

Graduate Seminar

An introduction to stochastic calculus with applications to finance and SPDEs

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Calculus

In undergraduate Calculus, we focus only on real valued measurable on \mathbb{R}^d :

$$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R},$$

and the measure space that is implied is the following:

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda),$$

where λ is Lebesgue measure. Recall the average value of f on (a, b) :

$$\frac{1}{\lambda[(a, b)]} \int_a^b f(x) \, dx = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Stochastic Calculus

When studying Probability Theory, we consider measurable functions (random variables),

$$X : \Omega \rightarrow \mathbb{R},$$

and we consider the measure spaces

$$(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda),$$

where Ω and \mathcal{F} are referred to as the sample space and set of possible events and

$$\mathbb{P}(\Omega) = 1.$$

The average or expected value of X is

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}} x dF(x) = \int_{\mathbb{R}} xf(x) dx.$$

Stochastic Processes

Definition (Stochastic Process)

A stochastic process is a random variable that is indexed by set, T , which usually represents time:

$$X_t(\omega), \quad t \in T, \omega \in \Omega.$$

One can think a stochastic processes as a two parameter function

$$X_t(\omega) = X(t, \omega) : T \times \Omega \rightarrow \mathbb{R}.$$

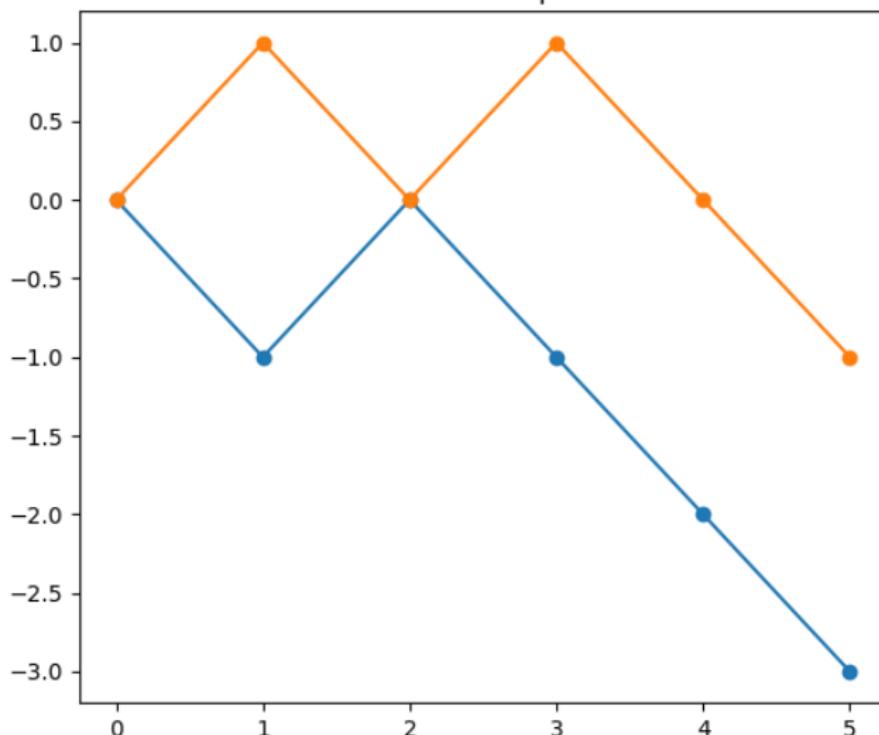
Definition (Sample Path)

The variable ω represents the idea of a single possible outcome. The following function is then referred to as a *sample path*:

$$t \mapsto X_t(\omega).$$

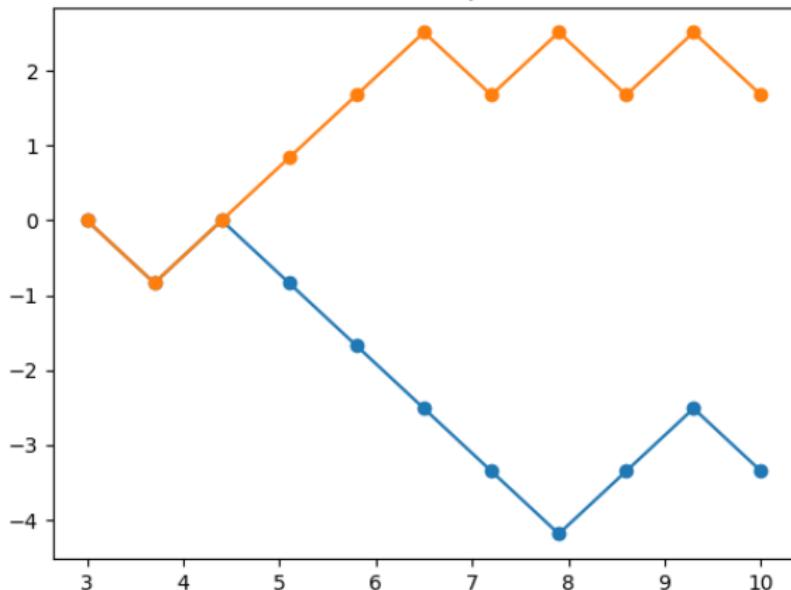
Random Walk Example

2 sample paths of a random walk
jump size = 1.0 and step size = 1.0
number of steps = 5



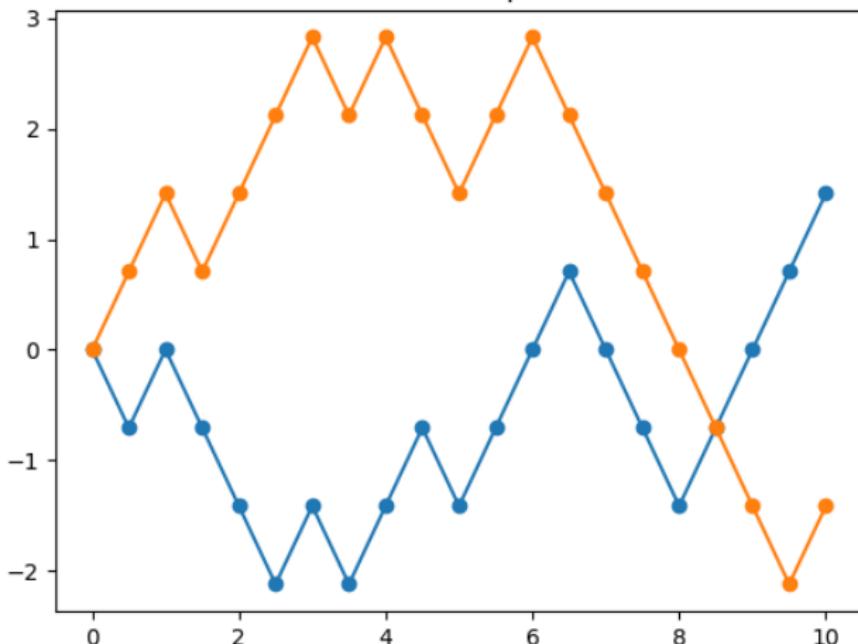
Random Walk Example

2 Sample paths of a random walk
jump size = 0.837 and step size = 0.7
number of steps = 10



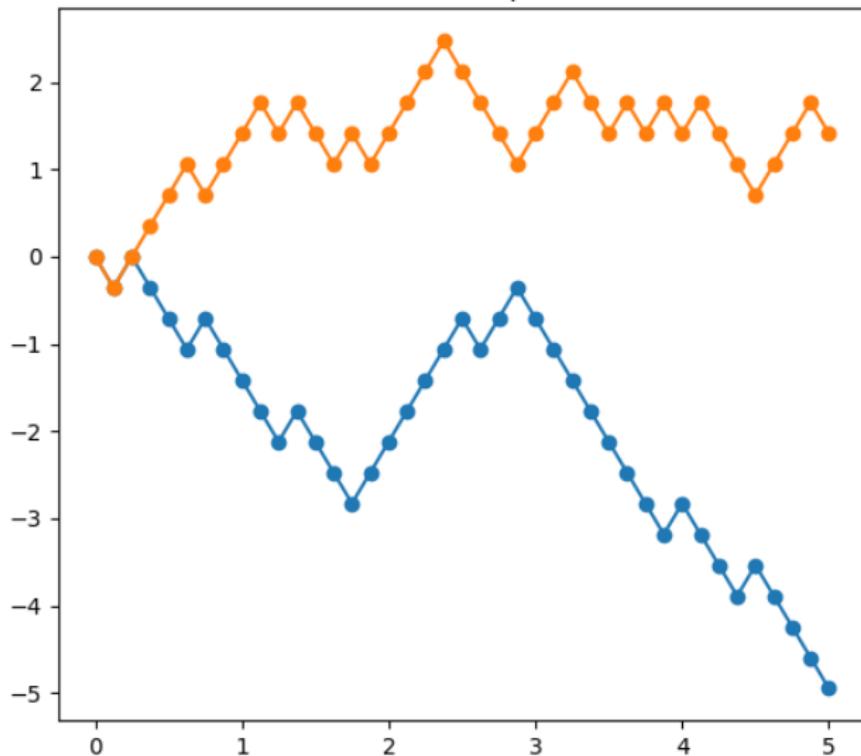
Random Walk Example

2 sample paths of a random walk
jump size = 0.707 and step size = 0.5
number of steps = 20

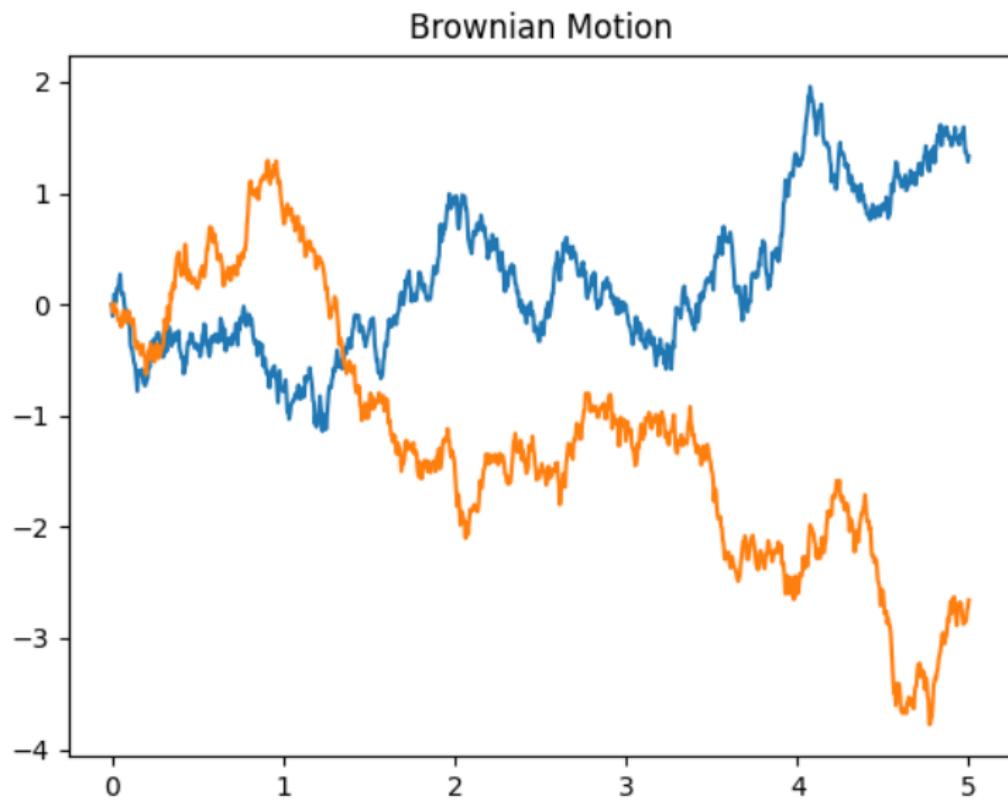


Random Walk Example

2 sample paths of a random walk
jump size = 0.354 and step size = 0.12
number of steps = 40



Random Walk Example



Brownian Motion.

Theorem

Consider a random walk $X = \{X_0 = 0, X_{t_1}, \dots, X_{t_n}\}$. Suppose that for $k \in \{1, \dots, n\}$ that

$$\mathbb{P}[X_{t_k} = -h] = \mathbb{P}[X_{t_k} = h] = 1/2.$$

and suppose that the time increments are all constant:

$$t_k - t_{k-1} = d \quad \forall k \in \{1, \dots, n\}.$$

Suppose that $h^2 = d$. Now define

$$B_n(t_k) = \sum_{i=0}^k X_{t_i}.$$

Then B_n converges to a Brownian motion.

The Ito integral (special case)

Suppose that f is a nice continuous function. Then we can define:

$$\int_0^t f(s)dB_s(\omega) := \lim_{\|P\| \rightarrow 0} \sum_{i \in P} f(t_{i-1})(B_{t_i}(\omega) - B_{t_{i-1}}(\omega)),$$

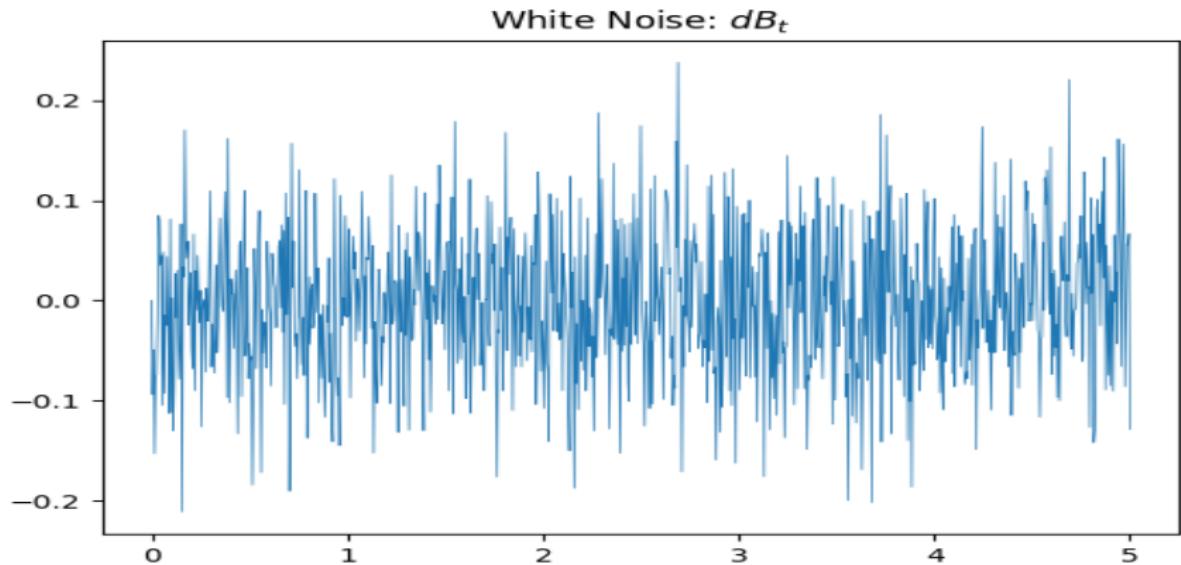
where $P = t_0 = 0, t_1, t_2, \dots, t_n = t$ is a partition of $[0, t]$. In other words, the Ito integral in this case the left point Riemann sum.

In general, this formula is not true because the increments

$$dB_{t_i}(\omega) := B_{t_i}(\omega) - B_{t_{i-1}}(\omega)$$

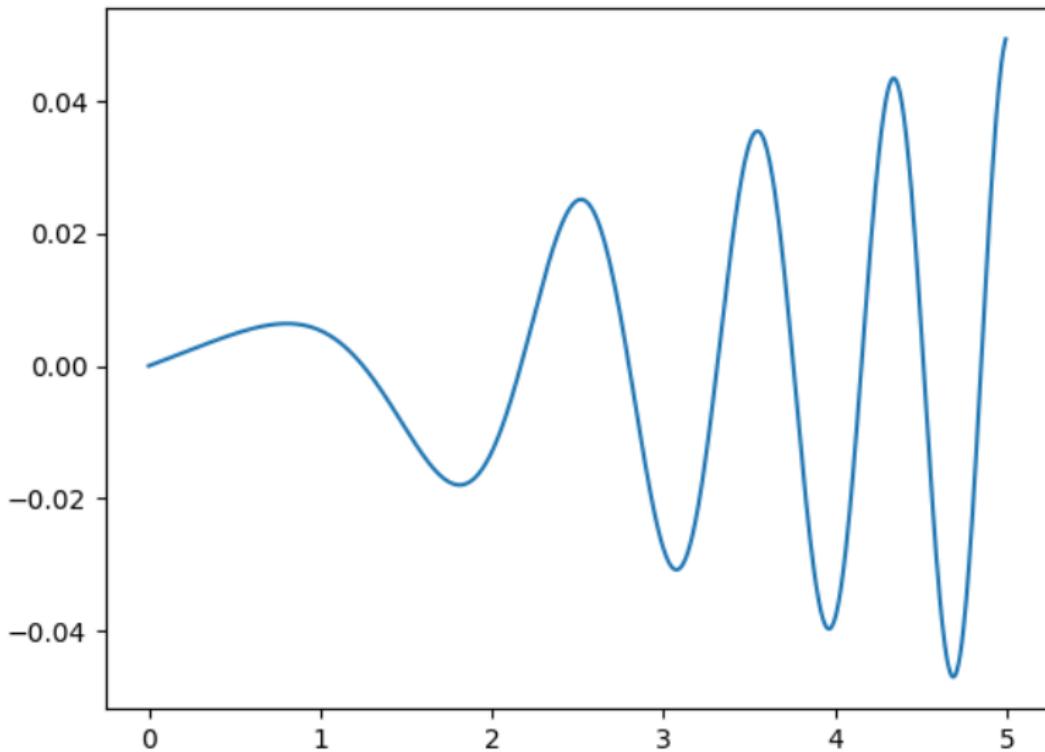
are very bad.

White Noise



Increments of a smooth function for reference

Increments, $df(x_i) = f(x_i) - f(x_{i-1})$, for $f(x) = \sin(x^2)$



A 'Fundamental Theorem of Ito Calculus'

If f is twice continuously differentiable then we have the following
"Fundamental Theorem of Ito Calculus"

$$\int_a^t \frac{d}{ds} f(B_s) dB_s = f(B_t) - f(B_a) - \frac{1}{2} \int_a^t \frac{d^2}{ds^2} (B_s) ds.$$

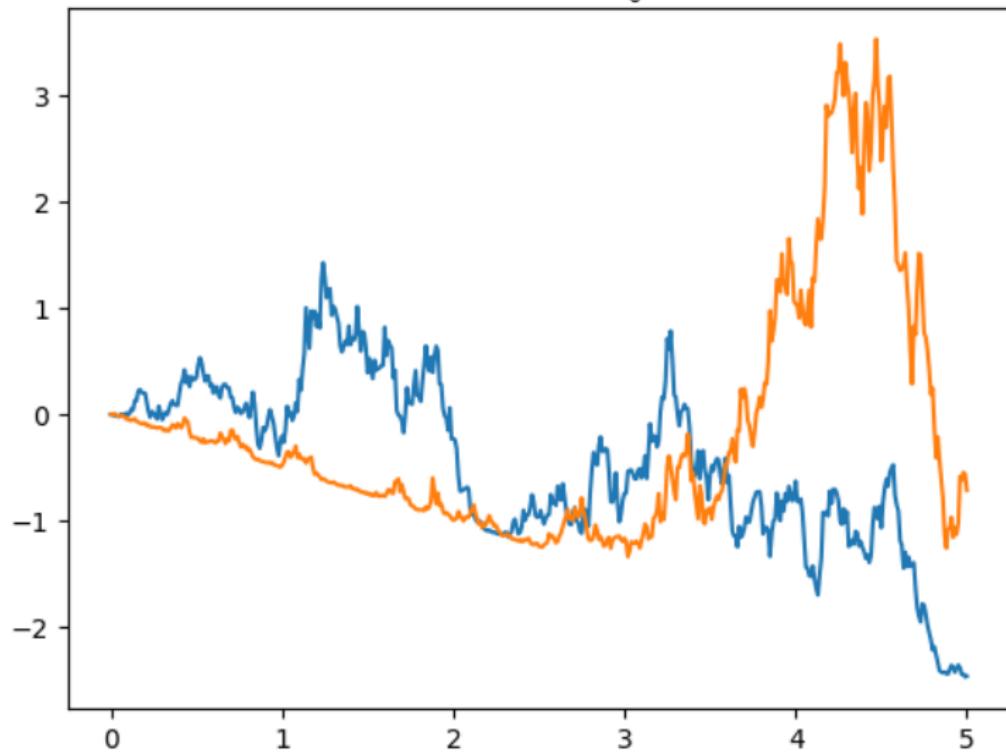
Example

If we apply the above to $f(x) = x^2/2$ with $a = 0$ then we have

$$\int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2}.$$

Two sample paths

Sample paths for $\int_0^t B_s dB_s$



One more example

For a nice function $f(t, x)$ we can generalize Itos formula to

$$f(t, B_t) - f(a, B_a) = \int_a^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \int_a^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds.$$

Example

Now consider the function $f(t, x) = tx$. Then

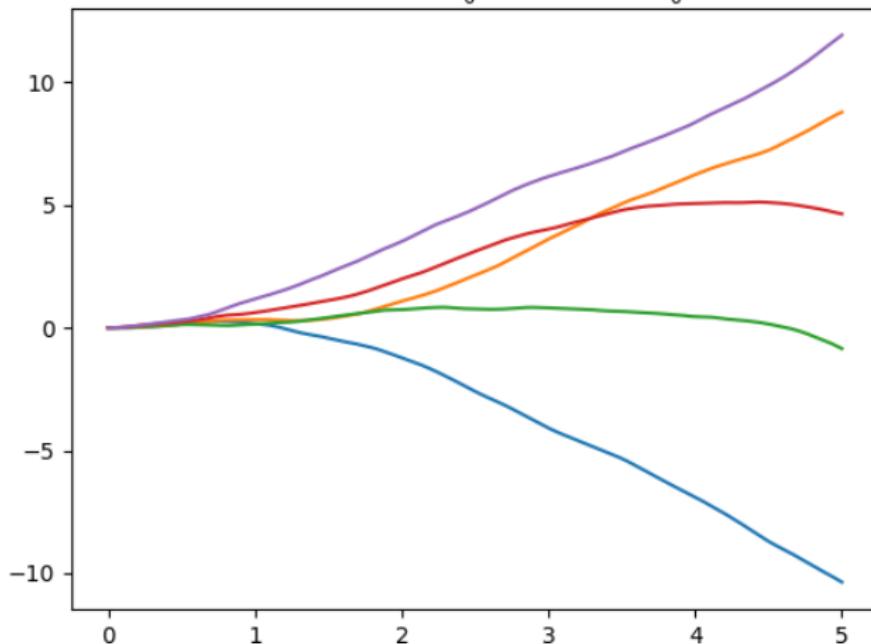
$$\frac{\partial f}{\partial t}(t, x) = x, \quad \frac{\partial f}{\partial x}(t, x) = t, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = 0$$

Plugging this in above for $a = 0$ gives us

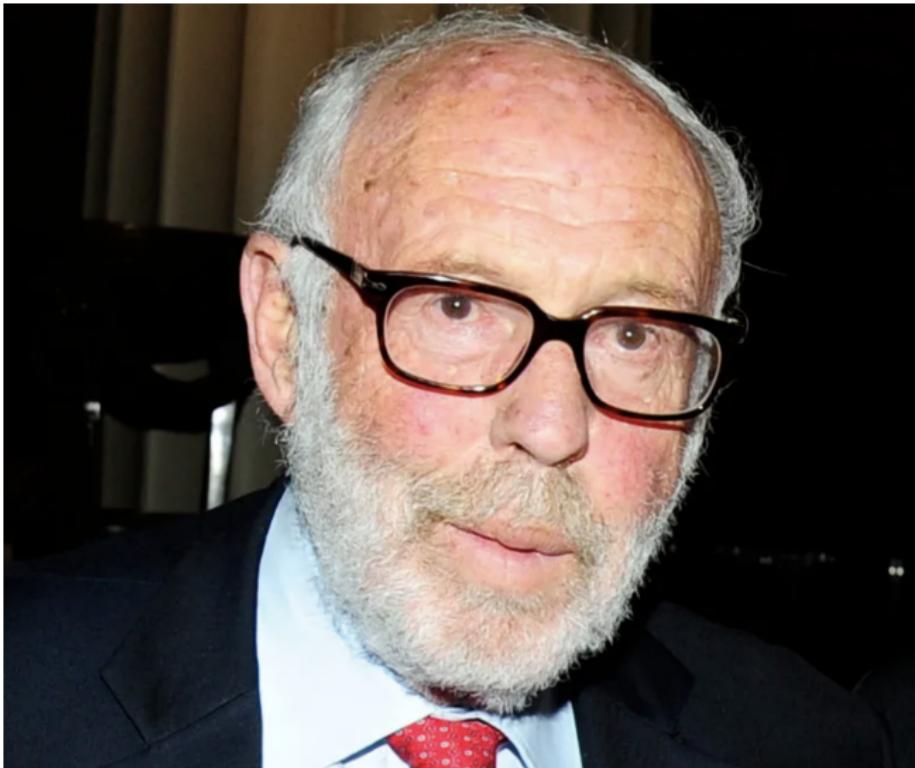
$$tB_t = \int_0^t s dB_s + \int_0^t B_s ds, \quad \Rightarrow \quad \int_0^t B_s ds = tB_t - \int_0^t s dB_s.$$

Smoothness?!

5 sample paths of $\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s$



Why do we care about Brownian Motion?



An application to finance.

Consider a stock that has value S_t at time t . Consider $t = 0$ to be the point of entry into the investment. We can define a return rate as follows:

$$S_t = S_0(1 + R_t), \quad R_t := \frac{S_t - S_0}{S_0} \text{ is the return rate over the period } [0, t].$$

We want to make predictions of our return. In a very simple model, we can assume that

$$R_t \sim \text{Normal}(\mu_t, \sigma^2), \quad \mu \in \mathbb{R}, \sigma \in (0, \infty).$$

In terms of Stochastic Calculus, we express this as the following stochastic differential equation:

$$R_t = \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad \text{where } B_t \text{ is a Brownian motion.}$$

Modeling a stock price with geometric brownian motion

The notation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad \text{where } B_t \text{ is a Brownian motion}$$

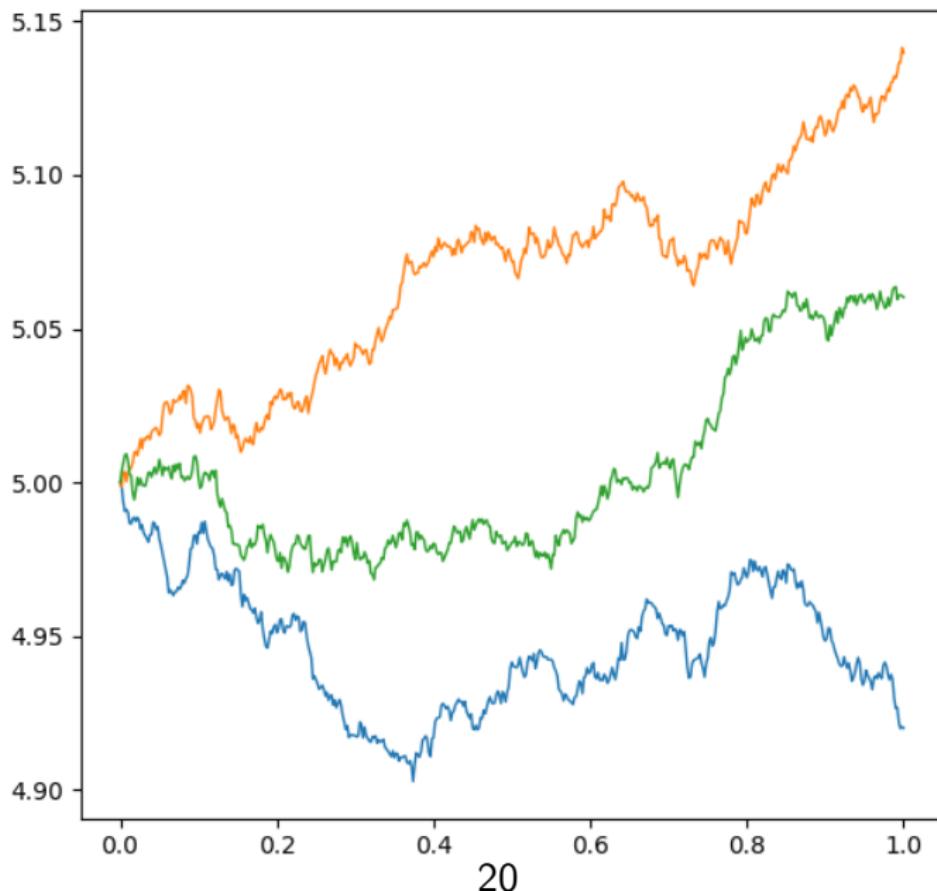
is only symbolic as dB_t does not exist. The above can be legitimately viewed as the following integral equation:

$$S_t - S_0 = \int_0^t \mu S_k dk + \int_0^t \sigma S_k dB_k.$$

The stochastic processes S_t that solves the above equation is called a Geometric Brownian Motion. Notice that we have came up with a random variable that models the stock price:

$$S_t(\omega) = S_0 + \int_0^t \mu S_k(\omega) dk + \int_0^t \sigma S_k(\omega) dB_k(\omega).$$

Geometric Brownian Motion: $dX_t = \sigma X_t dW_t + \mu X_t dt$
 $(a, b) = (0, 1)$, $X_a = 5$, $\sigma = 0.01$, and $\mu = 0.001$
we subdivide the domain n = 500 times



Black-Scholes option pricing model for European calls

Suppose at time $t = 0$ you buy a contract from a seller, allowing you to purchase 1 share of a stock at time $t = T$ and a set price of K , regardless of what the stock price at time T is.

$$\begin{aligned}\text{Payout and time } T &= \begin{cases} S_T - K & \text{if } S_T > k \\ 0 & \text{otherwise} \end{cases} \\ &= (S_T - K)^+\end{aligned}$$

What is a fair price for this contract?

For this we need a self-financing strategy.

We assume that the stocks price follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Let a_{t_i} be the amount of shares owned over the period $t \in (t_{i-1}, t_i]$. Then at any time $t \in [0, T]$, the amount of shares owned is

$$a_t = \sum_{i=1}^n a_{t_i} \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

and the realized over $[0, T]$

$$\begin{aligned} a_T S_T - a_0 S_0 &= \sum_{i=1}^n a_{t_i} (S_{t_i} - S_{t_{i-1}}) = \int_0^T a_t dS_t \\ &= \mu \int_0^T a_t S_t dt + \sigma \int_0^T a_t S_t dB_t \end{aligned}$$

Summarizing the last slide...

$$\text{Our stock profit on time } [0, T] = a_T S_T - a_0 S_0 = \underbrace{\int_0^T a_t dS_t}_{\text{ }}$$

Now suppose we also invest in a bond with interest rate, r , that has price β_t at time t . Similarly...

$$\text{Our bond profit on time } [0, T] = b_T S_T - b_0 S_0 = \underbrace{\int_0^T b_t d\beta_t}_{\text{ }}$$

Definition (Self Financing Strategy)

A pair of stocks and bonds, (a, b) , is self financing if

$$a_T S_T + b_T S_T = a_0 S_0 + b_0 S_0 + \int_0^T a_t dS_t + \int_0^T b_t d\beta_t$$

Fair price for a European call option

A fair price for a European call option that allows the buyer to buy 1 share at price K at time T regardless of the price of the stock S_T at time T is the amount one would need to invest in a self financing strategy,

$$\underbrace{a_T S_T + b_T S_T}_{V_T} = \underbrace{a_0 S_0 + b_0 S_0}_{V_0} + \int_0^T a_t dS_t + \int_0^T b_t d\beta_t,$$

such that the value of the strategy at time T is equal to $(S_T - K)^+$:

$$V_T = (S_T - K)^+$$

Let $f(S_t, T - t) = V_t$. So we see that

$$f(S_T, 0) = V_T = (S_T - K)^+ \quad \text{and} \quad f(S_0, T) = V_0$$

So we need to find f . Moreover, $f(S_0, T)$ is the fair price to pay.

It can be shown that f solves the following partial differential equation

$$\begin{cases} \frac{\partial f}{\partial s}(s, x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(s, x) + rx \frac{\partial f}{\partial x}(s, x) - rf(s, x) & (s, x) \in [0, T] \times (0, \infty) \\ f(0, x) = (x - K)^+ & x \in (0, \infty). \end{cases}$$

The solution is given by

$$f(t, x) = x\Phi(g(t, x)) - Ke^{-rt}\Phi(h(t, x)),$$

where

$$g(t, x) = \left[\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)t \right] / (\sigma\sqrt{t}),$$

and

$$h(t, x) = g(t, x) - \sigma/\sqrt{t},$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp\left(-\frac{|x|^2}{2}\right) dx$$

Potential for unlimited profit

In practice, option contracts are priced by "the market". This means that the cost of a contract is literally what the buyer is willing to pay or what the seller is willing to sell at.

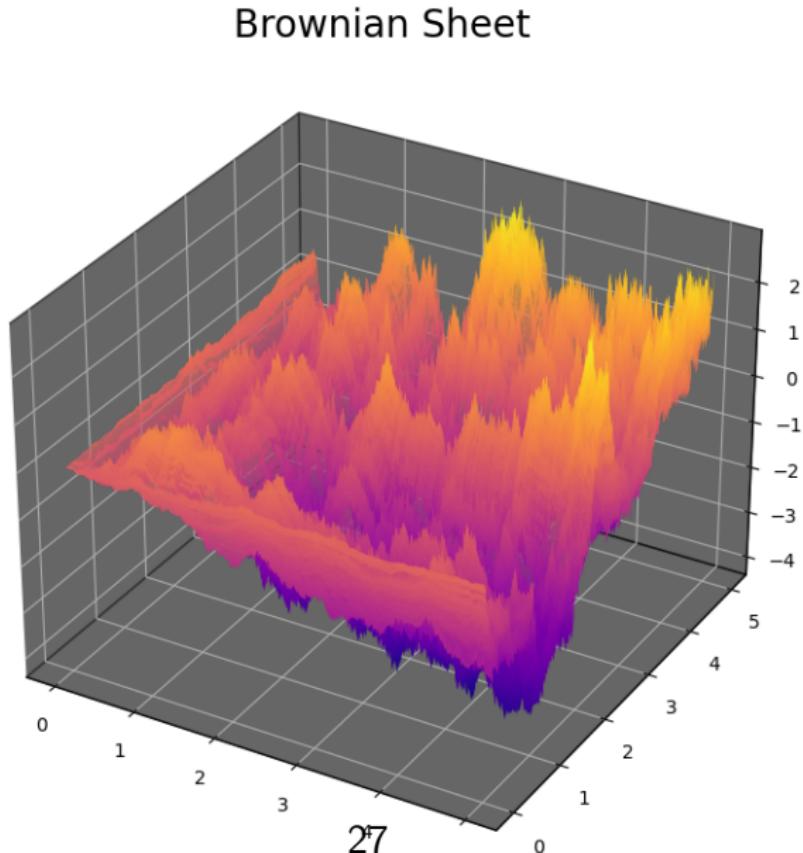
This gives the educated buyer (or seller) a chance to make profit.

Suppose the fair price for a call contract with strike K is $\$q$ but we find a buyer, who does not know about the Black Scholes model, who wants to buy the contract for $\$p > \q . We can do the following:

- ▶ sell a contract for $\$p$,
- ▶ invest $\$q$ in stock and bond at time $t = 0$ in a self financing strategy (HARD TO DO),
- ▶ keep the remaining $p - q$ dollars.
- ▶ pay the buyer $(S_T - K)^+$ at expiration.

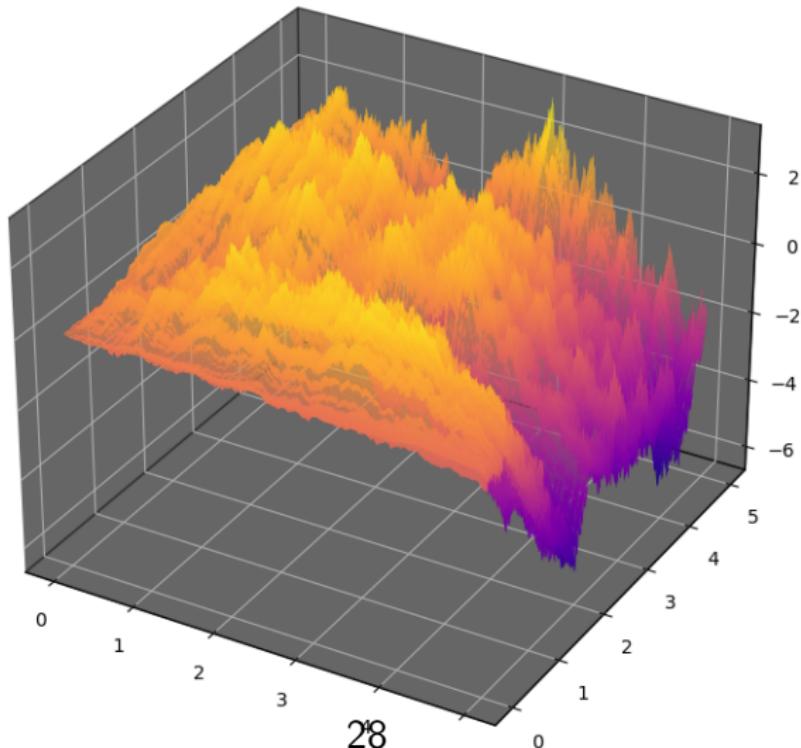
No matter what, we make $p - q$ dollars. If we sell 100 contracts then we make $100(p - q)\dots$

Sample path of a Brownian Sheet, $W(t, x)$



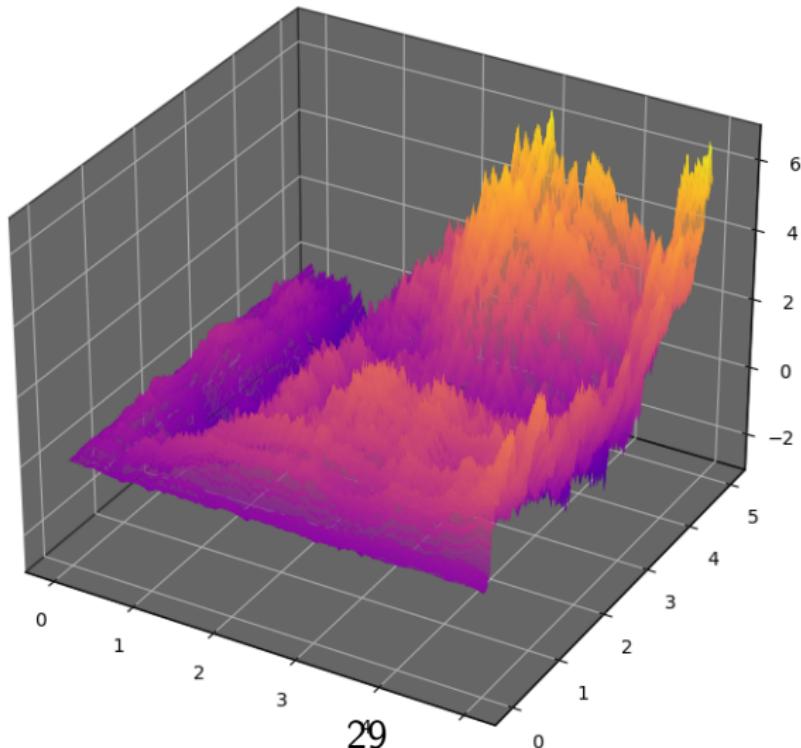
Sample path of a Brownian Sheet, $W(t, x)$

Brownian Sheet



Sample path of a Brownian Sheet, $W(t, x)$

Brownian Sheet



Stochastic Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = \lambda u(t, x) \frac{\partial^2}{\partial t \partial x} W(t, x) & t \geq 0, x \in \mathbb{R} \\ u(0, x) = f(x) & x \in \mathbb{R} \end{cases}$$

The solution to the stochastic heat equation is given through the following integral equation:

$$u(t, x) = \int_{\mathbb{R}} G(t, x - y) f(y) dy + \lambda \int_0^t \int_{\mathbb{R}} G(t - s, x - y) u(s, y) W(ds, dy),$$

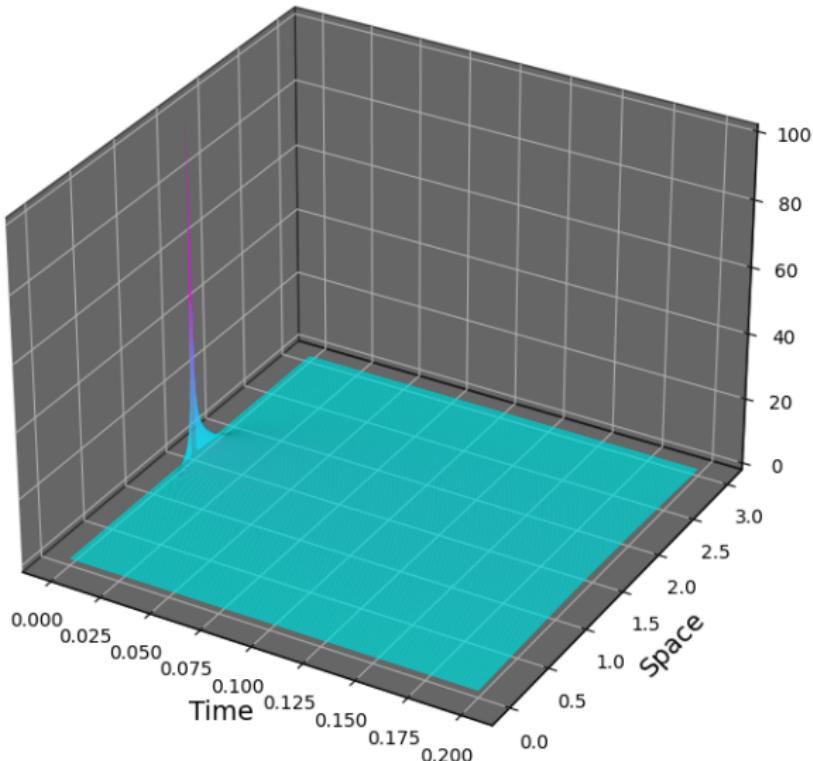
where

$$G(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right)$$

A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 0u(t, x)\dot{W}(t, x)$$

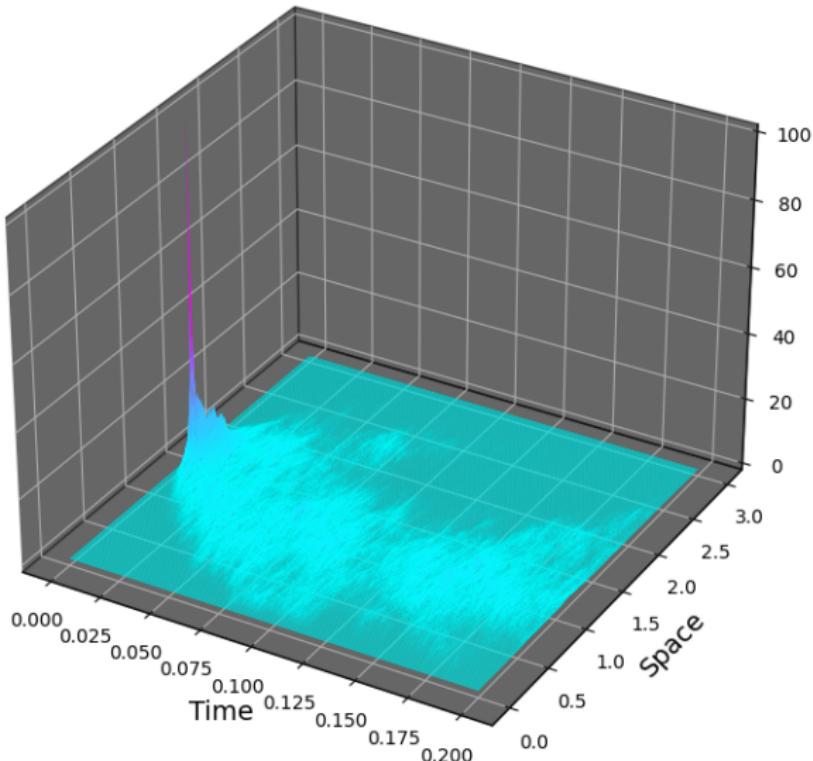
$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 1u(t, x)\dot{W}(t, x)$$

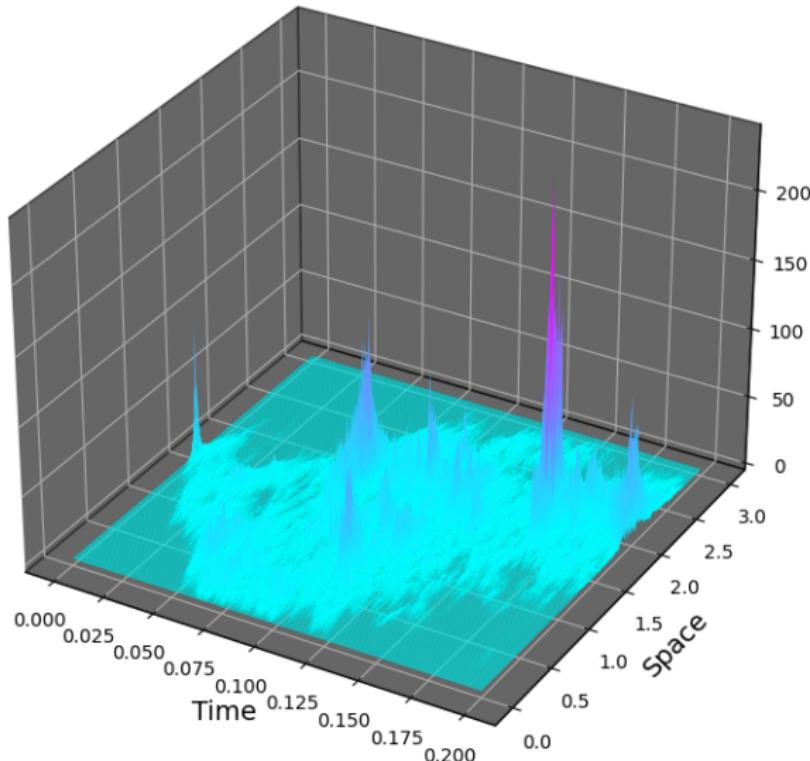
$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 2u(t, x)\dot{W}(t, x)$$

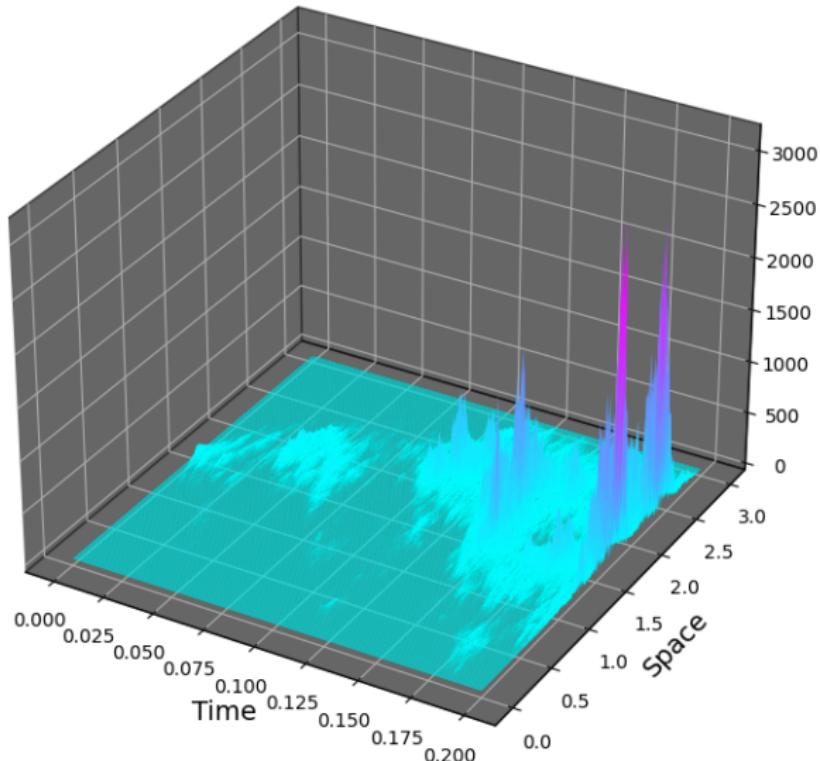
$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 2.15u(t, x)\dot{W}(t, x)$$

$$u(x, 0) = \delta_0(x)$$



Growth of Moments

We have a quantity that describes the growth of the p^{th} moments of the solutions to the stochastic heat question:

$$\lim_{t \rightarrow \infty} \frac{\log [\mathbb{E}(|u(t, x)|^p)]}{t}.$$

What does this limit say about the growth or decay of the moments?

Growth of Moments for the Stochastic Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) = \sqrt{\lambda} u(t, x) \frac{\partial}{\partial x} W(x) & t \geq 0, x \in \mathbb{R} \\ u(0, x) = 1 & x \in \mathbb{R} \end{cases}$$

Theorem (Hu 2002)

For the above equation, we can prove that

$$\lim_{t \rightarrow \infty} \frac{\log [\mathbb{E}(|u(t, x)|^p)]}{t^3} = \frac{p(p-1)\lambda^2}{24}$$

which means that

$$t \mapsto \mathbb{E}(|u(t, x)|^p) = O \left[\exp \left(\frac{p(p-1)\lambda^2}{24} t^3 \right) \right], \quad \text{as } t \rightarrow \infty.$$

So the moments grow exponentially in time.

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