

# Gravity as a Relativistic Effect

Nick van Eijndhoven

[nickve.nl@gmail.com](mailto:nickve.nl@gmail.com)

<http://sites.google.com/site/nickveweb>

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## Problems with Newtonian gravity

- Consider two pointmasses  $m_1$  and  $m_2$  at resp. locations  $\vec{r}_1$  and  $\vec{r}_2$

Gravity as described by Newton :

Instantaneous gravitational force  $\vec{F}_{12}$  of  $m_1$  exerted on  $m_2$  at time  $t$

$$|\vec{F}_{12}| = \frac{Gm_1m_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^2}$$

- \* **Clash with relativity** : Different notion of simultaneity in different inertial frames

→ The above Newtonian formula is valid in only 1 frame

- \* **Laws of physics should be identical in all inertial frames**

→ **New theory of gravity needed which is consistent with relativity**

- Another puzzle :  $\vec{F} = \frac{d\vec{p}}{dt} = (\text{constant mass}) = m\vec{a}$

Gravitation :  $\vec{F}_{grav} = m_G \cdot \vec{g} = m_I \cdot \vec{a}$

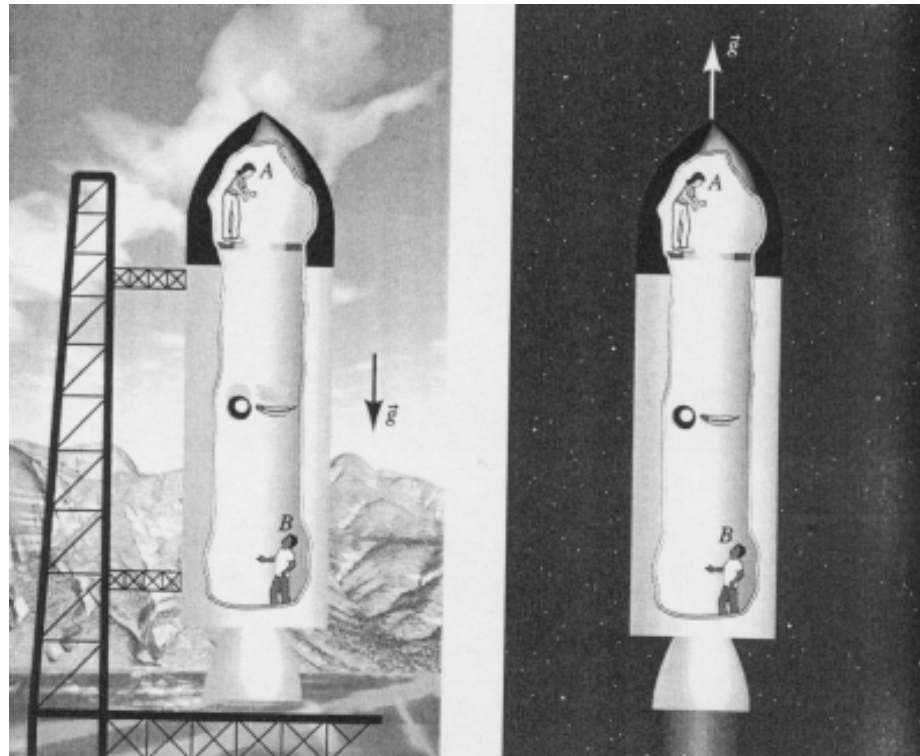
Why is **gravitational mass**  $m_G$  equal to the **inertial mass**  $m_I$  ?

Or : How does gravity "know" how strong to pull such that all objects get the same  $\vec{a}$  ?

## Einstein's insight

- Einstein : Observer in free fall to the earth → No experience of the gravitational field  
→ An accelerated frame can transform gravity away  
Objects at rest close to the observer will stay at rest → Local inertial frame  
→ **The gravitational field has only a relative existence**
- \* Consequence : **Gravity can be induced by accelerating the reference frame**

### The equivalence principle



# Effects in a uniform gravitational field

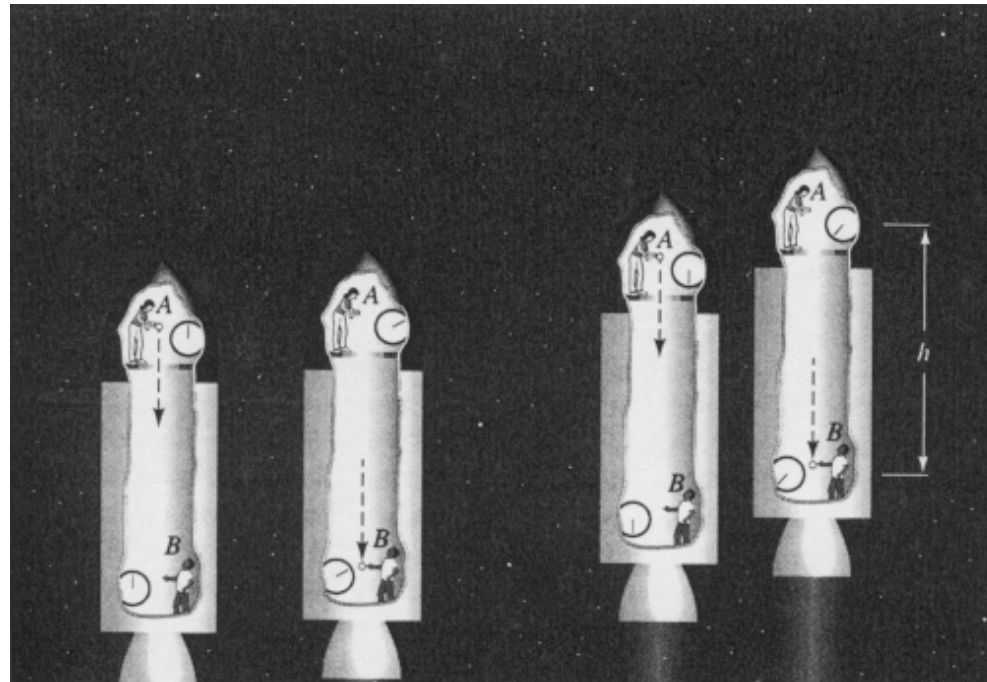
## Gravitational time dilation

- Consider a rocket in empty space with constant vertical acceleration  $g \hat{z}$

Nose of the rocket : Observer A with clock A emitting light signals at intervals  $\Delta\tau_A$

Tail of the rocket : Observer B with clock B receiving light signals at intervals  $\Delta\tau_B$

\* Distance A-B  $\equiv h$  : **At what time intervals does observer B receive the signals ?**



## Effects in a uniform gravitational field

- Enable simple Newtonian mechanics by selecting an inertial frame such that

$V \ll c$  ( $V$  is rocket velocity at signal emission)  $\rightarrow$  non-relativistic

$gh/c \ll c \rightarrow$  No acceleration to relativistic  $V$  while light travels nose-tail

- Choose origin of time : First pulse emitted at  $t = 0$  and  $z_B(t = 0) \equiv 0$

$V(t = 0) \equiv 0 \rightarrow$  Observer locations :  $z_B(t) = \frac{1}{2}gt^2$        $z_A(t) = h + \frac{1}{2}gt^2$

First pulse received at  $t = t_1$       second pulse emitted at  $t = \Delta\tau_A$

and second pulse received at  $t = t_1 + \Delta\tau_B$

- Approximation which is accurate to leading order in  $gh/c^2$  :

Distance traveled by first pulse :  $z_A(0) - z_B(t_1) = ct_1$

Distance traveled by second pulse :  $z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A)$

Using the observer locations and neglecting higher orders of  $\Delta\tau_A$  and  $\Delta\tau_B$  :

$$h - \frac{1}{2}gt_1^2 = ct_1 \quad \text{and} \quad h - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B = c(t_1 + \Delta\tau_B - \Delta\tau_A)$$

$$\rightarrow \Delta\tau_B = \frac{c\Delta\tau_A}{c + gt_1} \approx \frac{c\Delta\tau_A}{c + gh/c} = \frac{\Delta\tau_A}{1 + gh/c^2}$$

## Effects in a uniform gravitational field

Equivalence principle : The same must happen in a uniform gravitational field

- Gravitational time dilation

$$\Delta\tau_B \approx \frac{\Delta\tau_A}{1 + gh/c^2}$$

Emission and reception rates  $\nu$  :

$$\nu_B \approx \nu_A (1 + gh/c^2)$$

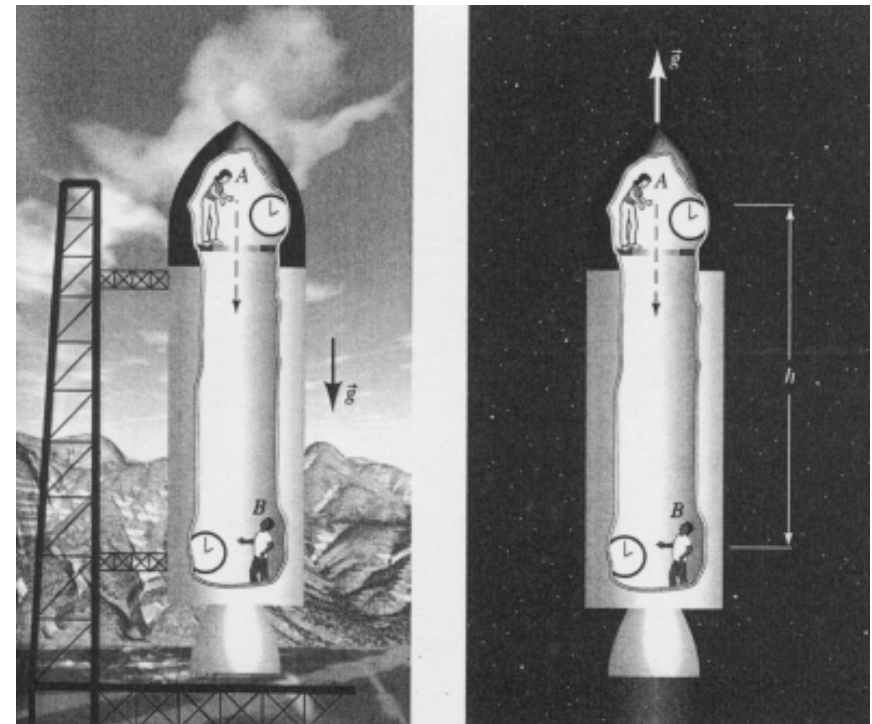
- In terms of the gravitational potential  $\Phi$  :

$$\nu_B \approx \nu_A \left( 1 + \frac{\Phi_A - \Phi_B}{c^2} \right)$$

Also valid in non-uniform grav. fields

### Gravitational blue/red shift

$$\text{Redshift : } z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}}$$

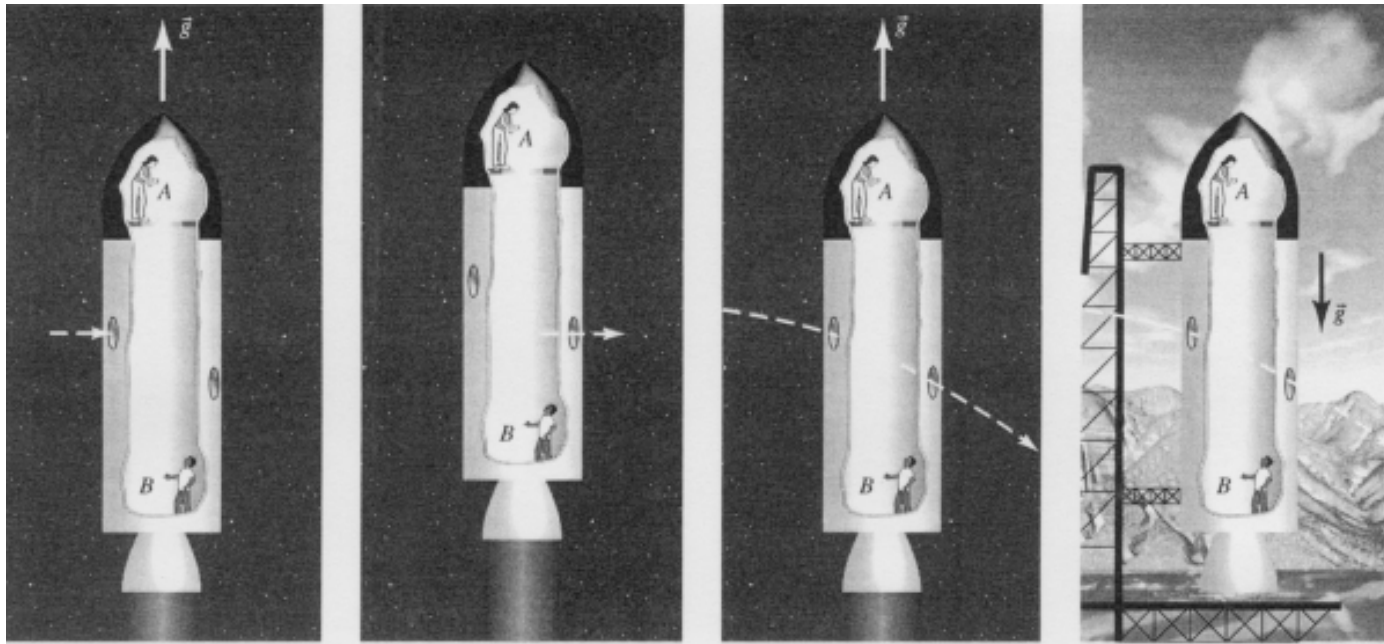


\* Exercise : Determine the redshift at  $r = \infty$  for the light emitted from the edge of a white dwarf with mass  $M = M_{\odot}$  and radius  $R = 10^3$  km

# Effects in a uniform gravitational field

## Gravitational deflection of light

- Consider a rocket in empty space with constant vertical acceleration  $g \hat{z}$   
Light ray enters (upper window) and exits (lower window)  
Outside inertial frame : light ray describes straight path  
Accelerated rocket frame : light ray falls down with acceleration  $g$



**A light ray in a gravitational field must fall with the same acceleration as other objects**

## Gravity as a geometrical effect

- Deflection of light in a gravitational field

Not corresponding to Newtonian gravity formula

Ultimate proof that all particles experience the same  $\vec{a}$  irrespective of  $m$

- Einstein's idea concerning gravitational path deflection

Consider a particle traveling in a straight line over a flat rubber sheet

Put a heavy object on the rubber sheet → sheet stretches and curves

→ The particle will now follow a curved path on the sheet

Gravitational path deflection is due to curvature of space

- Similar reasoning for the time coordinates

Gravitational time dilation is due to curvature of time

### Relativistic viewpoint on gravity

Gravity is a geometric effect due to curvature of space-time

The presence of mass introduces a curvature in space-time



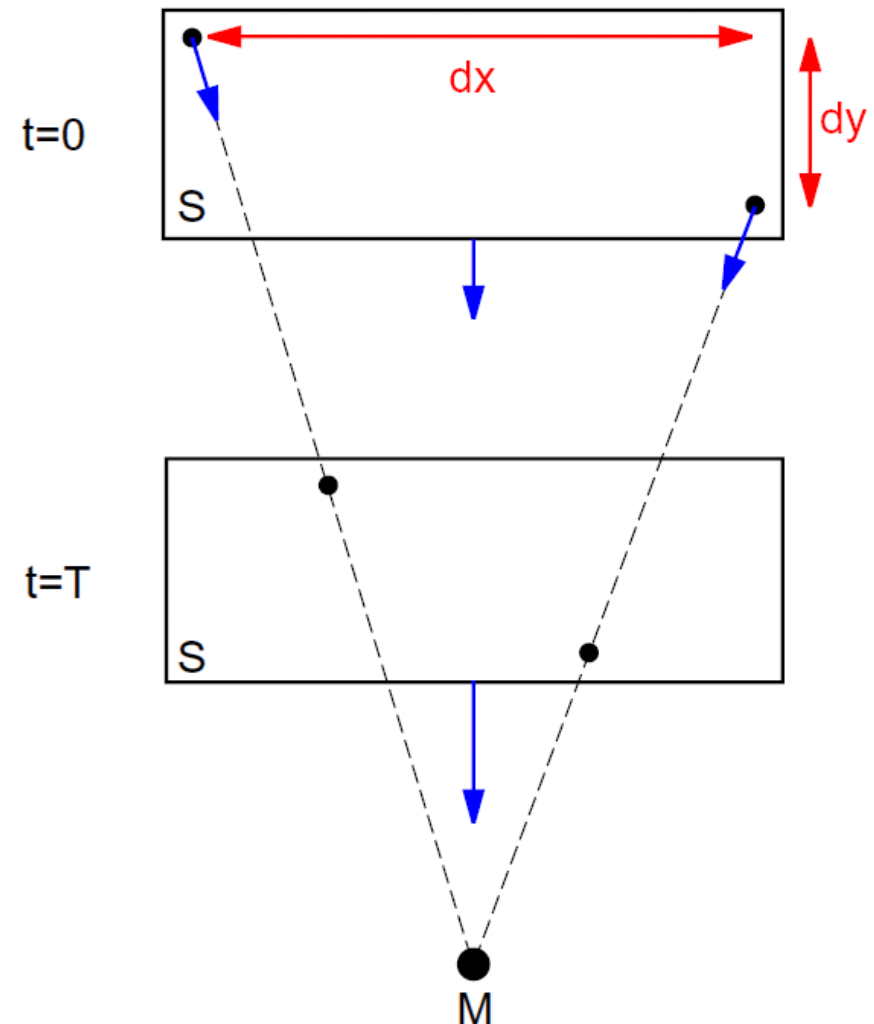
# Gravity as a geometrical effect

How to introduce curvature in space-time in accordance with observations ?

- Detailed look at the equivalence principle  
Frame  $S$  in free fall with two objects at rest
- \* Can the earth gravity stay "hidden" ?  
 $|\vec{g}|$  must be constant in  $S \rightarrow dy$  small  
 $g_x$  must be small  $\rightarrow dx$  small  
Observation time  $T$  must be small

## The Equivalence Principle

Experiments performed in a sufficiently small freely falling laboratory, over a sufficiently short time, yield results that are indistinguishable from those of the same experiments performed in an inertial frame in empty space.



## Linking the Equivalence Principle with relativity

- Relativistic description of an **inertial frame**

It's all included in the metric

$$ds^2 = (cdt)^2 - d\vec{r}^2$$

As 4-vectors :  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

with  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

- In the presence of gravity :

Freely falling frame is only locally inertial

→ All other locations in space-time have :

$$ds^2 = g_{\mu\nu}(\tilde{x}) dx^\mu dx^\nu$$

$\tilde{x} \equiv$  location in space-time

## Description of space-time curvature

Introduce metric tensor  $g_{\mu\nu}(\tilde{x})$  of which the components depend on the location in space-time

- Equivalence Principle :  $S \rightarrow S'$

$$g_{\mu\nu}(\tilde{x}) \rightarrow g'_{\mu\nu}(\tilde{x}') = \eta_{\mu\nu}$$

- Consequences :

$g_{\mu\nu}(\tilde{x})$  must be a symmetric 4x4 matrix

Always 1 time and 3 space coordinates

- \* This is the basis of **General Relativity**

What are the components of  $g_{\mu\nu}(\tilde{x})$  ?

→ Need to investigate curvature

- Investigate spatial curvature by considering familiar 2-dim. surfaces
  - Can be looked upon as embedded in 3-dim. space → easy to catch the picture
- \* 2-dimensional plane : Obviously flat
- \* Surface of a sphere : probably curved
  - Can't coincide with a plane without stretching or tearing
- \* Surface of a cylinder : probably flat
  - Can be unrolled onto a plane without distortion
- Gauss : The curvature of a surface can be determined intrinsically
  - Make measurements and apply the theorems of Euclidean plane geometry
  - In case of contradiction : Geometry is non-Euclidean → curved surface
- \* Example : Triangle in a plane → sum of the angles adds up to  $180^\circ$ 
  - Triangle on surface of a sphere : sum of angles  $> 180^\circ$  → spherical surface is curved
- Need for a precise definition of curvature
  - Use distance  $ds$  between any 2 points on the surface

- Curvature in a 2-dim. plane : Cartesian grid  $x^1 \equiv x \quad x^2 \equiv y$   
 $ds^2 = (dx)^2 + (dy)^2 = (dx^1)^2 + (dx^2)^2 \rightarrow g_{\mu\nu} = \text{diag}(1, 1)$  **No curvature**
- \* Use polar coordinates in the same plane :  $x^1 = r \quad x^2 = \varphi$   
 $ds^2 = (dr)^2 + (r d\varphi)^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 \rightarrow g_{\mu\nu} = \text{diag}(1, (x^1)^2)$   
**If started with polar coord., how would we know whether the surface is curved or not ?**  
 Try to find a coord. trafo such that  $g_{\mu\nu} \rightarrow \text{diag}(1, 1)$   
 Obviously such a trafo is :  $(x^1)' = x^1 \cos(x^2) \quad (x^2)' = x^1 \sin(x^2)$
- Curvature on the surface of a cylinder with radius  $R$  : Cyl. coordinates  $(r, \varphi, z)$   
 $ds^2 = (R d\varphi)^2 + dz^2$  where  $R$  is a constant  $\rightarrow$  Use  $x^1 = R\varphi \quad x^2 = z$   
 $ds^2 = (dx^1)^2 + (dx^2)^2 \rightarrow g_{\mu\nu} = \text{diag}(1, 1)$  **No curvature**
- Curvature on the surface of a sphere with radius  $R$  : Spherical coordinates  $(r, \theta, \varphi)$   
 $ds^2 = (R d\theta)^2 + (R \sin(\theta) d\varphi)^2$  where  $R$  is a constant  $\rightarrow$  Use  $x^1 = \theta \quad x^2 = \varphi$   
 $ds^2 = R^2 (dx^1)^2 + R^2 \sin^2(x^1) (dx^2)^2 \rightarrow g_{\mu\nu} = \text{diag}(R^2, R^2 \sin^2(x^1))$   
 No trafo possible to get  $g_{\mu\nu} = \text{diag}(1, 1) \rightarrow$  **Spherical surface is intrinsically curved**

## The curvature formula of Gauss

- From the previous : **The curvature is contained in  $g_{\mu\nu}$**

The **curvature  $K$**  is an invariant quantity

It should be possible to obtain the curvature  $K$  from the tensor  $g_{\mu\nu}$

- The recipe provided by Gauss to determine the curvature (limited to 2-dim. surfaces)

$$K = \frac{1}{2g_{11}g_{22}} \left\{ \frac{-\partial^2 g_{11}}{(\partial x^2)^2} + \frac{-\partial^2 g_{22}}{(\partial x^1)^2} + \frac{1}{2g_{11}} \left[ \frac{\partial g_{11}}{\partial x^1} \cdot \frac{\partial g_{22}}{\partial x^1} + \left( \frac{\partial g_{11}}{\partial x^2} \right)^2 \right] + \frac{1}{2g_{22}} \left[ \frac{\partial g_{11}}{\partial x^2} \cdot \frac{\partial g_{22}}{\partial x^2} + \left( \frac{\partial g_{22}}{\partial x^1} \right)^2 \right] \right\}$$

- \* For the surface of a sphere with radius  $R$  this yields :  $K = 1/R^2$
- Higher dimensional surfaces : Can't be embedded in 3-dim. space  $\rightarrow$  imagination fails  
Equivalent of curvature  $K$  for higher dim. surfaces involves a **curvature tensor  $R_{\alpha\beta\gamma\delta}$**
- \* General treatment of Einstein's theory  $\rightarrow$  **Tensor calculus**  
Our cases contain symmetries  $\rightarrow$  we can avoid tensor calculus and use  $K$  instead

## Curvature in space-time : The Schwarzschild metric

- Consider an isolated point mass  $M$  at the origin  $O$  in empty space

\* Two effects of curvature in space-time

Time is distorted by the gravitational time dilation

3-dim. position space becomes curved due to the presence of the mass  $M$

- Isolated point mass  $M$  in  $O \rightarrow$  Space is isotropic w.r.t.  $O \rightarrow$  Use spherical coord.

Curvature can only depend on  $r$  and should vanish when  $r \rightarrow \infty$

- General expression for a curved isotropic metric in spherical coordinates :

$$ds^2 = \alpha(r)(c dt)^2 - [f(r)(dr)^2 + (r d\theta)^2 + (r \sin(\theta) d\varphi)^2]$$

\* **Determination of the time distortion  $\alpha(r)$  using  $ds = c d\tau$  when  $d\vec{r} = \vec{0}$**

$$\text{From before we have seen : } d\tau_{obs} \approx d\tau_{emit} \left( 1 + \frac{\Phi_{emit} - \Phi_{obs}}{c^2} \right)^{-1}$$

Putting the emitter at coordinates  $(t, r, 0, 0)$  and the receiver at  $(t, \infty, 0, 0)$  we obtain :

$$\frac{(d\tau_{emit})^2}{(d\tau_{obs})^2} = \frac{\alpha(r)(dt)^2}{\alpha(\infty)(dt)^2} = \alpha(r) \approx \left( 1 + \frac{\Phi_{emit}}{c^2} \right)^2 \approx \left( 1 + \frac{2\Phi_{emit}}{c^2} \right)$$

## Curvature in space-time : The Schwarzschild metric

- Using  $\Phi_{emit} = \frac{-GM}{r}$  directly yields  $\alpha(r) = \left(1 - \frac{2GM}{c^2 r}\right)$

The same result is obtained from a rigorous treatment of Einstein's equations !

### \* Determination of the spatial deformation $f(r)$

Spatial curvature can only depend on  $r \rightarrow K(r)$  and  $K(\infty) \equiv 0$

- Let's try to "guess" the most simple form of  $K(r)$

$$K(r) \rightarrow 0 \text{ when } r \rightarrow \infty \Rightarrow K(r) \propto r^{-n}$$

Intuition :  $K \propto M$  and also  $G$  should be in the game

Use  $c$  to get dimensions right and use  $\lambda = \pm 1$  to allow positive and negative curvature

- Intuitive guess :  $K(r) = \lambda M G^k c^m r^{-n}$

$\rightarrow$  Dimensionless for  $(k, m, n) = (1, -2, 3)$

Spherical surface :  $K = 1/R^2 > 0 \rightarrow$  mass  $M$  : "rubber sheet" gets  $K < 0$

- \* So, for the simplest form we obtain  $K(r) = \frac{-GM}{c^2 r^3}$

The same result is obtained from a rigorous treatment of Einstein's equations !

## Curvature in space-time : The Schwarzschild metric

- Determination of  $f(r)$  from the Gauss curvature formule using our metric and  $K(r)$

For simplicity use  $\theta = \pi/2$  and  $d\theta = d\varphi = 0$  and of course  $dt = 0$

$$\text{This yields : } K(r) = \frac{1}{2r f^2(r)} \cdot \frac{df(r)}{dr}$$

$$\text{Using our curvature } K(r) = \frac{-GM}{c^2 r^3} \text{ we obtain } \frac{1}{f^2(r)} \frac{df(r)}{dr} = \frac{-2GM}{c^2 r^2}$$

$$\ast \text{ Solution of this differential equation : } \frac{-1}{f(r)} = \frac{2GM}{c^2 r} + C \quad (C = \text{constant})$$

$$\text{Boundary condition : } f(r = \infty) = 1 \rightarrow C = -1 \Rightarrow f(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

- The final metric we obtain is called the **Schwarzschild metric**

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) (c dt)^2 - \left[ \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + (r d\theta)^2 + (r \sin(\theta) d\varphi)^2 \right]$$



## Curvature in space-time : The Schwarzschild metric

- The Schwarzschild metric describes the space-time around an isolated point mass  $M$

→ Prediction of worldlines of test bodies which can be experimentally verified

\* Special case when  $\left(1 - \frac{2GM}{c^2 r}\right) = 0 \quad \rightarrow \quad g_{00} = 0 \quad g_{11} = -\infty$

Define the Schwarzschild radius  $R_s = 2GM/c^2$

- Consider a material body at rest at  $r > R_s \rightarrow ds^2 > 0 \Rightarrow$  normal (timelike) situation

At  $r < R_s$  the  $ds^2 > 0$  means the body HAS to move → it falls into  $M$

- Consider observation of a light ray emitted radially from  $r_{emit}$  to  $r = \infty$

Schwarzschild metric :  $\nu_{obs} = \nu_{emit} \left(1 - \frac{2GM}{c^2 r_{emit}}\right)^{1/2}$

If  $r_{emit} > R_s \rightarrow$  light is redshifted  $\Rightarrow$  normal situation

At  $r_{emit} = R_s$  the redshift becomes  $\infty \rightarrow$  No light is observed (infinite time dilation)

At  $r = R_s \rightarrow$  infinite time dilation  $\Rightarrow$  Events are observed as "frozen"

- \* Mass  $M$  contained within a sphere of radius  $R_s \rightarrow$  Nothing can escape the surface

An object which is smaller than its Schwarzschild radius is called a Black Hole

## Exercises

- Consider an isolated point mass  $M$  with Schwarzschild radius  $R_s$ , located at the origin  $O$   
From a distance  $r_e > R_s$  a light ray is radially emitted and observed at a distance  $r_o > r_e$
- \* Show that the exact formula for the gravitational redshift  $z$  is given by :

$$z = \sqrt{\frac{1 - R_s/r_o}{1 - R_s/r_e}} - 1$$

- Consider a proton as a spherical object with a radius of 1 fm.
  - \* Determine from this the density of normal nuclear matter in  $\text{GeV}/\text{fm}^3$
  - The mass of the Earth is  $M = 5.975 \cdot 10^{24} \text{ kg}$
  - \* Determine the Schwarzschild radius  $R_s$  of the Earth
  - Imagine that all the mass of the Earth is concentrated in a sphere with radius  $R_s$
  - \* Determine the density in  $\text{GeV}/\text{fm}^3$  of this "Earth black hole" object
- For comparison : QGP phase transition is expected to happen at about  $3 \text{ GeV}/\text{fm}^3$