APMA 2550, Homework 1: Due Oct. 1. Please turn in by dropping it in the course mailbox. Write your solutions using latex. Include your code at the end of the file (e.g. like an appendix).

Problem 1 Consider the problem

$$u_t(x,t) = u_x(x,t) \tag{0.1a}$$
$$u(x,0) = f(x) \tag{0.1b}$$

Consider the method

$$v_j^{n+1} = v_j^n + \frac{k}{h}(v_\ell^n - v_{\ell-1}^n)$$
$$v_j^0 = f_j$$

In order for $|\hat{Q}| \leq 1$, should $\ell = j+1$ or $\ell = j$? Show the analysis. If $\ell = j+1$ this will be the upwind method. If If $\ell = j$ this will be the downwind method.

Problem 2 Investigate the truncation error the upwind method. That is, if u solves (0.1) and is smooth find τ_j^n such that

$$u_j^{n+1} = u_j^n + \frac{k}{h}(u_{j+1}^n - u_j^n) + k\tau_j^n$$

What are the leading terms of τ_j^n? Could this method be more accurcate than the Lax-Wendroff method, in your opinion?

Problem 3 Consider a general explicit one step finite difference method for the transport probelm of the form

$$v_j^{n+1} = Qv_j^n,$$
$$v_j^0 = f_j.$$

where $Q = \sum_{\nu=-r}^{s} A_\nu(k,h)E^\nu$. Let N be even then we can write:

$$f_j = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} e^{i\omega x_j} \tilde{f}(\omega). \tag{0.2}$$

Show that

$$v_j^n = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N/2}^{N/2} [\hat{Q}(\omega)]^n e^{i\omega x_j} \tilde{f}(\omega).$$

What is $\hat{Q}(\omega)$?

Problem 4 Consider the θ scheme

$$(1-\theta k D_0)v_j^{n+1} = \Big(I + (1-\theta)kD_0\Big)v_j^n,$$
$$v_j^0 = f_j.$$

Note that when $\theta = 1$ we obtain the Backward Euler method and when $\theta = \frac{1}{2}$ you obtain the Crank-Nicholson method. Using the representation of f, (0.2), derive an explicit representation for v_j^n. In particular, what is $\hat{Q}$? Show that the method is unconditionally stable for $\frac{1}{2} \leq \theta \leq 1$.

Problem 5 Derive the linear system arising from the Crank-Nicholson method.

Problem 6 Code the following methods: Upwind Scheme, the naive method first introduced in class (which we can call the centered scheme), Lax-Wendroff scheme, Lax-Friedrichs scheme, Backward Euler scheme, and the Crank-Nicholson method.. Keeping $\lambda = k/h = 1/2$, apply these methods to the first order wave equation for $h = 1/2^j$ for $j = 5, 6, 7$. Let your initial condition be:

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \pi \\ 2\pi - x & \text{for } \pi < x < 2\pi \end{cases}$$

Plot your solutions at time $t_n = 1$. Also, give the error of each method measured in the $\|\cdot\|_h$ norm at time $t_n = 1$ when $h = 1/2^7$. Which methods do better?

A practical note: notice that $N + 1 = \frac{2\pi}{h}$ but with the choices I gave above N will not be an integer. Instead you should use the nearest integer of $\frac{2\pi}{h}$ to define N. Keep everything else as is.