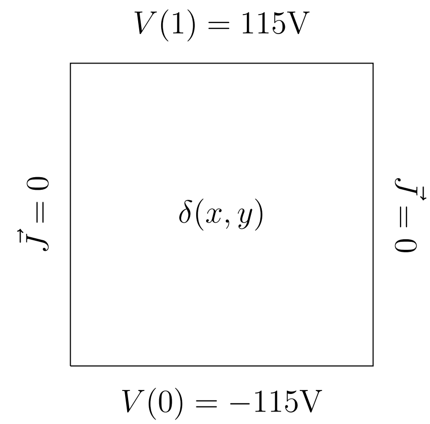
1 Attempt 1

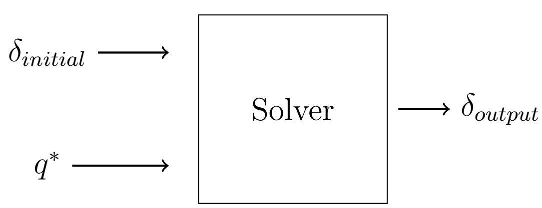
We have the following scenario.



With the problem setup we then continue onto the equations to solve.

$$\nabla \cdot (\sigma \delta(x, y) \nabla V) = 0 \tag{1}$$
$$q = \sigma |\nabla V|^2 \delta \tag{2}$$

The overall system is defined as follows



Suppose we had the following sample data for our q^* desired profile and the guess initial thickness profile which is constant.

x	y	$q^*[\text{W/m}^2]$	$\delta_1[\text{m}]$
0	0	1000	$4.1 \cdot 10^{-8}$
.5	0	1000	$4.1 \cdot 10^{-8}$
1	0	1000	$4.1 \cdot 10^{-8}$
0	0.5	2000	$4.1 \cdot 10^{-8}$
.5	0.5	2000	$4.1 \cdot 10^{-8}$
1	0.5	2000	$4.1 \cdot 10^{-8}$
0	1	3000	$4.1 \cdot 10^{-8}$
.5	1	3000	$4.1 \cdot 10^{-8}$
1	1	3000	$4.1 \cdot 10^{-8}$

Since this is a simple thickness profile ($\delta(x,y) = 4.1 \cdot 10^{-8}$) we can find an analytical expression.

$$
\begin{aligned}
&\nabla \cdot (\sigma\delta(x,y)\nabla V) = \sigma\delta(x,y)\frac{\partial^2 V}{\partial^2 y} = 0 \\
&\Longrightarrow \sigma\delta(x,y)\frac{\partial V}{\partial y} = c \quad \text{(some constant c)} \\
&\Longrightarrow \partial V = \frac{c}{\sigma\delta(x,y)}\partial y \\
&\Longrightarrow \int_{y=0}^{y} \partial V = \int_{y=0}^{y} \frac{c}{\sigma\delta(x,y)}\partial y \\
&\Longrightarrow V(y) - V(0) = \left. \frac{c}{\sigma\delta(x,y)}y \right|_{y=0}^{y} \\
&\Longrightarrow V(y) = V(0) + \frac{c}{\sigma\delta(x,y)}y
\end{aligned}
$$

From our boundary conditions we see that $V(1) = 115$ and $V(0) = -115$. Assume $\sigma = 10^6$.

$$
\begin{aligned}
V(1) = 115 = -115 + \frac{c}{10^6 \cdot (4.1 \cdot 10^{-8})} \cdot 1 \\
\Rightarrow c = 9.43
\end{aligned}
$$

Therefore for the initial iteration we get that our voltage function is

$$
V_1(y) = -115 + \frac{9.43 \cdot y}{10^6 \cdot (4.1 \cdot 10^{-8})}
$$

We can also calculate our surface joule heating function for this iteration

$$
\begin{aligned}
q_1 &= \sigma|\nabla V_1|^2\delta_1 \\
&= \sigma\left(\frac{\partial V_1}{\partial y}\right)^2 \delta_1 \\
&= 10^6\left(\frac{9.43}{10^6 \cdot (4.1 \cdot 10^{-8})}\right)^2 (4.1 \cdot 10^{-8}) \\
&= \frac{9.43^2}{10^6 \cdot (4.1 \cdot 10^{-8})} = 2168.9[w/m^2]
\end{aligned}
$$

So if we perform our typical division to update the thicknesses for the next iteration as follows

$$\delta_{i+1} = \frac{q^*}{Q_i} = \frac{q^*}{\frac{q_i}{\delta_i}} = \frac{q^* \cdot \delta_i}{q_i}$$

We then get the following new table of values

x	y	q^*[W/m^2]	δ_2[m]
0	0	1000	$1.89 \cdot 10^{-8}$
.5	0	1000	$1.89 \cdot 10^{-8}$
1	0	1000	$1.89 \cdot 10^{-8}$
0	0.5	2000	$3.78 \cdot 10^{-8}$
.5	0.5	2000	$3.78 \cdot 10^{-8}$
1	0.5	2000	$3.78 \cdot 10^{-8}$
0	1	3000	$5.67 \cdot 10^{-8}$
.5	1	3000	$5.67 \cdot 10^{-8}$
1	1	3000	$5.67 \cdot 10^{-8}$

We then perform a linear fit on the δ_2 values to get a function that is linear in y or that $\delta_2(x,y) \propto y$. However when we solve for this joule heating we find that a linear fit in the thicknesses correspond to a inverse relationship for the surface joule heating.

$$\delta_2(x,y) \propto y \rightarrow q_2 \propto \frac{1}{y}$$

So when we use the values associated with this new q_2 to do our division update, it makes the region near $y = 0$ have a smaller thickness. Decreasing the thickness will actually raise the joule heating in that section and this is not what we want since $y(0)$ should have the lowest joule heating.
We are concerned that the expression

$$\delta_{i+1} = \frac{q^*}{Q_i}$$

is not doing what we want.

2 Attempt 2

We will try to linearize some of our quantities and use first order approximations.

$$\delta = \overline{\delta} + \widetilde{\delta}$$
$$V = \overline{V} + \widetilde{V}$$
$$q = \overline{q} + \widetilde{q}$$

So we start with our two original equations

$$\nabla \cdot (\sigma\delta\nabla V) = 0$$
$$q = \sigma|\nabla V|^2\delta$$

then we take derivatives and appropriately substitute our first order derivatives. Doing this we get

$$\widetilde{q} = 2(\nabla\overline{V} \cdot \nabla\widetilde{V})\overline{\delta} + |\nabla\overline{V}|^2\widetilde{\delta}$$
$$\nabla \cdot (\widetilde{\delta}\nabla\overline{V} + \overline{\delta}\nabla\widetilde{V}) = 0$$

We use the first equation to solve for the error in thickness, $\widetilde{\delta}$.

$$\widetilde{\delta} = \frac{\widetilde{q} - 2(\nabla\overline{V} \cdot \nabla\widetilde{V})\overline{\delta}}{|\nabla\overline{V}|^2}$$

We substitute this into the second equation to get

$$\nabla \cdot \left(\frac{\widetilde{q} - 2\left(\nabla\overline{V} \cdot \nabla\widetilde{V}\right)\overline{\delta}}{|\nabla\overline{V}|^2}\nabla\overline{V} + \overline{\delta}\nabla\widetilde{V} \right) = 0$$

We then separate the term and simplify yielding

$$-\nabla \cdot \left(\widetilde{q}\frac{\nabla\overline{V}}{|\nabla\overline{V}|^2}\right) = \nabla \cdot \left(\overline{\delta}\nabla\widetilde{V} - 2\overline{\delta}\frac{\nabla\overline{V} \cdot \nabla\widetilde{V}}{|\nabla\overline{V}|^2}\nabla\overline{V}\right)$$
$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\nabla_i Q_{i,j}\nabla_j\widetilde{V}$$
$$= \frac{\partial}{\partial x}\left(Q_{1,1}\frac{\partial\widetilde{V}}{\partial x}\right) + \frac{\partial}{\partial x}\left(Q_{1,2}\frac{\partial\widetilde{V}}{\partial y}\right) + \frac{\partial}{\partial y}\left(Q_{2,1}\frac{\partial\widetilde{V}}{\partial x}\right) + \frac{\partial}{\partial y}\left(Q_{2,2}\frac{\partial\widetilde{V}}{\partial y}\right)$$

where

$$Q = \overline{\delta}\begin{bmatrix} 1 - \dfrac{2V_x^2}{|\nabla V|^2} & -\dfrac{2V_xV_y}{|\nabla V|^2} \\ -\dfrac{2V_xV_y}{|\nabla V|^2} & 1 - \dfrac{2V_y^2}{|\nabla V|^2} \end{bmatrix}$$

So the idea is that we follow the following steps

1. Choose δ_n

2. Solve for V_n using the equation $\nabla \cdot (\sigma\delta\nabla V) = 0$ and FEA in MATLAB

3. Calculate the updated Joule heating

$$q_n = \sigma|\nabla V_n|^2\delta_n$$

4. Calculate the error in joule heating

$$\widetilde{q} = q_n - \overline{q}$$

   where $\overline{q}$ is our desired Joule heating. If this difference is within some tolerance, terminate. Else continue on to the next step.

5. We then solve the new PDE here

$$-\nabla \cdot \left(\widetilde{q}\frac{\nabla V_n}{|\nabla V_n|^2}\right) = \nabla \cdot \left(\delta_n \nabla\widetilde{V} - 2\overline{\delta}\frac{\nabla V_n \cdot \nabla\widetilde{V}}{|\nabla V_n|^2}\nabla V_n\right)$$

   and this calculates an error in the voltage given by $\widetilde{V}$.

6. Calculate the new change in delta using the expression

$$\widetilde{\delta} = \frac{\widetilde{q}}{|\nabla V_n|^2} - \frac{2\nabla V_n \cdot \nabla\widetilde{V}}{|\nabla V_n|^2}\delta_n$$

7. We can then calculate the new thickness using

$$\delta_{n+1} = \delta_n + \widetilde{\delta}$$

   and use this as our new thickness then go back to step 2 and repeat through.