Chapters 3: Discrete Random Variables

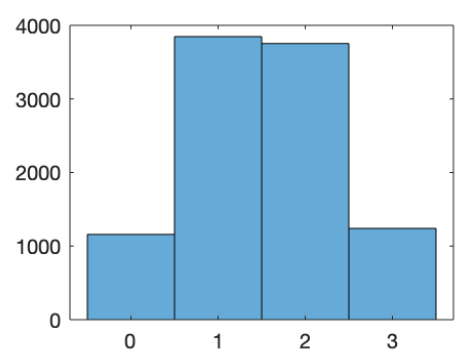
Discrete Random Variables (3.1)

Definition: a random variable (R.V.) is a function that assigns (maps) the sample space, S, to a real number

Notations:

- R.V. is always capitalzed
- The values that R.V. can take on are in lower case
- For a discrete R.V, $S_X = \{x_1, x_2, \dots\}$

- **Example**: consider the experiment involving 3 coin tosses in sequence
  - Sample space
  - R.V: $X =$ "number of heads"



  - Example of 10k experiments
    - Each outcome of sample space may be obtained from *MATLAB*: *randi([0 1],1,3)*
    - We can characterize randomness of the R.V. (X) with the probability measure of each possible value (x)
    - Probability Mass Function (PMF)

Probability Mass Function (3.2)

Definition: the probability mass function (PMF) of a discrete R.V. (X) is

$$P_X(x) = \mathrm{P}[X = x]$$

- Example: toss coin three times in sequence

- Properties of PMF (Thm 3.1): for a random variable X and range S_X
  - For any x, $P_X(x) \geq 0$
  - $\sum_{x \in S_X} P_X(x) = 1$
  - For any event $B \subset S_X$, the probability that X is in the set B is

$$\mathrm{P}[B] = \sum_{x \in B} P_X(x)$$

Families of Discrete Random Variables (3.3)

Definition: Family of R.V. refers to groups of R.V. that have PMFs with same *mathematical* form

- **Bernoulli Random Variable:** X is a *Bernoulli (p)* R.V. if the PMF of X has the form

$$P_X = \begin{cases} 1 - p, x = 0 \\ p, \quad\quad x = 1 \\ 0, \quad \text{otherwise} \end{cases}$$

  - Example: coin tossed three times. $X =$ # of heads is less than two

- **Geometric RV**: X is a *geometric (p)* random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

  Sequence of independent Bernoulli trials performed until the first observation of the outcome related to probability p.

  - Example: Sequence of tests with pass/fail outcomes.
    $p = \text{P[fail]}, \qquad (1-p) = \text{P[Pass]}, \qquad X =$ # of tests until first failure

- **Binomial RV**: X is a *binomial (n,p)* random variable if the PMF of X has the form

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 < p < 1$ and n is an integer such that $n \geq 1$

  - Example: Sequence of n independent Bernoulli trials is performed and x observations of one of the 2 outcomes are made

    $\mathrm{P[fail]} = p$, $\quad\quad$ $\mathrm{P[pass]} = 1 - p$, $\quad\quad$ $X =$ # of failures in n tests

- **Pascal RV:** X is a *Pascal (k,p)* random variable if the PMF of X has the form

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

where $0 < p < 1$ and k is an integer such that $k \geq 1$

  - Example: sequence of Bernoulli trials is performed until k observations of one of the outcomes.

    $p = \mathrm{P[fail]}$, $\quad$ $(1-p) = \mathrm{P[Pass]}$, $\quad$ $X =$ # of tests until *k* failure

- **<u>Uniform RV:</u>** X is a *uniform (k,l)* random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{1}{l-k+1} & x = k, k+1, k+2, \dots, l \\ 0 & \text{otherwise} \end{cases}$$

  - <u>Example:</u> roll a fair die. $X =$ # on side facing up

- **<u>Poisson RV:</u>** X is a *Poisson (α)* random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0,1,2, \dots \\ 0 & \text{otherwise} \end{cases}$$

  - <u>Example:</u> consider a transmitter that is transmitting at a bit rate of 10^9 bits/sec. What is the probability of x transmission errors in 1 second (or 10^9 transmissions)

$$\text{P[bit error]} = 10^{-9}$$

Cumulative Distribution Function (CDF) (3.4)

Definition: the cumulative distribution function (CDF) of a random variable X is

$$F_X(x) = \mathrm{P}[X \leq x]$$

- Note: CDF is another way to express the probability model of an experiment and carries the same information as PMF
  - PMF → CDF:
  - CDF → PMF:

- Probability of RV between two points (*Thm 3.3*): for all $b \geq a$

$$F_X(b) - F_X(a) = \mathrm{P}[a < X \leq b]$$

- Example: considering the following CDF, find the probabilities
  a) $\mathrm{P}[Y < 1]$, b) $\mathrm{P}[Y \leq 1]$, c) $\mathrm{P}[Y > 2]$, d) $\mathrm{P}[Y = 3]$

Expected Value (3.5)

Definition: the expected value of a random variable X is a measure of "central tendency"

$$\mathrm{E}[X] = \mu_X = \sum_{x \in S_X} x\, P_X(x)$$

- Example: consider an experiment that produces a random variable X and we perform n independent trials with observations x_i.

- A reasonable calculation of the expected value of X relates to frequency of a value occurring

- Generalizing expression and allowing trial number $n \to \infty$

- <u>$\mathrm{E}[X]$ for a Bernoulli RV (*Thm 3.4*)</u>: for a *Bernoulli (p)* RV:

- <u>$\mathrm{E}[X]$ for a uniform RV (*Thm 3.4*)</u>: for a *uniform (k,l)* RV

- E$[X]$ for a geometric RV (*Thm 3.5*): for a *geometric (p)* RV

- Other E$[X]$ (*Thms 3.6, 3.7*):
    - *Poisson* (α): $\mathrm{E}[X] = \alpha$
    - *Binomial (n,p):* $\mathrm{E}[X] = np$
    - *Pascal (k,p)*: $\mathrm{E}[X] = k/p$

Functions of Random Variables (3.6, 3.7)

Definition: a function $Y = g(X)$ of a random variable X is also a random variable. The *derived random variable* Y has PMF $P_Y(y)$ that is dependent on $P_X(x)$ and $g(X)$

- **Theorem 3.9:** PMF of derived random variable

$$P_Y(y) = \mu_Y = \sum_{x:g(x)=y} P_X(x)$$

- Example: consider the following PMF

- What is $P_Y(y)$ given that $Y = g(X) = X^2$?

- What is $\mathrm{E}[Y]$?

- **<u>Theorem 3.10:</u>** expected value of derived random variable

$$\mathrm{E}[Y] = \sum_{x \in S_X} g(x) P_X(x)$$

- **<u>Note:</u>** in general, $\mathrm{E}[g(X)] \neq g(\mathrm{E}[X])$

<u>One exception</u>: $g(x) = ax + b$

- **<u>Theorem 3.11:</u>** expected value of linear transformation

$$\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$$

Variance and Standard Deviation (3.8)

<u>Definition:</u> variance $\mathrm{Var}[X]$ is a measure of dispersion (spread) of sample values of X around the expected value $E[X]$

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[(X - \mu_X)^2] \\ &= \mathrm{E}[X^2] - \mu_X^2 = \mathrm{E}[X^2] - (\mathrm{E}[X])^2\end{aligned}$$

<u>Note:</u> also known as the *2nd central moment* of X

<u>Definition:</u> standard derivation σ_X of a random variable X

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

<u>Note</u>: standard variation have the same units as X

- Example: calculate the variance of the random variable with following PMFs

- <u>Property of variance (*Thm 3.15*):</u> consider $Y = aX + b$

$\mathrm{E}[Y^2] = \mathrm{E}[a^2X^2 + b^2 + 2abX] =$

$\mu_x^2 = \mathrm{E}[Y]^2 =$

$\mathrm{Var}[Y] =$