$\frac{dy}{dx} + \frac{1}{x}y = 4, \quad y(1) = 3$

This 1st order linear ODE already has a 1 in front of $\frac{dy}{dx}$, so $p(x) = \frac{1}{x}$, and the integrating factor is

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

So multiply the ODE by $\mu(x) = x$.

$$x\frac{dy}{dx} + y = 4x$$

LHS is $\frac{d}{dx}(\mu(x)y(x))$

$$\frac{d}{dx}(x\,y(x)) = 4x \qquad \text{(confirmed by differentiation on LHS)}$$

$$\int \frac{d}{dx}(x\,y(x))\,dx = \int 4x\,dx$$

$$x\,y(x) = 2x^2 + c$$

For $x \neq 0$, $y(x) = 2x + cx^{-1}$

$y(1) = 3 \Rightarrow 3 = 2 + c$, so $c = +1$

$$\boxed{y(x) = 2x + x^{-1}}$$

Solution interval is

$$\boxed{0 < x < \infty}$$

$\frac{dy}{dx} = y^2 \cos 2x, \quad y(0) = -1$

$y(x) = 0$ is one solution of the ODE, but that doesn't satisfy $y(0) = -1$

For $y \neq 0$,

$$y^{-2}\frac{dy}{dx} = \cos 2x$$

$$\int y^{-2}\frac{dy}{dx}\,dx = \int \cos 2x\,dx$$

$$-y^{-1} = \frac{1}{2}\sin 2x + c$$

$$y(x) = -\frac{1}{c + \frac{1}{2}\sin 2x}$$

$$y(0) = -1 = -\frac{1}{c}, \text{ so } c = 1$$

$$y(x) = -\frac{1}{1 + \frac{1}{2}\sin 2x}$$

or

$$\boxed{y(x) = -\frac{2}{2 + \sin 2x}}$$

$\underbrace{2xy + y^3}_{M(x,y)} + \underbrace{(x^2 + 3xy^2 + 1)}_{N(x,y)}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = 2x + 3y^2 \quad \frac{\partial N}{\partial x} = 2x + 3y^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so eqn is exact, and there exists an $f(x,y)$ such that $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

So $\frac{\partial f}{\partial x} = 2xy + y^3$.

$$f(x,y) = \int (2xy + y^3)\,dx$$

$$f(x,y) = x^2y + xy^3 + g(y)$$

Then $\partial f/\partial y = N(x,y)$ gives

$$x^2 + \cancel{3xy^2} + g'(y) = x^2 + \cancel{3xy^2} + 1$$

$$g'(y) = 1, \text{ so } g(y) = y$$

Thus $f(x,y) = x^2y + xy^3 + y$, and solutions of the ODE are level curves of $f(x,y)$, i.e. implicit solutions of

$$\boxed{x^2y + xy^3 + y = c}$$

Can easily verify this solution by differentiating both sides w.r.t. x.

$y'' + 2y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = 12$

Ansatz $y = e^{\lambda x}$ leads to

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

General solution in real form is then

$$y(x) = c_1 e^{-x}\cos 3x + c_2 e^{-x}\sin 3x$$

$$y'(x) = -c_1 e^{-x}\cos 3x - 3c_1 e^{-x}\sin 3x - c_2 e^{-x}\sin 3x + 3c_2 e^{-x}\cos 3x$$

$$= (3c_2 - c_1)e^{-x}\cos 3x - (3c_1 + c_2)e^{-x}\sin 3x$$

$y(0) = 3 \Rightarrow c_1 = 3$

$y'(0) = 12 \Rightarrow 3c_2 - c_1 = 12$

$3c_2 - 3 = 12$, so $c_2 = 5$

$$\boxed{\begin{array}{l} y(x) = 3e^{-x}\cos 3x + 5e^{-x}\sin 3x \\ \text{interval } -\infty < x < \infty \end{array}}$$

$y'' - 10y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = 17$

ansatz $y = e^{\lambda x}$ leads to

$\lambda^2 - 10\lambda + 25 = 0$, $(\lambda - 5)^2 = 0$, repeated roots $\lambda = 5$.

So gen'l solution is

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}$$

$$y'(x) = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x}$$

$$= (5c_1 + c_2)e^{5x} + 5c_2 x e^{5x}$$

$y(0) = 3 \Rightarrow 3 = c_1$

$y'(0) = 17 \Rightarrow 17 = 15 + c_2$, so $c_2 = 2$

$$\boxed{\begin{array}{l} y(x) = 3e^{5x} + 2xe^{5x} \\ \text{interval } -\infty < x < \infty \end{array}}$$

$y'' - 4y = 3e^{2x}$ (+5)

homog eqn $y'' - 4y = 0$, ansatz $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 - 4 = 0$, $\lambda = \pm 2$ $\quad y_1(x) = e^{2x}$ (+5) $\quad y_2(x) = e^{-2x}$ (+5)

guess $y_p(x) = Axe^{2x}$ (+10)

$$y_p' = A(e^{2x} + 2xe^{2x})$$

$$y_p'' = A(2e^{2x} + 2e^{2x} + 4xe^{2x}) = 4A(e^{2x} + xe^{2x})$$

$$4A(e^{2x} + \cancel{xe^{2x}}) - \cancel{4Axe^{2x}} = 3e^{2x}$$

$4A = 3$, $A = 3/4$ (+10) or $y_p(x) = \frac{3}{4}xe^{x}$

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{3}{4}xe^{2x} \qquad -\infty < x < \infty}$$

(+10) (+5)

$x^2y'' - 6y = 2x^4$

homog eqn $x^2y'' - 6y = 0$, ansatz $y = x^\lambda$

$$x^2\lambda(\lambda - 1)x^{\lambda - 2} - 6x^\lambda = 0$$

$$(\lambda^2 - \lambda - 6)x^\lambda = 0$$

for $x \neq 0$ $\quad (\lambda - 3)(\lambda + 2) = 0$, $\lambda_1 = 3$, $\lambda_2 = -2$ (+10)

$y_1(x) = x^3 \quad y_2(x) = x^{-2}$

nonhomog eqn $y'' - 6x^{-2}y = 2x^2$ (+5)

$$W = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -2 - 3 = -5$$

$u_1' = -\frac{1}{5}\begin{vmatrix} 0 & x^{-2} \\ 2x^2 & -2x^{-3} \end{vmatrix} = +\frac{2}{5} \qquad u_1(x) = +\frac{2}{5}x$ (+10)

$u_2' = -\frac{1}{5}\begin{vmatrix} x^3 & 0 \\ 3x^2 & 2x^2 \end{vmatrix} = -\frac{2}{5}x^5 \qquad u_2(x) = -\frac{2}{30}x^6$ (+10)

$y_p(x) = \frac{2}{5}x \cdot x^3 - \frac{1}{15}x^6 \cdot x^{-2} = \frac{1}{3}x^4$ (+5)

$y(x) = c_1x^3 + c_2x^{-2} + \frac{1}{3}x^4 \qquad -\infty < x < 0 \text{ or } 0 < x < \infty$

(+5) (+5)

$$\mathcal{L}\{e^{-5t}\cos 3t\} = [\mathcal{L}\{\cos 3t\}]_{s \to s+5} \quad (+5)$$

$$= \left[\frac{s}{s^2 + 9}\right]_{s \to s+5} \quad (+5)$$

$$\boxed{= \frac{s + 5}{s^2 + 10s + 34}} \quad (+5)$$

$$\left(= \frac{s+5}{(s+5)^2 + 9} \quad \text{also ok} \right)$$

$$\mathcal{L}\{\mathscr{U}(t-2)(t^2 - 1)\} = e^{-2s}\mathcal{L}\{[t^2 - 1]_{t \to t+2}\} \quad (+5)$$

$$= e^{-2s}\mathcal{L}\{t^2 + 4t + 3\} \quad (+5)$$

$$\boxed{= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right]} \quad (+5)$$

(c) $\mathcal{L}^{-1}\left\{\frac{7s}{(s-3)(s^2+2s+6)}\right\} =$ (20 pts)

$\frac{7s}{(s-3)(s^2+2s+6)} = \frac{A}{s-3} + \frac{Bs + C}{s^2 + 2s + 6}$ (+5)

cover-up: $A = 1$

$$7s = s^2 + 2s + 6 + (Bs + C)(s - 3)$$

$$7s = s^2 + 2s + 6 + Bs^2 + Cs - 3Bs - 3C$$

s^2 terms: $0 = 1 + B$, $B = -1$

s terms: $7 = 2 + C + 3$, $C = 2$ (+5) (consts)

$$\mathcal{L}^{-1}\left\{\frac{7s}{(s-3)(s^2+2s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+2s+6}\right\}$$

$$= e^{3t} - \mathcal{L}^{-1}\left\{\frac{(s+1) - 3}{(s+1)^2 + 5}\right\}$$

$$= e^{3t} - \mathcal{L}^{-1}\left\{\left[\frac{s-3}{s^2+5}\right]_{s \to s+1}\right\} \quad (+5)$$

$$= e^{3t} - e^{-t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} - \frac{3}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\}\right)$$

$$\boxed{= e^{3t} - e^{-t}\left(\cos\sqrt{5}t - \frac{3}{\sqrt{5}}\sin\sqrt{5}t\right)} \quad (+5)$$

Simplify!

$y'' + 4y = -\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = -e^{-\pi s}$$

$$(s^2 + 4)Y(s) = 1 - e^{-\pi s}$$

$$Y(s) = \frac{1}{s^2+4} - e^{-\pi s}\frac{1}{s^2+4} \quad (+10)$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+4}\right\}$$

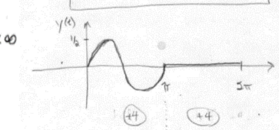
$y(t) = \frac{1}{2}\sin 2t - \mathscr{U}(t-\pi)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right]_{t \to t-\pi}$ (+10) 1st term

$= \frac{1}{2}\sin 2t - \mathscr{U}(t-\pi)\left[\frac{1}{2}\sin 2t\right]_{t \to t-\pi}$ (+15) 2nd term

$= \frac{1}{2}\sin 2t - \frac{1}{2}\mathscr{U}(t-\pi)\sin(2t - 2\pi)$

$= \frac{1}{2}\sin 2t - \frac{1}{2}\mathscr{U}(t-\pi)\sin 2t$ (+5) simplification

$$\boxed{y(t) = \frac{1}{2}(1 - \mathscr{U}(t-\pi))\sin 2t}$$



(+2) for labels

(+4) (+4)

$x' = x + 2y$
$y' = 4x + 3y$

SOLUTION:

In matrix-vector form, this system of differential equations is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.1}$$

We expect a solution of the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}, \tag{4.2}$$

where $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ are the eigenvalue-eigenvector pairs of the coefficient matrix above. So, the bulk of solving this system is completing the corresponding eigenvalue problem. We find the eigenvalues by

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)(3-\lambda) - 2(4) &= 0 \\ \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0. \end{aligned} \tag{4.3}$$

The eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$. We find the associated eigenvector by

$$\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.4}$$

Taking $\lambda = \lambda_1 = 5$, we have

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.5}$$

As expected, we have linear dependence between our rows and columns. We may choose any eigenvector that satisfies $-4v_1 + 2v_2 = 0$. One choice may be

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Taking $\lambda = \lambda_2 = 5$, (4.4) becomes

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.6}$$

which requires $v_1 = -v_2$, so a choice of eigenvector may be

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

In combination, we obtain the vector form of our solution by substituting into (4.2):

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{5t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t}. \tag{4.7}$$

We can rewrite this in scalar form by just reading the vector addition element-wise, giving

$$x(t) = c_1e^{5t} - c_2e^{-t}, \tag{4.8}$$

$$y(t) = 2c_1e^{5t} + c_2e^{-t}. \tag{4.9}$$

$x' = x/2, \quad x(0) = 3,$
$y' = x - 2y, \quad y(0) = 5.$

SOLUTION:

In matrix-vector form, this system can be written

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{6.1}$$

We find the eigenvalues by

$$\begin{aligned} \det\begin{pmatrix} 1/2-\lambda & 0 \\ 1 & -2-\lambda \end{pmatrix} &= 0 \\ (\tfrac{1}{2} - \lambda)(-2 - \lambda) &= 0, \end{aligned} \tag{6.2}$$

so we have eigenvalues of $\lambda_1 = \frac{1}{2}, \lambda_2 = -2$. Starting with $\lambda_1 = \frac{1}{2}$, we find the associated eigenvector:

$$\begin{pmatrix} 0 & 0 \\ 1 & -5/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.3}$$

This corresponds to the algebraic equation $v_1 - \frac{5}{2}v_2 = 0$, so one choice for an eigenvector might be

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Now, for $\lambda_2 = -2$,

$$\begin{pmatrix} 5/2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.4}$$

This requires that $v_1 = 0$. Otherwise, v_2 may be whatever we want. One choice for the eigenvector may be

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Altogether, the scalar form of the solution is

$$x(t) = 5c_1e^{t/2}, \tag{6.5}$$

$$y(t) = 2c_1e^{t/2} + c_2e^{-2t}. \tag{6.6}$$

We have initial conditions, so we may solve for c_1, c_2. Taking $x(0) = 3$ in (6.5), we find $c_1 = \frac{3}{5}$. Taking $y(0) = 5$ in (6.6),

$$\begin{aligned} y(0) = 5 &= 2\left(\frac{3}{5}\right) + c_2 \\ \Rightarrow \frac{19}{5} &= c_2, \end{aligned} \tag{6.7}$$

so the scalar form of the solution becomes

$$x(t) = 3e^{t/2}, \tag{6.8}$$

$$y(t) = \frac{6}{5}e^{t/2} + \frac{19}{5}e^{-2t}, \tag{6.9}$$

and the vector form is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 5 \\ 2 \end{pmatrix}e^{t/2} + \frac{19}{5}\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-2t}. \tag{6.10}$$

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$
$\mathscr{U}(t-a)f(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$y(t)$	$Y(s)$
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$
$y^{(n)}(t)$	$s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots y^{(n-1)}(0)$
$\mathcal{L}^{-1}\{F(s)\}$	$F(s)$

(1) $\frac{d}{dx}(x) = 1$	(1') $\int 1\,dx = \int dx = x + c$
(2) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, $n \neq -1$	(2') $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ $n \neq -1$
(3) $\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	(3') $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$
(4) $\frac{d}{dx}(\sin x) = \cos x$	(4') $\int \cos x\,dx = \sin x + c$
(5) $\frac{d}{dx}(-\cos x) = \sin x$	(5') $\int \sin x\,dx = -\cos x + c$
(6) $\frac{d}{dx}(\tan x) = \sec^2 x$	(6') $\int \sec^2 x\,dx = \tan x + c$
(7) $\frac{d}{dx}(-\cot x) = \csc^2 x$	(7') $\int \csc^2 x\,dx = -\cot x + c$
(8) $\frac{d}{dx}(\sec x) = \sec x \tan x$	(8') $\int \sec x \tan x\,dx = \sec x + c$
(9) $\frac{d}{dx}(-\csc x) = \csc x \cot x$	(9') $\int \csc x \cot x\,dx = -\csc x + c$
(10) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	(10') $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
(11) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	(11') $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$
(12) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	(12') $\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$
(13) $\frac{d}{dx}(e^x) = e^x$	(13') $\int e^x\,dx = e^x + c$
(14) $\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = a^x$	(14') $\int a^x\,dx = \frac{a^x}{\ln a} + c$ $(a > 0, a \neq 1)$

$$\frac{8x^2 - 12}{x(x^2 + 2x - 6)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x - 6}$$

$$\frac{3x^3 + 7x - 4}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$

$$\frac{9x + 25}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2}$$

$$\frac{9 - 9x}{(2x - 1)(x + 4)} = \frac{A}{2x - 1} + \frac{B}{x + 4}$$