

This 1st order linear ODE already has a 1 in front of $\frac{dy}{dx}$, so $p(x) = \frac{1}{x}$, and the integrating factor is $\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

So multiply the ODE by $\mu(x) = x$.

$$x \frac{dy}{dx} + y = 4x$$

LHS is $\frac{d}{dx}(x y(x))$ (confirmed by differentiation on LHS)

$$\frac{d}{dx}(x y(x)) = 4x$$

$$\int \frac{d}{dx}(x y(x)) dx = \int 4x dx$$

$$x y(x) = 2x^2 + C$$

For $x > 0$, $y(x) = 2x + C/x$

$y(1) = 3 \Rightarrow 3 = 2 + C$, so $C = 1$

$$y(x) = 2x + \frac{1}{x}$$

Solution interval is $0 < x < \infty$

$\frac{dy}{dx} = \cos 2x$, $y(0) = -1$

$y(x) = 0$ is one solution of the ODE, but that doesn't satisfy $y(0) = -1$

For $y \neq 0$,

$$y^2 \frac{dy}{dx} = \cos 2x$$

$$\int y^2 \frac{dy}{dx} dx = \int \cos 2x dx$$

$$-\frac{1}{3} y^3 = \frac{1}{2} \sin 2x + C$$

$$y(x) = -\frac{1}{\sqrt[3]{\frac{1}{2} \sin 2x + C}}$$

$y(0) = -1 \Rightarrow C = 1$

$$y(x) = -\frac{1}{1 + \frac{1}{2} \sin 2x}$$

interval $-\infty < x < \infty$

$2xy + y^2 + (x^2 + 3y^2 + 1) \frac{dy}{dx} = 0$

$\frac{dy}{dx} = 2x + 3y^2$, $\frac{dy}{dx} = 2x + 3y^2$

so eqn is exact, and there exists an $f(x, y)$

such that $\frac{\partial f}{\partial x} = M(x, y)$ and $\frac{\partial f}{\partial y} = N(x, y)$

So $\frac{\partial f}{\partial x} = 2x + 3y^2$

$$f(x, y) = \int (2x + 3y^2) dx = x^2 + 3xy^2 + g(y)$$

$$f(x, y) = x^2 + 3xy^2 + g(y)$$

Then $\frac{\partial f}{\partial y} = N(x, y)$ gives

$$2xy + 3y^2 + g'(y) = 2x + 3y^2 + 1$$

$$g'(y) = 1$$
, so $g(y) = y$

Thus $f(x, y) = x^2 + 3xy^2 + y$, and solutions of the ODE are level curves of $f(x, y)$, i.e. implicit solutions of

$$x^2 + 3xy^2 + y = C$$

Can easily verify this solution by differentiating both sides w.r.t. x .

$y'' + 2y' + 10y = 0$, $y(0) = 3$, $y'(0) = 12$

Ansatz $y = e^{\lambda x}$ leads to

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

General solution in real form is then

$$y(x) = c_1 e^{-x} \cos 3x + c_2 e^{-x} \sin 3x$$

$$y'(x) = -c_1 e^{-x} \cos 3x - 3c_1 e^{-x} \sin 3x - c_2 e^{-x} \sin 3x + 3c_2 e^{-x} \cos 3x$$

$$= (3c_2 - c_1) e^{-x} \cos 3x - (3c_1 + c_2) e^{-x} \sin 3x$$

$y(0) = 3 \Rightarrow c_1 = 3$

$y'(0) = 12 \Rightarrow 3c_2 - c_1 = 12$

$$3c_2 - 3 = 12$$
, so $c_2 = 5$

$$y(x) = 3e^{-x} \cos 3x + 5e^{-x} \sin 3x$$

interval $-\infty < x < \infty$

$y'' - 10y' + 25y = 0$, $y(0) = 3$, $y'(0) = 17$

Ansatz $y = e^{\lambda x}$ leads to

$$\lambda^2 - 10\lambda + 25 = 0$$
, $(\lambda - 5)^2 = 0$, repeated roots $\lambda = 5$.

So genl solution is

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}$$

$$y'(x) = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$$

$$= (5c_1 + c_2) e^{5x} + 5c_2 x e^{5x}$$

$y(0) = 3 \Rightarrow 3 = c_1$

$y'(0) = 17 \Rightarrow 17 = 5c_1 + c_2$, so $c_2 = 2$

$$y(x) = 3e^{5x} + 2xe^{5x}$$

interval $-\infty < x < \infty$

$y'' - 4y = 0$, Ansatz $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$$\lambda^2 - 4 = 0$$
, $\lambda = \pm 2$, $y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$

guess $y_0(x) = A x e^{2x}$

$$y_0' = A(e^{2x} + 2x e^{2x})$$

$$y_0'' = A(2e^{2x} + 2e^{2x} + 4x e^{2x}) = 4A(e^{2x} + x e^{2x})$$

$$4A(e^{2x} + x e^{2x}) - 4A x e^{2x} = 3e^{2x}$$

$$4A = 3$$
, $A = 3/4$, or $y_0(x) = \frac{3}{4} x e^{2x}$

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{3}{4} x e^{2x}$$

interval $-\infty < x < \infty$

homog eqn $x^2 y'' - 6y = 0$, Ansatz $y = x^{\lambda}$

$$\lambda(\lambda-1)x^{\lambda} - 6x^{\lambda} = 0$$

$$(\lambda^2 - \lambda - 6)x^{\lambda} = 0$$

for $x \neq 0$, $(\lambda-3)(\lambda+2) = 0$, $\lambda_1 = 3$, $\lambda_2 = -2$

$y_1(x) = x^3$, $y_2(x) = x^{-2}$

nonhomog eqn $y'' - 6x^2 y = 2x^2$

$$W = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -2 - 3 = -5$$

$u_1' = \frac{1}{-5} \begin{vmatrix} x^3 & 0 \\ 3x^2 & -2x^{-3} \end{vmatrix} = \frac{2}{5}$, $u_1(x) = \frac{2}{5} x$

$u_2' = \frac{1}{-5} \begin{vmatrix} 0 & x^3 \\ -2x^{-3} & x^2 \end{vmatrix} = -\frac{2}{5} x^5$, $u_2(x) = -\frac{2}{30} x^6$

$y_0(x) = \frac{2}{5} x^3 - \frac{1}{15} x^6$

$$y(x) = c_1 x^3 + c_2 x^{-2} + \frac{1}{3} x^4$$

interval $-\infty < x < 0$ or $0 < x < \infty$

$\mathcal{L}\{e^{-5t} \cos 3t\} = \mathcal{L}\{s \cos 3t\} s \rightarrow s+5$

$$= \frac{s}{s^2 + 9}$$

$$= \frac{s+5}{s^2 + 10s + 34}$$

also ok

$\mathcal{L}\{y(t) - 2(t-1)\} = e^{-2s} \mathcal{L}\{t^2 - 1\} \rightarrow t \rightarrow 3$

$$= e^{-2s} \mathcal{L}\{t^2 + 4t + 3\}$$

$$= e^{-2s} \left[\frac{s}{s^2} + \frac{4}{s} + \frac{3}{s^2} \right]$$

(c) $\mathcal{L}^{-1}\left\{\frac{7s}{(s-3)(s^2+2s+6)}\right\} = \frac{A}{s-3} + \frac{Bs+C}{s^2+2s+6}$

cover-up: $A = 1$

$$7s = s^2 + 2s + 6 + (Bs+C)(s-3)$$

$$7s = s^2 + 2s + 6 + Bs^2 + Cs - 3Bs - 3C$$

$$0 = s^2 + (B-2)s + (C-3)$$

$$B = 2, C = 3$$

$$\mathcal{L}^{-1}\left\{\frac{7s}{(s-3)(s^2+2s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3} + \frac{2s+3}{s^2+2s+6}\right\}$$

$$= e^{3t} - \mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+2s+6}\right\}$$

$$= e^{3t} - e^{-(t+1)} \left(\cos \sqrt{5}t - \frac{3}{\sqrt{5}} \sin \sqrt{5}t \right)$$

$y'' + 4y = -4(t-1)$, $y(0) = 0$, $y'(0) = 1$

$\mathcal{L}\{y(t) - 5y(t-1) + 4y(t-2)\} = -e^{-s}$

$(s^2 + 4)y(s) = -e^{-s}$

$$y(s) = \frac{-e^{-s}}{s^2 + 4} = -\frac{e^{-s}}{s^2 + 4}$$

$$y(t) = -\frac{1}{4} \int_0^{t-1} \cos \frac{t-\tau}{2} d\tau$$

$$= \frac{1}{4} \sin 2t - \frac{1}{4} \cos 2(t-1)$$

$$= \frac{1}{4} \sin 2t - \frac{1}{4} \cos(2t - 2\pi)$$

$$= \frac{1}{4} \sin 2t - \frac{1}{4} \cos(2t - \pi) \sin 2t$$

$$y(t) = \frac{1}{4} (1 - \cos(2t - \pi)) \sin 2t$$

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