CS 234r: Markets for Networks and Crowds **Lecture 5**
Bayes-Nash Equilibria, Revelation Principle, Optimal Auctions

Revenue Maximization

Settings:

- eBay
- Ad auctions
- Cloud pricing

Incomplete Information

Def. An incomplete information game with n players has

- *Common priors* $v_i \sim F_i$ for each player
- *Private values* v_i known only to i
- *Strategy* $s_i(\cdot)$ mapping values v_i to bids b_i
- *Strategy profile* $(s_1, \ldots, s_n)$

Dominant Strategy Equilibrium

One strategy maximizes expected value, regardless of others' strategies.

Notation

For strategy profile $(s_1, \ldots, s_n)$ and values $(v_1, \ldots, v_n)$,

- bid profile $s_{-i}(v_{-i})$ without i is

  $$(s_1(v_1), \ldots, s_{i-1}(v_{i-1}), ?, s_{i+1}(v_{i+1}), \ldots, s_n(v_n))$$

- if i bids b_i, bid profile $(b_i, s_{-i}(v_{-i}))$ becomes

  $$(s_1(v_1), \ldots, s_{i-1}(v_{i-1}), b_i, s_{i+1}(v_{i+1}), \ldots, s_n(v_n))$$

- allocation of i when deviating to b_i is $x_i(b_i, s_{-i}(v_{-i}))$
- payment of i when deviating to b_i is $p_i(b_i, s_{-i}(v_{-i}))$

Def: s Dominant strategy equilibrium if for all i, v_i, and b_{-i}, agent i maximizes utility by bidding $s_i(v_i)$.

Example: Second-price auction game, 2 players

- *Common priors* $v_i \sim U[0, 1]$
- *Strategy* $s_i(v_i) = v_i$ (truthful)

Truthful strategies are dominant strategy equilibrium.

Note: Expected revenue is second-highest value, 1/3

Bayes Nash Equilibrium

Players use knowledge of others' strategies and prior distribution to maximize expected value.

Notation

For strategy profile $(s_1, \ldots, s_n)$,

- expected utility $u(v_i, b_i)$ of type v_i from bidding b_i:

$$E_{v_{-i}}[v_i x_i(b_i, s_{-i}(v_{-i})) - p_i(b_i, s_{-i}(v_{-i}))]$$

Def: s is a BNE if for all i and v_i, assuming that other players bid according to s_{-i}, agent i maximizes expected utility by bidding $s_i(v_i)$:

$$u(v_i, s_i(v_i)) \geq u(v_i, b_i).$$

Example: First price auction game, 2 players.

- *Common priors* $v_i \sim U[0,1]$
- *Strategy* $s_i(v_i) = v_i/2$

These strategies are a BNE, i.e., if $s_2(v_2) = v_2/2$, player 1's optimal bid b_1 is $v_1/2$:

- if $b_1 > 1/2$, win and pay b_1. So best to bid $b_1 \leq 1/2$.
- If $b_1 \leq 1/2$,
  - utility is $Pr[win] \times [v_1 - b_1]$
  - $Pr[win] = Pr[b_2 \leq 2b_1] = 2b_1$
  - utility $= 2b_1(v_1 - b_1) = 2v_1 b_1 - 2b_1^2$
  - first order condition (set derivative equal to zero)

$$2v_1 - 4b_1 = 0 \rightarrow b_1 = v_1/2$$

Note: Expected revenue is half highest value, $1/2 \times 2/3 = 1/3$.

BNE Characterization

Claim: Assume $s_i(\cdot)$ is onto. Let

- probability type v_i wins when following strategy s_i is

$$x_i(v_i) = E_{v_{-i}}[x_i(s_i(v_i), s_{-i}(v_{-i}))]$$

- expected payment of type v_i when following strategy s_i

$$p_i(v_i) = E_{v_{-i}}[p_i(s_i(v_i), s_{-i}(v_{-i}))]$$

Then $(s_1, \ldots, s_n)$ is BNE if and only if

1. monotonicity:

   $x_i(v_i)$ is monotone non-decreasing

2. payment identity:

$$p_i(v_i) = v_i x_i(v_i) - \int_{z=0}^{v_i} x_i(z)dz + c$$

   (where $c = p_i(0)$, the payment of type 0, is usually 0).

Proof: *Sketch.*

- Sufficiency: must check

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(v_i') - p_i(v_i')$$

  Draw picture.

- Necessity of monotonicity: know

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(v_i') - p_i(v_i')$$

$$v_i' x_i(v_i') - p_i(v_i') \geq v_i' x_i(v_i) - p_i(v_i)$$

  adding these, conclude

$$(v_i - v_i')(x_i(v_i) - x_i(v_i')) \geq 0$$

  which implies monotonicity.

- Necessity of payment identity: solve for $p(v_i) - p(v_i')$ in above, get
  - upper bound
  $$v_i(x_i(v_i) - x_i(v_i')) \geq p_i(v_i) - p_i(v_i')$$
  - lower bound
  $$p_i(v_i) - p_i(v_i') \geq v_i'(x_i(v_i) - x_i(v_i'))$$

  Draw picture.

□

Revenue equivalence

Claim: If two auctions have same allocation rule, and zero types pay zero, then they have the same revenue.

Example: Use revenue equivalence to calculate first price equilibrium strategies, 2 players $v_i \sim U[0,1]$

- assume bid $b_i(v_i)$ monotone
- $x_i(v_i) = Pr[v_i \text{ highest}]$, same as second price
- payment of 0 type is 0 in both
- $\rightarrow$ payment in first price = second price
- payment in first price
  $$Pr[v_i \text{ highest}] \cdot b_i(v_i)$$
- payment in second price
  $$Pr[v_i \text{ highest}] \cdot E[\text{second highest} | v_i \text{ highest}]$$
- So $b_i(v_i) = E[\text{second highest} | v_i \text{ highest}]$

Revenue Maximization

The Revelation Principle

Claim: Take any mechanism M with BNE $(s_1, \ldots, s_n)$. Exists $\hat{M}$ in which truthtelling is BNE, and outcome is the same.

Proof. Consider mechanism:

- Agent i bids b_i
- outcome is $M(s_1(b_1), \ldots, s_n(b_n))$.

Check: $\hat{x}_i(v_i) = E_{v_{-i}}[s_i(v_i), s_{-i}(v_{-i})] = x_i(v_i)$, and $\hat{p}_i(v_i) = p_i(v_i)$.

Implications. Only need to consider design of truthtelling mechanisms.

Example: First price auction w/2 players becomes

- allocate to highest bidder
- charge $1/2 \times$bid

Optimal Auction Examples

Example: optimal price, 1 agent, $v \sim U[0,1]$

- opt price p maximizes
  $$p(1-p)$$
- $p = 1/2$

Example: 2 agents, $v_i \sim U[0,1]$

- second price revenue 1/3
- improved revenue?
  - set reserve 1/2

– revenue 5/12

$$\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{2}{3}\right) = \frac{5}{12}$$

Will show this is optimal.

Note: Different priors give different opt mechanisms

- surplus maximized pointwise
- revenue maximized in expectation

Optimal Auction

Recall: payment identity

$$p_i(v_i) = v_i x_i(v_i) - \int_{z=0}^{v_i} x_i(z)dz$$

Idea: use payment identity to write revenue of allocation rule $x(\cdot)$ in terms of distribution

Note: increasing allocation to a type v

- increases revenue of v
- decreases revenue of v' for $v' > v$

Draw picture

Idea: Start from high types, and increase allocation iteratively to $x(\cdot)$, track changes in revenue.

Change in revenue from increasing types $[v, v+dv]$'s allocation from 0 to $x(\cdot)$:

- gain is $vx(v)$ for $F(v+dv) - F(v) = f(v)dv$ types
- loss is $x(v)dv$ for $1 - F(v)$ types
- total is

$$vx(v)f(v)dv - (1-F(v))x(v)dv$$

- integrating over all types and rearranging

$$\int_v \left(v - \frac{1-F(v)}{f(v)}\right) x(v)f(v)dv$$

Def: The *virtual value* of type v drawn from distribution F is

$$\phi(v) = \left(v - \frac{1-F(v)}{f(v)}\right)$$

Claim: $E_v[p(v)] = E_v[\phi(v)x(v)]$

Proof: Recall integration by parts

$$\int_{v=0}^{\bar{v}} g(v)h'(v) = g(v)h(v)|_0^{\bar{v}} - \int_{v=0}^{\bar{v}} g'(v)h(v).$$

Use definition of expectation, sub in payment identity

$$\begin{aligned} E_v[p(v)] &= \int_{v=0}^{\bar{v}} p(v)f(v)dv \\ &= \int_{v=0}^{\bar{v}} [vx(v) - \int_{z=0}^{v} x(z)dz]f(v)dv \\ &= \int_{v=0}^{\bar{v}} vx(v)f(v)dv - \int_{v=0}^{\bar{v}}\int_{z=0}^{v} x(z)dzf(v)dv \end{aligned}$$

Now apply integration by parts with $g(v) = \int_{z=0}^{v} x(z)dz$, $h'(v) = f(v)$

$$\begin{aligned} &= \int_{v=0}^{\bar{v}} vx(v)f(v)dv - \int_{z=0}^{\bar{v}} x(z)dz + \int_{v=0}^{\bar{v}} x(v)F(v)dv \\ &= \int_{v=0}^{\bar{v}} vx(v)f(v)dv - \int_{v=0}^{\bar{v}} x(v)(1-F(v))dv \\ &= \int_{v=0}^{\bar{v}} \left(vx(v)f(v) - x(v)(1-F(v))\right)dv \\ &= \int_{v=0}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v)f(v)dv \\ &= E_v\left[\left(v - \frac{1-F(v)}{f(v)}\right)x(v)\right]. \end{aligned}$$

□

Finding Optimal Auctions

Example: 1 agent, $v \sim U[0,1]$

- $\phi(v) = v - \frac{1-F(v)}{f(v)} = v - (1-v) = 2v - 1$
  Draw picture.
- $x(v) = 0$ for $\phi(v) \leq 0$ (equiv. $v \leq 1/2$)
- $x(v) = 1$ for $\phi(v) > 0$ (equiv. $v > 1/2$)
- $p(v) = 0$ for $v \leq 1/2$, $p(v) = 1/2$ for $v > 1/2$ by payment identity

Question: Are we done? *No! Must check* $x(\cdot)$ *is monotone!*