Energy conservation and dissipation properties for elastodynamics with contact impact and friction

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Abstract The objective of this paper is to propose a time integration scheme for nonsmooth mechanical systems involving one-sided contact, impact and Coulomb friction, that respects the principles of discrete-time energy balance with positive dissipation. To obtain energetic consistency in the continuous time model, we work with an impact law inspired by the work of M. Frémond (Frémond, 2001, 2002, 2017) which ensures that dissipation is positive, *i.e.* that the Clausius–Duhem inequality is satisfied. On this basis, we propose a time integration method based on the Moreau–Jean scheme (Jean and Moreau, 1987; Moreau, 1988) and show that this method has correct dissipation properties.

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Notation The following notation is used throughout the paper. The Euclidean norm for a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$. For two vectors $x, y \in \mathbb{R}^n$, the Hadamard product is denoted by $x \circ y$. For a positive definite (respectively positive semi-definite) matrix $M \in \mathbb{R}^{n\times n}$, $\|x\|_M$ denotes the norm (respectively the semi-norm) in the metric defined by M. Let I denote a real time interval of any sort. For a function $f : I \to \mathbb{R}^n$ of Bounded Variation (BV), we denote the right-limit function by $f^+(t) = \lim_{s\to t, s>t} f(s)$, and respectively the left-limit by $f^-(t) = \lim_{s\to t, s<t} f(s)$. We denote by $0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$ a finite partition (or a subdivision) of the time interval $[0, T]$, where $(T > 0)$. The integer N is the number of time intervals in the subdivision. The length of a time step is denoted by $h_k = t_{k+1} - t_k$. For simplicity's sake, the schemes are presented in the following with a time step denoted by h for short. The value of a real function $x(t)$ at the time t_k, is approximated by x_k. In the same way, the notation $x_{k+\theta} = (1-\theta)x_k + \theta x_{k+1}$ is used for $\theta \in [0, 1]$. The notation $\mathrm{d}t$ defines the Lebesgue measure on $\mathbb{R}$.

1 Introduction and motivations

The objective of this paper is to propose a time integration scheme for nonsmooth mechanical systems involving one-sided contact, impact and Coulomb friction, that respects the principles of discrete-time energy balance with positive dissipation. For simplicity, we consider systems that have been spatially discretised (by the finite element method (FEM) for example), and that have dynamics that are linear, but with possible nonlinear constraints that model contact. The question of developing a consistent integration scheme with correct energy and dissipation properties implies that the continuous-time model also has these properties. It is known that discrete systems with Coulomb friction and a kinematic impact law can generate energy, in particular, when the direction of sliding velocity changes during impact (Brogliato, 2016; Glocker, 2013). In this work, we therefore propose to work with an impact law inspired by the work of M. Frémond (Frémond, 2001, 2002, 2017) which ensures that dissipation is positive, *i.e.* that the Clausius–Duhem inequality (and therefore the second law of thermodynamics) is satisfied. On this basis, we propose a time integration method based on the Moreau–Jean scheme (Jean and Moreau, 1987; Moreau, 1988) and show that this method has good dissipation properties.

Much work in the literature has focused on conservative or energy-dissipating time integration schemes for frictionless one-sided contact systems. These studies are mainly interested in this property because it confers stability properties to the schemes, which are often difficult to obtain. For a review of the literature, please refer to Acary (2016). On the other hand, there is very little work on contact with Coulomb friction. This may be due to the inherent difficulty of Coulomb friction for discrete systems that exhibit impulses. It is well known that the Coulomb friction model in percussion can be inconsistent.

The structure of the article is as follows. We begin in Section 2 by presenting the equations of motion of a discrete system with one-sided constraints, Coulomb friction and impacts. After a discussion of the energy balance of the system, we find that the system with Newton's law of impact and Coulomb friction can be non-dissipative. Finally, we propose an impact model inspired by the work of Frémond which guarantees that the system is dissipative. In Section 3, we propose a new scheme that is dissipative in discrete time, with dissipative terms that are numerical approximations of frictional and impact dissipation. Section 4 conclude the article and lists some perspectives.

2 Nonsmooth mechanical systems with unilateral contact, Coulomb's friction and impact

2.1 Smooth equations of motion for linear viscoelasticity and constitutive laws

The equations of motion of a discrete or discretised Lagrangian mechanical system in the linear viscoelastic case are

$$\begin{cases} q(t_0) = q_0,\ v(t_0) = v_0, & \text{(1a)} \\ \dot{q}(t) = v(t), & \text{(1b)} \\ M\dot{v}(t) + Kq(t) + Cv(t) = F(t) + R(t) & \text{(1c)} \end{cases}$$

where $q(t) \in \mathbb{R}^n$ is the vector of generalised coordinates and $v(t) = \dot{q}(t)$ is the corresponding vector of generalised velocities, the initial conditions are $q_0 \in \mathbb{R}^n$ and $v_0 \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ is the symmetric mass matrix that is assumed to be positive definite, $K \in \mathbb{R}^{n\times n}$ is the positive semi-definite stiffness matrix and $C \in \mathbb{R}^{n\times n}$ the positive semi-definite damping matrix, $F(t)$ is the vector of external applied forces and $R(t)$ is the generalised contact forces. The system is now subjected to a finite set of m unilateral contact that defines the admissible set for the configuration

$$\mathcal{C} = \{q \in \mathbb{R}^n \mid g_{\mathrm{N}}^{\alpha}(q) \geqslant 0, \alpha \in [\![1, m]\!]\} \subset \mathbb{R}^n \tag{2}$$

where $g^\alpha_{\mathrm{N}} : \mathbb{R}^n \to \mathbb{R}$ is assumed to smooth function with non-vanishing gradients. We denote by the function $g_{\mathrm{N}} : \mathbb{R}^n \to \mathbb{R}^m$ the function with g^α_{N} as components. For the perfect unilateral constraints, the Signorini condition is written as

$$0 \leqslant g_{\mathrm{N}}(q(t)) \perp \lambda(t) \geqslant 0, \tag{3}$$

where the inequalities involving vectors are understood to hold component-wise and the $\perp$ symbol means that $y^\top \lambda = 0$. The Lagrange multiplier λ is related to the generalised reaction forces R by

$$R(t) = \nabla_q g_{\mathrm{N}}(q(t))\lambda(t). \tag{4}$$

The Signorini condition at the velocity level can be also be defined as

$$0 \leqslant u_{\mathrm{N}}(t) \perp \lambda(t) \geqslant 0, \text{ if } g_{\mathrm{N}}(q(t)) \leqslant 0. \tag{5}$$

where the local relative velocity $u_{\mathrm{N}}(t)$ is defined by

$$u_{\mathrm{N}}(t) = \dot{g}(q(t)) = \nabla_q^\top g_{\mathrm{N}}(q(t))\, v(t), \qquad R(t) = G_{\mathrm{N}}(q)\lambda(t). \tag{6}$$

The Moreau's viability Lemma ensures that (5) implies (3).

Coulomb's friction For Coulomb's friction, the definition of the gradients of the constraints is not sufficient and an orthonormal local basis $(\mathrm{n}^\alpha, \mathrm{t}^\alpha_1, \mathrm{t}^\alpha_2)$ at contact composed of the inward normal vector $n^\alpha \in \mathbb{R}^3$ to set $\mathcal{C}$ co-linear to the $\nabla_q^\top g^\alpha_{\mathrm{N}}$ and the tangential vectors $\mathrm{t}^\alpha_1, \mathrm{t}^\alpha_2 \in \mathbb{R}^3$ is needed. For a contact α, the relative velocity and the reaction force r at contact are denoted in this local contact frame as

$$u^\alpha = \begin{pmatrix} u^\alpha_{\mathrm{N}} \\ u^\alpha_{\mathrm{T}} \end{pmatrix}, \quad r^\alpha = \begin{pmatrix} r^\alpha_{\mathrm{N}} \\ r^\alpha_{\mathrm{T}} \end{pmatrix}, \tag{7}$$

where $u^\alpha_{\mathrm{N}} \in \mathbb{R}$, $u^\alpha_{\mathrm{T}} = \begin{pmatrix} u^\alpha_{\mathrm{T}1}, & u^\alpha_{\mathrm{T}2} \end{pmatrix}^\top \in \mathbb{R}^2$, $r^\alpha_{\mathrm{N}} \in \mathbb{R}$, $r^\alpha_{\mathrm{T}} = \begin{pmatrix} r^\alpha_{\mathrm{T}1}, & r^\alpha_{\mathrm{T}2} \end{pmatrix}^\top \in \mathbb{R}^2$. The local variables are are related to the generalised variables by

$$u^\alpha = H^{\alpha,\top}(q)\, v, \qquad R^\alpha = H^\alpha(q)\, r^\alpha. \tag{8}$$

Coulomb's friction together with the Signorini at the velocity level is given by

$$-\tilde{u}^\alpha \in N_{K^\alpha}(r^\alpha), \qquad \text{if } g^\alpha_{\mathrm{N}}(q) \leqslant 0, \text{ else } r^\alpha = 0 \tag{9}$$

where $\tilde{u}$ is the modified De Saxcé velocity expressed as

$$\tilde{u}^\alpha = u^\alpha + \begin{pmatrix} \mu \|u^\alpha_{\mathrm{T}}\| \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

and K^α is the Coulomb's cone

$$K^\alpha = \{r, \|r_{\mathrm{T}}\| \leqslant \mu^\alpha r_{\mathrm{N}}\}, \tag{11}$$

with $\mu^\alpha \geqslant 0$ the friction coefficient.

Let $\mathcal{I} = [\![1, m]\!] \in \mathbb{N}$ be the set of indices of constraints. Let us define now the index of constraint at the velocity level by

$$\mathcal{I}^1 = \{\alpha \in \mathcal{I} \mid g^\alpha_{\mathrm{N}}(q(t)) \leqslant 0\}. \tag{12}$$

By collecting all the variables for each contact in the set $\mathcal{I}^1$, and implicitly introducing the notation dropping α ($x = [x^{\alpha,\top}, \alpha \in \mathcal{I}^1]^\top$), we get for the frictional contacts

$$-\tilde{u} \in \mathrm{N}_K(r), \tag{13}$$

considering that $r^\alpha = 0$ for $\alpha \notin \mathcal{I}^1$ and K is the Cartesian product of the cone K^α. Using the dual cone of K denoted as

$$K^* = \{u, \mu\|u_{\mathrm{T}}\| \leqslant u_{\mathrm{N}}\} \tag{14}$$

the relation (13) can be written as

$$-r \in \mathrm{N}_{K^*}(\tilde{u}). \tag{15}$$

or as a complementarity condition

$$K^* \ni \tilde{u} \perp r \in K. \tag{16}$$

To be self-contained, the equivalence with the standard form of the Coulomb friction in given in the following Lemma 1.

Lemma 1 ((Acary et al., 2011)) *For one contact point, the Coulomb's friction law with the Signorini condition at the velocity level given the three following conditions*

$$\left\{\begin{array}{llll} \textit{either} & r=0, & u_{\mathrm{N}} \geqslant 0 & \textit{(take-off)} \\ \textit{or} & u=0, & \|r_T\| \leqslant \mu r_n & \textit{(sticking)} \\ \textit{or} & \|r_T\| = \mu r_n, u_{\mathrm{N}} = 0 & \|u_T\| r_T = -\|r_T\| u_T & \textit{(sliding)} \end{array}\right. \tag{17}$$

is equivalent to the complementarity problem

$$K^* \ni \tilde{u} \perp r \in K \tag{18}$$

with

$$\tilde{u} = u + \begin{pmatrix} \mu\|u_T\| \\ 0 \\ 0 \end{pmatrix}. \tag{19}$$

Proof. ($\Longrightarrow$) The conditions (17) implies that $r \in K$ and $u_{\mathrm{N}} \geqslant 0$. Then $\tilde{u}_n \geqslant \mu\|u_{\mathrm{T}}\|$ and we conclude that $\tilde{u} \in K^*$. If $u = 0$, then $\tilde{u} = 0$ and the relation $\tilde{u}^\top r = 0$ holds trivially. The same applies for $r = 0$. In the sliding case, we have

$$\begin{array}{rcl} \tilde{u}^\top r & = & (u_{\mathrm{N}} + \mu\|u_{\mathrm{T}}\|) r_{\mathrm{N}} + u_{\mathrm{T}}^\top r_{\mathrm{T}} \\ & = & u_{\mathrm{N}} r_{\mathrm{N}} + \mu\|u_{\mathrm{T}}\| r_{\mathrm{N}} + u_{\mathrm{T}}^\top r_{\mathrm{T}} \\ & = & \|u_{\mathrm{T}}\|\|r_{\mathrm{T}}\| + u_{\mathrm{T}}^\top r_{\mathrm{T}} \text{ since } u_{\mathrm{N}} = 0 \text{ and } \mu r_{\mathrm{N}} = \|r_{\mathrm{T}}\| \end{array} \tag{20}$$

Since $\|u_{\mathrm{T}}\| r_{\mathrm{T}} = -\|r_{\mathrm{T}}\| u_{\mathrm{T}}$, the vector u_{T} and r_{T} are collinear and opposite direction, we have $u_{\mathrm{T}}^\top r_{\mathrm{T}} = -\|u_{\mathrm{T}}\|\|r_{\mathrm{T}}\|$ and hence $\tilde{u}^\top r = 0$.

($\Longleftarrow$) Conversely, if (18) holds, we have three separate cases:

1. $r = 0$. The relation $\tilde{u} \in K^*$. This implies $u_{\mathrm{N}} \geqslant 0$. We have the take-off case.
2. $\tilde{u} = 0$. This implies $u = 0$ and we have $r \in K$.
   We have the sticking case.
3. $\tilde{u} \neq 0$ and $r \neq 0$. Using $\tilde{u} \in K^*$ and $r \in K$, we get

   $$\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| \leqslant \mu \tilde{u}_{\mathrm{N}} r_{\mathrm{N}} \tag{21}$$

   The relation $\tilde{u}^\top r = 0$ implies $\tilde{u}_{\mathrm{N}} r_{\mathrm{N}} = -\tilde{u}_{\mathrm{T}}^\top r_{\mathrm{T}}$ so we get

   $$\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| \leqslant -\mu \tilde{u}_{\mathrm{T}}^\top r_{\mathrm{T}} \tag{22}$$

   The Cauchy-Schwarz inequality gives

   $$\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| \geqslant -\mu \tilde{u}_{\mathrm{T}}^\top r_{\mathrm{T}} \tag{23}$$

   so we conclude that

   $$\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| = -\mu \tilde{u}_{\mathrm{T}}^\top r_{\mathrm{T}} \tag{24}$$

   When $\mu \neq 0$, the vector $\tilde{u}_{\mathrm{T}}$ and r_{T} collinear and opposite direction since the Cauchy-Schawarz inequality is an equality. Let us assume that $r \in \operatorname{int}(K)$, then $\|r_{\mathrm{T}}\| < \mu r_{\mathrm{N}}$. Since $u \in K^*$, we have

   $$\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| < \mu r_{\mathrm{N}} \tilde{u}_{\mathrm{N}} = -\mu \tilde{u}_{\mathrm{T}}^\top r_{\mathrm{T}} \tag{25}$$

   which contradicts (24). Hence, we conclude $r \in \partial K$ and hence $\|r_{\mathrm{T}}\| = \mu r_{\mathrm{N}}$. Let us assume that $\tilde{u} \in \operatorname{int}(K^*)$, then $\mu\|\tilde{u}_{\mathrm{T}}\| < \tilde{u}_{\mathrm{N}}$. Since $r \in K$, $\mu\|\tilde{u}_{\mathrm{T}}\|\|r_{\mathrm{T}}\| < \mu r_{\mathrm{N}} \tilde{u}_{\mathrm{N}}$ contradicts (24). Hence, we conclude $\tilde{u} \in \partial K^*$. From $\tilde{u} \in \partial K^*$, we conclude that $u_{\mathrm{N}} = 0$. □

Smooth dynamics with Coulomb's friction For the sake of simplicity, we note by

$$-\tilde{u} = \Phi(u) \tag{26}$$

the De Saxcé changes of variable defined in (10) for a contact α. The smooth dynamics with Coulomb friction is then given by

$$\begin{cases} q(t_0) = q_0,\, v(t_0) = v_0, & \text{(27a)} \\ \dot{q}(t) = v(t), & \text{(27b)} \\ M\dot{v}(t) + Kq(t) + Cv(t) = F_{\mathrm{ext}}(t) + H(q(t))r(t) & \text{(27c)} \\ u(t) = H^\top(q(t))v(t) & \text{(27d)} \\ -\Phi(u) \in \mathrm{N}_K(r), & \text{(27e)} \end{cases}$$

2.2 Nonsmooth dynamics

If a contact α is closing at time t_i with a negative relative velocity, that is $g^\alpha_{\rm N}(q(t_i)) = 0$ and $u^\alpha_{\rm N}(t_i) < 0$, we have an impact at t_i. The velocity must jump to satisfy the unilateral constraint right after the impact $g^\alpha_{\rm N}(q(t_i+\varepsilon)) \geqslant 0, \varepsilon > 0$. The velocity v is usually assumed to be a function of bounded variations. The same applies for the relative velocity at contact. With finite dimensional systems, the smooth dynamics (27) is generally not sufficient to characterize the solution (infinitely many solutions after impacts). To close the system at the impact time t_i when $v^+(t_i) \neq v^-(t_i)$, an impact law must be added. In this work, we consider the Newton's impact law:

$$u^{\alpha,+}_{\rm N}(t_i) = -e^\alpha u^{\alpha,-}_{\rm N}(t_i) \text{ if } g^\alpha_{\rm N}(q(t_i)) \leqslant 0 \text{ and } u^{\alpha,-}_{\rm N}(t_i) \leqslant 0, \tag{28}$$

where $e^\alpha \in [0,1]$ is the Newton coefficient of restitution. To address the case of multiple contacts, the Newton impact law is written in terms of complementarity as

$$0 \leqslant u^{\alpha,+}_{\rm N}(t_i) + e^\alpha u^{\alpha,-}_{\rm N}(t_i) \perp p^\alpha_i \geqslant 0, \text{ if } \alpha \in \bar{\mathcal{I}}^1, \tag{29}$$

where p_i is the reaction impulse at time t_i and $\bar{\mathcal{I}}^1$ is the index set defined by

$$\bar{\mathcal{I}}^1 = \{\alpha \in \mathcal{I} \mid g^\alpha_{\rm N}(q(t)) \leqslant 0 \text{ and } u^{\alpha,-}_{\rm N}(t) \leqslant 0\}. \tag{30}$$

In the sequel, we will assume that the local variables at contact are collected for all indices belonging to $\bar{\mathcal{I}}^1$.

The nonsmooth equations of motion are written in terms of differential measure $\mathrm{d}v$ associated with v and the local impulse measure $\mathrm{d}i$ as follows

$$\begin{cases} q(t_0) = q_0,\ v^-(t_0) = v_0, & \text{(31a)}\\ \dot{q} = v, & \text{(31b)}\\ M\,\mathrm{d}v + Kq\ \ \mathrm{d}t + Cv\ \ \mathrm{d}t = F\,\mathrm{d}t + H(q)\,\mathrm{d}i, & \text{(31c)}\\ u = H^\top(q)v, & \text{(31d)}\\ -\Psi(u^+) \in \mathrm{N}_K(di), & \text{(31e)} \end{cases}$$

with the definition of the following function that takes into account the impact law (29)

$$\Psi^\alpha(u^\alpha) = u^\alpha + \begin{pmatrix} \mu\|u^\alpha_{\rm T}\| + e^\alpha u^{\alpha,-}_{\rm N} \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

Interpretation of the contact law (31e) **in terms of measures.** The equation (31e) deserves some explanations based on the definition of an normal cone inclusion with measures. Almost everywhere with respect to the Lebesgue measure $\mathrm{d}t$, we have

$$-\Psi(u^+) \in \mathrm{N}_K(r), \tag{33}$$

where r is the density of $\mathrm{d}i$ with respect to $\mathrm{d}t$. Since we have $u^- = u^+ = u$ almost everywhere, the relation (33) can be written for a contact α as

$$\bar{u}^\alpha = \begin{pmatrix} (1+e^\alpha)u^\alpha_{\rm N} + \mu\|u^\alpha_{\rm T}\| \\ u^\alpha_{\rm T} \end{pmatrix} \in \mathrm{N}_{K^\alpha}(r^\alpha), \tag{34}$$

Three cases are possible following the proof of the Lemma 1:

1. take-off case: $r^\alpha = 0$. In that case, we have $u^\alpha_{\rm N} \geqslant 0$.
2. sticking case: $\bar{u}^\alpha = 0$. In that case, we have $u^\alpha_{\rm T} = 0$ and $(1+e^\alpha)u^\alpha_{\rm N} = 0 = u^\alpha_{\rm N}$ and $r^\alpha \in K^\alpha$.
3. sliding case: $r^\alpha \neq 0$ and $\bar{u}^\alpha \neq 0$. Then we have $r^\alpha \in \partial K^\alpha$ and $\bar{u}^\alpha \in \partial K^{\alpha,*}$. This implies $(1+e)u^\alpha_{\rm N} = 0 = u^\alpha_{\rm N}$, $\|r_{\rm T}\| = \mu r_n$ and $\|u_{\rm T}\| r_{\rm T} = -\|r_{\rm T}\| u_{\rm T}$.

To conclude, we retrieve almost everywhere the Coulomb's friction with the Signorini condition at the velocity level. At any time t_i, we have

$$-\Psi(u^+) \in \mathrm{N}_K(p_i), \tag{35}$$

where p_i is the density of $\mathrm{d}i$ with respect to the Dirac atom at t_i, δ_{t_i}. We have again three possible cases for a contact α

1. take-off case: $p_i^\alpha = 0$. In that case, we have $u_{\rm N}^{\alpha,+} + e u_{\rm N}^{\alpha,-} \geqslant 0$.
2. sticking case: $\Psi(u^{\alpha,+}) = 0$. In that case, we have $u_{\rm T}^{\alpha,+} = 0$ and $u_{\rm N}^{\alpha,+} + e u_{\rm N}^{\alpha,-} = 0$ and $p_i^\alpha \in K^\alpha$. The impact law is satisfied since $u_{\rm N}^{\alpha,+} = -e u_{\rm N}^{\alpha,-}$
3. sliding case: $p_i^\alpha \neq 0$ and $\Psi(u^{\alpha,+}) \neq 0$. Then we have $p_i^\alpha \in \partial K^\alpha$ and $\Psi(u^{\alpha,+}) \in \partial K^{\alpha,*}$. This implies $u_{\rm N}^{\alpha,+} + e u_{\rm N}^{\alpha,-} = 0$, $\|p_{i,{\rm T}}^\alpha\| = \mu p_{i,{\rm N}}^\alpha$ and $\|u_{\rm T}^{\alpha,+}\| p_{i,{\rm T}}^\alpha = -\|p_{i,{\rm T}}^\alpha\| u_{\rm T}^{\alpha,+}$.

To conclude, the impact law is satisfied in sticking and sliding case and the Coulomb's friction is written in terms of impulses and of the right limit of the relative velocity. We will see in the sequel that is has an consequence on the dissipativity of the model.

2.3 Energy balance analysis

A detailed analysis of the energy balance for nonsmooth systems can be found in Lozano et al. (2013) and Leine and Wouw (2008). The energy balance is usually obtained by multiplying the equation of motion by $v^+ + v^-$. After some algebraic manipulations (see Acary (2016)), we obtain

$$2\mathrm{d}\mathcal{E} := \mathrm{d}(v^\top M v) + 2q^\top K \mathrm{d}q = 2v^\top F \ \mathrm{d}t - 2v^\top C v \ \mathrm{d}t + (v^+ + v^-)^\top H(q)\,\mathrm{d}i, \tag{36}$$

where the standard definition of the total mechanical energy of the system is

$$\mathcal{E} := \frac{1}{2} v^\top M v + \frac{1}{2} q^\top K q, \tag{37}$$

and $\mathrm{d}q = v^+(t)\,\mathrm{d}t = v^-(t)\,\mathrm{d}t$.

Energy balance almost everywhere Almost everywhere with respect to the Lebesgue measure $\mathrm{d}t$, the energy balance (36) reads as

$$\frac{d}{dt}\mathcal{E}(t) = v^\top(t) F(t) - v^\top(t) C v(t) + v^\top(t) H(q(t)) r(t), \tag{38}$$

which is the classical energy balance for a smooth dynamical system subjected to some constraints. The term $v^\top F$ is the power of external forces and $-v^\top C v$ is the power of viscous forces. The last term corresponds to the power of reaction forces. For a contact α, it can be decomposed as

$$v^\top H^\alpha(q) r^\alpha = u^{\alpha,\top} r^\alpha = u_{\rm N}^\alpha r_{\rm N}^\alpha + u_{\rm T}^{\alpha,\top} r_{\rm T}^\alpha. \tag{39}$$

As we assumed that the constitutive law (33) is satisfied almost everywhere, an examination of the three possibles cases for the contact law yields

$$v^\top H^\alpha(q) r^\alpha = -\mu^\alpha r_{\rm N}^\alpha \|u_{\rm T}^\alpha\|, \tag{40}$$

which the standard dissipated power by a sliding contact with Coulomb friction. Note that the dissipated power is always non-positive, which is consistent with the principles of thermodynamics.

Energy balance at any time At any time t_i, the energy balance (36) reads as

$$\mathcal{E}^+(t_i) - \mathcal{E}^-(t_i) = \frac{1}{2}(v^+(t_i) + v^-(t_i))^\top H(q(t_i)) p_i. \tag{41}$$

For a contact α, it can be evaluated as

$$\frac{1}{2}(v^+(t_i) + v^-(t_i))^\top H^\alpha(q(t_i)) p_i^\alpha = \frac{1}{2}(u^{\alpha,+}(t_i) + u^{\alpha,-}(t_i))^\top p_i^\alpha. \tag{42}$$

With the constitutive law given by (33), the dissipated power in (42) vanishes in the take-off case. In the sticking case, the sliding velocity vanishes and we get

$$\begin{aligned}\frac{1}{2}(v^+(t_i) + v^-(t_i))^\top H^\alpha(q(t_i)) p_i^\alpha &= \frac{1}{2}(u_{\rm N}^{\alpha,+}(t_i) + u_{\rm N}^{\alpha,-}(t_i)) p_{{\rm N},i}^\alpha + \frac{1}{2} u_{\rm T}^{\alpha,-}(t_i) p_{{\rm T},i}^\alpha \\ &= \frac{1}{2}(1-e) u_{\rm N}^{\alpha,-}(t_i) p_{{\rm N},i}^\alpha + \frac{1}{2} u_{\rm T}^{\alpha,-}(t_i) p_{{\rm T},i}^\alpha.\end{aligned} \tag{43}$$

Since $u_{\rm N}^{\alpha,-}(t_i) \leqslant 0$ and $e \in [0,1]$, the first term of the dissipated power by an impact is non-positive, but we cannot conclude a priori on the sign of the second term. In the sliding case, we get

$$\begin{aligned}\frac{1}{2}(v^+(t_i) + v^-(t_i))^\top H^\alpha(q(t_i)) p_i^\alpha &= \frac{1}{2}(1-e) u_{\rm N}^{\alpha,-}(t_i)) p_{{\rm N},i}^\alpha + \frac{1}{2}(u_{\rm T}^{\alpha,+}(t_i) + u_{\rm T}^{\alpha,-}(t_i))^\top p_{{\rm T},i}^\alpha \\ &= \frac{1}{2}(1-e) u_{\rm N}^{\alpha,-}(t_i)) p_{{\rm N},i}^\alpha - \frac{1}{2}\mu^\alpha p_{{\rm N},i}^\alpha \|u_{\rm T}^{\alpha,+}(t_i)\| + \frac{1}{2}(u_{\rm T}^{\alpha,-}(t_i))^\top p_{{\rm T},i}^\alpha\end{aligned} \tag{44}$$

The first two terms in (44) are non-positive. For the last terms, we cannot conclude a priori on its sign. For instance, if there is a reversal of the sliding direction during the impact, $u_{\mathrm{T}}^{\alpha,-}(t_i) = -u_{\mathrm{T}}^{\alpha,-}(t_i)$, the power of the frictional impulse may generate energy. In other words, the coulomb's law in terms of impulses together with the Newton's impact is not dissipative and does not respect the second principle of thermodynamics.

The fact is not new. The famous Kane (1984) example (see also Brogliato (2016) and Glocker (2013)) shows that some amount of energy can be generated using such type of law. For our purpose, it seems to be difficult to propose a consistent numerical scheme that dissipates energy based on such a law.

2.4 Frémond's model of frictional impact

To obtain a constitutive law that is consistent with the thermodynamics and the Newton's impact law, we propose to use

$$-\Xi\left(\frac{1}{2}(u^+ + u^-)\right) \in N_K(\mathrm{d}i) \tag{45}$$

with

$$\Xi^\alpha\left(\frac{1}{2}(u^{\alpha,+} + u^{\alpha,-})\right) = \begin{pmatrix} \frac{1}{2}(u_{\mathrm{N}}^{\alpha,+} + u_{\mathrm{N}}^{\alpha,-}) + \frac{1}{2}(e^\alpha - 1)u_{\mathrm{N}}^{\alpha,-} + \mu^\alpha \|\frac{1}{2}(u_{\mathrm{T}}^{\alpha,+} + u_{\mathrm{T}}^{\alpha,-})\| \\ \frac{1}{2}(u_{\mathrm{T}}^{\alpha,+} + u_{\mathrm{T}}^{\alpha,-}) \end{pmatrix} \tag{46}$$

or

$$\Xi^\alpha(u^\alpha) = \begin{pmatrix} u_{\mathrm{N}}^\alpha + \frac{1}{2}(e^\alpha - 1)u_{\mathrm{N}}^{\alpha,-} + \mu^\alpha \|u_{\mathrm{T}}^\alpha\| \\ u_{\mathrm{T}}^\alpha \end{pmatrix} \tag{47}$$

This law is inspired by the energy balance (42) and the work of Frémond Frémond, 2001, 2002, 2017. We will called the Frémond's model in the sequel. Let us show that the Frémond always dissipates energy

Lemma 2 *The Frémond's model of contact, impact and friction given by* (45) *and* (46) *satisfies the dissipation inequality, that is*

$$(v^+ + v^-)^\top H(q)\,\mathrm{d}i \leqslant 0. \tag{48}$$

Proof. Almost everywhere with respect to the Lebesgue measure $\mathrm{d}t$, we have $u^+ = u^- = u$ the constitutive law (45) reduces to

$$-\begin{pmatrix} \frac{1}{2}(1 + e^\alpha)u_{\mathrm{N}}^\alpha + \mu^\alpha \|u_{\mathrm{T}}^\alpha\| \\ u_{\mathrm{T}}^\alpha \end{pmatrix} \in N_K(r). \tag{49}$$

The three possible cases of (49) coincide with those of (33). In other words, the law in (45) is identical for the smooth dynamics to the standard Coulomb's law with the Signorini condition at the velocity level given by (33). At any time t_i, we obtain

$$-\Xi\left(\frac{1}{2}(u^+(t_i) + u^-(t_i))\right) \in N_K(p_i). \tag{50}$$

To be explicit, let us examine the three possible cases for a contact α:

1. take-off case: $p_i^\alpha = 0$. In that case, we have $\frac{1}{2}(u_{\mathrm{N}}^{\alpha,+} + u_{\mathrm{N}}^{\alpha,-}) + \frac{1}{2}(e^\alpha - 1)u_{\mathrm{N}}^{\alpha,-} \geqslant 0$ that is $u_{\mathrm{N}}^{\alpha,+} + eu_{\mathrm{N}}^{\alpha,-} \geqslant 0$.
2. sticking case: $\Xi(\frac{1}{2}(u^{\alpha,+}(t_i) + u^{\alpha,-}(t_i))) = 0$. In that case, we have $\frac{1}{2}(u_{\mathrm{T}}^{\alpha,+} + u_{\mathrm{T}}^{\alpha,-}) = 0$ and $u_{\mathrm{N}}^{\alpha,+} + eu_{\mathrm{N}}^{\alpha,-} = 0$ and $p_i^\alpha \in K^\alpha$. The impact law is satisfied since $u_{\mathrm{N}}^{\alpha,+} = -eu_{\mathrm{N}}^{\alpha,-}$. Note that in the sticking case, we have $u_{\mathrm{T}}^{\alpha,+} = -u_{\mathrm{T}}^{\alpha,-}$. Only the velocity average over the time of impact vanishes.
3. sliding case: $p_i^\alpha \neq 0$ and $\Xi(\frac{1}{2}(u^{\alpha,+}(t_i)+u^{\alpha,-}(t_i))) \neq 0$. Then we have $p_i^\alpha \in \partial K^\alpha$ and $\Xi(\frac{1}{2}(u^{\alpha,+}(t_i)+u^{\alpha,-}(t_i))) \in \partial K^{\alpha,*}$. This implies $u_{\mathrm{N}}^{\alpha,+}+eu_{\mathrm{N}}^{\alpha,-} = 0$, $\|p_{i,\mathrm{T}}^\alpha\| = \mu p_{i,\mathrm{N}}^\alpha$ and $\|\frac{1}{2}(u^+(t_i)+u^-(t_i))\|p_{i,\mathrm{T}}^\alpha = -\|p_{i,\mathrm{T}}^\alpha\|\frac{1}{2}(u^+(t_i) + u^-(t_i))$.

In the take-off case, the impulse p_i vanishes there the dissipated power vanishes. In the sliding case, we obtain

$$\begin{aligned}\frac{1}{2}(v^+(t_i)+v^-(t_i))^\top H^\alpha(q(t_i))p_i^\alpha &= \frac{1}{2}(u_{\rm N}^{\alpha,+}(t_i)+u_{\rm N}^{\alpha,-}(t_i))p_{{\rm N},i}^\alpha \\ &= \frac{1}{2}(1-e)u_{\rm N}^{\alpha,-}(t_i)p_{{\rm N},i}^\alpha \leqslant 0\end{aligned} \tag{51}$$

since $\frac{1}{2}(u_{\rm T}^{\alpha,+}(t_i)+u_{\rm T}^{\alpha,-}(t_i))$ vanishes. Since $u_{\rm N}^{\alpha,-}(t_i)\leqslant 0$ and $e\in[0,1]$, the dissipated power by an impact is non-positive. In the sliding case, we get

$$\begin{aligned}\frac{1}{2}(v^+(t_i)+v^-(t_i))^\top H^\alpha(q(t_i))p_i^\alpha &= \frac{1}{2}(1-e)u_{\rm N}^{\alpha,-}(t_i)p_{{\rm N},i}^\alpha+\frac{1}{2}(u_{\rm T}^{\alpha,+}(t_i)+u_{\rm T}^{\alpha,-}(t_i))^\top p_{{\rm T},i}^\alpha \\ &= \frac{1}{2}(1-e)u_{\rm N}^{\alpha,-}(t_i)p_{{\rm N},i}^\alpha-\frac{1}{2}\mu^\alpha p_{{\rm N},i}^\alpha\|\frac{1}{2}(u_{\rm T}^{\alpha,+}(t_i)+u_{\rm T}^{\alpha,-}(t_i))\|\leqslant 0.\end{aligned} \tag{52}$$

□

We conclude that the dissipated power related to the reaction forces is always non-negative. In other words, the model satisfies the principles of the thermodynamics at the impact time.

Remark 1 *The model proposed by Frémond is not only consistent with the principles of thermodynamics, but has also been validated by experimental tests. In (Cholet, 1998; Frémond, 2001, 2002, 2017), we find experimental results showing that percussions belong to the Coulomb cone. Even more interestingly, these experimental tests also show that Coulomb's relationship between impact and velocity is valid for the averaged velocity* $\frac{1}{2}(u^+(t_i)+u^-(t_i)$ *and not for the velocity after impact* $u^+(t_i)$.

3 A new time-stepping scheme with dissipation properties

3.1 Flaws in the standard Moreau–Jean scheme in the frictional case.

Formulation of the Moreau–Jean scheme The Moreau–Jean scheme (Jean, 1999; Jean and Moreau, 1987; Moreau, 1988) on an interval $(t_k, t_{k+1}]$ of length h is as follows

$$\begin{cases} M(v_{k+1}-v_k)+hKq_{k+\theta}+hCv_{k+\theta}-hF_{k+\theta}=H(q_k)p_{k+1}, & \text{(53a)}\\ q_{k+1}=q_k+hv_{k+\theta}, & \text{(53b)}\\ u_{k+1}=H^\top(q_k)\,v_{k+1}, & \text{(53c)}\\ -\Gamma(u_{k+1})\in {\rm N}_K(p_{k+1}) & \text{(53d)}\end{cases}$$

with $\theta\in[0,1]$. The following approximations are considered:

$$v_{k+1}\approx v(t_{k+1});\quad u_{k+1}\approx u^+(t_{k+1});\quad p_{k+1}\approx {\rm d}i((t_k,t_{k+1}]), \tag{54}$$

The index set $\bar{\mathcal{I}}^1$ is approximated at each time step by

$$\bar{\mathcal{I}}^1_k=\{\alpha\in\mathcal{I}\mid g^\alpha(q_k+hv_k)\leqslant 0 \text{ and } U^\alpha_k\leqslant 0\}. \tag{55}$$

The discrete variables at contact are collected in the unknown variables u_{k+1} and p_{k+1} for the indices such that $\alpha\in\bar{\mathcal{I}}^1_k$.

Discrete dissipation analysis Following the Lemma 5.1 in Acary (2016), the discrete-time dissipation equality of the Moreau–Jean scheme (53) over a time-step $[t_k,t_{k+1}]$ is given by

$$\Delta\mathcal{E}-W^{\sf ext}_{k+1}-W^{\sf damping}_{k+1} = (\frac{1}{2}-\theta)\left[\|v_{k+1}-v_k\|^2_M+\|q_{k+1}-q_k\|^2_K\right]+u^\top_{k+\theta}p_{k+1}, \tag{56}$$

where the discrete approximation of the work done by the external forces within the step by

$$W^{\sf ext}_{k+1}=hv^\top_{k+\theta}F_{k+\theta}\approx\int_{t_k}^{t_{k+1}}Fv\;{\rm d}t, \tag{57}$$

and the discrete approximation of the work done by the damping term by

$$W^{\sf damping}_{k+1}=-hv^\top_{k+\theta}Cv_{k+\theta}\approx-\int_{t_k}^{t_{k+1}}v^TCv\;{\rm d}t. \tag{58}$$

To ensure that the scheme dissipates energy, the first condition is $\theta \geqslant \dfrac{1}{2}$. The second condition is related to the sign of the discrete work of the reaction impulse given by

$$u_{k+\theta}^\top p_{k+1} = u_{\mathrm{N},k+\theta}^\top p_{\mathrm{N},k+1} + u_{\mathrm{T},k+\theta}^\top p_{\mathrm{T},k+1}. \tag{59}$$

For a contact α, the normal term $u^{\alpha,\top}_{\mathrm{N},k+\theta} p^\alpha_{\mathrm{N},k+1}$ vanishes in the take-off case, and is given by

$$u^{\alpha,\top}_{\mathrm{N},k+\theta} p^\alpha_{\mathrm{N},k+1} = (1-\theta(1+e^\alpha))u^\alpha_{\mathrm{N},k} p^\alpha_{\mathrm{N},k+1} \tag{60}$$

in the sliding and sticking cases. With the condition

$$\theta \leqslant \frac{1}{1+e^\alpha} \leqslant 1, \text{ for all } \alpha \in \mathcal{I}. \tag{61}$$

we conclude that $u_{\mathrm{N},k+\theta}^\top p_{\mathrm{N},k+1} \leqslant 0$. For the frictional term, $u^{\alpha,\top}_{\mathrm{T},k+\theta} p^\alpha_{\mathrm{T},k+1}$, we get

$$u^{\alpha,\top}_{\mathrm{T},k+\theta} p^\alpha_{\mathrm{T},k+1} = \theta u^{\alpha,\top}_{\mathrm{T},k+1} p^\alpha_{\mathrm{T},k+1} + (1-\theta) u^{\alpha,\top}_{\mathrm{T},k}\, p^\alpha_{\mathrm{T},k+1} \tag{62}$$

The first term is nonnegative but nothing can be said about the sign of the second term. We will see that on a simple example this term can positive and then the scheme will generate energy.

This fact is not really surprising. If we accept that the scheme is consistent and converges towards the continuous-time model, then we must converge towards a model that can generate energy. In practice, energy can be generated even by reducing the time steps to obtain a more accurate solution.

3.2 Principles of the proposed scheme

The following proposal for a new time-stepping scheme differs in the way the contact law is discretised to obtain contact dissipation that is always positive. The new Moreau–Jean time-stepping is defined as follows:

$$\begin{cases} M(v_{k+1}-v_k) + hKq_{k+\theta} + hCv_{k+\theta} - hF_{k+\theta} = H(q_k)p_{k+1}, & \text{(63a)}\\ q_{k+1} = q_k + hv_{k+\theta}, & \text{(63b)}\\ u_{k+1} = H^\top(q_k)\, v_{k+1}, & \text{(63c)}\\ -\Theta(u_{k+\theta}) \in \mathrm{N}_K(p_{k+1}) & \text{(63d)} \end{cases}$$

with the function Θ as

$$\Theta^\alpha(u^\alpha) = u^\alpha + \begin{pmatrix} (\theta(1+e^\alpha)-1)u^\alpha_{\mathrm{N},k} + \mu^\alpha \|u^\alpha_{\mathrm{T}}\| \\ 0 \\ 0. \end{pmatrix} \tag{64}$$

Let us first details what is solved in the three possible cases of the contact law.

1. take-off case: $p^\alpha_{k+1} = 0$. In that case, we have

$$u^\alpha_{\mathrm{N},k+\theta} + (\theta(1+e^\alpha)-1)u^\alpha_{\mathrm{N},k} + \mu^\alpha\|u^\alpha_{\mathrm{T},k+\theta}\| \geqslant \mu^\alpha\|u^\alpha_{\mathrm{T},k+\theta}\| \tag{65}$$

   that is

$$u^\alpha_{\mathrm{N},k+1} + e^\alpha u^\alpha_{\mathrm{N},k} \geqslant 0. \tag{66}$$

2. sticking case: $\Theta^\alpha(u^\alpha_{k+\theta}) = 0$. In that case, we have $u^\alpha_{\mathrm{T},k+\theta} = 0$ and

$$u^\alpha_{\mathrm{N},k+\theta} + (\theta(1+e^\alpha)-1)u^\alpha_{\mathrm{N},k} = 0, \tag{67}$$

   that is

$$u^\alpha_{\mathrm{N},k+1} = -e^\alpha u^\alpha_{\mathrm{N},k} \tag{68}$$

3. sliding case: $p^\alpha_{k+\theta} \neq 0$ and $\Theta^\alpha(u^\alpha_{k+\theta}) \neq 0$. Then we have $p^\alpha_{k+\theta} \in \partial K^\alpha$ and $\Theta^\alpha(u^\alpha_{k+\theta}) \in \partial K^{\alpha,*}$. This implies

$$u^\alpha_{\mathrm{N},k+\theta} + (\theta(1+e^\alpha)-1)u^\alpha_{\mathrm{N},k} + \mu^\alpha\|u^\alpha_{\mathrm{T},k+\theta}\| = \mu^\alpha\|u^\alpha_{\mathrm{T},k+\theta}\| \tag{69}$$

   that is

$$u^\alpha_{\mathrm{N},k+\theta} + (\theta(1+e^\alpha)-1)u^\alpha_{\mathrm{N},k} = 0 \tag{70}$$

   and then

$$u^\alpha_{\mathrm{N},k+1} = -e^\alpha u^\alpha_{\mathrm{N},k}. \tag{71}$$

   For the frictional part, we have

$$\|p^\alpha_{\mathrm{T},k+1}\| = \mu p^\alpha_{\mathrm{N},k+1} \text{ and } \|u^\alpha_{\mathrm{T},k+\theta}\| p^\alpha_{\mathrm{T},k+1} = -\|p^\alpha_{\mathrm{T},k+1}\| u^\alpha_{\mathrm{T},k+\theta}. \tag{72}$$

To summarize, the Newton impact law is written in a similar way as in the Moreau–Jean time-stepping scheme: for the normal part of the law, the pre-impact velocity it the velocity at the beginning pf the time-step $u^\alpha_{\textsc{n},k}$ and the post -impact velocity is the velocity at the end of the time step $u^\alpha_{\textsc{n},k+1}$. The Coulomb's friction law is written with the average velocity $u^\alpha_{\textsc{t},k+\theta}$.

When $\theta = 1/2$, the function Θ^α simplifies in

$$\Theta^\alpha(u^\alpha) = u^\alpha + \begin{pmatrix} u^\alpha_{\textsc{n}} + \dfrac{1}{2}(e^\alpha - 1)u^\alpha_{\textsc{n},k} + \mu^\alpha \|u^\alpha_{\textsc{t}}\| \\ u^\alpha_{\textsc{t}}. \end{pmatrix} \tag{73}$$

and the discrete contact law is

$$-\Theta\left(\frac{1}{2}(u_{k+1} + u_k)\right) \in \mathrm{N}_K(p_{k+1}) \tag{74}$$

We can observe that the expression (73) is very similar to the expression of the function Ξ in (47). In practice, if the scheme converges, we can expect to approximate the Frémond impact law.

When $\theta = 1$, the function Θ^α is

$$\Theta^\alpha(u^\alpha) = u^\alpha + \begin{pmatrix} u^\alpha_{\textsc{n}} + e^\alpha u^\alpha_{\textsc{n},k} + \mu^\alpha \|u^\alpha_{\textsc{t}}\| \\ u^\alpha_{\textsc{t}}. \end{pmatrix} = \Gamma(u^\alpha) \tag{75}$$

and the discrete contact law is

$$-\Theta(u_{k+1}) = -\Gamma(u_{k+1}) \in \mathrm{N}_K(p_{k+1}) \tag{76}$$

We observe that the scheme if similar to the Moreau–Jean time-stepping for the case $\theta = 1$.

3.3 Discrete dissipation properties and energy balance

Let us give now a result concerning the dissipation of this new Moreau–Jean scheme.

Proposition 1 *The new Moreau–Jean scheme dissipates energy in the sense that*

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) \leqslant W^{\mathsf{ext}}_{k+1} + W^{\mathsf{damping}}_{k+1}, \tag{77}$$

if

$$\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+\bar{e}} \leqslant 1. \tag{78}$$

where $\bar{e} = \max \rho^\alpha, \alpha \in \mathcal{I}$.

In other words, providing that (78) is satisfied, the variation of the total mechanical energy of the system is always less than the energy supplied by the external and damping forces.

Proof The discrete-time dissipation equality of the new Moreau–Jean scheme (63) over a time-step $[t_k, t_{k+1}]$ is also given by (56). Obviously, we have

$$(\frac{1}{2} - \theta)\left[\|v_{k+1} - v_k\|^2_M + \|q_{k+1} - q_k\|^2_K\right] \leqslant 0, \text{ if and only if } \theta \geqslant 1/2. \tag{79}$$

It remains to prove that $u^\top_{k+\theta} p_{k+1} \leqslant 0$. Let us examine the three possible cases for a contact α:

1. take-off case: $p^\alpha_{k+1} = 0$. In that case, we have $u^{\alpha,\top}_{k+\theta} p^\alpha_{k+1} = 0$.
2. sticking case: $\Theta^\alpha(u^\alpha_{k+\theta}) = 0$. In that case, we have $u^\alpha_{\textsc{t},k+\theta} = 0$ and then $u^{\alpha,\top}_{k+\theta} p^\alpha_{k+1} = u^\alpha_{\textsc{n},k+\theta} p^\alpha_{\textsc{n},k+1}$. From (67), we conclude that $u^\alpha_{\textsc{n},k+\theta} \leqslant 0$ since $\theta \leqslant \dfrac{1}{1+\bar{e^\alpha}}$ and $u^\alpha_{\textsc{n},k} \leqslant 0$ and then $u^{\alpha,\top}_{k+\theta} p^\alpha_{k+1} \leqslant 0$ since $p^\alpha_{\textsc{n},k+1} \geqslant 0$.
3. sliding case: $p^\alpha_{k+\theta} \neq 0$ and $\Theta^\alpha(u^\alpha_{k+\theta}) \neq 0$. Then we have $p^\alpha_{k+\theta} \in \partial K^\alpha$ and $\Theta^\alpha(u^\alpha_{k+\theta}) \in \partial K^{\alpha,*}$. From (70), we conclude that $u^\alpha_{\textsc{n},k+\theta} \leqslant 0$ since $\theta \leqslant \dfrac{1}{1+\bar{e^\alpha}}$ and $u^\alpha_{\textsc{n},k} \leqslant 0$,, and then $u^\alpha_{\textsc{n}k+\theta} p^\alpha_{\textsc{n},k+1} \leqslant 0$. For the frictional part, we have from (72)

$$u^{\alpha,\top}_{\textsc{t},k+\theta} p^\alpha_{\textsc{t},k+1} = -\mu^\alpha p^\alpha_{\textsc{n},k+1} \|u^\alpha_{\textsc{t},k+\theta}\| \leqslant 0 \tag{80}$$

□

4 Conclusion

In this article, we propose

- to use the model of frictional contact with impact inspired by the work of Frémond, which we show is always dissipative,
- to modify Moreau-Jean's time integration method in order to approximate Frémond's model and we show that the scheme is dissipative in discrete time.

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