Efficient Partnership Dissolution

Nick Cao

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A partnership is jointly owned by agents $1, \ldots, n$, each of whom own shares $\mathbf{r} = (r_1, \ldots, r_n)$ respectively. Their valuations $\mathbf{v} = (v_1, \ldots, v_n)$ are independently distributed according to a distribution F over $[\underline{v}, \overline{v}]$. Social welfare under the ex-post efficient outcome is given by $W^*(\mathbf{v}) = \max\{v_1, \ldots, v_n\}$.

The expected ex-post efficient social welfare is

$$\mathbf{E}[W^*(\mathbf{v})] = \mathbf{E}[\max\{v_1, \ldots, v_n\}] = n \int_{\underline{v}}^{\overline{v}} vF(v)^{n-1} f(v)dv \tag{1}$$

where the probability density function of the maximum valuation is $nF(v)^{n-1}f(v)$. On the other hand, we have

$$H_i(v_i) = \mathbf{E}[\max\{v_1, \ldots, v_n\}|v_i] - r_i v_i = v_i F(v_i)^{n-1} + (n-1) \int_{v_i}^{\overline{v}} vF(v)^{n-2} f(v)dv - r_i v_i$$

where: (1) the expectation is now conditioned on knowing the value of v_i; (2) evaluating the maximum otherwise remains the same; and (3) an allowance must be made for the non-zero payoff of non-participation. By the fundamental theorem of calculus, we have

$$H_i'(v_i) = F(v_i)^{n-1} + (n-1)v_i F(v_i)^{n-2} f(v_i) - (n-1)v_i F(v_i)^{n-2} f(v_i) - r_i = F(v_i)^{n-1} - r_i$$

At $v_i = v_i^*$, the first order condition stipulates that $H_i'(v_i^*) = 0$. Therefore, we must have

$$F(v_i^*)^{n-1} = r_i \iff v_i^* = F^{-1}(\sqrt[n-1]{r_i}) \tag{2}$$

where F^{-1} can be interpreted as the quantile function of the distribution. Then we may evaluate the function H_i at v_i^* to obtain

$$H_i(v_i^*) = (n-1) \int_{v_i^*}^{\overline{v}} vF(v)^{n-2} f(v)dv \tag{3}$$

The condition for dominant incentive compatible, ex-post budget balanced, interim individually rational, and ex-post efficient partnership dissolution is that $\sum_{i=1}^n H_i(v_i^*) \geq (n-1)\mathbf{E}[W^*(\mathbf{v})]$, which is equivalent to

$$(n-1) \sum_{i=1}^n \int_{v_i^*}^{\overline{v}} vF(v)^{n-2} f(v)dv \geq n(n-1) \int_{\underline{v}}^{\overline{v}} vF(v)^{n-1} f(v)dv \tag{4}$$

Integrating by parts, this condition becomes

$$\sum_{i=1}^n \left(\left[vF(v)^{n-1}\right]_{v_i^*}^{\overline{v}} - \int_{v_i^*}^{\overline{v}} F(v)^{n-1} dv \right) \geq (n-1) \left(\left[vF(v)^n\right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F(v)^n dv \right)$$

Evaluating the terms (note that $F(\overline{v}) = 1$ and $F(\underline{v}) = 0$), we obtain

$$\sum_{i=1}^n \left(\overline{v} - v_i^* F(v_i^*)^{n-1} - \int_{v_i^*}^{\overline{v}} F(v)^{n-1} dv \right) \geq (n-1) \left(\overline{v} - \int_{\underline{v}}^{\overline{v}} F(v)^n dv \right)$$

Then recalling that $F(v_i^*)^{n-1} = r_i$, the condition becomes

$$n\overline{v} + \sum_{i=1}^n \left(-r_i v_i^* - \int_{v_i^*}^{\overline{v}} F(v)^{n-1} dv \right) \geq (n-1)\overline{v} - (n-1) \int_{\underline{v}}^{\overline{v}} F(v)^n dv$$

which further simplifies to:

$$\overline{v} + \sum_{i=1}^n \left(-r_i v_i^* - \int_{v_i^*}^{\overline{v}} F(v)^{n-1} dv \right) + (n-1) \int_{\underline{v}}^{\overline{v}} F(v)^n dv \geq 0 \tag{5}$$

1 Equal Ownership

Here, $r_i = 1/n$ for all $i = 1, \ldots, n$. As ownerships are equal, equation (2) shows that v_i^* must be equal across all partners. The left hand side of (5) thus simplifies to

$$\overline{v} - v_i^* - n \int_{v_i^*}^{\overline{v}} F(v)^{n-1} dv + (n-1) \int_{\underline{v}}^{\overline{v}} F(v)^n dv$$

This is equal to

$$\overline{v} - v_i^* + (n-1) \int_{\underline{v}}^{v_i^*} F(v)^n dv - \int_{v_i^*}^{\overline{v}} \left(nF(v)^{n-1} - (n-1)F(v)^n \right) dv \tag{6}$$

Taking the derivative of $nF(v)^{n-1} - (n-1)F(v)^n$, we obtain

$$n(n-1)F(v)^{n-2}f(v) - n(n-1)F(v)^{n-1}f(v) = n(n-1)F(v)^{n-2}f(v)(1 - F(v)) \geq 0$$

as $F(v) \leq 1$ by the axioms of probability. The maximum point is exactly at $F(v) = 1$, which is where $v = \overline{v}$. Here, $nF(\overline{v})^{n-1} - (n-1)F(\overline{v})^n = 1$. Hence we have shown that $nF(v)^{n-1} - (n-1)F(v)^n \leq 1$ for all $v \in [\underline{v}, \overline{v}]$, and thus we can bound the expression (6) from above by

$$\overline{v} - v_i^* + (n-1) \int_{\underline{v}}^{v_i^*} F(v)^n dv - \int_{v_i^*}^{\overline{v}} dv$$

This is equal to

$$\overline{v} - v_i^* + (n-1) \int_{\underline{v}}^{v_i^*} F(v)^n dv - (\overline{v} - v_i^*)$$

and is equal to

$$(n-1) \int_{\underline{v}}^{v_i^*} F(v)^n dv > 0$$

Hence condition (5) is satisfied and efficient surplus division is possible.

2 Concentrated Ownership

Say if partner 1 currently owns the entire partnership: $r_1 = 1$ and $r_i = 0$ for $i = 2, \ldots, n$. Then according to equation (2), we have that $v_1^* = F^{-1}(1) = \overline{v}$ and $v_i^* = F^{-1}(0) = \underline{v}$ for all $i = 2, \ldots, n$. The left hand side of (5) therefore becomes

$$\overline{v} - v_1^* - (n-1) \int_{\underline{v}}^{\overline{v}} F(v)^{n-1} dv + (n-1) \int_{\underline{v}}^{\overline{v}} F(v)^n dv$$

which simplifies simply to

$$(n-1) \int_{\underline{v}}^{\overline{v}} F(v)^{n-1}(F(v) - 1) dv < 0$$

As the condition fails, it is impossible to dissolve the partnership in a way that is ex-post efficient, ex-post budget balanced, interim individually rational, and Bayesian incentive compatible.