1. $y' - y = 1 \qquad y(0) = 1 \qquad \mathcal{L}\{y' - y\} = \mathcal{L}\{1\}$

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$

~~$sY(s) + Y(s) = 1/s \qquad Y(s) = \frac{1}{s} \cdot \frac{1}{s+1} = \frac{1}{s+s}$~~

~~$Y(s)(s+1) = 1/s \qquad Y(s) = \frac{1}{s} \cdot \frac{1}{s} \qquad Y(s) = \frac{1}{s+5}$~~

~~$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2+s}\}\ 1/(s^2+5)$~~

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$
$\quad\downarrow \qquad\qquad \downarrow \qquad\quad \downarrow$

$sY(s) - y(0) \quad - \qquad Y(s) \quad = \quad 1$

$sY(s) - 1 - Y(s) = 1 \qquad sY(s) - Y(s) = 2 \qquad Y(s)(s-1) = 2 \qquad Y(s) = \frac{2}{s-1}$
$\quad\quad\;+1 \qquad\quad +1$

$\frac{2}{s-1} = \frac{1}{s-1} + \frac{1}{s-1} \qquad \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{2}{s-1}\}$
$\qquad\;\; e^t + e^t \qquad\qquad\qquad \boxed{y(t) = 2e^t} \qquad a = 1$

2. $y' + 7y = e^{3t} \qquad y(0) = 2$

$\mathcal{L}\{y'\} + 7\mathcal{L}\{y\} = \{e^{3t}\}$
$\quad\downarrow$

$sY(s) + 7Y(s) = \frac{1}{s-3}$

$Y(s)(s+7) = \frac{1}{s-3} \qquad\qquad Y(s) = \frac{1}{(s-3)} \cdot \frac{1}{(s+7)}$

$1 = A(s-3) + B(s+7) \qquad s=3: \; 1 = A(3-3) + B(10) \qquad 1 = 10B \quad B = 1/10$

$1 = A(s-3) + \frac{1}{10}(s+7) \qquad s=-7: \; 1 = A(-10) + \frac{1}{10}(0) \qquad 1 = -10A \quad A = -1/10$

$Y(s) = \frac{1}{10(s+7)} - \frac{1}{10(s-3)}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{10(s+7)}\} - \mathcal{L}^{-1}\{\frac{1}{10(s-3)}\}$

$\boxed{y(t) = \frac{1}{10}e^{7t} - \frac{1}{10}e^{3t}}$

3. $y'' + 5y' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$

$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$

$(s^2 Y(s) - sy(0) - y'(0)) + 5(sY(s) - y(0)) + 4Y(s) = 0$

$s^2 Y(s) - 2s + 5sY(s) + 10 + 4Y(s) = 0$

$Y(s)(s^2 - 5s + 4) = 2s - 10$

$$Y(s) = \frac{2s-10}{s^2 - 5s + 4} = \frac{2s-10}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4}$$

$2s - 10 = A(s-4) + B(s-1)$

$s = 4:\ 8 - 10 = 0 + B(3)\quad B = -2/3$

$s = 1:\ -8 = A(-3) + 0\quad A = 8/3$

$Y(s) = \frac{8/3}{s-1} + \frac{-2/3}{s-4}$

$\mathcal{L}^{-1}\{Y(s)\} = \frac{8}{3}\mathcal{L}^{-1}\frac{1}{s-1} - \frac{2}{3}\mathcal{L}^{-1}\frac{1}{s-4}$

$$\boxed{y(t) = \frac{8}{3}e^t - \frac{2}{3}e^{4t}}$$

4. $y'' + y = 3\sin(\sqrt{2}t)$ $y(0) = 0$ $y'(0) = 5$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\sin\sqrt{2}t\}$

$(s^2 Y(s) - sy(0) - y'(0)) + Y(s) = 3\frac{\sqrt{2}}{s^2 + 2}$

$Y(s)(s^2 + 1) = 3\frac{\sqrt{2}}{s^2+2} + 5$

$$Y(s) = \frac{3\sqrt{2}}{(s^2+2)} + \frac{5}{s^2+1}$$

$\mathcal{L}^{-1}\{Y(s)\} = \frac{3\sqrt{2}}{s^2-s^2-2} + \cdots$

$$\boxed{y(t) = \frac{\sqrt{2}+5}{2\sqrt{2}}(e^{\sqrt{2}t} + e^{-\sqrt{2}t}) - \sqrt{2}\sin t}$$

5. $y'' + 2y' + 5y = 1$ $y(0) = 3$ $y'(0) = 1$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{1\}$

$(s^2 Y(s) - sy(0) - y'(0)) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{s}$

$Y(s)(s^2 + 2s + 5) = \frac{1}{s}(-3s^2 - 6s - 2)$

$Y(s) = \left(\frac{-3s^2 - 6s - 2}{s}\right) \cdot \frac{1}{s^2 + 2s + 5}$

6. $y'' - 6y' + 9y = t$ $y(0) = 0$ $y'(0) = -1$

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t\}$

$(s^2 Y(s) - sy(0) - y'(0)) - 6(sY(s) - y(0)) + 9Y(s) = \frac{1}{s^2}$

$Y(s)(s^2 - 6s + 9) = \frac{1}{s^2} + 1$

$$Y(s) = \frac{1 + s^2}{s^2} \cdot \frac{1}{s^2 - 6s + 9} = \frac{1 + s^2}{s^2(s-3)^2}$$

$Y(s) = \frac{2}{9s} + \frac{1}{9s^2} - \frac{2}{27(s-3)} + \frac{10}{9(s-3)^2}$

$\mathcal{L}^{-1}\{Y(s)\} = \frac{2}{9}\mathcal{L}^{-1}\frac{1}{s} + \frac{1}{9}\mathcal{L}^{-1}\frac{1}{s^2} - \frac{2}{27}\mathcal{L}^{-1}\frac{1}{s-3} + \frac{10}{9}\mathcal{L}^{-1}\frac{1}{(s-3)^2}$