# The Frequency-Domain Bootstrap of Dahlhaus & Janas

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Final Project for Prof. Aue's STA 237A, Time Series Analysis

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# Bootstrapping and Time Series

- Bootstrapping is based on resampling i.i.d. variables
- How to deal with the dependence structure of time series?
- Need (approximately) i.i.d. variables
  - Time Domain: residuals, innovations
  - Frequency Domain: (Studentized) periodogram ordinates
- Paper describes favorable properties of the latter

# Spectral Mean

- $\phi = (\phi^{(1)}, \cdots, \phi^{(d)})$, each $\phi^{(i)}$ of bounded variation
- $I_T$ - tapered periodogram
- Spectral mean $A(\phi, f) = \int_0^\pi \phi(\alpha) f(\alpha) d\alpha$
- Canonical estimate $A(\phi, I_T) = \int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha$

# Spectral Mean - Important Examples

- Spectral mean $A(\phi, f) = \int_0^\pi \phi(\alpha) f(\alpha) d\alpha$
- Autocovariance estimate $\hat{\gamma}(u), u \in \mathbb{Z}$
  $\phi(\alpha) = 2\cos(\alpha u)$
- Spectral distribution function estimate $\hat{F}_T(\lambda)$
  $\phi(\alpha) = \mathbb{1}_{[0,\lambda]}(\alpha)$

# Ratio Statistics - Definition

- Normalized spectral density: $g(\alpha) = \frac{f(\alpha)}{F(\pi)}$
- Estimate: $J_T(\alpha) = \frac{I_T(\alpha)}{\hat{F}_T(\pi)}$
- Ratio statistic: $A(\phi, J_T) = \frac{\int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha}{\int_0^\pi I_T(\alpha) d\alpha} = \frac{A(\phi, I_T)}{A(1, I_T)}$

# Ratio Statistics - Important Examples

- Ratio statistic: $A(\phi, J_T) = \frac{\int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha}{\int_0^\pi I_T(\alpha) d\alpha} = \frac{A(\phi, I_T)}{A(1, I_T)}$
- Autocorrelation estimate $\hat{\rho}_T(u), u \in \mathbb{Z}$
  $\phi(\alpha) = \cos(\alpha u)$
- Normalized sdf estimate $\hat{F}_T(\lambda)/\hat{F}_T(\pi)$
  $\phi(\alpha) = \mathbb{1}_{[0,\lambda]}(\alpha)$

# Whittle Estimators

- Parametric family of spectral densities:
  $\mathcal{F} = \{f_\theta : \theta \in \Theta\}, \Theta \in \mathbb{R}$
- Whittle's likelihood: $\mathcal{L}_T(\theta) = \frac{1}{2\pi} \int_0^\pi \left[\log f_\theta(\alpha) + \frac{I_T(\alpha)}{f_\theta(\alpha)}\right] d\alpha$
- $\hat{\theta}$ obtained by minimizing $\mathcal{L}_T(\theta)$
- Spectral mean with $\phi(\alpha) = \nabla \frac{1}{f_\theta}(\alpha)$

# Spectral Mean Distribution

If we were to Studentize our periodogram values $\{I_j\}$ with estimated spectral density $\hat{f}$ in order to obtain bootstrap values $\{I_j^*\}$, we could form the bootstrap approximation

$$B(\phi, I_T^*) := \frac{\pi}{n} \sum_{j=1}^{n} \phi_j I_j^* \quad \text{, where} \quad \phi_j := \phi(2\pi j/T) \quad .$$

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Under appropriate conditions ...

- $\sqrt{T}\left(A(\phi, I_T) - A(\phi, f)\right)$ is asymptotically normal with variance $= \quad 2\pi \int \phi^2 f^2 + (\kappa_4/\sigma^4)\left(\int \phi f\right)^2$ .
- $\sqrt{T}\left(B(\phi, I_T^*) - B(\phi, \hat{f})\right)$ is asymptotically normal with variance $\propto \quad 2\pi \int \phi^2 f^2$ .

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For ratio statistics, second term vanishes!

# Algorithm

Let $n = \lceil T/2 \rceil$ and $I_j := I_T(2\pi j/T)$.

1. Obtain periodogram sample values $\{I_j\}$, $j = 1 : n$.
2. Obtain estimate $\hat{f}$ of spectral density $f$ (e.g., kernel estimate).

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3. Obtain Studentized periodogram samples $\hat{\epsilon}_j := I_j/\hat{f}_j$.
4. Rescale the $\hat{\epsilon}_j$'s by their mean, so that $\tilde{\epsilon}_j := \hat{\epsilon}_j/\{(1/n)\sum_j \hat{\epsilon}_j\}$.

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4. Rescale the $\hat{\epsilon}_j$'s by their mean, so that $\tilde{\epsilon}_j := \hat{\epsilon}_j/\{(1/n)\sum_j \hat{\epsilon}_j\}$.
5. Sample the empirical distribution of $\{\tilde{\epsilon}_j\}$ to obtain a bootstrap sample $\{\epsilon_j^*\}$ (whose size may differ from $n$).
6. Define bootstrap periodogram values via $I_j^* := \hat{f}_j \epsilon_j^*$.

# Some Assumptions

Performance guarantee of the bootstrap approximation requires ...

- $\{X_t : t \in \mathbb{Z}\}$ is a real-valued linear process: $X_t = \sum_{u \in \mathbb{Z}} a_u \xi_{t-u}$
  The $\{\xi_t\}$ satisfy $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 = 1$, $\mathbb{E}\xi_1^8 < \infty$, $\mathbb{E}\xi_t^3 = 0$.
  The $\{a_t\}$ satisfy $\sum_t t^2 |a_t| < \infty$.
- $\sup_{\alpha \in [0,\pi]} |\hat{f}(\alpha) - f(\alpha)| \to 0$ almost surely.
- $\phi$ is a $d$-dimensional vector of BV functions $\phi^{(r)} : [0, \pi] \to \mathbb{R}$, each with even periodic extension to all of $\mathbb{R}$.
- The taper $h_t$ has the form $h_t := h(t/T)$, with $h : \mathbb{R} \to [0, 1]$ a BV function with support in $(0, 1]$.

# More Technical Assumptions

Edgeworth expansions are used to show that we outperform the normal approximation. For these we need ...

- The $\{a_t\}$ and Fourier coefficients $\{\hat{\phi}(\omega)\}$ vanish exponentially. That is, for a fixed $\tau \in (0, 1)$, $|a_t| \leq \tau^{|t|}$ and $\|\hat{\phi}(\omega)\| \leq \tau^{|\omega|}$.
- several other detailed technical assumptions on $h$, $\xi$, and $\phi$ (more than five minutes allow! See the paper).

# Convergence Rate

- To bootstrap the distribution of $A(\phi, J_T) - A(\phi, g)$, use the statistic $B(\phi, J_T^*) - B(\phi, \hat{g})$, where $\hat{g}_j := \hat{f}_j / \{(\pi/n) \sum_k \hat{f}_k\}$.
- Let $D_T^2 = V_T^{-1}$, where $V_T$ is the covariance matrix of $\sqrt{T} A(\phi, J_T)$. Define $\hat{D}_T$ analogously via $\sqrt{T} B(\phi, J_T^*)$.
- Let $P^*$ denote the conditional distribution given the data.

**Theorem**

Suppose the assumptions stated in the paper hold. Then for almost all samples $\{I_j\}$,

$$\sup_C \Big| P(\sqrt{T} D_T (A(\phi, J_T) - A(\phi, g)) \in C) -$$

$$P^*(\sqrt{T} \hat{D}_T (B(\phi, J_T^*) - B(\phi, \hat{g})) \in C) \Big| = o(T^{-1/2}) \quad .$$

The sup is taken over convex measurable subsets of $\mathbb{R}^d$.

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The sup is taken over convex measurable subsets of $\mathbb{R}^d$.

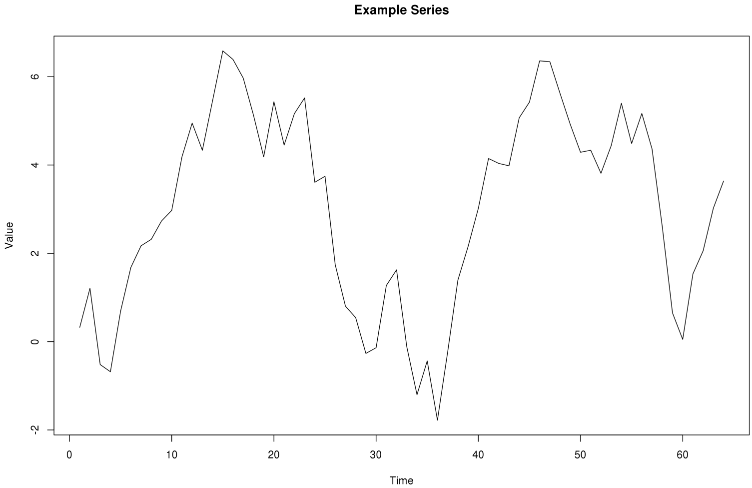
In short, estimating the distribution of the standardized spectral density by resampling from the standardized periodogram is consistent, and the convergence rate is asymptotically better than that achieved by the normal approximation.

# Setup

Simulate the $\mathrm{AR}(1)$ process $X_t = 0.9X_{t-1} + Z_t$.

- Innovations $\{Z_t\} \overset{\mathrm{iid}}{\sim} \mathrm{U}(-\sqrt{3}, \sqrt{3})$ are mean 0, variance 1 white noise.
- Draw 64 observations.
- Recall that $\hat{\rho}(1)$ is the Yule-Walker estimate for the AR parameter 0.9.

# Series Plot



# Practical Details

- Computed the periodogram with a 10% Tukey-Hanning taper.
- Smoothed the log-periodogram using the Epanechnikov kernel

$$K(x) = \frac{3}{4}\pi\left[1 - \left(\frac{x}{\pi}\right)^2\right].$$

and bandwidth 0.1 to obtain an estimate of the spectral density.

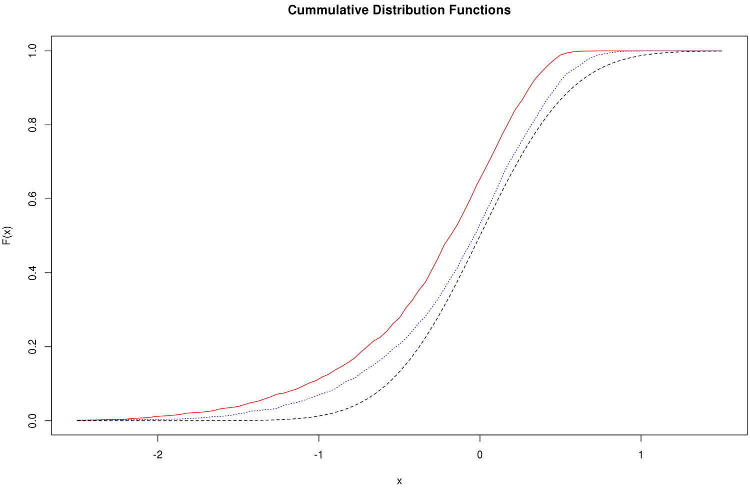
# Error Distribution

- Since $\hat{\rho}(1)$ is the Yule-Walker estimate for the AR parameter, then as $n \to \infty$,

$$\sqrt{n}\left(\frac{\hat{\rho}(1) - 0.9}{\sqrt{c}}\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1 - 0.9^2),$$

  where $c$ is a correction for the taper.
- 2000 bootstrapped estimates (from one simulation) produce a bootstrapped distribution.
- 2000 simulations approximate the true distribution.

# Distributions Plot



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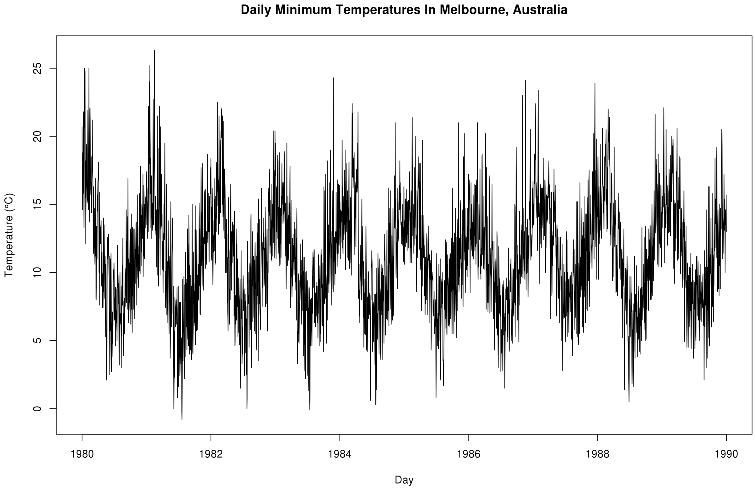
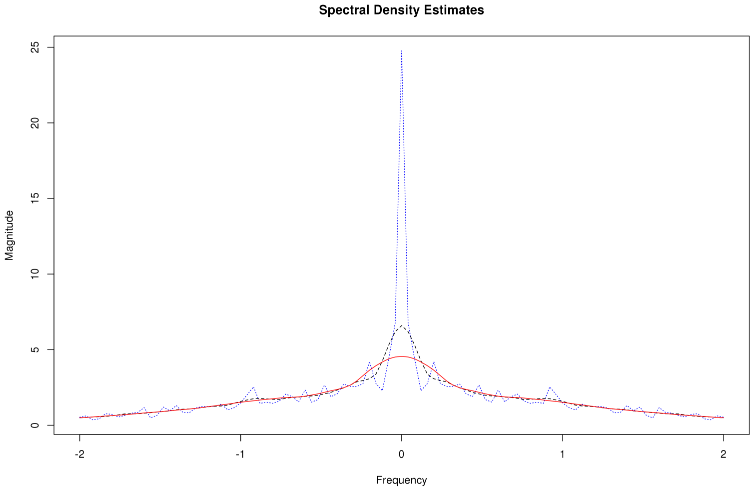

Daily Minimum Temperatures In Melbourne, Australia
Temperature (°C)
0
5
10
15
20
25
1980
1982
1984
1986
1988
1990
Day

# Spectral Density Estimate



# Autocorrelation Estimates

- Lag-1 autocorrelation confidence interval $(0.5234, 0.5787)$.
- Lag-2 autocorrelation confidence interval $(0.2041, 0.2917)$.