The Frequency-Domain Bootstrap of Dahlhaus & Janas

Dmitriy Izyumin, Eugene Shvarts, Nick Ulle

Final Project for Prof. Aue's STA 237A, Time Series Analysis

June 5, 2014

Bootstrapping and Time Series

- Bootstrapping is based on resampling i.i.d. variables
- How to deal with the dependence structure of time series?
- Need (approximately) i.i.d. variables
  - Time Domain: residuals, innovations
  - Frequency Domain: (Studentized) periodogram ordinates
- Paper describes favorable properties of the latter

Spectral Mean

- $\phi = (\phi^{(1)}, \cdots, \phi^{(d)})$, each $\phi^{(i)}$ of bounded variation
- I_T - tapered periodogram
- Spectral mean $A(\phi, f) = \int_0^\pi \phi(\alpha) f(\alpha) d\alpha$
- Canonical estimate $A(\phi, I_T) = \int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha$

Spectral Mean - Important Examples

- Spectral mean $A(\phi, f) = \int_0^\pi \phi(\alpha) f(\alpha) d\alpha$
- Autocovariance estimate $\hat{\gamma}(u), u \in \mathbb{Z}$
  $\phi(\alpha) = 2\cos(\alpha u)$
- Spectral distribution function estimate $\hat{F}_T(\lambda)$
  $\phi(\alpha) = \mathbb{1}_{[0,\lambda]}(\alpha)$

Ratio Statistics - Definition

- Normalized spectral density: $g(\alpha) = \frac{f(\alpha)}{F(\pi)}$
- Estimate: $J_T(\alpha) = \frac{I_T(\alpha)}{\hat{F}_T(\pi)}$
- Ratio statistic: $A(\phi, J_T) = \frac{\int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha}{\int_0^\pi I_T(\alpha) d\alpha} = \frac{A(\phi, I_T)}{A(1, I_T)}$

Ratio Statistics - Important Examples

- Ratio statistic: $A(\phi, J_T) = \frac{\int_0^\pi \phi(\alpha) I_T(\alpha) d\alpha}{\int_0^\pi I_T(\alpha) d\alpha} = \frac{A(\phi, I_T)}{A(1, I_T)}$
- Autocorrelation estimate $\hat{\rho}_T(u), u \in \mathbb{Z}$
  $\phi(\alpha) = \cos(\alpha u)$
- Normalized sdf estimate $\hat{F}_T(\lambda)/\hat{F}_T(\pi)$
  $\phi(\alpha) = \mathbb{1}_{[0, \lambda]}(\alpha)$

Whittle Estimators

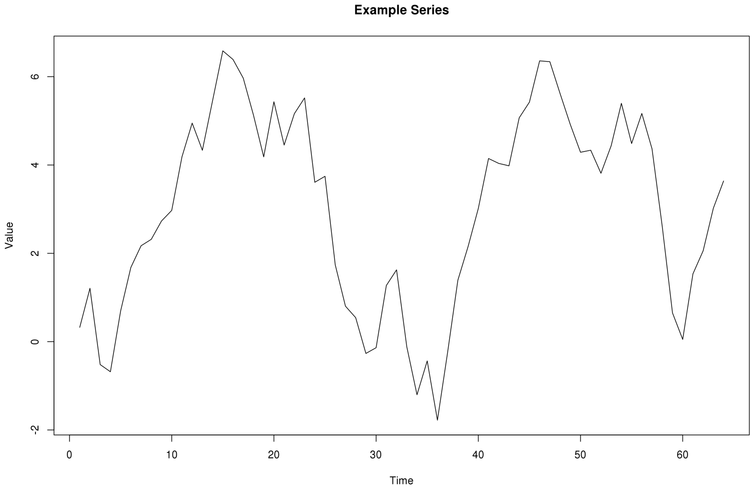
- Parametric family of spectral densities:
  $\mathcal{F} = \{f_\theta : \theta \in \Theta\}, \Theta \in \mathbb{R}$
- Whittle's likelihood: $\mathcal{L}_T(\theta) = \frac{1}{2\pi} \int_0^\pi \left[\log f_\theta(\alpha) + \frac{I_T(\alpha)}{f_\theta(\alpha)}\right] d\alpha$
- $\hat{\theta}$ obtained by minimizing $\mathcal{L}_T(\theta)$
- Spectral mean with $\phi(\alpha) = \nabla \frac{1}{f_\theta}(\alpha)$

Setup

Simulate the $\mathrm{AR}(1)$ process $X_t = 0.9X_{t-1} + Z_t$.

- Innovations $\{Z_t\} \overset{\text{iid}}{\sim} \mathsf{U}(-\sqrt{3}, \sqrt{3})$ are mean 0, variance 1 white noise.
- Draw 64 observations.
- Recall that $\hat{\rho}(1)$ is the Yule-Walker estimate for the AR parameter 0.9.

Series Plot



Practical Details

- Computed the periodogram with a 10% Tukey-Hanning taper.
- Smoothed the log-periodogram using the Epanechnikov kernel

$$K(x) = \frac{3}{4}\pi\left[1 - \left(\frac{x}{\pi}\right)^2\right].$$

and bandwidth 0.1 to obtain an estimate of the spectral density.

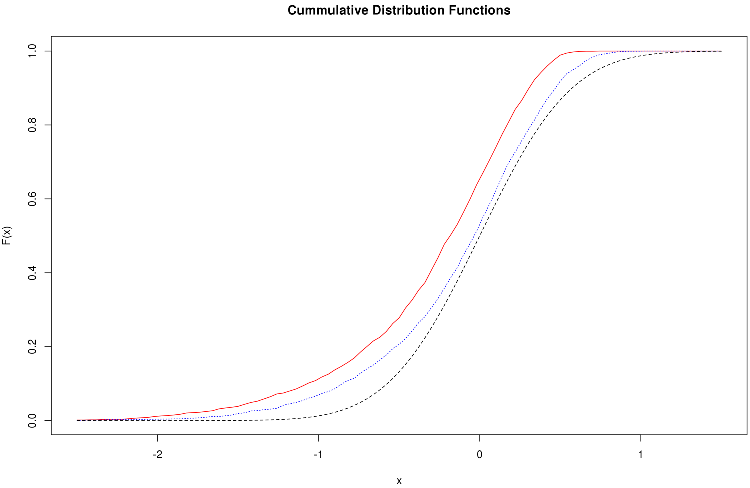
Error Distribution

- Since $\hat{\rho}(1)$ is the Yule-Walker estimate for the AR parameter, then as $n \to \infty$,

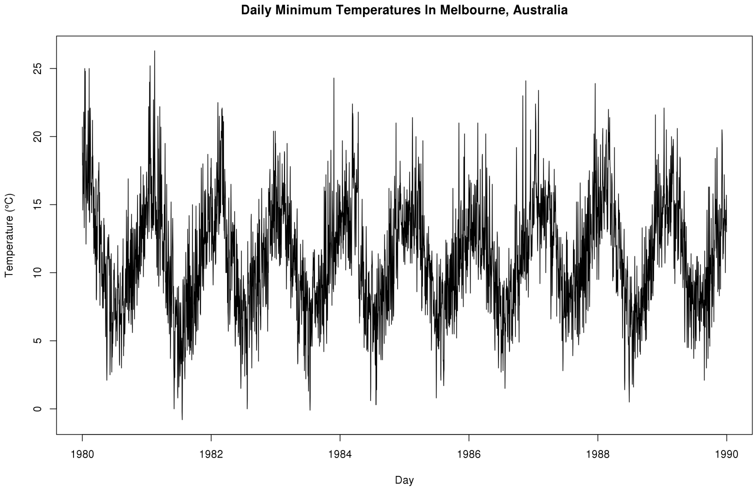
$$\sqrt{n}\left(\frac{\hat{\rho}(1) - 0.9}{\sqrt{c}}\right) \xrightarrow{\text{d}} \mathrm{N}(0, 1 - 0.9^2),$$

  where c is a correction for the taper.
- 2000 bootstrapped estimates (from one simulation) produce a bootstrapped distribution.
- 2000 simulations approximate the true distribution.

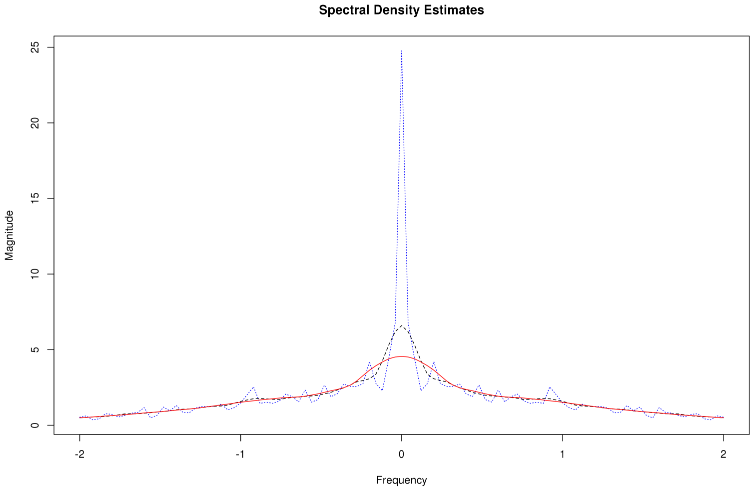
Distributions Plot



Series Plot


Daily Minimum Temperatures In Melbourne, Australia
Temperature (°C)
0
5
10
15
20
25
1980
1982
1984
1986
1988
1990
Day

Spectral Density Estimate



Autocorrelation Estimates

- Lag-1 autocorrelation confidence interval $(0.5234, 0.5787)$.
- Lag-2 autocorrelation confidence interval $(0.2041, 0.2917)$.