# Solving Trig Equations

A lot of students made mistakes solving trig equations so I'll work through a few examples here and then give you some to try on your own.

**Example 1: Find all $t$ in the interval $[0, 2\pi]$ satisfying**

$$\cos^2(t) - 5\cos(t) - 6 = 0$$

This looks very similar to a quadratic equation and indeed it is if we let $x = \cos(t)$ then we get the following:

$$x^2 - 5x - 6 = 0$$

So we can solve this using the same ways we already know. There are three ways we can solve a quadratic equation. They are

1. Factoring
2. Quadratic Formula
3. Completing the Square

We can factor this so let's just do that...

$$x^2 - 5x - 6 = (x - 6)(x + 1) = 0$$

And remember we let $x = \cos(t)$ so substitute $\cos(t)$ in for $x$ to get...

$$\Big(\cos(t) - 6\Big)\Big(\cos(t) + 1\Big) = 0$$

So we set both terms equal to 0 and solve for $t$ to get

$$\cos(t) = 6 \qquad \text{AND} \qquad \cos(t) = -1$$

Well there is **no solution** to $\cos(t) = 6$, but for $\cos(t) = -1$ in the interval $[0, 2\pi]$ we get $t = \pi$ as our only solution and we are done.

**Example 2: Find all values of $t$ in the interval $[0, 2\pi]$ satisfying the given equation:**

$$\Big(6\cot(t)\Big)^2 = 108$$

First, we square the left hand side to get...

$$36\cot^2(t) = 108$$

Then we divided by 36 on both sides to get...

$$\cot^2(t) = 3$$

Then write $\cot(t)$ in terms of both $\sin(t)$ and $\cos(t)$ to get

$$\frac{\cos^2(t)}{\sin^2(t)} = 3$$

Multiply across by $\sin^2(t)$ to get

$$\cos^2(t) = 3\sin^2(t)$$

And then using an identity for $\sin^2(t) = 1 - \cos^2(t)$ we get...

$$\cos^2(t) = 3\Big(1 - \cos^2(t)\Big)$$

Now we distribute the 3 to get...

$$\cos^2(t) = 3 - 3\cos^2(t)$$

Combining like terms we get

$$4\cos^2(t) = 3$$

Divide by 4 on both sides to get...

$$\cos^2(t) = \frac{3}{4}$$

Now take the square root of both sides to get...

$$\cos(t) = \pm\frac{\sqrt{3}}{2}$$

So now we ask ourselves when is $\cos(t)$ equal to $\pm\frac{\sqrt{3}}{2}$?

We will find that $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ and these are all of our $t$ in the interval $[0, 2\pi]$ so those are our answers for $t$ and we are done.

**Example 3: Find all values of $t$ in the interval $[0, 2\pi]$ satisfying the given equation:**

$$4\sin(2t) - 2\tan(2t) = 0$$

Before we begin, let's note that $2t$ is in the interval from $[0, 4\pi]$ since it is given that $0 < t < 2\pi$ then this implies that $0 < 2t < 4\pi$. So at the end you will see that we have a long list of answers for $t$ because we must solve not for $t$ in $[0, 2\pi]$, but we must solve for $2t$ in $[0, 4\pi]$. It's a subtle aspect of this problem that can easily go overlooked.

Rewriting $\tan(2t)$ in terms of $\sin(2t)$ and $\cos(2t)$ we get...

$$4\sin(2t) - 2\frac{\sin(2t)}{\cos(2t)} = 0$$

Multiplying across the entire equation by $\cos(2t)$ to get rid of the denominator we get...

$$4\sin(2t)\cos(2t) - 2\sin(2t) = 0$$

Now we see that $\sin(2t)$ is common to both terms so we factor that out to get...

$$\sin(2t)\Big(4\cos(2t) - 2\Big) = 0$$

So we set both terms in the above product equal to 0 to get

$$\sin(2t) = 0 \qquad \text{AND} \qquad \cos(2t) = \frac{1}{2}$$

So we let $x = 2t$ and ask ourselves two questions:

1. When is $\sin(x) = 0$ in $[0, 4\pi]$?

2. When is $\cos(x) = \dfrac{1}{2}$ in $[0, 4\pi]$?

**(1.)** We know $\sin(x) = 0$ for $x = 0, \pi, 2\pi, 3\pi, 4\pi$ for $x$ in $[0, 4\pi]$

Now since $x = 2t$ then we know $2t = 0, \pi, 2\pi, 3\pi, 4\pi$

So $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

**(2.)** We know $\cos(x) = \dfrac{1}{2}$ for $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$ for $x$ in $[0, 4\pi]$

Now since $x = 2t$ then we know $2t = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$

So $t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

So our answers for $t$ are...

**Solution:** $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

**Example 4: Find all values of $x$ in the interval $[0, 2\pi]$ that satisfy the given equation:**

$$\sin(2x) = \sqrt{3}\cos(x)$$

We know $\sin(2x) = 2\sin(x)\cos(x)$ because it's one of the trig formulas we went over in class so let's replace $\sin(2x)$ with $2\sin(x)\cos(x)$ in the above equation to get...

$$2\sin(x)\cos(x) = \sqrt{3}\cos(x)$$

Now we subtract $\sqrt{3}\cos(x)$ over to the left hand side to get...

$$2\sin(x)\cos(x) - \sqrt{3}\cos(x) = 0$$

And we notice that $\cos(x)$ is common to both terms so let's factor that out to get...

$$\cos(x)\left(2\sin(x) - \sqrt{3}\right) = 0$$

And now we set both terms in the product equal to 0 to get

$$\cos(x) = 0 \qquad \text{AND} \qquad \sin(x) = \frac{\sqrt{3}}{2}$$

So we ask ourselves two questions:

1. When is $\cos(x) = 0$ for $x$ in $[0, 2\pi]$?
2. When is $\sin(x) = \dfrac{\sqrt{3}}{2}$ for $x$ in $[0, 2\pi]$?

**(1.)** We know that $\cos(x) = 0$ for $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ for $x$ in $[0, 2\pi]$

**(2.)** We know that $\sin(x) = \dfrac{\sqrt{3}}{2}$ for $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ for $x$ in $[0, 2\pi]$

So our answers for $x$ are $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ and we are done.

**Example 5: Find all values of $x$ in the interval $[0, 2\pi]$ that satisfy the given equation:**

$$2\sin^2(x) - \cos(x) - 1 = 0$$

We know $\sin^2(x) = 1 - \cos^2(x)$ so we replace $\sin^2(x)$ with $1 - \cos^2(x)$ in the above equation to get...

$$2\Big(1 - \cos^2(x)\Big) - \cos(x) - 1 = 0$$

Distributing the 2 we get...

$$2 - 2\cos^2(x) - \cos(x) - 1 = 0$$

And we combine like terms to get...

$$-2\cos^2(x) - \cos(x) + 1 = 0$$

This looks just like a quadratic equation and indeed it is if we let $y = \cos(x)$ to get...

$$-2y^2 - y + 1 = 0$$

So we can solve this using the same ways we already know. There are three ways we can solve a quadratic equation. They are

1. Factoring
2. Quadratic Formula
3. Completing the Square

I factored in Example 1 so let's use the quadratic formula here

So then we have that

$$\begin{aligned} y = \cos(x) &= \frac{1 \pm \sqrt{1-(4)(-2)(1)}}{2(-2)} \\ &= \frac{1 \pm \sqrt{9}}{-4} \\ &= \frac{1 \pm 3}{-4} \\ &= -1, \frac{1}{2} \end{aligned}$$

So we solve these two equations for $x$ in $[0, 2\pi]$:

$$\cos(x) = -1 \qquad \text{AND} \qquad \cos(x) = \frac{1}{2}$$

So our solutions for $x$ is $x = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$ and we are done.

**Try Some Yourself:**

1. $\cos^2(t) + 9\cos(t) + 8 = 0$
2. $2\sin^2(t) - 3\sin(t) + 1 = 0$
3. $18\tan(t) - 18 = 0$
4. $\Big(3\cot(t)\Big)^2 = 27$
5. $2\sqrt{3}\sin(2t) - \sqrt{3}\tan(2t) = 0$
6. $-\sin(2x) = \sqrt{2}\cos(x)$
7. $2\sin^2(x) + 3\cos(x) - 3 = 0$