1

Q. What limits # stable mates?
- correlated prefs
- short pref lists
- market imbalance. (# women $\neq$ # men)

Probabilistic Model (n women/men)

$\mathcal{D}$ = arbitrary fixed dist. over women s.t. $\forall w,\ p_w > 0$

Label women so $p_i \geq p_j \ \forall i \geq j$ (ie order of popularity).

Setting:

men's lists: $\ell_1 \succ_m \cdots \succ_m \ell_k$
- at step i, repeatedly select $w \sim \mathcal{D}$ until $w \notin \{\ell_1, \ldots, \ell_{i-1}\}$
- set $\ell_i = w$; repeat until $i = k$

short lists of size k

women's lists: arbitrary complete lists.

2

$C_k(n) = E[\#\text{ women w/} > 1 \text{ stable mate}]$

Theorem [I.-Mahdian] $\forall$ fixed k,

$$\lim_{n \to \infty} \frac{C_k(n)}{n} = 0$$

Remark: Arbitrary dist. not true

Example. n even

$w_1, \ldots, w_{n/2}: m_1 \succ m_2 \succ \cdots \succ m_n$

$w_{n/2+1}, \ldots, w_n: m_n \succ m_{n-1} \succ \cdots \succ m_1$

Each man m: pick $i \in \{1, \ldots, n/2\}$ uar,
- $w_i \succ w_{n/2+i}$ w/prob 1/2
- $w_{n/2+i} \succ w_i$ "

3

Suppose ≥ 2 men pick i & choose opposite orderings. Let m, m' "earliest men" s.t.

$m: w_i \succ w_{i+n/2}$ $\quad w_i: \cdots \succ m' \succ \cdots \succ m \succ \cdots$

$m': w_{i+n/2} \succ w_i$ $\quad w_{n/2+i}: \cdots \succ m \succ \cdots \succ m' \succ \cdots$

So $w_i, w_{i+n/2}$ each have ≥ 2 stable mates.

$$\text{Prob}[\cdot] = 1 - \sum_{j=0}^{n} \underbrace{\binom{n}{j}\left(\frac{1}{n/2}\right)^j\left(1-\frac{1}{n/2}\right)^{n-j}}_{\text{prob exactly } j \text{ men pick } i} \underbrace{\min\left(1, \left(\tfrac{1}{2}\right)^j \cdot 2\right)}_{\text{prob all choose same } \succ}$$

$$= 1 - \sum_{j=0}^{n} \binom{n}{j}\left(\frac{2}{n}\right)^j\left(1-\frac{2}{n}\right)^{n-j}\left[2 \cdot \left(\tfrac{1}{2}\right)^j\right] + \left(1-\frac{2}{n}\right)^n$$

$$= 1 - 2\sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{n}\right)^j\left(1-\frac{2}{n}\right)^{n-j} + \left(1-\frac{2}{n}\right)^n$$

$$= 1 - 2\left(1-\frac{2}{n}+\frac{1}{n}\right)^n + \left(1-\frac{2}{n}\right)^n$$

$$= 1 - 2\left(1-\frac{1}{n}\right)^n + \left(1-\frac{2}{n}\right)^n = 1 - 2e^{-1} + e^{-2}$$

$\approx 40\%$

4

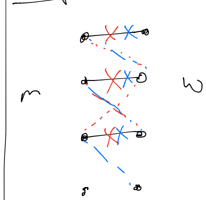
Economic Implications

1. For a random participant x, the prob. truthtelling is not a best response when others are truthful is $o(1)$.

Prf. By thm., random person has ≤ 1 stable partner whp.

2. $\exists$ NE w/ $(1-o(1))$ frac strategies are truthful
3. Truthtelling is a $(1+\varepsilon)$-approx BNE when men/women have uniformly rand. $\succ$.

Rejection Chains



1. ends in a cycle
2. ends in a single person/path

can't be stable! (rural hospital thm.)

Proof Sketch: (5)

① Define rejection chain alg that counts # achievable mates of some w.

② analyze prob. alg outputs > 1 in terms of # of singles.

③ Bound # singles.

Step 2.

Rejection Chain Alg. A

1. Run M-proposing. If w^* unmatched output $\emptyset$.
2. Output $m = \mathcal{M}(w^*)$ as achievable mate. Remove (m, w^*) from matching + set proposer $= m$.
3. If proposer has run through list, terminate. Else let $w =$ fav. woman not yet proposed to. draw from $\mathcal{D}$
4. a. If w has offer she prefers to proposer, she rejects him, continue at 3.
   b. Else w accepts, if $w = w^*$ go to 2, otherwise
   - w previously married $\to$ her mate is suitor, go to 2.
   - w previously single $\to$ terminate.

Claim. Alg outputs all acheivable mates of w in reverse order of pref. (6)

Proof. By induction.

- 1^{st} output M-opt / W-pessimal.
- Suppose output m_i is w's i^{th} worst.
- Consider running M-proposing with w's list truncated just before m_i and order of proposals same as in alg. A
- Then tentative matching same in 2 algs except sometimes w matched in A until w accepts a man (call him m_{i+1}) in M-proposing DA.
- When m_{i+1} accepted, M-proposing terminates (all other women who will get matched have mates since they do in A)

$\Rightarrow m_{i+1}$ her $(i+1)^{st}$ worst mate.

Principle of Deferred Decisions: (7)

"Men only need to select i^{th} favorite woman when asked to propose."

Analyzing expectation $C_K(n)$

Let $X_{w^*} =$ # acheivable mates of w^*.

$$C_K(n) = \sum_w \Pr[X_w > 1]$$

Let $\mathcal{M} = 1^{st}$ (man-opt) matching A finds.

$$S_{\mathcal{M}}(w) = \{\text{single woman } w' \text{ in } \mathcal{M} \text{ w/ } p_{w'} > p_w\}$$
$$= \text{singles more popular than } w.$$

$$\overline{X}_{\mathcal{M}}(w) = |S_{\mathcal{M}}(w)|$$

Note: $X_{w^*} > 1$ iff w^* gets another proposal before women in $S_{\mathcal{M}}(w^*) \Longrightarrow$

$$\Pr[X_{w^*} > 1 \mid \mathcal{M}] \leq \frac{1}{\overline{X}_{\mathcal{M}}(w^*)+1}$$

so

$$\Pr[X_{w^*} > 1] \leq E_{\mathcal{M}}\left[\frac{1}{\overline{X}_{\mathcal{M}}(w^*)+1}\right]$$

Claim (# singles) $\forall\, w > 4K$, (8)

$$E\left[\frac{1}{\overline{X}_{\mathcal{M}}(w)+1}\right] \leq \frac{12e^{8nK/w}}{w}$$

where women named in order of popularity.

Prf. (that $C_K(n) = o(n)$)

$$C_K(n) = \sum_{w \leq \ell} \Pr[X_w > 1] + \sum_{w > \ell} \Pr[X_w > 1]$$
$$\leq \ell + \sum_{w > \ell} \frac{1}{w} 12e^{8nK/w} \qquad (\ell \geq 4K)$$
$$\leq \ell + (n-\ell)\frac{1}{\ell} 12e^{8nK/\ell} \qquad \left(\ell = \frac{16nK}{\ln n}\right)$$
$$\leq \frac{16nK}{\ln n} + n\frac{\ln n}{16nK} 12e^{\frac{1}{2}\ln n}$$
$$= \frac{16nK}{\ln n} + \frac{\ln n}{4K} 3\sqrt{n} = o(n)$$

Pf. (of # singles) (9)

Let $Y_w = |\{w' < w : w' \text{ not on any man's list}\}|$

Note $\forall m, w,\ X_m(w) \geq Y_w$.

Goal: Bound $E\left(\frac{1}{Y_w+1}\right)$.

① $E[Y_w]$ is large.

Let $Q = \sum_{w=1}^{K} p_w$ = total prob. K most popular women.

— $\text{Prob}[w \text{ not listed by } m \text{ as } i^{th} \text{ choice given } w_1, \ldots, w_{i-1}\ 1^{st}\ i-1 \text{ choices}]$

$$= 1 - \frac{p_w}{1-\sum_{j=1}^{i-1} p_{w_j}} \geq 1 - \frac{p_w}{1-Q}$$

$(w > K)$

— $\text{Prob}[w \text{ not listed by any man}]$

$$\geq \left(1 - \frac{p_w}{1-Q}\right)^{nK}$$

(10)

— $p_w \leq \frac{1-Q}{w-k}$ (if remaining mass uniformly distributed)

$$\Rightarrow \Pr[w \text{ not listed}] \geq \left(1 - \frac{1}{w-k}\right)^{nk}$$

$$\geq e^{-4\cancel{2}nk/(w\cancel{-k})}$$

$(K < w/2)$

$$\Rightarrow E[Y_w] = \sum_{w'=1}^{w} \Pr[w' \text{ not listed}]$$

$$\geq \sum_{w'=2k}^{w} e^{-4nk/w'}$$

$$\geq \sum_{w'=w/2}^{w} e^{-4nk/w'} \quad (2K < w/2)$$

$$\geq (w/2)\, e^{-8nK/w}$$

(11)

② $\text{Var}[Y_w]$ is small.

Let E_w = event w not listed by any man.

F_w^m = " " man m.

$$Y_w = \sum_{w'=1}^{w} 1_{E_{w'}}$$

$$\text{Var}[Y_w] = E[Y_w^2] - E^2[Y_w]$$

$$= E\left[\left(\sum_{i=1}^{w} 1_{E_i}\right)^2\right] - \left(\sum_{i=1}^{w} E[1_{E_i}]\right)^2$$

$$= \sum_{i=1}^{w} E[1_{E_i}^{\cancel{2}}] - \sum_{i=1}^{w} E^2[1_{E_i}] \quad \text{neg.}$$

$$+ 2\sum_{i=1}^{w}\sum_{j=1}^{i-1} E[1_{E_i} 1_{E_j}] - 2\sum_{i,j} E[1_{E_i}]E[1_{E_j}]$$

⇐ show this is neg.

$$\Rightarrow \text{Var}[Y_w] \leq E[Y_w]$$

Want: $E[1_{E_i} 1_{E_j}] \leq E[1_{E_i}]\,E[1_{E_j}]$

$\quad\ \ \| \qquad\qquad\qquad \| \qquad\quad \|$

$\Pr[E_i \wedge E_j] \qquad \Pr[E_i]\ \Pr[E_j]$

$\quad\ \ \| \qquad\qquad\qquad \| \qquad\quad \|$

$\Pr[F_i^m \wedge F_j^m]^n \qquad \Pr[F_i^m]^n\ \Pr[F_j^m]^n$

$\quad\ \ \|$

$\left(\Pr[F_i^m \mid F_j^m]\Pr[F_j^m]\right)^n$

(12)

Will argue $\Pr[F_i^m \mid F_j^m] \leq \Pr[F_i^m]$

(ie events are negatively correlated).

Consider perm π of multiset of names Σ containing $\lfloor M p_i \rfloor$ copies of name i for large M.

— F_i^m = π's w/ K distinct names 6/ i

— $F_i^m \mid F_j^m = \sigma$'s of $\Sigma \setminus \{j\}$ " "

— but then each σ in $F_i^m \mid F_j^m$ has corresponding π in F_i^m.

③ Apply Chebyshev.

$$\Pr[Y_g < E[Y_g]/2] \leq \Pr\left[|Y_g - E[Y_g]| > \frac{E(Y_g)}{2}\right]$$

$$\leq \frac{\text{Var}[Y_g]}{(E[Y_g]/2)^2} \leq \frac{4}{E[Y_g]}$$

④ Conclude

$$E\left[\frac{1}{X_m(w)+1}\right] \leq E\left[\frac{1}{Y_w+1}\right]$$

$$\leq \Pr[Y_w < E[Y_w]/2] \cdot 1 + (1-\Pr[\cdot])\frac{1}{E[Y_w]/2+1}$$

$$\leq \frac{4}{E[Y_w]} + \frac{2}{E[Y_w]} = \frac{6}{E[Y_w]} \leq \frac{12}{w} e^{8Kn/w}$$

Small core: unbalanced market. (13)

Setting: n men, $n+k$ women, random complete prefs.

Let $R_M(\mu)$ = average rank of matched men

$R_W(\mu)$ = " " women

$k=0$: $\Pr[R_M(\mu^M) > \frac{3}{2}\ln n] \longrightarrow 0$

$E[R_M(\mu^M)] \leq \ln n$

$E[R_M(\mu^W)] \geq n/\ln n$

$E[C_n(n)] = n$

$k>0$: with high probability:

(i) $\forall$ stable μ:

$R_M(\mu) \leq 3\ln(n/k)$

$R_W(\mu) \geq n/3\ln(n/k)$

(ii) $R_M(\mu^W) \leq (1+o(1))R_M(\mu^M)$

$R_W(\mu^M) \geq (1-o(1))R_W(\mu^W)$

(iii) $C_n(n) \longrightarrow 0$

Prf. $k=0$ (Balanced Markets) (14)

Alg (Deferred Acceptance)

While $\exists$ unmarried man m,

- m proposes to ~~favorite~~ random $w \in W \setminus R(m)$
- let $m' \leftarrow \mu(w)$.
  if $m' \succ_w m$, w rejects m (add w to $R(m)$)
  else w rejects m' (add w to $R(m')$)
  & marries m (set $\mu(w) = m$)

Want: # proposals in random market

Principle of deferred decisions: select i^{th} woman when alg requests it.

Thm. Total # proposals $\leq 10 n\log n$ whp.

Cor. $R_M(\mu^M) \leq \log n$ whp.

Prf. (of thm)

- A woman, once married, stays so
- A woman is married iff she's received a proposal
- Alg stops iff $\forall w$ are married (iff $\forall w$ have received a proposal)

Q. How long until $\forall w$ received a proposal?

Coupon Collector Easy [Skip if following] (15)

- n bins (women), m balls (proposals)
- Want: how big should m be s.t. when throwing balls uar, each bin has a ball whp
- Let E_i = event i^{th} bin empty

$\Pr[E_i] = (1-\frac{1}{n})^m$

- Let E = event some bin empty

$\Pr[E] = \Pr[E_1 \vee \cdots \vee E_n] \leq \sum_{i=1}^{n}(1-\frac{1}{n})^m$

$= n e^{-m/n} \leq \frac{1}{n}$

$\Rightarrow e^{-m/n} \leq \frac{1}{n^2}$, or $-\frac{m}{n} \leq -2\ln n$, or $m \geq 2n\ln n$

Coupon Collector Harder

- X = # balls when last bin hit

$$\left.\begin{array}{l} X_0 = \text{\# balls until } 1^{st} \text{ bin hit} \\ X_1 = \text{\# after } 1^{st} \text{ until } 2^{nd} \text{ bin hit} \\ \vdots \\ X_{n-1} = \text{\# after } (n-1)^{st} \text{ until } n^{th} \text{ hit} \end{array}\right\} \text{epochs}$$

$\therefore X = \sum_{i=0}^{n-1} X_i$

In epoch i, prob hit next bin $p_i = \frac{\cancel{n}-i}{\cancel{n}}$ $\frac{n+k-i}{n+k}$

$E[X_i] = \frac{1}{p_i}$, $E[X] = \sum_{i=0}^{n-1}\frac{\cancel{n}}{\cancel{n-i}} \frac{n+k}{n+k-i} \approx n\ln n \approx n\ln(\frac{n}{k})$

(16)

$\Rightarrow E[R_M(\mu^M)] = \ln n$

$Var[X_i] = \frac{1-p_i}{p_i^2}$

$Var[X] = \sum_{i=0}^{n-1}\frac{1-p_i}{p_i^2} = \sum_{i=0}^{n-1}\frac{i/n}{(\frac{n-i}{n})^2} = \sum_{i=0}^{n-1}\frac{in}{(n-i)^2}$

$\leq \sum_{i=0}^{n-1}\frac{n^2}{(n-i)^2} = n^2\sum_{i=1}^{n}\frac{1}{i^2} \leq 2n^2$

Chebyshev: $\Pr[|X - E[X]| > t] \leq \frac{2Var[X]}{t^2}$

$\Pr[|X - n\ln n| > n\ln\ln n] \leq \frac{2n^2}{n^2(\ln\ln n)^2} \longrightarrow 0$

$\Rightarrow$ Whp, $\frac{\ln n}{2} \leq \ln n - \ln\ln n \leq R_M(\mu)$

$R_M(\mu^M) \leq \ln n + \ln\ln n \leq \frac{3}{2}\ln n$

Random Perm.

Let $r(w)$ = # proposals w received

- $\frac{1}{n}\sum_w r(w) = R_M(\mu^M) \leq \frac{3}{2}\ln n$ whp

$\Rightarrow \Pr_w[r(w) \geq 3\ln n] \leq \frac{1}{2}$

- w marries lowest-ranked man who proposes

$\Rightarrow E_w[R_w(\mu^M)] = E_w\left[\frac{n}{r(w)+1}\right]$

$\geq (1/2)\cdot 1 + (1/2)\frac{n}{3\ln n + 1} \geq \frac{n}{\ln n}$

or Jensen's ineq.

Unbalanced Markets

Note $E[R_M(\mu^M)] = \ln\left(\frac{n}{k}\right)$

Claim: $R_m(\mu) \le 3\ln(n/k)$ $\forall$ stable μ!

Why might this be true?

① suppose $k=1$ + let w be woman unmatched at every stable μ
[[by rural hospital thm]]

② $\Rightarrow$ μ stable for balanced market $W \setminus \{w\}$, M

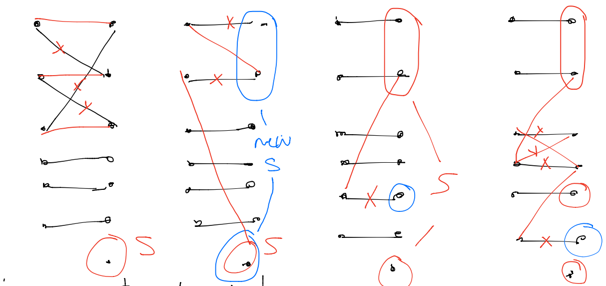
③ so start w/ balanced market, stable μ, add $w \Rightarrow \mu$ stays stable if $R_m(w) > R_m(\mu(w))$ $\forall m$

④ $\Rightarrow R_m(\mu(m))$ must be low.

Prf. Idea

- use rejection chain alg to find stable μ
- calculate prob ends in single woman
  - if n men, $(1+\lambda)n$ women,
    prob[m proposes to w] $\sim 1/n$
    prob[" single] $\sim \frac{\lambda n}{(1+\lambda)n}$

- if n men, $n+1$ women
  bound total rank by # proposals in rejection chain only
  (in improvement phases or IIC.)
  - Let $S = \{w :$ current mate best achievable$\}$
  - When alg finds μ^M, $S \supseteq \{$single $w\}$



improvement phase | terminal phase | terminal phase | terminal w/ internal improv. cycle

- terminal if ends in S.
- can add women from chain in terminal phase to S
- helps us improve bound on # phases
- must be careful of internal improvements

Consider starting rejection chain alg w/ woman w who received most proposals

- $\Pr[m$ picks $w] \approx \Pr[m$ picks single$]$ $= 1/n$.
- phase likely to be terminal since single accepts all proposals, w accepts only if better than the $\log n$ proposals she already received.

What is length of chain (- IIC)?

- $\Pr[\text{end}] \approx \frac{1}{n}$ $\Rightarrow$ # accepted proposals $\approx n$
- $\Rightarrow \sim \sqrt{n}$ after removing IIC (birthday paradox)

Now S large $\Rightarrow$ all remaining phases terminal

Prf. $K=1$

Part I: # proposals in M-proposing DA is $\le 3n\ln n$ whp. ✓

Part II: until end 1st terminal phase whp
- # proposals $\le n \ln n$
- $|S| \ge n^{(1-\varepsilon)/2}$

Part III: until $|S| \ge n^{0.7}$ whp
- # proposals $o(n)$
- # phases $O(n^{0.47})$

Part IV: until alg ends or $\ge 50 n \log n$ proposal
- whp alg ends
- # proposals in improvement / IIC $o(n)$

Part II (21)

① $\Pr[\text{phase ends}] = \Pr[\text{proposal to single } w]$

$\geq \frac{1}{n+1} \geq \frac{1}{2n}$

$\Rightarrow \Pr[\#\text{proposals} > n \ln n] \leq (1-\frac{1}{2n})^{n \ln n}$

$= e^{-\frac{1}{2}\ln n} = \frac{1}{\sqrt{n}}$

② Whp $\forall m, R_m(M^M) \leq 3(\ln n)^2$. ??

(i) $\max_m |\text{Reject}(m)| < n^{\varepsilon}$:

- Say m starts run of proposals when rejected by mate or forcibly divorced.
- Failure if m starts $\geq (\log n)^2$ runs or length of run $\geq (\log n)^3$ proposals
- Starts ≤ 1 run by divorce
- other runs when m' propose to m's mate + she prefers m'
  - m' equally likely to propose to m's mate as to single woman
  - $\Pr[m \text{ starts run}] = 1/2$
    $\Rightarrow \Pr[>(\log n)^2 \text{ runs}] \leq (1/2)^{(\log n)^2} \leq \frac{1}{n^2}$
    $\Rightarrow \Pr[\text{some } m \ \text{"} \ ] \leq n \cdot \frac{1}{n^2} \leq \frac{1}{n}$

(22)

- assuming no failure
  $\#$ proposals m makes $\leq (\log n)^5$
- in each proposal before failure, m picks random $w \in W \setminus \text{Reject}(m)$

$\Pr[\text{accepted}] = \frac{1}{|W \setminus R(m)|} \sum_{w \in W \setminus R} \frac{1}{r(w)+1} \geq \frac{1}{\frac{1}{1\cdot 1} r(1\cdot 1)+1}$

$|W \setminus R(m)| \geq n - 3(\ln n)^2 - (\ln n)^5$
$\geq n/2$

$r(W \setminus R(m)) \leq \#$ proposals so far
$\leq n \ln n + n \ln n = 2n \ln n$

$\Rightarrow \geq \frac{1}{4 \ln n + 1} \geq \frac{1}{5 \ln n}$

- $\Pr[m \text{ makes } > (\log n)^3 \text{ proposals before acc}]$
  $\leq (1 - \frac{1}{5 \ln n})^{(\log n)^3} \leq e^{-(\log n)^2} \leq \frac{1}{n^3}$
- $\Pr[\exists \text{ run where } m \text{ fails}] \leq n^2 \cdot \frac{1}{n^3} = \frac{1}{n}$

$\Rightarrow |R(m)| \leq (\log n)^2 (\log n)^3 = n^{\varepsilon}$ whp.

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② $|S| > n^{(1-\varepsilon)/2}$

Let V = rejection chain

- if $|V| < n^{(1-\varepsilon)/2}$, for next proposal, prob that:
  - goes to $w \in V \leq 1/n^{(1+\varepsilon)/2}$
  - goes to single $\leq \frac{1}{n+1} \leq 1/n^{(1+\varepsilon)/2}$
  - goes to $w \in W \setminus (V \cup \{\text{single}\})$ + w accepts
    $\geq \frac{1}{1\cdot 1} \underbrace{\sum_w \frac{1}{r(w)+1} \geq \frac{1}{\frac{1}{1\cdot 1} r(1\cdot 1)+1}}_{\text{Jensen's inequality}} \geq \underbrace{\frac{1}{2 \log n}}_{\#\text{proposals so far} \sim n \ln n}$
- If $|V| < n^{(1-\varepsilon)/2}$, prob(reach $|V| > n^{(1-\varepsilon)/2}$ before proposal to single + before $\sqrt{n}$ proposals]:
  - proposal to single in next $\sqrt{n}$ w/prob $\leq 1/\sqrt{n}$
  -

$X_i = 1$ if step i doesn't end build

$\Pr[X_i = 1] = \Pr[\neg 1 \wedge \neg 2] = 1 - \Pr[1 \vee 2]$
$\geq 1 - \Pr[1] - \Pr[2] \geq 1 - \frac{2}{\sqrt{n}}$

$E[X] \geq \sum_{i=1}^{\sqrt{n}} (1 - \frac{2}{\sqrt{n}}) \geq \sqrt{n}/2$

$E[|V| \mid \neg\text{end}] \geq \sum_{i=1}^{\sqrt{n}/2} \Pr[|V| \geq i \mid \neg\text{end}]$
$\geq \sum_{i=1}^{\sqrt{n}/2} \left(\frac{1}{2\log n}\right)^i$

$$\frac{1 - \left(\frac{1}{2\log n}\right)^{\sqrt{n}/2+1}}{1 - \left(\frac{1}{2\log n}\right)}$$