Math for Business Study Guide Midterm 2

Fall 2014

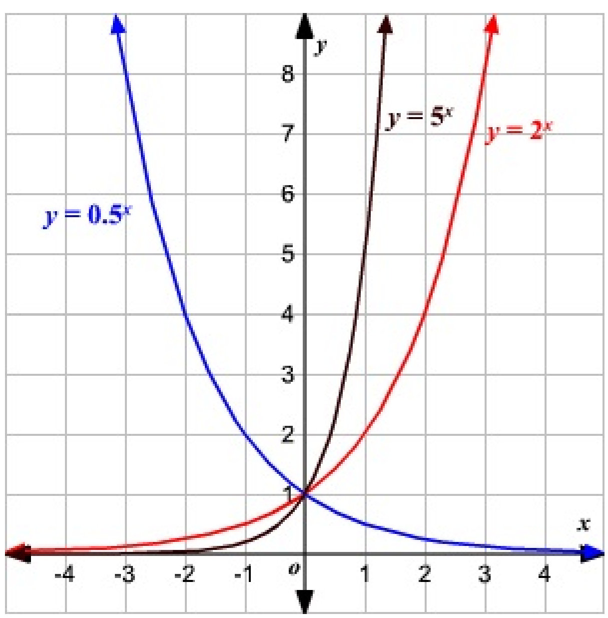
NOTE: This is intended as a study aid for the midterm. This is not an official study guide. It is just something that I create on my own in the hopes that you find it useful. The supervising professor for the course has nothing to do with the creation of this study guide. There are likely things that will be covered in the midterm that I don't put on here so it's ultimately up to you as the student to make sure you're properly prepared. However, that being said, I do try to cover the main topics and it's helpful to have one document to look at that lists most of the formulas you need to know for the midterm.

1 Chapter 3: Exponential and Logarithm Functions

1.1 Exponential Functions and Their Graphs

Exponential Function

$$f(x) = b^x \qquad b \neq 1 \text{ and } b > 0$$



We say b is the **base** of the exponential function.

There is a special kind of exponential function that we single out because of its significance and we call it the **Natural Exponential Function**. It is the function

$$f(x) = e^x$$

Evaluating Logarithms:

Evaluate the following expressions:

1. $\log_2(8)$
2. $\log_{32}(2)$
3. $\log_7(-3)$
4. $6^{\log_6(8)}$

(1) We ask ourselves the following question: When does $2^y = 8$? Well we know $2^3 = 8$ so our answer is $\log_2(8) = 3$

(2) We ask ourselves the following question: When does $32^y = 2$? Well we know $32^{\frac{1}{5}} = 2$ so our answer is $\log_{32}(2) = \dfrac{1}{5}$

(3) We ask ourselves the following question: When does $7^y = -3$? If we think about it we should realize this can never happen so there is no solution

(4) There is a property in the book that says that $a^{\log_a(x)} = x$ so our answer is $6^{\log_6(8)} = 8$

Compound Interest:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

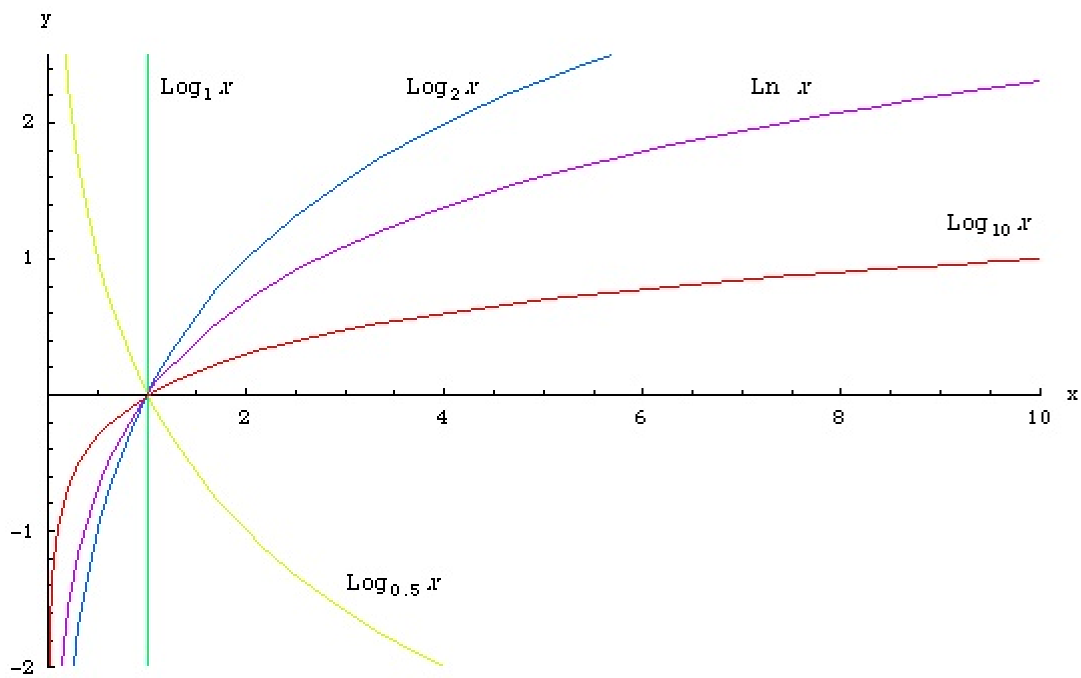
Continously Compounded Interest:

$$A = Pe^{rt}$$

1.2 Logarithm Functions and Their Graphs

Logarithm Function

$$f(x) = \log_b(x) \qquad\qquad b \neq 1 \text{ and } b > 0$$



Again, we call b the **base** of the logarithm function.

There is a special kind of logarithm function that we single out becaue of its significance. It is the **Natural Log Function**. It is the function

$$f(x) = \ln(x)$$

The Natural Log Function is the logarithm function with **base e** $\Big(f(x) = \log_e(x)\Big)$ and it is the **inverse** of the natural exponential function $f(x) = e^x$

Note: If you see something like $f(x) = \log(x)$ with no base, then we assume it is **log base 10**, or written out, we assume $f(x) = \log_{10}(x)$

Logarithm to Exponential Conversion:

$$y = \log_b(x) \Leftrightarrow x = b^y$$

Finding Domain of a Log Function:

$$\text{Given } \log_b(\text{STUFF}) \text{ we set STUFF} > 0$$

Example: Find the domain of $\log_3(x - 3)$

- Set $x - 3 > 0 \implies x > 3$ is the domain

1.3 Properties of Logarithms

Important Properties of Exponents:

1. $b^m \cdot b^n = b^{m+n}$
2. $(b^m)^n = b^{mn}$
3. $b^1 = b$
4. $b^{-m} = \dfrac{1}{b^m}$
5. $(ab)^m = a^m \cdot b^m$
6. $\dfrac{b^m}{b^n} = b^{m-n}$
7. $b^0 = 1$

Important Properties of Logs:

1. $\log_b(1) = 0$
2. $\log_b(b) = 1$
3. $\log_b(b^x) = x$
4. $b^{\log_b(x)} = x$
5. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
6. $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
7. $\log_b(x^p) = p \cdot \log_b(x)$

Change of Base:

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}$$

1.4 Exponential and Logarithm Equations

- Refer to the textbook and homework/quizzes (Sorry I had no idea how to explain this in a PDF without just typing up a bunch of examples which I didn't feel like doing)

1.5 Exponential and Logarithm Models

- Refer to the textbook and homework/quizzes

2 Chapter 8: Systems of Linear Equations and Inequalities

2.1 Systems of Linear Equations in Two Variables

- We learned two methods for solving systems of equations in two variables:
  1. **Substitution Method**
  2. **Elimination Method**
- See the following examples for the difference between these two methods

Substitution Method:

Example 2: Solve

$$x + y = 12$$
$$2x + y = 5$$

Original system	$x + y = 12$ $2x + y = 5$
Step 1: Solve the first equation for x.	$x + y = 12 \Rightarrow x = 12 - y$ $2x + y = 5$
Step 2: Substitute the expression for x into the second equation.	$2(12 - y) + y = 5$
Step 3: Solve the equation.	$2(12 - y) + y = 5$ $24 - 2y + y = 5$ $24 - y = 5$ $24 - 24 - y = 5 - 24$ $-y = -19$ $\frac{-1y}{-1} = \frac{-19}{-1}$ $y = 19$
Step 4: Substitute the value into one of the original equations and solve.	$x + y = 12$ $x + 19 = 12$ $x + 19 - 19 = 12 - 19$ $x = -7$
Solution	$(-7, 19)$
Step 5: Check	$x + y = 12$ $\quad$ $2x + y = 5$ $-7 + 19 = 12$ $\quad$ $2(-7) + 19 = 5$ $12 = 12$ $\quad$ $-14 + 19 = 5$ $5 = 5$

NOTE: To solve the system, for Step 1, we could have solved the first equation for y or we could have solved the second equation for y and we would have obtained the same result.

Elimination Method:

Example 3: Solve

$$3x + 5y = -6$$
$$-2x + 7y = 4$$

Original system	$3x + 5y = -6$ $-2x + 7y = 4$
Step 2: Multiply the first equation by 2 and the second equation by 3.	$2(3x + 5y = -6) \Rightarrow 6x + 10y = -12$ $3(-2x + 7y = 4) \Rightarrow -6x + 21y = 12$
Step 3: Add the equations together.	$6x + 10y = -12$ $\underline{-6x + 21y = 12}$ $31y = 0$
Step 4: Solve	$\frac{31y}{31} = \frac{0}{31}$ $y = 0$
Step 5: Substitute the value into one of the original equations and solve.	$3x + 5y = -6$ $3x + 5(0) = -6$ $3x = -6$ $\frac{3x}{3} = \frac{-6}{3}$ $x = -2$
Solution	$(-2, 0)$
Step 6: Check	$3x + 5y = -6$ $-2x + 7y = 4$ $3(-2) + 5(0) = -6$ $-2(-2) + 7(0) = 4$ $-6 = -6$ $4 = 4$

NOTE: To solve this system, for step 2, we could have multiplied the first equation by -7 and the second equation by 5 and we would have obtained the same result.

2.2 Systems of Linear Equations in Three Variables

- In this section we just used the elimination method for systems with three equations and three unknowns. Please refer to the textbook for this section or just go look at the Matrix Algebra section of this review. I cover the three methods we learned to solving systems of equations

2.3 Systems of Linear Equations and Matrices

Matrix:

- a *matrix* is a rectangular array of numbers
- the *dimension* of a matrix is the number of rows (m) by the number columns (n). A general matrix with m rows and n columns has dimension $m \times n$.

  Dimension: (# of Rows) $\times$ (# of Columns)
- matrices allow us to succinctly represent a large amount of information

Augmented Matrix:

- In order to solve a linear system using matrices we write a system of equations as an *augmented matrix*
- See the picture below to see how we write a system of equations as an augmented matrix:

system
$$\begin{cases} x+4y+3z=12 \\ 2x-3y-z=20 \\ 4x+5y=15 \end{cases}$$

augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 12 \\ 2 & -3 & -1 & 20 \\ 4 & 5 & 0 & 15 \end{array}\right]$$

Now we write the system as an augmented matrix to help solve the system for x, y, and z. In this section we learned 2 methods for solving systems of equations:

1. **Gaussian Elimination with Back-Substitution**
2. **Gauss-Jordan Elimination**

- Please refer to the next section for examples on Gaussian Elimination with Back-Substitution and Gauss-Jordan Elimination

2.4 Matrix Algebra

- **Addition:** Given two matrices A, B of the same dimension then the sum $A + B$ is the matrix obtained by adding the corresponding entries in the two matrices

  Example:
$$\begin{bmatrix} 1 & 2 & 0 \\ -5 & 0 & 3 \\ 2 & -3 & -6 \end{bmatrix} + \begin{bmatrix} -3 & 7 & -2 \\ 0 & 4 & -2 \\ 5 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 9 & -2 \\ -5 & 4 & 1 \\ 7 & -4 & -4 \end{bmatrix}$$

- **Subtraction:** Given two matrices A, B of the same dimension then the difference $A - B$ is the matrix obtained by subtracting the corresponding entries in the two matrices

- **Scalar Multiplication:** Given a matrix A and a number c then the product cA is the matrix obtained by multiplying each entry of A by c

  Example:

$$2 \cdot \begin{bmatrix} 1 & 2 & 0 \\ -5 & 0 & 3 \\ 2 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -10 & 0 & 6 \\ 4 & -6 & -12 \end{bmatrix}$$

Matrix Multiplication:

- Given two matrices, A, B, we can also find the matrix product AB, but we need to be careful. Matrix multiplication is NOT ALWAYS possible.
- Given two matrices A, B we can find the matrix product, AB, if the **number of columns of A is equal to the number of rows in B**
- Given a matrix A of dimension $(m \times n)$ and a matrix B of dimension $(n \times k)$ AB will be a matrix of dimension $(m \times k)$
- So if we are multiplying a $({\color{red}m} \times {\color{blue}n})$ matrix A by a $({\color{blue}n} \times {\color{red}k})$ matrix B. The inner numbers (the numbers in blue) will tell you whether or not matrix multiplication is possible. If it is possible, the outer numbers (numbers in red) tell you the dimension of the matrix AB

  Example:

$$\text{A} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 1 & 4 \\ 3 & 0 & -3 \\ 2 & -4 & -6 \end{bmatrix} \qquad \text{B} = \begin{bmatrix} -1 & 2 & 0 \\ -5 & 2 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\text{AB} = \begin{bmatrix} 1(-1)+3(-5)+-2(0) & 1(2)+3(2)+-2(-1) & 1(0)+3(3)+-2(-3) \\ -1(-1)+1(-5)+4(0) & -1(2)+1(2)+4(-1) & -1(0)+1(3)+4(-3) \\ 3(-1)+0(-5)+-3(0) & 3(2)+0(2)+-3(-1) & 3(0)+0(3)+-3(-3) \\ 2(-1)+-4(-5)+-6(0) & 2(2)+-4(2)+-6(-1) & 2(0)+-4(3)+-6(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 10 & 15 \\ -4 & -4 & -9 \\ -3 & 9 & 9 \\ 18 & 2 & 6 \end{bmatrix}$$

Note: To find the entry in the **first row**, **first column** of AB we multiply the corresponding entries of the **first row of A** by the **first column of B** and add them all up.

To find the entry in the **first row**, **second column** of AB we multiply the corresponding entries of the **first row of A** by the **second column of B** and add them all up

We can continue on in this fashion:

i.e. To find the...

First row, **third column** of AB → multiply the **first row of A** by the **third column of B** and add

First row, **fourth column** of AB → multiply the **first row of A** by the **fourth column of B** and add

Second row, **first column** of AB → multiply the **second row of A** by the **first column of B** and add

etc...

Identity Matrix:

- the **Identity Matrix** is a special type of matrix with $1's$ along the main diagonal and zeros everywhere else
- it is always a square matrix (the number of rows = number of columns of the identity matrix)
- given a matrix A and the identity matrix I, the product of $AI = A$ (provided they are of appropriate dimension). So to give an analogy, multiplying a matrix by the identity matrix is like multiplying a number times 1

3x3 Identity Matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Linear Systems

- We learned how to solve two types of linear systems:
  1. 2 equations and 2 unknowns

$$a_{1,1}x + a_{1,2}y = b_1$$
$$a_{2,1}x + a_{2,2}y = b_2$$

     Note: These are merely equations of lines. (Any two points x and y determine a line)

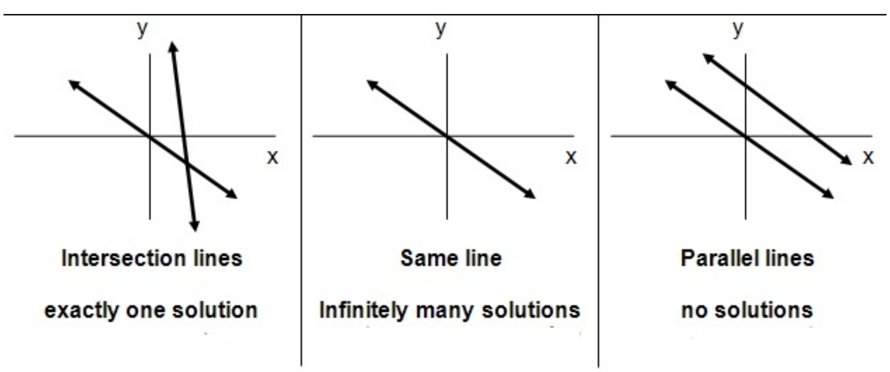
  2. 3 equations and 3 unknowns:

$$a_{1,1}x + a_{1,2}y + a_{1,3}z = b_1$$
$$a_{2,1}x + a_{2,2}y + a_{2,3}z = b_2$$
$$a_{3,1}x + a_{3,2}y + a_{3,3}z = b_3$$

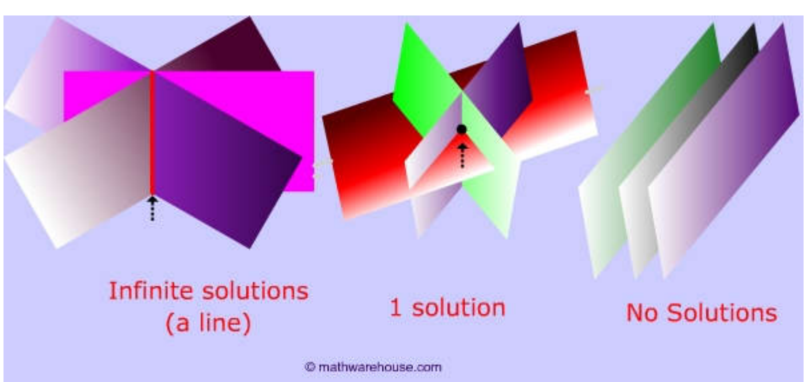
     Note: These are equations of planes. (Any three points x, y, and z determine a plane)

- For a (2 by 2) or a (3 by 3) system the solution will be one of three possibilities:
  1. There will be one and only one solution
  2. There will be an infinite number of solutions
  3. There will be no solution

  Visually, this is what is going on in the (2 by 2) case:



  Visually, this is what is going on in the (3 by 3) case:



- to solve a linear system we learned three methods:
  1. **Gaussian Elimination with Back-Substitution**
  2. **Gaussian-Jordan Elimination**
  3. **Inverse Matrix Method**

  All methods are very similar and they involve augmented matrices and performing row operations on the augmented matrix

Row Operations

1. Interchange any two rows
2. Multiply any row by a nonzero number
3. Add or subtract a multiple of any row to another row

Example of Gaussian Elimination with Back-Substitution:

The system of equations $\begin{cases} x+y+z=3 \\ 2x+3y+7z=0 \\ x+3y-2z=17 \end{cases}$ has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in row-echelon form as shown below

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system has become $\begin{cases} x+y+z=3 \\ y+5z=-6 \\ z=-2 \end{cases}$. By back sunstitution

we find that $x=1$, $y=4$, and $z=-2$.

Example of Gaussian-Jordan Elimination:

Example: The system of equations $\begin{cases} x+y+z=3 \\ 2x+3y+7z=0 \\ x+3y-2z=17 \end{cases}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The augmented matrix now says that $x=1$, $y=4$, and $z=-2$.

Inverse Matrix Method:

- Given a linear system of 3 equations and 3 unknowns:

$$a_{1,1}x + a_{1,2}y + a_{1,3}z = b_1$$
$$a_{2,1}x + a_{2,2}y + a_{2,3}z = b_2$$
$$a_{3,1}x + a_{3,2}y + a_{3,3}z = b_3$$

- we can write this in **matrix form** $AX = B$:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3.2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- to solve this we want to find the inverse matrix of A which we denote by A^{-1}. Once we have A^{-1} we multiply $A^{-1}B$ which gives us X.
- the general idea is to start with an augmented matrix with the matrix of coefficients (A) on the left and the (3×3) Identity matrix on the right:

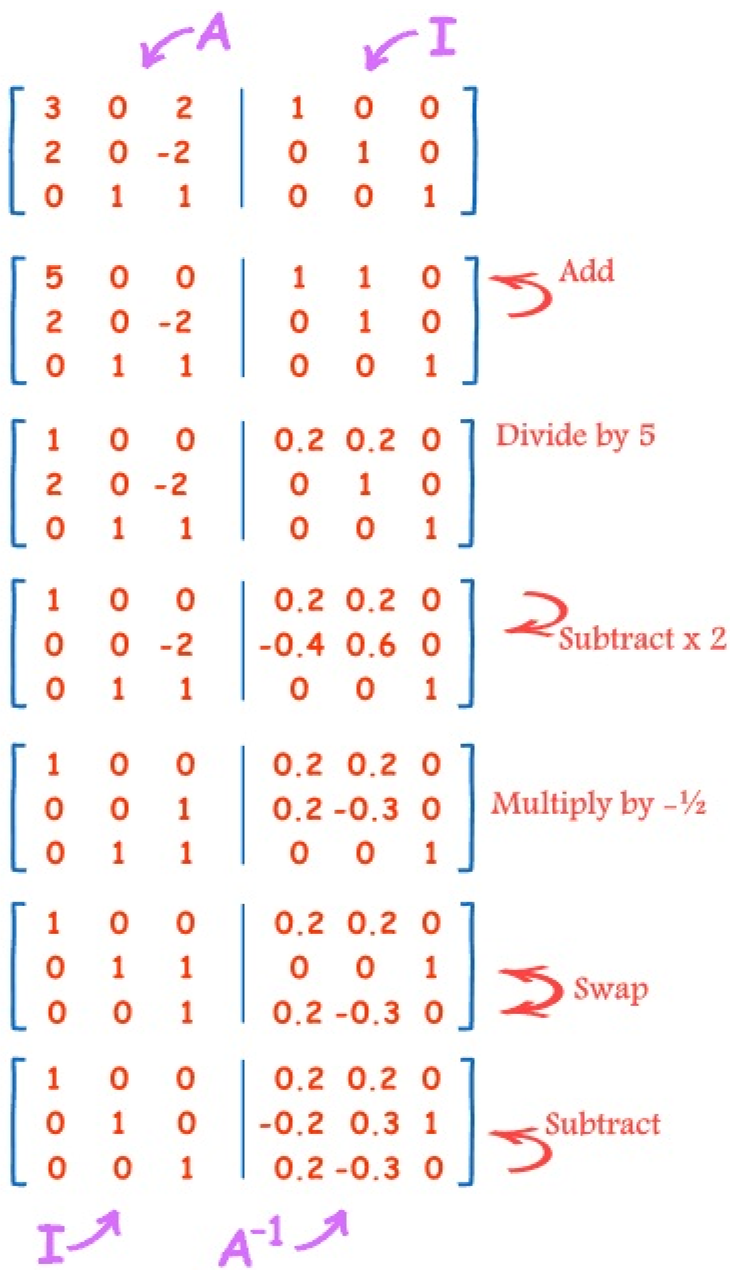
$$\left[\begin{array}{ccc|ccc} a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \end{array}\right]$$

and then perform a bunch of row operations to get it in the form:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{1,1}^{-1} & a_{1,2}^{-1} & a_{1,3}^{-1} \\ 0 & 1 & 0 & a_{2,1}^{-1} & a_{2,2}^{-1} & a_{2,3}^{-1} \\ 0 & 0 & 1 & a_{3,1}^{-1} & a_{3,2}^{-1} & a_{3,3}^{-1} \end{array}\right]$$

where we now have the (3×3) Identity matrix on the left and the inverse matrix (A^{-1}) is on the right. See the following example of how to find the inverse of a matrix A.

Example of Finding Matrix Inverse:



Internet Links:

- Matrix Inverse Method Example
- PatrickJMT 2x2 Matrix Inverse Method Example
- PatrickJMT 3x3 Matrix Inverse Method Example

Formula for Inverse of a 2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.5 The Determinant of a Square Matrix and Cramer's Rule

Determinant of a 2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Det}(A) = |A| = ad - bc$$

Determinant of a 3x3 Matrix: (Expanded along Row 1)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Determinant of a 4x4 Matrix: (Expanded along Row 1)

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Example of Finding the Determinant of a 2x2 Matrix:

$$D=\begin{bmatrix}4 & -3\\ 6 & 5\end{bmatrix}\xrightarrow{\text{find its determinant}}|D|=\begin{vmatrix}4 & -3\\ 6 & 5\end{vmatrix}=(4)(5)-(-3)(6)$$

$$=20-(-18)$$

$$=20+18$$

$$=38$$

Example of Finding the Determinant of a 3x3 Matrix:

Example #5: Find the determinant of the given 3x3 matrix

$$\begin{bmatrix}5 & 3 & -4\\ 2 & 0 & -2\\ 2 & 5 & -1\end{bmatrix}$$

Solution:

$$\det\begin{bmatrix}5 & 3 & -4\\ 2 & 0 & -2\\ 2 & 5 & -1\end{bmatrix} = 5\det\begin{bmatrix}0 & -2\\ 5 & -1\end{bmatrix} - 3\det\begin{bmatrix}2 & -2\\ 2 & -1\end{bmatrix} + (-4)\det\begin{bmatrix}2 & 0\\ 2 & 5\end{bmatrix}$$

$$\begin{bmatrix}5 & 3 & -4\\ 2 & 0 & -2\\ 2 & 5 & -1\end{bmatrix}\quad\begin{bmatrix}5 & 3 & -4\\ 2 & 0 & -2\\ 2 & 5 & -1\end{bmatrix}\quad\begin{bmatrix}5 & 3 & -4\\ 2 & 0 & -2\\ 2 & 5 & -1\end{bmatrix}$$

$$= 5\left[0-(-10)\right]-3\left[-2-(-4)\right]-4\left[10-0\right]$$

$$= 5(10)-3(2)-4(10)$$

$$= 50-6-40$$

$$= \boxed{4}$$

Example 2 of Finding the Determinant of a 3x3 Matrix:

Example #2: Find the determinant of the given 3x3 matrix

$$\begin{bmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution:

$$\det \begin{bmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix} = -4 \det \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} - 5 \det \begin{bmatrix} -3 & 2 \\ -1 & 5 \end{bmatrix} + 2 \det \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

$$= -4\left[20-4\right] - 5\left[-15-(-2)\right] + 2\left[-6-(-4)\right]$$

$$= -4(16) - 5(-13) + 2(-2)$$

$$= \boxed{-3}$$

Cramer's Rule:

- Another method for solving systems of equations
- We learned Cramer's Rule for 2x2 matrices and for 3x3 matrices

Cramer's Rule for 2x2 Matrices:

Cramer's Rules for the Systems of Linear Equations with Two Variables	
Given a linear system	x-column constant column $a_1 x + b_1 y = c_1$ $a_2 x + b_2 y = c_2$ y-column
Assign names for each matrix	**coefficient matrix** $D = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ **X - matrix** $D_x = \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}$ **Y - matrix** $D_y = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}$
Solve variable x by	$x = \dfrac{D_x}{D} = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$
Solve variable y by	$y = \dfrac{D_y}{D} = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

Cramer's Rule for 3x3 Matrices:

Cramer's Rules for the Systems of Linear Equations with Three Variables	
Given a linear system	x-column, z-column $\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$ y-column, constant-column
Labeling each of the four matrices	**coefficient matrix** $D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ **X - matrix** $D_x = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$ **Y - matrix** $D_y = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$ **Z - matrix** $D_z = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$

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To solve for x:	$x=\dfrac{\lvert D_x\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$
To solve for y:	$y=\dfrac{\lvert D_y\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$
To solve for z:	$z=\dfrac{\lvert D_z\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$

Internet Links:

- Guru Cramer's Rule 2x2 Example
- Cramer's Rule for 2x2 and 3x3 Matrices Examples and Solutions
- PatrickJMT Cramer's Rule 3x3 Example 1
- PatrickJMT Cramer's Rule 3x3 Example 2