# The Interpolated Stochastic Heat and Wave Equation: Solvability and Exact Moment Asymptotics

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figs/AU.png

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# The Stochastic Heat and Wave Equations

$$\begin{cases} \left(\partial_t^b - \Delta\right) u(t,x) = u(t,x)\, \dot{W}(x) & x \in \mathbb{R}^d,\ t > 0 \\ u(0,\cdot) = 1 & b = 1 \quad \text{(SHE)} \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b = 2 \quad \text{(SWE)} \end{cases}$$

- $W = \{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\}$ is a centered and time-independent Gaussian noise

- The choice of initial condition is such that the solution to the homogeneous equation is constant one.

- The solution is understood in the *Skorohod* sense.

For the SHE, i.e., $b = 1$ $(a = 2, \nu = 2)$
[X. Chen '17]

$$\lim_{t \to \infty} t^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{2}{2-\alpha}}(2-\alpha)\left(\frac{2\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2-\alpha}}$$

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For the SWE, i.e., $b = 2$ $(a = 2, \nu = 2)$
[Balan, L. Chen, and X. Chen '21]

$$\lim_{t \to \infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2(3-\alpha)}} \frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

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For the SHE, i.e., $b = 1$ ($a = 2$, $\nu = 2$)

[X. Chen '17]

$$\lim_{t \to \infty} t^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{2}{2-\alpha}}(2-\alpha)\left(\frac{2\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2-\alpha}}$$

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$b \in (0,2)$ and $a \in (0,2]$ ??

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For the SWE, i.e., $b = 2$ ($a = 2$, $\nu = 2$)

[Balan, L. Chen, and X. Chen '21]

$$\lim_{t \to \infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2(3-\alpha)}}\frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

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# The Interpolated Stochastic Heat and Wave Equation

$$
\begin{cases}
\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0 \\
u(0,\cdot) = 1 & b \in (0,1] \\
u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2)
\end{cases}
$$

$$\text{(ISHWE)}$$

- $\partial_t^b$ is the *Caputo* fractional derivative

- $(-\Delta)^{a/2}$ is the fractional Laplacian of order $a \in (0,2]$

- $I_t^r$ is the Riemann-Liouville fractional integral of order $r \geq 0$

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# The Noise, $\dot{W}$.

The time independent noise informally satisfies

$$\mathbb{E}(\dot{W}(x)\dot{W}(y)) = \gamma(x - y).$$

The spatial correlation and spectral density, $\gamma$ and $\varphi$, for the noise $\dot{W}$ can be assumed to be any of the following:

$$\gamma(x) = |x|^{-\alpha}, \qquad \varphi(\xi) = C_1|\xi|^{d-\alpha} \qquad \alpha \in (0, d)$$

$$\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\alpha_i}, \qquad \varphi(\xi) = C_2 \prod_{i=1}^{d} |\xi_i|^{1-\alpha_i} \qquad \alpha_i \in (0, 1),\ \alpha = \sum_i \alpha_i$$

## Mild Solution

### Definition

For $T > 0$, a random field $u = \{u(t, x) : t \in (0, T), x \in \mathbb{R}^d\}$ is called a *mild solution* if $G(t - s, x - \cdot)u(s, \cdot)1_{\{s<t\}}$ is Skorohod integrable and the following holds almost surely:

$$u(t, x) = 1 + \sqrt{\theta} \int_0^t \left( \int_{\mathbb{R}^d} G(t - s, x - y)u(s, y)W(\delta y) \right) \mathrm{d}s$$

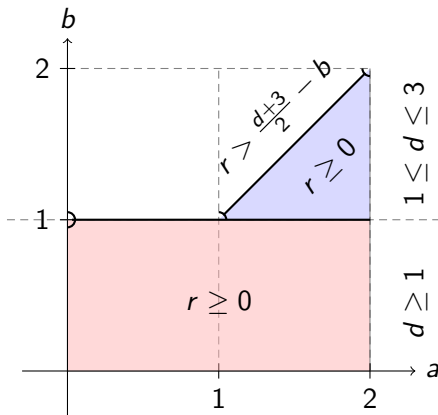where $G$ is defined through the *Fox-H function*.

An important characteristic of $G$ is that

$$\mathcal{F}\mathcal{L}G(s, \xi) = \frac{s^{-r}}{s^b + \frac{\nu}{2}|\xi|^a}$$

## Nonnegativity assumption on $G$

Under any of the following cases, $G$ is nonnegative [Chen, Hu, Nualart '19]:

- $d \geq 1$, $b \in (0, 1]$, $a \in (0, 2]$, $r \geq 0$;
- $1 \leq d \leq 3$, $1 < b < a \leq 2$, $r > 0$;
- $1 \leq d \leq 3$, $1 < b = a < 2$, $r > \dfrac{d+3}{2} - b$.

# Global Solution

$$\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right]$$

### Definition
$u(t,x)$ is a *global solution* to the (ISHWE) if $\|u(t,x)\|_p < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$.

### Theorem (Chen-E. '21+)
*A global solution exists provided G is nonnegative and*

$$0 < \alpha < \min\left(\frac{a}{b}[2(b+r)-1], 2a, d\right)$$

## Local Solution

### Definition

$u(t,x)$ is a *local solution* to the (ISHWE) if there exists $0 < T_a \leq T_b < \infty$ such that $\|u(t,x)\|_2 < \infty$ when $0 < t < T_a$ and $\|u(t,x)\|_2$ D.N.E. (does not exist) for $t > T_b$.

### Theorem (Chen-E. '21+)

*A local solution exists provided $G$ is nonnegative and if*

$$r \in [0, 1/2] \qquad and \qquad 0 < \alpha = \frac{a}{b}[2(b+r) - 1] \leq d.$$
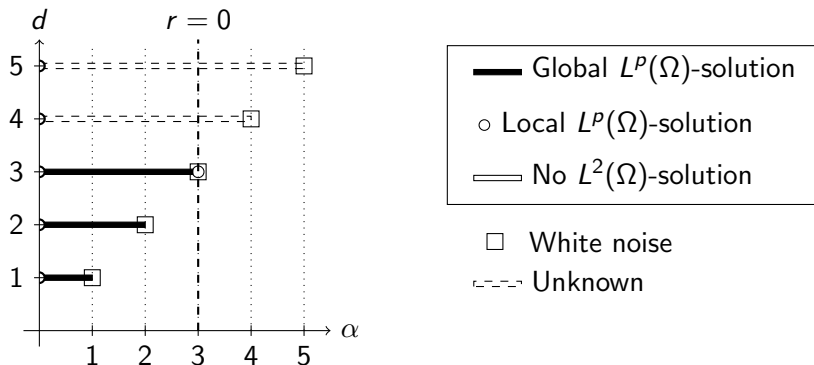
*In this case, a unique $L^p(\Omega)$ solution exists for $t \in (0, T_p)$ where*

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}$$

*and the solution does not exist for $t > T_2$.*

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## Example: Solvability for the SWE ($a = b = 2$)

A local solution only exists when $\alpha = 3 + 2r \leq d \leq 3$.
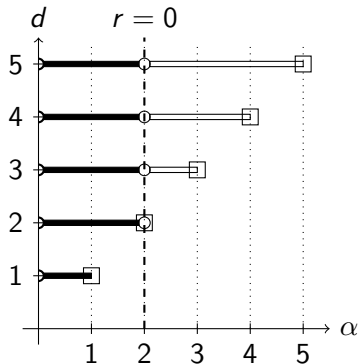


By replacing $\nu = 2$ in the following, we recover (1.12) [Balan, L. Chen & X. Chen '21]

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \quad p \geq 2$$

# Example: Solvability for the SHE ($a = 2$ and $b = 1$)

By setting $a = 2$, $b = 1$ and $r = 0$, we obtain the following condition for existence of a local solution: $\alpha = 2 \leq d$.



- When $\alpha = d = 2$, the critical time becomes
  $$T_p = \frac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}.$$
- Theorem 4.1 [Y. Hu, '01] proves that an $L^2(\Omega)$ solution exists for $t < 2$ but not for $t > 2\pi$.
- $T_2$ being precise implies $2 \leq T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \leq 2\pi$

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## Wiener Chaos Expansion

Through a standard procedure, we may define

$$f_n(x_1, \cdots, x_n; x, t) = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^n G(t_{k+1} - t_k, x_{k+1} - x_k) \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

where $t = t_{n+1}$ and $x = x_{n+1}$, and say that

1. $u(t, x) = 1 + \sum_{k=1}^\infty \theta^{k/2} I_k(f_k(\cdot, x, t)), \quad (t, x) \in (0, T) \times \mathbb{R}^d$
2. $\mathbb{E}(u(t, x)^2) = \sum_{k=0}^\infty \theta^k \|f_k(\cdot, x, t)\|_{\mathcal{H}^{\otimes n}}^2, \quad (t, x) \in (0, T) \times \mathbb{R}^d.$

Through a change of variable, one can easily show that

1. $\mathcal{F}G(t, \cdot)(c\xi) = c^{-\frac{a}{b}(b+r-1)} \mathcal{F}G\left(c^{\frac{a}{b}} t, \cdot\right)(\xi)$
2. $\left\|\widetilde{f}_n(\cdot, 0, t)\right\|_{\mathcal{H}^{\otimes n}}^2 = t^{[2(b+r)-b\alpha/a]n} \left\|\widetilde{f}_n(\cdot, 0, 1)\right\|_{\mathcal{H}^{\otimes n}}^2$

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## How to Find the Critical $\alpha$

For simplicity we consider the SWE ($a = b = 2$ and $r = 0$).

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} \theta^n n! \overbrace{t^{[4-\alpha]n} \left\|\widetilde{f_n}(\cdot, 0, 1)\right\|_{\mathcal{H}^{\otimes n}}^2}^{\left\|\widetilde{f_n}(\cdot, 0, t)\right\|_{\mathcal{H}^{\otimes n}}^2}$$

$$= \sum_{n\geq 0} \frac{\theta^n t^{(4-\alpha)n}}{(n!)^{3-\alpha}} R_n, \quad R_n = (n!)^{4-\alpha} \left\|\widetilde{f_n}(\cdot, 0, 1)\right\|_{\mathcal{H}^{\otimes n}}^2$$

When $\alpha = 3$, then the above reduces down to

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} (\theta t)^n n! \left\|\widetilde{f_n}(\cdot, 0, 1)\right\|_{\mathcal{H}^{\otimes n}}^2.$$

and we lose the $n!$ term in the demonstrator.

$\rho$

We define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1 + \frac{\nu}{2}|x+y|^a}\sqrt{1 + \frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x)$$

Theorem (X. Chen '07)

$$\lim_{n\to\infty} \frac{1}{n} \log \left[ \frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left( \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \frac{1}{1 + \frac{\nu}{2}|\sum_{j=k}^{n} \xi_{\sigma(j)}|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}) \right]$$
$$= \log\left( \rho_{\nu,a}\left(\gamma\right) \right).$$

where $\gamma$ is the spatial correlation function.

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## Connection of the SPDE with $\rho$

Recall from above that

$$\mathcal{FLG}(1,\xi) = \frac{1}{1 + \frac{\nu}{2}|\xi|^a}.$$

By using this relation, it can be shown that

$$\mathcal{FL}(\tilde{f}_n)(1,\xi) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \frac{1}{1 + \frac{\nu}{2}|\sum_{j=k}^{n} \xi_{\sigma(j)}|^a}.$$

Applying this to the limit representation of $\rho$ above yields that

$$\lim_{n \to \infty} \frac{1}{n} \log \left[ \int_{(\mathbb{R}^d)^n} \left| \mathcal{FL}(\tilde{f}_n)(1,\xi) \right|^2 \mu(\mathrm{d}\vec{\xi}) \right] = \log \left( \rho_{\nu,a}(\gamma) \right).$$

Recall the scaling property

$$\left\| \tilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} = t^{(4-\alpha)n} \left\| \tilde{f}_n(\cdot, 0, 1) \right\|^2_{\mathcal{H}^{\otimes n}}.$$

Using this scaling property we see that

$$\int_0^\infty e^{-t} \left\| \tilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = \Gamma((4-\alpha)n + 1) \left\| \tilde{f}_n(\cdot, 0, 1) \right\|^2_{\mathcal{H}^{\otimes n}}.$$

Lemma (R. Balan, L. Chen & X. Chen '21)

$$\lim_{n\to\infty} \frac{1}{n} \log\left( \Gamma((4-\alpha)n + 1) \left\| \tilde{f}_n(\cdot, 0; 1) \right\|^2_{\mathcal{H}^{\otimes n}} \right) = \log\left[ 2^{4-\alpha} \rho \right]$$

$$\lim_{n\to\infty} \frac{1}{n} \log\left( (n!)^{4-\alpha} \left\| \tilde{f}_n(\cdot, 0; 1) \right\|^2_{\mathcal{H}^{\otimes n}} \right) = \log\left[ \left( \frac{2}{4-\alpha} \right)^{4-\alpha} \rho \right]$$

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## How to Find the Blowup Time (SWE)

Recall from above that the critical case corresonds to $\alpha = 3 = d$.

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} \theta^n n! \left\|\tilde{f}(\cdot,0,t)\right\|_{H^{\otimes n}}^2$$

$$= \sum_{n\geq 0}(\theta t)^n n! \left\|\tilde{f}(\cdot,0,1)\right\|_{H^{\otimes n}}^2$$

$$=: \sum_{n\geq 0}(\theta t)^n R_n, \quad R_n = n! \left\|\tilde{f}(\cdot,0,1)\right\|_{H^{\otimes n}}^2.$$

The above lemma says that $\frac{1}{n}\log R_n \to \log(2\rho_c)$. Now the Cauchy-Hadamard theorem can be directly applied to see that the radius of convergence is precisely $(2\theta\rho_c)^{-1} = T_2$.

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## $\mathcal{M}$

We define

$$
\begin{aligned}
\mathcal{M}_{a,d}(\gamma,\theta) &:= \sup_{g \in \mathcal{F}_a} \left\{ \left( \iint_{\mathbb{R}^{2d}} g^2(x) g^2(y) \gamma(x+y) \mathrm{d}x \mathrm{d}y \right)^{1/2} - \frac{\theta}{2} \, \mathcal{E}_a(g,g) \right\} \\
&= \sup_{g \in \mathcal{F}_a} \left\{ \langle g^2 * g^2, \gamma \rangle_{L^2(\mathbb{R}^d)}^{1/2} - \frac{\theta}{2} \, \mathcal{E}_a(g,g) \right\},
\end{aligned}
$$

where

$$
\mathcal{E}_a(g,g) := (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^a |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi \quad \text{and}
$$

$$
\mathcal{F}_a := \left\{ f \in L^2(\mathbb{R}^d) : \ \|f\|_{L^2(\mathbb{R}^d)} = 1, \ \mathcal{E}_a(f,f) < \infty \right\}.
$$

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# The Connection between $\rho$ and $\mathcal{M}$

Recall that

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x).$$

Theorem (Bass, X. Chen and Rosen '09)

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a} \mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty$$

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## The Moment Asymptotic

### Theorem (Chen-E. '21+)

*Suppose a global solution to the SPDE in question exists and $\dot{W}$ is given through the generalized Riesz kernel defined above. Then,*

$$
\lim_{t_p \to \infty} t_p^{-\beta} \log \|u(t,x)\|_p
$$
$$
= \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r) - b\alpha}\right)^{\beta}
$$
$$
\times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r) - b\alpha - a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right),
$$

*where*

$$
\beta := \frac{2(b+r) - \dfrac{b\alpha}{a}}{2(b+r) - \dfrac{b\alpha}{a} - 1} \qquad \text{and} \qquad t_p := (p-1)^{1-1/\beta} t.
$$

# Exact Moment Lyapunov Exponent

(Recall $\beta := \frac{2(b+r) - \frac{b\alpha}{a}}{2(b+r) - \frac{b\alpha}{a} - 1}$)

## Corollary (Chen-E. '21+)

*For $p \geq 2$ fixed, we have that*

$$
\lim_{t \to \infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right)
$$
$$
= p(p-1)^{\frac{1}{2(b+r) - \frac{b\alpha}{a} - 1}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r) - b\alpha}\right)^{\beta}
$$
$$
\times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r) - b\alpha - a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right).
$$

From this we can deduce that $t \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_1 t^{\beta}\right)$, in other words, the moments will grow exponentially in time, and we obtain the exact expression for the constant $K_1$.

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# Exact Large Moment Asymptotics

(Recall $\beta := \frac{2(b+r)-\frac{b\alpha}{a}}{2(b+r)-\frac{b\alpha}{a}-1}$)

## Corollary (Chen-E. '21+)

*For $t > 0$ fixed , we have that*

$$
\lim_{p \to \infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right)
$$
$$
= t^\beta \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r) - b\alpha}\right)^\beta
$$
$$
\times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right).
$$

From this we can deduce that $p \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_2 p^\beta\right)$, in other words, the moments will grow exponentially in $p$ as well.

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# End

Thank you for listening!

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## Anticipated Question: Finiteness of $\rho$

Recall $\varphi(x) = \prod_{i=1}^{n} |x_{(i)}|^{d_i - \alpha_i}$. By examining Lemma 1.6 (X. Chen 2007), to show $\rho$ is finite for the generalized Riesz kernel, then we need to prove the following:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(y) G(x) \varphi(x - y) \mathrm{d}y \mathrm{d}x \leq C \left\| F \right\|_{2d/(d+\alpha)} \left\| G \right\|_{2d/(d+\alpha)}$$

This can be done by using weak Young's inequality

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x) b(x - y) c(y) \mathrm{d}x \mathrm{d}y \right| \leq K_{p,q,r,d} \left\| a \right\|_p \left\| b \right\|_{q,w} \left\| c \right\|_r$$

where $q^{-1} + q'^{-1} = 1$ and $p^{-1} + q^{-1} + r^{-1} = 2$ and

$$\left\| b \right\|_{q,w} = \sup_A |A|^{-1/q'} \int_A |b(x)| \mathrm{d}x, \quad |A| < \infty.$$

We apply this with

$$a = F, \quad b = \varphi, \quad c = G$$

and we show that $\left\| \varphi \right\|_{q,w} < \infty$ with $q = d/(d - \alpha)$.

## Anticipated Question: Why does $\rho$ appear in the Lemma.

For simplicity we will consider the second moments of stochastic wave equation ($a = b = 2$ and $r = 0$). We will also only consider the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \Gamma((4 - \alpha)n + 1) \left\| \tilde{f}_n(\cdot, 0; 1) \right\|^2_{\mathcal{H}^{\otimes n}} \right) = \log \left[ 2^{4-\alpha} \rho \right]$$

Recall that

$$\int_0^\infty e^{-t} \left\| \tilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = \Gamma((4 - \alpha)n + 1) \left\| \tilde{f}_n(\cdot, 0, 1) \right\|^2_{\mathcal{H}^{\otimes n}}.$$

Define $T_n := \int_{(\mathbb{R}^d)^n} \left| \mathcal{F}\mathcal{L}\tilde{f}_n(1, \xi) \right|^2 \mu(\mathrm{d}\vec{\xi})$. Through direct calculation and Sterlings formula:

$$\frac{c_\alpha}{C_n} 2^{(4-\alpha)n} T_n \leq \int_0^\infty e^{-t} \left\| \tilde{f}(\cdot, 0, t) \right\|^2_{H^{\otimes n}} \mathrm{d}t \leq 2^{(4-\alpha)n} T_n$$

Where $\log(C_n)/n \to 0$ and $c_\alpha > 0$.

## Anticipated Question: Finding the moment asymptotic

$$\mathbb{E}(|u(t,x)|^2) = \sum_{n \geq 0} z_n R_n t^{(2(b+r)-b\alpha/a)n}$$

$$R_n = (n!)^{2(b+r)-b\alpha/a} \left\| \widetilde{f_n}(\cdot, 0, 1) \right\|_{\mathcal{H}^{\otimes n}}^2 \quad \text{and} \quad z_n = \frac{\theta^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}.$$

$$\frac{1}{n} \log(R_n) \to \log \left( \left( \frac{2}{2(b+r) - \frac{b\alpha}{a}} \right)^{2(b+r)-\frac{b\alpha}{a}} \rho \right) \quad \text{as } n \to \infty.$$

If we find a $\beta$ and $A$ so that

$$\lim_{t \to \infty} \frac{1}{t^\beta} \log \sum_{n \geq 0} z_n R^n \left( t^{(2(b+r)-b\alpha/a)} \right)^n = A.$$

then this is implies

$$\lim_{t \to \infty} \frac{1}{t^\beta} \log \sum_{n \geq 0} z_n R_n \left( t^{(2(b+r)-b\alpha/a)} \right)^n = A.$$