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MATH 527,02
HW4

1. $y''+3y'+2y=10\cos(2x)$ Guess: $y_p(x)=A\cos(Bx)$

$y_p'(x)=-AB\sin(Bx)$

$y_p''(x)=-AB^2\cos(Bx)$

$-AB^2\cos(Bx)+3(-AB\sin(Bx))+2(A\cos(Bx))=10\cos(2x)$

$-AB^2\cos(Bx)-3AB\sin(Bx)+2A\cos(Bx)=10\cos(2x)$

for $A=0$, $0-0+0=10\cos(2x)$

for $A=1$, $-B^2\cos(Bx)-3B\sin(Bx)+2\cos(Bx)=10\cos(2x)$

$y_p(x)=A\cos 2x+B\sin 2x$

$y_p'(x)=-2A\sin 2x+2B\sin 2x$

$y_p''(x)=-4\cos 2x-4B\sin 2x$

-60 $+60$ $+20$ $+20$

$(-40A\cos 2x-40B\sin 2x)+3(-20A\sin 2x+20B\sin 2x)+2(10A\cos 2x+10B\sin 2x)$
$=10\cos 2x$

$-40A\cos 2x+20B\cos 2x=(-40A+20B)\cos 2x$

$-40B\sin 2x-60A\sin 2x+60B\sin 2x+10B\sin 2x$
$=(-4B-6A+6B+1B)\sin 2x=(-6A-2B)\sin 2x$ $A=\frac{-1}{2}$, $B=3/2$

Ansatz: $e^{\lambda x}$ $\lambda=-1,2$ $y_h=C_1e^{-x}+C_2e^{2x}$

$y_p(x)=\frac{-1}{2}\cos 2x+\frac{3}{2}\sin 2x$ $\boxed{y(x)=C_1e^{-x}+C_2e^{2x}+\frac{3}{2}\sin 2x-\frac{1}{2}\cos 2x}$

2. $y''-2y'-8y=e^{4x}$ $y_h(x)=C_1e^{-2x}+C_2e^{4x}$

$(\lambda-4)(\lambda+2)=0$ $\lambda=-2,4$

$y_p(x)=Axe^{4x}$ $y_p'(x)=4Axe^{4x}+Ae^{4x}$

$y_p''(x)=16Axe^{4x}+8Ae^{4x}$

$[16Axe^{4x}+8Ae^{4x}]-2[4Axe^{4x}+Ae^{4x}]-8[Axe^{4x}]=e^{4x}$

$16Axe^{4x}-8Axe^{4x}-8Axe^{4x}=0$ $8Ae^{4x}-2Ae^{4x}=e^{4x}$

$6Ae^{4x}=e^{4x}$ $6A=1$ $A=1/6$ $y_p(x)=\frac{1}{6}xe^{4x}$

$\boxed{y(x)=C_1e^{-2x}+C_2e^{4x}+\frac{1}{6}xe^{4x}}$

3. $y''+y'-6y=\sin t+e^{2t}$ $y_h(x)=C_1e^{-3x}+C_2e^{2x}$

$(\lambda-2)(\lambda+3)$

$y_p(x)=A\cos t+B\sin t+Cte^{2t}$

$y_p'(x)=-A\sin t+B\cos t+2Cte^{2t}+Ce^{2t}$

$y_p''(x)=-A\cos t-B\cos t+4Cte^{2t}+4Ce^{2t}$

$[-A\cos t-B\cos t+4Cte^{2t}+4Ce^{2t}]+[-A\sin t+B\cos t+2Cte^{2t}+Ce^{2t}]$
$-6[A\cos t+B\sin t+Cte^{2t}]=\sin t+e^{2t}$

$A=-1/50$, $B=-7/50$, $C=1/5$ $y_p(x)=\frac{-1}{50}\cos t+\frac{-7}{50}\sin t+\frac{1}{5}te^{2t}$

$y(x)=C_1e^{-3x}+C_2e^{2x}-\frac{1}{50}\cos t-\frac{7}{50}\sin t+\frac{1}{5}te^{2t}$

4. $2y'' - 3y' + y = (t^2+1)e^{2t}$ $\quad y_h(x) = C_1 e^{1/2 x} + C_2 e^{x}$

$y_p(x) = (At^2 + Bt + C)e^{2t}$

$2[4(At^2+Bt+C)e^{2t} + 4(2At+B)e^{2t} + 2Ae^{2t}]$
$-3[2(At^2+Bt+C)e^{2t} + (2At+B)e^{2t}]$
$+[(At^2+Bt+C)e^{2t}]$ $\quad A = 1/3,\ B = -10/9,\ C = \frac{47}{27}$

$y_p(x) = \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}$

$y(x) = C_1 e^{1/2 t} + C_2 e^{t} + \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}$

5. $2y'' - 3y' + y = (t^2+1)e^{2t} \rightarrow y'' - \frac{3}{2}y' + \frac{1}{2}y = \frac{1}{2}(t^2+1)e^{2t}$

$y_h(x) = C_1 e^{1/2 t} + C_2 e^{t}$ $\quad y(x) = u_1 y_1 + u_2 y_2$

$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix}$

$W = \begin{vmatrix} e^{1/2 t} & e^{t} \\ \frac{1}{2}e^{\frac{1}{2}t} & e^{t} \end{vmatrix} = e^{\frac{1}{2}t}e^{t} - e^{t}\left(\frac{1}{2}e^{\frac{1}{2}t}\right) = \frac{1}{2}e^{3/2 t}$

$W_1 = \begin{vmatrix} 0 & e^{t} \\ (t^2+1)e^{2t} & e^{t} \end{vmatrix} = -\frac{1}{2}(t^2+1)e^{3t}, \quad W_2 = \begin{vmatrix} e^{\frac{1}{2}t} & 0 \\ \frac{1}{2}e^{\frac{1}{2}t} & (t^2+1)e^{2t} \end{vmatrix} = \frac{1}{2}(t^2+1)e^{5/2 t}$

$u_1' = \frac{-\frac{1}{2}(t^2+1)e^{3t}}{\frac{1}{2}e^{3/2 t}} \quad u_1 = \left(-\frac{2}{3}t^2 + \frac{8}{9}t - \frac{34}{27}\right)e^{3/2 t} + C_1$

$u_2' = \frac{\frac{1}{2}(t^2+1)e^{5/2 t}}{\frac{1}{2}e^{3/2 t}} \quad u_2 = (t^2 - 2t + 3)e^{t} + C_2$

$y(x) = C_1 e^{1/2 t} + C_2 e^{t} + \left(\frac{1}{3}t^2 - \frac{10}{9}t + \frac{47}{27}\right)e^{2t}$

6. $y'' + y = \sec t$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad y_1 = \cos t \quad y_2 = \sin t$

$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \quad u_1' = \frac{W_1}{W} \quad W_1 = \begin{vmatrix} 0 & \sin t \\ \sec t & \cos t \end{vmatrix} = -\tan t \quad u_1' = -\tan t$

$u_2' = \frac{W_2}{W} \quad W_2 = \begin{vmatrix} \cos & 0 \\ -\sin & \sec t \end{vmatrix} = 1 \quad u_2' = 1$

$u_1 = -\int \tan t\, dt = \ln(\cos t) \quad u_2 = t \quad y_p = \ln(\cos t)\cos t + t\sin t$

$y(t) = C_1 \cos t + C_2 \sin t + \ln(\cos t)\cos t + t\sin t$

7. $y'' - 3y' + 2y = te^{3t} + 1$ $\quad y_h(t) = C_1 \underbrace{e^{t}}_{y_1} + C_2 \underbrace{e^{2t}}_{y_2}$

$W = \begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix} = 2e^{3t} - e^{3t} = e^{3t}$

$W_1 = \begin{vmatrix} 0 & e^{2t} \\ te^{3t}+1 & 2e^{2t} \end{vmatrix} = (te^{3t}+1)e^{2t} \quad W_2 = \begin{vmatrix} e^{t} & 0 \\ e^{t} & te^{3t}+1 \end{vmatrix} = (te^{3t}+1)e^{t}$

$u_1' = \frac{(te^{3t}+1)e^{2t}}{e^{3t}} = \frac{te^{3t}+1}{e^{t}} \quad u_2' = \frac{(te^{3t}+1)e^{t}}{e^{3t}} = \frac{te^{3t}+1}{e^{2t}}$

$u_1 = \int \frac{te^{3t}+1}{e^{t}}\, dt = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + e^{-t}$

$u_2 = \int \frac{te^{3t}+1}{e^{2t}}\, dt = te^{t} - e^{t} - \frac{1}{2}e^{-2t} \quad y_p(t) = \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \frac{1}{2}$

$y(t) = C_1 e^{t} + C_2 e^{2t} + \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} + \frac{1}{2}$