APMA 2550, Homework 3: Due Nov. 10

Problem 1 Consider the equation (here $d = 1$ and $m = 1$).

$$\begin{aligned} u_t(x,t) =& a\partial_x^s u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where is s is a positive integer and a is a constant. What does a have to be for the problem to be stable?

Problem 2

a) Suppose that $d > 1$ and $m = 1$ and consider the problem

$$\begin{aligned} u_t(x,t) =& Pu = \sum_{\ell=1}^{d}\sum_{j=1}^{d} \partial_{x_\ell}(a_{\ell j}(x)\partial_{x_j} u(x,t)) \\ u(x,0) =& f(x). \end{aligned}$$

Let A the $d \times d$ matrix with entries $(a_{\ell j})$. We assume that the coefficients are 2π-periordic in each variable. Assume that A is symmetric and uniformly positive definite. That is, it is symmetric and there exists $\theta > 0$ for every $v \in \mathbb{R}^d$

$$v^T A(x) v \geq \theta v^T v \quad \forall v \in \mathbb{R}^d, \forall x.$$

Show that Pu is semi-bounded. Hint: you should show that

$$\langle v, Av\rangle \geq \theta\langle v, v\rangle \quad \forall v \in \mathbb{C}^d, \forall x.$$

b) Again assuming that $d > 1$ and $m = 1$ consider the problem:

$$\begin{aligned} u_t(x,t) =& \sum_{\ell=1}^{d}\sum_{j=1}^{d} \partial_{x_\ell}(a_{\ell j}(x)\partial_{x_j} u(x,t)) + \sum_{j=1}^{d} \beta_j \partial_{x_j} u. \\ u(x,0) =& f(x). \end{aligned}$$

Where the coefficients are 2π periodic and uniformly bounded. That is, there exists a constant C so that $|\beta_j(x)| \leq C$, $\forall x$. Show that this problem is stable.

Problem 3 Consider the Laplace operator $\Delta = \partial_{x_1}^2 + \cdots \partial_{x_d}^2$. Let $d \geq 1$ and $m = 1$.

a) Show that $(\Delta u, w) = -\sum_{j=1}^{d}(\partial_{x_j} u, \partial_{x_j} w)$ when u and w are smooth 2π in each variable. Also, show that $(\Delta u, w) = (u, \Delta w)$.

b) Let s be a positive integer. For what value r is $P = (-1)^r \Delta^s$ semi-bounded?

c) Can you find $\hat{P}(i\omega)$ where $P = (-1)^r \Delta^s$?

Problem 4 Let $d = 1$ and $m \geq 1$. Consider the problem

$$\begin{aligned} u_t(x,t) =& A\partial_x u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

Here A is a constant $m \times m$ matrix. Suppose we can write $A = \Psi\Lambda\Psi^{-1}$ where Λ is a diagonal matrix.

a) In the case $A = \Lambda$ find the solution operator $S(t, \tau)$.

b) Again in the case $A = \Lambda$ consider the problem

$$
\begin{aligned}
u_t(x,t) =& A\partial_x u(x,t) + F(x,t) \\
u(x,0) =& f(x)
\end{aligned}
$$

Using Duhamel's principle write out the solution for $u(x,t)$. In particular, what is the representation of $u_1(x,t)$?

c) In the more general case when $A = \Psi\Lambda\Psi^{-1}$. Find the solution operator $S(t,\tau)$. Hint: Let $v(x,t) = \Psi^{-1}u(x,t)$. Which equation does $v(x,t)$ satisfy?