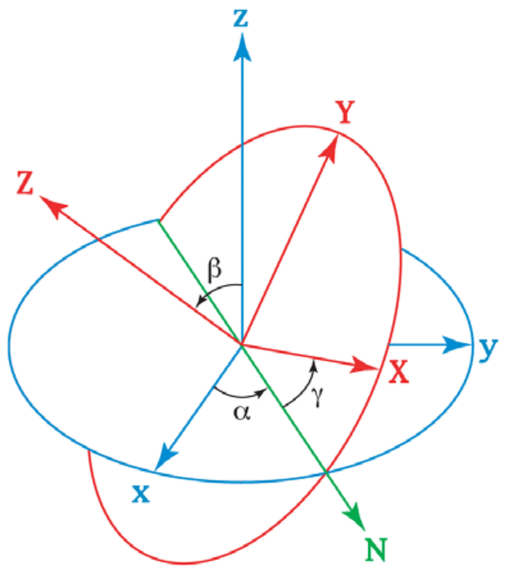
$$\begin{tikzcd} \mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\ & \mathrm{SO}(3) \end{tikzcd}$$



The Algebra of Rotations in $\mathbb{R}^3$

An Exploration of Representations by Quaternions, SU(2), and SO(3)

Nicholas Meyer
nmeyer14@winona.edu
Winona State University

DEPARTMENT OF MATHEMATICS AND STATISTICS
College of Science and Engineering

Acknowledgements

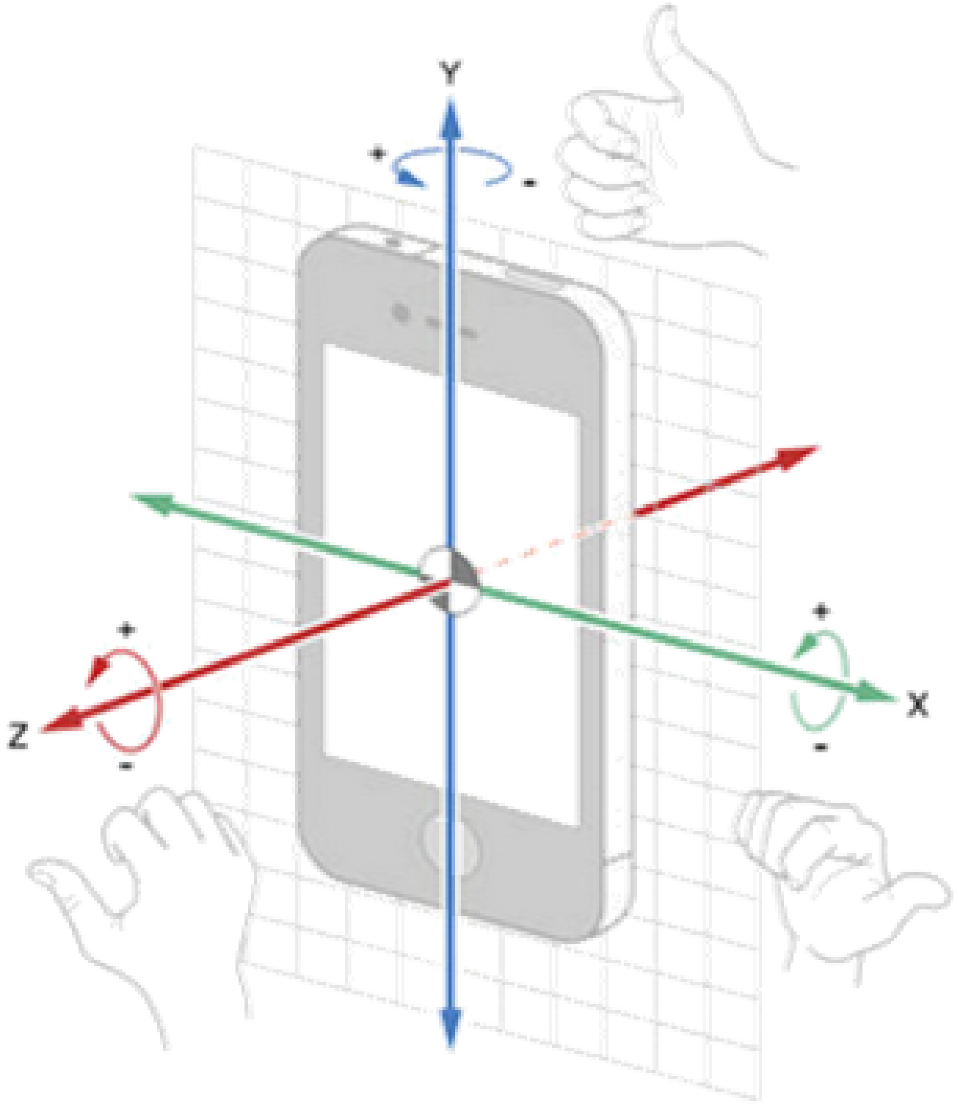
› "This project has been sponsored by a student project grant from Winona State University."

› Advisors: Dr. Aaron Wangberg, Dr. Joyati Debnath

› Special Thanks: Dr. Tevian Dray, Dr. Sarah Phan-Budd

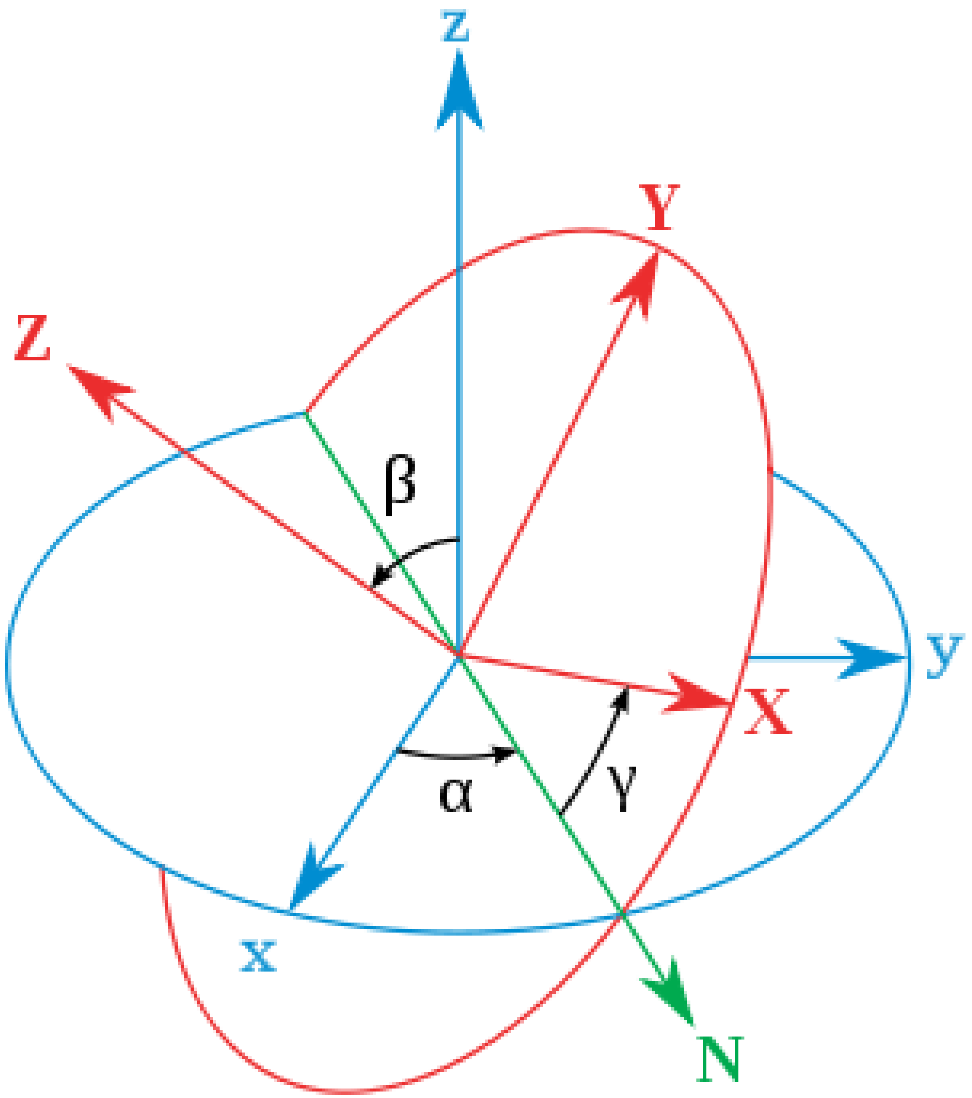
Outline

- Euler Angles
- Rodrigues Rotation Formula
- Hamilton's Quaternions
- The Versor Convention
- Matrices, Maps, and Morphisms!
- Comparison of Representations
- Practical Application (Computer Animation!)



Euler Angles

- 1765- Euler studied the mechanics of rotating bodies
- Decomposition of Motion
- Need 6 parameters to rotate a on object
- Many constraints on what the parameters can be
- Hard to do computations with

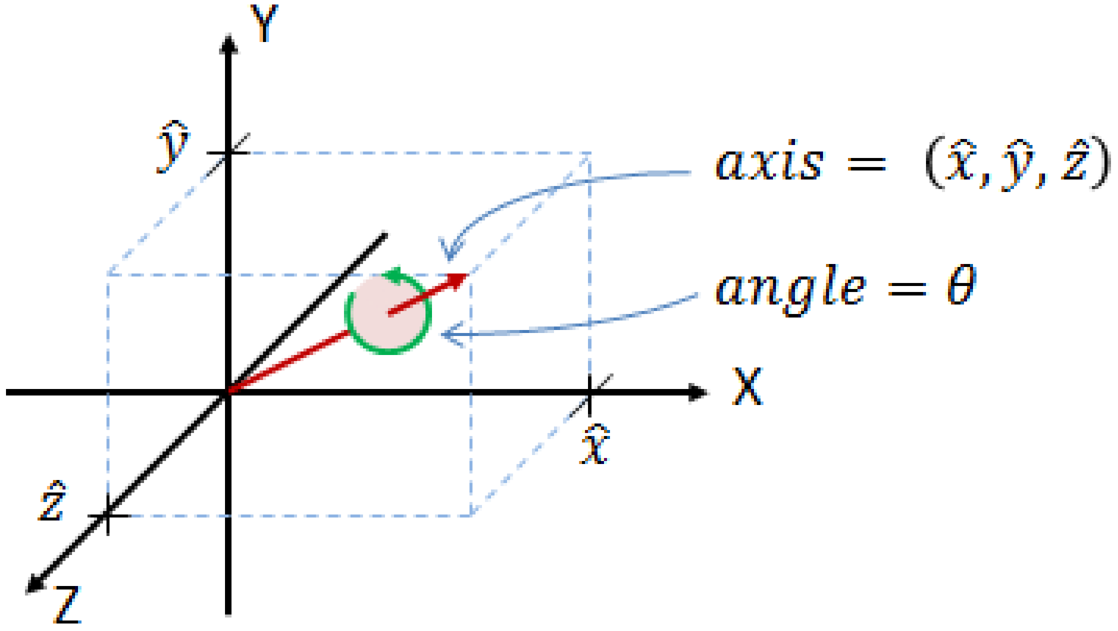


The Rodrigues Rotation Formula (RRF)

- 1840- Published paper using Euler's Four Square Identity

- If $\vec{\boldsymbol{v}}, \vec{\boldsymbol{e}} \in \mathbb{R}^3$ and $|\vec{\boldsymbol{e}}| = 1$, to rotate $\vec{\boldsymbol{v}}$ about $\vec{\boldsymbol{e}}$ by some angle

$$\rho = 2\theta\colon \vec{\boldsymbol{v}}_{rot} = \vec{\boldsymbol{v}}\cos\theta + (\vec{\boldsymbol{e}} \times \vec{\boldsymbol{v}})\sin\theta + \vec{\boldsymbol{e}}(\vec{\boldsymbol{e}} \cdot \vec{\boldsymbol{v}})\sin\theta$$



The Rodrigues Rotation Formula (Cont'd)

› In matrix form, letting $\hat{\mathbf{e}} = (x, y, z),\ \lambda = \cos\theta\,,\quad \mu = \sin\theta$:

$$\boldsymbol{M}_{rot} = \begin{pmatrix} x^2(1-\lambda)+\lambda & xy(1-\lambda)-z\mu & xz(1-\lambda)+y\mu \\ xy(1-\lambda)+z\mu & y^2(1-\lambda)+\lambda & yz(1-\lambda)-x\mu \\ xz(1-\lambda)-y\mu & yz(1-\lambda)+x\mu & z^2(1-\lambda)+\lambda \end{pmatrix}$$

› The above matrix has eigenvalues $1, e^{i\theta}, e^{-i\theta}$.

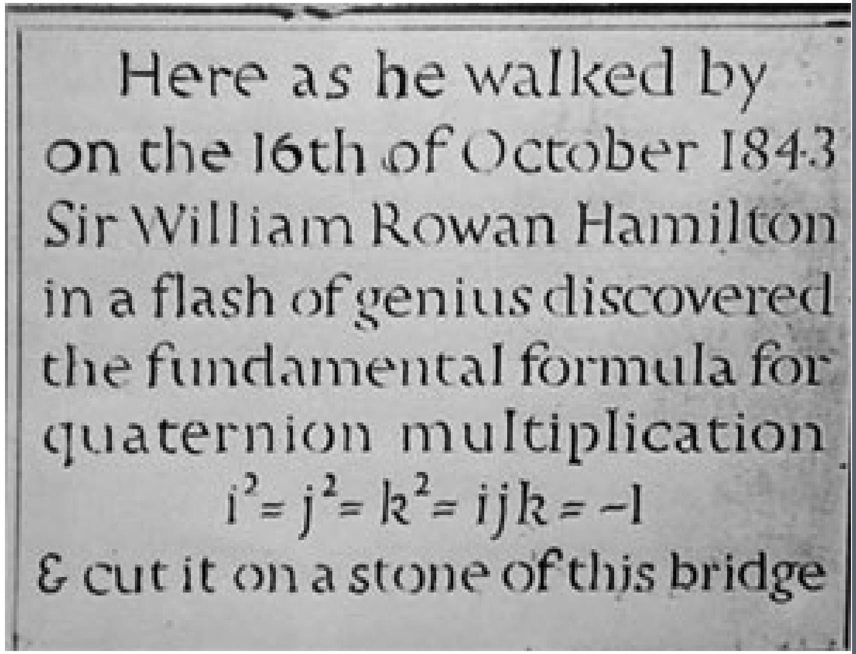
Hamilton’s Quaternions

› 1843 – W. R. Hamilton discovered quaternions

› $i^2 = j^2 = k^2 = ijk = -1$

› Form a group under multiplication

› Not Commutative



π

The Quaternions as Rotations (Versors!)

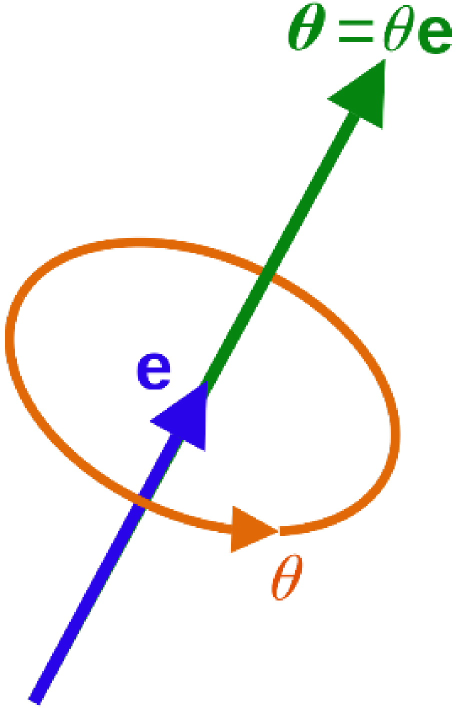
› Versor Conventions:

– $\mathbb{H}^* = \{a + xi + yj + zk \mid a^2 + x^2 + y^2 + z^2 = 1\}$

– All $q \in \mathbb{H}^*$ can be represented as

$q = [\lambda, \vec{\boldsymbol{e}}\mu]$, where $\vec{\boldsymbol{e}} = (x, y, z)$,

$\lambda = \cos\theta$, and $\mu = \sin\theta$



π Matrices, Maps, and Morphisms

$$
\begin{tikzcd}
\mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\
 & \mathrm{SO}(3)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathrm{SU}(2) \arrow[d, "\psi"] \\
& \mathrm{SO}(3)
\end{tikzcd}
$$

Quaternions and SU(2)

- From our diagram, we can see that $\tilde{\varphi}\colon\ \mathbb{H}^* \to$ S
- We define the isomorphism $\tilde{\varphi}$ as:

$$\tilde{\varphi}(a + xi + yj + zk) = \begin{pmatrix} a + ix & y + iz \\ -y + iz & a - ix \end{pmatrix}$$

- Upon inspection, we see that the above map can be given in terms of a linear combination of simpler matrices

The Quaternion Matrices

THE MATRICES

- $\gamma_0 = ID_2$
- $\gamma_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
- $\gamma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\gamma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

THEIR PRODUCTS

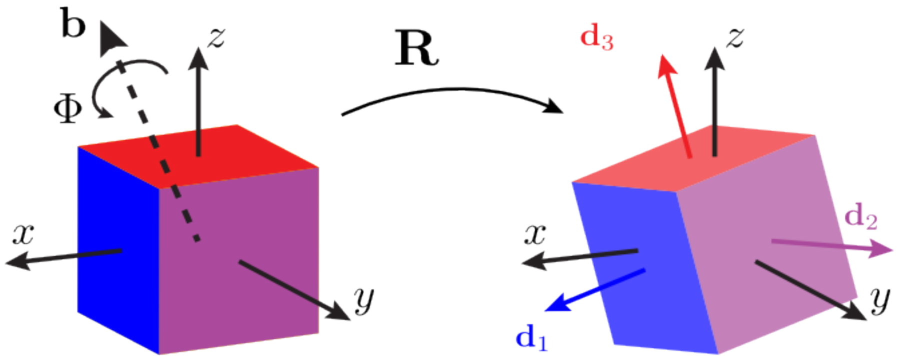
- γ_0 commutes with the other γ
- $\gamma_x\gamma_y = -\gamma_y\gamma_x = \gamma_z$
- $\gamma_y\gamma_z = -\gamma_z\gamma_y = \gamma_x$
- $\gamma_z\gamma_x = -\gamma_x\gamma_z = \gamma_y$
- $\gamma_x^2 = \gamma_y^2 = \gamma_z^2 = \gamma_x\gamma_y\gamma_z = -\gamma_0$
- $|\gamma_x| = |\gamma_y| = |\gamma_z| = 1$

Quaternions and SO(3)

$$\begin{tikzcd} \mathbb{H}^* \arrow[r, hook, two heads, "\widetilde{\varphi}"] \arrow[dr, "\eta"'] & \mathbf{SU}(2) \arrow[d, "\psi"] \\ & \mathbf{SO}(3) \end{tikzcd}$$

- So what about η?
- Just use a versor!
- $\eta([\lambda, \vec{\boldsymbol{e}}\mu])$ is just the Rodrigues Matrix for the versor$(\lambda, \mu\vec{\boldsymbol{e}})$!
- η is a 2:1 Homomorphism, since a rotation about $\vec{\boldsymbol{e}}$ by $\rho = 2\theta$ is equivalent to a rotation about -$\vec{\boldsymbol{e}}$ by $-\rho = -2\theta$
- Since $\mathbb{H}^* \cong SU(2)$ we use $\psi = \eta \circ \tilde{\varphi}^{-1}$, so ψ must also be a 2:1 Homomorphism

π Which Representation to Use?



› Quaternions ($\mathbb{H}^*$):

– Four Float components

– 24 Additions, 32 Multiplications for standard

– 17 Additions, 24 Multiplications when optimized

– Smallest Memory Use

– Efficient when using Multiple Transformations

Which Representation to Use? (Cont'd)

SO(3)

- 9 Float components
- Initial creation requires a fair amount of computation
- 6 Additions and 9 Multiplications
- Repeated use of a single transformation
- Easy to Parallelize

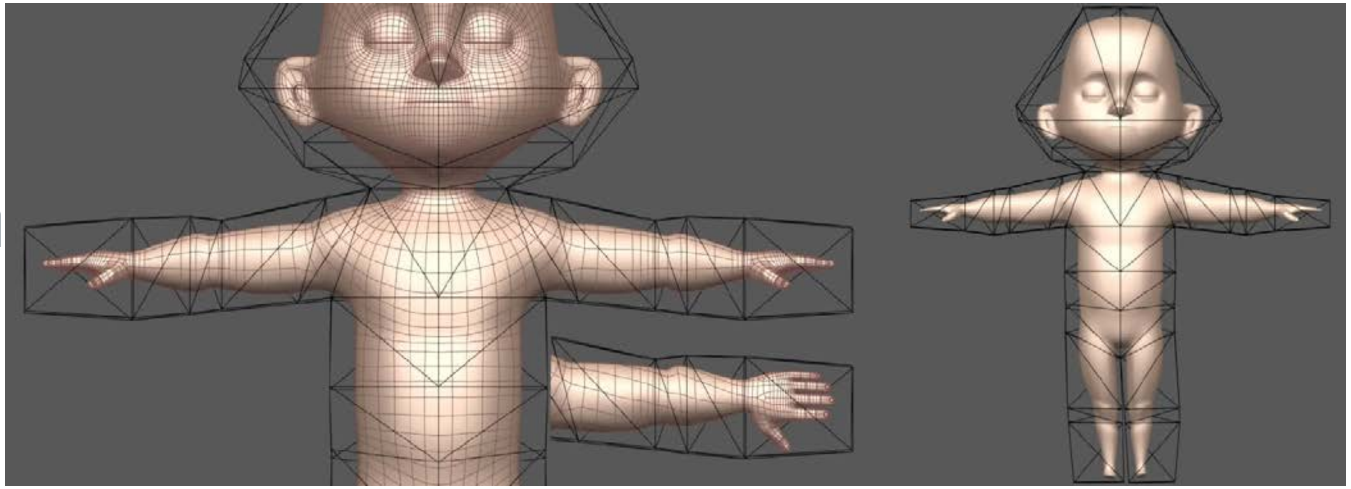
SU(2)

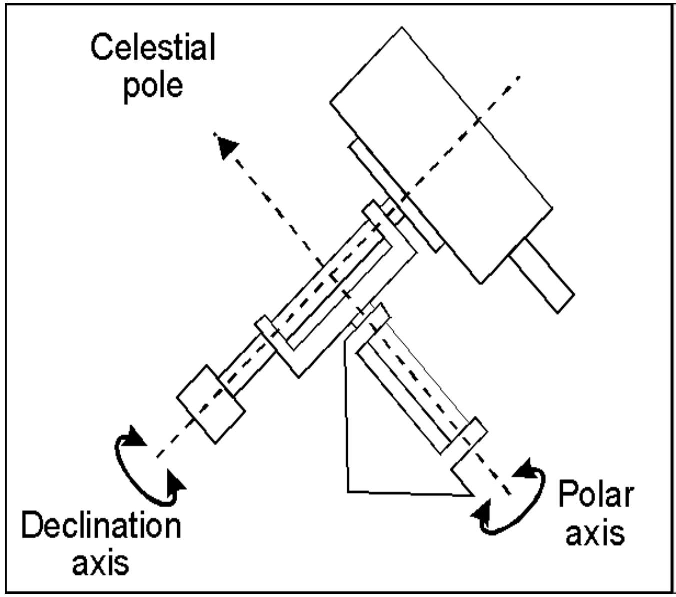
- 4 complex Float components / 8 real Float components
- Memory use depends on if language implements Complex Floats.
- No easy way to compute, transform to Quaternion or SO(3)
- Natural to use for Quantum Mechanics

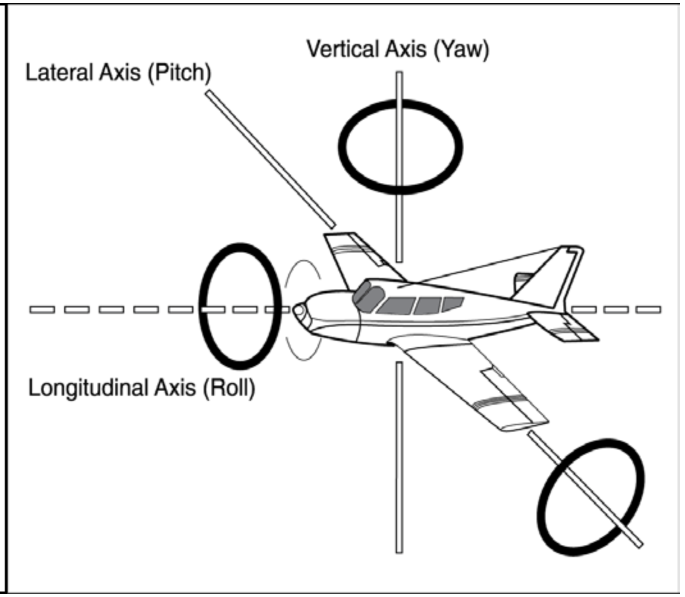
Demo

Applications

- Computer Animation
- Telescopy
- Medical Imaging
- Robotics
- CAD Software
- Statistics







Further Studies:

- Expansion of this work to $\mathbb{R}^4$:
  - How does $\mathbb{H}^* \otimes \mathbb{H}^*$ relate to $SU(2) \otimes SU(2)$?
  - 4-D Versor Convention?
- How about $\mathbb{R}^7 / \mathbb{R}^8$ via octonions?
  - How do we generalize our results in such a way that ignores associativity?
- Lie Theory, Quantum Groups, and Spin Groups!
- How do we unify translational motion and rotational motion in terms of groups?

References:

- Simon L. Altmann. *Rotations, Quaternions, and Double Groups*. Mineola, NY: Dover, 2005.
- John H. Conwway and Derek A. Smith. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*. A K Peters, Ltd., 2003.
- Jian S. Dia. “Euler-Rodrigues formula variations, quaternion conjugation and intrinsic connections”. *Mechanism and Machine Theory* 92 (2015), 144-152.
- Govind S. Krishnaswami and Sonakshi Sachdev. “Algebra and Geometry of Hamilton’s quaternions”. *Resonance – Journal of Science Education* 21.6 (2016), pp. 529–433.
- Cibelle Celestino Silva and Roberto de Andrade Martins. “Polar and axial vectors versus quaternions”. *American Journal of Physics* 70.9 (2002), pp. 958–963.
- David Eberly. “Rotation Representations and Performance Issues”. Online. 2008.
- Rhett Allain. “How Olympic Gymnasts Use Physics to Pull Off Those Crazy Twists”. *Wired* (August 2016). Online.

Thank You!