Oral Exam

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Projects

I will briefly discuss the three following projects:

- Invariant measures for the Stochastic Heat Equation with rough initial conditions,
- Exact moment asymptotics and Solvability for the Interpolated Stochastic Heat and Wave Equation,
- Global solutions for the Interpolated Stochastic Heat and Wave Equation with a super linear diffusion term and no drift.

Project 1

Invariant Measures for the Stochastic Heat Equation

Invariant Measures for the Stochastic Heat Equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \frac{1}{2}\Delta_x u(t,x) = b(x,u(t,x))\dot{W}(t,x) \\ u(0,\cdot) = \mu(\cdot) \end{cases}$$

We assume that:

- $|b(x,u) - b(x,v)| < L_b|u-v|$ and $|b(x,0)| \leq L_0$ for $u, v \in \mathbb{R}$ and $x \in \mathbb{R}^d$,
- W is a centered Gaussian noise that is homogeneously colored in space,

$$J(\phi,\psi) := \mathbb{E}\left[W(\phi)W(\psi)\right] = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\phi(s,\cdot) * \tilde{\psi}(s,\cdot))(x),$$

where $\tilde{\psi}(s,x) = \psi(s,-x)$ and Γ is a non-negative and nonegative definite measure referred to as the *spatial correlation measure*.

Mild Solution

We interpret the solution of the Stochastic Heat Equation as the following stochastic integral equation:

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} G(t, x-y)\mu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) b(u(s,y)) W(\mathrm{d}s, \mathrm{d}y) \\ &= J(t,x) + I(t,x), \end{aligned}$$

where $G(t,x)$ is the heat kernel

$$G(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$$

A processes, $\{u(t,x) : t \geq 0,\ x \in \mathbb{R}^d\}$, that satisfies this relation is referred to as the *mild solution*.

Existence and Uniqueness

Existence and uniqueness of a mild solution of the Stochastic Heat Equation has been proven under the following two conditions:

- the initial measure, μ, satisfies

$$\int_{\mathbb{R}^d} \exp(-a|x|^2)\mu(\mathrm{d}x) < \infty \quad \text{for any } a > 0,$$

- the Fourier transform of the spatial correlation measure, $\hat{f} := \mathcal{F}\Gamma$ satisfies *Dalang's condition:*

$$\Upsilon(\beta) := \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{\beta + |\xi|^2} < \infty.$$

We refer to $\hat{f}$ as the spectral measure of the noise $\dot{W}$.

Admissible Weight

Since we work in the whole space $\mathbb{R}^d$, we consider weights, ρ such that

- $\rho \in L^1(\mathbb{R}^d)$ is positive, continuous and bounded,
- for any $T > 0$, there exists a $C(T) > 0$ such that

$$\sup_{x\in\mathbb{R}^d,\ t\in[0,T]} \frac{(G(t,\cdot) * \rho)(x))}{\rho(x)} \leq C(T).$$

We then work in the weighted Hilbert Space, $L^2_\rho(\mathbb{R}^d)$, with norm defined as

$$\|\phi(\cdot)\|^2_{L^2_\rho(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \phi(x)^2 \rho(x) \mathrm{d}x.$$

Some examples of admissible weights include the following:

$$\frac{1}{1 + |x|^a},\ a > d \qquad \text{and} \qquad \exp(-a|x|),\ a > 0.$$

Intermittency

We define the upper an lower Lyapunov exponents

$$\underline{m}_2 := \liminf_{t>0} \frac{1}{t} \log \left(\mathbb{E}(|u(t,x)|^2) \right), \text{ and } \overline{m}_2 := \limsup_{t>0} \frac{1}{t} \log \left(\mathbb{E}(|u(t,x)|^2) \right).$$

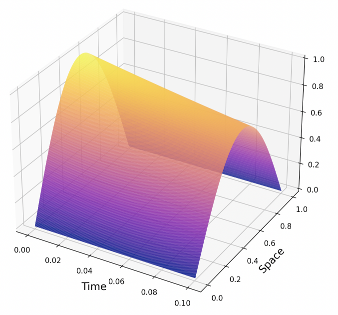
The solution is said to be *fully intermittent* when

$$\inf_{x\in\mathbb{R}^d} \underline{m}_2 > 0.$$

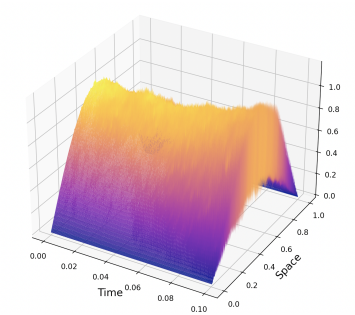
We say a *phase transition* happens when there exists $\underline{\lambda} \leq \overline{\lambda}$ such that

$$\begin{cases} \sup_{x\in\mathbb{R}^d} \overline{m}_2 = 0 & L_b < \underline{\lambda} \\ \inf_{x\in\mathbb{R}^d} \underline{m}_2 = 0 & \overline{\lambda} < l_b \end{cases}$$

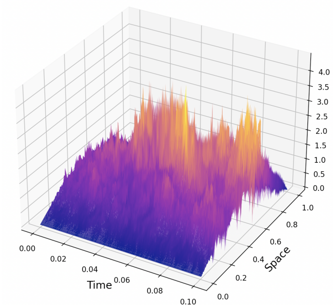
Large Peaks: $u_t - \frac{1}{2} u_{xx} = \lambda u \dot{W}$ with $u_0(x) = \sin(\pi x)$



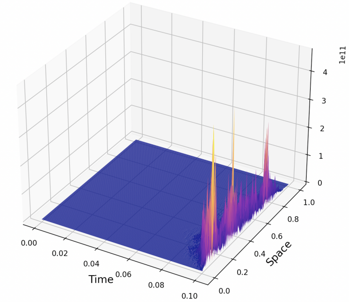
$\lambda = 0$



$\lambda = .1$



$\lambda = 1$



$\lambda = 4$

Bounded Moments

Under the assumptions for existence and uniqueness, we have the following moment bound

$$\|u(t,x)\|_p \leq C[1 + J_+(t,x)]H(t;\gamma_p)^{1/2}, \quad \gamma_p = 32pL_b^2,$$

where $t \mapsto H(t,\gamma_p)$ is nondecreasing. Following a strategy laid out by Daprato and Zabczyk, we wish to prove that

$$\sup_{t\geq 0} \mathbb{E}\left(\|u(t,x)\|^2_{L^2_\rho(\mathbb{R}^d)}\right) < \infty.$$

Hence, a phase transition must occur. By Theorem 1.3, this forces us to require $d \geq 3$ and $\Upsilon(0) < \infty$. If we further assume that L_b is small then we may also say

$$\|u(t,x)\|_p \leq C[1 + J_+(t,x)].$$

Invariant Measures

Consider a measure space $(E, \mathscr{E})$ and denote $\mathcal{M}_1(E)$ to be the set of probability measure on this space.

Definition (Invariant Measure)

A measure $\eta(\cdot) \in \mathcal{M}_1(E)$ is invariant for transition functions $\{P_t(x, \Gamma)\}_{t \geq 0}$ if for any $\Gamma \in \mathscr{E}$ we have that

$$\eta(\Gamma) = \int_E P_t(x, \Gamma)\eta(\mathrm{d}x) \quad \text{for any } t \geq 0$$

Recall from Calculus that the average of a function $f : [a, b] \to \mathbb{R}$ is defined as

$$\mathrm{AVG}(f) = \frac{1}{a-b}\int_a^b f(x)\mathrm{d}x$$

Thus, $\eta(\Gamma)$ is the average the family of probability measures $\{P_t(x, \Gamma)\}_{x \in E}$ for any $t \geq 0$.

Theorem: Existence of an Invariant measure

A processes $u(t,\cdot)(\omega)$ is bounded in probability in $L^2_\rho(\mathbb{R}^d)$ if

$$\forall \epsilon > 0,\ \exists R > 0 :\ \forall t > 0,\ \mathbb{P}\left\{\|u(t,\cdot)\|_{L^2_\rho(\mathbb{R}^d)} \geq R\right\} \leq \epsilon.$$

Tessitore and Zabczyk proved the following:

Theorem

Suppose $d \geq 3$ and that there exists a $p \in [1, d/(d+2))$ such that the spatial correlation density $\gamma \in L^p(\mathbb{R}^d)$. Suppose that there exists two weights, ρ and $\hat{\rho}$ and an element $\phi \in L^2_\rho(\mathbb{R}^d) \cap L^2_{\hat{\rho}}(\mathbb{R}^d)$ such that

$$\frac{\rho}{\hat{\rho}} \in L^1(\mathbb{R}^d) \quad \textit{and} \quad u^\phi(t,\cdot) \textit{is bounded in probability in} L^2_{\hat{\rho}}.$$

Then there exists an invariant measure in $L^2_\rho(\mathbb{R}^d)$.

Sketch of the Proof of Existence

It has been shown by DaPrato and Zabczyk that $P_t(\phi, \Gamma) := \mathscr{L}(u^\phi(t,\cdot))(\Gamma)$ defines a Markov transition function on $(L^2_\rho(\mathbb{R}^d), \mathscr{L})$ where,

$$\mathscr{L}(u^\phi(t,\cdot))(\Gamma) = \mathbb{P}\left[\omega \in \Omega : u^\phi(t,\cdot)(\omega) \in \Gamma\right], \quad \Gamma \in \mathscr{L}.$$

We then prove for any $\epsilon \in (0,1)$ and $R > 0$, the existence of a compact set, $\mathcal{K} \in \mathscr{L}$ such that

$$\mathscr{L}(u^\phi(t,\cdot))(\mathcal{K}) \geq (1-\epsilon)\mathbb{P}\left[\omega \in \Omega : \left\|u^\phi(t-1,\cdot)(\omega)\right\|_{L^2_\rho(\mathbb{R}^d)} < R\right], \quad t \geq 1.$$

If we assume $u^\phi(t,\cdot)$ is bounded in probability in $L^2_\rho(\mathbb{R}^d)$ then one can argue the existence of a sequence $\{T_n\}$ such that $0 < T_n \uparrow \infty$

$$\frac{1}{T_n}\int_{t_0}^{T_n+t_0} \mathscr{L}(u^\phi(t,\cdot))(\Gamma) \xrightarrow[weakly]{} \eta(\Gamma) \quad \forall \Gamma \in \mathscr{L},$$

and moreover, η is invariant in $L^2_\rho(\mathbb{R}^d)$.

Tessitore and Zabczyk's Further Assumption

If $B := \sup_{t\geq 0} \mathbb{E}\left(\|u(t,x)\|^2_{L^2_\rho(\mathbb{R}^d)}\right) < \infty$ then by Chebyshev's inequality

$$\mathbb{P}[|u(t,\cdot)|_\rho \geq R] \leq R^{-2}\mathbb{E}|u(t,\cdot)|^2_\rho \leq R^{-2}B < \epsilon, \quad \text{for any } R \geq \sqrt{B/\epsilon}.$$

Theorem

Suppose that $d \geq 3$ and that $L < L_b^{-2}$ where

$$\tilde{\Gamma} := \left|\widehat{\gamma^{1/2}}\right| * \left|\widehat{\gamma^{1/2}}\right| \quad \textit{and} \quad L := \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \int_{\mathbb{R}^d} \tilde{\Gamma}(\zeta)|\zeta|^{2-d}\mathrm{d}\zeta < \infty.$$

Then for any admissible weight ρ, we have that

$$\sup_{t\geq 0} \mathbb{E}\left(\left\|u^{\mathbf{1}}(t,x)\right\|^2_{L^2_\rho(\mathbb{R}^d)}\right) < \infty.$$

Moment Bound: Measure Valued Initial Conditions

Theorem

Suppose that the initial measure, μ, and spectral measure, $\hat{f}$:

$$\sup_{x\in\mathbb{R}^d} \left(G(t,\cdot) * |\mu|)(x)\right) < \infty \quad \forall t > 0 \quad \textit{and} \quad \Upsilon(0) < \infty. \tag{0.1}$$

Then if $\mu(\mathrm{d}x) = \phi(x)\mathrm{d}x$ with $\phi \in L^2_\rho(\mathbb{R}^d)$,

$$\sup_{t\geq 0} \mathbb{E}\left(\left\| u^\phi(t,x)\right\|^2_{L^2_\rho(\mathbb{R}^d)}\right) < \infty.$$

Further, if μ has no density but we strengthen (0.1) to include

$$\int_0^s \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi) \exp\left(-r|\xi|^2\right) < \infty$$

then for any $t_0 > 0$,

$$\sup_{t\geq t_0} \mathbb{E}\left(\| u^\mu(t,x)\|^2_{L^2_\rho(\mathbb{R}^d)}\right) < \infty.$$

Invariant Measures: Measure Valued Initial Conditions

We wish prove the existence of an invariant measure for measure valued initial conditions such as the Dirac delta measure, $\delta_0(\cdot)$. Our plan is to consider

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \frac{1}{2}\Delta_x u(t,x) = b(x,u(t,x))\dot{W}(t,x) \\ u(0,\cdot) = \mu(\cdot) \end{cases}$$

and then restart our system. In other words, we will consider

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \frac{1}{2}\Delta_x v(t,x) = b(x,v(t,x))\dot{W}(t,x) \\ v(0,\cdot) = u(t_0,\cdot) \end{cases} .$$

With the absence of noise, we know that $v(t,x) = u(t+t_0,x)$ and we wish to show that this property holds with the addition of noise. We also know that $\sup_{t\geq 0}\|u(t+t_0,\cdot)\|_{L^2_\rho(\mathbb{R}^d)} < \infty$. Thus we need to figure out how extend the known proof to handle random initial conditions.

Project 2:
The Interpolated Stochastic Head and Wave Equation: Sovability and Exact Moment Asymptotics

The Stochastic Heat and Wave Equations

$$\begin{cases} \left(\partial_t^b - \Delta\right)\ u(t,x) = u(t,x)\ \dot{W}(x) & x \in \mathbb{R}^d,\ t > 0 \\ u(0,\cdot) = 1 & b = 1 \quad \text{(SHE)} \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b = 2 \quad \text{(SWE)} \end{cases}$$

- $W = \{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\}$ is a centered and time-independent Gaussian noise
- The choice of initial condition is such that the solution to the homogeneous equation is constant one.
- The solution is understood in the *Skorohod* sense.

Known Results

For the SHE, i.e., $b = 1$ ($a = 2$, $\nu = 2$)
[X. Chen '17]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{2}{2-\alpha}}(2-\alpha)\left(\frac{2\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2-\alpha}}$$

For the SWE, i.e., $b = 2$ ($a = 2$, $\nu = 2$)
[Balan, L. Chen, and X. Chen '21]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2(3-\alpha)}}\frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

The Interpolated Stochastic Heat and Wave Equation

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r\left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0 \\ u(0,\cdot) = 1 & b \in (0,1] \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2) \end{cases} \tag{ISHWE}$$

- ∂_t^b is the *Caputo* fractional derivative
- $(-\Delta)^{a/2}$ is the fractional Laplacian of order $a \in (0,2]$
- I_t^r is the Riemann-Liouville fractional integral of order $r \geq 0$

The Noise, $\dot{W}$.

The time independent noise informally satisfies

$$\mathbb{E}(\dot{W}(x)\dot{W}(y)) = \gamma(x-y).$$

The spatial correlation and spectral density, γ and φ, for the noise $\dot{W}$ can be assumed to be any of the following:

$$\gamma(x) = |x|^{-\alpha}, \qquad \varphi(\xi) = C_1|\xi|^{d-\alpha} \qquad \alpha \in (0,d)$$

$$\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\alpha_i}, \qquad \varphi(\xi) = C_2 \prod_{i=1}^{d} |\xi_i|^{1-\alpha_i} \qquad \alpha_i \in (0,1),\ \alpha = \sum_i \alpha_i$$

Mild Solution

Definition

For $T > 0$, a random field $u = \{u(t,x) : t \in (0,T), x \in \mathbb{R}^d\}$ is called a *mild solution* if $G(t-s, x-\cdot)u(s,\cdot)1_{\{s<t\}}$ is Skorohod integrable and the following holds almost surely:

$$u(t,x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y)u(s,y)W(\delta y) \right) \mathrm{d}s$$

where G is defined through the *Fox-H function*.

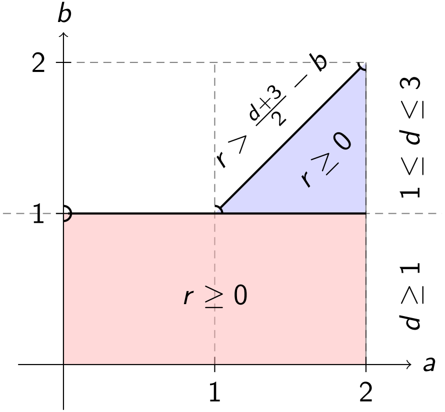
An important characteristic of G is that

$$\mathcal{F}\mathcal{L}G(s,\xi) = \frac{s^{-r}}{s^b + \frac{\nu}{2}|\xi|^a}$$

Nonnegativity assumption on G

Under any of the following cases, G is nonnegative [Chen, Hu, Nualart '19]:

- $d \geq 1,\ b \in (0,1],\ a \in (0,2],\ r \geq 0$;
- $1 \leq d \leq 3,\ 1 < b < a \leq 2,\ r > 0$;
- $1 \leq d \leq 3,\ 1 < b = a < 2,\ r > \dfrac{d+3}{2} - b$.



Global Solution

$\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) \, u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right]$

Definition

$u(t,x)$ is a *global solution* to the (ISHWE) if $\|u(t,x)\|_p < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$.

Theorem (Chen-E. '21+)

A global solution exists provided G is nonnegative and

$$0 < \alpha < \min\left(\frac{a}{b}[2(b+r)-1], 2a, d\right)$$

Local Solution

Definition

$u(t,x)$ is a *local solution* to the (ISHWE) if there exists $0 < T_a \le T_b < \infty$ such that $\|u(t,x)\|_2 < \infty$ when $0 < t < T_a$ and $\|u(t,x)\|_2$ D.N.E. (does not exist) for $t > T_b$.

Theorem (Chen-E. '21+)

A local solution exists provided G is nonnegative and if

$$r \in [0, 1/2] \qquad \textit{and} \qquad 0 < \alpha = \frac{a}{b}[2(b+r)-1] \le d.$$

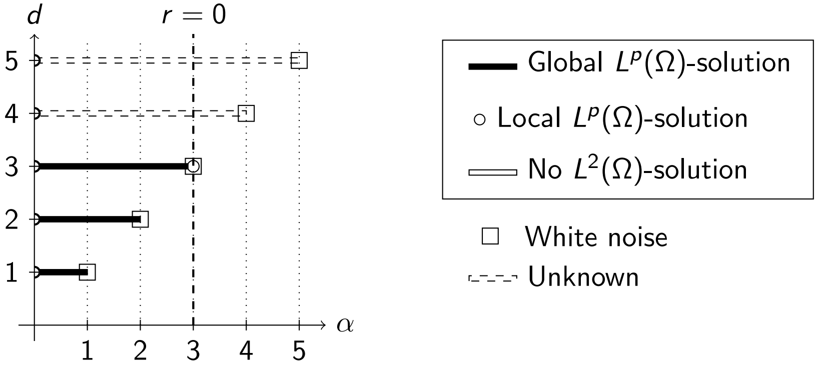
In this case, a unique $L^p(\Omega)$ solution exists for $t \in (0, T_p)$ where

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}$$

and the solution does not exist for $t > T_2$.

Example: Solvability for the SWE ($a = b = 2$)

A local solution only exists when $\alpha = 3 + 2r \leq d \leq 3$.

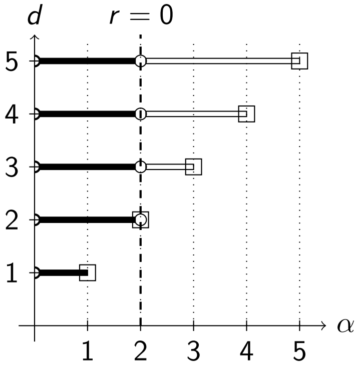


By replacing $\nu = 2$ in the following, we recover (1.12) [Balan, L. Chen & X. Chen '21]

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \quad p \geq 2$$

Example: Solvability for the SHE ($a = 2$ and $b = 1$)

By setting $a = 2$, $b = 1$ and $r = 0$, we obtain the following condition for existence of a local solution: $\alpha = 2 \le d$.



- When $\alpha = d = 2$, the critical time becomes
  $$T_p = \frac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}.$$
- Theorem 4.1 [Y. Hu, '01] proves that an $L^2(\Omega)$ solution exists for $t < 2$ but not for $t > 2\pi$.
- T_2 being precise implies $2 \le T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \le 2\pi$

Wiener Chaos Expansion

Through a standard procedure, we may define

$$f_n(x_1, \cdots, x_n; x, t) = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^{n} G(t_{k+1} - t_k, x_{k+1} - x_k) \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

where $t = t_{n+1}$ and $x = x_{n+1}$, and say that

1. $u(t,x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot, x, t)), \quad (t,x) \in (0,T) \times \mathbb{R}^d$
2. $\mathbb{E}(u(t,x)^2) = \sum_{k=0}^{\infty} \theta^k \|f_k(\cdot, x, t)\|^2_{\mathcal{H}^{\otimes n}}, \quad (t,x) \in (0,T) \times \mathbb{R}^d$.

Through a change of variable, one can easily show that

1. $\mathcal{F}G(t,\cdot)(c\xi) = c^{-\frac{a}{b}(b+r-1)} \mathcal{F}G\left(c^{\frac{a}{b}} t, \cdot\right)(\xi)$
2. $\left\|\widetilde{f}_n(\cdot, 0, t)\right\|^2_{\mathcal{H}^{\otimes n}} = t^{[2(b+r)-b\alpha/a]n} \left\|\widetilde{f}_n(\cdot, 0, 1)\right\|^2_{\mathcal{H}^{\otimes n}}$

How to Find the Critical α

For simplicity we consider the SWE ($a = b = 2$ and $r = 0$).

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} \theta^n n! \overbrace{t^{[4-\alpha]n} \left\| \widetilde{f}_n(\cdot,0,1) \right\|_{\mathcal{H}^{\otimes n}}^2}^{\left\| \widetilde{f}_n(\cdot,0,t) \right\|_{\mathcal{H}^{\otimes n}}^2}$$

$$\leq \sum_{n\geq 0} \frac{(\theta t^{4-\alpha} c_1 2^{4-\alpha} C_\mu)^n}{(n!)^{3-\alpha}}, \quad C_\mu = \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^2 \mu(\mathrm{d}\xi) < \infty$$

When $\alpha = 3$, then the above reduces down to, we lose the $n!$ term in the denominator.

How to Find the Blowup Time (SWE)

Recall from above that the critical case corresonds to $\alpha = 3 = d$.

$$\begin{aligned}
\|u(t,x)\|_2^2 &= \sum_{n\geq 0} \theta^n n! \left\| \tilde{f}(\cdot, 0, t) \right\|_{H^{\otimes n}}^2 \\
&= \sum_{n\geq 0} (\theta t)^n n! \left\| \tilde{f}(\cdot, 0, 1) \right\|_{H^{\otimes n}}^2 \\
&=: \sum_{n\geq 0} (\theta t)^n R_n, \quad R_n = n! \left\| \tilde{f}(\cdot, 0, 1) \right\|_{H^{\otimes n}}^2 .
\end{aligned}$$

The lemma below says that $\frac{1}{n} \log R_n \to \log(2\rho)$. Now the Cauchy-Hadamard theorem can be directly applied to see that the radius of convergence is precisely $(2\theta\rho)^{-1} = T_2$.

Recall the scaling property

$$\left\|\tilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}} = t^{(4-\alpha)n}\left\|\tilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}.$$

Using this scaling property we see that

$$\int_0^\infty e^{-t}\left\|\tilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}}\mathrm{d}t = \Gamma((4-\alpha)n+1)\left\|\tilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}.$$

Lemma (R. Balan, L. Chen & X. Chen '21)

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\Gamma((4-\alpha)n+1)\left\|\tilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}\right) = \log\left[2^{4-\alpha}\rho\right]$$

$$\lim_{n\to\infty}\frac{1}{n}\log\left((n!)^{4-\alpha}\left\|\tilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}\right) = \log\left[\left(\frac{2}{4-\alpha}\right)^{4-\alpha}\rho\right]$$

ρ

We define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x)$$

Theorem (X. Chen '07)

$$\lim_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} \frac{1}{1+\frac{\nu}{2}\left|\sum_{j=k}^{n} \xi_{\sigma(j)}\right|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}) \right] = \log\left(\rho_{\nu,a}(\gamma)\right).$$

where γ is the spatial correlation function.

Connection of the SPDE with ρ

Recall from above that

$$\mathcal{F}\mathcal{L}G(1,\xi) = \frac{1}{1+\frac{\nu}{2}|\xi|^a}.$$

By using this relation, it can be shown that

$$\mathcal{F}\mathcal{L}(\tilde{f}_n)(1,\xi) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n}\xi_{\sigma(j)}|^a}.$$

Applying this to the limit representation of ρ above yields that

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\int_{(\mathbb{R}^d)^n}\left|\mathcal{F}\mathcal{L}(\tilde{f}_n)(1,\xi)\right|^2\mu(\mathrm{d}\vec{\xi})\right] = \log\left(\rho_{\nu,a}\left(\gamma\right)\right).$$

$\mathcal{M}$

We define

$$\begin{aligned}\mathcal{M}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a}\left\{\left(\iint_{\mathbb{R}^{2d}} g^2(x)g^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y\right)^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\}\\ &= \sup_{g\in\mathcal{F}_a}\left\{\langle g^2 * g^2, \gamma\rangle^{1/2}_{L^2(\mathbb{R}^d)} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\},\end{aligned}$$

where

$$\mathcal{E}_a(g,g) := (2\pi)^{-d}\int_{\mathbb{R}^d} |\xi|^a |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi \quad \text{and}$$

$$\mathcal{F}_a := \left\{f \in L^2(\mathbb{R}^d):\ \|f\|_{L^2(\mathbb{R}^d)} = 1,\ \mathcal{E}_a(f,f) < \infty\right\}.$$

Theorem (Bass, X. Chen and Rosen '09)

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty$$

The Moment Asymptotic

Theorem (Chen-E. '21+)

Suppose a global solution to the SPDE in question exists and $\dot{W}$ is given through the generalized Riesz kernel defined above. Then,

$$
\begin{aligned}
&\lim_{t_p\to\infty} t_p^{-\beta} \log \|u(t,x)\|_p \\
&\quad = \left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\qquad\qquad \times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r)-\frac{b\alpha}{a}-1\right),
\end{aligned}
$$

where

$$
\beta := \frac{2(b+r)-\dfrac{b\alpha}{a}}{2(b+r)-\dfrac{b\alpha}{a}-1} \qquad \text{and} \qquad t_p := (p-1)^{1-1/\beta}\, t.
$$

Exact Moment Lyapunov Exponent

(Recall $\beta := \frac{2(b+r)-\frac{b\alpha}{a}}{2(b+r)-\frac{b\alpha}{a}-1}$)

Corollary (Chen-E. '21+)

For $p \geq 2$ fixed, we have that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) \\
&\quad = p(p-1)^{\frac{1}{2(b+r)-\frac{b\alpha}{a}-1}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\qquad \times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r)-\frac{b\alpha}{a}-1\right).
\end{aligned}$$

From this we can deduce that $t \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_1 t^{\beta}\right)$, in other words, the moments will grow exponentially in time, and we obtain the exact expression for the constant K_1.

Exact Large Moment Asymptotics

(Recall $\beta := \frac{2(b+r)-\frac{b\alpha}{a}}{2(b+r)-\frac{b\alpha}{a}-1}$)

Corollary (Chen-E. '21+)

For $t > 0$ fixed , we have that

$$\lim_{p\to\infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right)$$

$$= t^{\beta}\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}$$

$$\times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r)-\frac{b\alpha}{a}-1\right).$$

From this we can deduce that $p \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_2 p^{\beta}\right)$, in other words, the moments will grow exponentially in p as well.

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Project 3: Global Solutions for the Interpolated Stochastic Heat and Wave Equation with a Superlinear Drift Term

The Interpolated Stochastic Heat and Wave Equation

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sigma(u(t,x))\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0 \\ u(0,\cdot) = 1 & b \in (0,1] \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2) \end{cases} \tag{0.2}$$

The drift coefficient has super-linear growth:

$$|\sigma(x)| \le \sigma_1 + \sigma_2 |x| [\ln(|x|)]^\delta \quad \text{as } |x| \to \infty.$$

We first consider the noise to be space-time white. In other words, the covariance structure is defined for $\phi, \psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ as

$$J(\phi,\psi) := \mathbb{E}(W(\phi)W(\psi)) = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x\ \phi(s,x)\psi(s,x).$$

Mild Solution

For any $t \in [0, T]$ and $x \in \mathbb{R}^d$, we write

$$u(t,x) = 1 + I(t,x)$$

where

$$I(t,x) = \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y).$$

The function Y is defined through the Fox-H function and its Fourier transform can be shown to be given as

$$\mathcal{F}Y(t,\cdot)(\xi) = t^{\beta+\gamma-1}E_{\beta.\beta+\gamma}(-2^{-1}\nu t^{\beta}|\xi|^{\alpha}),$$

where

$$E_{\alpha,\beta}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Existence and Uniqueness

Because of the bounded initial conditions, [1, Theorem 3.1] implies that the following condition on Y will imply existence and uniqueness:

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y |Y(s,y)|^2 < \infty, \quad \text{for all } t > 0,$$

which is equivalent to

$$d < 2\alpha + \frac{\alpha}{\beta} \min\{2\gamma - 1, 0\} =: \Theta,$$

which is further equivalent to

$$\rho(d) > 0 \quad \text{and} \quad d < 2\alpha,$$

where

$$\rho(x) := 2\beta + 2\gamma - 1 - \beta x/\alpha.$$

Some Results for a Globally Lipschitz Drift Term

We first prove some results assuming that the drift term, σ is globally Lipschitz:

$$|\sigma(x) - \sigma(y)| \leq L(\sigma)|x - y|, \quad x, y \in \mathbb{R}^d.$$

We also define

Proposition

Suppose that $d < 2\alpha$ *and* $\rho = \rho(d) > 0$. *Then there exists a universal constant* $K := K(\alpha, \beta, \gamma, d) > 0$ *such that for any*

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \textit{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

Then we have that

$$N_{a,p}(u) \leq 2\mathcal{T}_0 + \frac{c(\sigma)}{L(\sigma)} \tag{0.3}$$

where

$$\mathcal{T}_0 = \begin{cases} (\lceil\beta\rceil - 1)^{\lceil\beta\rceil - 1}(ea)^{1-\lceil\beta\rceil} \|u_0\|_\infty & \beta \in (0, 1] \\ (\lceil\beta\rceil - 1)^{\lceil\beta\rceil - 1}(ea)^{1-\lceil\beta\rceil} \|v_0\|_\infty + \|u_0\|_\infty & \beta \in (1, 2). \end{cases}$$

Moreover for any $T > 0$,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}(|u(t,x)|^p) \leq \exp(apt) \left[2\mathcal{T}_0 + \frac{c(\sigma)}{L(\sigma)}\right]^p, \ t \in [0, T]. \tag{0.4}$$

Suppose that $0 < \theta < (\Theta - d) \wedge 2$ and $\rho = \rho(d) > 0$. Let $T > 0$ and $t, r \in [0, T]$. Then for any

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \textit{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

and any $x, z \in \mathbb{R}^d$ we have that

$$\frac{\|u(t,x) - u(r,z)\|_p}{\left(|t-r|^q + |x-z|^\theta\right)^{1/2}} \leq C(p, \theta, T)\left[\mathcal{M}_1 + \mathcal{M}_2 e^{aT}\left(2 + \frac{c(\sigma)}{L(\sigma)}\right)\right]. \quad (0.5)$$

where,

1. $0 < q \leq \rho$ under $\beta \in (0,1]$ and $\gamma \in (0, 1-\beta]$,
2. $0 < q < \rho$ under $\beta \in (0,2)$ and $\gamma > 0$,

and

$$\mathcal{M}_1 = \sqrt{p}c(\sigma) \quad \text{and} \quad \mathcal{M}_2 = \sqrt{p}L(\sigma).$$

Holder Continuity

The solution, u, has a version, still denoted by u, that is η_1-Hölder continuous in time and η_2-Hölder continuous in space with $0 < \eta_1 < q/2$ and $0 < \eta_2 < \theta/2$ where $0 < \theta < (\Theta - d) \wedge 2$ and

1. $0 < q \le \rho$ under $\beta \in (0, 1]$ and $\gamma \in (0, 1 - \beta]$,
2. $0 < q < \rho$ under $\beta \in (0, 2)$ and $\gamma > 0$,

Moreover, if we consider

$$a > \left(8L^2(\sigma)K^2\right)^{1/\rho} \quad \text{and} \quad p \in \left[2, \frac{a^\rho}{4L^2(\sigma)K^2}\right],$$

then we have that

$$\mathbb{E}\left[\sup_{t\in[0,T],\ |x|\le R} |u(t,x)|^p\right] \\ \le 2^{p-1} + C(p, \theta, T, R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p e^{apT}\left(2 + \frac{c(\sigma)}{L(\sigma)}\right)^p\right].$$

Locally Lipschitz Diffusion Term

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sigma(u(t,x))\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0 \\ u(0,\cdot) = 1 & b \in (0,1] \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2) \end{cases}$$

$$|\sigma(x)| \leq \sigma_1 + \sigma_2 |x| [\ln(|x|)]^{\delta},$$

where $\sigma_1 = \sigma(0)$ and $\sigma_2 > 0$.

Theorem

Let $M, T > 0$. Under Dalang's condition, $\rho(d) > 0$ and $d < 2\alpha$, there exists a random field solution for $|x| \leq M$ denoted as $(u(t,x) : (t,x) \in [0,T] \times [-M,M])$. This solution is unique and satisfies

$$\sup_{t\in[0,T],\, x\in[-R,R]} |u(t,x)| < \infty, a.s.$$

Proof Sketch

We consider a truncated version of the equation where σ replaced by σ_N:

$$\sigma_N(x) = \sigma(x)\mathbf{1}_{\{|x|\le N\}} + \sigma(N)\mathbf{1}_{\{x>N\}} + \sigma(-N)\mathbf{1}_{\{x\le -N\}}.$$

We see that σ_N is Lipschitz:

$$|\sigma_N(x)| \le \sigma_1 + \sigma_2 \ln(2N)^{\delta}|x|,$$

so there exists a unique solution which is Holder continuous, which we denote as $u_N := \{u_N(t,x) : (t,x) \in [0,T] \times \mathbb{R}\}$. We define,

$$\tau_N := \inf\left\{t > 0 : \sup_{|x|\le R} |u_N(t,x)| \ge N\right\} \wedge T.$$

We see that on $\{t < \tau_N\}$, $u_N(t,x) = u_{N+k}(t,x)$ for any $k \in \mathbb{N}$.

Proof Sketch

We can show that $\sup_N \tau_N = T$ which implies that $\{t < \tau_N\} \uparrow \Omega$. On each $\{t < \tau_N\}$, define $(u(t,x) : (t,x) \in [0,T) \times \mathbb{R})$ by $u(t,x) = u_N(t,x)$ and hence $u(t,x) = u_{N+k}(t,x)$ for any $k \in \mathbb{N}$. This implies that on $\{t < \tau_N\}$

$$u(t,x) = 1 + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \; Y(t-s, x-y)\sigma_N(u(s,y))W(\mathrm{d}s, \mathrm{d}y).$$

However, on $\{t < \tau_N\}$, we have that $\sigma_N(u_N(t,x)) = \sigma(u(t,x))$ and so u satisfies

$$u(t,x) = 1 + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \; Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y), \quad \{t < \tau_N\}.$$

Lastly, since $\{t < \tau_N\} \uparrow \Omega$, we conclude that

$$u(t,x) = 1 + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \; Y(t-s, x-y)\sigma(u(s,y))W(\mathrm{d}s, \mathrm{d}y), \quad (t,x) \in [0,T)\times \tag{0.6}$$

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