: SD is strategyproof. ③

Proof: Available set at step i doesn't depend on i.

Random Serial Dictatorship (RSD)

Algorithm: Run SD for π chosen (uniformly) at random.

Note: RSD is a distribution over SD $\Rightarrow$ PE & strategyproof
(= ex post eff)

Example:

$A, B : a \succ b \succ c \succ d$

$C, D : b \succ a \succ d \succ c$

ex ante allocation

	a	b	c	d
A,B	5/12	1/12	5/12	1/12
C,D	1/12	5/12	1/12	5/12

Randomized Mechs: ④

Notation/Defns.

- Matching $\mu \sim$ <u>Permutation Matrix</u> P

$$\begin{array}{cc} & \text{items} \\ \text{agents} & \\ P = & \left[\ P_{ia} = \begin{cases} 1 & \text{if } \mu(i) = a \\ 0 & \text{oth.} \end{cases}\ \right] \end{array}$$

(Perm P iff all entries 0/1 & exactly one 1 in each row/col)

- <u>Random assignment</u> is a distribution $\alpha : \{\text{Perm } P\} \to [0,1]$ st $\sum_{\text{Perm } P} \alpha_P = 1$.
- <u>Lottery</u> for i is distribution $p : \{\text{items}\} \to [0,1]$ st $\sum_{\text{items } a} p_a = 1$

5

Example:

	a	b	c	d
A	1/2	0	1/2	0
B	1/2	0	1/2	0
C	0	1/2	0	1/2
D	0	1/2	0	1/2

$\equiv M$

Q. Is this implementable by a random assignment?

A. Yes.

$$M = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Defn. Matrix is doubly stochastic if all row/col sums equal one (all entries non-neg)

$\forall i, \sum_j M_{ij} = 1, \forall j \sum_i M_{ij} = 1$

Note: Perm $\underline{P}$ are doubly stochastic.

6

Main Theorem (Birkoff von Neumann)

Any doubly stochastic matrix M can be written as a convex combination of perm. matrices (i.e., it's a random assignment!).

Recall Hall's Thm. Bipartite graph G has a perfect matching iff every set of girls likes at least as large a set of boys.

Proof (of BvN): Induction on # non-0 entries

- Girls = rows, Boys = cols
  edge (i, a) if $M_{ia} > 0$.
- $S \subseteq$ Girls; $T = \{$boys they like$\}$
  Goal (Hall): $|S| \leq |T|$

$$|S| = \underbrace{\sum_{i \in S} \sum_{a \in Boys} M_{ai}}_{\text{M doubly stochastic}} \underbrace{=}_{\text{defn of graph}} \sum_{i \in S} \sum_{a \in T} M_{ai}$$

$$= \sum_{a \in T} \sum_{i \in S} M_{ai} \leq \sum_{a \in T} \sum_{i \in Girls} M_{ai} = |T|$$

- Select perfect matching $\underline{P}$ & let $\delta = \min_{(a,i) \in \underline{P}} M_{ia}$

7

- Let $M^\delta = \left(\frac{1}{1-\delta}\right)\left(M - \delta \underline{P}\right)$

normalization factor; row/col sums $= 1-\delta$ since $\underline{P}$ perm. matrix

$\Rightarrow M^\delta$ doubly stochastic

- Let $M^\delta = \sum_{i=1}^{K} \alpha_i \underline{P}_i$ be decomposition of M^δ (induction)
- Then $M = \sum_{i=1}^{K} (1-\delta)\alpha_i \underline{P}_i + \delta \underline{P}$ is decomposition of M.

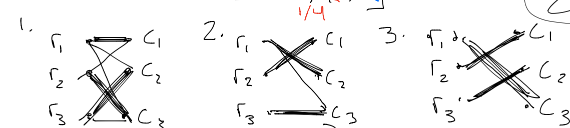
$\left(\sum_{i=1}^{K} (1-\delta)\alpha_i + \delta = (1-\delta)\sum_{i=1}^{K} \alpha_i + \delta = (1-\delta)(1) + \delta = 1 \checkmark \right)$

Q. Base case?

Notes: constructive! decomposition is polynomial!

8

Example. $M = \begin{bmatrix} \cancel{1/2} & \cancel{1/4} & 1/4 \\ 1/4\ \cancel{1/2} & 0 & \cancel{1/2} \\ 0 & \cancel{3/4}\ 1/4 & \cancel{1/4} \end{bmatrix}$

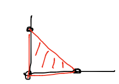


$$\frac{1}{2} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4} \times \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{4} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matching Polytope

defn. Polytope = convex hull of a set of pts. in $\mathbb{R}^n$.

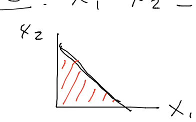
Example $\{(0,0), (0,1), (1,0)\}$



defn. Polyhedron = set of pts in $\mathbb{R}^n$ satisfying a collection of linear inequalities $\{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_j \leq b_j\}$

equivalently $\{x : Ax \leq b\}$ (usually $x \geq 0$)

Example: $x_1 + x_2 \leq 1, x_1, x_2 \geq 0$



Comment: Any bounded polyhedron P is a polytope. Vertex is $x \in P$ that can't be written as a convex comb. of other pts in P.

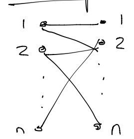
Why polytopes?

Describe solns to comb. opt. problems

Why polyhedra?

Can optimize over them (linear prog)

Example



variables:

x_{ij} for edge (i,j)

$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ in soln} \\ 0 & \text{otherwise} \end{cases}$

Perfect Matching Polytope $P_M = conv(\{x : x \text{ is perfect matching}\})$

Polyhedron $P^* = \{x : \forall i, \sum_j x_{ij} = 1$
$\forall j, \sum_i x_{ij} = 1\}$

Claim: $P_M = P^*$

Proof: $P_M \subseteq P^*$ ✓; $P^* \subseteq P_M$ by BvN ✓

Cor. Alg. for max weight PM:

max $\sum_{ij} w_{ij} x_{ij}$ st $x \in P^*$

Efficient if poly # constraints/vars.

General Theory: $P = \{x : Ax \le b\}$ integral if A TUM.

{Random Assignments} = {doubly-stochastic matrices}

Q. Which RAs do we want?

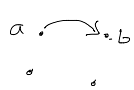
defn RA M is ordinally efficient if $\nexists$ RA M' st. $\forall i, a$

$$\sum_{b \succ_i a} M'_{ib} \ge \sum_{b \succ_i a} M_{ib}$$

stochastic dominance

Note: M' preferred to M for all consistent utilities.

Claim: Given M, define graph G_M:



nodes = items

edges = $\{a \rightsquigarrow b : \exists i$ w/ $a \succ_i b$ and $M_{ib} > 0\}$

Then M is OE iff G_M is acyclic.

Proof:

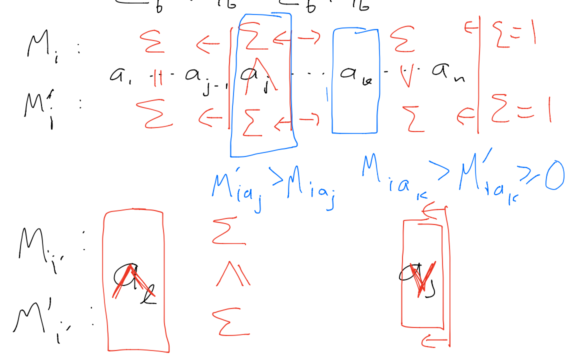
$\Rightarrow$: G_M has cycle $(a_1, \dots, a_k)$, $a_{k+1} = a_1$

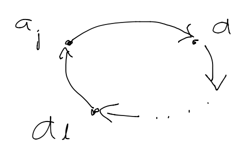
- let i_j be agent who prefers a_j to a_{j+1}
- let $\delta = \min_j M_{i_j a_{j+1}}$
- define $M'_{i_j a_j} = M_{i_j a_j} + \delta$, $M'_{i_j a_{j+1}} = M_{i_j a_{j+1}} - \delta$
- $\forall i_j, \sum_a M'_{i_j a} = 1$, $\forall a_j \sum_i M'_{i a_j} = 1$
- $\forall i, a \sum_{b \succ_i a} M'_{ib} \ge \sum_{b \succ_i a} M_{ib}$, $>$ for $i_j, a = a_{j+1}$

$\Leftarrow$: M not OE, let M' dominate it.

- $\exists i, a$ st. $\sum_{b \succ_i a} M'_{ib} > \sum_{b \succ_i a} M_{ib}$

and $\sum_b M'_{ib} = \sum_b M_{ib}$





Interpretation: No one wants to trade probability shares.

defn. ex ante eff at u if $\nexists M'$ st. $\forall i$, $\sum_a M'_{ia} u(a) \ge \sum_a M_{ia} u(a)$ and strict somewhere.

Homework:

ex ante : wrt u

$\Downarrow$ $\not\Uparrow$

OE : wrt $\succ$

$\Downarrow$ $\not\Uparrow$

ex post : wrt u or $\succ$

Q. How to find OE M?

Hw. Clear cycles.

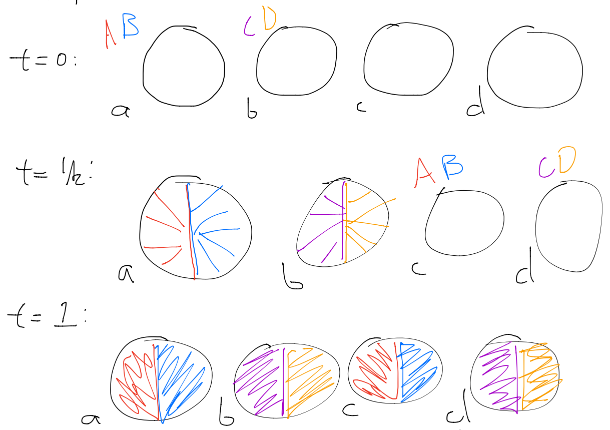
Algorithm: Probabilistic Serial (Eating Mech)

Let $\{w_i(t)\}$ be st $\int_{t=0}^{1} w_i(t)\, dt = 1$.

- n items are pies of size 1
- while pies remain: agents simultaneously eat from fav pie at rates $\{w_i(t)\}$
- output pie shares as RA.

Example $\omega_i(t) = 1$:

$A, B: a \succ b \succ c \succ d$; $C, D: b \succ a \succ d \succ c$

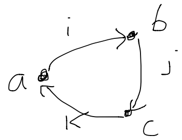


$$M(\omega_i(t) = t) = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

Claim. PS is OE.

Proof. Suppose not.

- Consider cycle in G_M.



Suppose 1^{st} item to finish is b at time s.

- Then $a \succ_i b$ and $M_{ib} > 0$ so i ate b before time s.
- But a was available. Contradiction.

Claim. Any OE RA can be output by PS w/ right choice of $\{\omega_i(t)\}$.

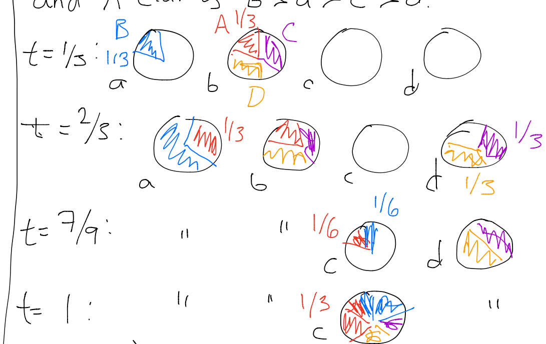
Proof. In reading.

Note: $PS(\omega_i(t) = 1)$ satisfies equal treatment of equals, envy-free...

Claim. OE not strategyproof!

Example. $A, B: a \succ b \succ c \succ d$
$C, D: b \succ a \succ d \succ c$

Suppose $u_A(a) \approx u_A(b) \gg u_A(c) > u_A(d)$ and A claims $b \succ' a \succ' c \succ' d$.

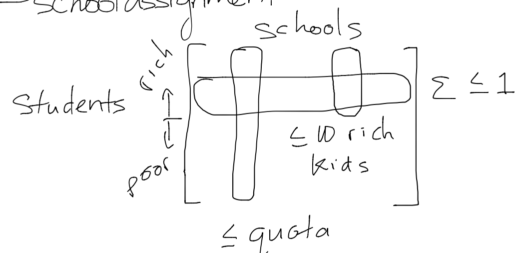


$M_A(\succ) = (1/2, 0, 1/2, 0)$

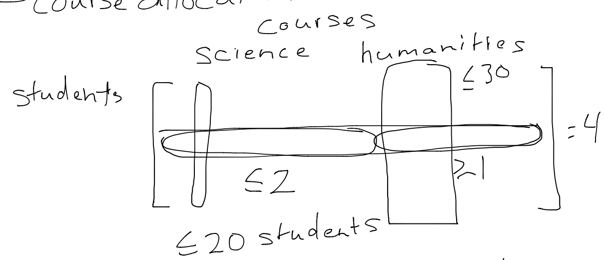
$M_A(\succ'_A, \succ_{-A}) = (1/3, 1/3, 1/3, 0)$

Extensions:

- School assignment



- Course allocation



defn. Constraint set $(\underline{q}_S, \bar{q}_S, S)$ where $S \subseteq$ agents $\times$ items and

$$\underline{q}_S \le \sum_{(i,a)} x_{ia} \le \bar{q}_S$$

Constraint structure $= \{(\underline{q}_S, \bar{q}_S, S)\}$

defn. Constraints are a bihierarchy if $\exists$ partition H_1, H_2 s.t. S in each family is laminar (if 2 sets have non-empty intersection, one is subset of other)

Example
- Standard matching: rows $\cup$ cols
- School assign: rows $\cup$ school quota + diversity constraints
- Course alloc: ....

Claim: All above extends to alloc problems w/ bihierarchy constraints.
- Find OE using PS, "stop" when a constraint goes tight.
- Show "can be rounded" (TUM)

Endowments

Agents $\{a_i\}$, houses $\{h_i\}$
- a_i owns house h_i

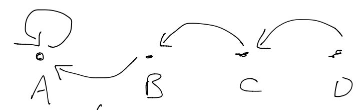
defn Matching M in core if no ν and coalition A of agents s.t. $\forall a \in B$
- $\nu(a) \succsim_a M(a)$
- $\exists a \in A$, $\nu(a) \succ_a M(a)$
- $\nu(a)$ initially owned by some $a' \in A$

Claim Core exists for every housing market.

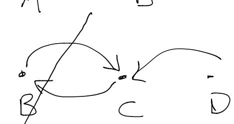
Proof. Top Trading Cycles (TTC)

Alg.
- ask each agent to point to favorite remaining house
- choose cycle, perform trades, remove match
- repeat.

Example: $A: a \succ b \succ c \succ d$
$B: a \succ c \succ b \succ d$
$C: b \succ c \succ d \succ a$
$D: c \succ b \succ d \succ a$

(1) 

$\Rightarrow M(A) = a$

(2) 

$\Rightarrow M(B) = c,\ M(C) = b$

(3) 

$\Rightarrow M(D) = d$

Note. Alg well defined (always $\exists$ a cycle). Why?

Claim: Individually rational (no worse off by participating) [Keeps market thick!] and strategy proof.

Claim: In core

Proof:
- No one in 1^{st} cycle can be in a coalition (all get top choice).
- Suppose no one in i^{th} cycle can be in a coalition.
- Options for people in $(i+1)^{st}$ cycle all available when cycle selected.

Claim Unique

Prf. Consider core matching ν + TTC matching M.
- Let i be 1^{st} agent w/ $M(i) \neq \nu(i)$
- Let C_ℓ be cycle including i.
- Then $\nu(j) = M(j)$ for all j matched before i.
- And $M(j) \succsim_j \nu(j)$ for all $j \in C_\ell$ and $M(i) \succ_i \nu(i)$ (since $\neq$).
- Furthermore $\forall i \in C_\ell$, $M(i)$ initially owned by some $j \in C_\ell \Rightarrow \nu$ not core.

Extension

(dorm assignment)

existing tenants + empty houses.

I-~~get~~-your-house-you-get-my-turn.