Graduate Seminar
An introduction to stochastic calculus with applications to finance and SPDEs

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March 29, 2022

Stochastic Calculus

In Real Analysis we study measurable functions:

$$f : D \to R,$$

where both D and R are measurable spaces:

$$(D, \sigma(D), \mu_1) \quad \text{and} \quad (R, \sigma(R), \mu_2).$$

In undergraduate Calculus, we focus only on real valued measurable on $\mathbb{R}^d$:

$$f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d,$$

and the measure space that is implies is the following:

$$(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \lambda),$$

where λ is Lebesgue measure.

Stochastic Calculus

When studying Stochastic Calculus, and in general Probability Theory, we consider measurable functions,

$$X : \Omega \to \mathbb{R}^d,$$

and we consider the measure spaces

$$(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \lambda),$$

where Ω and $\mathcal{F}$ are referred to as the sample space and set of possible events and

$$\mathbb{P}(\Omega) = 1.$$

Definition (Random Variable)

A measurable function $X : \Omega \to \mathbb{R}^d$ is called a $\mathbb{R}^d$-*valued Random Variable*.

So essentially, Stochastic Calculus and Probability is just Real Analysis on Random Variables.

Stochastic Processes

Definition (Stochastic Process)

A stochastic process is a random variable that is indexed by set, T, which usually represents time:

$$X_t(\omega), \quad t \in T,\ \omega \in \Omega.$$

One can think a stochastic processes as a two parameter function

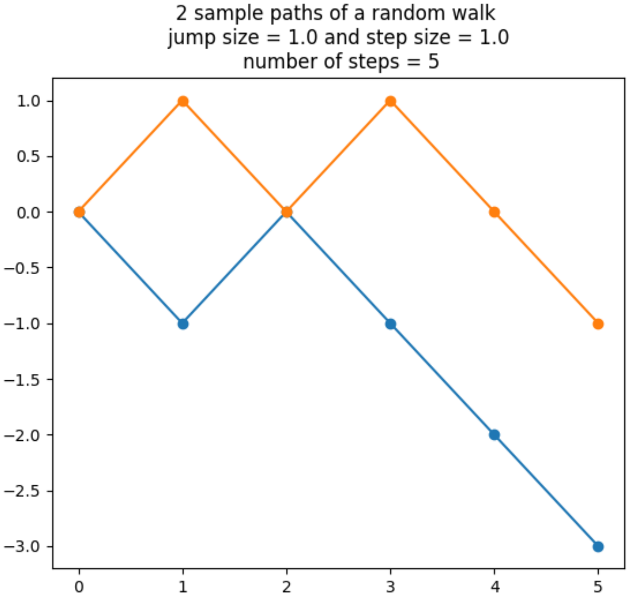
$$X_t(\omega) = X(t, \omega) : T \times \Omega \to \mathbb{R}^d.$$

Definition (Sample Path)

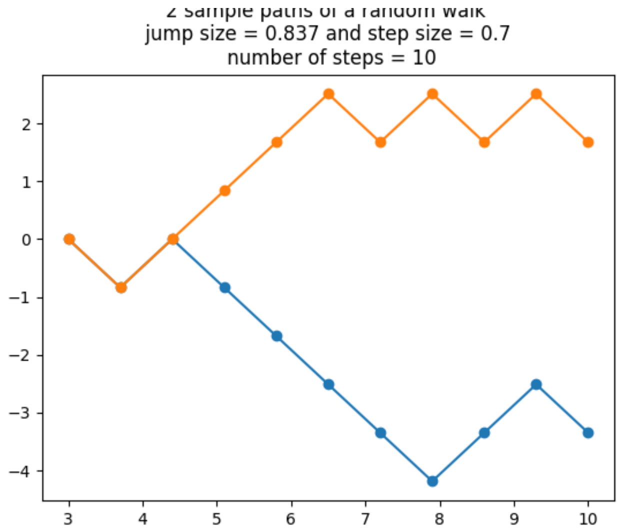
The variable ω represents the idea of a single possible outcome. The following function is then referred to as a *sample path*:

$$t \mapsto X_t(\omega).$$

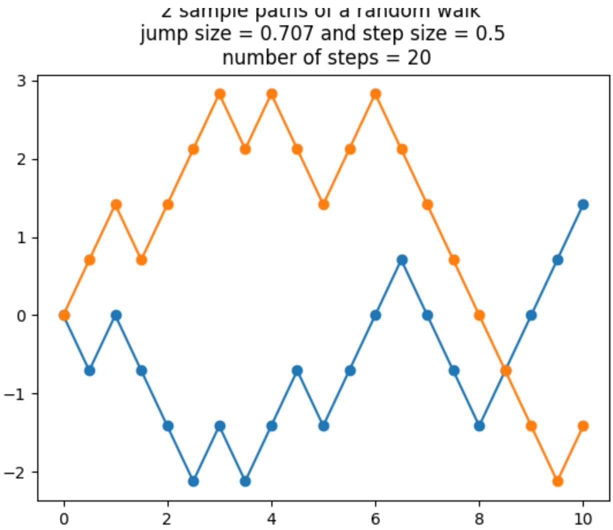
Random Walk Example



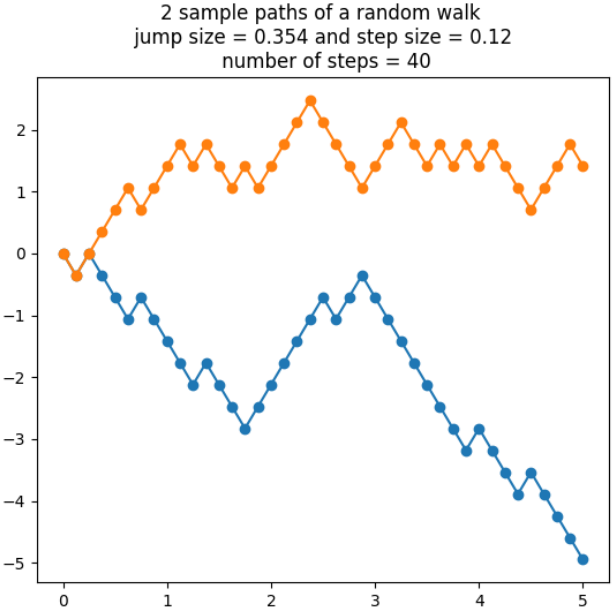
Random Walk Example



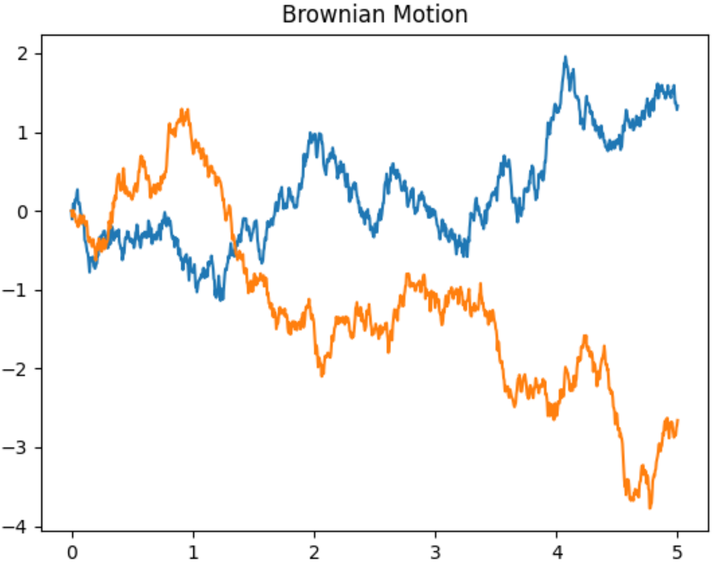
Random Walk Example



Random Walk Example



Random Walk Example



Brownian Motion.

Theorem

Consider a random walk $X = \{X_0 = 0, X_{t_1}, \cdots, X_{t_n}\}$. *Suppose that for* $k \in \{1, \cdots, n\}$ *that*

$$\mathbb{P}\left[X_{t_k} = -h\right] = \mathbb{P}\left[X_{t_k} = h\right] = 1/2.$$

and suppose that the time increments are all constant:

$$t_k - t_{k-1} = d \quad \forall k \in \{1, \cdots, n\}.$$

Suppose that $h^2 = d$. *Now define*

$$B_n(t_k) = \sum_{i=0}^{k} X_{t_i}.$$

Then B_n *converges to a Brownian motion.*

The Ito integral (special case)

Suppose that f is a nice continuous function. Then we can define:

$$\int_0^t f(s)\mathrm{d}B_s(\omega) := \lim_{\|P\| \to 0} \sum_{i \in P} f(t_{i-1})(B_t(\omega) - B_{t_{i-1}}(\omega)),$$

where $P = t_0 = 0, t_1, t_2, \cdots, t_n = t$ is a partition of $[0, t]$. In other words, the Ito integral in this case the left point Riemann sum.

Theorem (Ito's Formula)

If f is twice continuously differentiable then we have the following "Fundamental Theorem of Ito Calculus"

$$\int_a^t \frac{d}{ds} f(B_s)\mathrm{d}B_s = f(B_t) - f(B_a) - \frac{1}{2}\int_a^t \frac{d^2}{ds^2}(B_s)\mathrm{d}s.$$

Example

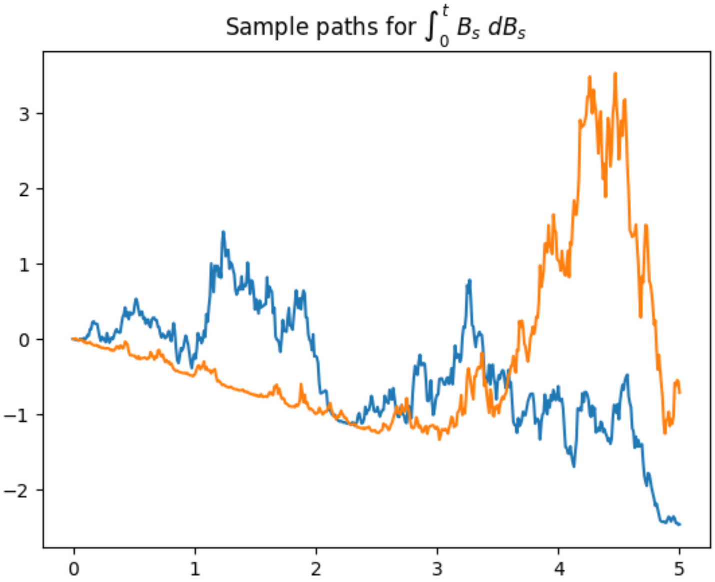
If we apply the above to $f(x) = x^2/2$ with $a = 0$ then we have

$$\int_0^t B_s \, \mathrm{d}B_s = \frac{B_t^2}{2} - \frac{t}{2},$$

which is not quite the same thing as

$$\int_0^{B_t} x \, \mathrm{d}x = \frac{B_t^2}{2}$$

Two sample paths



One more example

For a nice function $f(t,x)$ we can generalize Itos formula to

$$f(t,B_t)-f(a,B_a)=\int_a^t \frac{\partial f}{\partial x}(s,B_s)\mathrm{d}B_s+\int_a^t \frac{\partial f}{\partial t}(s,B_s)+\frac{\partial^2 f}{\partial x^2}(s,B_s)\mathrm{d}s.$$

Now consider the function $f(t,x)=tx$. Then

$$\frac{\partial f}{\partial t}(t,x)=x,\quad \frac{\partial f}{\partial x}(t,x)=t,\quad \frac{\partial^2 f}{\partial x^2}(t,x)=0$$

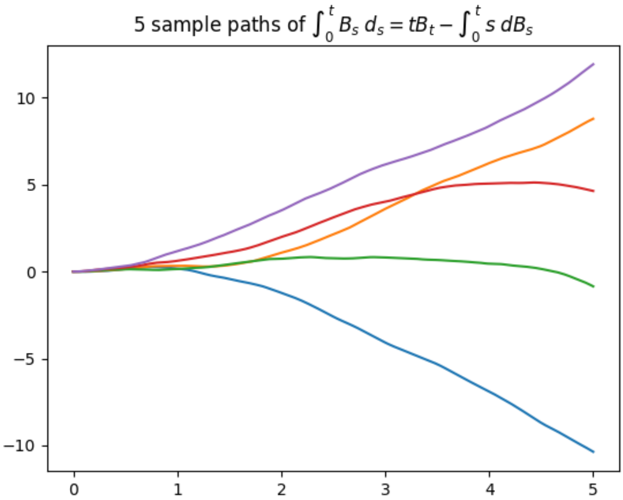
Plugging this in above for $a=0$ gives us

$$tB_t=\int_0^t s\mathrm{d}B_s+\int_0^t B_s\mathrm{d}s,$$

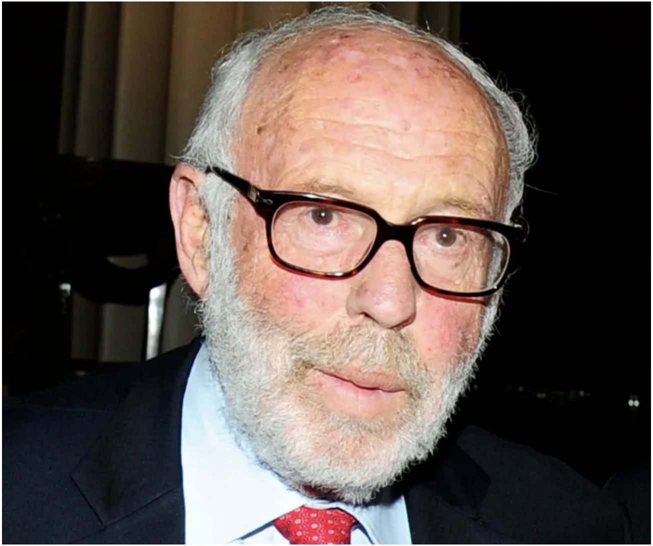
So we see that the Riemann integral of a Brownian motion is

$$\int_0^t B_s\mathrm{d}s=tB_t-\int_0^t s\mathrm{d}B_s.$$

Continuity?!



Why do we care about Brownian Motion?



An application to finance.

Consider a stock that has value S_t at time t. Consider $t = 0$ to be the point of entry into the investment. We can define a return rate as follows:

$$S_t = S_0(1 + R_t), \quad R_t := \frac{S_t - S_0}{S_0} \text{ is the return rate over the period } [0, t].$$

We want to make predictions of our return. In a very simple model, we can assume that

$$R_t \sim \text{Normal}(\mu_t, \sigma^2), \quad \mu \in \mathbb{R},\ \sigma \in (0, \infty).$$

In terms of Stochastic Calculus, we express this as the following stochastic differential equation:

$$R_t = \frac{dS_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t, \quad \text{where } B_t \text{ is a Brownian motion.}$$

Modeling a stock price with geometric brownian motion

The notation

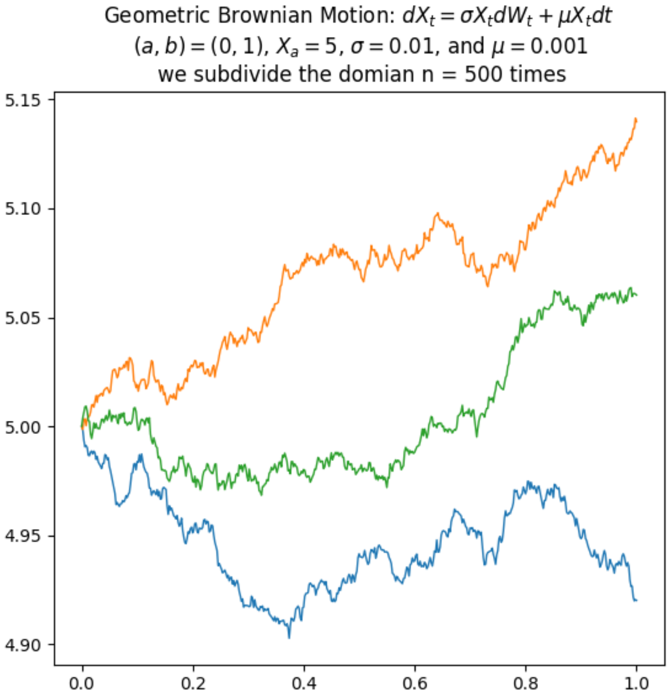
$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad \text{where } B_t \text{ is a Brownian motion}$$

is only symbolic as dB_t does not exist. The above can be legitimately viewed as the following integral equation:

$$S_t - S_0 = \int_0^t \mu S_k \, \mathrm{d}k + \int_0^t \sigma S_k \, \mathrm{d}B_k.$$

The stochastic processes S_t that solves the above equation is called a Geometric Brownian Motion. Notice that we have came up with a random variable that models the stock price:

$$S_t(\omega) = S_0 + \int_0^t \mu S_k(\omega) \, \mathrm{d}k + \int_0^t \sigma S_k \, \mathrm{d}B_k(\omega).$$


Geometric Brownian Motion: $dX_t = \sigma X_t dW_t + \mu X_t dt$
$(a, b) = (0, 1)$, $X_a = 5$, $\sigma = 0.01$, and $\mu = 0.001$
we subdivide the domian n = 500 times
5.15
5.10
5.05
5.00
4.95
4.90
0.0
0.2
0.4
0.6
0.8
1.0

Black-Scholes option pricing model for European calls

Suppose at time $t = 0$ you buy a contract from a seller, allowing you to purchase 1 share of a stock at time $t = T$ and a set price of K, regardless if what the stock price at time T is.

$$\text{Payout and time } T = \begin{cases} S_T - K & \text{if } S_T > k \\ 0 & \text{otherwise} \end{cases}$$
$$= (S_T - K)^+$$

What is a fair price for the this contract?
For this we need a to create a self-financing strategy.

We assume that the stocks price follows a geometric Brownian motion,

$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t.$$

Let a_{t_i} be the amount of shares owned over the period $t \in (t_{i-1}, t_i]$. Then at any time $t \in [0, T]$, the amount of shares owned is

$$a_t = \sum_{i=1}^{n} a_{t_i} \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

and the realized over $[0, T]$

$$\begin{aligned} a_T S_T - a_0 S_0 = \sum_{i=1}^{n} a_{t_i}(S_{t_i} - S_{t_{i-1}}) &= \int_0^T a_t \, \mathrm{d}S_t \\ &= \mu \int_0^T a_t S_t \, \mathrm{d}t + \sigma \int_0^T a_t S_t \, \mathrm{d}B_t \end{aligned}$$

Summarizing the last slide...

$$\text{Our stock profit on time } [0, T] = \underbrace{a_T S_T - a_0 S_0 = \int_0^T a_t \, \mathrm{d}S_t}$$

Now suppose we also invest in a bond with interest rate, r, that has price β_t at time t. Similarly...

$$\text{Our bond profit on time } [0, T] = \underbrace{b_T S_T - b_0 S_0 = \int_0^T b_t \, \mathrm{d}\beta_t}$$

Definition (Self Financing Strategy)

A pair of stocks and bonds, (a, b), is self financing if

$$a_T S_T + b_T S_T = a_0 S_0 + b_0 S_0 + \int_0^T a_t \, \mathrm{d}S_t + \int_0^T b_t \, \mathrm{d}\beta_t$$

Fair price for a European call option

A fair price for a European call option that allows the buyer to buy 1 share at price K at time T regardless of the price of the stock S_T at time T is the amount one would need to invest in a self financing strategy,

$$\underbrace{a_T S_T + b_T S_T}_{V_T} = \underbrace{a_0 S_0 + b_0 S_0}_{V_0} + \int_0^T a_t \, \mathrm{d}S_t + \int_0^T b_t \, \mathrm{d}\beta_t,$$

such that the value of the strategy at time T is equal to $(S_T - K)^+$:

$$V_T = (S_T - K)^+$$

Let $f(S_t, T - t) = V_t$. So we see that

$$f(S_T, 0) = V_T = (S_T - K)^+ \quad \text{and} \quad f(S_0, T) = V_0$$

So we need to find f. Moreover, $f(S_0, T)$ is the fair price to pay.

It can be shown that f solves the following partial differential equation

$$\begin{cases} \frac{\partial f}{\partial s}(s,x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(s,x) + rx\frac{\partial f}{\partial x}(s,x) - rf(s,x) & (s,x) \in [0,\, T] \times (0,\infty) \\ f(0,x) = (x-K)^+ & x \in (0,\infty). \end{cases}$$

The solution is given by

$$f(t,x) = x\Phi\big(g(t,x)\big) - Ke^{-rt}\Phi\big(h(t,x)\big),$$

where

$$g(t,x) = \left[\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) t\right] / (\sigma\sqrt{t}),$$

and

$$h(t,x) = g(t,x) - \sigma/\sqrt{t},$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp\left(-\frac{|x|^2}{2}\right) \mathrm{d}x$$

Potential for unlimited profit

In practice, option contracts are priced by "the market". This means that the cost of a contract is literally what the buyer is willing to pay or what the seller is willing to sell at.

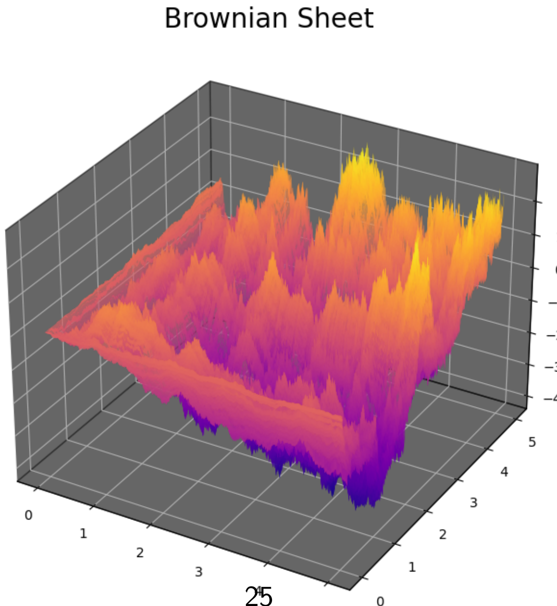
This gives the educated buyer (or seller) a chance to make profit.

Suppose the fair price for a call contract with strike K is $\$q$ but we find a buyer, who does not know about the Black Scholes model, who wants to buy the contract for $\$p > \$q$. We can do the following:

- sell a contract for $\$p$,
- invest $\$q$ in stock and bond at time $t = 0$ in a self financing strategy (HARD TO DO),
- keep the remaining $p - q$ dollars.
- pay the buyer $(S_T - K)^+$ at expiration.

No matter what, we make $p - q$ dollars. If we sell 100 contracts then we make $100(p - q)$...

Sample path of a Brownian Sheet, $W(t, x)$



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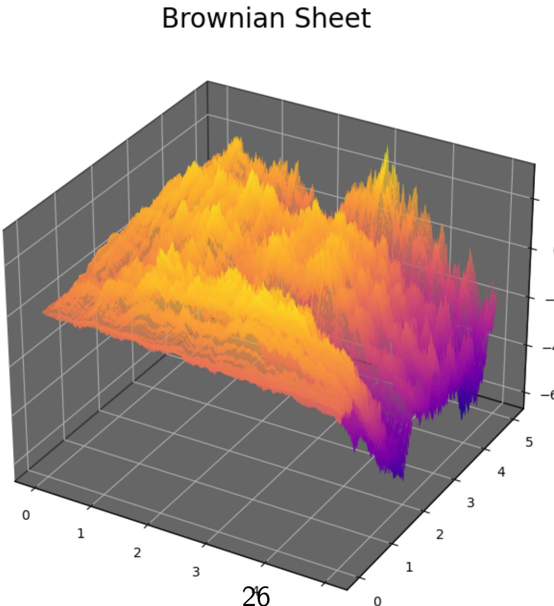
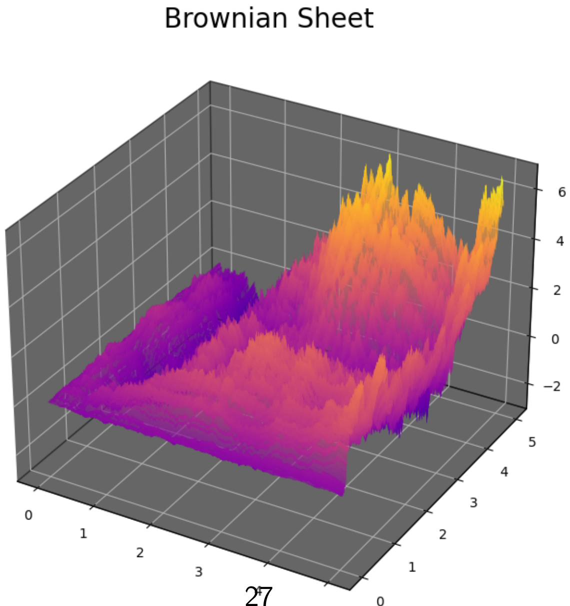


Sample path of a Brownian Sheet, $W(t, x)$



Stochastic Heat Equation

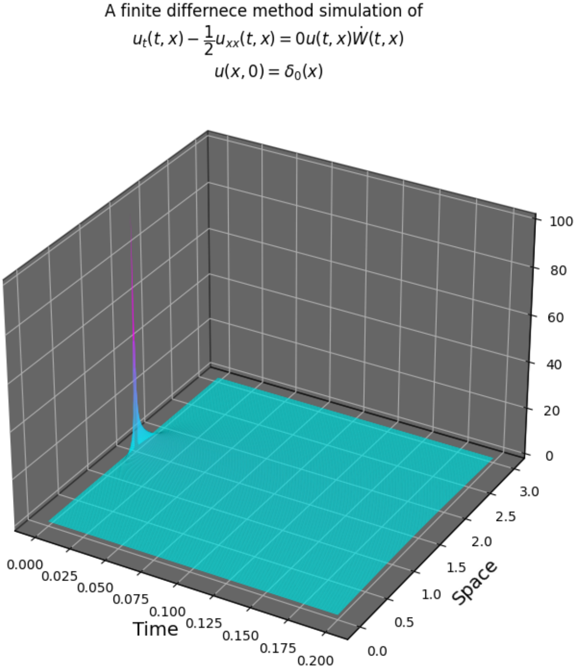
$$\begin{cases} \frac{\partial}{\partial t} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) = \lambda u(t,x) \frac{\partial^2}{\partial t \partial x} W(t,x) & t \geq 0,\ x \in \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R} \end{cases}$$

The solution to the stochastic heat equation is given through the following integral equation:

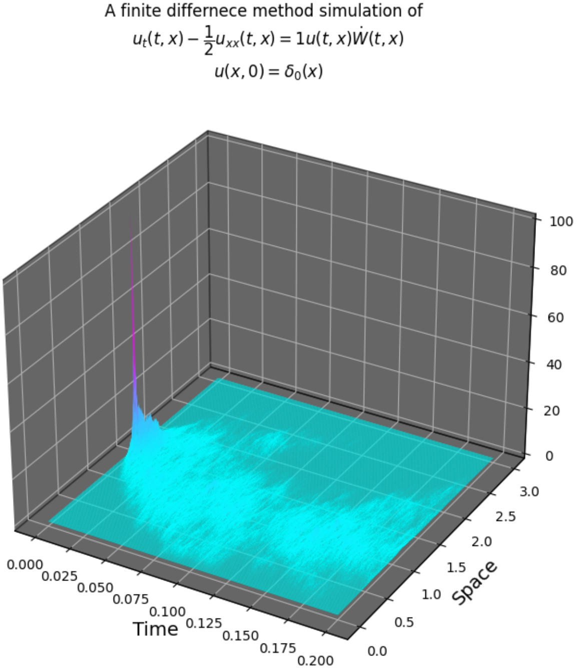
$$u(t,x) = \int_{\mathbb{R}} G(t, x-y) f(y) \mathrm{d}y + \lambda \int_0^t \int_{\mathbb{R}} G(t-s, x-y) u(s,y) W(\mathrm{d}s, \mathrm{d}y),$$

where

$$G(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t} \right)$$



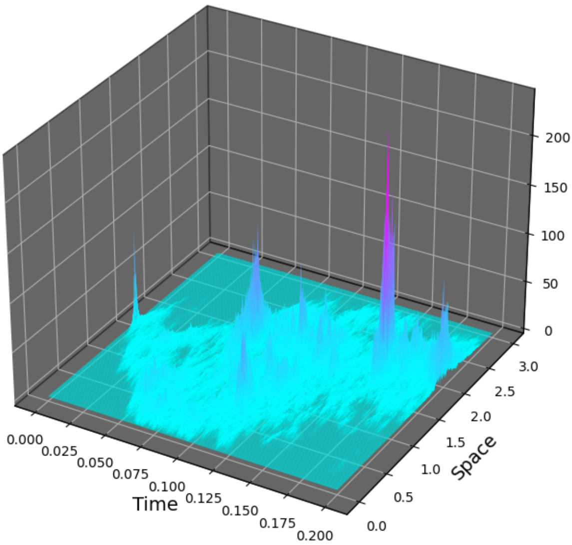
A finite differnece method simulation of
$u_t(t,x) - \frac{1}{2}u_{xx}(t,x) = 0u(t,x)\dot{W}(t,x)$
$u(x,0) = \delta_0(x)$
100
80
60
40
20
0
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Space
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
Time


A finite differnece method simulation of
$u_t(t,x) - \frac{1}{2}u_{xx}(t,x) = 1u(t,x)\dot{W}(t,x)$
$u(x,0) = \delta_0(x)$
Time
Space
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0
20
40
60
80
100

A finite differnece method simulation of

$$u_t(t,x) - \frac{1}{2}u_{xx}(t,x) = 2u(t,x)\dot{W}(t,x)$$

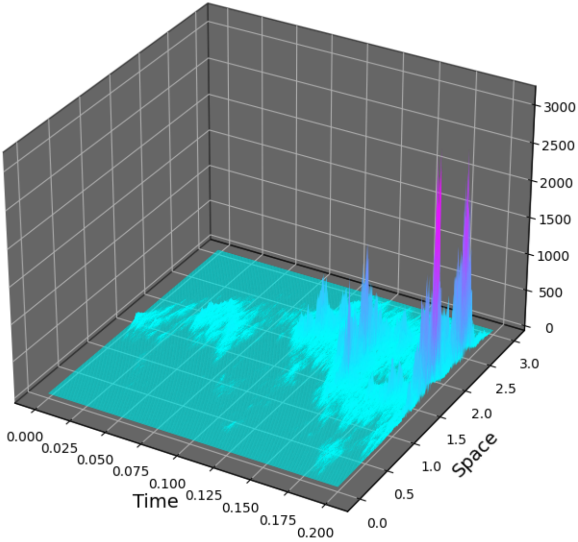
$$u(x,0) = \delta_0(x)$$



A finite differnece method simulation of

$$u_t(t,x) - \frac{1}{2}u_{xx}(t,x) = 2.15u(t,x)\dot{W}(t,x)$$

$$u(x,0) = \delta_0(x)$$



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