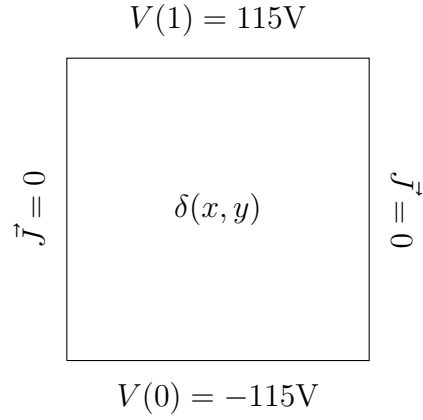


1 Attempt 1

We have the following scenario.

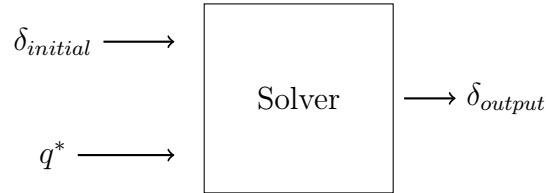


With the problem setup we then continue onto the equations to solve.

$$\nabla \cdot (\sigma \delta(x, y) \nabla V) = 0 \quad (1)$$

$$q = \sigma |\nabla V|^2 \delta \quad (2)$$

The overall system is defined as follows



Suppose we had the following sample data for our q^* desired profile and the guess initial thickness profile which is constant.

x	y	$q^* [\text{W}/\text{m}^2]$	$\delta_1 [\text{m}]$
0	0	1000	$4.1 \cdot 10^{-8}$
.5	0	1000	$4.1 \cdot 10^{-8}$
1	0	1000	$4.1 \cdot 10^{-8}$
0	0.5	2000	$4.1 \cdot 10^{-8}$
.5	0.5	2000	$4.1 \cdot 10^{-8}$
1	0.5	2000	$4.1 \cdot 10^{-8}$
0	1	3000	$4.1 \cdot 10^{-8}$
.5	1	3000	$4.1 \cdot 10^{-8}$
1	1	3000	$4.1 \cdot 10^{-8}$

Since this is a simple thickness profile ($\delta(x, y) = 4.1 \cdot 10^{-8}$) we can find an analytical expression.

$$\begin{aligned}
\nabla \cdot (\sigma \delta(x, y) \nabla V) &= \sigma \delta(x, y) \frac{\partial^2 V}{\partial^2 y} = 0 \\
\implies \sigma \delta(x, y) \frac{\partial V}{\partial y} &= c \quad (\text{some constant } c) \\
\implies \partial V &= \frac{c}{\sigma \delta(x, y)} \partial y \\
\implies \int_{y=0}^y \partial V &= \int_{y=0}^y \frac{c}{\sigma \delta(x, y)} \partial y \\
\implies V(y) - V(0) &= \frac{c}{\sigma \delta(x, y)} y \Big|_{y=0}^y \\
\implies V(y) &= V(0) + \frac{c}{\sigma \delta(x, y)} y
\end{aligned}$$

From our boundary conditions we see that $V(1) = 115$ and $V(0) = -115$. Assume $\sigma = 10^6$.

$$\begin{aligned}
V(1) = 115 &= -115 + \frac{c}{10^6 \cdot (4.1 \cdot 10^{-8})} \cdot 1 \\
&\Rightarrow c = 9.43
\end{aligned}$$

Therefore for the initial iteration we get that our voltage function is

$$V_1(y) = -115 + \frac{9.43 \cdot y}{10^6 \cdot (4.1 \cdot 10^{-8})}$$

We can also calculate our surface joule heating function for this iteration

$$\begin{aligned}
q_1 &= \sigma |\nabla V_1|^2 \delta_1 \\
&= \sigma \left(\frac{\partial V_1}{\partial y} \right)^2 \delta_1 \\
&= 10^6 \left(\frac{9.43}{10^6 \cdot (4.1 \cdot 10^{-8})} \right)^2 (4.1 \cdot 10^{-8}) \\
&= \frac{9.43^2}{10^6 \cdot (4.1 \cdot 10^{-8})} = 2168.9 [w/m^2]
\end{aligned}$$

So if we perform our typical division to update the thicknesses for the next iteration as follows

$$\begin{aligned}\delta_{i+1} &= \frac{q^*}{Q_i} \\ &= \frac{q^*}{\frac{q_i}{\delta_i}} \\ &= \frac{q^* \cdot \delta_i}{q_i}\end{aligned}$$

We then get the following new table of values

x	y	$q^*[\text{W}/\text{m}^2]$	$\delta_2[\text{m}]$
0	0	1000	$1.89 \cdot 10^{-8}$
.5	0	1000	$1.89 \cdot 10^{-8}$
1	0	1000	$1.89 \cdot 10^{-8}$
0	0.5	2000	$3.78 \cdot 10^{-8}$
.5	0.5	2000	$3.78 \cdot 10^{-8}$
1	0.5	2000	$3.78 \cdot 10^{-8}$
0	1	3000	$5.67 \cdot 10^{-8}$
.5	1	3000	$5.67 \cdot 10^{-8}$
1	1	3000	$5.67 \cdot 10^{-8}$

We then perform a linear fit on the δ_2 values to get a function that is linear in y or that $\delta_2(x, y) \propto y$. However when we solve for this joule heating we find that a linear fit in the thicknesses correspond to a inverse relationship for the surface joule heating.

$$\delta_2(x, y) \propto y \rightarrow q_2 \propto \frac{1}{y}$$

So when we use the values associated with this new q_2 to do our division update, it makes the region near $y = 0$ have a smaller thickness. Decreasing the thickness will actually raise the joule heating in that section and this is not what we want since $y(0)$ should have the lowest joule heating.

We are concerned that the expression

$$\delta_{i+1} = \frac{q^*}{Q_i}$$

is not doing what we want.

2 Attempt 2

We will try to linearize some of our quantities and use first order approximations.

$$\begin{aligned}\delta &= \bar{\delta} + \tilde{\delta} \\ V &= \bar{V} + \tilde{V} \\ q &= \bar{q} + \tilde{q}\end{aligned}$$

So we start with our two original equations

$$\begin{aligned}\nabla \cdot (\sigma \delta \nabla V) &= 0 \\ q &= \sigma |\nabla V|^2 \delta\end{aligned}$$

then we take derivatives and appropriately substitute our first order derivatives. Doing this we get

$$\begin{aligned}\tilde{q} &= 2(\nabla \bar{V} \cdot \nabla \tilde{V}) \bar{\delta} + |\nabla \bar{V}|^2 \tilde{\delta} \\ \nabla \cdot (\tilde{\delta} \nabla \bar{V} + \bar{\delta} \nabla \tilde{V}) &= 0\end{aligned}$$

We use the first equation to solve for the error in thickness, $\tilde{\delta}$.

$$\tilde{\delta} = \frac{\tilde{q} - 2(\nabla \bar{V} \cdot \nabla \tilde{V}) \bar{\delta}}{|\nabla \bar{V}|^2}$$

We substitute this into the second equation to get

$$\nabla \cdot \left(\frac{\tilde{q} - 2(\nabla \bar{V} \cdot \nabla \tilde{V}) \bar{\delta}}{|\nabla \bar{V}|^2} \nabla \bar{V} + \bar{\delta} \nabla \tilde{V} \right) = 0$$

We then separate the term and simplify yielding

$$\begin{aligned}-\nabla \cdot \left(\tilde{q} \frac{\nabla \bar{V}}{|\nabla \bar{V}|^2} \right) &= \nabla \cdot \left(\bar{\delta} \nabla \tilde{V} - 2\bar{\delta} \frac{\nabla \bar{V} \cdot \nabla \tilde{V}}{|\nabla \bar{V}|^2} \nabla \bar{V} \right) \\ &= \sum_{i=1}^2 \sum_{j=1}^2 \nabla_i Q_{i,j} \nabla_j \tilde{V} \\ &= \frac{\partial}{\partial x} \left(Q_{1,1} \frac{\partial \tilde{V}}{\partial x} \right) + \frac{\partial}{\partial x} \left(Q_{1,2} \frac{\partial \tilde{V}}{\partial y} \right) + \frac{\partial}{\partial y} \left(Q_{2,1} \frac{\partial \tilde{V}}{\partial x} \right) + \frac{\partial}{\partial y} \left(Q_{2,2} \frac{\partial \tilde{V}}{\partial y} \right)\end{aligned}$$

where

$$Q = \tilde{\delta} \begin{bmatrix} 1 - \frac{2V_x^2}{|\nabla V|^2} & -\frac{2V_x V_y}{|\nabla V|^2} \\ -\frac{2V_x V_y}{|\nabla V|^2} & 1 - \frac{2V_y^2}{|\nabla V|^2} \end{bmatrix}$$

So the idea is that we follow the following steps

1. Choose δ_n
2. Solve for V_n using the equation $\nabla \cdot (\sigma \delta \nabla V) = 0$ and FEA in MATLAB
3. Calculate the updated Joule heating

$$q_n = \sigma |\nabla V_n|^2 \delta_n$$

4. Calculate the error in joule heating

$$\tilde{q} = q_n - \bar{q}$$

where \bar{q} is our desired Joule heating. If this difference is within some tolerance, terminate. Else continue on to the next step.

5. We then solve the new PDE here

$$-\nabla \cdot \left(\tilde{q} \frac{\nabla V_n}{|\nabla V_n|^2} \right) = \nabla \cdot \left(\delta_n \nabla \tilde{V} - 2\tilde{\delta} \frac{\nabla V_n \cdot \nabla \tilde{V}}{|\nabla V_n|^2} \nabla V_n \right)$$

and this calculates an error in the voltage given by \tilde{V} .

6. Calculate the new change in delta using the expression

$$\tilde{\delta} = \frac{\tilde{q}}{|\nabla V_n|^2} - \frac{2\nabla V_n \cdot \nabla \tilde{V}}{|\nabla V_n|^2} \delta_n$$

7. We can then calculate the new thickness using

$$\delta_{n+1} = \delta_n + \tilde{\delta}$$

and use this as our new thickness then go back to step 2 and repeat through.