

Since Y has the standard normal distribution and T has the chi-square distribution with $IJ - 1$ degrees of freedom, the random variable $(Y + \delta)/\sqrt{T/(IJ - 1)}$ has the noncentral t distribution with $IJ - 1$ degrees of freedom and noncentrality parameter δ . Hence, we have

$$k = \frac{t_{IJ-1}(\delta, \gamma)\sigma_B}{\sqrt{IJ(IJ - 1)}}, \quad (27)$$

where $t_{IJ-1}(\delta, \gamma)$ denotes the $100\gamma^{th}$ percentile of the noncentral t distribution with $IJ - 1$ degrees of freedom and noncentrality parameter δ . Let τ denote the ratio of the expected mean squares (EMS), that is

$$\tau = \frac{\sigma_B^2}{\sigma_w^2} = JR + 1. \quad (28)$$

Set

$$\hat{\mu} - k\sqrt{T} = \hat{\mu} - k_T \cdot s_w, \quad (29)$$

where

$$k_T = \frac{t_{IJ-1}(\delta, \gamma)}{\sqrt{IJ(IJ - 1)}} \sqrt{(I - 1)\frac{s_B^2}{s_w^2} + I(J - 1)\tau}. \quad (30)$$

Therefore, the tolerance limit can be written as

$$\hat{\mu} - \frac{t_{IJ-1}(\delta, \gamma)}{\sqrt{IJ(IJ - 1)}} \sqrt{(I - 1)\frac{s_B^2}{s_w^2} + I(J - 1)\tau} \cdot s_w. \quad (31)$$

Referring to the noncentrality parameter δ in (25), let

$$\frac{\sigma_x}{\sigma_B} = \left[\frac{1}{J} \left(1 + \frac{J - 1}{\tau} \right) \right]^{1/2}. \quad (32)$$

The noncentrality parameter δ and the tolerance limit factor k_T depend on the parameter τ , which is unknown. Consequently it is natural to replace the parameter τ by a suitable estimator. Let $\hat{\tau} = F_\eta Z$, where Z denotes the

mean square ratio s_B^2/s_w^2 , and F_η the $100\eta^{th}$ percentile of an F -distribution with degrees of freedom $\nu_1 = I(J - 1)$ and $\nu_2 = I - 1$. We next study the relationship between η and (I, J, R) .

Substituting the estimator $\hat{\tau}$ into (30), we set

$$\begin{aligned}\hat{k}_T(\eta) &= \frac{t_{IJ-1}(\hat{\delta}, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1)F_\eta Z} \\ &= \frac{t_{IJ-1}(\hat{\delta}, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1) + I(J-1)F_\eta} \cdot \frac{s_B}{s_w}\end{aligned}\quad (33)$$

The properties of the coverage rate,

$$\gamma(\eta) = P_{\hat{\mu}, T, \hat{\tau}}\{\hat{\mu} - \hat{k}_T(\eta)s_w \leq u_p\}, \quad (34)$$

was studied by simulation for various values of I, J, η . Specifically, the coverage rate was generated for 10,000 simulations for $P = .90, \gamma = .95; I = 4, 8; J = 3, 5, 7, 9, 13; \eta = .50, .60, .70, .85$. For the sake of brevity, we provide only a summary of the conclusions.

$\gamma(\eta)$ is less than γ for $\eta = .50, .60, .70$, except for small values of R . $\gamma(\eta)$ is larger than γ for $\eta = .85$, confirming the conservative property of the Meek-Owen procedure. For fixed I and J , $\gamma(\eta)$ decreases as R increases and increases with I for fixed J and R .

A second set of simulations was carried out to ascertain the behavior of η as a function of I, J and R . For $P = .90, \gamma = .95$, 10,000 replications were calculated for $I = 4$ and various values of R from 0.00 to 20.00 and $J = 3, 5, 7, 9$. It was noted that η exhibits very regular behavior with a maximum (for the selected values) of .84. Thus regularity will be subsequently exploited.

We now describe the determination of η . Let I and J be fixed. If $R = 0$, $\tau = 1$ and the noncentrality parameter $\delta = \sqrt{IJ}K_p$. Hence the tolerance limit

(31) is given by

$$\hat{\mu} - t_{IJ-1}(\delta, \gamma)s_t/\sqrt{IJ}, \quad (35)$$

where s_t^2 is defined in the ANOVA table (Table 1). That is, if $R = 0$, there is no variation between batches and the model reduces to a normal random sample of size IJ . Then (35) is the tolerance limit obtained by Owen [1963].

However, we have assumed that R is unknown and will still require an estimate of τ . Since Z has the F -distribution with $I - 1$ and $I(J - 1)$ degrees of freedom, when $R = 0$, we determine η by

$$\eta = P\{Z \leq 1\}. \quad (36)$$

Next, we determine η for large values of R ($R \rightarrow \infty$). Writing (25) as

$$\delta = \sqrt{IJ}K_p \sqrt{\frac{1}{J} \left(1 + \frac{J-1}{\tau} \right)}, \quad (37)$$

Since $R \rightarrow \infty$ implies $\tau \rightarrow \infty$, $\delta \rightarrow \delta^* = \sqrt{I}K_p$. Then (31) becomes

$$\begin{aligned} & \hat{\mu} - \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1)F_\eta Z \cdot s_w} \\ &= \hat{\mu} - \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1) + I(J-1)F_\eta} \cdot s_B. \end{aligned} \quad (38)$$

We can now solve for F_η using (24) and (38), determining

$$P_{Y, s_B} \left\{ \frac{Y + \delta}{s_B/\sigma_B} \leq \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ-1}} \sqrt{(I-1) + I(J-1)F_\eta} \right\} \geq \gamma, \quad (39)$$

where $Y = \sqrt{IJ}(\hat{\mu} - \mu)/\sigma_B$, $\delta = \sqrt{IJ}K_p\sigma_x/\sigma_B$. Y has the standard normal distribution and $(I-1)s_B^2/\sigma_B^2$ has the chi-square distribution with $I-1$ degrees of freedom. Then $(Y + \delta)/(s_B/\sigma_B)$ has the noncentral t distribution with $I-1$ degrees of freedom and noncentrality parameter δ .

Writing

$$\int_{-\infty}^c f_{I-1}(t; \delta) dt = \gamma \quad (40)$$

where $f_{I-1}(t; \delta)$ is the probability density function of the noncentral t distribution with $I - 1$ degrees of freedom and noncentrality parameter δ . Then F_η can be determined by solving (40) for c .

Consequently we write

$$t_{I-1}(\delta, \gamma) = \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ-1}} \sqrt{(I-1) + I(J-1)} F_\eta, \quad (41)$$

where $t_d(\delta, \gamma)$ denotes the $100\gamma^{th}$ percentile for a noncentral t distribution with d degrees of freedom and noncentrality parameter δ . As $R \rightarrow \infty$, $\sigma_x/\sigma_B \rightarrow 1/\sqrt{J}$, we have $\delta \rightarrow \delta^*$. Therefore

$$F_\eta \rightarrow F_\eta^* = \frac{IJ-1}{I(J-1)} \left(\frac{t_{I-1}(\delta^*, \gamma)}{t_{IJ-1}(\delta^*, \gamma)} \right)^2 - \frac{I-1}{I(J-1)}. \quad (42)$$

Regarding F_η as the $100\eta^{th}$ percentile of the F distribution with $I(J-1)$ and $(I-1)$ degrees of freedom, η is ready available. We now proceed to determine η for all R , $0 \leq R < \infty$. For various I, J values and $P = 0.9, \gamma = 0.95$, η values were simulated for $R = 0, .1, 1, 5, 40$. $R \rightarrow \infty$ and $R = 0$ were determined analytically. The relationship,

$$\eta = \eta_\infty - (\eta_\infty - \eta_0) \exp(-\theta R^\zeta), \quad (43)$$

where η_∞ and η_0 are the η values when $R = \infty$ and $R = 0$ respectively. (43) satisfies the plots of the simulations with negligible error. The parameters θ and ζ can be determined by a curve fitting algorithm. Some η values for $P = .90, \gamma = .95$ are listed in Table 2. Simulation study shows that linear interpolation in η would give satisfactory results for values of I, J and R being not tabulated.

We now proceed to a verification of the efficiency of this procedure and a comparison with the Mee-Owen [1983] method.

Table 2. η values for $P = .90, \gamma = .95$

R	J	I						
		2	3	4	5	6	8	15
0	2	.58	.54	.52	.52	.51	.51	.50
	4	.64	.60	.57	.56	.55	.54	.53
	6	.66	.61	.59	.57	.57	.55	.54
	8	.67	.62	.59	.58	.57	.56	.54
	16	.67	.62	.50	.59	.58	.57	.55
	32	.68	.63	.60	.59	.58	.57	.55
.1	2	.63	.63	.62	.61	.60	.59	.55
	4	.63	.63	.63	.61	.60	.59	.56
	6	.67	.63	.63	.63	.61	.60	.57
	8	.68	.65	.63	.63	.62	.61	.61
	16	.74	.68	.67	.66	.64	.64	.63
	32	.78	.71	.70	.71	.66	.66	.63
1	2	.75	.75	.74	.72	.70	.71	.64
	4	.80	.78	.76	.74	.74	.72	.71
	6	.83	.79	.79	.77	.76	.74	.72
	8	.84	.80	.79	.77	.76	.75	.73
	16	.85	.82	.80	.78	.76	.76	.73
	32	.86	.82	.80	.79	.78	.78	.73
5	2	.84	.81	.79	.79	.76	.76	.70
	4	.87	.84	.82	.80	.78	.76	.76
	6	.89	.84	.83	.81	.80	.78	.76
	8	.89	.84	.83	.81	.81	.79	.77
	16	.89	.85	.84	.82	.81	.80	.77
	32	.89	.85	.85	.83	.82	.81	.77
∞	2	.86	.83	.81	.80	.78	.77	.74
	4	.88	.86	.83	.82	.81	.79	.77
	6	.89	.86	.84	.83	.82	.80	.78
	8	.89	.87	.85	.83	.82	.81	.79
	16	.90	.87	.85	.84	.83	.82	.79
	32	.90	.87	.86	.84	.83	.82	.79

Table 3. Simulated Coverage for $P = .90$, $\gamma = .95$
 (10,000 observations)

η	Mee-Owen		Present	
	max	min	max	min
0	.990	.965	.952	.948
.1	.988	.961	.952	.948
1	.978	.948	.952	.948
5	.965	.946	.952	.948
40	.965	.948	.954	.948

3 Comparison of Tolerance Limits

Extensive numerical computations were made for the purpose of evaluating the present procedure and providing comparisons of the coverage and the actual tolerance limit with the Mee-Owen [1983] technique. Table 3 and 4 exhibit the results.

The Mee-Owen procedure as noted in the preceding section is most conservative for low values of R , particularly for small values of I and J . The coverage of the present procedure is particularly uniform across the range of values where the sampling error of the simulation is taken into account. Finally, their tolerance limits are also substantially higher for the present procedure. Some sample values are given below.

Note that the Mee-Owen procedure is less conservative for large values of R . The standard errors of the tolerance limits have also been calculated. The present procedure has been shown to have consistently smaller standard errors over most of the range of values and about the same standard error as the Mee-Owen procedure for large R .

Table 4. Tolerance limits for $P = .90, \gamma = .95$

R	I	J	Mee-Owen	Present
0	2	2	-14.11	-4.57
	2	8	-8.33	-2.33
	8	2	-2.14	-2.01
	8	8	-1.67	-1.60
.1	2	2	-15.14	-5.16
	2	8	-11.30	-3.16
	8	2	-2.27	-2.15
	8	8	-1.84	-1.76
1	2	2	-21.38	-10.62
	2	8	-21.40	-11.34
	8	2	-3.26	-3.17
	8	8	-3.00	-2.88
5	2	2	-38.05	-30.43
	2	8	-39.04	-34.47
	8	2	-5.96	-5.93
	8	8	-5.83	-5.77
40	2	2	-97.57	-99.75
	2	8	-99.85	-105.79
	8	2	-15.93	-15.95
	8	8	-16.02	-15.82

4. Numerical Illustration.

To provide a numerical comparison with the results of Lemon (1977) and Mee and Owen (1983), we will apply the present procedure to data analyzed in the above papers. Lemon's data consisted of five batches with six observations per batch. The data and specifications for the tolerance limit are summarized as follows.

$$P = .9, \gamma = .95, \hat{\mu} = 186, s_b = 6.87, s_w = 5.86, s_x = 9.04,$$

$$\text{hence } \hat{R} = 1.37.$$

Lemon obtained a lower tolerance limit of 156.3. The procedure of Mee and Owen resulted in a lower tolerance limit of 160.0, which is less conservative than Lemon's procedure. The present methodology utilizes the non-central t distribution with non-centrality parameter $\delta = 3.56$ and 29 degrees of freedom, resulting in a lower tolerance limit of 165.0, which is less conservative than the result of Mee and Owen..

REFERENCES

- Aspin, A.A. (1948) "An Examination and Further Development of a Formula Arising in the Problem of Comparing Two Mean Values," *Biometrika*, 35, 88-96.
- Bagui, S.C., Bhaumik, D.K. and Parnes, M. (1996) "Two-Sided Tolerance Limits for Unbalanced m-Way Random-Effects ANOVA Models," *Journal of Applied Statistical Science*, 3, 135-148.
- Beckman, R.J., and Tietjen, G.L. (1989) "Two-Sided Tolerance Limits for Balanced Random-Effects ANOVA Models," *Technometrics*, 31, 185-197.
- Bhaumik, D.K. and Kulkarni, P.M. (1991) "One-Sided Tolerance Limits for Unbalanced One-Way ANOVA Random-Effects Model," *Communications in Statistics: Theory and Methods*, 20, 1665-1675.
- Lemon, G. H. (1977). "Factors for One-Sided Tolerance Limites for Balanced One-Way-ANOVA Random-Effects Model," *Journal of the American Statistical Association*, 72, 676-680.
- Mee, R.W., and Owen,D. B. (1983). "Improved Factors for One-Sided Tolerance Limits for Balanced One-Way-ANOVA Random Model," *Journal of the American Statistical Association*, 78, 901-905.
- Trickett, W.H. and Welch, B.L. (1954) "On the Comparison of Two Means: Further Discussion of Iterative Methods for Calculating Tables," *Biometrika*, 41, 361-374.
- Vangel, M.G. (1992) "New Methods for One-Sided Tolerance Limits for a One-Way Balanced Random-Effects ANOVA Model," *Technometrics*, 34, 176-185.
- Welch, B.L. (1947) "The Generalization of Student's Problem When Several Different Population Variances Are Involved," *Biometrika*, 34, 28-35.

WEDNESDAY, OCTOBER 18

CONTRIBUTED SESSION IV (1530 - 1700)