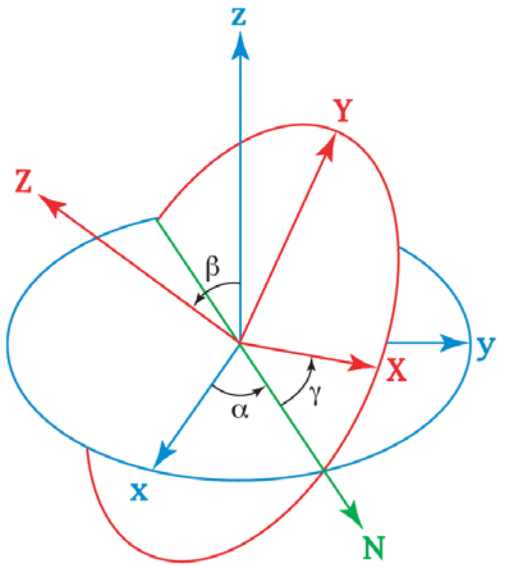
$$\begin{tikzcd}
\mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\
& \mathrm{SO}(3)
\end{tikzcd}$$



The Algebra of Rotations in $\mathbb{R}^3$

An Exploration of Representations by Quaternions, $SU(2)$, and $SO(3)$

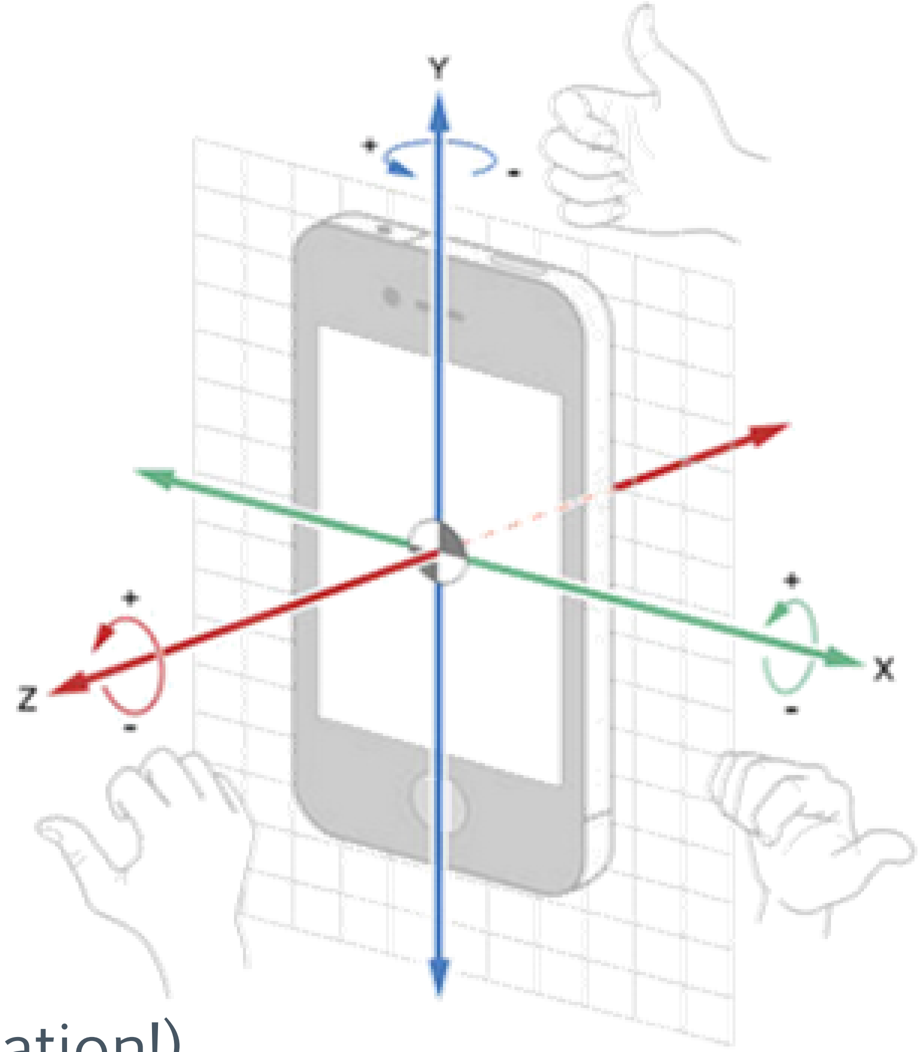
π

Nicholas Meyer
nmeyer14@winona.edu
Winona State University

DEPARTMENT OF MATHEMATICS AND STATISTICS
College of Science and Engineering

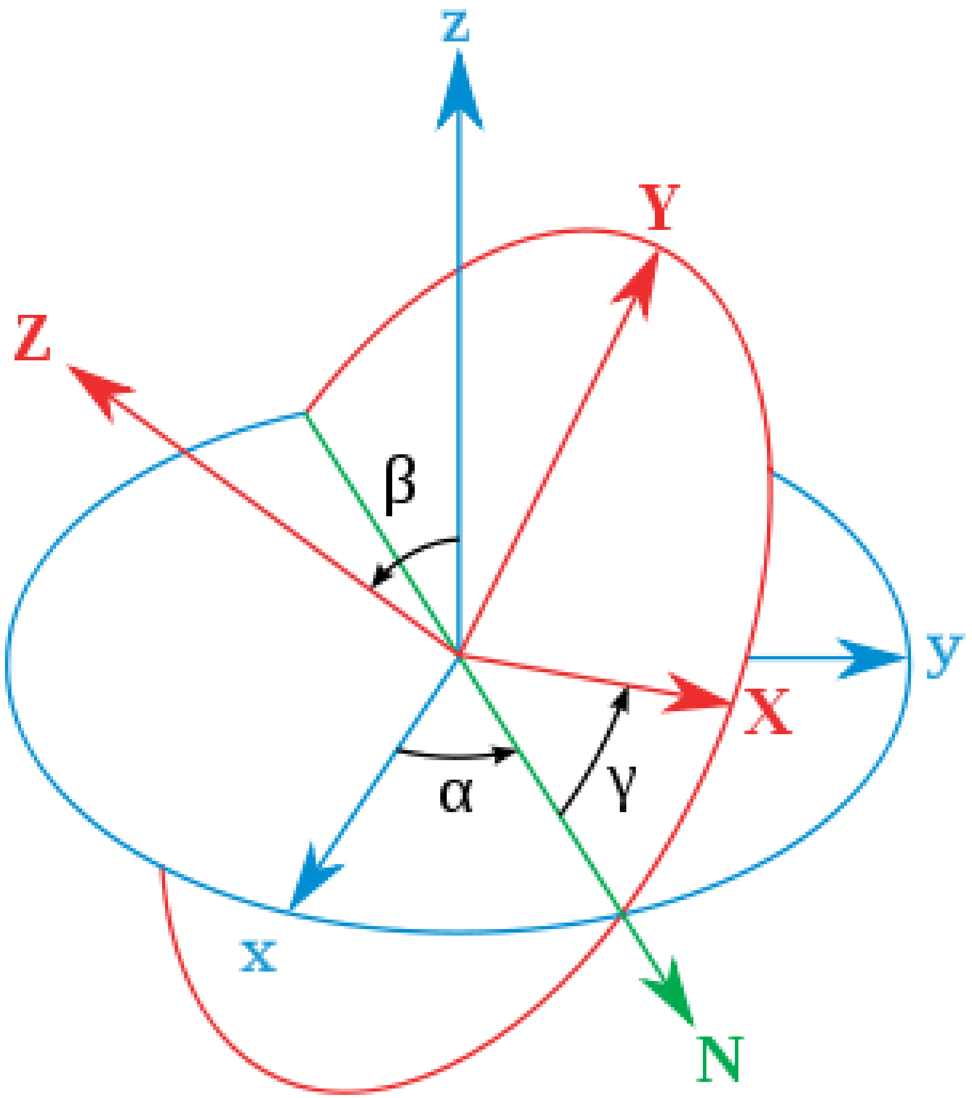
Outline

- Euler Angles
- Rodrigues Rotation Formula
- Hamilton’s Quaternions
- The Versor Convention
- Matrices, Maps, and Morphisms!
- Practical Application (Computer Animation!)



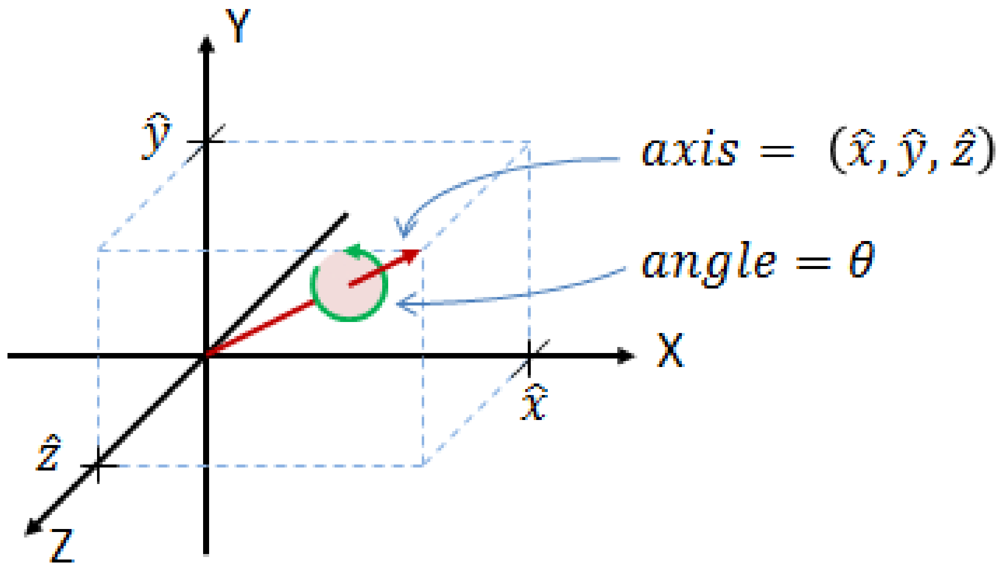
Euler Angles

- 1765- Euler studied the mechanics of rotating bodies
- Decomposition of Motion
- Need 6 parameters to rotate a on object
- Many constraints on what the parameters can be
- Suffers gimbal lock (local singularity in the rotation sequence)



The Rodrigues Rotation Formula (RRF)

- 1840- Published paper using Euler's Four Square Identity
- If $\vec{\boldsymbol{v}}, \vec{\boldsymbol{e}} \in \mathbb{R}^3$ and $|\vec{\boldsymbol{e}}| = 1$, to rotate $\vec{\boldsymbol{v}}$ about $\vec{\boldsymbol{e}}$ by some angle $\rho = 2\theta$: $\vec{\boldsymbol{v}}_{rot} = \vec{\boldsymbol{v}}\cos\theta + (\vec{\boldsymbol{e}} \times \vec{\boldsymbol{v}})\sin\theta + \vec{\boldsymbol{e}}(\vec{\boldsymbol{e}} \cdot \vec{\boldsymbol{v}})\sin\theta$



The Rodrigues Rotation Formula (Cont'd)

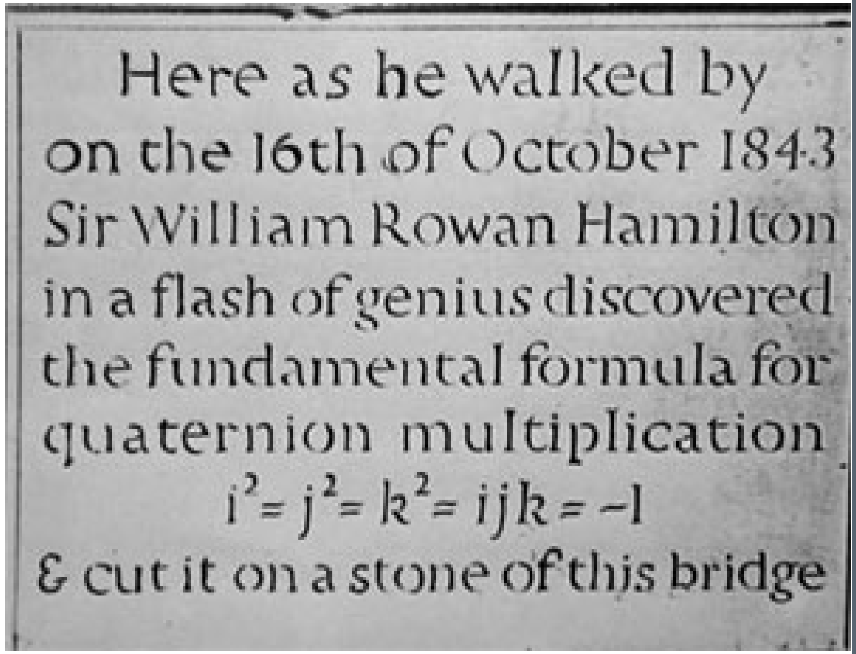
- In matrix form, letting $\hat{\mathbf{e}} = (x, y, z),\ \lambda = \cos\theta\,,\quad \mu = \sin\theta$:

$$\boldsymbol{M}_{rot} = \begin{pmatrix} x^2(1-\lambda)+\lambda & xy(1-\lambda)-z\mu & xz(1-\lambda)+y\mu \\ xy(1-\lambda)+z\mu & y^2(1-\lambda)+\lambda & yz(1-\lambda)-x\mu \\ xz(1-\lambda)-y\mu & yz(1-\lambda)+x\mu & z^2(1-\lambda)+\lambda \end{pmatrix}$$

- The above matrix has eigenvalues $1, e^{i\theta}, e^{-i\theta}$.

Hamilton's Quaternions

- 1843 – W. R. Hamilton discovered quaternions
- $i^2 = j^2 = k^2 = ijk = -1$
- Form a group under multiplication
- **Not** Commutative

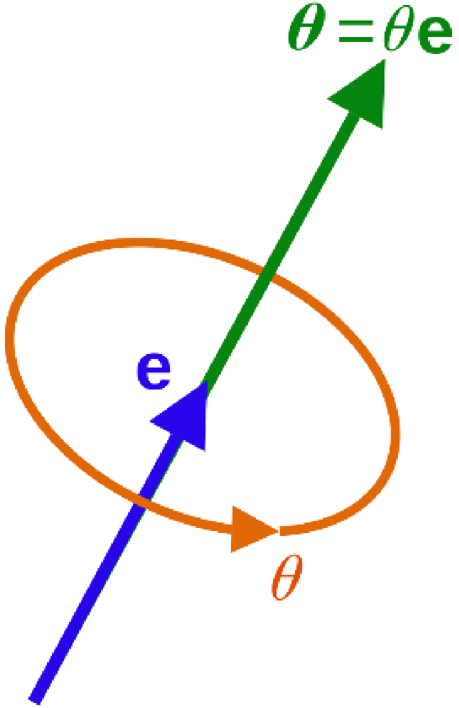


The Quaternions as Rotations (Versors!)

- Versor Conventions:
  - $\mathbb{H}^* = \{a + xi + yj + zk \mid a^2 + x^2 + y^2 + z^2 = 1\}$
  - All $q \in \mathbb{H}^*$ can be represented as

    $q = [\lambda, \vec{\boldsymbol{e}}\mu]$, where $\vec{\boldsymbol{e}} = (x, y, z)$,

    $\lambda = \cos\theta$, and $\mu = \sin\theta$



Matrices, Maps, and Morphisms

$$\begin{tikzcd}
\mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\
& \mathrm{SO}(3)
\end{tikzcd}$$

$$\begin{tikzcd} \mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\ & \mathrm{SO}(3) \end{tikzcd}$$

Quaternions and SU(2)

- From our diagram, we can see that $\tilde{\varphi}$: $\mathbb{H}^*$
- We define the isomorphism $\tilde{\varphi}$ as:

$$\tilde{\varphi}(a + xi + yj + zk) = \begin{pmatrix} a + ix & y + iz \\ -y + iz & a - ix \end{pmatrix}$$

- Upon inspection, we see that the above map can be given in terms of a linear combination of simpler matrices

The Quaternion or “Rhotation” Matrices

THE MATRICES

- $\rho_0 = ID_2$
- $\rho_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
- $\rho_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\rho_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

THEIR PRODUCTS

- ρ_0 commutes with the other ρ
- $\rho_x\rho_y = -\rho_y\rho_x = \rho_z$
- $\rho_y\rho_z = -\rho_z\rho_y = \rho_x$
- $\rho_z\rho_x = -\rho_x\rho_z = \rho_y$
- $\rho_x^2 = \rho_y^2 = \rho_z^2 = \rho_x\rho_y\rho_z = -\rho_0$
- $|\rho_x| = |\rho_y| = |\rho_z| = 1$

Quaternions and SO(3)

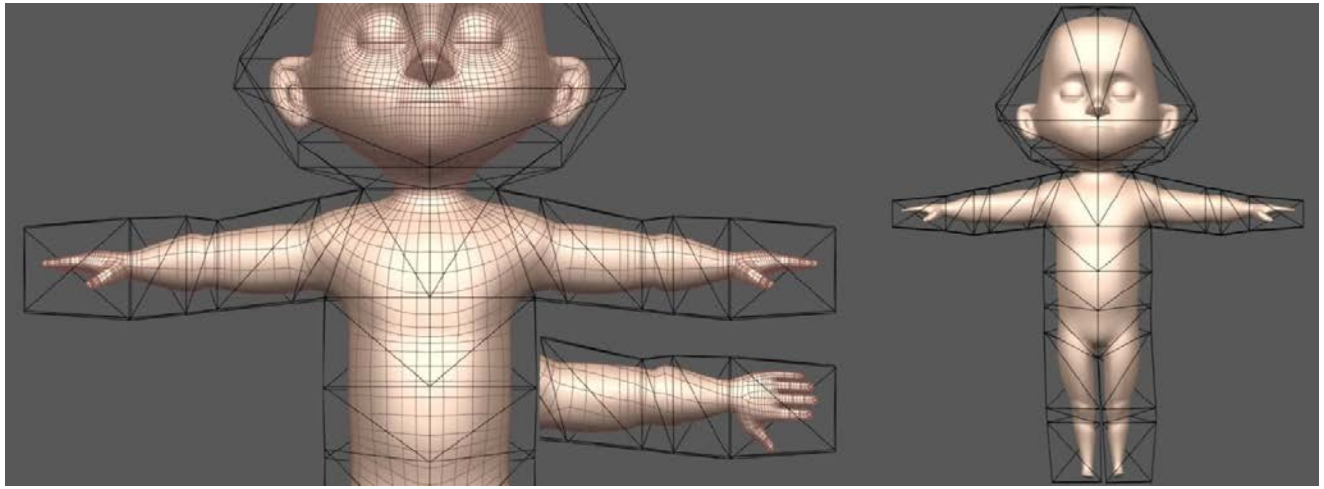
$$\begin{tikzcd} \mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathbf{SU(2)} \arrow[d, "\psi"] \\ & \mathbf{SO(3)} \end{tikzcd}$$

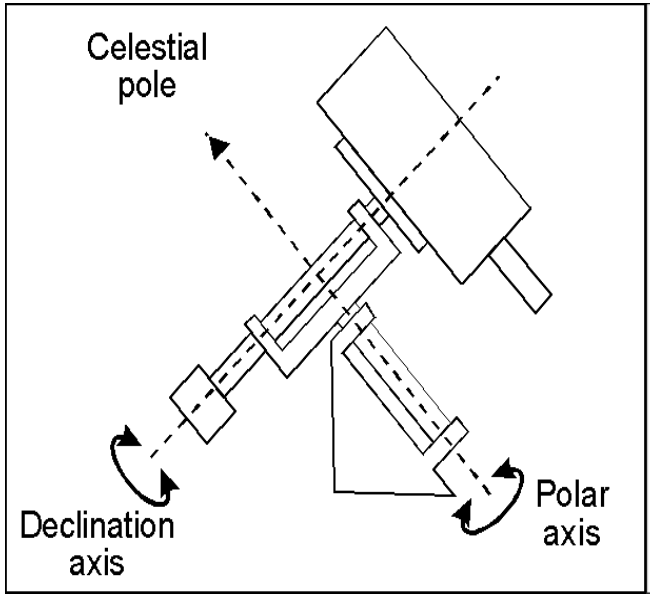
- So what about η?
- Just use a versor!
- $\eta([\lambda, \vec{\boldsymbol{e}}\mu])$ is just the Rodrigues Matrix for the versor $\mathrm{q} = (\lambda, \mu\vec{\boldsymbol{e}})$!
- η is a 2:1 Homomorphism, since a rotation about $\vec{\boldsymbol{e}}$ by $\rho = 2\theta$ is equivalent to a rotation about -$\vec{\boldsymbol{e}}$ by $-\rho = -2\theta$
- Since $\mathbb{H}^* \cong SU(2)$ we choose ψ such that our diagram commutes.
- Hence $\psi = \eta \circ \tilde{\varphi}^{-1} \Rightarrow \psi$ must be a 2:1 Homomorphism

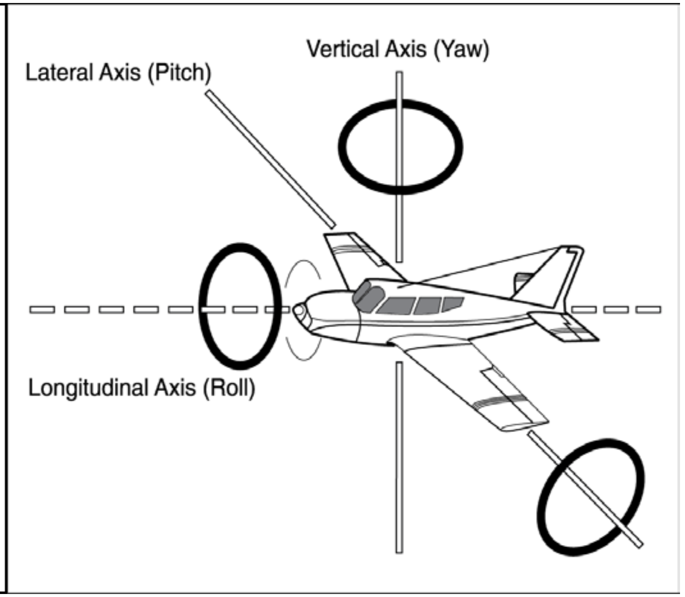
Demo

Applications

- Computer Animation
- Telescopy
- Medical Imaging
- Building Stab
- Robotics
- CAD Software
- Statistics







Further Studies:

- Expansion of this work to $\mathbb{R}^4$:
  - How does $\mathbb{H}^* \otimes \mathbb{H}^*$ relate to $SU(2) \otimes SU(2)$?
  - 4-D Versor Convention?
- How about $\mathbb{R}^7 / \mathbb{R}^8$ via octonions?
  - How do we generalize our results in such a way that ignores associativity?
- Lie Theory, Quantum Groups, and Spin Groups!
- How do we unify translational motion and rotational motion in terms of groups?

References:

- Simon L. Altmann. *Rotations, Quaternions, and Double Groups*. Mineola, NY: Dover, 2005.
- John H. Conwway and Derek A. Smith. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*. A K Peters, Ltd., 2003.
- Jian S. Dia. "Euler-Rodrigues formula variations, quaternion conjugation and intrinsic connections". *Mechanism and Machine Theory* 92 (2015), 144-152.
- Govind S. Krishnaswami and Sonakshi Sachdev. "Algebra and Geometry of Hamilton's quaternions". *Resonance – Journal of Science Education* 21.6 (2016), pp. 529–433.
- Cibelle Celestino Silva and Roberto de Andrade Martins. "Polar and axial vectors versus quaternions". *American Journal of Physics* 70.9 (2002), pp. 958–963.
- David Eberly. "Rotation Representations and Performance Issues". Online. 2008.
- Rhett Allain. "How Olympic Gymnasts Use Physics to Pull Off Those Crazy Twists". *Wired* (August 2016). Online.

Acknowledgements

- "This project has been sponsored by a student project grant from Winona State University."
- Advisors: Dr. Aaron Wangberg, Dr. Joyati Debnath
- Special Thanks: Dr. Sarah Phan-Budd

Thank You!