

Cauchy-Schwarz Inequality. Suppose $u, v \in V$. Then $|\langle u, v \rangle| \leq \|u\| \|v\|$ with equality $\Leftrightarrow u = \lambda v$ for $\lambda \in \mathbb{F}$.

Over \mathbb{C} : Let a_1, \dots, a_n and b_1, \dots, b_n be complex numbers. Then $\left| \sum_{i=1}^n a_i b_i \right|^2 \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2$. Why?

$$(a_1, \dots, a_n) \quad (b_1, \dots, b_n)$$

Claim: Inner product over \mathbb{R}^n : $\vec{a} \cdot \vec{b} = (a_1 b_1, \dots, a_n b_n)$

Axioms: 1) $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ (complex conjugate)

2) $\langle c\vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (linear in 1st argument)

3) $\langle \vec{u}, \vec{u} \rangle \geq 0$ w/ equality iff $\vec{u} = 0$ (positive definiteness).

Proof: $\vec{a} = \vec{a} \Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

$$((c(\vec{u}) + \vec{v}) \cdot \vec{w}) \cdot \vec{w} \Rightarrow (cu_1 + v_1, \dots, cu_n + v_n) \cdot (w_1, \dots, w_n)$$

$$= ((cu_1 + v_1)w_1, \dots, (cu_n + v_n)w_n)$$

$$\stackrel{\text{dist. of } \mathbb{R}^n}{=} (cu_1 w_1 + v_1 w_1, \dots, cu_n w_n + v_n w_n)$$

$$= (cu_1 w_1, \dots, cu_n w_n) + (v_1 w_1, \dots, v_n w_n)$$

$$= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(u_1, u_1, \dots, u_n, u_n) = (0, \dots, 0) \Leftrightarrow u_i u_i = 0 \quad \forall 1 \leq i \leq n$$

$$\Leftrightarrow u_i = 0$$

> 0 by pos. of squares. \square

Lemma. (Orthogonal Decomposition). Suppose $\vec{u}, \vec{v} \in V$, $\vec{v} \neq 0$. Then there exists $c = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$ and $\vec{w} = \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$ such that $\vec{u} = c\vec{v} + \vec{w}$ and $\langle \vec{w}, \vec{v} \rangle = 0$.

Proof: Let $\vec{u}, \vec{v} \in V$. By closure under scalar multiplication

$$\vec{u} \in V, c \in \mathbb{F} \Rightarrow c\vec{u} \in V.$$

Then $\vec{u} = c\vec{v} + (\vec{u} - c\vec{v})$. We want c such that $\langle \vec{u} - c\vec{v}, \vec{v} \rangle = 0$. Suppose so. Then

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle - c \langle \vec{v}, \vec{v} \rangle &= 0 \\ \Rightarrow \langle \vec{u}, \vec{v} \rangle - c \|\vec{v}\|^2 &= 0 \end{aligned}$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle = c/\|\vec{v}\|^2.$$

$$\Rightarrow \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} < c.$$

So

$$\vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} + \left(\vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right).$$

And since

$$\begin{aligned} \langle \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}, \vec{v} \rangle &\stackrel{\text{lin.}}{=} \langle \vec{u}, \vec{v} \rangle - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \langle \vec{v}, \vec{v} \rangle \\ &= \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

$$\text{let } c = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}, \quad w = \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}. \quad \text{Then } = 0.$$

$$\vec{u} = c\vec{v} + w \quad \text{and } \langle \vec{w}, \vec{v} \rangle = 0. \quad \blacksquare$$

Proof of Cauchy-Schwarz. (Everything from Orthogonal decompr.) Then

$$\begin{aligned} u &= \frac{\langle u, v \rangle}{\|v\|^2} v + w \\ \Rightarrow \|u\|^2 &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\| \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\| + \|w\|^2 \\ &= \left| \frac{\langle u, v \rangle}{\|v\|^2} \right|^2 \|v\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \\ &\geq |\langle u, v \rangle|^2 \\ \Rightarrow |\langle u, v \rangle|^2 &\leq \|u\|^2 \|v\|^2 \\ \Rightarrow |\langle u, v \rangle| &\leq \|u\| \|v\|. \end{aligned}$$

Now, equality holds $\Leftrightarrow w = 0$. Thus $w = 0$. But $w = 0 \Leftrightarrow u = \lambda v, \lambda \in \mathbb{R}$.