Self-Exciting Spatio-Temporal Models for Count Data

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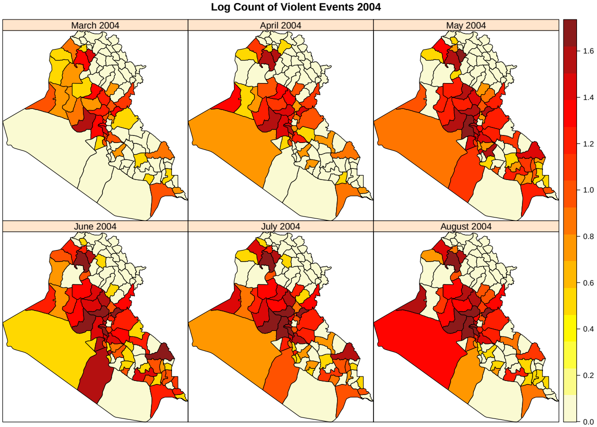
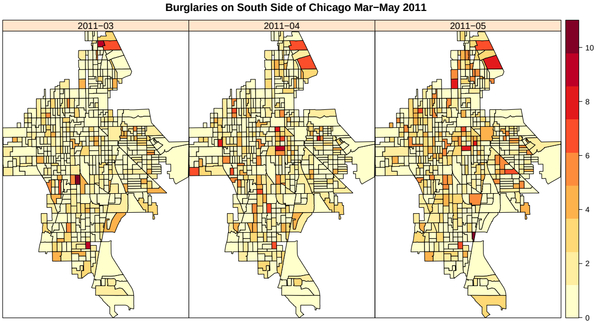
Motivation: The Evolution of Violence in Space and Time

At present, the most under-researched area of spatial criminology is that of spatio-temporal crime patterns... the temporal component of the underlying crime distributions has languished as a largely ignored area of study - **Crime mapping: Spatial and Temporal Challenges**, Ratcliffe (2010)

The Spread of Violence in Iraq 2004



Burglaries South Side of Chicago



Goals

- General statistical model for diffusion of violence in space-time
  - Accurately reflects beliefs on how violence/crime evolves
  - Extends traditional statistical models for count data
  - Stationary with extremely flexible second order properties
  - Inference via traditional MCMC techniques

Overview

1. Mathematical Model for Diffusion of Crime and Related Statistical Models
   - Issues with INGARCH (1,1) Model
2. SPINGARCH Model
3. SPINGARCH Stationarity and Model Properties
4. Inference
5. Simulation
6. Burglaries in South Side of Chicago

A Model of Criminal Behavior (Short et al. 2008)

- $Z(s_i, t)$ - number of observed burglaries from $(t - \Delta t, t)$
- $s_i \in \{s_1, \cdots, s_n\}$ - fixed regions in $\mathbb{R}^2$
- $t \in \{1, \cdots, T\}$ - discrete time
- Define $A(s_i, t) \equiv A(s_i, 0) + B(s_i, t)$ as attractiveness

$$B(s_i, t) = (1 - \chi \Delta t)\, B(s_i, t - \Delta t) + \eta Z(s_i, t - \Delta t) \tag{1}$$

- Probability of occurrence at each time interval, $(t, t + \Delta t)$ is Poisson with rate, $A(s_i, t)$
- Three factors impact change in crime rate, base attractiveness , decay χ, and repeat victimization, η

Relationship to INGARCH Model

Integer Auto-Regressive Conditionally Heteroskedastic, INGARCH (1,1), or Poisson Auto-Regression Model

$$Z(s_i, t) \sim \text{Pois}\left(\lambda(s_i, t)\right) \tag{2}$$

$$\lambda(s_i, t) = d + a\lambda(s_i, t-1) + bZ(s_i, t-1) \tag{3}$$

- Unlike GARCH, not solely a variance property
- Short model is similar to INGARCH(1,1) with $A(s_i, 0) = \sum_{k=0}^{t} a^k d$, $a = (1 - \chi\Delta t)$, and $b = \eta$

Relationship to Self-Exciting Models

- Point process introduced by Alan Hawkes with intensity

$$\lambda(t) = \nu(t) + \int_0^t g(t-u)N(ds) \tag{4}$$

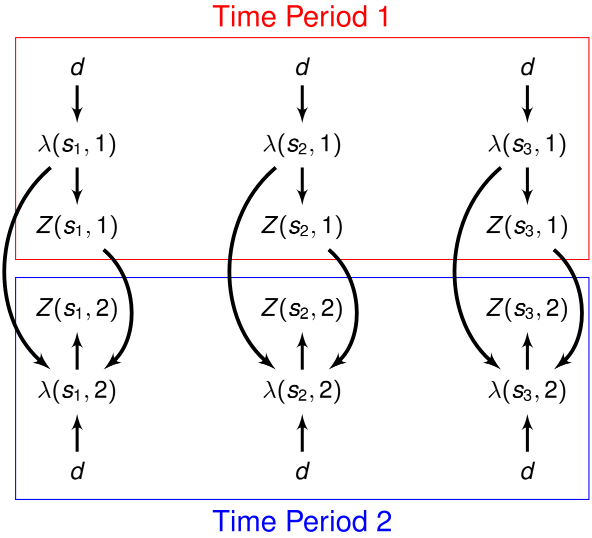
- Commonly discretized as

$$Z(s_i, t) \sim \text{Pois}\left(\lambda(s_i, t)\right) \tag{5}$$

$$\lambda(s_i, t) = \nu + \sum_{j<t} \eta^{t-j} Z(s_i, t-j) \tag{6}$$

- Equivalent to stationary INGARCH(1,1)

Structural Diagram - INGARCH(1,1)



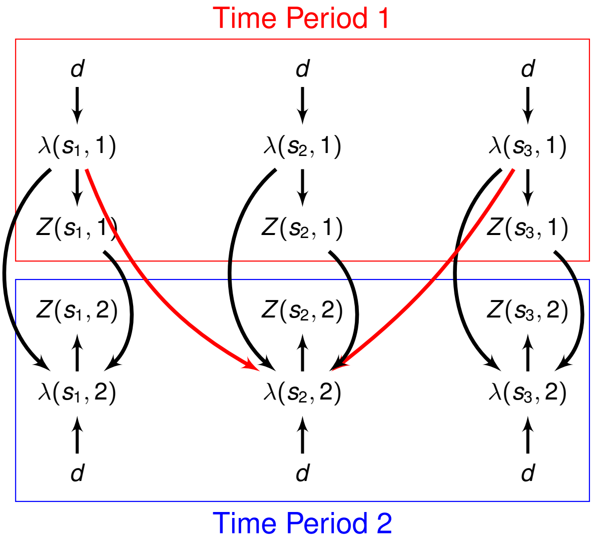
Short (2008) Extension - Spatial Spread

- Motivated by Reaction-Diffusion PDE
- Recall $A(s_i, t) \equiv A(s_i, 0) + B(s_i, t)$ as attractiveness, let N_{s_i} be spatial adjacency matrix

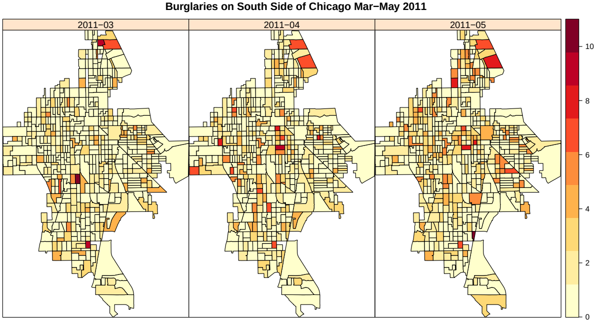
$$B(s_i, t) = \kappa B(s_i, t - \Delta t) + \frac{\psi}{|N_{s_i}|} \sum_{s_j \in N_{s_i}} \left[B(s_j, t - \Delta t) - B(s_i, t - \Delta t)\right] + \eta Z(s_i, t - \Delta t) \quad (7)$$

- Four factors impact change in crime rate, base attractiveness, decay χ, and repeat victimization, η, and spatial spread ψ
- Resulting model is MINGARCH (1,1)

Short (2008) Extension - Spatial Spread



Applied to Residential Burglaries in Chicago



552 Spatial Locations, 72 Months, residential burglaries

Applied to Residential Burglaries in Chicago

$$
\begin{aligned}
B(s_i,t) &= \kappa B(s_i,t-\Delta t) \\
&+ \frac{\psi}{|N_{s_i}|}\sum_{s_j\in N_{s_i}}\left[B(s_j,t-\Delta t)-B(s_i,t-\Delta t)\right]+\eta Z(s_i,t-\Delta t) \qquad (8)
\end{aligned}
$$

- Further structure $A(s_i,0)$ to account for socio-economic factors
- MLEs are $\hat{\psi}=.02$, $\hat{\eta}=.155$ $\hat{\kappa}=.773$
- Simulate data from asymptotic distribution - unable to replicate lag-one autocorrelation, spatial correlation, or variance to mean ratio of original data

Properties of INGARCH Model

$$Z(s_i, t) \sim \text{Pois}\left(\lambda(s_i, t)\right)$$
$$\lambda(s_i, t) = d + a\lambda(s_i, t-1) + bZ(s_i, t-1)$$

Stationarity yields:

$$E[Z(s_i, t)] = \frac{d}{1-(a+b)} \tag{9}$$

$$\text{Var}[Z(s_i, t)] = \frac{1-(a+b)^2+b^2}{1-(a+b)^2} E[Z(s_i, t)] \tag{10}$$

$$\text{Cov}[Z(s_i, t), Z(s_i, t-h)] = \frac{b(1-a(a+b))(a+b)^h}{1-(a+b)^2} E[Z(s_i, t)] \tag{11}$$

$$\text{Var-Mean Ratio}[Z(s_i, t)] = 1 + \frac{b^2}{1-(a+b)^2} \tag{12}$$

Issues

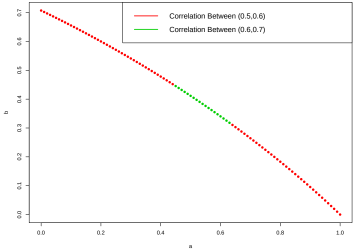
Allows for Overdispersion... But at a cost!

$$\text{Cor}[Z(s_i,t),Z(s_i,t-1)] = \frac{b(a+b)(a^2+ab-1)}{a^2+2ab-1} \tag{13}$$

$$\text{Var-Mean Ratio}[Z(s_i,t)] = 1 + \frac{b^2}{1-(a+b)^2} \tag{14}$$

$$\tag{15}$$

For fixed Var-Mean Ratio at 2 $\implies b = 1/2(-a+\sqrt{2-a^2})$.



$Y(s_i, t)$ - Spatially Correlated Latent Gaussian

Time Period 1

$$
\begin{tikzcd}
\exp(Y(s_1,1)) \arrow[r, leftrightarrow] \arrow[d] & \exp(Y(s_2,1)) \arrow[r, leftrightarrow] \arrow[d] & \exp(Y(s_3,1)) \arrow[d] \\
\lambda(s_1,1) \arrow[d] & \lambda(s_2,1) \arrow[d] & \lambda(s_3,1) \arrow[d] \\
Z(s_1,1) \arrow[dd, bend right] & Z(s_2,1) \arrow[dd, bend right] & Z(s_3,1) \arrow[dd, bend right] \\
Z(s_1,2) & Z(s_2,2) & Z(s_3,2) \\
\lambda(s_1,2) \arrow[u] & \lambda(s_2,2) \arrow[u] & \lambda(s_3,2) \arrow[u] \\
\exp(Y(s_1,2)) \arrow[r, leftrightarrow] \arrow[u] & \exp(Y(s_2,2)) \arrow[r, leftrightarrow] \arrow[u] & \exp(Y(s_3,2)) \arrow[u]
\end{tikzcd}
$$

Time Period 2

- **Theory:** Exist common cause between geographically similar locations, regions that experience uptick in violence likely to have short term self-excitation

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- Mixture of two processes that influence expectation : LGCP and Hawkes process
- Hawkes process letting $g(t-j) = \eta$ if $(t-j) = 1$, 0 otherwise

$$
\begin{aligned}
Z(\boldsymbol{s_i}, t) | \lambda(\boldsymbol{s_i}, t) &\sim \text{Pois}\left(\lambda(\boldsymbol{s_i}, t)\right) \\
\lambda(\boldsymbol{s_i}, t) &= \exp(Y(\boldsymbol{s_i}, t)) + \eta Z(\boldsymbol{s_i}, t-1) \\
Y(\boldsymbol{s_i}, t) &= \theta_1 \sum_{\boldsymbol{s_j} \in N(\boldsymbol{s_i})} Y(\boldsymbol{s_j}, t) + \epsilon(\boldsymbol{s_i}, t) \\
\epsilon(\boldsymbol{s_i}, t) &\sim Gau(0, \sigma^2)
\end{aligned}
\tag{16}
$$

$Y(s_i, t)$ - Spatially Correlated Latent Gaussian

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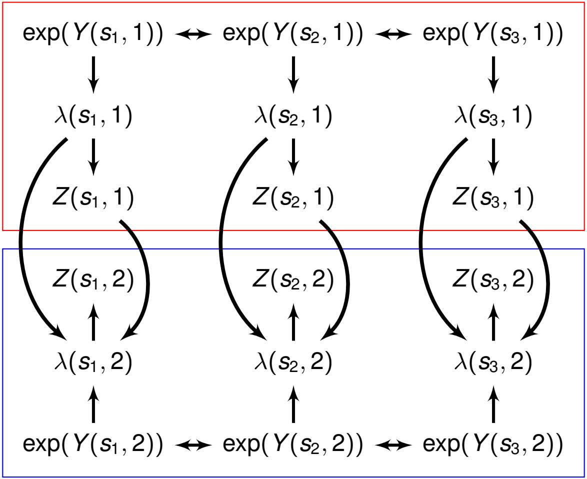
Z(s_1,2) & Z(s_2,2) & Z(s_3,2) \\
\lambda(s_1,2) \arrow[u] & \lambda(s_2,2) \arrow[u] & \lambda(s_3,2) \arrow[u] \\
\exp(Y(s_1,2)) \arrow[r, leftrightarrow] \arrow[u] & \exp(Y(s_2,2)) \arrow[r, leftrightarrow] \arrow[u] & \exp(Y(s_3,2)) \arrow[u]
\end{tikzcd}
$$

Time Period 2

SPINGARCH(1,1) Model

Spatially Correlated INGARCH(1,1) Model

Time Period 1



Time Period 2

SPINGARCH Model

Theory: Exist common cause between geographically similar locations, regions that experience uptick in violence likely to have short term self-excitation, ***absence of violence or exogeneous effects reduces tension***

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- $Z(s_i, t)|Y(s_i, t), \mathcal{H}_{Z(s_i)} \sim \text{Pois}\,(\lambda(s_i, t))$ where $\mathcal{H}_{Z(s_i)}$ is history of process at location s_i

$$\lambda(s_i, t) = \exp\left[Y(s_i, t)\right] + \eta Z(s_i, t-1) + \kappa\lambda(s_i, t-1) \tag{17}$$

- Define: $N_i = \{s_j : s_j \text{ is a spatial neighbor of } s_i\}$

$$Y(s_i, t)|\boldsymbol{Y}(N_i) \sim N(\mu(s_i, t), \sigma^2_{sp}) \tag{18}$$

$$\mu(s_i, t) = \alpha(s_i) + \zeta \sum_{s_j \in N_i} \{Y(s_j, t) - \alpha(s_j)\}.$$

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- $\eta = 0, \kappa = 0$ Poisson - CAR, $\sigma^2_{sp} \to 0$, INGARCH(1,1)/Short model

SPINGARCH Model as Stochastic Difference Equation

$$\frac{\lambda(s_i, t) - \lambda(s_i, t-1)}{\Delta t} = d - \chi\lambda(s_i, t-1) + \eta Z(s_i, t-1) \tag{19}$$

- Change in violence due to exogeneous d, natural decay, χ, and excitement, η
- Assume each time period, exogeneous impact is stochastic and spatially correlated yields SPINGARCH

$$\frac{\lambda(s_i, t) - \lambda(s_i, t-1)}{\Delta t} = \exp(Y(s_i, t)) - \chi\lambda(s_i, t-1) + \eta Z(s_i, t-1) \tag{20}$$

- Change in intensity due to three factors, CAR, natural decay, and excitement

SPINGARCH Model - Parameter Space

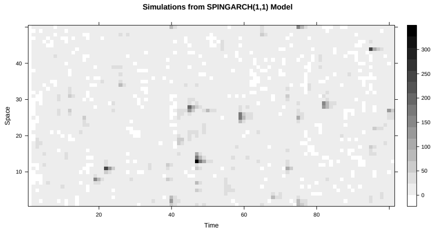
- $Z(s_i,t)|Y(s_i,t),\mathcal{H}_{Z(s_i)} \sim$ Pois $(\lambda(s_i,t))$ where $\mathcal{H}_{Z(s_i)}$ is history of process at location s_i

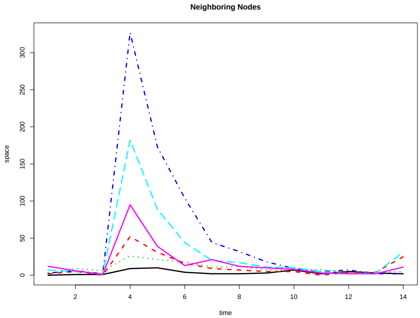
$$\lambda(s_i,t) = \exp\left[Y(s_i,t)\right] + \eta Z(s_i,t-1) + \kappa\lambda(s_i,t-1)$$
$$Y(s_i,t)|\boldsymbol{Y}(N_i) \sim N(\mu(s_i,t),\sigma^2_{sp})$$
$$\mu(s_i,t) = \alpha(s_i) + \zeta \sum_{s_j \in N_i} \{Y(s_j,t) - \alpha(s_j)\}.$$

- $\zeta \in (\psi_{(1)}^{-1}, \psi_{(n)}^{-1})$ where $\psi_{(i)}$ is the ith largest eigenvalue of adjacency matrix
- For stationarity, $\eta > 0$, $\kappa > 0$, $\eta + \kappa < 1$

Data Realizations

50 Spatial Observations on $\mathbb{R}^1$, 100 Temporal Observations





$$\lambda(s_i, t) = \exp\left[Y(s_i, t)\right] + 0.1\, Z(s_i, t-1) + 0.4\, \lambda(s_i, t-1)$$
$$Y(s_i, t) | \boldsymbol{Y}(N_i) \sim \text{Gau}(\mu(s_i, t), 0.5)$$
$$\mu(s_i, t) = 0 + 0.49 \sum_{s_j \in N_i} \{Y(s_j, t)\}. \tag{21}$$

SPINGARCH Model as Markov Chain

- Let $\boldsymbol{\lambda_t} = (\lambda(s_1, t), \lambda(s_2, t), \cdots, \lambda(s_{n_d}, t))^T$

$$\begin{aligned}
Z(s_i, t)|\lambda(s_i, t) &\sim \text{Pois}(\lambda(s_i, t)) \\
E[Z(s_i, t)] &= \lambda(s_i, t) \\
\boldsymbol{\lambda_t} &= \exp(\boldsymbol{Y_t}) + \eta \boldsymbol{Z_{t-1}} + \kappa \boldsymbol{\lambda_{t-1}} \\
\boldsymbol{Y_t} &\sim \text{Gau}(\boldsymbol{\alpha_t}, (I_{n_d,n_d} - \boldsymbol{C})^{-1}\boldsymbol{M})
\end{aligned} \tag{22}$$

- Markov chain for $\boldsymbol{\lambda_t}$ on State space, $(\mathbb{R}^+)^{n_d}$

Impact of Initial Condtions and Recursion

By recursion

$$
\begin{aligned}
&[\lambda(s_i, t)|\lambda(s_i, 0) = B] = \exp(Y(s_i, t)) + \kappa\lambda(s_i, t-1) + \eta Z(s_i, t-1) \\
&= \exp(Y(s_i, t)) + \kappa\,[\exp(Y(s_i, t-1)) + \kappa\lambda(s_i, t-2) \\
&\quad + \eta Z(s_i, t-2)] + \eta Z(s_i, t-1) \\
&\quad \ldots \\
&= \sum_{k=0}^{t-1} \kappa^k \exp(Y(s_i, t-k)) + \sum_{k=0}^{t-1} \kappa^k \eta Z(s_i, t-k-1) + \kappa^t B.
\end{aligned}
$$

Geometric Ergodicity with Finite Moments

Under the parameter space restriction, $\eta, \kappa \geq 0$ and $\eta + \kappa < 1$, the SPINGARCH (1,1) is geometrically ergodic and admits a unique stationary distributions that has finite first two moments.

Sketch of Proof.

Meyn and Tweedie (15.0.1) need to show aperiodic, ϕ-irreducible and $\exists$ test function $V(.)$ such that
$E[V(\boldsymbol{\lambda}_{t+1})|\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_*] \leq \psi V(\boldsymbol{\lambda}_*) + L\, \mathrm{I}(\boldsymbol{\lambda}_* \in C)$ holds where $\psi \in (0,1)$, $L \in (0, \infty)$ and $I(.)$ is the indicator function and C is a petite set.

Basic Idea: With positive probability, $\exists$ a realization $Z(s_i, 1) = Z(s_i, 2) = \cdots = Z(s_i, t-1) = 0$. Along that chain, $P(\lambda(s_i, t)) \in A = P(\exp(Y(s_i, t)) \in A - \kappa^t B)$. If $\kappa^T B > \sup A$ run chain longer. $\square$

Geometric Ergodicity with Finite Moments

Under the parameter space restriction, $\eta, \kappa \geq 0$ and $\eta + \kappa < 1$, the SPINGARCH (1,1) is geometrically ergodic and admits a unique stationary distributions that has finite first two moments.

Sketch of Proof Cont.

Test function $V(\lambda) = 1 + \lambda^2$ works for
$E[V(\boldsymbol{\lambda}_{t+1}) | \boldsymbol{\lambda}_t = \boldsymbol{\lambda}_*] \leq \psi V(\boldsymbol{\lambda}_*) + L\, \mathrm{I}(\boldsymbol{\lambda}_* \in C)$.

$\implies$ Unique stationary distribution, goes to geometrically fast.
Specific choice of $V(.)$ gives (at least) finite first two moments (can be extended likely as in Fokianos, 2009.)

$\square$

Increased Modeling Flexibility with SPINGARCH(1,1)

$$
\begin{aligned}
&Z(s_i,t) \sim \text{Pois}\left(\lambda(s_i,t)\right) \\
&\lambda(s_i,t) = \exp\left[Y(s_i,t)\right] + \eta Z(s_i,t-1) + \kappa E\left[Z(s_i,t-1)\right] \\
&Y(s_i,t)|\boldsymbol{Y}(N_i) \sim N(\mu(s_i,t),\sigma^2_{sp}) \\
&\mu(s_i,t) = \alpha(s_i) + \zeta \sum_{s_j \in N_i} \left\{Y(s_j,t) - \alpha(s_j)\right\}
\end{aligned}
$$

Define $\Sigma_{i,j}$ as i,j entry of $(I_{n_d,n_d} - \boldsymbol{C})^{-1}\boldsymbol{M}$

$$
E\left[Z(s_i,t)\right] = \frac{1}{1-\eta-\kappa}\exp\left(\alpha + \frac{\Sigma_{1,1}}{2}\right) \tag{23}
$$

$$
\text{Var}\left(Z(s_i,t)\right) = \frac{1}{1-(\kappa+\eta)^2}\text{Var}\left(\exp(Y(s_i,t))\right) + \frac{1-\kappa^2-2\kappa\eta}{1-(\kappa+\eta)^2}E(Z(s_i,t)) \tag{24}
$$

Increased Modeling Flexibility with SPINGARCH(1,1)

Temporal Covariance:

$$\text{Cov}\left(Z(s_i,t), Z(s_i,t-1\right) = (\eta+\kappa)\text{Var}(Z(s_i,t)) - \kappa E\left[Z(s_i,t)\right] \tag{25}$$

Increased Modeling Flexibility with SPINGARCH(1,1)

Temporal Covariance:

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Recall: Var-Mean Ratio at 2 $\implies$ Lag-1 correlation is between 0.5 and $\sqrt{1/2}$ for INGARCH(1,1)

Increased Modeling Flexibility with SPINGARCH(1,1)

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Recall: Var-Mean Ratio at 2 $\implies$ Lag-1 correlation is between 0.5 and $\sqrt{1/2}$ for INGARCH(1,1)

Let $\kappa = 0$ (SPINGARCH(0,1)), Var-Mean Ratio at 2

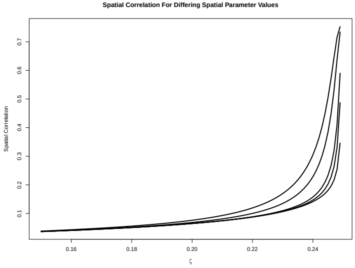
$$\implies 2 = \frac{\text{Var}\left(\exp(Y(s_i,t))\right)}{(1-\eta)^2 E\left[Z(s_i,t)\right]} + \frac{1}{1-\eta^2} \quad (26)$$

$$\text{Cor}\left(Z(s_i,t), Z(s_i,t-1\right) = \eta \quad (27)$$

$\forall \eta \in (0, \sqrt{1/2}) \quad \exists \alpha, \sigma^2_{sp}$ such that equality holds

Spatial Correlation

$$\mathrm{Corr}(Z(s_i,t),Z(s_j,t)) = \frac{(\exp(\Sigma_{i,i}+\Sigma_{i,j})-\exp(\Sigma_{i,i}))}{\exp(2\Sigma_{i,i})-\exp(\Sigma_{i,i})+\exp(-\alpha+\frac{\Sigma_{i,i}}{2})\frac{1}{1-(\kappa+\eta)}}$$



$\eta = .3$, $\sigma^2_{sp} = .5$, 4×4 to 15×15 size lattice

Inference

- Likelihood roots for INGARCH(1,1) easily found, asymptotically Gaussian
- Inclusion of latent process in SPINGARCH(1,1) complicates
- $\boldsymbol{\theta} \equiv (\eta, \alpha, \zeta, \sigma_{sp}^2)$

$$\pi(\boldsymbol{\theta}|\boldsymbol{Z}, \boldsymbol{Y}) \propto \prod_t \pi(\boldsymbol{Z}_t|\boldsymbol{\lambda}_t)\pi(\boldsymbol{\lambda}_t|\boldsymbol{\lambda}_{t-1}, \boldsymbol{Z}_{t-1}, \boldsymbol{\theta}, \boldsymbol{Y}_t)\pi(\boldsymbol{Y}_t|\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \quad (28)$$

$$\pi(\boldsymbol{Y}|\boldsymbol{Z}, \boldsymbol{\theta}) \propto \prod_t \pi(\boldsymbol{Z}_t|\boldsymbol{\lambda}_t)\pi(\boldsymbol{\lambda}_t|\boldsymbol{\lambda}_{t-1}, \boldsymbol{Z}_{t-1}, \boldsymbol{\theta}, \boldsymbol{Y}_t)\pi(\boldsymbol{Y}_t|\boldsymbol{\theta}). \quad (29)$$

Efficient Bayesian Inference

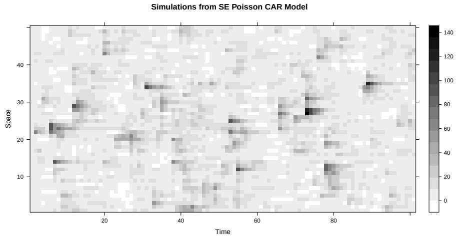
$$
\begin{aligned}
\log(\boldsymbol{Y}|\boldsymbol{\alpha}, \sigma_{sp}, \zeta) \propto & \frac{1}{2}\log|\Sigma_f^{-1}(\theta)| \\
& - \frac{1}{2}(Y-\alpha)^T \Sigma_f^{-1}(\theta)(Y-\alpha),
\end{aligned} \tag{30}
$$

- $\Sigma_f^{-1} \equiv \left(I_{n_d \times T, n_d \times T} - I_{t,t} \otimes \boldsymbol{C}\right)^{-1} I_{t,t} \otimes \boldsymbol{M}$
- $\log|\Sigma^{-1}(\theta)| = \frac{n_d}{2\log\sigma_{sp}^2} + \log|I_{n_d,n_d} - \zeta N|$
- Letting $V\Lambda V^T$ be the spectral decomposition of N we have $|I_{n_d,n_d} - \zeta N| = |V||I_{n_d,n_d} - \zeta\Lambda||V^T| = \prod_{j=1}^{n_d}\left(1-\zeta\chi_j\right)$ where χ_j are the eigenvalues of the neighborhood matrix

$$
\begin{aligned}
\log|\Sigma_f^{-1}(\theta)| &= T\log|\Sigma^{-1}(\theta)| && (31)\\
&\propto \frac{n_d \times T}{\log\sigma_{sp}^2} + T\sum_{j=1}^{n_d}(1-\zeta\chi_j) && (32)
\end{aligned}
$$

Simulation and Estimation

$$
\begin{aligned}
Z(s_i, t) &\sim \text{Pois}(\lambda(s_i, t)) \\
\lambda(s_i, t) &= \exp\left[Y(s_i, t)\right] + 0.66 Z(s_i, t-1) \\
Y(s_i, t) | \boldsymbol{Y}(N_i) &\sim \text{Gau}(\mu(s_i, t), 0.5) \\
\mu(s_i, t) &= 0 + 0.49 \sum_{s_j \in N_i} \{Y(s_j, t)\}.
\end{aligned} \tag{33}
$$

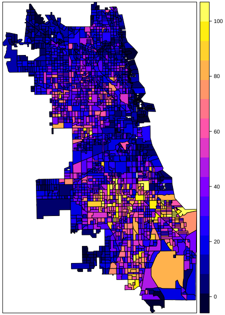


Simulation and Estimation

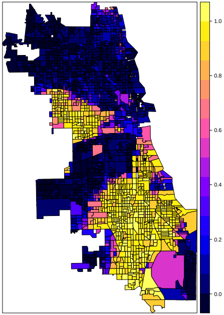
- SPINGARCH(0,1) 95% credible intervals: $\alpha \in (-0.24, 0.1)$, $\sigma^2 \in (0.46, 0.59)$, $\zeta \in (0.486, 0.492)$, and $\eta \in (0.64, 0.66)$
- SPINGARCH(1,0) 95% credible intervals: $\alpha \in (-0.54, -0.2)$, $\sigma^2 \in (0.96, 1.2)$, $\zeta \in (0.47, 0.48)$, and $\kappa \in (0.65, 0.67)$

	SPINGARCH(1,0)	SPINGARCH(0,1)
p_1 - Moran's I	.05	.46
p_2 - Var to Mean	.99	.65
p_3 - Lag 1 Corr	.45	.7

Burglaries in South Side of Chicago

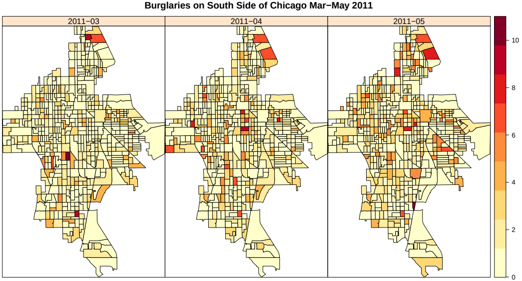


Aggregated Burglaries



Racial Segregation

Burglaries South Side of Chicago



- Crime data from city of Chicago
- 72 months (2010-2015), 552 locations (Census block groups)
- Demographic data from Census bureau

SPINGARCH(1,1) Model

$$
\begin{aligned}
Z(s_i,t) &\sim \text{Pois}(\lambda(s_i,t)) \\
E[Z(s_i,t)] &= \lambda(s_i,t) \\
\boldsymbol{\lambda_t} &= \exp(\boldsymbol{Y_t} + \boldsymbol{U}) + \eta \boldsymbol{Z_{t-1}} + \kappa \boldsymbol{\lambda_{t-1}} \\
\boldsymbol{Y_t} &\sim \text{Gau}(\boldsymbol{\alpha}, \sigma^2_{ind} \boldsymbol{I}_{n_d,n_d}) \\
\boldsymbol{U} &\sim \text{Gau}(\boldsymbol{0}, [\sigma^2_{sp}(\boldsymbol{N} - \boldsymbol{C})]^{-1})
\end{aligned}
\tag{34}
$$

- Removed temporal trend and seasonality
- Single spatial effect, ζ fixed near edge of parameter space
- Additional small scale effect captured in σ^2_{ind}

$$
\begin{aligned}
\alpha_{s_i} = &\exp\Big(\beta_0 + \beta_{pop}\log(\text{Pop}_{s_i}) \\
&+ \beta_{ym}\text{Young Men}_{s_i} + \beta_{wealth}\text{Wealth}_{s_i} + \beta_{unemp}\text{Unemp}_{s_i}\Big)
\end{aligned}
\tag{35}
$$

Impacts of Including Spatial Correlation

Parameter	SPINGARCH(1,1)	INGARCH(1,1)
β_0	(-3.3,-1.0)	(-4.2,-3.4)
β_{pop}	(0.11,0.34)	(0.33,0.46)
β_{ym}	$(-0.75, 0.17)$	$(0.06, 0.09)$
β_{wealth}	$(0.05, 0.16)$	$(-0.04, 0.01)$
β_{unemp}	(0.006,0.07)	(0.002,0.03)
η	$(0.04, 0.07)$	$(0.22, 0.24)$
κ	(0.31,0.39)	(0.44,0.48)
σ^2_{sp}	(0.40,0.54)	-
σ^2_{ind}	(0.40,0.47)	-

Model Assessment - Posterior Predictive Checks

	SPINGARCH(1,1)	INGARCH(1,1)
p_1 - Moran's I Statistic	0.43	0
p_2 - Variance to Mean Ratio	0.62	0
p_3 - Lag 1 Auto Correlation	0.67	0.74

- SPINGARCH(1,1) - observed maximum (p=.67), number of zeros (p=.49)
- Conclusions on repeat-victimization

Summary

- Addition of spatially correlated effects naturally extends INGARCH model
- Both Poisson-CAR and INGARCH arise from SPINGARCH in limit
- Failure to specify random structure may result in differing conclusions
- Precomputing eigenvalues allows for relatively efficient Bayesian inference (8 hrs for 552 spatial locations, 104 time locations)

Future Work

- Impacts of aggregation
- Laplace approximations greatly speed up SPINGARCH(0,1) - Can extend to SPINGARCH(1,1)?
- Reaction Diffusion Self-Exciting Model from (Clark & Dixon, 2018) does not fit in framework (temporally correlated errors)
    - RDSEM captures reaction diffusion process of Short in Latent Process
- Slides and code available at `https://github.com/nick3703/Talk`
- Questions?