The Simultaneous Truel

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Abstract: The simultaneous truel is a three-person game which is a generalization of the simple duel. The players' positions are fixed and their firing is simultaneous. Each player's only decision is which of his opponents will be his target. The (simultaneous) firing continues until there is at most one survivor or until all survivors have fired a specified number of times. Each player is assumed to be concerned only with his own survival; he is indifferent to the fate of his opponents. These games (parametrized by the maximum possible number of shots by each player) are examined for equilibrium points. It is found that, in many cases, the truel has a unique equilibrium point at which the player who is the poorest marksman has the greatest chance of survival.

The duel is interesting from the game-theoretic point of view as a game of timing. Since each adversary has only one target to shoot at, his choice of strategy is limited to the decision of when to shoot. If the two duelists advance upon each other at a known rate, the probability of scoring a hit increases as the distance between them decreases, and hence as time increases. Given these probabilities, the problem is to determine the optimum time of firing for each party.

When there are more than two adversaries, each against all others, the choice of target enters as an additional strategic consideration. In this paper one imagines three opponents in fixed positions, so that the question of timing does not arise. This truel will be modelled by a three-person non-cooperative game in which the choice of target is the only relevant strategic problem.

The players will be called 1, 2, and 3, with respective markmanship ratings of $1 - a$, $1 - b$, and $1 - c$, where $0 < a \leq b \leq c < 1$. A player's marksmanship rating represents the probability that he hits his chosen target. It is assumed that there is no possibility of a player hitting a target other than the one at which he aims. For $i = 1, 2, 3$, player i's strategies will be denoted j_i and k_i, where $\{i,j,k\} = \{1,2,3\}$. The shooting occurs simultaneously, at a signal. This game, played once, will be called Γ^0, the simultaneous truel. (SHUBIK [1954] has studied a similar game in which the players fire sequentially in one of six equiprobable orders. A game related to SHUBIK's is considered by GARDNER [1958].)

It is easily seen that if Γ^0 is played with survival probabilities taken as payoffs, each player is indifferent between his two available strategies. This is because a player's survival probability does not depend on his own choice of strategy, but only on the choices of the other two players. Therefore, in the $2 \times 2 \times 2$

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normal form representation of Γ^0, every point is an equilibrium point. Thus Γ^0 is a trivial game.

The truel becomes interesting if it is repeated. Assume that each player has $m + 1$ bullets, where $m \geq 0$. Thus he may, if he is not killed, shoot up to $m + 1$ times. This iterated game will be called Γ^m. The rules of Γ^m are as follows:

(1) Γ^0 is played first. If, after a play of Γ^0, all three players have survived and have at least one bullet remaining, Γ^0 is replayed.

(2) A player's only information is the identity of the players alive at the end of each play. In particular, no player is aware of the strategy choices of either of the other two players in any previous play of Γ^0.

(3) If two players j and k have survived and have at least one bullet remaining, and if player i has been killed, the simultaneous duel Γ_i is played by j and k. In Γ_i, j and k have only one strategy (viz. to shoot at each other) and the marksmanship of j and k is as given for Γ^0.

(4) A payoff of 0 is awarded to any player who is killed. A payoff of 1 is awarded to any player who is alive when both other players are dead, or who is alive and has no bullets remaining.

If the stopping rule is removed (each player is provided with an infinite collection of bullets), the stochastic game Γ^∞ is defined by the above rules. (Rule (2) is included in order to restrict the strategy sets in Γ^m and Γ^∞. Note, however, that this restriction is innocuous as a player's payoff depends on others' choices of strategy only through their successful shots.)

Let Σ_i be player i's strategy set for Γ^0. Formally speaking, each of i's strategies for Γ^m specifies i's target in each of the $m + 1$ plays of Γ^m, should it occur, contingent on the results of all previous plays. (Because of rule (2), explicit contingencies on the other's previous choices of target are not included.) However, if any player is killed in play $r < m + 1$, i has no further choice in the selection of a target. Therefore, the only strategies we need consider for i in Γ^m are those in the Cartesian product $\bigtimes_1^{m+1} \Sigma_i$. Moreover, if all three players survive m plays of Γ_m, we have seen that i is indifferent between his two choices of target in the $m + 1^{\text{th}}$ play. Thus i's strategy set for Γ^m can be reduced to $\Sigma_i^m = \bigtimes_1^m \Sigma_i$. Similarly, i's strategy set in Γ^∞ is essentially $\Sigma_i^\infty = \bigtimes_1^\infty \Sigma_i$. In particular, in Γ^1 player i is concerned only with the selection of a point of $\Sigma_i^1 = \Sigma_i$.

We now examine Γ^1 for equilibrium points. First we consider player 1's problem. Suppose that players 2 and 3 have fixed their strategy choices at $\sigma_2 \in \Sigma_2^1$ and $\sigma_3 \in \Sigma_3^1$, respectively. As a result of these strategies, there are numbers x, y, and z which represent the probabilities that players 1, 2, and 3, respectively, would survive the first play of Γ^1, if player 1 were not to shoot. Furthermore, $xyz = bc$. For $\sigma_1 \in \Sigma_1^1$, let $V(\sigma_1)$ be player 1's expected payoff, given that he chooses σ_1 on the first play of Γ^1. If x_2 is the probability that 1 survives the second play of Γ^1, given that all three players survive the first, then

$$V(2_1) = abcx_2 + x(1-ay)zc + xay(1-z)b + x(1-ay)(1-z),$$

and

$$V(3_1) = abcx_2 + x(1-y)azc + xy(1-az)b + x(1-y)(1-az).$$

Therefore,

$$\frac{V(2_1) - V(3_1)}{x(1-a)} = y(1-b) - z(1-c),$$

and $V(2_1) > V(3_1)\ \forall\, \sigma_2 \in \Sigma_2^1,\ \sigma_3 \in \Sigma_3^1$, if and only if $c > \dfrac{1}{2-b}$, because $y \geq c$ and $z \leq 1$. Define $S(1) = c(1-b) - (1-c) = c(2-b) - 1$; then it is easy to show that

(A) If $b \neq c$, and if $\sigma_2 \neq 1_2$ or $\sigma_3 \neq 2_3$, 1 prefers[1] 2_1 to 3_1; if $\sigma_2 = 1_2$ and $\sigma_3 = 1_3$, 1 prefers 2_1 to 3_1 if $S(1) > 0$; 1 is different between 2_1 and 3_1 if $S(1) = 0$; 1 prefers 3_1 to 2_1 if $S(1) < 0$.

If $b = c$, and if $\sigma_2 = 1_2$ and $\sigma_3 = 1_3$, or if $\sigma_2 = 3_2$ and $\sigma_3 = 2_3$, 1 is indifferent between 2_1 and 3_1; if $\sigma_2 = 3_2$ and $\sigma_3 = 1_3$, 1 prefers 2_1 to 3_1; if $\sigma_2 = 1_2$ and $\sigma_3 = 2_3$, 1 prefers 3_1 to 2_1.

Note that if $b = c$, $S(1) < 0$.

Now we consider player 2's problem. If 1 and 3 have chosen $\sigma_1 \in \Sigma_1^1$ and $\sigma_3 \in \Sigma_3^1$, respectively, we can define x, y, z and V analogously to the above and obtain

$$\frac{V(1_2) - V(3_2)}{y(1-b)} = x(1-a) - z(1-c).$$

Therefore $V(1_2) > V(3_2)$ if and only if $S(2) > 0$, where $S(2) = c(2-a) - 1$. Then we have that

(B) If $a \neq c$, and if $\sigma_1 \neq 2_1$ or $\sigma_3 \neq 1_3$, 2 prefers 1_2 to 3_2; if $\sigma_1 = 2_1$ and $\sigma_3 = 1_3$, 2 prefers 1_2 to 3_2 if $S(2) > 0$; 2 is indifferent between 1_2 and 3_2 if $S(2) = 0$; 2 prefers 3_2 to 1_2 if $S(2) < 0$.

If $a = c$, and if $\sigma_1 = 2_1$ and $\sigma_3 = 2_3$, or if $\sigma_1 = 3_1$ and $\sigma_3 = 1_3$, 1 is indifferent between 1_2 and 3_2; if $\sigma_1 = 3_1$ and $\sigma_3 = 2_3$, 2 prefers 1_2 to 3_2; if $\sigma_1 = 2_1$ and $\sigma_3 = 1_3$, 2 prefers 3_2 to 1_2.

By our assumptions on a, b, and c, $a = c$ implies $a = b = c$. If $a = c$, $S(2) < 0$.

Similarly, for player 3 we find that

$$\frac{V(1_3) - V(2_3)}{z(1-c)} = x(1-a) - y(1-b).$$

[1]) Here and throughout this paper, "i prefers j_i to k_i" is to be read as "i strictly prefers j_i to k_i", and not as "i prefers j_i to k_i, or is indifferent between j_i and k_i".

Therefore, $V(1_3) > V(2_3)$ if and only if $S(3) > 0$, where $S(3) = b(2 - a) - 1$; and it can be shown that

(C) If $a \neq b$, and if $\sigma_1 \neq 3_1$ or $\sigma_2 \neq 1_2$, 3 prefers 1_3 to 2_3; if $\sigma_1 = 3_1$ and $\sigma_2 = 1_2$, 3 prefers 1_3 to 2_3 if $S(3) > 0$; 3 is indifferent between 1_3 and 2_3 if $S(3) = 0$; 3 prefers 2_3 to 1_3 if $S(3) < 0$.

If $a = b$, and if $\sigma_1 = 3_1$ and $\sigma_2 = 3_2$, or if $\sigma_1 = 2_1$ and $\sigma_2 = 1_2$, 3 is indifferent between 1_3 and 2_3; if $\sigma_1 = 2_1$ and $\sigma_2 = 3_2$, 3 prefers 1_3 to 2_3; if $\sigma_1 = 3_1$ and $\sigma_2 = 1_2$, 3 prefers 2_3 to 1_3.

Note that if $a = b$, $S(3) < 0$.

Theorem 1:

Γ^1 always has at least one equilibrium point in pure strategies. Indeed, all of the pure strategy equilibrium points of Γ^1 and the conditions under which they exist are given in the following list:

$$
\begin{array}{llll}
(1) \text{ If } a = b = c, & (2_1, 3_2, 1_3) & (1a) & \\
 & (3_1, 1_2, 2_3) & (1b) & \\
(2) \text{ If } a = b < c, & (2_1, 1_2, 1_3) & (2a) & \rbrace \text{ if } S(1) \geq 0 \\
 & (2_1, 1_2, 2_3) & (2b) & \\
 & (2_1, 3_2, 1_3) & (2c) & \rbrace \text{ if } S(1) \leq 0 \\
 & (3_1, 1_2, 2_3) & (2d) & \\
(3) \text{ If } a < b = c, & (2_1, 1_2, 1_3) & (3a) & \rbrace \text{ if } S(3) \geq 0 \\
 & (3_1, 1_2, 1_3) & (3b) & \\
 & (2_1, 3_2, 1_3) & (3c) & \rbrace \text{ if } S(3) \leq 0 \\
 & (3_1, 1_2, 2_3) & (3d) & \\
(4) \text{ if } a < b < c, & (2_1, 1_2, 1_3) & (4a) & \text{if } S(2) \geq 0 \\
 & (2_1, 3_2, 1_3) & (4b) & \text{if } S(2) \leq 0 \\
 & (3_1, 1_2, 2_3) & (4c) & \text{if } S(1) \leq 0 \text{ and } S(3) \leq 0 .
\end{array}
$$

Proof:

First observe that

$$S(1) \leq S(2), \text{ with equality if and only if } a = b;$$
$$S(3) \leq S(2), \text{ with equality if and only if } b = c.$$

The proof will be presented only for case (4). In the other three cases the proof is similar.

By (A), if $\sigma_3 = 1_3$, 1 prefers 2_1. By (C), if $\sigma_1 = 2_1$, 3 prefers 1_3. By (B), if $\sigma_1 = 2_1$ and $\sigma_3 = 1_3$, 2 prefers 1_2 if $S(2) > 0$, is indifferent between 1_2 and 3_2 if $S(2) = 0$, and prefers 3_2 if $S(2) < 0$. This proves that (4a) and (4b) are equilibrium points under precisely the conditions stated.

Suppose that $\sigma_2 = 1_2$ and $\sigma_3 = 2_3$ and $S(1) \leq 0$. Then by (A), 1 has no preference to 3_1. If $\sigma_1 = 3_1$ and $\sigma_2 = 1_2$ and $S(3) \leq 0$, then by (C) 3 has no preference to 2_3. If $\sigma_1 = 3_1$ and $\sigma_3 = 2_3$, then 2 prefers 1_2 by (B). Thus, if $S(1) \leq 0$ and $S(3) \leq 0$, (4c) is an equilibrium point. The converse follows from (A) and (C).

The proof (of case (4)) will be completed by showing that if $a < b < c$, there are no equilibrium points other than (4a), (4b), and (4c). We distinguish four possibilities, which, by the observation above, are exhaustive and mutually exclusive. They are

(i) $S(1) > 0$ or $S(3) > 0$,

(ii) $S(2) > 0$ and $S(1) \leq 0$ and $S(3) \leq 0$,

(iii) $S(2) < 0$,

(iv) $S(2) = 0$.

If (i) holds, then if $\sigma_1 = 2_1$ and $\sigma_2 = 1_2$, 3 prefers 1_3 to 2_3, yielding (4a). If $\sigma_1 = 2_1$ and $\sigma_2 = 3_2$, 3 prefers 1_3 to 2_3, and by (B), $(2_1, 3_2, 1_3)$ is not an equilibrium point. If $\sigma_1 = 3_1$ and $\sigma_2 = 1_2$, 3 prefers 1_3 if $S(3) > 0$; but $(3_1, 1_2, 1_3)$ is not an equilibrium point by (A). Otherwise we must have that 3 has no preference to 2_3, and $S(1) > 0$. But then $(3_1, 1_2, 2_3)$ is not an equilibrium point by (A). If $\sigma_1 = 3_1$ and $\sigma_2 = 3_2$, 3 prefers 1_3, and $(3_1, 3_2, 1_3)$ is not an equilibrium point by (A). If (ii) holds, the analysis is exactly as for (i), except now the point $(3_1, 1_2, 2_3)$ ((4c)) is also an equilibrium point.

Suppose (iii) holds. If $\sigma_1 = 2_1$ and $\sigma_2 = 1_2$, 3 prefers 1_3; but $(2_1, 1_2, 1_3)$ is not an equilibrium point by (B). If $\sigma_1 = 2_1$ and $\sigma_2 = 3_2$, 3 prefers 1_3, yielding (4b). If $\sigma_1 = 3_1$ and $\sigma_2 = 1_2$, 3 prefers 2_3, yielding (4c). If $\sigma_1 = 3_1$ and $\sigma_2 = 3_2$, 3 prefers 1_3; but $(3_1, 3_2, 1_3)$ is not an equilibrium point by (A). In case (iv), the only equilibrium points are (4a), (4b), and (4c); the proof of this is a combination of the proofs for (ii) and (iii). QED

Of course the expected payoff of each player depends on all the choices of strategy for the second play of Γ^1, as well as those for the first. At each equilibrium point given by Theorem 1, an arbitrary strategy for the second play of Γ^1 can be assigned to each player; all of the equilibrium points of Γ^1 can be obtained in this way. Maximin values for each player can be calculated in terms of a, b, and c.

We now proceed to calculate the expected payoffs, in the game Γ^m, associated with the points of the diagonals of the strategy sets (Σ_i^m) of this game. Any equilibrium point so obtained can be considered as a stationary strategy equilibrium point[1,2]. Suppose that players 2 and 3 have chosen diagonal points from Σ_2^m and Σ_3^m respectively. Let x, y, and z represent the survival probabilities of players 1, 2, and 3, respectively, on the first play of Γ^m, if 1 were not to shoot. For $1 \leq r \leq m$ and given that 1 plays $\sigma_1^m \in \Sigma_1^m$, let $P_r(\Gamma | \sigma_1^m)$ represent the probability that all three players survive r plays of Γ^m; let $P_r(\Gamma_i | \sigma_1^m)$ represent the probability that players j and k survive r plays of Γ^m and that player i does not; and let $P_r(W_1 | \sigma_1^m)$ represent the probability that players 2 and 3 have been killed and that player 1 has not, at or before the end of the r^{th} play of Γ^m. Then 1's expected payoff is

$$V_{m+1}(\sigma_1^m) = x P_m(\Gamma | \sigma_1^m) + c P_m(\Gamma_2 | \sigma_1^m) + b P_m(\Gamma_3 | \sigma_1^m) + P_m(W_1 | \sigma_1^m) \quad \ldots (1)$$

[1]) This concept was introduced by SHAPLEY [1953].

[2]) In Γ^m, we shall always assume that the players are restricted to diagonal strategies.

Assume that σ_1^m is a diagonal point of Σ_1^m, and let σ_1 be its first component. It is immediate that $P_r(\Gamma|\sigma_1^m) = (abc)^r$. By induction it can be shown that

$$P_r(\Gamma_2|2_1^m) = x(1 - ay)z(ac)^{r-1}\sum_{t=0}^{r-1} b^t,$$

$$P_r(\Gamma_3|2_1^m) = xay(1 - z)(ab)^{r-1}\sum_{t=0}^{r-1} c^t, \text{ and}$$

$$\begin{aligned} P_r(W_1|2_1^m) = {} & x(1 - ay)(1 - z)\sum_{t=0}^{r-1} (abc)^t \\ & + x(1 - ay)z(1 - a)c\sum_{t=0}^{r-2} (ac)^t \sum_{s=0}^{t} b^s \\ & + xay(1 - z)(1 - a)b\sum_{t=0}^{r-2} (ab)^t \sum_{s=0}^{t} c^s, \text{ for } r \geq 2. \end{aligned}$$

"3_1^m" can be substituted for "2_1^m" in the left sides of these three equations if "ay" is replaced by "y" and "z" by "az" in the right sides. Then, after some manipulation, (1) yields

$$\frac{V_{m+1}(2_1^m) - V_{m+1}(3_1^m)}{x(1 - a)} = yF_m(a,b,c) - zF_m(a,c,b) \qquad \ldots (2)$$

where

$$F_m(\alpha,\beta,\gamma) = (1 - \beta)\sum_{t=0}^{m-1} (\alpha\beta)^t \sum_{s=0}^{t} \gamma^s.$$

In the case $m = 1$, (2) yields the result already obtained in deriving (A).

Let $\sigma_1^\infty \in \Sigma_1^\infty$ be a diagonal strategy for 1 in Γ^∞, and let σ_1^m be its first m components. Define $V_\infty(\sigma_1^\infty) = \lim_{m\to\infty} V_{m+1}(\sigma_1^m)$, and note that

$$F_\infty(\alpha,\beta,\gamma) = \lim_{m\to\infty} F_m(\alpha,\beta,\gamma) = \frac{(1 - \beta)}{(1 - \alpha\beta)(1 - \alpha\beta\gamma)}.$$

From (2) follows

$$\frac{(1 - abc)}{x(1 - a)}(V_\infty(2_1^\infty) - V_\infty(3_1^\infty)) = y\left[\frac{1 - b}{1 - ab}\right] - z\left[\frac{1 - c}{1 - ac}\right]. \qquad \ldots (3)$$

All of the above statements about player 1 can be converted into statements about player 2 by the permutations $1 \to 2 \to 1, 3 \to 3; x \to y \to x, z \to z; a \to b \to a, c \to c$. Similarly, statements about player 3 can be obtained from the corresponding statements about player 1 by the permutations $1 \to 3 \to 2 \to 1$; $x \to z \to y \to x$; $a \to c \to b \to a$.

Lemma 1:

For any positive m or for $m = \infty$, if $\beta \leq \gamma$, then $F_m(\alpha,\gamma,\beta) \geq F_m(\alpha,\beta,\gamma)$, with equality if and only if $\beta = \gamma$.

Proof:

Assume $\beta < \gamma$. For the case $m = \infty$, it can be shown by differentiation that $\frac{1-\beta}{1-\alpha\beta} > \frac{1-\gamma}{1-\alpha\gamma}$. For $m < \infty$, observe that,

$$F_m(\alpha,\beta,\gamma) = \sum_{t=0}^{m-1} (\alpha\beta\gamma)^t - \alpha^{m-1}\beta^m \sum_{t=0}^{m-1} \gamma^t - (1-\alpha)\beta \sum_{t=0}^{m-2} (\alpha\beta)^t \sum_{s=0}^{t} \gamma^s .$$

This yields

$$F_m(\alpha,\beta,\gamma) - F_m(\alpha,\gamma,\beta) = \alpha^{m-1} \sum_{t=0}^{m-1} (\beta\gamma)^t (\gamma^{m-t} - \beta^{m-t}) + (1-\alpha) \sum_{t=0}^{m-2} \alpha^t \sum_{s=0}^{t} (\beta\gamma)^s (\gamma^{t-s+1} - \beta^{t-s+1}),$$

which is certainly positive. QED

We now consider diagonal strategy equilibrium points in Γ^∞. To simplify notation, σ_i will denote both the diagonal point $\sigma_i^\infty \in \Sigma_i^\infty$ and its projection on Σ_i, and $V_\infty(\sigma_i^\infty)$ will be written $V(\sigma_i)$. Equilibrium points will be denoted by Roman numerals, and the payoff to player i at the equilibrium point Y will be written w_i^Y, a function of a, b, and c. A player's expectation in Γ^∞ represents his probability of eventual survival; the probability that no player survives at Y is $w_0^Y = 1 - \sum_{i=1}^{3} w_i^Y$.

Inasmuch as Γ^∞ resembles a 3-person stochastic game, diagonal strategies in Σ_i^∞ resemble stationary strategies in such a stochastic game. However there is no existence theorem for stationary strategy equilibrium points which is applicable to Γ^∞ or to games of similar structure [1]). It is conjectured that such a theorem is available; in any case (pure) diagonal strategy equilibrium points will be shown to always exist in Γ^∞ (Theorem 2).

Now, in analogy with the treatment of Γ^1, we consider player 1's problem in Γ^∞. In (3), we must have $y \geq c$ and $z \leq 1$. Consequently, 2_1 dominates 3_1 provided that $b < c$ and

$$c\frac{1-b}{1-ab} > \frac{1-c}{1-ac}.$$

If we define

$$S_\infty(1) = c\frac{1-b}{1-ab} - \frac{1-c}{1-ac},$$

then from (3) it can be verified that (A) holds with $S(1)$ replaced by $S_\infty(1)$. Furthermore, $b = c$ implies that $S_\infty(1) < 0$.

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[1]) Neither the theorem of Fink [1964] nor of Takahashi [1964] can be applied to Γ^∞.

For player 2, we obtain from (3) and the permutations described above the relation

$$\frac{(1-abc)}{y(1-b)}\{V(1_2)-V(3_2)\} = x\left[\frac{1-a}{1-ab}\right] - z\left[\frac{1-c}{1-bc}\right].$$

From this and Lemma 1 it follows that 1_2 dominates 3_2 if $a < c$ and $S_\infty(2) > 0$, where

$$S_\infty(2) = c\frac{1-a}{1-ab} - \frac{1-c}{1-bc}.$$

It can be shown that (B) holds with $S(2)$ replaced by $S_\infty(2)$. If $a = c$, $S_\infty(2) < 0$.

Applying a similar procedure to player 3, we obtain that 1_3 dominates 2_3 if $a < b$ and $S_\infty(3) > 0$, where

$$S_\infty(3) = b\frac{1-a}{1-ac} - \frac{1-b}{1-bc}.$$

In addition, it follows that (C) holds with $S(3)$ replaced by $S_\infty(3)$. If $a = b$, $S_\infty(3) < 0$.

In analogy with the result for Γ^1, we have the following lemma:

Lemma 2:

$S_\infty(1) \le S_\infty(2)$, with equality if and only if $a = b$. $S_\infty(3) \le S_\infty(2)$, with equality if and only if $b = c$.

Proof:

If $a = b$, it can be checked that $S_\infty(1) = S_\infty(2)$. If $a < b$,

$$S_\infty(1) < S_\infty(2) \Leftrightarrow c\frac{1-a}{1-ab} - c\frac{1-b}{1-ab} > \frac{1-c}{1-bc} - \frac{1-c}{1-ac}$$

$$\Leftrightarrow \frac{c(b-a)}{1-ab} > \frac{c(1-c)(b-a)}{(1-ac)(1-bc)}$$

$$\Leftrightarrow ab + c(1-a-b-ab) + abc^2 > 0.$$

But since $(1-a)(1-b) > 0$, $1-a-b-ab > -2ab$, so that

$$ab + c(1-a-b-ab) + abc^2 > ab - 2abc + abc^2 = ab(1-c)^2 > 0.$$

This shows that $a < b \Rightarrow S_\infty(1) < S_\infty(2)$.
If $b = c$, it can be checked that $S_\infty(3) = S_\infty(2)$. If $b < c$,

$$S_\infty(3) < S_\infty(2) \Leftrightarrow \frac{1-b}{1-bc} - \frac{1-c}{1-bc} > b\frac{1-a}{1-ac} - c\frac{1-a}{1-ab}$$

$$\Leftrightarrow \frac{1}{1-bc} > \frac{(1-a)(ac+ab-1)}{(1-ac)(1-ab)} \qquad \ldots (4)$$

Differentiation with respect to c shows that

$$\frac{1}{1-bc} > \frac{1}{1-b^2}.$$

Differentiation with respect to c shows that

$$\frac{(1-a)(ab+ac-1)}{(1-ab)(1-ac)} < \frac{ab+a-1}{1-ab} = \frac{a}{1-ab} - 1\,.$$

Thus to show that (4) holds and complete the proof it is sufficient to show that

$$\frac{a}{1-ab} < \frac{1}{1-b^2} + 1 = \frac{2-b^2}{1-b^2}\,.$$

But

$$\frac{a}{1-ab} \leq \frac{b}{1-b^2} < \frac{1}{1-b^2} < \frac{2-b^2}{1-b^2}\,. \qquad \text{QED}$$

Theorem 2:

Γ^∞ always has at least one equilibrium point in pure diagonal strategies. The following is a complete list of such equilibrium points:

(I) $(2_1, 1_2, 1_3)$ if and only if $S_\infty(2) \geq 0$.
(II) $(2_1, 3_2, 1_3)$ if and only if $S_\infty(2) \leq 0$.
(III) $(3_1, 1_2, 2_3)$ if and only if $S_\infty(1) \leq 0$ and $S_\infty(3) \leq 0$.
(IV) $(2_1, 1_2, 2_3)$ if and only if $a = b < c$ and $S_\infty(2) \geq 0$.
(V) $(3_1, 1_2, 1_3)$ if and only if $a < b = c$ and $S_\infty(2) \geq 0$.

Proof:

Using Lemma 2 and replacing S by S_∞, Theorem 1 can be reproven. Such a theorem is easily seen to be equivalent to the above formulation, again using Lemma 2. QED

In connection with Theorem 2, note that

$$(a,b,c) = (\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}) \text{ yields } S_\infty(1) < 0,\ S_\infty(2) > 0,\ S_\infty(3) < 0\,.$$

Thus it is possible that the only equilibrium points of Γ^∞ are I and III. Of course, if II is an equilibrium point, so is III. It is clear that the points IV and V occur only when these points are, to players 3 and 1, respectively, indistinguishable from I. In this sense IV and V can be regarded as degenerate cases.

An expression for player 1's expectation could be obtained by substituting and taking the limit in (1). Here we give an alternate derivation. If the duel game Γ_2 is played, we can calculate its value to 1, w_{21}, by the identity

$$w_{21} = acw_{21} + c(1-a)(1) + a(1-c)(0) + (1-a)(1-c)(0)\,,$$

which yields $w_{21} = \dfrac{c(1-a)}{1-ac}$. Similarly, $w_{31} = \dfrac{b(1-a)}{1-ab}$. Now suppose that players 2 and 3 have chosen diagonal strategies for Γ^∞ so that the survival probabilities of 1, 2, and 3 would be x, y, and z, respectively, if 1 were not to shoot. Then we have that

$$V(2_1) = xayzV(2_1) + x(1-ay)zw_{21} + xay(1-z)w_{31} + x(1-ay)(1-z)\,,$$

or

$$V(2_1) = \frac{x}{1-abc}\left[(1-ay)z\frac{c(1-a)}{1-ac} + ay(1-z)\frac{b(1-a)}{1-ab} + (1-ay)(1-z)\right].$$

As noted previously, an expression for $V(3_1)$ can be obtained from the above expression for $V(2_1)$ by substitution of "y" for "ay" and of "az" for "z" throughout. Then (3) can be obtained from these two relations. Similar expressions for the values of the various strategies of 2 and 3 may be obtained by means of the permutations given above.

At the equilibrium point I, we have

$$w_1^{\mathrm{I}} = \frac{(1-a)^2bc^2}{(1-abc)(1-ac)}, \qquad w_2^{\mathrm{I}} = \frac{a(1-b)c}{1-abc}$$

$$w_3^{\mathrm{I}} = \frac{ab(1-c)(1-ac) + abc(1-a)(1-c) + (1-a)(1-ac)(1-bc)}{(1-abc)(1-ac)}.$$

When II is an equilibrium point,

$$w_1^{\mathrm{II}} = \frac{(1-a)c}{1-abc}\left[\frac{(1-a)bc}{1-ac} + \frac{ab(1-b)}{1-ab} + (1-b)\right],$$

$$w_2^{\mathrm{II}} = \frac{a(1-b)}{1-abc}\left[\frac{bc(1-c)}{1-bc} + \frac{a(1-b)c}{1-ab} + (1-c)\right],$$

$$w_3^{\mathrm{II}} = \frac{b(1-c)}{1-abc}\left[\frac{ab(1-c)}{1-bc} + \frac{a(1-a)c}{1-ac} + (1-a)\right].$$

When III is an equilibrium point,

$$w_1^{\mathrm{III}} = \frac{(1-a)b}{1-abc}\left[\frac{ac(1-c)}{1-ac} + \frac{(1-a)bc}{1-ab} + (1-c)\right],$$

$$w_2^{\mathrm{III}} = \frac{(1-b)c}{1-abc}\left[\frac{a(1-b)c}{1-bc} + \frac{a(1-a)b}{1-ab} + (1-a)\right],$$

$$w_3^{\mathrm{III}} = \frac{a(1-c)}{1-abc}\left[\frac{b(1-b)c}{1-bc} + \frac{ab(1-c)}{1-ac} + (1-b)\right].$$

When IV is an equilibrium point, then I is also an equilibrium point and

$$w_1^{\mathrm{IV}} = w_2^{\mathrm{I}},\; w_2^{\mathrm{IV}} = w_1^{\mathrm{I}},\; w_3^{\mathrm{IV}} = w_3^{\mathrm{I}}.$$

When V is an equilibrium point, then I is also an equilibrium point and

$$w_1^{\mathrm{V}} = w_1^{\mathrm{I}},\; w_2^{\mathrm{V}} = w_3^{\mathrm{I}},\; w_3^{\mathrm{V}} = w_2^{\mathrm{I}}.$$

Several inequalities between payoffs can be derived from these expressions. If $S_\infty(2) \le 0$, both II and III are equilibrium points and it is meaningful to compare payoffs at these two points. If is easy to show that

$$\frac{(1-abc)}{(1-a)}[w_1^{\mathrm{II}} - w_1^{\mathrm{III}}] = bc\left[\frac{c-a}{1-ac} - \frac{b-a}{1-ab}\right] + (c-b).$$

Differentiation of $\frac{c-a}{1-ac}$ with respect to c shows that the first term on the right side of this equation is positive; consequently, $w_1^{\mathrm{II}} \geq w_1^{\mathrm{III}}$. Furthermore, equality holds if and only if $b = c$. Similarly, $w_2^{\mathrm{II}} \leq w_2^{\mathrm{III}}$, with equality if and only if $a = c$, and $w_3^{\mathrm{II}} \geq w_3^{\mathrm{III}}$, with equality if and only if $a = b$. By choosing appropriate values of a, b, and c, satisfying $S_\infty(2) \geq 0$, it can be shown that any conceivable inequality, between any pair of w_1^{I}, w_2^{I}, and w_3^{I}, can fail.

In the special case $a = b = c$, $S_\infty(1) = S_\infty(2) = S_\infty(3) < 0$, and the only equilibrium points are II and III. All payoffs coincide;

$$w_i^Y = \frac{a(1+a^2)}{(1+a+a^2)(1+a)}, \quad \text{for } Y = \mathrm{II} \text{ or } \mathrm{III},\ i = 1, 2, \text{or } 3\,.$$

Note that $w_i^Y \to 0$ as $a \to 0$, that $w_i^Y \to 1/3$ as $a \to 1$, and that w_0^Y is a strictly decreasing function of a. These results verify our intuitive notions about this "fair" version of Γ^∞.

Several numerical examples will illustrate the properties of the equilibrium point payoffs as functions of the variables a, b, and c. In the example used by SHUBIK, $(a,b,c) = (1/5, 3/10, 2/5)$. For these values, $S_\infty(2) < 0$, so that the only equilibrium points are II and III. The above formulae yield

$$w_0^{\mathrm{II}} = 0.448,\ w_1^{\mathrm{II}} = 0.278,\ w_2^{\mathrm{II}} = 0.106,\ w_3^{\mathrm{II}} = 0.168;$$
$$w_0^{\mathrm{III}} = 0.449,\ w_1^{\mathrm{III}} = 0.186,\ w_2^{\mathrm{III}} = 0.262,\ w_3^{\mathrm{III}} = 0.103.$$

If $(a,b,c) = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4})$, $S_\infty(1) < 0$, $S_\infty(2) > 0$, and $S_\infty(3) < 0$, so that the only equilibrium points are I and III, and the payoffs are

$$w_0^{\mathrm{I}} = 0.122,\ w_1^{\mathrm{I}} = 0.200,\ w_2^{\mathrm{I}} = 0.167,\ w_3^{\mathrm{I}} = 0.511;$$
$$w_0^{\mathrm{III}} = 0.189,\ w_1^{\mathrm{III}} = 0.344,\ w_2^{\mathrm{III}} = 0.333,\ w_3^{\mathrm{III}} = 0.133.$$

If $(a,b,c) = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $S_\infty(1) > 0$, so that the only equilibrium point is I and the payoffs are

$$w_0^{\mathrm{I}} = 0.107,\ w_1^{\mathrm{I}} = 0.215,\ w_2^{\mathrm{I}} = 0.103,\ w_3^{\mathrm{I}} = 0.575\,.$$

The "strength through weakness" phenomenon noted by SHUBIK is especially apparent in the last example.

We return briefly to the games Γ^m. Define

$$S_m(1) = cF_m(a,b,c) - F_m(a,c,b)$$

and use the permutations given above to define $S_m(2)$ and $S_m(3)$. Note that the previous definitions of $S_\infty(i)$ imply that $S_\infty(i) = (1 - abc)\lim_{m\to\infty} S_m(i)$. Equation (2) and Lemma 1 now show that (A), (B), and (C), with S replaced by S_m, are applicable to players 1, 2, and 3, respectively, in Γ_m, providing we let $\sigma_i^m \in \Sigma_i^m$ be denoted by σ_i. We can now obtain a complete characterization of the diagonal equilibrium points of Γ^m.

Theorem 3:

Γ^m always has at least one equilibrium point in pure diagonal strategies. Indeed, the equilibrium points of Γ^m are precisely those for Γ^∞ given in Theorem 2, with S_∞ replaced by S_m.

Proof:

Because of Lemma 1, the method of Theorem 1 can be applied to Γ^m (parallel to its use in Theorem 2) if it is shown that

$$\left.\begin{aligned} S_m(1) &\le S_m(2), \text{ with equality if and only if } a = b; \\ S_m(3) &\le S_m(2), \text{ with equality if and only if } b = c. \end{aligned}\right\} \qquad \ldots(5)$$

It is easy to see that $a = b$ implies $S_m(1) = S_m(2)$, and that $b = c$ implies $S_m(3) = S_m(2)$. Thus it suffices to prove that

$$a < b \Rightarrow S_m(1) < S_m(2);$$
$$b < c \Rightarrow S_m(3) < S_m(2).$$

By substitution from the definitions of S_m and F_m, this is equivalent to

$$\left.\begin{aligned} a < b &\Rightarrow c(b-a)\sum_{t=0}^{m-1}(ab)^t\sum_{s=0}^{t}c^s > (1-c)\sum_{t=0}^{m-1}c^t\left[b^t\sum_{s=0}^{t}a^s - a^t\sum_{s=0}^{t}b^s\right]; \\ b < c &\Rightarrow (c-b)\sum_{t=0}^{m-1}(bc)^t\sum_{s=0}^{t}a^s > (1-a)\sum_{t=0}^{m-1}a^t\left[c^t\sum_{s=0}^{t}b^{s+1} - b^t\sum_{s=0}^{t}c^{s+1}\right]. \end{aligned}\right\} \qquad \ldots(6)$$

Suppose that $0 < \alpha < \beta < 1$ and that $0 < \gamma < 1$. Consider the inequality

$$(\beta-\alpha)\sum_{t=0}^{m-1}(\alpha\beta)^t\sum_{s=0}^{t}\gamma^s > \begin{cases} 0, \text{ if } m = 1 \\ (1-\gamma)\displaystyle\sum_{t=1}^{m-1}\gamma^{t-1}\left[\beta^t\sum_{s=0}^{t-1}\alpha^s - \alpha^t\sum_{s=0}^{t-1}\beta^s\right], \text{ if } m > 1. \end{cases} \qquad \ldots(7)$$

The choice of $\alpha = a$, $\beta = b$, and $\gamma = c$ converts (7) into precisely the first statement of (6). By choosing $\alpha = b$, $\beta = c$, and $\gamma = a$, it is easy to see that (7) also implies the second statement of (6). Therefore we need only show that (7) holds. If can be checked that (7) is true for $m = 1$ and $m = 2$. For $m > 2$, rather than proving (7) directly, we will show

$$\begin{aligned} (\beta-\alpha)\sum_{t=0}^{m-2}(\alpha\beta)^t\sum_{s=0}^{t}\gamma^s + \sum_{t=1}^{m-2}\gamma^t\left[\beta^t\sum_{s=0}^{t-1}\alpha^s - \alpha^t\sum_{s=0}^{t-1}\beta^s\right] & \\ - \sum_{t=1}^{m-1}\gamma^{t-1}\left[\beta^t\sum_{s=0}^{t-1}\alpha^s - \alpha^t\sum_{s=0}^{t-1}\beta^s\right] &> 0. \end{aligned} \qquad (8)$$

Clearly (8) implies (7). If $m = 3$, the left side of (8) is

$$(\beta-\alpha)(1+\gamma+\alpha\beta+\alpha\beta\gamma-1-\alpha\gamma-\beta\gamma-\alpha\beta\gamma) = (\beta-\alpha)[\alpha\beta+\gamma(1-\alpha-\beta)] > 0,$$

so that (8) is true when $m = 3$.

Now let $m > 3$ and suppose that (8) is known to hold for all integers m' with $3 \leq m' < m$. To show that (8) holds for m, it suffices to prove that

$$(\beta - \alpha)(\alpha\beta)^{m-2}\sum_{s=0}^{m-2}\gamma^s + \gamma^{m-2}\left[\beta^{m-2}\sum_{s=0}^{m-3}\alpha^s - \alpha^{m-2}\sum_{s=0}^{m-3}\beta^s\right]$$
$$- \gamma^{m-2}\left[\beta^{m-1}\sum_{s=0}^{m-2}\alpha^s - \alpha^{m-1}\sum_{s=0}^{m-2}\alpha^s\right] > 0.$$

The left side of this inequality is a polynomial of degree $m - 2$ in γ,

$$P(\gamma) = \sum_{k=0}^{m-2} q_k \gamma^{m-2-k}.$$

For $1 \leq k \leq m - 2$, $q_k = (\beta - \alpha)(\alpha\beta)^{m-2}$. We calculate q_0, the coefficient of γ^{m-2} in P.

$$q_0 = (\beta - \alpha)(\alpha\beta)^{m-2} + \left[\beta^{m-2}\sum_{s=0}^{m-3}\alpha^s - \alpha^{m-2}\sum_{s=0}^{m-3}\beta^s\right]$$
$$- \left[\beta^{m-1}\sum_{s=0}^{m=3}\alpha^s - \alpha^{m-1}\sum_{s=0}^{m-3}\beta^s\right] - (\beta^{m-1}\alpha^{m-2} - \alpha^{m-1}\beta^{m-2})$$
$$= (\beta - \alpha)\sum_{s=0}^{m-3}(\alpha\beta)^s\sum_{r=0}^{m-3-s}\alpha^r\beta^{m-3-s-r} - (\beta - \alpha)\sum_{s=0}^{m-3}(\alpha\beta)^s\sum_{r=0}^{m-2-s}\alpha^r\beta^{m-2-s-r}$$
$$= (\beta - \alpha)\sum_{s=0}^{m-3}(\alpha\beta)^s\left[\sum_{r=0}^{m-3-s}\alpha^r\beta^{m-3-s-r} - \sum_{r=0}^{m-2-s}\alpha^r\beta^{m-2-s-r}\right].$$

Now if $0 \leq s \leq m - 4$, then

$$(\alpha\beta)^{s+1}\sum_{r=0}^{m-3-(s+1)}\alpha^r\beta^{m-3-(s+1)-r} = (\alpha\beta)^s\sum_{r=0}^{m-4-s}\alpha^{r+1}\beta^{m-3-s-r}$$
$$= (\alpha\beta)^s\sum_{r=1}^{m-3-s}\alpha^r\beta^{m-2-s-r}$$
$$= (\alpha\beta)^s\left[\sum_{r=0}^{m-2-s}\alpha^r\beta^{m-2-s-r} - (\beta^{m-2-s} + \alpha^{m-2-s})\right]$$

Therefore,

$$q_0 = (\beta - \alpha)\left[\sum_{r=0}^{m-3}\alpha^r\beta^{m-3-r} - \sum_{s=0}^{m-3}(\alpha\beta)^s\beta^{m-2-s} - \sum_{s=0}^{m-3}(\alpha\beta)^s\alpha^{m-2-s}\right]$$
$$= (\beta - \alpha)\left[\sum_{s=0}^{m-3}\alpha^s\beta^{m-3-s} - \beta^{m-2}\sum_{s=0}^{m-3}\alpha^s - \alpha^{m-2}\sum_{s=0}^{m-3}\beta^s\right].$$

For $1 \leq k \leq m-2$, $q_k > 0$ so that $q_k \gamma^{m-2-k} > q_k \gamma^{m-2}$. Then

$$\begin{aligned}
P(\gamma) &= \sum_{k=0}^{m-2} q_k \gamma^{m-2-k} > \gamma^{m-2} \sum_{k=0}^{m-2} q_k \\
&= (\beta-\alpha)\gamma^{m-2}\Bigg[(m-2)(\alpha\beta)^{m-2} + \sum_{s=0}^{m-3} \alpha^s \beta^{m-3-s} - \beta^{m-2} \sum_{s=0}^{m-3} \alpha^s \\
&\quad - \alpha^{m-2} \sum_{s=0}^{m-3} \beta^s\Bigg] \\
&= (\beta-\alpha)\gamma^{m-2} \sum_{s=0}^{m-3} \alpha^s (\beta^{m-3-s} - \beta^{m-2}) - \sum_{s=0}^{m-3} \alpha^{m-2} (\beta^s - \beta^{m-2}) \\
&= (\beta-\alpha)\gamma^{m-2} \sum_{s=0}^{m-3} \alpha^{m-3-s} (\beta^s - \beta^{m-2}) - \sum_{s=0}^{m-3} \alpha^{m-2} (\beta^s - \beta^{m-2}) \\
&= (\beta-\alpha)\gamma^{m-2} \sum_{s=0}^{m-3} (\alpha^{m-3-s} - \alpha^{m-2})(\beta^s - \beta^{m-2}) > 0\,.
\end{aligned}$$

Thus $P(\gamma) > 0$, proving (8) and therefore (5). QED

Theorems 2 and 3 show that the games Γ^m and Γ^∞ have a similar equilibrium point structure. In comparing these two theorems, it should be noted that the functions S_m are not in general monotone in m for fixed a, b, and c. However their convergence guarantees that, if the number of bullets provided the players is sufficiently large, any equilibrium point reached in the game will remain an equilibrium point if the restriction on the length of the game is removed.

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