

Cauchy-Schwarz Inequality. Suppose $u, v \in V$. Then $|\langle u, v \rangle| \leq \|u\| \|v\|$ with equality $\Leftrightarrow u = \lambda v$ for $\lambda \in \mathbb{F}$.

Over \mathbb{C} : Let a_1, \dots, a_n and b_1, \dots, b_n be complex numbers. Then $|\sum_{i=1}^n a_i b_i| \leq \sqrt{\sum_{i=1}^n |a_i|^2} \sqrt{\sum_{i=1}^n |b_i|^2}$. Why?

$$(a_1, \dots, a_n) \quad (b_1, \dots, b_n)$$

Claim: Inner product over \mathbb{R}^n : $\vec{a} \cdot \vec{b} = (a_1 b_1, \dots, a_n b_n)$

Axioms: 1) $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ (complex conjugate)

2) $\langle c\vec{u} + \vec{v}, \vec{w} \rangle = c\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (linear in 1st argument)

3) $\langle \vec{u}, \vec{u} \rangle \geq 0$ w/ equality iff $\vec{u} = 0$ (positive definiteness).

Proof: $\vec{a} = \vec{a} \Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

$$(c(\vec{u}) + \vec{v}) \cdot \vec{w} \Rightarrow (cu_1 + v_1, \dots, cu_n + v_n) \cdot (w_1, \dots, w_n)$$

$$= ((cu_1 + v_1)w_1, \dots, (cu_n + v_n)w_n)$$

$$\stackrel{\text{dist. of } \mathbb{R}^n}{=} (cu_1 w_1 + v_1 w_1, \dots, cu_n w_n + v_n w_n)$$

$$= (cu_1 w_1, \dots, cu_n w_n) + (v_1 w_1, \dots, v_n w_n)$$

$$= c\vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(u_1^2, \dots, u_n^2) = (0, \dots, 0) \Leftrightarrow u_i^2 = 0 \quad \forall 1 \leq i \leq n$$

$$\Leftrightarrow u_i = 0$$

> 0 by pos. of squares. \square

Lemma. (Orthogonal Decomposition). Suppose $\vec{u}, \vec{v} \in V$, $\vec{v} \neq 0$. Then there exists $c = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$ and $\vec{w} = \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$ such that $\vec{u} = c\vec{v} + \vec{w}$ and $\langle \vec{w}, \vec{v} \rangle = 0$.

Proof: Let $\vec{u}, \vec{v} \in V$. By closure under scalar multiplication

$$\vec{u} \in V, c \in \mathbb{F} \Rightarrow c\vec{u} \in V.$$

Then $\vec{u} = c\vec{v} + (\vec{u} - c\vec{v})$. We want c such that $\langle \vec{u} - c\vec{v}, \vec{v} \rangle = 0$. Suppose so. Then

$$\langle \vec{u}, \vec{v} \rangle - c\langle \vec{v}, \vec{v} \rangle = 0$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle - c\|\vec{v}\|^2 = 0$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle = c \|\vec{v}\|^2.$$

$$\Rightarrow \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} = c.$$

So

$$\vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} + \left(\vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right).$$

And since

$$\begin{aligned} \left\langle \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}, \vec{v} \right\rangle &\stackrel{\text{lin.}}{=} \langle \vec{u}, \vec{v} \rangle - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \langle \vec{v}, \vec{v} \rangle \\ &= \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle \\ &= 0. \end{aligned}$$

let $c = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$, $\vec{w} = \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$. Then

$$\vec{u} = c\vec{v} + \vec{w} \text{ and } \langle \vec{w}, \vec{v} \rangle = 0.$$

Proof of Cauchy-Schwarz. (Everything from orthogonal decomp.) Then

$$\begin{aligned} \vec{u} &= \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} + \vec{w} \\ \Rightarrow \|\vec{u}\|^2 &= \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\|^2 + \|\vec{w}\|^2 \\ &= \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\| \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\| + \|\vec{w}\|^2 \\ &= \left| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \right|^2 \|\vec{v}\|^2 + \|\vec{w}\|^2 \\ &= \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|\vec{v}\|^2} + \|\vec{w}\|^2 \\ &\geq \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|\vec{v}\|^2} \\ \Rightarrow |\langle \vec{u}, \vec{v} \rangle|^2 &\leq \|\vec{u}\|^2 \|\vec{v}\|^2 \\ \Rightarrow |\langle \vec{u}, \vec{v} \rangle| &\leq \|\vec{u}\| \|\vec{v}\|. \end{aligned}$$

Now, equality holds $\Leftrightarrow \vec{u} = \vec{v}$. Thus $\vec{w} = 0$. But $\vec{w} = 0 \Leftrightarrow \vec{u} = \lambda \vec{v}, \lambda \in \mathbb{F}$.