+2 statement

Cauchy-Schwarz Inequality. Suppose $u, v \in V$. Then $|\langle u,v\rangle| \leq \|u\| \|v\|$ with equality $\iff$ $u = \lambda v$ for $\lambda \in \mathbb{F}$.

Over $\mathbb{C}^n$: Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be complex numbers. Then $\left|\sum_{i=1}^n a_i b_i\right|^2 \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2$. Why?

$$\underset{\shortparallel}{(a_1, \dots, a_n)} \quad \underset{\shortparallel}{(b_1, \dots, b_n)}$$

Claim: Inner product over $\mathbb{R}^n$: $\vec{a} \cdot \vec{b} = (a_1 b_1, \dots, a_n b_n)$

Axioms:
1) $\langle \vec{u}, \vec{v}\rangle = \overline{\langle \vec{v}, \vec{u}\rangle}$ (complex conjugate)
2) $\langle c\vec{u} + \vec{v}, \vec{w}\rangle = c\langle \vec{u}, w\rangle + \langle \vec{v}, w\rangle$ (linear in 1st argument)
3) $\langle \vec{u}, \vec{u}\rangle \geq 0$ w/ equality iff $\vec{u} = 0$ (positive definiteness).

Proof: $\vec{a} = \vec{a} \Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

$$(c(\vec{u}) + \vec{v}) \cdot \vec{w} \Rightarrow (cu_1 + v_1, \dots, cu_n + v_n) \cdot (w_1, \dots, w_n)$$
$$= ((cu_1 + v_1)w_1, \dots, (cu_n + v_n)w_n)$$
$$\overset{\text{dist. of } \mathbb{R}^n}{=} (cu_1 w_1 + v_1 w_1, \dots, cu_n w_n + v_n w_n)$$
$$= (cu_1 w_1, \dots, cu_n w_n) + (v_1 w_1, \dots, v_n w_n)$$
$$= c\vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(u_1 u_1, \dots, u_n u_n) = (0, \dots, 0) \iff u_i u_i = 0 \ \forall\, 1 \leq i \leq n$$
$$\iff u_i = 0$$

≥ 0 by pos. of squares. ∎

Lemma. (Orthogonal Decomposition). Suppose $\vec{u}, \vec{v} \in V$, $\vec{v} \neq 0$. Then there exists $c = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{v}\|^2}$ and $\vec{w} = \vec{u} - \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{v}\|^2} v$ such that $\vec{u} = c\vec{v} + \vec{w}$ and $\langle \vec{w}, \vec{v}\rangle = 0$. Don't need to state.

Proof: Let $\vec{u}, \vec{v} \in V$. By closure under scalar multiplication

$$\vec{u} \in V, c \in \mathbb{F} \Rightarrow c\vec{u} \in V.$$ +2 setup

+2 setup

Then $\vec{u} = c\vec{v} + (\vec{u} - c\vec{v})$. We want c such that $\langle \vec{u} - c\vec{v}, \vec{v}\rangle = 0$.
Suppose so. Then

$$\langle \vec{u}, \vec{v}\rangle - c\langle \vec{v}, \vec{v}\rangle = 0$$
$$\Rightarrow \langle \vec{u}, \vec{v}\rangle - c\|\vec{v}\|^2 = 0$$

$$\Rightarrow \langle \vec{u},\vec{v}\rangle = c\|v\|^2.$$

$$\Rightarrow \frac{\langle \vec{u},\vec{v}\rangle}{\|v\|^2} = c.$$ +2 correct

So

$$\vec{u} = \frac{\langle \vec{u},\vec{v}\rangle}{\|v\|^2}\vec{v} + \left(\vec{u} - \frac{\langle \vec{u},\vec{v}\rangle}{\|v\|^2} v\right).$$

And since

$$\left\langle \vec{u} - \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{v}\|^2}\vec{v}, \vec{v} \right\rangle \overset{\text{lin.}}{=} \langle \vec{u},\vec{v}\rangle - \frac{\langle \vec{u},\vec{v}\rangle}{\|v\|^2}\langle \vec{v},\vec{v}\rangle$$

$$= \langle \vec{u},\vec{v}\rangle - \langle \vec{u},\vec{v}\rangle$$

+2 Correct

$$= 0,$$

let $c = \frac{\langle \vec{u},\vec{v}\rangle}{\|v\|^2}$, $W = \vec{u} - \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{v}\|^2}$. Then

$$\vec{u} = c\vec{v} + \vec{w} \text{ and } \langle \vec{w},\vec{v}\rangle = 0. \blacksquare$$

Proof of Cauchy-Schwarz. (Everything from orthogonal decomp.) Then

$$u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$$

+1 Correct

Pythagorean Theorem $\Rightarrow \|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$ +1 Correct

$$= \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\| \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\| + \|w\|^2$$

$$= \left| \frac{\langle u,v\rangle}{\|v\|^2} \right|^2 \|v\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2$$ +2 correct

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\Rightarrow |\langle u,v\rangle|^2 \leq \|u\|^2\|v\|^2$$

$$\Rightarrow |\langle u,v\rangle| \leq \|u\|\|v\|.$$ +2 correct

+2 Correct

Now, equality holds $\Leftrightarrow u = v$. Thus $w = 0$. But $w = 0 \Leftrightarrow u = \lambda v, \lambda \in \mathbb{F}$.

+0 Poor style

+1 good style w/ some unclear spots.

+2 good style