A back test to see how accurate the black scholes formula is.

It is known that the price of an American and European call on a non-dividend paying stock is the same. For that reason we should be able to use the Black-Scholes formula to price an American call on an arbitrary non-dividend paying stock. We will need the following prelims for the set up of the Black-Scholes formula

1 prelims

1.1 geometric brownian motion

Denote the price of a stock at time t by S_t. We assume that the price of the stock follows a geometric Brownian motion:

$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{1.1}$$ {E: geo_b}

where B_t is a standard Brownian motion. Through the use of Ito's formula, it can be shown that the solution to SDE (1.1) is given by the following:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right). \tag{1.2}$$ {E: geo_b_sol

Note that this heuristically suggests that the average rate of return on the stock is μ and that the variance is σ^2. This is suggested by the following:

$$\frac{dS_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t. \tag{1.3}$$ {E: geo_b_roc

Suppose we denote by R_x, the rate of return of S_t over the period $[t, t+x]$. In other words,

$$R_x = \frac{S_{t+x} + S_t}{S_t}.$$

Thus on the interval $[t, t+x]$, (1.3) becomes

$$R_x = \mu \cdot x + \sigma(B_{t+x} - B_t), \tag{1.4}$$

which implies that,

$$\mathbb{E}\left(\frac{R_x}{x}\right) = \mu \tag{1.5}$$

since the expected value of a Brownian motion, $\mathbb{E}(B_t) = 0$. This suggestes that the average rate of change of R over $[t, t+x]$ is μ. Furthermore, since $\mathbb{E}(B_t^2) = t$, then

$$V(R_x) = \mathbb{E}(R_x^2) - \mathbb{E}(R_x)^2 = \mu^2 x^2 + \sigma^2(x) - \mu^2 x^2 = \sigma^2 x,$$

and thus,

$$\frac{V(R_x)}{x} = \sigma^2.$$

1.2 self-financing strategy

Consider a bond with price at time t given by β_t with risk free rate r. Next, consider a portfolio consisting only of shares of one stock and bonds. Let a_t be the amount of shares at time t and b_t be the amount of bonds at time t. Then the values of the portfolio at time t is given by,

$$V_t := a_t S_t + b_t \beta_t,$$

where the initial value of the portfolio is given by V_0. Consider of the period $[0, t]$, the capital gains of the stock given by,

$$a_t(S_t - S_0).$$

Next, suppose that we partition out interval $[0, t]$ by $0 = t_0, \cdots, t_i, \cdots, t_n = t$ and that we actively manage our stock holdings over this time period such that out current stock holding is given by the following process,

$$a_t = \sum_{i=0}^{n} a_{t_i} \mathbf{1}_{[t_{i-1}, t_i]}(t). \tag{1.6}$$ {E: holdings}

Then the capital gains take the following form,

$$\sum_{i=0}^{n} a_{t_i}(S_{t_i} - S_{t_{i-1}}), \tag{1.7}$$ {E: captiol_g

which is just the stochastic integral of (1.6). In other words, (1.7) can be written as the following:

$$\int_0^T a_t \mathrm{d}S_t = \mu \int_0^T a_t S_t \mathrm{d}t + \sigma \int_0^t a_t S_t \mathrm{d}B_t. \tag{1.8}$$

The same can be done with the bonds and $\int_0^T b_t \mathrm{d}\beta_t$ can be defined equivalently.

With this, we can define a self financing strategy as a pair (a, b) such that

$$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^T a_t \mathrm{d}S_t + \int_0^T b_t \mathrm{d}\beta_t. \tag{1.9}$$

In other words, the value at time t is equal to the initial value of the portfolio plus all capital gains.

2 Black-Scholes

A European call option on a stock that does not pay any dividends that has a strike, K, and expires at time T will pay $(S_t - K)^+$ at time T. The fair value of this option is defined to be the amount of money that can be invested into a self financing strategy at time 0 that at time T will be worth $(S_t - K)^+$. This fair value only depends on the price S_0 and the time to expire T. It is derived to be the following:

$$f(S_0, T) = S_0 \phi(g(S_0, T)) - K e^{-rt} \phi(h(S_0, T)), \tag{2.1}$$

where $\phi(t)$ is the standard normal cumulative distribution and

$$g(x, t) = \left[\ln(x/K) + \left(r + \frac{1}{2}\sigma^2 \right) t \right] / \sigma\sqrt{t},$$

$$h(x, t) = g(x, t) - \sigma\sqrt{t}.$$

Furthermore, it can be shown that the amount of shares owned a_t and bonds owned b_t are given by the following:

$$a_t = \frac{\partial}{\partial x} f(S_t, T - t) \quad \text{and} \quad b_t = \frac{f(S_t, T - t) - a_t S_t}{\beta_t}.$$

3 back-test

Historical end of day option prices are freely available though Polygon I believe. Here is what we can try.

1. Pick a non dividend paying stock, AMZN for example.

2. Pick a date in the past, 11-1-2022. This will be time $t = 0$.
3. Pick a call in the future. There was a 95 strike call expiring on 12-2-2022.
4. Find the price at $t = 0$. According to Polygon, this call opened at \$7 and closed at \$6.04.
5. Since we are back testing, S_t is not a random variable, we can treat it as a deterministic sequence and then calculate a_t and b_t explicitly. To do this we can partition the time interval 2022-11-1 to 2022-12-2 and then do find the fair price of the option discretely. We can see if the fair price is higher or lower than the close price of the option on 2022-11-1. We also run a monte carlo simulation for AMZN by simulation a geometric brownian motion with μ and σ correctly chosen. With this simulations, we can calculate the fair price of the option and see how theses prices compare.