NUMERICAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS 1, HW 3

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Problem 1 Consider the equation (here $d = 1$ and $m = 1$).

$$\begin{aligned} u_t(x,t) =& a\partial_x^s u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where is s is a positive integer and a is a constant. What does a have to be for the problem to be stable?

Solution To determine what a has to be for this problem to be stable we will do a Fourier Analysis. Recall for problems of the form,

$$\begin{aligned} \partial_t u(x,t) &= P(x,t,\partial_x)u(x,t) \\ u(x,0) =& f(x). \end{aligned}$$

.

The solution in Fourier Space is

$$u(\omega,t) = e^{\hat{P}(i\omega)t}\hat{f}(\omega)$$

.

Since $P(x,t,\partial_x) = \partial_x^s$, we then know that $\hat{P}(i\omega) = (i\omega)^s$. For the solution to the PDE to be stable, we need the following condition to hold

$$\|u(\cdot,t)\| \leq Ke^{\alpha t}\|f\|$$

which simplifies to $\|e^{\hat{P}(i\omega)t}\| \leq Ke^{\alpha t}$ when we use the fact that $u(x,t) = \frac{1}{\sqrt{2\pi}}\hat{u}(\omega,t)e^{i\omega x}$.

This yields the following cases for $\hat{P}(i\omega)$.

$$\hat{P}(i\omega) = \begin{cases} (i\omega)^s & \text{for } s \textit{ odd} \\ (-1)^{s/2}\omega^s & \text{for } s \textit{ even} \end{cases}$$

So if s is odd, this problem is automatically stable for any a since it will be bounded by 1, which makes since given that the coefficient for the transport equation can be positive or negative. For s even, if $s = 2n$, with n being an odd number, then a must be positive, else this problem will blow up for large values of t. Likewise if n is even, then a must have a negative sign, else the same problem occurs.

Problem 2

a) Suppose that $d > 1$ and $m = 1$ and consider the problem

$$\begin{aligned} u_t(x,t) =& Pu = \sum_{\ell=1}^{d}\sum_{j=1}^{d}\partial_{x_\ell}(a_{\ell j}(x)\partial_{x_j}u(x,t)) \\ u(x,0) =& f(x). \end{aligned}$$

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Let A the $d \times d$ matrix with entries $(a_{\ell j})$. We assume that the coefficients are 2π-periordic in each variable. Assume that A is symmetric and uniformly positive definite. That is, it is symmetric and there exists $\theta > 0$ for every $v \in \mathbb{R}^d$

$$v^T A(x) v \geq \theta v^T v \quad \forall v \in \mathbb{R}^d, \forall x.$$

Show that Pu is semi-bounded. Hint: you should show that

$$\langle v, Av \rangle \geq \theta \langle v, v \rangle \quad \forall v \in \mathbb{C}^d, \forall x.$$

b) Again assuming that $d > 1$ and $m = 1$ consider the problem:

$$\begin{aligned} u_t(x,t) =& \sum_{\ell=1}^{d} \sum_{j=1}^{d} \partial_{x_\ell}(a_{\ell j}(x) \partial_{x_j} u(x,t)) + \sum_{j=1}^{d} \beta_j \partial_{x_j} u. \\ u(x,0) =& f(x). \end{aligned}$$

Where the coefficients are 2π periodic and uniformly bounded. That is, there exists a constant C so that $|\beta_j(x)| \leq C$, $\forall x$. Show that this problem is stable.

Solution

a). We want to show that P is semi-bounded which is equivalent to saying that

$$(Pu, u) + (u, Pu) \leq 2\alpha \|u\|$$

From class, recall that $(\partial_{x_j} u, w) = -(u, \partial_{x_j} w)$. This will then yield that

$$\begin{aligned} (Pu, u) &= (\sum_{\ell=1}^{d} \sum_{j=1}^{d} \partial_{x_\ell}(a_{\ell j}(x) \partial_{x_j} u(x,t)), u(x,t)) \\ &= \sum_{\ell=1}^{d} \sum_{j=1}^{d} (\partial_{x_\ell}(a_{\ell j}(x) \partial_{x_j} u(x,t)), u(x,t)) \\ &= -\sum_{\ell=1}^{d} \sum_{j=1}^{d} (a_{\ell j}(x) \partial_{x_j} u(x,t), \partial_{x_\ell} u(x,t)) \end{aligned}$$

Likewise for (u, Pu). This gives us that

$$\begin{aligned} (u, Pu) + (Pu, u) &= -\sum_{\ell=1}^{d} \sum_{j=1}^{d} (\partial_{x_\ell} u(x,t), a_{\ell j}(x) \partial_{x_j} u(x,t)) - (a_{\ell j}(x) \partial_{x_j} u(x,t), \partial_{x_\ell} u(x,t)) \\ &\leq -2\theta \|\partial_{x_\ell} u\|^2 \end{aligned}$$

This more or less completes the proof as the last part follows from the uniform positive definiteness of A. However, to complete the proof we must extend $v^T A(x) v \geq \theta v^T v$ to the complex numbers.

$$(w, v) = \int_0^{2\pi} \cdots \int_0^{2\pi} \langle w(x), v(x) \rangle dx_1 \cdots dx_d.$$

$$\langle w, v \rangle = \overline{w_1} v_1 + \cdots \overline{w_m} v_m.$$

To that extent, let us show that

$$\langle v, Av \rangle \geq \theta \langle v, v \rangle \quad \forall v \in \mathbb{C}^d, \forall x.$$

.

$\forall v \in \mathbb{C}^d$, we can write $v = u + iw$, where $u, v \in \mathbb{R}^d$. This will yield

$$
\begin{aligned}
\langle Av, v\rangle &= \overline{v^T A^T} v \\
&= \overline{v^T} Av, \; \textit{since } A \textit{ is symmetric} \\
&= (u - iw)^T A(u + iw) \\
&= u^T Au + w^T Aw \\
&\geq \theta \|v\|^2
\end{aligned}
$$

b). We want to show under these conditions, then the problem is stable. To simplify this problem, let's define $Pu = P_1u + P_2u$, where

$$
\begin{aligned}
P_1 u &= \sum_{\ell=1}^{d} \sum_{j=1}^{d} \partial_{x_\ell}(a_{\ell j}(x)\partial_{x_j} u(x,t)) \\
P_2 u &= \sum_{j=1}^{d} \beta_j(x)\partial_{x_j} u
\end{aligned}
$$

We already know that P_1u is semi-bounded from part (a). To show P_2u is semi-bounded as well, notice

$$
\begin{aligned}
(u, P_1u) + (P_1u, u) + (u, P_2u) + (P_2u, u) &\leq -\theta \sum_{j=1}^{d} \|\partial_{x_j} u\|^2 + \sum_{j=1}^{d} \Big((\beta_j(x)\partial_{x_j} u, u) + (u, \beta_j(x)\partial_{x_j} u) \Big) \\
&\leq -\theta \sum_{j=1}^{d} \|\partial_{x_j} u\|^2 + \sum_{j=1}^{d} \Big(2\|\beta_j(x)\partial_{x_j} u\| \|u\| \Big) \\
&\leq -\theta \sum_{j=1}^{d} \|\partial_{x_j} u\|^2 + \sum_{j=1}^{d} \Big(2C\|\partial_{x_j} u\| \|u\| \Big)
\end{aligned}
$$

Where $(u, \beta_j(x)\partial_{x_j} u) \leq \|\beta_j(x)\partial_{x_j} u\|$ follows from the Cauchy-Schwarz inequality. We recall the following inequality

$$
2\alpha\beta \leq \epsilon\alpha^2 + \frac{\beta^2}{\epsilon} \qquad \text{for all } , \theta, \beta \geq 0, \epsilon > 0.
$$

Choosing $\epsilon = \frac{\theta}{C}, \alpha = \|\partial_{x_j} u\|, \beta = \|u\|$, we then find that

$$
\begin{aligned}
(u, P_1u) + (P_1u, u) + (u, P_2u) + (P_2u, u) &\leq -\theta \sum_{j=1}^{d} \|\partial_{x_j} u\|^2 + \sum_{j=1}^{d} \Big(\theta\|\partial_{x_j} u\|^2 + \frac{C^2\|u\|^2}{\theta}) \\
&= 2\frac{dC^2}{2\theta}\|u\|^2
\end{aligned}
$$

which shows the operator P is semi-stable, which in turn implies the problem is stable.

Problem 3 Consider the Laplace operator $\Delta = \partial_{x_1}^2 + \cdots \partial_{x_d}^2$. Let $d \geq 1$ and $m = 1$.

a) Show that $(\Delta u, w) = -\sum_{j=1}^{d} (\partial_{x_j} u, \partial_{x_j} w)$ when u and w are smooth 2π in each variable. Also, show that $(\Delta u, w) = (u, \Delta w)$.

b) Let s be a positive integer. For what value r is $P = (-1)^r \Delta^s$ semi-bounded?

c) Can you find $\hat{P}(i\omega)$ where $P = (-1)^r \Delta^s$?

Solution

a). Recall from class we showed $(u+v,w) = (u,w) + (v,w)$ which establishes linearity, and that $(\partial_{x_j}^2 u, w) = -(\partial_{x_j} u, \partial_{x_j} w)$. As $\Delta = \partial_{x_1}^2 + \cdots \partial_{x_d}^2$, we can see that

$$\begin{aligned}(\Delta u, w) &= (\sum_{j=1}^{d} \partial_{x_j}^2 u, w) \\ &= \sum_{j=1}^{d} (\partial_{x_j}^2 u, w) \\ &= \sum_{j=1}^{d} -(\partial_{x_j} u, \partial_{x_j} w) \\ &= \sum_{j=1}^{d} (u, \partial_{x_j}^2 w) \\ &= (u, \Delta w)\end{aligned}$$

b). Intuition says that $r = s+1$ since the problem $\partial_t u = \kappa \partial_x^2 u$ is stable only for $\kappa > 0$, and $\partial_t u = \kappa \partial_x^4 u$ is stable only for $\kappa < 0$.

Using the multinomial theorem, we can write $(-1)^r \Delta^s$ more succinctly as

$$(-1)^r \Delta^s = (-1)^r \sum_{|\alpha|=s} \frac{s!}{\alpha_1! \alpha_2! \cdots \alpha_d!} \prod_{j=1}^{d} \partial_{x_j}^{2\alpha_j}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $|\alpha| = \sum_{j=1}^{d} \alpha_j$. Using this form of P, we can see that $\hat{P}(i\omega)$ is simply just

$$\hat{P}(i\omega) = (-1)^{r+s} \sum_{|\alpha|=s} \frac{s!}{\alpha_1! \alpha_2! \cdots \alpha_d!} \prod_{j=1}^{d} (\omega_j)^{2\alpha_j}$$

So to make sure $\|e^{\hat{P}(i\omega)t}\| \leq K e^{\alpha t}$ for the problem $\partial_t u(x,t) = (-1)^r \Delta^s u(x,t)$, we need to make sure $(-1)^{r+s}$ is always negative, which can be done by letting $r = s+1$

c). Using the form of P from the previous problem, we can see that $\hat{P}(i\omega)$ is simply just

$$\hat{P}(i\omega) = (-1)^{r+s} \sum_{|\alpha|=s} \frac{s!}{\alpha_1! \alpha_2! \cdots \alpha_d!} \prod_{j=1}^{d} (\omega_j)^{2\alpha_j}$$

Problem 4 Let $d = 1$ and $m \geq 1$. Consider the problem

$$\begin{aligned}u_t(x,t) =& A \partial_x u(x,t) \\ u(x,0) =& f(x)\end{aligned}$$

Here A is a constant $m \times m$ matrix. Suppose we can write $A = \Psi \Lambda \Psi^{-1}$ where Λ is a diagonal matrix.

a) In the case $A = \Lambda$ find the solution operator $S(t,\tau)$.

b) Again in the case $A = \Lambda$ consider the problem

$$\begin{aligned}u_t(x,t) =& A \partial_x u(x,t) + F(x,t) \\ u(x,0) =& f(x)\end{aligned}$$

Using Duhamel's principle write out the solution for $u(x,t)$. In particular, what is the representation of $u_1(x,t)$?

c) In the more general case when $A = \Psi\Lambda\Psi^{-1}$. Find the solution operator $S(t,\tau)$. Hint: Let $v(x,t) = \Psi^{-1}u(x,t)$. Which equation does $v(x,t)$ satisfy?

Solution

a). Given the system of equations

$$\begin{aligned} u_t(x,t) =& \Lambda\partial_x u(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

for a diagonal $m \times m$ matrix Λ, we can uncouple the transport equations so that they are only dependent upon the same indices. For example

$$\begin{aligned} \partial_t u_j(x,t) =& \lambda_j \partial_x u_j(x,t) \\ u_j(x,0) =& f_j(x) \end{aligned}$$

$\forall j \in \{1, 2, \cdots, m\}$, where λ_j is the j-th eigenvalue of the matrix Λ. This will yield m solutions of the form $u_j(x,t) = f_j(x + \lambda_j t)$. We can then define the solution operator $S(t,\tau)$ to be

$$u(x, t-\tau) = \Big(S(t,\tau)f\Big)(x,t) = \begin{bmatrix} f_1(x + \lambda_1(t-\tau)) \\ f_2(x + \lambda_2(t-\tau)) \\ \vdots \\ f_m(x + \lambda_m(t-\tau)) \end{bmatrix}$$

for $t > \tau$.

b). Now consider the problem

$$\begin{aligned} u_t(x,t) =& A\partial_x u(x,t) + F(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

where A is of the same form in (a). Using Duhamel's Principle we find the solution to be the same as before, with an added integral for the inhomogeneous portion of the equation.

$$u(x,t) = \Big(S(t,0)f\Big)(x,t) + \int_0^t \Big(S(t,\tau)F(x,\tau)\Big)(x,t)d\tau = \begin{bmatrix} f_1(x + \lambda_1 t) + \int_0^t F_1(x + \lambda_1(t-\tau), \tau)d\tau \\ f_2(x + \lambda_2 t) + \int_0^t F_2(x + \lambda_2(t-\tau), \tau)d\tau \\ \vdots \\ f_m(x + \lambda_m t) + \int_0^t F_m(x + \lambda_m(t-\tau), \tau)d\tau \end{bmatrix}$$

c). Now assume with $A = \Psi\Lambda\Psi^{-1}$ Again in the case $A = \Lambda$ consider the problem

$$\begin{aligned} u_t(x,t) =& A\partial_x u(x,t) + F(x,t) \\ u(x,0) =& f(x) \end{aligned}$$

If we let $v(x,t) = \Psi^{-1}u(x,t)$, then this will simplify the problem substantially as $v(x,t)$ will satisfy the equation

$$\begin{aligned} v_t(x,t) =& \Lambda\partial_x u(x,t) + G(x,t) \\ v(x,0) =& g(x) \\ g(x) =& \Psi^{-1}f(x) \\ G(x,t) =& \Psi^{-1}F(x,t) \end{aligned}$$

Given this formulation, v(x,t) will simply be the solution from (b), which is

$$v(x,t) = \Big(S(t,0)g\Big)(x,t) + \int_0^t \Big(S(t,\tau)G(x,\tau)\Big)(x,t)d\tau = \begin{bmatrix} g_1(x+\lambda_1 t) + \int_0^t G_1(x+\lambda_1(t-\tau),\tau)d\tau \\ g_2(x+\lambda_2 t) + \int_0^t G_2(x+\lambda_2(t-\tau),\tau)d\tau \\ \vdots \\ g_m(x+\lambda_m t) + \int_0^t G_m(x+\lambda_m(t-\tau),\tau)d\tau \end{bmatrix}$$

so we can then see that our solution operator $S(t,\tau)$ for u(x,t) is just going to be

$$u_j(x,t) = \sum_{\ell=1}^{m} \psi_{\ell j}\Big(f_\ell(x+\lambda_\ell t) + \int_0^t F_\ell(x+\lambda_\ell(t-\tau),\tau)d\tau\Big)$$