Things I Don't Know About Physics

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1 Electrostatics

This is a brief and limited description of electrostatics as studied by an undergraduate from California State University Sacramento. I expect this to be an informal rough outline of my understanding of the field as I go through the course and attempt to learn Latex as well. Let's get started.

1.1 Introduction

It seems that this field can be boiled down to a single simple question, which is "What force does one or more electric charges, $q_1, q_2, q_3, ...$ impose upon another charge Q which can be called the **source charge** and **test charge** respectively. It seemed the most fundamental solution to this problem is the **principle of superposition** which claims that the interaction, or force, between two charges completely ignores the forces of any other charges. This is interesting to me since in mechanics, two bodies that come into contact with each other have some influence over the behavior of the other.

A ball rolling down a hill may be slowed or shifted by friction or some other object. The only thing that comes to mind that is similar to an electric field then is gravity. Since planets also interact with each other at a distance with an attraction that is not affected by other planets albeit the net force will induce motion. So then similar to mechanics we calculate the force between two charges (ignoring all others), F_1, and find the force between each source charge q and test charge Q, and find the vector sum $F = F_1 + F_2 + F_3 ...$ just as we do when finding the net force in mechanics.

It seems that this is just a part of the interaction between these particles or charges. Later I will learn more about these charges moving with some acceleration and velocity, but since electromagnetic information travels at the speed of light each case being studied will be at an instant in time or with all source charges held stationary.

1.1.1 Coulomb's Law

The answer to the question asked above is Coulomb's Law, the force of one charge acting on another, or vice verse. This expression was found empirically and has no known method of derivation as it merely describes the nature of reality as we have observed it. This law was plucked out of the observation and not mathematically constructed.

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_s^2}\hat{r} \tag{1}$$

The constant, ϵ_0 is called the **permittivity of free space** and has so far been nothing I have needed to understand more than the fact that it is an experimental constant needed in the equation.

$$\epsilon_0 = 8.85X10^{-12}\frac{C^2}{Nm^2}$$

This expression shows that the force is proportional to the product of the charges and inversely proportional to the square of the distance between them. While $r_s = r_q - r_Q$ the the magnitude r used in the equation and $\hat{r}$ is the unit vector or direction. The force points along the line from q to Q and their charge will determine the sign of the expression. If they are the same charge they will repel each other and if they are of different charge they'll attract.

Griffiths mentions that Coulomb's law and the principle of superposition are the physical aspects of electrostatics, everything else besides some special properties of matter is "mathematical elaboration" of the fundamental physical rules. I assume a clear understanding of these concepts will become important when solving problems so understanding them is important.

1.1.2 The Electric Field

Following what was just claimed about the principle of superposition and Coulomb's law, if we had several point charges $q_1, q_2, q_3, ...$ at distances $r_1, r_2, r_3, ...$ from some test point Q, then the total force on Q would be

$$\vec{F} = \vec{F_1} + \vec{F_2} + ... = \frac{1}{4\pi\epsilon_0}(\frac{q_1 Q}{r_1^2}\hat{r_1} + \frac{q_2 Q}{r_2^2}\hat{r_2} + ...)$$
$$= \frac{Q}{4\pi\epsilon_0}(\frac{q_1}{r_1^2}\hat{r_1} + \frac{q_2}{r_2^2}\hat{r_2} + ...)$$

shortened to

$$\vec{F} = Q\vec{E}$$

where

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r_i}$$

In this case $\vec{E}$ is the **electric field** of all the source charges q_i. The electric field is a function of a position vector because the separation vectors, $\hat{r}$ depend on the location of the test point. The electric field function makes no use of the test charge Q, this is because the electric is a vector quantity that varies from each point q and is determined by the arrangement of the source charges. In a physical sense, the electric field is the force per unit charge that would be impressed upon a test charge if placed at some point P.

Griffiths has started the text off simple and explains that this does not fully define what an electric field is. This is the "minimal interpretation" of the electric field in the next step for calculating electric forces. We are encouraged to continue thinking of the field as a "real" physical entity that fills the space around electric charges. He takes a moment here to talk on Maxwell discussing the electric and magnetic fields providing stresses and strains in an invisible "ether" which special relativity has forced us to abandon. This prevents us from using Maxwell's mechanical interpretation of the electric field, but I believe Griffiths is encouraging students to think of it in this fashion for the time being, "I can't tell you, then, what a field is–only how to calculate it and what it can do for you once you've got it."

This is strange to me.

Example 1.1.2 Find the electric feild a distance z above the midpoint between two equal charges (q), a distance d apart.

Let $\vec{E_1}$ be the field of the left charge alone, and $\vec{E_2}$ that of the right charge alone. Adding them (vectorially), the horizontal components cancel and the vertical components conspire:

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{r_s^2}\cos\theta$$

Here $r_s = \sqrt{z^2 + (\frac{d}{2})^2}$ and $\cos\theta = \frac{z}{r_s}$, so

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{[z^2 + (\frac{d}{2})^2]^(3/2)}\hat{z}$$

Check: When $z >> d$ you're so far away that it just looks like a single charge $2q$, so the field should reduce to $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\hat{z}$. And it does (just set d approaching 0 in the formula).

1.1.3 Continuous Charge Distributions

Earlier we made the assumption that our electric field was a sum of discrete source charges, but if we had a continuous distribution over some region, then we have an integral!

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{q}{r^2}\hat{r}\ dq \tag{2}$$

There are three things to consider here when integrating over a continuous charge distribution and that is the dimension of distribution. If the charge is spread along a line, with a charge per unit length λ, then $dq = \lambda dl'$ where dl is the element of length along the line. If the charge is "smeared" out over a surface with a charge per unit area σ, then $dq = \sigma la'$ where da' is an element of area on the surface. Lastly, if the charge fills a volume with a charge per unit volume ρ, then $dq = \rho d\tau'$ where $d\tau'$ is an element of volume.

$$dq \to \lambda dl' \;\; \sigma da' \;\; \rho d\tau'$$

Thus the electric field of a line is,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\hat{r_q})}{r^2}\hat{r}\; dl' \tag{3}$$

For a surface charge,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\hat{r_q})}{r^2}\hat{r}\; da' \tag{4}$$

and for volumes,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\hat{r_q})}{r^2}\hat{r}\; d\tau' \tag{5}$$

The volume distribution being the most general and is often the one referred to as "Coulomb's Law" because it is essentially the original equation 1 and is also the most realistic case. Griffiths makes a note to remind us to be careful to remember the meaning of r_s in these formulas. Oringinally in 1 $r_s i$ was the vecot from the source charge q_i to the field point r_Q. In the new equations r_s is the vector from dq (which means from dl', da', or $d\tau'$) to the field point r_Q.

Example 1.1.3.2 Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ.

The simplest method is to chop the line into symmetrically placed pairs (at $+x$ o4 $-x$), quote the result of Example **??**.1 (with $\frac{d}{2} \to x, q \to \lambda dx$), and integrate $(x : 0 \to L)$. But here's a more general approach:

$$r_Q = z\hat{z}, \qquad r_q = x\hat{x}, \qquad dl' = dx;$$

$$\vec{r_s} = r_q - r_Q = z\hat{z} - x\hat{x}, \qquad r_s = \sqrt{z^2+x^2}, \qquad \hat{r_s} = \frac{\vec{r_s}}{r_s} = \frac{z\hat{z}-x\hat{x}}{\sqrt{z^2+x^2}}$$

$$E = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda}{z^2+x^2}\frac{z\hat{z}-x\hat{x}}{\sqrt{z^2+x^2}}dx$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[z\hat{z}\int_{-L}^{L}\frac{1}{(z^2+x^2)^{\frac{3}{2}}}dx - \hat{x}\int_{-L}^{L}\frac{x}{z^2+x^{2\frac{3}{2}}}dx\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[z\hat{z}\frac{x}{z^2\sqrt{z^2+x^2}}\Big|_{-L}^{L} - \hat{x}\frac{1}{\sqrt{z^2+x^2}}\Big|_{-L}^{L}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{z}$$

Now is a great time to tear this equation open and look at the physical results. If the position P were to be taken infinitely far away from this line of charge $z >> L$,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2}$$

Figure 1

Figure 2

the line begins to look like a point charge as $q = 2\lambda L$ and as $L \to \infty$ we see,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}$$

1.2 Divergence and Curl of Electrostatic Fields

1.2.1 Field Lines, Flux, Gauss's Law

What we saw in the previous section was a way to calculate the field of a charge distribution with Eq. 2, and the force of a charge Q placed within that field. It becomes clear that these calculations lead to painful integrals but the rest of this study of electrostatics is devoted to learning the tools that allows the integrals to be avoided. Throughout the rest of this approach to electrostatics, the fundamentals of Coulomb's Law and the principle of superposition will be revisited and repackaged. The table has been set, now it is time to feast upon the finer details. In this section the divergence and curl of the electric field will be explored. Griffiths mentions wanting to take a moment before this to show a more illuminating or intuitive approach.

To develop this intuition we begin with the simplest possible case and place a point charge at the origin:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

a basic diagram of this field in two dimensions is shown in figure 1 where the field lines are always pointing radially outward. This figure is to show the shift from vectors of magnitude with arrows of size proportional to the magnitude or strength, into field lines whose strength depend on the density instead of the length. The field lines have greater charge density toward the center and decrease as the field lines become farther apart. Giffiths makes a note that this is "deceptive" since in two dimensions the strength of the lines goes as $\frac{1}{r}$ instead of inversely squared. The number of field lines passing through a circle is $\frac{n}{2\pi r}$ this is fixed if you imagine field lines in 3 dimensions radiating from a sphere which would be $\frac{n}{4\pi r^2}$ preserving the inverse squared. I do not see why it is important physically to think of n number of field lines since a charge of any kind would have an infinite amount, but never the less it plays to the point.

Griffiths shows that these two dimensional diagrams can be useful for more complicated fields. The number of field lines chosen for a diagram can be arbitrary, but must be enough to visualize the field and must be consistent once chosen. As a general rule for consistency, field lines will start on positive charges and end on negative charges, and the must complete travel between two points without crossing. If the two field lines were to cross, they would have two different directions at that intersection point, which is not going to happen. We can construct this model to look like figure 2 and notice the lines go off to infinity.

In this model the flux of $\vec{E}$ through a surface S is the number of field lines that move through the surface.

$$\phi_E = \int_S \vec{E} \cdot d\vec{a} \tag{6}$$

It might be prudent here to emphasize again that the number of field lines to draw doesn't matter, there will be infinite. What does matter, is that the flux will be proportional to the number of lines

Figure 3

Figure 4

drawn, because the field strength is proportional to the density of the field lines, and furthermore the flux ($\vec{E} \cdot d\vec{a}$) is proportional to the number of lines passing through the infinitesimal are $d\vec{a}$. Recall that the dot product is going to pick out the direction of $\vec{E}$ as shown in figure (). Specifically the area perpendicular to the electric field is found when discussing the density of field lines being the number per unit area. There is an infinitesimal amount of field lines moving through an infinitesimally small area! This is difficult to imagine.

The essence of **Gauss's Law** is to say that the flux through any *closed* surface is a measure of the total charge inside. Charges originating inside will either pass through the surface, or cancel with an opposite charge while charges outside the surface will enter and exit thus contributing nothing to the flux. It seems all that is noteworthy here are the field lines that pass through the surface a single time.

If we had a spherical surface or some radius r and a point charge within, the flux of the electric field would be

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\epsilon_0}(\frac{q}{r^2}) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}) = \frac{1}{\epsilon_0} q \tag{7}$$

with $\hat{r}$ fixed so no dr needed here as we just care about the surface area of the sphere, $d\theta d\phi$ in spherical coordinates keeping the $\hat{r}$ unit vectors in mind when setting up the integral and allowing the dot product to take care of them as parallel unit vectors. Something interesting to note here is that the radius from both vectors cancel. As the radius increases as r^2 is is also decreasing as $\frac{1}{r^2}$ and so is constant. This makes sense because the same number of field lines will pass through the spherical area if the charge is centered at the origin no matter the size of the sphere. Interesting too, is that this is not spherically dependent, the charge could be enclosed by *any surface shape* like in figure 4 and the flux would still be $\frac{q}{\epsilon_0}$.

To generalize this, if a bunch of charges scattered instead of a lone charge, well using the principle of superposition they can each be added together to find the vector sum of all the fields:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

Therefore if we wanted the flux through a surface that encloses them all would be,

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \Big(\oint \vec{E}_i \cdot d\vec{a} \Big) = \sum_{i=1}^{n} \frac{1}{\epsilon_0} Q_{enc}$$

So then,

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} \tag{8}$$

This is a quantitative statement of **Gauss's Law** with Q_{enc} being the total charge enclosed within the surface. It is interesting that this holds no new information. Everything derived so far was done so from Coulomb's Law and the principle of superposition. Griffiths calls this magical and makes sure to mention that the power of Gauss's Law comes from the beautiful cancellation of the squared radius, r^2, if that had not taken place, then Eq.6 would not have taken place and the flux would depend on the shape of the surface instead of the total charge enclosed. Apparently there are other forces that obey an inverse squared law (like the universal law of gravitation) and have their own versions of Gauss's Law that later will be shown to directly carry over!

To say again *the flux through any surface enclosing the charge is* $\frac{q}{\epsilon_0}$. Okay, so I have made Gauss's Law an integral equation but it can be turned into a *differential* one by applying the divergence theorem,

$$\oint_S \vec{E}\, d\vec{a} = \int_V (\nabla \cdot \vec{E})\, d\tau$$

rewrite Q_{enc} in terms of the charge density ρ, to get

$$Q_{enc} = \int_V \rho\, d\tau$$

So Gauss's Law then turns into,

$$\int_V (\nabla \cdot \vec{E})\, d\tau = \int_V (\frac{\rho}{\epsilon_0})\, d\tau$$

Since this holds for any volume, the integrands must be equal to each other. Pull a little fundamental theorem of calculus trick and voila!

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{9}$$

This equation says the same thing as the integral form of Gauss's Law, but the integral form is nice for point, line, and surface charges while the differential version just looks cleaner.

1.2.2 The Divergence of the Electric Field

One way to calculate the divergence of the electric field is to begin with Eq.5,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{allspace} (\frac{\hat{r}}{r^2}) \rho(\vec{r_q})\, d\tau \tag{10}$$

Normally this is integrated throughout the volume of some shape, but in this case "all space" is used since $\rho = 0$ in the exterior region and this is encapsulated within the equation. Since an $\vec{r}$ dependence occurs within the $\vec{r_s}$, it must remain inside the integral.

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot (\frac{\hat{r}}{r^2}) \rho(\vec{r_q})\, d\tau$$

This is the divergence that Griffiths calculates in his first chapter, but these notes to not cover this material, but it is listed here.

$$\nabla \cdot (\frac{\hat{r}}{r^2}) = 4\pi\delta^3(\vec{r_s})$$

Putting it all together,

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(\vec{r} - \vec{r_q}) \rho(\vec{r_q})\, d\tau = \frac{1}{\epsilon_0}\rho(\vec{r})$$

This is Gauss's Law in differential form again. The integral form can be recovered by just tracing your steps.

$$\int_V \nabla \cdot \vec{E}(\vec{r})\, d\tau = \oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_V \rho\, d\tau = \frac{1}{\epsilon_0} Q_{enc}$$

Figure 5

1.2.3 Applications of Gauss's Law

This will just be a series of examples used to demonstrate how useful Gauss's Law is when symmetry permits. It has already been shown how Coulomb's Law was dramatically simplified by the symmetry of Gauss's Law, but some applications may be helpful in furthering understanding.

Example 1.2.3.1 Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Imagine a spherical surface at radius $r > R$; this is called a **Gaussian surface** in the trade and is shown at figure 5. Gauss's Law says that

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$

,

and in this case $Q_{enc} = q$. At first glance this doesn't seem to get us very far because the quantity we want, is buried inside the surface integral. Luckily, symmetry allows us to extract from under the integral sign. The key here is that points radially outwards at all points, and so does d, they are parallel so the dot product can be excused. Giving,

$$\oint_S \vec{E} \cdot d\vec{a} = \oint_S |\vec{E}| d\vec{a}$$

and the magnitude of is constant over the Gaussian surface as well, so it comes outside the integral:

$$\oint_S |\vec{E}| d\vec{a} = |\vec{E}| \oint_S d\vec{a} = |\vec{E}|(4\pi r^2)$$

Therefore, we get

$$|\vec{E} 4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{\hat{r}}$$

Something of importance to notice here is that the electric field is exactly the same outside the sphere as it would have been if the charge were simply concentrated at a point in the center or at the origin. Remember that Gauss's Law deals with the **flux**, which doesn't matter how far from the source of the field lines you are, all of them will be captured.

This makes me think of using a net to catch all the fishies swimming outward radially. It is always the same number of fish that leave the starting point, the more spread out they are and the larger my net is makes no difference since all the fish are still captured.

Griffiths makes a distinction here and says Gauss's Law is *always true*, but it is not *always useful.* The catch here is that if the charge distribution ρ was not uniform, or if the surface was not completely symmetrical. It would still be true that he flux of the electric field is the charge over the permiativity of free space, but if the surface were to lose its symmetry then $\vec{E}$ would not point radially outward everywhere parallel with $d\vec{a}$ and we would lose the simplicity of pulling E out of the integral as a constant. It is the symmetry that makes Gauss's Law significant and effective.

As far as Griffiths knows, there are only three kinds of symmetrical shapes that work for Gauss's Law,

- Spherical symmetry: Using a sphere as the Gaussian surface
- Cylindrical symmetry: Using a coaxial cylinder

Figure 6

Figure 7

- Plane symmetry: Using a Gaussian "pillbox" that straddles the surface

The cylinder and plane symmetry seen in figure 6 require the surface to be infinitely long cylinders and planes, but they can be used to approximate answers for "long" cylinders and "large" planes at point far from the edges. The sphere, as we've seen can be of any radius that encapsulates the charge. This can be just slightly more volume than the point charge, or very large volumes less than infinity. Whatever the shape, the Gaussian surface *needs to be closed*.

Another example to look at involving the cylindrical symmetry,
Example 1.2.1.2 A long cylinder in figure 7 carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

Drawing a Gaussian cylinder of some length l and radius s. For this surface, Gauss's Law states:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$

The enclosed charge is

$$Q_{enc} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz)$$

$$Q_{enc} = k \int_0^{2\pi} \int_0^{l} \int_0^{s} s'^2 \, ds' \, d\phi \, dz$$

$$Q_{enc} = 2\pi kl \int_0^{s} s'^2 \, ds' = \frac{2}{3}\pi kls^3.$$

Like mentioned before, the electric field points radially outward along all of the curved portion of the cylinder, giving

$$\int \vec{E} \cdot d\vec{a} = \int |\vec{E}| d\vec{a} = |\vec{E}| \int da = |\vec{E}| 2\pi sl$$

The two end caps of the cylinder contribute nothing to the flux since $\vec{E}$ and $d\vec{a}$ are perpendicular to each other and the dot product is zero at every point along the end caps. Therefore our electric field is found to be,

$$|\vec{E}| 2\pi sl = \frac{1}{\epsilon_0} \frac{2}{3} \pi kls^3$$

$$\vec{E} = \frac{1}{3\epsilon_0} ks^2 \hat{s}$$

The next example is to demonstrate planar symmetry. This is a limited symmetry since this only applies to an infinite plane, but is useful for approximating the electric field at any surface that can be zoomed in on and made to seem like an infinite plane.
Example 1.2.3.3 An infinite plane carries a uniform surface charge σ. Find its electric field. Drawing a "Gaussian pillbox", extending equal distances above and below the plane in figure 8. Apply Gauss's Law to this surface:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$

Figure 8

Figure 9

In this case, the charge enclosed $Q_{enc} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\vec{E}$ points normal to the plane on both bottom and top of the plane. The top and bottom surfaces are equal and opposite and yield,

$$\int \vec{E} \cdot d\vec{a} = 2A|\vec{E}|$$

Whereas the sides contribute nothing again since the area vector is normal to the sides and perpendicular with the electric field from the plane thus capturing no charge (or fish) in the flux (or net). Giving,

$$2A|\vec{E}| = \frac{1}{\epsilon_0}\sigma A$$
$$\vec{E} = \frac{1}{2\epsilon_0}\sigma\hat{n}$$

Griffiths takes a moment to make note of the surprising fact that the electric field of the infinite plane is independent of how far away you are. Yes, Coulombs Law says the field vanishes with respect to the inverse square law, but with the plane, the farther from the plane you are, the more of the plane enters your field of view. This compensates for the diminishing influence of any particular piece. The field of a line vanishes with $\frac{1}{r}$, the field of the sphere vanishes with $\frac{1}{r^2}$, and the field of an infinite plane does not vanish."You cannot escape an infinite plane."

We are limited to these three shapes, planes, cylinders, and spheres, but we can also put these shapes together in a system that is not so symmetrical and still use Gauss's Law to find the electric field between them. This will work for a sphere above an infinite plane or two planes parallel. This invokes the power of super position principle to work. Lets try this now in an example.

Example1.2.3.4 Two infinite parallel planes carry equal and opposite uniform charge densities, figure 9. Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

The left plate produces a field $\frac{\sigma}{2\epsilon_0}$ as we found earlier, and this points away from it to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $\frac{\sigma}{2\epsilon_0}$ as well and this field will point *toward* it to the right in regions (i) and (ii) and to the left in region (iii) this analysis can also be seen in figure 9 The two fields cancel in regions (i) and (iii), they point in the same direction in region (ii). The fields are zero then outside of the two plates, and $2(\frac{\sigma}{2\epsilon_0}$ both halves come together in between the plates be make a stronger field. The principle of superposition allowed this problem to be so simple, that and Gauss's Law.

1.2.4 The Curl Of The Electric Field

Just as we calculated the divergence of the electric field in section 1.2.2 we will now calculate the curl. The simplest possible model will be the point charge centered at the origin.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Looking at figure 1 we can intuit that the curl of a point charge is zero. We can be a little more rigorous and begin by calculating a line integral from some point a to another point b as in figure .

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical coordinates, $d\vec{l} = dr\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi}$, so this gives us

$$\vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

therefore,

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr$$

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{-1}{4\pi\epsilon_0}\frac{q}{r^2}\Big|_{r_a}^{r_b}$$

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{-1}{4\pi\epsilon_0}\left(\frac{q}{r^2} - \frac{q}{r^2}\right)$$

$$\int_a^b \vec{E} \cdot d\vec{l} = 0 \tag{11}$$

and applying Stoke's Theorem,

$$\nabla \times \vec{E} = 0 \tag{12}$$

This only proves these equations for the field of a single point charge at the origin, but these results make no reference to what is, after all a perfectly arbitrary choice of coordinates. These equations will hold no matter where the charge is located. The principle of superposition states that the total field is a vector sum of their individual fields, $\vec{E} = \vec{E_1} + \vec{E_2} + \vec{E_3} + ...$, so

$$\nabla \times \vec{E} = \nabla \times (\vec{E_1} + \vec{E_2} + \vec{E_3} + ...) = (\nabla \times \vec{E_1}) + (\nabla \times \vec{E_2}) + ... = 0$$

These equations hold for *any static charge distribution whatever.*

1.3 Electric Potential

The fastest way to sum up this section is by stating,

$$\vec{E} = -\nabla V \tag{13}$$

This equation states that the electric field is equal to the gradient of the potential scalar function. This is just the theor

1.4 Capacitors

Capacitance is a way of storing charges. Charge can be stored on two oppositely charged objects that come in a variety of shapes and sizes that are conventionally called "plates". There is a potential difference between the two plates proportional to the charge such that,

$$\frac{q}{V} = C. \tag{14}$$

Where the proportionality of charge and potential equal a constant value called "Capacitance", and this is dependent on the geometry of the plates. V is not volume, so maybe one might think, how is C dependent on the geometry of the plates if this proportionality does not include area or any spatial dimensions? One way to satisfy these thoughts are to think of how to calculate the electric potential, where the integral depends on the electric field and a path, where the electric field might be found using Gauss' Law which depends on enclosed charge and the surface area of a shape. But we won't get into the weeds with this concept just yet! For now, it is enough to know that more charge, and/or less potential difference is more capacitance. A battery in a circuit can provide this potential difference and pushing the charges along the circuit to collect on one of the plates, causing a collection of opposite charges along the other plate. In this way, the battery provides the proportionality between potential and charge to cause capacitance.

A process to keep in mind when calculating capacitance is the assumption of charge collecting on plates, an electric field that is generated from these charges, a potential difference generated from the electric field, and the capacitance is calculated from knowledge of the charges and potential.

1. Assume a charge on plates
2. Calculate electric field $\vec{E}$ using Gauss' Law, which is proportional to q
$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$
$$Q = \epsilon_0 EA$$
3. Calculate electric potential V using $\vec{E}$, which is also proportional to q
$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$
4. Calculate capacitance knowing q and V.
$$C = \frac{q}{V}$$

1.4.1 Parallel Plate Capacitor

In this example the plates are so large and so close together that nothing complicated happens with the electric field at the edges, allowing $\vec{E}$ to be constant everywhere between the plates. Using Gauss' Law to enclose the charge on a single plate, say positive gives, $q = \epsilon_0 EA$.

With constant E our potential becomes,

$$V = \int_i^f \vec{E} \cdot d\vec{s} = \int_-^+ \vec{E} \cdot d\vec{s} = \int_0^d \vec{E} \cdot d\vec{s} = Ed.$$

Substitute q and V into C to give,

$$C = \frac{q}{V} = \frac{\epsilon_0 EA}{Ed} = \frac{\epsilon_0 A}{d}.$$

This demonstrates the geometric dependence of capacitance on the surface area of the plates and the distance between them. Both our geometric dependencies came from calculating the electric field and potential so they are all connected.

1.5 Magnetic Fields

1.5.1 Charged Particle Moving In A Field

Thinking of a charged particle moving in a circular path within a uniform magnetic field. The Lorentz force will act upon the particle curving its trajectory perpendicular to the direction of motion like,

$$\vec{F} = q\vec{v} \times \vec{B}.$$

The force applied to the charged particle is analogous to a stone swinging on a rope, or a satellite orbiting in space around Earth. In the satellite's case, the field of gravity is acting on the object with a centripetal force pulling inwards. The same is happening with the charged particle in a magnetic field. The particle has a velocity and is moving in a direction, but the force from the magnetic field acts perpendicular to the direction of motion, creating a circle and inwards to that circle.

Recall Newton's Laws and the centripetal acceleration,

$$\vec{F} = m\vec{a} = m\frac{v^2}{r}.$$

Substituting the Lorentz force when $\vec{v}$ and $\vec{B}$ are perpendicular into Newton's Law gives,

$$q\vec{v}\vec{B} = m\frac{v^2}{r}.$$

Solving for r,

$$r = \frac{mv^2}{q\vec{v}\vec{B}} = \frac{mv}{q\vec{B}}.$$

Now we can see that fast particles will have a greater radius, and slow particles will have a small radius in their circular trajectory, but the time it takes for the particle to revolve around that circular path is *not* dependent on the speed!

The period is some circumference divided by the speed,$T = \frac{2\pi r}{\vec{v}}$, but if the radius we just calculated is plugged into our equation for period the velocity drops out. So frequency, angular frequency, and period all don't care how fast the particle is traveling (at non-relativistic speeds).

$$T = \frac{2\pi m}{q\vec{B}}, \qquad f = \frac{1}{T} = \frac{q\vec{B}}{2\pi m}, \qquad \omega = 2\pi f = \frac{q\vec{B}}{m}.$$

1.5.2 Biot-Savart Law

2 Electrodynamics

2.1 Introduction

2.2 Maxwell's Equations

2.3 Magneto-hydrodynamics

2.3.1 Example: Dispersion Relation of Unmagnetized Plasma

Consider a transverse electromagnetic wave mode propagating in an unmagnetized, partially ionized gas in which the electron-neutral collision frequency is ν_e. Include the effects of collisions in the electron equation of motion, by introducing a term $n_e m_e \nu_e \mathbf{u}_e$ on the right hand side. Ignore ion motion and electron-ion and electron-electron collisions.

Derive the dispersion relation using $L_{ij}E_j = 0$. In this case, $\omega >> \nu_e$ and the electron equation of motion is,

$$-i\omega n_e m_e \boldsymbol{u_e} = q_e n_e \boldsymbol{E} - n_e m_e \nu_e \boldsymbol{u}_e$$

Recall that the linearized current density is $\boldsymbol{j} = \sum_s n_s q_s \boldsymbol{u}_s = \sum_s \frac{i n_s q_s^2}{m_s \omega} \boldsymbol{E}$

Solution: Starting from the electron equation of motion and deriving the current density,

$$-i\omega n_e m_e \boldsymbol{u_e} - q_e n_e \boldsymbol{E} + n_e m_e \nu_e \boldsymbol{u}_e = 0$$

$$n_e m_e \boldsymbol{u}_e(-i\omega + \nu_e - \frac{q\boldsymbol{E}}{m_e \boldsymbol{u}_e}) = 0$$

From here we can make the assumption that $\omega >> \nu_e$ to simplify the equation. Then we redistribute and multiply by $\frac{iq}{\omega m_e}$,

$$q n_e \boldsymbol{u}_e - \frac{iq^2 n_e}{\omega m_e}\boldsymbol{E} = 0$$

Then substituting $\boldsymbol{j} = q n_e \boldsymbol{u}_e$, we get the current density,

$$\boldsymbol{J} = \frac{iq^2 n_e}{\omega m_e} = \boldsymbol{\sigma E}$$

$$k_{eij} = \sum_e \frac{iq^2 n_e}{\omega m_e}\delta_{ij} = \boldsymbol{\sigma}\delta_{ij}$$

From here the energy lost per unit volume is balanced by Ohmic heating in the perturbed plasma. We can use the conductivity tensor in the ohmic relation to find the plasma frequency ω_p which accounts for a tiny correction due to the movement of the protons.

2.4 Plasmas

2.5 Radiation From Moving Charge

2.6 Multipole Radiation

2.7 Scattering

3 Mechanics

3.1 Introduction

3.2 Laws of Motion

3.3 Oscillations

3.3.1 Introduction

3.3.2 Harmonic Motion

Oscillations with a linear restoring force is referred to as harmonic motion and is described by Hooke's Law. The most simple model that demonstrates harmonic motion is a block on a horizontal friction-less surface connected to a wall by a spring with some restoring force constant, k. Although the model is simple, this same motion is used to describe a broad variety of cyclic motion. Hooke's Law is where to start,

$$\vec{F}(x) = -\frac{dV(x)}{dx} = -(2a_2)x = -kx \tag{15}$$

where $k = 2a_2$ is the spring constant. This is how small displacements from equilibrium are defined, at least for displacements small enough to obey Hooke's Law and keep the restoring force linear. If the spring were to be stretched too far, this would no longer be a linear force. There is a requirement that the derivative of the potential energy function must be negative for positive displacements from equilibrium and vice versa. This gives us the negative sign in Hooke's Law, and can be added to the second law of motion,

$$m\ddot{x} + kx = 0 \tag{16}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

This is a second order differential equation with some constant coefficients and can be solved in many ways. The superposition principle holds for solutions to this equation. To expose some characteristics about this equations, the motion is both periodic and bounded. This means that a mass would vibrate back and forth between two limiting positions. Say the block on the spring is stretched to some positive (right) position x_{m1} and then released from rest. The restoring force, $-kx_{m1}$ pulls the mass back toward the left decreasing as the block approaches $x = 0$ where the equilibrium position resides. As the block moves past the equilibrium point due to momentum, and the spring begins to compress, the restoring force will act upon the block to the right and begin to increase as the spring compresses. This will slow the block down until $v = 0$ at a turning point, x_{m2}, where it stops for an instant and begins to move to the right. The restoring force always "pulling" toward equilibrium and momentum carrying the block through it. The spring continues to oscillate the block between the two x positions with these forces maintaining cyclic behavior. Functions that demonstrate this cyclic behavior are sines and cosines and these functions are the "real" solutions to the second order differential equation above. Later, exponential functions will be used in their place.

A solution to Eq.16 would be,

$$x = A\sin(\omega_0 t + \phi_0) \tag{17}$$

This can be checked by plugging it in to the differential equation. Another important equation is the angular frequency of the system,

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and is apart of the cyclic nature of Eq.17. We have a sinusoidal cyclic oscillation about an equilibrium. Three things that characterize this motion:

- It is characterized by a single angular frequency ω_0
- The motion is bounded
- The phase angle ϕ_0 is the initial value of the angular argument of the sine function

The motion is characterized by this angular frequency, ω_0, and after the sine function $(\omega_0 t + \phi_0)$ has advanced by 2π, or one cycle has finished. The time taken to complete one cycle or to advance by 2π is

$$\omega_0(t + T_0) + \phi_0 = \omega_0 t + \phi_0 + 2\pi$$

This gives the period of motion,

$$T_0 = \frac{2\pi}{\omega_0} \tag{18}$$

The bounded motion will just mean it is confined within the limits of the two extreme positions x_m earlier, this can also be called the *amplitude* bounding x to $-A \leq x \leq +A$. The amplitude is the maximum displacement from equilibrium and is independent of the angular frequency. The phase angle ϕ_0 determines the value of the displacement x at time $t = 0$. For example, $t = 0$ gives,

$$x(t = 0) = A \sin(\phi_0)$$

The max displacement from equilibrium will occur at a time t_m provided by the condition that the angular argument of the sine function is equal to $\frac{\pi}{2}$ or,

$$\omega_0 t_m = \frac{\pi}{2} - \phi_0$$

Common expressions are:

- frequency: $f_0 = \frac{1}{T_0}$
- angular frequency: $\omega_0 = 2\pi f_0$
- combined: $f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

These help describe frequency, period, and angular frequency and show how they are related to each other. Frequency, f is in units s^{-1} called hertz, named after Heinrich Hertz credited with the discovery of radio waves.

3.3.3 Constants of the Motion and Initial Conditions

There are some arbitrary constants to flesh out of Eq.17, A and ϕ_0. The value of each constant are determined by the initial conditions that vary with each problem. The simplest initial conditions include having a block stretched out a maximum displacement x_m at $t = 0$, therefore the values of the constants would become $A = x_m$ and $\phi_0 = \frac{\pi}{2}$.

Another simple example is starting an oscillator at rest, $x = 0$ at $t = 0$ and giving it a "kick" that provides an initial velocity, v_0 in the positive x direction. In this case the phase hasn't been shifted and $\phi_0 = 0$. This automatically ensures the solution yields $x = 0$ and $t = 0$. To find the amplitude in this case, differentiate x to get the velocity of the oscillator as a function of time and call it v_0 at $t = 0$.

$$v(t) = \dot{x}(t) = \omega_0 A \cos(\omega_0 t + \phi_0)$$

$$v(0) = v_0 = \omega_0 A$$
$$A = \frac{v_0}{\omega_0} \tag{19}$$

This gives us the amplitude having known the initial conditions.

More generally, consider some mass moved some distance x_m and given an initial velocity v_0 the constants can be determined,

$$x(0) = A \sin(\phi_0) = x_0$$
$$\dot{x}(0) = \omega_0 A \cos(\phi_0) = v_0$$

therefore,

$$\tan(\phi_0) = \frac{\omega_0 x_0}{v_0}$$
$$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2} \tag{20}$$

by setting x_0 or v_0 equal to zero, this general solution should reduce to either of the equations above. To analyze this a little further for clarity, consider $x_0 = 0$,

$$A^2 = \frac{v_0^2}{\omega_0^2}$$

$$A = \frac{v_0}{\omega_0}$$

$$\tan \phi_0 = 0$$

This is useful for solving for the constants in the problem. This can be applied to v_0 as well but it is not necessary to type that here.

4 Quantum Mechanics

4.1 Time Independent Schrodinger Equation

4.2 Identical Particles

4.2.1 Two-Particle Systems

4.3 Atoms

4.4 Solids

Instead of focusing on elections "attached" to the nucleus of an atom, we now look at many electrons in a solid built of many atoms. In a solid, the electrons are allowed to "float" around as "valence" electrons sort of riding highways from one atom to the next. There are two ways to model a solid, the electron gas theory of Sommerfeld and Bloch's theory. Sommerfeld's ignores all forces but the boundaries which treats the wandering electrons as free particles in a box (3D infinite square well). Bloch's theory introduces periodic potential representing the electrical attraction of the regularly spaced, positively charged, nuclei while still ignoring electron pair repulsion. These are rudimentary quantum theories of solids, but they reveal the critical role of the Pauli exclusion principal in accounting for "solidity" and offer insight into the remarkable properties of conductors, semi-conductors, and insulators.

4.5 Statistical Quantum Mechanics

So we've been leading up to statistical quantum mechanics this whole chapter I think. There will be a lot of probabilities again and should be analogous to statistical mechanics. Starting from absolute zero we know that a physical system will occupy it's lowest allowed energy state configuration. As the temperature increases, random thermal activity or motion becomes more prominent and a system of particles would vibrate. The point of this section is to explore the probability of a system of N particles in equilibrium held at some temperature T having a specific particle with a specific energy E.
Remember that the **fundamental assumption of statistical mechanics** says that in thermal equilibrium, every distinct state of some total energy is equally probable. The random thermal motion mentioned above transfers energy between particles and also changes the type of energy considered like kinetic, vibrational, or rotational while keeping the total energy conserved. In all this transferring of energy and switching of state, no particular particle or state will experience favoritism! That's a pretty big assumption!
In the quantum version of this the state are discrete, which should make things easier to work with. The temperature will still be the total energy in a system and this assumption also calls for no external forces, but let's open this can of beans.

Example 4.5.1 Here is a simple example to get the ball rolling and develop some fundamental understanding of the process. Here we have three particles of equal mass in a 1D infinite square well, the total energy here would be,

$$E = E_1 + E_2 + E_3 = \frac{\pi^2\hbar^2}{2ma^2}(n_1^2 + n_2^2 + n_3^2)$$

This comes section 4.1 except it's being applied to three particles instead of one. Now say that the total energy for the system is known to be,

$$E = 363(\frac{\pi^2\hbar^2}{2ma^2})$$

$$n_1^2 + n_2^2 + n_3^2 = 363$$

.

This just says that the total of the squared states of the three particles must equal 363. Well, there are several different combinations to add up to 363 (13 exactly). Here we go,

$$(n_1, n_2, n_3)$$
$$(11, 11, 11)$$
$$(13, 13, 5), (13, 5, 13), (5, 13, 13)$$
$$(1, 1, 19), (1, 19, 1), (19, 1, 1)$$
$$(5, 7, 17), (5, 17, 7), (7, 5, 17), (7, 17, 5), (17, 5, 7), (17, 7, 5)$$

.

Yup, that's all of them. If these particles are distinguishable and these are discrete quantum states, then they are all equally probable according to the fundamental assumption. Griffiths takes a footnote here talking about the way in which these particles maintain thermal equilibrium is a problem not to worry much about. He mentions divine intervention as a possible way to think of this and accept it for what it is. The other option is weak interactions, just strong enough to thermalize the system but too weak to alter the stationary states more than negligibly. Alright, let's get some terminology right, what we care more about here is the number of particles in each state called the occupation number N_n, for the state of ψ_n. The collection of all the occupation numbers for this given three particle state will be called the configuration.

Using this, if all the particles were in state ψ_{11} the configuration would be,

$$N_{11} = 3 \rightarrow (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, ...)$$

$$N_5 = 1, N_{13} = 2 \rightarrow (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, ...)$$

$$N_1 = 2, N_{19} = 1 \rightarrow (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, ...)$$

$$N_5 = N_7 = N_{17} = 1 \rightarrow (0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, ...)$$

Of all of these configurations, the last one is the most probable because it can be achieved in 6 different ways showed above where as the first one can only be achieved a single way and is the least probable. So lets complete this though and find the probability of finding a particle with a specific energy. If we want to find the probability of E_1, we have to go to the third configuration with N_1 where the chances of being in this configuration are 3 of the total 13. The probability of getting E_1 in that configuration is $\frac{2}{3}$ since $N_1 = 2$ and $N_{19} = 1$. Great so the total probability of finding the that state is $P_1 = (\frac{3}{13})(\frac{2}{3}) = \frac{2}{13}$. I won't do it because I'm lazy but if you found the probability of each energy state in all configurations and added the probabilities together if you be, you guessed it, $P = 1$.

That is how to find the probabilities for distinguishable particles! Lets try our identical fermions and bosons. Right off the bat this gets easier knowing that the first three configurations just plain cannot happen due to the Pauli Exclusion Principle which leaves us with the fourth configuration. In which case $P_5 = P_7 = P_{17} = \frac{1}{3}$ and the sum is still $P = 1$. If they are identical bosons, the symmetrization requirement allows for 1 state in each configuration instead of all the distinguishable ones bringing us down to 4. So now the probability of ψ_1 is $P_1 = (\frac{1}{4})(\frac{2}{3}) = \frac{1}{6}$.

Okay that was a little tedious, but it was an important example to show because this is going to be what is generalized in the next section. textbfThe counting states depends on the nature of the particles. When the particles are distinguishable the most probable configuration is $N_5 = N_7 = N_{17} = 1$ and as the number of particles becomes a huge number that probability becomes overwhelmingly more probable than the other configurations leaving them negligible. *The distribution of individual particle energies, at equilibrium, is simple their distribution in the most probable configuration.*

4.5.1 The General Case

5 Quantum Field Theory

The study of a quantized system of fields.

5.1 Klein-Gordon Field

5.2 Lorentz Invariance

6 General Relativity

6.1 Special Relativity

6.2 Tensor Analysis

6.3 Differentiable Manifolds & Tensors

First things first is that a *manifold* is a zoomed in space to locally look like Euclidean space, but in this section the curviness will be added in such a way that allows for global curves. So parameters make up the dimensions of space in the manifold (like on the surface of a sphere) and locally the manifold is flat, but globally the surface curves.

6.4 Perfect Fluids In Special Relativity

This section focuses on relativistic fluids by first defining the fluid and the pieces used to model it. The fluid in this case is a simple "dust" because it meets the requirements of being as compact or spacious as needed in a fluid element. A "fluid element" is a section of the fluid that maintains some constant volume and density, but may change shape and pressure. A fluid element is large enough to include many particles that make up the fluid, but not so many that the volume or density can vary within the element, or "boxy" thing.

We also look at the number density of particles n within some frame with velocity in this section,

$$\frac{n}{\sqrt{1-v^2}}.$$

To talk about the flux, I would need to draw pictures and I'm not going to do that right now, but the equation is this,

$$(flux)^{\bar{x}} = \frac{nv^{\bar{x}}}{\sqrt{1-v^2}}$$

6.4.1 Number Flux Four-Vector

Beginning by defining a vector,

$$\vec{N} = n\vec{U}$$

Where $\vec{U}$ is the four-velocity of the particles. Therefore, in some frame where they have a velocity, the four-velocity vector would look like,

$$\vec{U} = \Big(\frac{1}{\sqrt{1-v^2}}, \frac{v^x}{\sqrt{1-v^2}}, \frac{v^y}{\sqrt{1-v^2}}, \frac{v^z}{\sqrt{1-v^2}}\Big)$$

And so the four-vector $\vec{N}$ looks like,

$$\vec{N} = \Big(\frac{n}{\sqrt{1-v^2}}, \frac{nv^x}{\sqrt{1-v^2}}, \frac{nv^y}{\sqrt{1-v^2}}, \frac{nv^z}{\sqrt{1-v^2}}\Big).$$

Schutz takes a moment here to mention that this is a very important conceptual result. In Galilean physics, number density was just a scalar that was frame independent and ignored Lorentz contraction. Flux was frame dependent since there was included a time dependency in the velocities that changed with each frame. This relativistic approach has combined them into a single frame independent four-vector. Schutz declares this as progress in our thinking.

Now we have some geometrical object that is the same in all frames. It's components will change with the frame, but the object itself is preserved. A three-vector of old could be frame independent, but had to be defined relative to some other inertial frame. Frames moving relative to each other would not preserve three-vector flux on their own, our four-vector is able to maintain its autonomy in all frames regardless of their motion.

Our number density used above is a scalar in the sense that,

$$\vec{N} \cdot \vec{N} = -n^2, \quad n = (-\vec{N} \cdot \vec{N})^2.$$

So in the same way rest mass and energy are scalars, number density is thought of as a rest density. However energy and inertial mass are frame dependent and our rest density is not. This same frame dependence will be done to pressure, temperature, and other properties of fluids so that we can construct a frame independent fluid element to be used in relativistic models.

6.4.2 One Forms And Surfaces

So apparently a surface is generally defined as a solution to an equation,

$$\phi(t, x, y, z) = C.$$

The gradient of the function ϕ, is $\tilde{d}\phi$ and is a normal one-form. In some sense, $\tilde{d}\phi$ defines the surface $\phi = C$ as it uniquely determines the directions normal to that surface. Any multiple of the one-form $\tilde{d}\phi$ also defines the surface, so it is customary to use what is called the "unit normal one-form" when the surface isn't zero,

$$\tilde{n} = \frac{\tilde{d}\phi}{|\tilde{d}\phi|}$$

Where $|\tilde{d}\phi|$ is the magnitude of $\tilde{d}\phi$ and equal to $|\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}|^{\frac{1}{2}}$. Also, it's important to note that $\tilde{n}$ and n are not the same, but historically use the same letter.

6.4.3 General Fluids

Our fluid element before this section was simplified to neglect variations in particle velocity and forces acting between them within the fluid element. Now we will include those.

6.5 Spherical Solutions For Stars

This will be the first look at strong gravitational fields in GR. Spherically symmetric systems are reasonably simple and many astrophysical objects have this approximate symmetry.

Coordinates For Spherically Symmetric Spacetimes

First we need to choose a coordinate system to reflect this symmetry we will look at. By defining the usual coordinates of (r, 0, ϕ), the Minkowski space can be written,

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Each surface of constant r or t is a sphere, Shutz calls it a "two sphere", which is a two dimensional spherical surface. Distances along this two dimensional sphere have dr = 0 and dt = 0. This simplifies the coordinates to,

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = r^2 d\Omega^2$$

Making $d\Omega^2$ the symbol for an element of a solid angle. A sphere has a circumference of $2\pi r$ and a surface area of $4\pi r^2$ still. Which can look like 2π times the square root of the coefficient of $d\Omega^2$ and 4π times the coefficient of $d\Omega^2$. So, any two surface with the line element above and r^2 independent of θ and ϕ will have the intrinsic geometry of a two sphere.

6.6 Schwartzchild Geometry And Black Holes