CS 234r: Markets for Networks and Crowds	Lecture 4
Auctions, Mechanisms, and Welfare Maximization	

Single-Item Auctions

Suppose we have one or more items to sell and a pool of potential buyers. How should we decide who gets which items? What protocol should we follow? This differs from our previous study of two-sided markets, since we are assuming the role of the seller.

We'll start with a simple setting, where we have just one item to sell. Many different procedures (e.g., auctions) can be used. For example:

- English Auction. Ask if anyone is willing to purchase at an initial, low price. If so, call out increasingly higher prices until only one bidder remains. That bidder wins, and pays the last amount called out.
- Dutch Auction. Start with a high price, and slowly lower the offered price until someone says they are willing to buy. That bidder wins at the final price.
- Silent Auction. Each bidder writes down a private bid and passes it to the auctioneer. The highest bid wins and pays their bid.

This lecture: a formal theory for reasoning about different auction formats, using the language of **Mechanism Design**.

Def: A *mechanism* is a protocol between a central authority (the auctioneer) and one or more agents (bidders). The protocol involves one or more rounds of communication between the agents and the central authority. At the conclusion of the protocol, the mechanism outputs an *allocation* and *payments* owed by the agents. The agents are assumed to have valuation functions that assign value to the possible allocations.

For example, an English auction involves many rounds of communication where the auctioneer raises the current price, and agents specify whether they are willing to bid at the current price. The auction concludes with a choice of who wins, and what payment the winner owes. A silent auction has just a single round of communication: the agents send bids to the auctioneer, who then determines the winner and payments. The silent auction is an example of a *direct* mechanism.

Def: In a *direct* (or *direct revelation*) mechanism, each agent declares a valuation, and the mechanism maps those reports to a choice of outcome and payments.

Some notation: we'll tend to write $\mathbf{b} = (b_1, \ldots, b_n)$ for the reports of the agents, to differentiate them from the real valuations $\mathbf{v}$. We write $\mathbf{b}_{-i}$ to mean the reports of everyone except agent i. We also tend to write $x_i(\mathbf{b})$ and $p_i(\mathbf{b})$ for the allocation assigned to agent i, and the payment of agent i, respectively, when agents declare valuations $\mathbf{b}$. The utility of agent i when the agents report $\mathbf{b}$ is the value of the allocation received minus the payment: $u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - p_i(\mathbf{b})$.

Def: A direct mechanism is *dominant strategy incentive compatible* (DSIC) if for each agent i, and for every possible report of the other agents $\mathbf{b}_{-i}$, agent i weakly maximizes utility by reporting $b_i = v_i$. That is, for all possible reports b_i^*,

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b_i^*, \mathbf{b}_{-i}).$$

We sometimes say "truthful" to mean the same thing as DSIC.

Note: if a mechanism is DSIC, then it is optimal for each agent to report their valuation truthfully, regardless of what the other agents are doing. In other words, agents do not need to know anything about the market or the preferences of others in order to bid optimally.

The following important result in auction theory implies that we can focus on DSIC direct mechanisms. This is useful because it means we can restrict our attention to simple mechanisms defined by allocation and payment functions, rather than arbitrarily complex multi-round protocols.

Theorem 1 (Revelation Principle) *Any outcome that can be implemented at equilibrium by some mechanism can be implemented by a DSIC direct mechanism.*

For example, the English auction for a single item is outcome-equivalent to the following *Second-Price Auction*: each agent declares their value to the auctioneer, the agent with the highest declared value wins, and they pay the second-highest declared value. This auction is also known as the *Vickrey Auction.*

Theorem 2 *The Vickrey auction is DSIC and implements the socially efficient outcome.*

This is a special case of a more general theorem we'll prove later for VCG auctions. But it's a good exercise to try to prove it directly.

Note: Why is the second-highest bid the "right" price? Note that this is the minimum amount that the winner could have bid and still won. We call this the *threshold* (or *critical*) payment, and it shows up frequently in the design of truthful auctions.

Note: The Vickrey auction implements the socially efficient outcome even if the auctioneer knows nothing about the bidders' values.

Note: The Vickrey auction isn't the only truthful single-item auction. For example, the auction that ignores the reported valuations and gives away the item to a random agent, for free, is also DSIC (but not socially efficient). Also, there are some allocation rules that cannot be made DSIC; for example, an auction that gives the item to the agent with the *smallest* bid is not truthful, regardless of what payment rule is used (exercise: why?). We'll revisit this question of what allocation rules can be made truthful, when we study revenue-maximizing auctions in a few weeks.

General Auctions

The Vickrey auction is DSIC and implements the socially efficient outcome, for a single-item auction. Can we hope for a similarly nice result in more general settings?

Here's a very general allocation problem setup:

- a space Ω of possible outcomes
- n agents, where agent i has a valuation function $v_i \colon \Omega \to \mathbb{R}_{\geq 0}$ that assigns a value to each outcome

For example, in a single item auction Ω is the set of possible winners of the item. In a combinatorial auction (recall lecture 2), Ω is the set of all partitions of m items among the bidders. Note that this formulation is general enough that an agent's value might depend not only what they win, but also what others win. We can also model public goods in this framework: for example, Ω could be a set of shared outcomes, like a set of possible locations for a bridge, and v_i encodes agent i's value for each possible bridge location. In each of these examples, a mechanism will choose an outcome $\omega \in \Omega$, and also choose payments to collect from each of the agents.

Surprisingly, we can extend the Vickrey auction to cover this extremely general setup, and obtain an auction that is DSIC and implements the socially efficient outcome. This is known as the Vickrey-Clarke-Groves (VCG) mechanism.

Def: The *VCG Mechanism* is a direct mechanism defined as follows. There are n bidders and a space Ω of potential outcomes.

- Agents declare valuation functions $(b_1, \ldots, b_n)$
- The mechanism returns an outcome $\omega^* \in \text{argmax}_{\omega \in \Omega}\{\sum_i b_i(\omega)\}$
- Agent i pays

$$p_i = \left(\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)\right) - \left(\sum_{j \neq i} b_j(\omega^*)\right)$$

Note: The outcome chosen, ω^*, maximizes the social welfare, given the declared valuations.

Note: The payment of agent i is the externality imposed by agent i on the other agents. In other words, it's how much happier the other agents would be if agent i were not present, and we instead selected an outcome that maximized the total welfare enjoyed by everyone other than i.

Example: Suppose Ω is a set of 3 possible bridge locations, $\{A, B, C\}$. There are $n = 3$ agents. The agents report the following valuations:

- $b_1(A) = 10, b_1(B) = 4, b_1(C) = 2$
- $b_2(A) = 3, b_2(B) = 5, b_2(C) = 6$
- $b_3(A) = 1, b_3(B) = 6, b_3(C) = 1$

Then the VCG mechanism would choose location B, since that is the choice that maximizes total declared value. The payments are as follows:

- If agent 1 were not present, the mechanism would still choose B, since (5+6) is bigger than either (3+1) or (6+1). So $p_1 = (5 + 6) - (5 + 6) = 0$.
- If agent 2 were not present, the mechanism would choose A, since $10+1 > 4+6$. So $p_2 = (10 + 1) - (4 + 6) = 1$.
- If agent 3 were not present, the mechanism would again choose A. So $p_3 = (10 + 3) - (4 + 5) = 4$.

Theorem 3 *The VCG mechanism is DSIC and implements the socially efficient outcome.*

Proof: The fact that the VCG mechanism maximizes social welfare follows immediately from truthfulness, since it is defined to maximize declared social welfare. So it suffices to prove DSIC.

Fix i and $\mathbf{b}_{-i}$. Suppose ω^* is the outcome when agent i bids b_i. Then agent i's utility when bidding b_i is

$$v_i(\omega^*) - p_i = v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)$$

The last term in this expression does not depend on b_i. So for agent i, maximizing utility is equivalent to maximizing

$$v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*). \tag{1}$$

But note that if i declares $b_i = v_i$, then the VCG mechanism chooses ω^* precisely to maximize expression (1), over all possible outcomes in Ω. It is therefore utility-maximizing for agent i to declare $b_i = v_i$. □

Critiques of VCG

The VCG mechanism is very general, but has numerous drawbacks — especially in practice. Some common critiques:

1. High communication cost: agents need to express values for every possible outcome, up front.
2. Computational cost of finding the welfare-optimal outcome. For some problems, such as combinatorial auctions, this is NP-hard.
3. Payments are unintuitive and difficult to explain. It's not "obviously" strategyproof.
4. Lack of privacy: auctioneer learns a lot about the agents' preferences.
5. Vulnerable to collusion between bidders.
6. Not "renegotiation proof" — a losing agent might be willing to offer the auctioneer more money than they collected from the winners.

Issues 1 and 2, the complexity of VCG, are central problems in computational economics. A high-level question is: when can a "simple" and approximately-optimal allocation algorithm be made into a DSIC mechanism? We will be exploring this in problem set 2.

Example: For an example of issue 5, collusion, suppose we have two cars for sale, one red and one blue, and two bidders. The first bidder bids \$4 on the red car and \$5 on the blue car, and \$9 on the pair. (I.e., his valuation is additive.) The second bidder bids \$5 on the red car and \$4 on the blue car, and is also additive across the cars. Then the VCG mechanism gives the blue car to the first bidder and the red car to the second bidder, and charges each of them \$4. But if the agents colluded with each other, they could each change their bids to \$0 on the car they like least, and then they would get the same outcome but each pay \$0.

Example: For an example of issue 6, renegotiation, suppose we have two cars for sale, one red and one blue, and three bidders. The first bidder bids \$5 on the red car and the second bidder bids \$5 on the blue car. The third bidder declares that they are willing to pay \$6 to get both cars; but they would not pay anything for either car by itself. Then the VCG mechanism would give the red car to the first bidder and the blue car to the second bidder, and each of them pays \$1. While this is DSIC, the third bidder might be unhappy with this outcome; after all, the seller refused to sell to the third bidder and received \$2 in total, but the third bidder would be willing to pay more than \$2 to get both cars!

Practical Example: Internet Advertising

Advertising space on search engines tends to be sold by auction. This has the advantage that advertisers can bid on keywords that target the users they want to reach, and the advertising platform (e.g., Google, Bing, etc.) does not have to predict how valuable each possible search query is in advance. But what auction rule should be used?

A toy model:

- n bidders (advertisers), k advertising slots.
- advertisers get assigned to slots, and a user might click on one or more of the ads shown. Advertisers want to be clicked.
- slot i has a "click rate" α_i, with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$.
- bidder i has value v_i per click, so the value of being assigned to slot i is $v_i\alpha_i$.
- we can assume $k \geq n$ without loss of generality, by adding slots with click rate 0.

We'll consider direct auctions, where the agents declare values $b_1, \ldots, b_n$. For convenience we'll order the bidders so that $b_1 \geq b_2 \geq \ldots \geq b_n$.

Each of the auctions we consider will return the allocation that maximizes declared social welfare. This allocation assigns each bidder i to slot i (recalling that slots are ordered from highest click rate to lowest, and bidders are ordered from highest bid to lowest).

What payments should be used? We'll consider three options:

- **Pay-your-bid:** bidder i pays b_i per click, for a total expected payment of $b_i\alpha_i$.
- **VCG:** each bidder i pays their externality. (Exercise: calculate the VCG payment of bidder i.)
- **Generalized Second-price Auction (GSP):** bidder i pays b_{i+1} per click, for a total expected payment of $b_{i+1}\alpha_i$.

The GSP auction is commonly used in practice, as an alternative to the first-price payment rule. (Some platforms use VCG instead.) While GSP looks similar to the Vickrey auction, it is not truthful.

Example: Suppose there are 3 slots and 3 bidders. The click rates are $\alpha_1 = 1.0$, $\alpha_2 = 0.9$, $\alpha_3 = 0.8$. The bidders have value-per-click of $v_1 = 20$, $v_2 = 19$, and $v_3 = 5$. If we run a GSP auction and the bidders bid truthfully, then bidder 1 gets the first slot for a value of 20×1.0, and pays 19×1.0, for a total utility of 1. Suppose bidder 1 instead changed her bid to 18. Then she would win the second slot for a value of 20×0.9, but would only pay 5×0.9, for a total utility of 13.5. So bidder 1 would be much better off by bidding 18 instead of truthfully bidding 20.

Does this mean that GSP is a bad auction format? Not necessary. To argue about what happens in a GSP auction, we need to reason about what bidders do in a non-truthful auction. One approach to this is to study equilibria of the GSP auction.

Def: Given valuations $\mathbf{v}$, a set of bids $\mathbf{b}$ forms a *(pure) Nash equilibrium* if for all i and all potential bids b_i^*,

$$u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b_i^*, \mathbf{b}_{-i}).$$

Note that, unlike the definition of DSIC, this definition fixes a particular profile of bids $\mathbf{b}$, rather than quantifying over all possible $\mathbf{b}_{-i}$.

Def: The *Price of Anarchy* (PoA) of an auction is

$$\max_{\mathbf{v}} \max_{\mathbf{b}\text{ a NE}} \frac{\max_{\mathbf{x}^*} \sum_i v_i(x_i^*)}{\sum_i v_i(x_i(\mathbf{b}))}$$

where the max over $\mathbf{x}^*$ is over all possible allocations, and the max over $\mathbf{b}$ is over all Nash Equilibria for valuation profile $\mathbf{v}$.

In other words, the price of anarchy is the worst-case ratio between the optimal social welfare achievable for a valuation profile, and the welfare actually achieved at a worst-case equilibrium. It's important to note that we take a worst case over equilibria, since there may be multiple Nash equilibria and the PoA always considers the worst one.

It turns out that GSP always has a welfare-optimal Nash equilibrium, but may have non-optimal Nash equilibria as well. It also turns out that, under some mild assumptions on the agents' bidding behavior, the price of anarchy of GSP is quite small; the current best-known bound is 1.282. We'll prove something simpler: that the auction with the pay-your-bid payment rule has PoA at most 2.

Theorem 4 *The advertising auction described above with pay-your-bid payment rule has PoA ≤ 2.*

Proof: Fix $\mathbf{v}$ and let $\mathbf{b}$ be a Nash Equilibrium. Order the agents so that $v_1 \geq \ldots \geq v_n$, so that in the optimal allocation agent i wins slot i.

Consider what would happen if bidder i changed their bid from b_i to $b_i^* = v_i/2$.

Case 1: bidder i wins slot i or better. That is, bidder i wins slot $j \leq i$. Then we have

$$u_i(b_i^*, \mathbf{b}_{-i}) = (v_i - v_i/2) \cdot \alpha_j \geq v_i\alpha_i/2.$$

Since $\mathbf{b}$ is a Nash equilibrium, it must be that $u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b_i^*, \mathbf{b}_{-i})$. So we therefore have that

$$u_i(\mathbf{b}) \geq v_i\alpha_i/2. \tag{2}$$

Case 2: bidder i does not win slot i or better. That means that there are at least i other bidders who are bidding $v_i/2$ or more in $\mathbf{b}$. In particular, the bidder who wins slot i under the bid profile $\mathbf{b}$ must be bidding at least $v_i/2$, so is paying at least $v_i\alpha_i/2$. Write $R_j(\mathbf{b})$ for the expected payment of the bidder who wins slot j when the bidders declare $\mathbf{b}$. Then we've argued that $R_i(\mathbf{b}) \geq v_i\alpha_i/2$.

Each bidder falls into either case 1 or case 2. Write W for the set of agents who fall into case 1. Note that

$$\begin{aligned}
&\sum_i v_i(x_i(\mathbf{b})) \\
&= \left(\sum_i v_i(x_i(\mathbf{b})) - \sum_i p_i(\mathbf{b})\right) + \left(\sum_i p_i(\mathbf{b})\right) \\
&= \sum_i u_i(\mathbf{b}) + \sum_i R_i(\mathbf{b})
\end{aligned}$$

which is to say that the total social welfare can be split into the utility obtained by the buyers, and the revenue obtained by the seller. In the second equality, we also used the fact that the total payment of all buyers is the same as the total payment obtained from each slot. But now, using our cases from above, we have

$$\begin{aligned}
\sum_i v_i(x_i(\mathbf{b})) &= \sum_i u_i(\mathbf{b}) + \sum_i R_i(\mathbf{b}) \\
&\geq \sum_{i \in W} u_i(\mathbf{b}) + \sum_{i \notin W} R_i(\mathbf{b}) \\
&\geq \sum_{i \in W} v_i\alpha_i/2 + \sum_{i \notin W} v_i\alpha_i/2 \\
&= \sum_i v_i\alpha_i/2
\end{aligned}$$

which is exactly half of the optimal social welfare. So the welfare obtained at equilibrium is at least half of the optimal social welfare, and hence the PoA is at most 2. □

Critiques of the model and PoA result

- Real advertising auctions are more complicated than this, with advertiser quality scores, ads of different types / sizes, etc.
- This analysis ignores budget constraints of advertisers
- PoA analysis assumes advertisers are playiing at equilibrium. Is this a reasonable assumption?
- The model considers only a single auction instance. In reality, advertisers are participating in many many auctions per day, often against the same competitors over and over. This can influence agent behavior in ways we aren't capturing here. E.g., bidding strategies that try to use up a competitor's budget, or otherwise influence competitor behavior in subsequent rounds.