Gravity as a Relativistic Effect

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Contents

Problems with Newtonian gravity

- Consider two pointmasses m_1 and m_2 at resp. locations $\vec{r}_1$ and $\vec{r}_2$

  Gravity as described by Newton :

  Instanteneous gravitational force $\vec{F}_{12}$ of m_1 exerted on m_2 at time t

$$|\vec{F}_{12}| = \frac{G m_1 m_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^2}$$

$*$ Clash with relativity : Different notion of simultaneity in different inertial frames

$\longrightarrow$ The above Newtonian formula is valid in only 1 frame

$*$ Laws of physics should be identical in all inertial frames

$\longrightarrow$ New theory of gravity needed which is consistent with relativity

- Another puzzle : $\vec{F} = \dfrac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \text{(constant mass)} = m\vec{a}$

  Gravitation : $\vec{F}_{grav} = m_G \cdot \vec{g} = m_I \cdot \vec{a}$

  Why is gravitational mass m_G equal to the inertial mass m_I ?

  Or : How does gravity "know" how strong to pull such that all objects get the same $\vec{a}$?

Einstein's insight

- Einstein : Observer in free fall to the earth $\longrightarrow$ No experience of the gravitational field

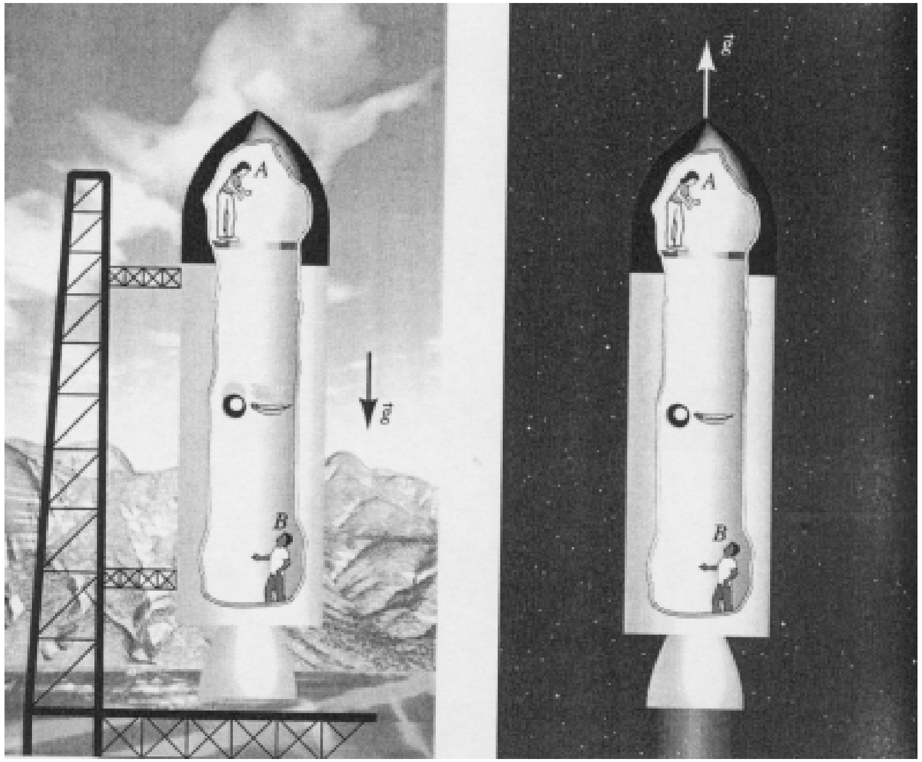
  $\longrightarrow$ An accelerated frame can transform gravity away

  Objects at rest close to the observer will stay at rest $\longrightarrow$ Local inertial frame

  $\longrightarrow$ The gravitational field has only a relative existence

$*$ Consequence : Gravity can be induced by accelerating the reference frame

The equivalence principle



Effects in a uniform gravitational field

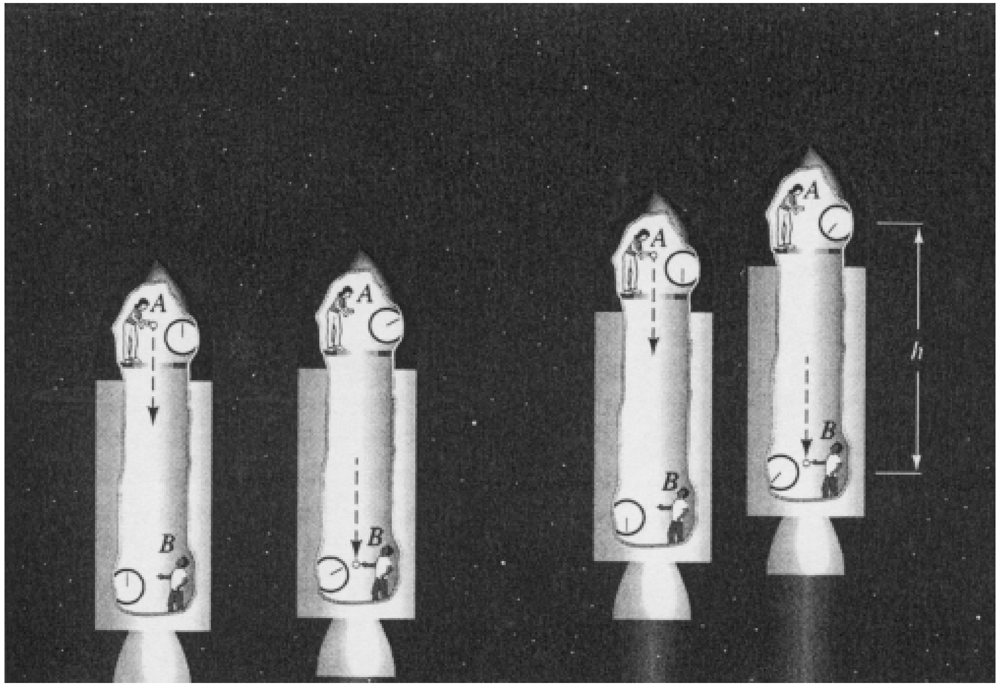
Gravitational time dilation

• Consider a rocket in empty space with constant vertical acceleration $g\,\hat{z}$

Nose of the rocket : Observer A with clock A emitting light signals at intervals $\Delta\tau_A$

Tail of the rocket : Observer B with clock B receiving light signals at intervals $\Delta\tau_B$

∗ Distance A-B $\equiv h$: At what time intervals does observer B receive the signals ?



Effects in a uniform gravitational field

- Enable simple Newtonian mechanics by selecting an inertial frame such that
  $V \ll c$ (V is rocket velocity at signal emission) $\longrightarrow$ non-relativistic
  $gh/c \ll c \longrightarrow$ No acceleration to relativistic V while light travels nose-tail
- Choose origin of time : First pulse emitted at $t = 0$ and $z_B(t=0) \equiv 0$
  $V(t=0) \equiv 0 \longrightarrow$ Observer locations : $z_B(t) = \frac{1}{2}gt^2 \qquad z_A(t) = h + \frac{1}{2}gt^2$
  First pulse received at $t = t_1$ second pulse emitted at $t = \Delta\tau_A$
  and second pulse received at $t = t_1 + \Delta\tau_B$
- Approximation which is accurate to leading order in gh/c^2 :
  Distance traveled by first pulse : $z_A(0) - z_B(t_1) = ct_1$
  Distance traveled by second pulse : $z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A)$
  Using the observer locations and neglecting higher orders of $\Delta\tau_A$ and $\Delta\tau_B$:

$$h - \tfrac{1}{2}gt_1^2 = ct_1 \quad \text{and} \quad h - \tfrac{1}{2}gt_1^2 - gt_1\Delta\tau_B = c(t_1 + \Delta\tau_B - \Delta\tau_A)$$

$$\longrightarrow \Delta\tau_B = \frac{c\Delta\tau_A}{c + gt_1} \approx \frac{c\Delta\tau_A}{c + gh/c} = \frac{\Delta\tau_A}{1 + gh/c^2}$$

Effects in a uniform gravitational field

Equivalence principle : The same must happen in a uniform gravitational field

- Gravitational time dilation

$$\Delta\tau_B \approx \frac{\Delta\tau_A}{1 + gh/c^2}$$

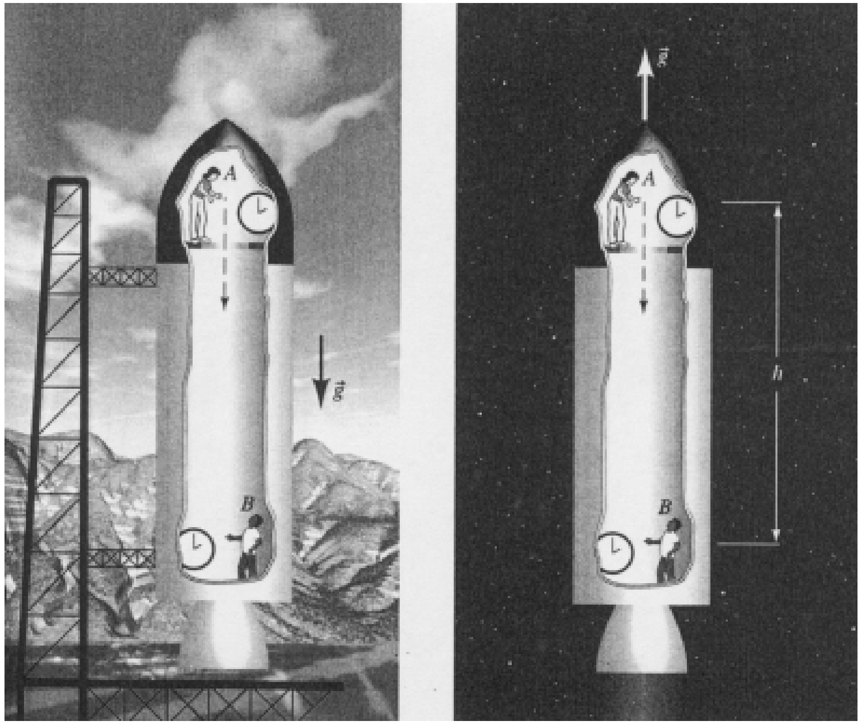
Emission and reception rates ν :

$$\nu_B \approx \nu_A \left(1 + gh/c^2\right)$$

- In terms of the gravitational potential Φ :

$$\boxed{\nu_B \approx \nu_A \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right)}$$

Also valid in non-uniform grav. fields



Gravitational red shift

$$z \equiv \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} = \frac{\lambda_{obs}}{\lambda_{emit}} - 1$$

($z < 0 \rightarrow$ blue shift)

∗ Exercise : Determine the redshift at $r = \infty$ for the light emitted from the edge of a white dwarf with mass $M = M_\odot$ and radius $R = 10^3$ km

Effects in a uniform gravitational field

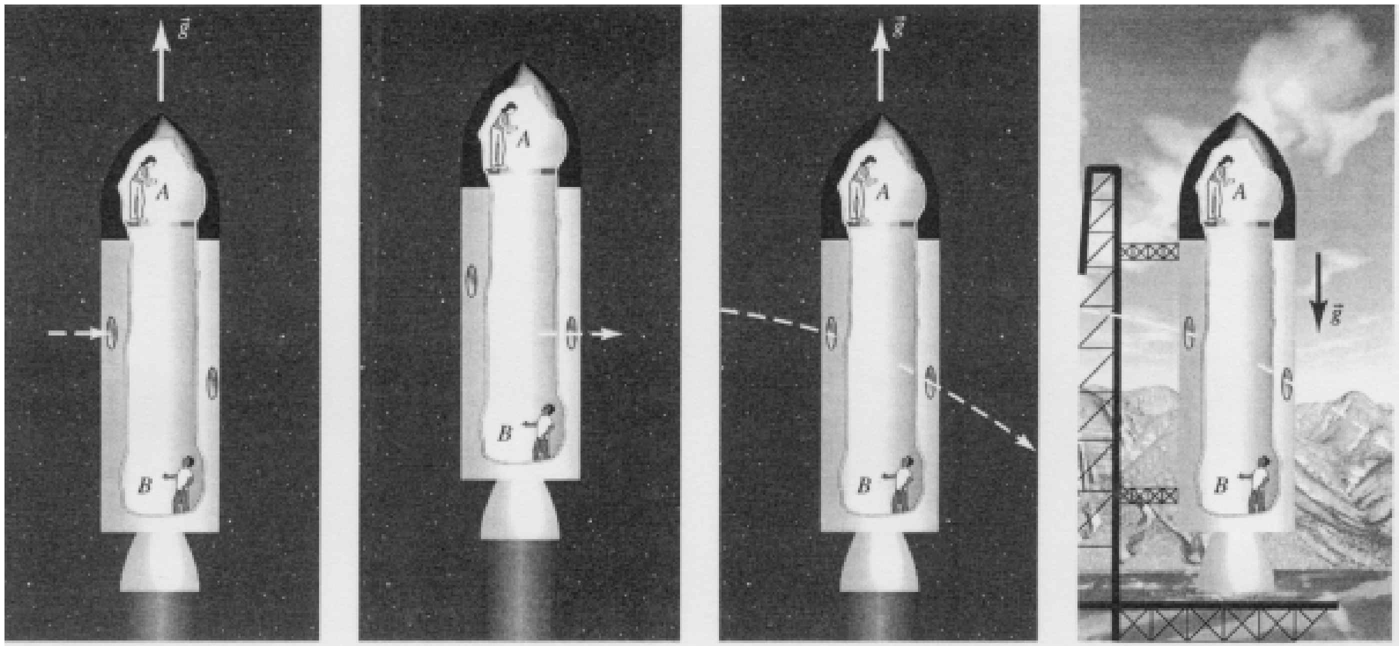
Gravitational deflection of light

- Consider a rocket in empty space with constant vertical acceleration $g\,\hat{z}$

  Light ray enters (upper window) and exits (lower window)

  Outside inertial frame : light ray describes straight path

  Accelerated rocket frame : light ray falls down with acceleration g



A light ray in a gravitational field must fall with the same acceleration as other objects

Gravity as a geometrical effect

- Deflection of light in a gravitational field

  Not corresponding to Newtonian gravity formula

  Ultimate proof that all particles experience the same $\vec{a}$ irrespective of m
- Einstein's idea concerning gravitational path deflection

  Consider a particle traveling in a straight line over a flat rubber sheet

  Put a heavy object on the rubber sheet $\longrightarrow$ sheet stretches and curves

  $\longrightarrow$ The particle will now follow a curved path on the sheet

  Gravitational path deflection is due to curvature of space
- Similar reasoning for the time coordinates

  Gravitational time dilation is due to curvature of time

Relativistic viewpoint on gravity

Gravity is a geometric effect due to curvature of space-time

The presence of mass introduces a curvature in space-time

Gravity as a geometrical effect

How to introduce curvature in space-time in accordance with observations ?

- Detailed look at the equivalence principle

  Frame S in free fall with two objects at rest

∗ Can the earth gravity stay "hidden" ?

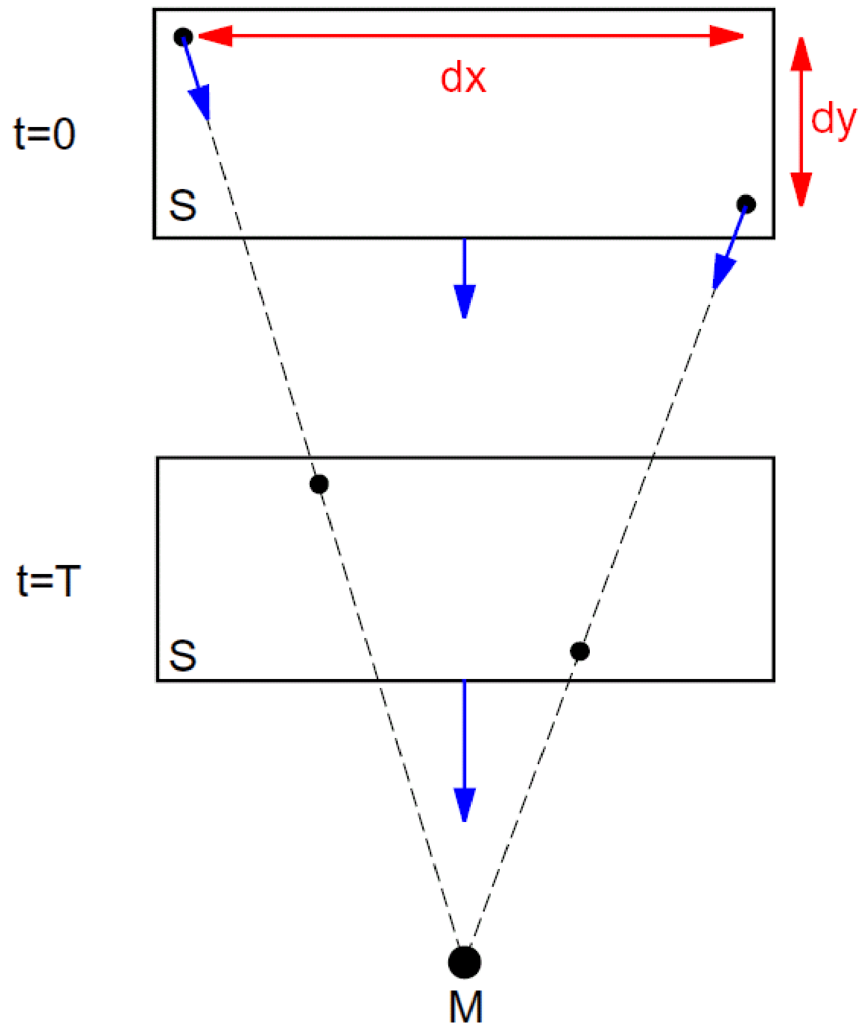
  $|\vec{g}|$ must be constant in $S \longrightarrow \mathrm{d}y$ small

  g_x must be small $\longrightarrow \mathrm{d}x$ small

  Observation time T must be small

The Equivalence Principle

Experiments performed in a sufficiently small freely falling laboratory, over a sufficiently short time, yield results that are indistinguishable from those of the same experiments performed in an inertial frame in empty space.



Gravity as a geometrical effect

Linking the Equivalence Principle with relativity

- Relativistic description of an inertial frame

  It's all included in the metric

  $ds^2 = (c\,dt)^2 - d\vec{r}^{\,2}$

  As 4-vectors : $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$

  with $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

- In the presence of gravity :

  Freely falling frame is only locally inertial

  $\longrightarrow$ All other locations in space-time have :

  $$ds^2 = g_{\mu\nu}(\tilde{x})\, dx^\mu\, dx^\nu$$

  $\tilde{x} \equiv$ location in space-time

Description of space-time curvature

Introduce metric tensor $g_{\mu\nu}(\tilde{x})$ of which the components depend on the location in space-time

- Equivalence Principle : $S \longrightarrow S'$

  $g_{\mu\nu}(\tilde{x}) \longrightarrow g'_{\mu\nu}(\tilde{x}\,') = \eta_{\mu\nu}$

- Consequences :

  $g_{\mu\nu}(\tilde{x})$ must be a symmetric 4x4 matrix

  Always 1 time and 3 space coordinates

$\ast$ This is the basis of General Relativity

What are the components of $g_{\mu\nu}(\tilde{x})$?

$\longrightarrow$ Need to investigate curvature

Curvature in space

- Investigate spatial curvature by considering familiar 2-dim. surfaces

  Can be looked upon as embedded in 3-dim. space $\longrightarrow$ easy to catch the picture

$*$ 2-dimensional plane : Obviously flat

$*$ Surface of a sphere : probably curved

  Can't coincide with a plane without stretching or tearing

$*$ Surface of a cylinder : probably flat

  Can be unrolled onto a plane without distortion

- Gauss : The curvature of a surface can be determined intrinsically

  Make measurements and apply the theorems of Euclidean plane geometry

  In case of contradiction : Geometry is non-Euclidean $\longrightarrow$ curved surface

$*$ Example : Triangle in a plane $\longrightarrow$ sum of the angles adds up to 180^0

  Triangle on surface of a sphere : sum of angles $> 180^0$ $\longrightarrow$ spherical surface is curved

- Need for a precise definition of curvature

  Use distance $\mathrm{d}s$ between any 2 points on the surface

Curvature in space

- Curvature in a 2-dim. plane : Cartesian grid $x_1 \equiv x \quad x_2 \equiv y$

  $ds^2 = (dx)^2 + (dy)^2 = (dx_1)^2 + (dx_2)^2 \longrightarrow g_{\mu\nu} = \text{diag}(1,1)$ No curvature

$\ast$ Use polar coordinates in the same plane : $x_1 = r \quad x_2 = \varphi$

  $ds^2 = (dr)^2 + (r d\varphi)^2 = (dx_1)^2 + (x_1)^2 (dx_2)^2 \longrightarrow g_{\mu\nu} = \text{diag}(1, (x_1)^2)$

  If started with polar coord., how would we know whether the surface is curved or not ?

  Try to find a coord. trafo such that $g_{\mu\nu} \longrightarrow \text{diag}(1,1)$

  Obviously such a trafo is : $(x_1)' = x_1 \cos(x_2) \quad (x_2)' = x_1 \sin(x_2)$

- Curvature on the surface of a cylinder with radius R : Cyl. coordinates (r, φ, z)

  $ds^2 = (R d\varphi)^2 + dz^2$ where R is a constant $\longrightarrow$ Use $x_1 = R\varphi \quad x_2 = z$

  $ds^2 = (dx_1)^2 + (dx_2)^2 \longrightarrow g_{\mu\nu} = \text{diag}(1,1)$ No curvature

- Curvature on the surface of a sphere with radius R : Spherical coordinates (r, θ, φ)

  $ds^2 = (R d\theta)^2 + (R \sin(\theta) d\varphi)^2$ where R is a constant $\longrightarrow$ Use $x_1 = \theta \quad x_2 = \varphi$

  $ds^2 = R^2 (dx_1)^2 + R^2 \sin^2(x_1)(dx_2)^2 \longrightarrow g_{\mu\nu} = \text{diag}(R^2, R^2 \sin^2(x_1))$

  No trafo possible to get $g_{\mu\nu} = \text{diag}(1,1) \longrightarrow$ Spherical surface is intrinsically curved

Curvature in space

The curvature formula of Gauss

- From the previous : The curvature is contained in $g_{\mu\nu}$

  The curvature K is an invariant quantity

  It should be possible to obtain the curvature K from the tensor $g_{\mu\nu}$
- The recipe provided by Gauss to determine the curvature (limited to 2-dim. surfaces)

$$K = \frac{1}{2g_{11}g_{22}} \left\{ \frac{-\partial^2 g_{11}}{(\partial x_2)^2} + \frac{-\partial^2 g_{22}}{(\partial x_1)^2} + \frac{1}{2g_{11}} \left[\frac{\partial g_{11}}{\partial x_1} \cdot \frac{\partial g_{22}}{\partial x_1} + \left(\frac{\partial g_{11}}{\partial x_2} \right)^2 \right] \right.$$
$$\left. + \frac{1}{2g_{22}} \left[\frac{\partial g_{11}}{\partial x_2} \cdot \frac{\partial g_{22}}{\partial x_2} + \left(\frac{\partial g_{22}}{\partial x_1} \right)^2 \right] \right\}$$

∗ For the surface of a sphere with radius R this yields : $K = 1/R^2$

- Higher dimensional surfaces : Can't be embedded in 3-dim. space ⟶ imagination fails

  Equivalent of curvature K for higher dim. surfaces involves a curvature tensor $R_{\alpha\beta\gamma\delta}$

∗ General treatment of Einstein's theory ⟶ Tensor calculus

  Our cases contain symmetries ⟶ we can avoid tensor calculus and use K instead

Curvature in space-time : The Schwarzschild metric

- Consider an isolated point mass M at the origin O in empty space

$*$ Two effects of curvature in space-time

Time is distorted by the gravitational time dilation

3-dim. position space becomes curved due to the presence of the mass M

- Isolated point mass M in O $\longrightarrow$ Space is isotropic w.r.t. O $\longrightarrow$ Use spherical coord.

Curvature can only depend on r and should vanish when $r \longrightarrow \infty$

- General expression for a curved isotropic metric in spherical coordinates :

$$\mathrm{d}s^2 = \alpha(r)(c\,\mathrm{d}t)^2 - [f(r)(\mathrm{d}r)^2 + (r\mathrm{d}\theta)^2 + (r\sin(\theta)\mathrm{d}\varphi)^2]$$

$*$ Determination of the time distortion $\alpha(r)$ using $\mathrm{d}s = c\,\mathrm{d}\tau$ when $\mathrm{d}\vec{r} = \vec{0}$

From before we have seen : $\mathrm{d}\tau_{obs} \approx \mathrm{d}\tau_{emit}\left(1 + \frac{\Phi_{emit} - \Phi_{obs}}{c^2}\right)^{-1}$

Putting the emitter at coordinates $(t, r, 0, 0)$ and the receiver at $(t, \infty, 0, 0)$ we obtain :

$$\frac{(\mathrm{d}\tau_{emit})^2}{(\mathrm{d}\tau_{obs})^2} = \frac{\alpha(r)(\mathrm{d}t)^2}{\alpha(\infty)(\mathrm{d}t)^2} = \alpha(r) \approx \left(1 + \frac{\Phi_{emit}}{c^2}\right)^2 \approx \left(1 + \frac{2\Phi_{emit}}{c^2}\right)$$

Curvature in space-time : The Schwarzschild metric

- Using $\Phi_{emit} = \dfrac{-GM}{r}$ directly yields $\alpha(r) = \left(1 - \dfrac{2GM}{c^2 r}\right)$

  The same result is obtained from a rigorous treatment of Einstein's equations !

* Determination of the spatial deformation $f(r)$

  Spatial curvature can only depend on $r \longrightarrow K(r)$ and $K(\infty) \equiv 0$

- Let's try to "guess" the most simple form of $K(r)$

  $K(r) \longrightarrow 0$ when $r \longrightarrow \infty \Rightarrow K(r) \propto r^{-n}$

  Intuition : $K \propto M$ and also G should be in the game

  Use c to get dimensions right and use $\lambda = \pm 1$ to allow positive and negative curvature

- Intuitive guess : $K(r) = \lambda M G^k c^m r^{-n}$

  $\longrightarrow$ Dimensionless for $(k, m, n) = (1, -2, 3)$

  Spherical surface : $K = 1/R^2 > 0 \longrightarrow$ mass M : "rubber sheet" gets $K < 0$

* So, for the simplest form we obtain $K(r) = \dfrac{-GM}{c^2 r^3}$

  The same result is obtained from a rigorous treatment of Einstein's equations !

Curvature in space-time : The Schwarzschild metric

- Determination of $f(r)$ from the Gauss curvature formule using our metric and $K(r)$

  Go 2-D in (r,φ) by using $\theta \equiv \pi/2 \rightarrow \mathrm{d}\theta = 0 \quad \sin(\theta) = 1$ and of course $\mathrm{d}t \equiv 0$

  This yields : $K(r) = \dfrac{1}{2rf^2(r)} \cdot \dfrac{\mathrm{d}f(r)}{\mathrm{d}r}$

  Using our curvature $K(r) = \dfrac{-GM}{c^2r^3}$ we obtain $\dfrac{1}{f^2(r)}\dfrac{\mathrm{d}f(r)}{\mathrm{d}r} = \dfrac{-2GM}{c^2r^2}$

∗ Solution of this differential equation : $\dfrac{-1}{f(r)} = \dfrac{2GM}{c^2r} + C \quad (C =$constant$)$

  Boundary condition : $f(r=\infty) = 1 \rightarrow C = -1 \Rightarrow f(r) = \left(1 - \dfrac{2GM}{c^2r}\right)^{-1}$

- The final metric we obtain is called the Schwarzschild metric

$$\boxed{\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2r}\right)(c\,\mathrm{d}t)^2 - \left[\frac{(\mathrm{d}r)^2}{\left(1 - \frac{2GM}{c^2r}\right)} + (r\mathrm{d}\theta)^2 + (r\sin(\theta)\mathrm{d}\varphi)^2\right]}$$

Curvature in space-time : The Schwarzschild metric

- The Schwarzschild metric describes the space-time around an isolated point mass M

  $\longrightarrow$ Prediction of worldlines of test bodies which can be experimentally verified

* Special case when $\left(1 - \frac{2GM}{c^2 r}\right) = 0 \quad \longrightarrow g_{00} = 0 \quad g_{11} = -\infty$

  Define the Schwarzschild radius $R_s = 2GM/c^2$

- Consider a material body at rest at $r > R_s \longrightarrow \mathrm{d}s^2 > 0 \quad \Rightarrow$ normal (timelike) situation

  At $r < R_s$ the $\mathrm{d}s^2 > 0$ means the body HAS to move $\longrightarrow$ it falls into M

- Consider observation of a light ray emitted radially from r_{emit} to $r = \infty$

  Schwarzschild metric : $\nu_{obs} = \nu_{emit} \left(1 - \frac{R_s}{r_{emit}}\right)^{1/2}$

  If $r_{emit} > R_s \longrightarrow$ light is redshifted $\Rightarrow$ normal situation

  At $r_{emit} = R_s$ the redshift becomes $\infty \longrightarrow$ No light is observed (infinite time dilation)

  At $r = R_s \longrightarrow$ infinite time dilation $\Rightarrow$ Events are observed as "frozen"

* Mass M contained within a sphere of radius $R_s \longrightarrow$ Nothing can escape the surface

  An object which is smaller than its Schwarzschild radius is called a Black Hole

Exercises

- Consider an isolated point mass M with Schwarzschild radius R_s, located at the origin O
  From a distance $r_e > R_s$ a light ray is radially emitted and observed at a distance $r_o > r_e$

* Show that the exact formula for the gravitational redshift z is given by :

$$z = \sqrt{\frac{1 - R_s/r_o}{1 - R_s/r_e}} - 1$$

- Consider a proton as a spherical object with a radius of 1 fm.

* Determine from this the density of normal nuclear matter in GeV/fm^3

- The mass of the Earth is $M = 5.975 \cdot 10^{24}$ kg

* Determine the Schwarzschild radius R_s of the Earth

- Imagine that all the mass of the Earth is concentrated in a sphere with radius R_s

* Determine the density in GeV/fm^3 of this "Earth black hole" object
  For comparison : QGP phase transition is expected to happen at about 3 GeV/fm^3