Final Exam Study Guide

Final Exam Date & Location: Thursday May 15 8:00pm - 10:00pm MacBride Auditorium

NOTE: The final exam is **CUMULATIVE**. Make sure you study everything we've done this semester and not just the material since the previous midterm.

1 Matrices & Solving Linear Systems

Matrix:

- a *matrix* is a rectangular array of numbers
- the *dimension* of a matrix is the number of rows (m) by the number columns (n). A general matrix with m rows and n columns has dimension $m \times n$. **Dimension:** (# of Rows) $\times$ (# of Columns)
- matrices allow us to succinctly represent a large amount of information

Matrix Algebra:

- **Addition:** Given two matrices A, B of the same dimension then the sum $A + B$ is the matrix obtained by adding the corresponding entries in the two matrices

  Example:

$$\begin{bmatrix} 1 & 2 & 0 \\ -5 & 0 & 3 \\ 2 & -3 & -6 \end{bmatrix} + \begin{bmatrix} -3 & 7 & -2 \\ 0 & 4 & -2 \\ 5 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 9 & -2 \\ -5 & 4 & 1 \\ 7 & -4 & -4 \end{bmatrix}$$

- **Subtraction:** Given two matrices A, B of the same dimension then the difference $A - B$ is the matrix obtained by subtracting the corresponding entries in the two matrices
- **Scalar Multiplication:** Given a matrix A and a number c then the product cA is the matrix obtained by multiplying each entry of A by c

  Example:

  $$2 \cdot \begin{bmatrix} 1 & 2 & 0 \\ -5 & 0 & 3 \\ 2 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -10 & 0 & 6 \\ 4 & -6 & -12 \end{bmatrix}$$

Matrix Multiplication:

- Given two matrices, A, B, we can also find the matrix product AB, but we need to be careful. Matrix multiplication is NOT ALWAYS possible.
- Given two matrices A, B we can find the matrix product, AB, if the **number of columns of A is equal to the number of rows in B**
- Given a matrix A of dimension $(m \times n)$ and a matrix B of dimension $(n \times k)$ AB will be a matrix of dimension $(m \times k)$
- So if we are multiplying a $(m \times n)$ matrix A by a $(n \times k)$ matrix B. The inner numbers (the numbers in blue) will tell you whether or not matrix multiplication is possible. If it is possible, the outer numbers (numbers in red) tell you the dimension of the matrix AB

  Example:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 1 & 4 \\ 3 & 0 & -3 \\ 2 & -4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 0 \\ -5 & 2 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(-1)+3(-5)+-2(0) & 1(2)+3(2)+-2(-1) & 1(0)+3(3)+-2(-3) \\ -1(-1)+1(-5)+4(0) & -1(2)+1(2)+4(-1) & -1(0)+1(3)+4(-3) \\ 3(-1)+0(-5)+-3(0) & 3(2)+0(2)+-3(-1) & 3(0)+0(3)+-3(-3) \\ 2(-1)+-4(-5)+-6(0) & 2(2)+-4(2)+-6(-1) & 2(0)+-4(3)+-6(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 10 & 15 \\ -4 & -4 & -9 \\ -3 & 9 & 9 \\ 18 & 2 & 6 \end{bmatrix}$$

Note: To find the entry in the **first row**, **first column** of AB we multiply the corresponding entries of the **first row of A** by the **first column of B** and add them all up.

To find the entry in the **first row**, **second column** of AB we multiply the corresponding entries of the **first row of A** by the **second column of B** and add them all up

We can continue on in this fashion:

i.e. To find the...

First row, **third column** of AB $\rightarrow$ multiply the **first row of A** by the **third column of B** and add

First row, **fourth column** of AB $\rightarrow$ multiply the **first row of A** by the **fourth column of B** and add

Second row, **first column** of AB $\rightarrow$ multiply the **second row of A** by the **first column of B** and add

etc...

Identity Matrix:

- the **Identity Matrix** is a special type of matrix with $1's$ along the main diagonal and zeros everywhere else
- it is always a square matrix (the number of rows = number of columns of the identity matrix)
- given a matrix A and the identity matrix I, the product of $AI = A$ (provided they are of appropriate dimension). So to give an analogy, multiplying a matrix by the identity matrix is like multiplying a number times 1

3x3 Identity Matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Linear Systems

- We learned how to solve two types of linear systems:

  1. 2 equations and 2 unknowns

$$a_{1,1}x + a_{1,2}y = b_1$$

$$a_{2,1}x + a_{2,2}y = b_2$$

  Note: These are merely equations of lines. (Any two points x and y determine a line)

2. 3 equations and 3 unknowns:

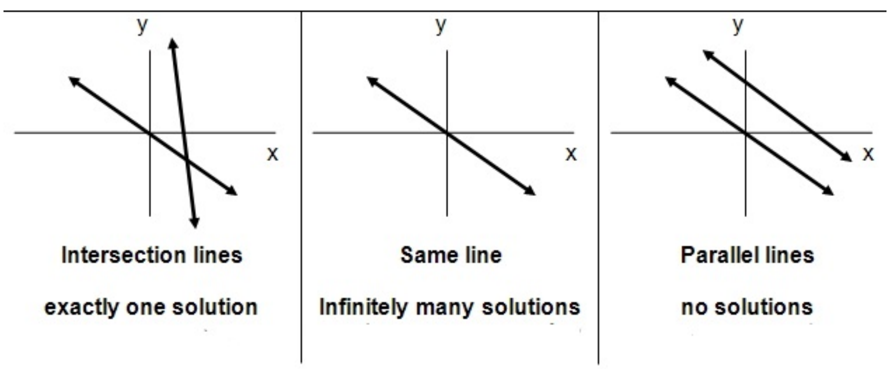
$$a_{1,1}x + a_{1,2}y + a_{1,3}z = b_1$$
$$a_{2,1}x + a_{2,2}y + a_{2,3}z = b_2$$
$$a_{3,1}x + a_{3,2}y + a_{3,3}z = b_3$$

   Note: These are equations of planes. (Any three points x, y, and z determine a plane)

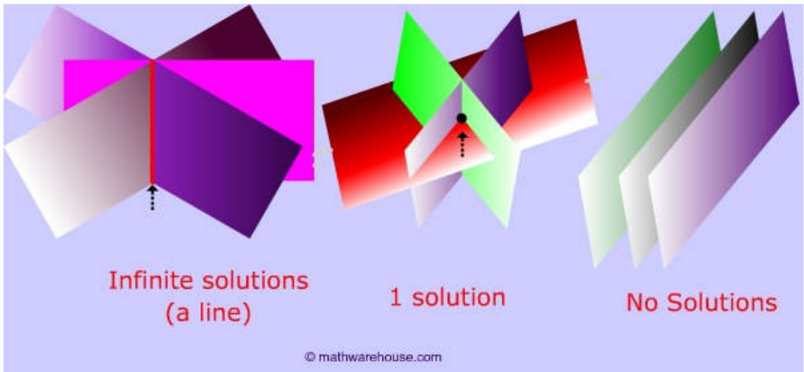
- For a (2 by 2) or a (3 by 3) system the solution will be one of three possibilities:

  1. There will be one and only one solution
  2. There will be an infinite number of solutions
  3. There will be no solution

  Visually, this is what is going on in the (2 by 2) case:



  Visually, this is what is going on in the (3 by 3) case:



- to solve a linear system we learned two methods:
  1. **Gaussian Elimination**
  2. **Inverse Matrix Method**

  Both methods are very similar and they involve augmented matrices and performing row operations on the augmented matrix

Row Operations

1. Interchange any two rows
2. Multiply any row by a nonzero number
3. Add or subtract a multiple of any row to another row

Example of Gaussian Elimination:

The system of equations $\begin{cases} x+y+z=3 \\ 2x+3y+7z=0 \\ x+3y-2z=17 \end{cases}$ has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in row-echelon form as shown below

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system has become $\begin{cases} x+y+z=3 \\ y+5z=-6 \\ z=-2 \end{cases}$. By back sunstitution

we find that $x=1$, $y=4$, and $z=-2$.

Inverse Matrix Method:

- Given a linear system of 3 equations and 3 unknowns:

$$a_{1,1}x + a_{1,2}y + a_{1,3}z = b_1$$

$$a_{2,1}x + a_{2,2}y + a_{2,3}z = b_2$$

$$a_{3,1}x + a_{3,2}y + a_{3,3}z = b_3$$

- we can write this in **matrix form** $AX = B$:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ \\ a_{2,1} & a_{2,2} & a_{2,3} \\ \\ a_{3,1} & a_{3.2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x \\ \\ y \\ \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ b_2 \\ \\ b_3 \end{bmatrix}$$

- to solve this we want to find the inverse matrix of A which we denote by A^{-1}. Once we have A^{-1} we multiply $A^{-1}B$ which gives us X.

- the general idea is to start with an augmented matrix with the matrix of coefficients (A) on the left and the (3×3) Identity matrix on the right:

$$\left[\begin{array}{ccc|ccc} a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \end{array}\right]$$

  and then perform a bunch of row operations to get it in the form:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{1,1}^{-1} & a_{1,2}^{-1} & a_{1,3}^{-1} \\ \\ 0 & 1 & 0 & a_{2,1}^{-1} & a_{2,2}^{-1} & a_{2,3}^{-1} \\ \\ 0 & 0 & 1 & a_{3,1}^{-1} & a_{3,2}^{-1} & a_{3,3}^{-1} \end{array}\right]$$

  where we now have the (3×3) Identity matrix on the left and the inverse matrix (A^{-1}) is on the right. See the following page for an example of how to find the inverse of a matrix A.

Example of Inverse Matrix Method:

$$\begin{array}{c} \begin{array}{cc} A & I \end{array} \\ \left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Add}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Divide by 5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Subtract x 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Multiply by } -\tfrac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \quad \text{Swap}$$

$$\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \quad \text{Subtract} \\ \begin{array}{cc} I & A^{-1} \end{array} \end{array}$$

2 Set Theory & Probability Theory

Sets

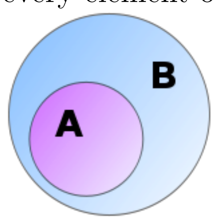
- a *set* is a collection of objects
- the **empty set** is the set containing no elements and is denoted as $\emptyset$

Subsets

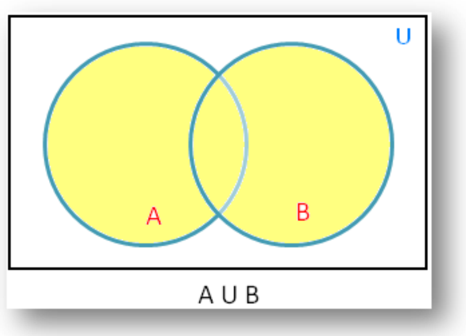
- a set A is a *subset* of a set B if every element of A is also an element of B
- if A is a subset of B then we denote it by $A \subseteq B$

Example: Let $A = \{a, c, e, g\}$ and $B = \{a, b, c, d, e, f, g\}$

Then $A \subseteq B$ since every element of A is also in B

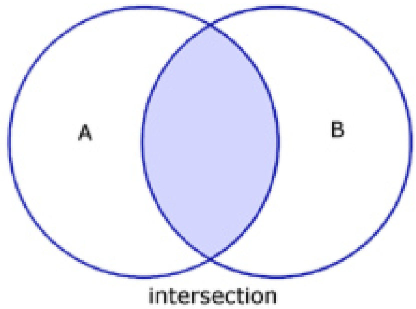


Union - given two sets A, B, the *union* is the collection of elements in both A **or** B denoted by $A \cup B$



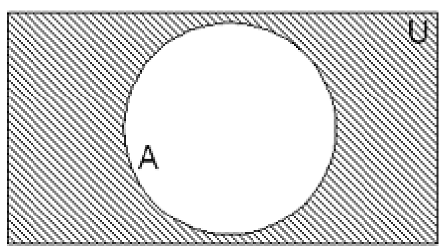
Note: Same definition holds for two events, E & F, in a sample space S

Intersection - given two sets A, B, the *intersection* is the collection of elements in both A **and** B denoted by $A \cap B$



Note: The same definition holds for two events, E & F, in a sample space S

Complement - given a set A the *complement* of A is the collection of elements **not in** A denoted by A^c



Note: The same definition holds for an event, E, in a sample space S

Properties of Set Operations:

1. $A \cup B = B \cup A$ (Union is commutative)
2. $A \cap B = B \cap A$ (Intersection commutative)
3. $A \cup (B \cup C) = (A \cup B) \cup C$ (Union is associative)
4. $A \cap (B \cap C) = (A \cap B) \cap C$ (Intersection is associative)

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law for Union)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law for Intersection)

De Morgan's Laws:

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Generalized Multiplication Principle:

Suppose a task T_1 can be performed in N_1 ways, a task T_2 can be performed in N_2 ways, a task T_3 can be performed in N_3 ways,..., and, finally, a task T_m can be performed in N_m ways. Then the number of ways of performing the tasks $T_1, T_2, T_3, ..., T_m$ in succession is given by:

$$N_1 \cdot N_2 \cdot N_3 \cdot \cdots \cdot N_m$$

n-Factorial: For any natural number n, the *n-factorial* is given by

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1$$
$$\text{with } 0! = 1$$

Permutation of n Distinct Objects:

The number of permutations of n objects taken r at a time is given by:

$$P(n, r) = \frac{n!}{(n-r)!}$$

With permutations, **ORDER MATTERS**

Permutation of n Objects (Not all Distinct):

Given a set of n objects in which n_1 objects of one kind are alike, n_2 objects of another kind are alike, n_3 objects of another kind are alike,...,n_m objects of another kind are alike then the number of permutations of these n taken n at a time is:

$$P(n, n_1, n_2, n_3, ..., n_m) = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_m!}$$

Combination of n Distinct Objects:

The number of combinations of n objects taken r at a time is given by:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

With combinations, **ORDER DOESN'T MATTER**

Probability of Simple Events:

Let S be a sample space consisting of the following **simple** events:

$$S = \{s_1, s_2, s_3, \ldots s_n\}$$

Then the following three conditions hold

1. $0 \leq P(s_i) \leq 1$ for $i = 1, 2, 3 \cdots, n$
2. $P(s_1) + P(s_2) + P(s_3) + \cdots + P(s_n) = 1$
3. $P(s_i \cup s_j) = P(s_i) + P(s_j)$ for $i \neq j$

   Note: The first two conditions ALWAYS hold for any sample space S. The probability of any one event will ALWAYS be between 0 and 1 (Condition 1) and the probability of the entire sample space will ALWAYS be equal to 1 (Condition 2).

   The last condition (Condition 3) only holds because the events are **simple** events which means they are **mutually exclusive** in that given two events of the sample space S only one can occur at a time. If two events, E & F, are mutually exclusive then $P(E \cap F) = 0$. This is not always the case.

Mutually Exclusive Events - two events E, F are *mutually exclusive* if they cannot occur simultaneously. In other words, this means $P(E \cap F) = 0$

Probability of an Event in Uniform Space - if S is a uniform sample space, then the probability that an event E will happen is given by

$$P(E) = \frac{\text{Number of Outcomes in E}}{\text{Total Number of Outcomes in S}}$$

Let's do some examples to highlight what this all means:

Example 1: Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ be the sample space associated with the experiment having the following probability distribution of simple events:

Outcome	s_1	s_2	s_3	s_4	s_5	s_6
Probability	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{12}$

(a) Find the probability of $A = \{s_1, s_3\}$

Since these are **simple events** which means they are **mutually exclusive** then the probability that A occurs is given by

$$P(A) = P(s_1 \cup s_3) = P(s_1) + P(s_3) = \frac{1}{12} + \frac{4}{12} = \frac{5}{12}$$

(b) Find the probability of $B = \{s_2, s_4, s_5, s_6\}$

Again these are simply events so that $P(B) = P(s_2) + P(s_4) + P(s_5) + P(s_6) = \frac{2}{12} + \frac{1}{12} + \frac{1}{12} + \frac{3}{12} = \frac{7}{12}$

(c) Find the probability of $C = S$.

Now S is the whole sample space and we know the probability of the entire sample space is ALWAYS equal to 1 so $P(C) = P(S) = 1$

Example 2: If a ball is selected at random from an urn containing six red balls, four white balls, and ten blue balls, what is the probability that it will be a white ball?

Answer: There are a total of 20 balls in the urn and there are

4 white balls in the bag so we know the probability of choosing a white ball out of the urn is given by the number of ways we can pick a white ball (4) over the total number of balls in the urn (20) so...

$P(\text{Choosing a White Ball}) = \frac{4}{20} = \frac{1}{5} = 0.2$

Conditional Probability: If A and B are events in an experiment and $P(A) \neq 0$ then the *conditional probabiity* that the event B will occur given that the event A has already occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Probability Rules:

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (**Addition Rule**)
2. $P(A^c) = 1 - P(A)$ (**Rule of Complements**)
3. $P(A \cap B) = P(A) \cdot P(B|A)$ (**Product Rule**)

Note: These rules ALWAYS hold for any events A and B whether they are mutually exclusive or not. Notice that for the addition rule if A and B are mutually exclusive, then $P(A \cap B) = 0$ and we get that $P(A \cup B) = P(A) + P(B)$.

Independent Events - two events A , B are said to be *independent* if the outcome of one does not affect the outcome of the other. Independence of events is equivalent to the following:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A) \cdot P(B)$

   Note: If $A_1, A_2, A_3, ..., A_n$ are all independent events then the following is true:

   $P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \cdots \cdot P(A_n)$