# Geodesics

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# Introduction

- Relativity provides a consistent framework to describe
  - High-Energy phenomena
  - Gravity

$*$ But how can we make predictions that can be tested experimentally ?

For example : The trajectory of a test body in curved space-time

- Test body :

  Object with negligible mass that does not significantly curve space-time itself

  For instance a satellite orbiting the Earth

- We need somehow to derive the equations governing the motion of test bodies

  Obviously Newtonian mechanics won't work

  We will limit ourselves to space-time curvature only (no Electric forces etc.)

  $\longrightarrow$ We will see that a Hamilton-Lagrange approach can be applied

- But first we will investigate a relatively simple measurement

  which is in fact applied to correct timing via the GPS system

## Exercise

- Consider a satellite moving eastwards over the Earth equator at a height $h$ above the Earth's surface with constant velocity $v$ w.r.t. the ground

  The satellite contains a clock which measures the satellite's proper time $\tau_s$

  In a ground station at the equator an identical clock measures the proper time $\tau_g$ of the station
- Consider the Earth as a perfect sphere with mass $M$, radius $R$ and Schwarzschild radius $R_s$
- The Earth rotates around its axis with a constant angular velocity $\Omega$
- When the satellite passes over the ground station, both clocks are synchronised
- When the satellite passes over the ground station after one circumnavigation, the elapsed proper times $\Delta\tau_s$ and $\Delta\tau_g$ are compared

$*$ By assuming that all velocities are non-relativistic, show that

$$\left(\frac{\Delta\tau_s}{\Delta\tau_g}\right)^2 = \frac{1 - R_s/(R+h) - ([(R+h)\Omega + v]/c)^2}{1 - R_s/R - (R\Omega/c)^2}$$

# The Classical Hamilton and Lagrange formalism

- Observation : Nature is lazy
  - Systems evolve performing the least action
- Could we use this feature to derive the laws of nature ?
  - $\star$ $\mathrm{d}[action]/\mathrm{d}[something] = 0 \longrightarrow$ physical laws ?
  - $\star$ Need to define action
  - $\star$ Which parameters to describe system and action ?
  - $\star$ Energy somehow involved $\longrightarrow$ description via scalars ?
- Observation : Physicists (and also students !) are lazy
  - Choose the most convenient parameters
  - Choose minimal amount of parameters
    - $\star$ Free motion $\longrightarrow (x, y, z)$
    - $\star$ Spherical symmetry $\longrightarrow (r, \theta, \varphi)$
    - $\star$ Planar motion $\longrightarrow (\rho, \varphi)$ or $(r, \theta)$

# The Classical Hamilton and Lagrange formalism

- Which and how many parameters to choose ?
  - We can measure coordinates and time
  - Velocity, acceleration etc... are derived observables
    but they describe the evolution of the system
  - Only independent parameters needed

## Generalized coordinates

- The configuration of a system is uniquely described by any complete set of independent coordinates $q_k$ $(k = 1, \ldots, n)$
  - One is free to choose which coordinates $q_k$
    The $q_k$ are called generalized coordinates
  - Easy to check if the $q_k$ are independent
  - But what is the value of $n$ ?

# The Classical Hamilton and Lagrange formalism

- **Free particle**

  $(x, y, z)$ or $(r, \theta, \varphi)$ uniquely determine the position $\rightarrow n = 3$

- **Particle confined to a plane** $(z = 0)$

  $(x, y)$, $(r, \theta)$ or $(\rho, \varphi)$ uniquely determine the position $\rightarrow n = 2$

- **Planar circular motion** $(\rho = R = constant)$

  $\varphi$ uniquely determines the position $\rightarrow n = 1$

- **System of $N$ particles** $\rightarrow n = 3N,\ 2N,\ N$ resp. in the above situations

  **Definition : Holonomic constraint**

  $f(x_i, y_j, z_k, t) = 0 \quad (i, j, k = 1, \ldots, N)$

$*$ Consider the case of $m$ holonomic constraints (all independent)

  $f_\alpha(x_i, y_j, z_k, t) = 0 \quad (i, j, k = 1, \ldots, N)\ (\alpha = 1, \ldots, m)$

  $\rightarrow n = (3N - m)$ generalized coordinates needed

## The Classical Hamilton and Lagrange formalism

- ⋆ Previous example of planar circular motion
  - Only 1 particle $\rightarrow N = 1$
  - Constraints : $z = 0$ and $x^2 + y^2 - R^2 = 0 \rightarrow m = 2$

    $\Rightarrow n = 3 - 2 = 1$ generalized coordinates needed
  - We had chosen $\varphi$ for this case
- ⋆ Add a second particle in the same plane at radius $R_2$

  Use cylindrical coordinates $(\rho, \varphi)$ for simplicity
  - Now we have 2 particles $\rightarrow N = 2$
  - Constraints : $z_1 = 0 \qquad \rho_1 - R = 0$

    $$z_2 = 0 \qquad \rho_2 - R_2 = 0$$

    $\rightarrow m = 4$

    $\Rightarrow n = 6 - 4 = 2$ generalized coordinates needed
  - Obviously we can choose $\varphi_1$ and $\varphi_2$

$\star$ **Assume both particles have fixed relative position**

$\longrightarrow$ **fixed angle $\alpha$ between them**

- **We still have 2 particles $\longrightarrow N = 2$**
- **Constraints :** $z_1 = 0 \qquad \rho_1 - R = 0$

$$z_2 = 0 \qquad \rho_2 - R_2 = 0$$

$$\varphi_1 - \varphi_2 - \alpha = 0$$

$\longrightarrow m = 5$

$\Rightarrow n = 6 - 5 = 1$ **generalized coordinates needed**

- **We can choose $\varphi_1$ in this case**

• **Note :**

$n$ **is called the number of degrees of freedom of the system**

## Hamilton's variational principle

- Systems evolve along a certain path in space-time
- Relate the action to the path-integral $\mathcal{P} = \int_{t_1}^{t_2} L \, \mathrm{d}t$

  with : $L \equiv$ some function related to the system's energy ($L \sim E_{kin}$ and $E_{pot}$)

  ⋆ Use generalized coordinates for convenience $\longrightarrow L = L(q, \dot{q}, t)$
- $\mathcal{P}$ is an extremum along the actual path

  Actual path $\longrightarrow \delta\mathcal{P} = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) \, \mathrm{d}t = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t) \, \mathrm{d}t = 0$

  with : $\delta \equiv$ infinitessimal variation of any system parameter
  away from the actual path value
- Note : The endpoints at $t_1$ and $t_2$ have to remain fixed on the actual path

⋆ Regard $\delta\mathcal{P}$ as an investigation of the value of $\mathcal{P}$

– Like estimating the weight of an object by trying to lift it

# The Classical Hamilton and Lagrange formalism

- Consider conservative systems with time-independent holonomic constraints
  - No explicit time dependence of $L \rightarrow L = L(q, \dot{q})$
- As usual we can now write :

$$\int_{t_1}^{t_2} \delta L \, \mathrm{d}t = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left( \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) \mathrm{d}t \equiv 0 \qquad \textbf{(1)}$$

⋆ Note that : $\delta \dot{q}_k = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q_k)$ **(2)**

Since : $\delta \dot{q} = \delta \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\delta q(t+\Delta t) - \delta q(t)}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q)$

- (1)&(2) yields : $\int_{t_1}^{t_2} \sum_{k=1}^{n} \left( \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{\mathrm{d}}{\mathrm{d}t}(\delta q_k) \right) \mathrm{d}t = 0$
- Integration by parts of the second term yields :

$$\left[ \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, \mathrm{d}t$$

⋆ Note : $\left[\sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$ (fixed endpoints)

• This finally yields : $\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[ \frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k \, \mathrm{d}t = 0$

⋆ Should hold for all independent $q_k$ and $\delta q_k$

In other words :

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = 0}$$

These equations are called the Lagrangian equations of motion and $L$ is called the Lagrangian of the system

# The Classical Hamilton and Lagrange formalism

- **But what is the expression for $L(q, \dot{q})$ ?**
  - **The Lagrange equations should yield Newton's laws**
  - **Guesswork, try and error by Lagrange $(\sim 1800)$**

$$\boxed{L = T - V}$$

⋆ **No deep reason behind expression for $L$, it just works !**

**Note :**

- **Lagrange equations only involve scalars $\longrightarrow$ easy**
- **Lagrange equations only valid for conservative systems**

## Recipe for Lagrangian analysis

1. Determine the number of degrees of freedom $(n)$ of the system
2. Select a suitable set of generalized coordinates $q_k$
3. Find relation between Cartesian- and generalized coordinates
4. Express $T$ and $V$ in terms of Cartesian coordinates
5. Express $T$ and $V$ in terms of generalized coordinates
6. Write $L$ in terms of generalized coordinates
7. Apply the Lagrange equations
8. Solve the diff. equations $\longrightarrow$ Problem solved !

$\star$ Steps (3) and (4) may be skipped in obvious situations

- **Example : Free particle**

1. Only 1 particle ($N = 1$), no constraints ($m = 0$) $\longrightarrow n = 3$
2. Just choose the Cartesian $(x, y, z)$
3. Trivial
4. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and $V = 0$
5. Same as above
6. $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$
7. Lagrange equations : $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$
   - ⋆ $\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = 0 \longrightarrow \ddot{x} = 0$
   - ⋆ $\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) = 0 \longrightarrow \ddot{y} = 0$
   - ⋆ $\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) = 0 \longrightarrow \ddot{z} = 0$
8. Usual procedure $\longrightarrow$ Newtonian results !

⋆ Note : Lagrange equations yield $m\dot{x} = p_x = constant$ etc.
Obviously due to the fact that $L$ independent of $q_k$

## Generalized momenta

Define : $\boxed{p_k \equiv \dfrac{\partial L}{\partial \dot{q}_k}}$ $\longrightarrow$ Lagrange equations : $\dot{p}_k = \dfrac{\partial L}{\partial q_k}$

- Fundamental physics law :

**Translational invariance $\longrightarrow$ Momentum conservation**

- From Quantum Mechanics or Special Relativity we have seen that also :

**Time invariance $\longrightarrow$ Energy conservation**

$*$ Reflects the Noether theorem : Some invariance $\Leftrightarrow$ Some Conservation

- **Example : Particle in central force field**

1. **Planar motion** $(z = 0) \rightarrow n = 2$
2. **Choose plane polar coordinates** $(r, \theta)$
3. **Coordinate relations :**

$$x = r\cos(\theta) \qquad \dot{x} = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)$$
$$y = r\sin(\theta) \qquad \dot{y} = \dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)$$

4. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad V = V(\sqrt{x^2 + y^2})$
5. $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad V = V(r)$
6. $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$
7. **Lagrange equations :**
   - ⋆ $mr\dot{\theta}^2 - \frac{\partial V}{\partial r} - m\ddot{r} = 0 \rightarrow m\ddot{r} = mr\dot{\theta}^2 + f(r)$
   - ⋆ $\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \rightarrow |\vec{L}| = constant$
8. **Identical to Kepler orbit situation** $\rightarrow$ **Same results !**

## Hamilton's equations of motion

- Define the **Hamiltonian** : $\boxed{H \equiv \sum_{k=1}^{n} \dot{q}_k p_k - L}$

$$\rightarrow \delta H = \sum_{k=1}^{n} \left[ p_k \delta \dot{q}_k + \dot{q}_k \delta p_k - \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k - \frac{\partial L}{\partial q_k} \delta q_k \right]$$

⋆ Using : $p_k = \frac{\partial L}{\partial \dot{q}_k}$ and $\dot{p}_k = \frac{\partial L}{\partial q_k} \rightarrow \delta H = \sum_{k=1}^{n} [\dot{q}_k \delta p_k - \dot{p}_k \delta q_k]$ **(1)**

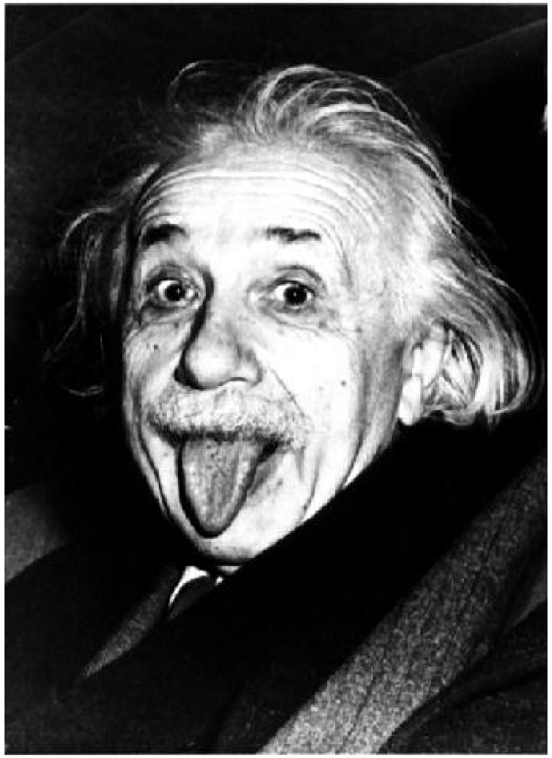
⋆ So : $H = H(p_k, q_k) \rightarrow \delta H = \sum_{k=1}^{n} \left[ \frac{\partial H}{\partial p_k} \delta p_k + \frac{\partial H}{\partial q_k} \delta q_k \right]$ **(2)**

- (1) & (2) directly yield : $\boxed{\frac{\partial H}{\partial p_k} = \dot{q}_k \quad \text{and} \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k}$

# The Classical Hamilton and Lagrange formalism

- **Hamilton's equations hold for more general systems (non-conservative, explicit time dependence of $L$, ...)**
- **Simple conservative systems $\rightarrow H = T + V$**

**And still something is wrong !**



# Going relativistic

- **In relativity everything is determined via the metric**

  **Special relativity :** $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$

  **General relativity :** $\mathrm{d}s^2 = g_{\mu\nu}(\tilde{x})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$

  **where** $\mathrm{d}\tilde{x} = (c\,\mathrm{d}t, \mathrm{d}\vec{r})$ **and** $\eta_{\mu\nu}$**=diag(1,-1,-1,-1)**

- **Consider two events $A$ and $B$ in space-time** $\rightarrow \Delta s = \int_A^B \mathrm{d}s$

  **where the integral follows some path (i.e. worldline) in space-time**

∗ **Hey, this looks like the classical Hamilton-Lagrange path integral**

  $\mathcal{P} = \int_{t_1}^{t_2} L\,\mathrm{d}t$ **where** $L = L(q_k, \dot{q}_k)$ **and** $q_k = q_k(t)$

- **Consider $A$ and $B$ as two fixed endpoints of some worldline in space-time**

  **Any worldline between $A$ and $B$ can be parametrized as** $x^\mu = x^\mu(\lambda)$

  **which yields :** $\mathrm{d}x^\mu = \dfrac{\partial x^\mu}{\partial\lambda}\mathrm{d}\lambda = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\mathrm{d}\lambda$

# Going relativistic

- This then leads to the following expressions :

Special relativity : $\Delta s = \int_{\lambda_1}^{\lambda_2} \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda$

General relativity : $\Delta s = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu}(\tilde{x}) \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda$

which can be written as $\Delta s = \int_{\lambda_1}^{\lambda_2} L\, \mathrm{d}\lambda$ with $L = L\left(x^\mu, \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)$

$*$ Compare this with $\mathcal{P} = \int_{t_1}^{t_2} L\, \mathrm{d}t$ with $L = L(q_k, \dot{q}_k)$

- If the actual worldline (called Geodesic) corresponds to an extremum of $\Delta s$ we obtain the Relativistic Lagrange equations

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial(\mathrm{d}x^\mu/\mathrm{d}\lambda)}\right) = \frac{\partial L}{\partial x^\mu}}$$

# The Geodesic equation

- The Relativistic Lagrange equations provide the so called Geodesic equation of which the solution is the actual worldline or Geodesic

  We will investigate free particles c.q. test bodies as examples

## Special Relativity

- $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ with $\eta_{\mu\nu}$=diag(1,-1,-1,-1) $\rightarrow L = \sqrt{\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} = \frac{c\,\mathrm{d}\tau}{\mathrm{d}\lambda}$
- Lagrange equations : $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial(\mathrm{d}x^\mu/\mathrm{d}\lambda)}\right) = \frac{\partial L}{\partial x^\mu}$

  Since $\frac{\partial L}{\partial x^\mu} = 0$ we get $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{2L}\frac{\partial}{\partial(\mathrm{d}x^\mu/\mathrm{d}\lambda)}\left[\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right]\right) = 0$

  Using $\mathrm{d}\lambda = \frac{c\,\mathrm{d}\tau}{L}$ and $\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \pm\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)^2$ we obtain : $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = 0$
- This is indeed the correct equation of motion for a free particle

# The Geodesic equation

## General Relativity

- $\mathrm{d}s^2 = g_{\mu\nu}(\tilde{x})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \longrightarrow L = \sqrt{g_{\mu\nu}(\tilde{x})\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} = \frac{c\,\mathrm{d}\tau}{\mathrm{d}\lambda}$
- **Lagrange equations :** $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial(\mathrm{d}x^\mu/\mathrm{d}\lambda)}\right) = \frac{\partial L}{\partial x^\mu}$

∗ **Writing out the equations as before we obtain the General Geodesic Equation**

$$\boxed{\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = \Gamma^\mu_{\nu\gamma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}}$$

- **Here the $\Gamma^\mu_{\nu\gamma}$ are called the Christoffel Symbols with**

$$\boxed{g_{\mu\delta}\Gamma^\delta_{\nu\gamma} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\nu} + \frac{\partial g_{\nu\gamma}}{\partial x^\mu}\right)}$$