CS234r: Markets for Networks and Crowds
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Problem Set #3
Due: **Oct. 20, 2017**

Answer 5 of the following 6 questions.

1. (20 points) This question tests your basic comprehension of the lectures.

   (a) (10 points) **Auction Design**. Consider selling a single item to two bidders. Bidder 1's value is drawn uniformly from the interval $[0, 1]$. Bidder 2's value is drawn uniformly from the interval $[0, 2]$.

   - Describe the allocation and payments of the revenue-optimal auction. You may express your answer graphically.
   - Find all value profiles for which the resulting allocation differs from the efficient allocation. You may again express your answer graphically.

   (b) (10 points) **Pricing**. Again consider selling a single item to two bidders. Bidder 1's value is drawn uniformly from the interval $[0, 1]$, and bidder 2's value is drawn uniformly from the interval $[0, 2]$.

   - Consider the second-price auction with monopoly reserves. Find all value profiles for which the resulting allocation differs from the allocation of the optimal mechanism from part (a). You may express your answer graphically.
   - Calculate the expected revenue of the second-price auction with monopoly reserves.
   - Suppose that instead of running a second-price auction, we instead offer a take-it-or-leave it price to bidder 1, then if she rejects we offer a take-it-or-leave-it price to bidder 2. Compute the optimal prices to offer, and calculate the expected revenue.

2. (20 points) Throughout this problem, assume there are n agents whose values for allocation are drawn IID from $U[0, 1]$.

   (a) (4 points) First consider an auction for an excludable public good like a golf course. There are n agents, and feasible allocations are $0 \leq x_i(v_i, v_{-i}) \leq 1$. The good costs c to produce. This cost of c must be paid by the seller unless $x_i(v_i, v_{-i}) = 0$ for all i. Describe the allocation and payments of the revenue-optimal mechanism.

   (b) (4 points) Exhibit a cost and value profile for $n = 3$ agents in which this mechanism loses money.

   (c) (4 points) Let $c = 0$. Express the expected allocation, $E[\sum_{i=1}^{n} x_i(v_i, v_{-i})]$, and the expected revenue $E[\sum_{i=1}^{n} p_i(v_i, v_{-i})]$ as a function of n, the number of agents.

   (d) (4 points) Now consider an auction for a non-excludable public good like a bridge. There are n agents, and feasible allocations are either

$$x_i(v_i, v_{-i}) = 1 \text{ for all agents } i,$$

or

$$x_i(v_i, v_{-i}) = 0 \text{ for all agents } i.$$

To simplify matters, assume there is no cost for producing the good. Describe the allocation and payments of the revenue-optimal mechanism.

(e) (4 points) Express the expected allocation, $E[\sum_{i=1}^{n} x_i(v_i, v_{-i})]$, as a function of n. Calculate the expected revenue $E[p(v)]$ if there is a single agent, $n = 1$. Then calculate the expected revenue $E[p_1(v_1, v_2) + p_2(v_1, v_2)]$ for $n = 2$ agents.[1] Compare this to the expected allocation and revenue of the excludable public good problem with $c = 0$.

3. (20 points) In this question we consider the lookahead auction for selling a single item to n bidders. (See Algorithm 1 below.) Each bidder i has a value v_i drawn independently from distribution F_i. The auction collects the reported values of all the bidders, and reindexes them so that $v_1 \geq v_2 \geq \ldots \geq v_n$. It then offers the highest bidder, agent 1, a take-it-or-leave-it price p. This price p is the Myerson optimal reserve price for a "perturbed" distribution F_1', which is just F_1 restricted to values greater than v_2.

(For example, if $F_1 = U[0, 1]$ and $v_2 = 0.6$, then F_1' is $U[0.6, 1]$, since this is F_1 restricted to values greater than v_2.)

Let $\mathbf{v} \leftarrow$ { the reported values from the agents };
Index the agents so that $v_1 \geq v_2 \geq \ldots v_n$;
Let F_1' be distribution F_1 restricted to values greater than v_2;
Let p be the monopoly (Myerson) reserve price for distribution F_1';
if $v_1 \geq p$ **then**
  | Allocate to agent 1, who pays p;
else
  | Do not allocate to any agent;
end
Algorithm 1: The Lookahead auction for a single item with value distributions $F_1, \ldots, F_n$.

(a) (2 points) Suppose that there are 3 bidders, and each value distribution is $U[0, 1]$. If the declared bids to the lookahead auction are 0.6, 0.2, and 0.8, which bidder wins and at what price?

(b) (5 points) Prove that the expected revenue of the lookahead auction is at least the expected payment of the highest-valued bidder in the revenue-optimal auction. HINT: it may be easier to prove a stronger claim — the expected revenue of the lookahead auction is at least the expected payment of the highest-valued bidder in ANY other truthful auction.

[1]Extra credit: express the expected revenue as a function of n, the number of agents.

(c) (10 points) Prove that the expected revenue of the lookahead auction is at least the expected payment from all other bidders besides the highest-valued bidder, in the revenue-optimal auction. HINT: show that the expected revenue of the lookahead auction must be at least the expected second-highest bid.

(d) (3 points) Conclude from the previous two parts that the lookahead auction is a 2-approximation to the optimal revenue.

4. (20 points) You are a customer trying to buy computing resources from a cloud provider. Each day, the available resources are sold as CPU units. For simplicity, assume that no customer (including you) can use more than one CPU unit in a day. Each day, you can purchase a CPU unit in one of two ways: you can pay a fixed price set in advance, or you can bid in a spot market where the provider sells any leftover units not sold by fixed prices. If there are k CPU units for sale in the spot market, the top k highest-bidding customers each win one unit, and pay the $(k+1)^{\text{st}}$ highest bid.

   There are 9 other customers, besides you, trying to buy CPU units. The total number of resources available each day is a uniformly random number between 2 and 8. Assume that every other customer has a value drawn uniformly between 0 and 10 for getting a computing resource, each day.

   In each of the following scenarios, describe a strategy for purchasing CPU units (take the fixed price or bid in the spot market, and what to bid in the latter case). Unless otherwise specified, your goal is to maximize expected utility (value minus costs). You will also **provide a written description** (1-2 paragraphs each) explaining why you chose the strategy you did. It is more important to explain your reasoning than to try to find a precisely optimal ranking – in many cases, there is not a single "best" strategy.

   (a) (4 points) You need a single CPU unit, and it must be on a specific day. The fixed price that day is 6. You have value 7 for obtaining the unit.

   (b) (4 points) You need a single CPU unit, and it must be on a specific day. But now you have a choice of 2 different datacenters from which to buy the CPU unit. You don't care which datacenter you get the CPU unit from, but you only get value for one, and each of their spot markets resolve simultaneously. Each datacenter has 9 other customers wanting to buy CPU units, as described above. You have value 10 for obtaining the unit, and the fixed price that day (for each datacenter) is 7.

   (c) (4 points) We are now back in the case of a single datacenter, but now you need 3 CPU units over the course of a week. You can use at most 1 CPU unit each day, so you need to win a unit on at least 3 days that week. Your value is 20 if you get at least 3 units, otherwise your value is 0. The fixed price each day is 6.

   (d) (4 points) This part is the same as the previous part, but now instead of having a value for getting at least 3 units, you are tasked with getting at least 3 units while paying at most your budget, which is 14. If you succeed, you get a payoff of 1; otherwise your payoff is 0. The fixed price each day is 5.

   (e) (4 points) You are now taking the role of the cloud provider. Suppose that all buyers have payoffs similar to the one described in the previous part, where the budget for each buyer is uniformly distributed between 10 and 20. You need to choose a single fixed price, which will be offered each day. Your goal is to maximize revenue – what price do you choose?

5. (20 points) Prepare a discussion of Bulow and Roberts' *The Simple Economics of Optimal Auctions*, Sections I-III. Summarize the contribution and any models described in this paper, discuss the key new ideas, and/or suggest some next steps that build upon the paper. Do you like the paper? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations of the paper?

6. (20 points) Prepare a discussion of Bulow and Klemperer's *Auctions Versus Negotiations*, Introduction and Sections I, IIc, and III. Summarize the models described in this paper, discuss the key new ideas, and/or suggest some next steps that build upon the paper. Do you like the paper? What are the important features of the market(s) being studied? Are the examples and discussions insightful? What are the limitations?