Math 418: Worksheet 8

November 1, 2020

Some Properties of Logarithms: Suppose $a > 0, a \neq 1, M, N > 0$ and n is a real number.

I) $\log_a(a^n) = n = a^{\log_a(n)}$

II) $\log_a(MN) = \log_a(M) + \log_a(N)$

III) $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$

IV) $\log_a(M) = \frac{\log_N(M)}{\log_N(a)}$

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1. Evaluate:
   a) $\log_4 64$
   b) $\log_{144}\left(\frac{1}{12}\right)$
   c) $\log .01$
   d) $\ln \sqrt[5]{e}$
2. Solve the equation $2^{(3x^2)} = 5$.
3. Answer the questions below.
   a) Evaluate $\log_9 27$ and $\log_{27} 9$. How are they related?
   b) Evaluate $\log_2 8$ and $\log_8 2$ How are they related?
   c) Suppose a and b are two positive numbers such that $\log_a b = 2.5$. Evaluate $\log_b a$.
4. Solve $9^x + 3^x - 6 = 0$.
5. Solve $16^x + 4^{x+1} - 12 = 0$.
6. Suppose $f(x) = 2(3^{5x})$.
   a) Prove that $y = \ln(f(x))$ is a linear function.
   b) What is the slope of the linear function in part a? What is the y-intercept?
7. a) Solve $\log_2\left(\frac{x+3}{2x-1}\right) = 2$.
   b) Solve $\frac{\log_2(x+3)}{\log_2(2x-1)} = 2$.

c) Suppose $0 < x < y$. Which of the following inequalities will hold? Fully explain your answer.

   I) $\log_3 x < \log_3 y$

   II) $\log_3 x > \log_3 y$

8. A rare radioactive isotope has a half-life of 24 days. If Dr. Quark has a sample of the isotope that initially masses 18kg, how long will it take before Dr. Neutral only has 2grams left?

9. Jim wants to buy a boat. Currently, Jim has \$500 in his wallet. The cheapest boat Jim can buy costs \$1,000,000. If Jim places all his money in a bank account that gets compounded three times a year with an interest rate of 2%, how long must Jim wait before he can buy a boat?

Read the three information below and answer the following questions.

Logarithms: One way of thinking about exponentiation is repeated multiplication. For example $2^3 = 2 * 2 * 2$ is two multiplied by itself three times. Similarly $5^4 = 5 * 5 * 5 * 5$. If exponentiation is repeated **multiplication** (by the base) then the inverse process should be repeated **division** (by the base). Of course, the inverse process is what we have been calling taking the log(arithm) of a number (with respect to a certain base). We divide by the base as many times as necessary to get 1 and then count the number of divisions. As an example, let's calculate $\log_2 8$.

A) Start with 8 and divide by the base (2). This gives us 4.

B) Divide 4 by two again, which is two.

C) Divide two by two which is one.

Now that we have reached one, let's count the number of times we had to divide by two to get to one. We divided 8 by the base (2) three times to reach one, and so $\log_2 8 = 3$. If the input isn't a 'nice' power of the base we can still proceed, though we lose our precision.
For example, let's calculate $\log_3 7$

A) Start with 7 and divide by three, giving $\frac{7}{3} = 2.333...$

B) Divide 2.333... by three, giving $\frac{2.333...}{3} = .777... < 1$.

Since two divisions takes us below one, it takes slightly less than two divisions (by 3) to get to 1 itself. This tells us $1 < \log_3 7 < 2$.

How many digits does it take to write an integer x? For example, the number 1234 has 4 digits and 16789 has 5 digits. It isn't always easy to simply count off the digits. Consider $x = 2^{17} + 32$, how can we measure the length just from the value of x? To be able to calculate this we must first introduce another notion. For any real number x, consider the nearest integer to x that is less than x. This is called the **floor** of x and is denoted $\lfloor x \rfloor$. Consider the following examples: $\lfloor 1.234 \rfloor = 1, \lfloor 4 \rfloor = 4, \lfloor -2.7 \rfloor = -3$. If we know that $4 < y < 5$ then we know that $\lfloor y \rfloor = 4$. Similarly, if $1 < y < 2$ then $\lfloor y \rfloor = 1$. Define a new function $length(x) = \lfloor (\log x) + 1 \rfloor$. As an example, $length(1234) = 4$.

More Questions:

13. Use Repeated Division to **evaluate** $\log_2 64$

14. Use Repeated Division to **estimate** $\log_2 100$

15. If we know $1 < y < 2$ then explain how we can conclude that $\lfloor y \rfloor = 1$. Fully justify your reasoning.

16. Verify $length(1234) = 4$ using repeated division.

17. Verify $length(101010) = 6$ using repeated division.