DIFFERENTIAL GEOMETRY: PROBLEM SET 8

NICHOLAS ARSENAULT

1. Using the symmetry of the connection and the fact that $\nabla_X X = 0$, we can say

$$\nabla_{X+Y}(X+Y) = 0$$

which combined with $\nabla_X Y - \nabla_Y X = [X,Y]$ gives $\nabla_X Y = \frac{1}{2}[X,Y]$. Secondly, using the defintion

$$\begin{aligned}
R(X,Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y + \nabla_{[X,Y]} Z \\
&= \frac{1}{2}\nabla_Y [X,Z] - \frac{1}{2}\nabla_X [Y,Z] + [[X,Y],Z] \\
&= \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z] \\
&= \frac{1}{4}\left([Y,[X,Z]] - [X,[Y,Z]] + [[X,Y],Z]\right) + \frac{1}{4}[[X,Y],Z] \\
&= \frac{1}{4}\left([[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]\right) + \frac{1}{4}[[X,Y],Z] \\
&= \frac{1}{4}[[X,Y],Z].
\end{aligned}$$

Finally,

$$\begin{aligned}
K(\sigma)(X,Y) &= \langle R(X,Y)X, Y)\rangle \\
&= \left\langle \frac{1}{4}[[X,Y],X],Y] \right\rangle \\
&= -\frac{1}{4}\left\langle [X,Y],[Y,X] \right\rangle \\
&= \frac{1}{4}\|[X,Y]\|^2.
\end{aligned}$$

The second to last equality holds by (3.) on Pg. 40, and the denominator of the sectional curvature is 1, since X, Y are assumed to be orthonormal.

7.