The Interpolated Stochastic Heat and Wave Equation: Solvability and Exact Moment Asymptotics

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The Interpolated Stochastic Heat and Wave Equation.

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) \; u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\; \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0, \\ u(0,\cdot) = 1 & b \in (0,1], \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2), \end{cases}$$

- ∂_t^b is the Caputo fractional derivative
- $(-\Delta)^{a/2}$ is the fractional Laplacian of order $a \in (0,2]$
- I_t^r is the Riemann-Liouville fractional integral of order $r \geq 0$
- $W = \{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\}$ is a centered Gaussian and time-independent noise

Case of $b = a = 2$ and $r = 0$ and the Riesz kernel noise.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sqrt{\theta}\, u(t,x)\, \dot{W}(x) & x \in \mathbb{R}^d,\ t > 0, \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2), \end{cases}$$

Theorem (Balan, Chen, L. and Chen, X.)

Suppose $\mu(\mathrm{d}x) = |x|^{-d+\alpha}\mathrm{d}x$ *and that* $0 < \alpha < d \leq 3$. *Let* $t_p := (p-1)^{1/(4-\alpha)} t$. *Then the following moment asymptotic holds:*

$$\lim_{t_p \to \infty} t_p^{-\frac{4-\alpha}{3-\alpha}} \log \|u(t,x)\|_p = \theta^{\frac{1}{3-\alpha}} 2^{\frac{-\alpha}{2(3-\alpha)}} \frac{3-\alpha}{2} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha} \right)^{\frac{4-\alpha}{3-\alpha}} .$$

In addition by fixing $p \geq 2$, *we get that*

$$\lim_{t \to \infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}} \theta^{\frac{1}{3-\alpha}} 2^{\frac{-\alpha}{2(3-\alpha)}} \frac{3-\alpha}{2} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha} \right)^{\frac{4-\alpha}{3-\alpha}} .$$

Gaussian Processes

A Gaussian Processes, $G := \{G_t : t \in T\}$ is a stochastic process such that for and finite set of $t_1, ..., t_n \in T$, the random vector $(G(t_1), ..., G(t_n))$ follows a n-dimensional Gaussian distribution. The processes G is uniquely determined by

1. $\mu(t) = \mathbb{E}(G(t))$
2. $C(s, t) = \mathbb{E}(G(t)G(s))$

Theorem (Important)

The space of all symmetric and non-negative definite functions, $f(s, t)$, on $T \times T$ matches the space of all covariance functions of all Gaussian processes on T. Thus, given a non-negative definite function on $T \times T$ then there exists a Gaussian Processes that possesses this covariance function.

The noise, $\dot{W}$.

We start with a nonnegative and nonnegative definite tempered measure Γ with density γ,

$$\Gamma(\mathrm{d}x) = \gamma(x)\mathrm{d}x.$$

Bochner's theorem guarantees the existence of a measure μ, the spectral measure, defined through the following:

$$\int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)\phi(x) = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(\mathrm{d}\xi) \qquad \phi \in \mathscr{S}(\mathbb{R}^d).$$

We now define a functional on $C_0^\infty(\mathbb{R}^d)$.

$$C(\phi, \psi) = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi)$$

The noise, $\dot{W}$.

Recalling the theorem mentioned above, we have a zero mean Gaussian processes, $W = \{W(\phi) : \phi \in C_0^\infty,\ \phi(x)\}$, with covariance given by $C(\cdot,\cdot)$. In other words,

$$\begin{aligned}\mathbb{E}(W(\phi)W(\psi)) &= \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(\mathrm{d}\xi) \\ &= \int_{\mathbb{R}^{2d}} \phi(x)\psi(y)\gamma(x-y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

When μ has a density, we denote it as $\varphi(x)$.

Spatially Correlated Versus White Verses Independent

- Correlated:

$$\mathbb{E}(W(\phi)W(\psi)) = \int_{\mathbb{R}^{2d}} \phi(x)\psi(y)\gamma(x-y)\mathrm{d}x\mathrm{d}y.$$

- White:

$$\begin{aligned}\mathbb{E}(W(\phi)W(\psi)) &= \int_{\mathbb{R}^{2d}} \phi(x)\psi(y)\delta_0(x-y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^{2d}} \phi(x)\psi(x)\mathrm{d}x\end{aligned}$$

  and informally we may think that $\mathbb{E}(\dot{W}(x)\dot{W}(y)) = \delta_0(x-y)$.

- Independent:

$$\mathbb{E}(W(\phi)W(\psi)) = \mathbb{E}(W(\phi))\mathbb{E}(W(\psi)) = 0 \quad \text{for all } \phi, \psi.$$

The noise, $\dot{W}$.

- Let $k \in \{1, \cdots, d\}$
- Partition the d-coordinates of $x = (x_1, \cdots, x_d)$ into k distinct groups of size d_i so that $d_1 + \cdots d_k = d$
- $x_{(i)} = (x_{i_1}, \cdots, x_{i_{d_i}})$ denotes the coordinates of the i^{th} partition.
- Choose $\alpha_i \in (0, d_i)$ and denote $\alpha = \sum_{i=1}^k \alpha_i$

We consider the following Riesz-type spatial correlation function and its spectral density:

$$\gamma(x) = \prod_{i=1}^{k} |x_{(i)}|^{-\alpha_i} \quad \varphi(\xi) = \prod_{i=1}^{k} C_{\alpha_i, d_i} |\xi_{(i)}|^{d_i - \alpha_i}.$$

Note that when $k = 1$, the above reduces to the Riesz kernel.

The fundamental solution.

$$\begin{cases} \left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) \ u(t,x) = I_t^r \left[f(t,x)\right] & x \in \mathbb{R}^d, \ t > 0, \\ u(0,\cdot) = 1 & b \in (0,1], \\ u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2), \end{cases}$$

When $f(t,x)$ is a deterministic function, the fundamental solution given through the triple

$$\{Z_{a,b,r,\nu,d}(t,x), Z^*_{a,b,r,\nu,d}(t,x), \, Y_{a,b,r,\nu,d}(t,x)\}$$

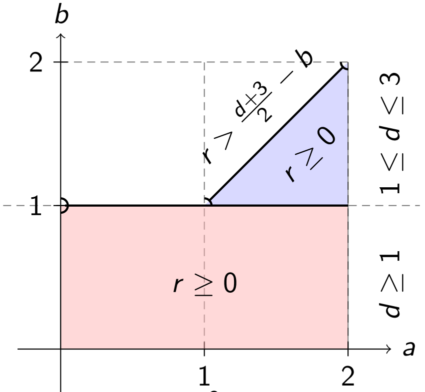
where each member is defined explicitly through the Fox-H function. We have for $b \in (0,1]$ and $b \in (1,2)$ respectively that

$$u(t,x) = \begin{cases} (Z(t,\cdot) * u(0,\cdot))(x) + (Y \star f)(t,x) \\ (Z^*(t,\cdot) * u(0,\cdot))(x) + (Z(t,\cdot) * \partial_t u(0,\cdot))(x) + (Y \star f)(t,x). \end{cases}$$

Assumption on G

We need to make an assumption that $G(t,x) = Y_{a,b,r,\nu,d}(t,x)$ is nonnegative. This is the case under any of the following:

- $d \geq 1$, $b \in (0,1]$, $a \in (0,2]$, $r \geq 0$;
- $1 \leq d \leq 3$, $1 < b < a \leq 2$, $r > 0$;
- $1 \leq d \leq 3$, $1 < b = a < 2$, $r > \dfrac{d+3}{2} - b$.



Mild solution

Considering the constant 1 initial condition, it can be shown the solution to the homogeneous equation is equal to 1.

Definition

Let $G(t, x) = Y_{a,b,r,\nu,d}(t, x)$. For $T > 0$, a random field $u = \{u(t, x) : t \in (0, T), x \in \mathbb{R}^d\}$ is called a *mild solution* if $G(t - s, x - \cdot)u(s, \cdot)1_{\{s<t\}}$ is Skorohod integrable and the following holds almost surely:

$$u(t, x) = 1 + \sqrt{\theta} \int_0^t \left(\int_{\mathbb{R}^d} G(t - s, x - y)u(s, y)W(\delta y) \right) \mathrm{d}s.$$

- *Global Solution:* $\|u(t, x)\|_p < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$.
- *Local Solution:* If there exists $0 < T_1 \leq T_2 < \infty$ such that $\|u(t, x)\|_2 < \infty$ when $0 < t < T_1$ and $\|u(t, x)\|_2$ D.N.E. for $t > T_2$.

Wiener Chaos Expansion:

If $t = t_{n+1}$ and $x = x_{n+1}$, then a Picard iteration scheme will suggest that for

$$f_n(x_1, \cdots, x_n; x, t) = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^{n} G(t_{k+1} - t_k, x_{k+1} - x_k) \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

the solution can be written as

$$u(t,x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot, x, t)), \quad (t,x) \in (0,T) \times \mathbb{R}^d. \tag{0.1}$$

This further suggests

$$\mathbb{E}(u(t,x)^2) = \sum_{k=0}^{\infty} \theta^k \left\| f_k(\cdot, x, t) \right\|_{\mathcal{H}^{\otimes n}}^2, \quad (t,x) \in (0,T) \times \mathbb{R}^d. \tag{0.2}$$

Theorem 3.3 (Chen and Eisenberg)

Theorem

Fix any $T \in (0, \infty]$. Suppose that $f_n(\cdot, x; t) \in \mathcal{H}^{\otimes n}$ for and $t \in (0, T)$ and $x \in \mathbb{R}^d$. Then the SPDE in question has a unique $L^2(\Omega)$ solution on $(0, T) \times \mathbb{R}^d$ if and only if the series

$$\sum_{k=0}^{\infty} \theta^k \|f_k(\cdot, x, t)\|^2_{\mathcal{H}^{\otimes n}}, \quad (t, x) \in (0, T) \times \mathbb{R}^d.$$

converges in $L^2(\Omega)$ for any $(t, x) \in (0, T) \times \mathbb{R}^d$. In this case, the second moment of the solution is given by this series.

Theorem 1.6 (Chen and Eisenberg)

Assuming $G(t,x)$ is nonnegative and the noise has spatial correlation $\gamma(x) = \prod_{i=1}^{k} |x_{(i)}|^{-\alpha_i}$:

- A *global solution* exists provided

$$0 < \alpha < \min\left(\frac{a}{b}[2(b+r)-1], 2a, d\right)$$

- Otherwise, if

$$r \in [0, 1/2] \qquad \text{and} \qquad 0 < \alpha = \frac{a}{b}[2(b+r)-1] \leq d,$$

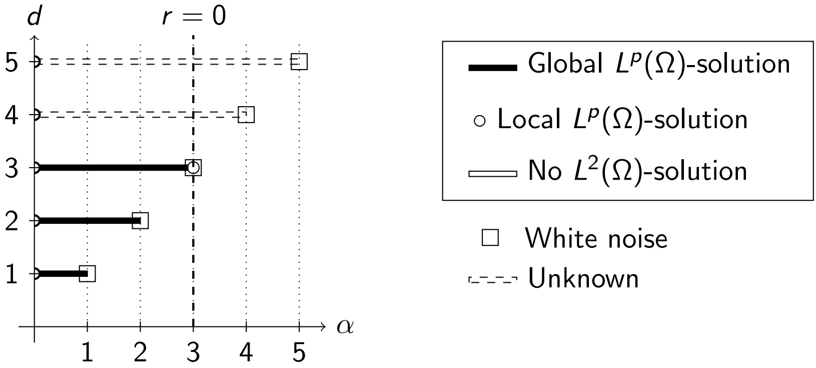
  then a *local solution* exists for $t \in (0, T_p))$ where

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}$$

  and the solution D.N.E. for $t > T_2$.

Example: Solvability for the SWE ($a = b = 2$)

A local solution only exists when $\alpha = 3 + 2r \leq d \leq 3$.

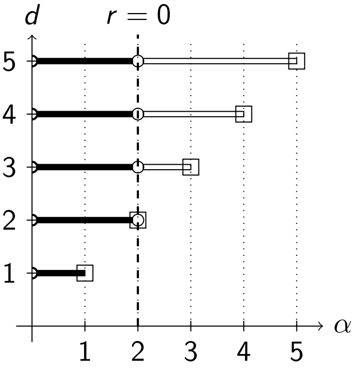


By replacing $\nu = 2$ in the following, we recover (1.12) Balan, Chen and Chen:

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \quad p \geq 2$$

Example: Solvability for the SHE ($a = 2$ and $b = 1$)

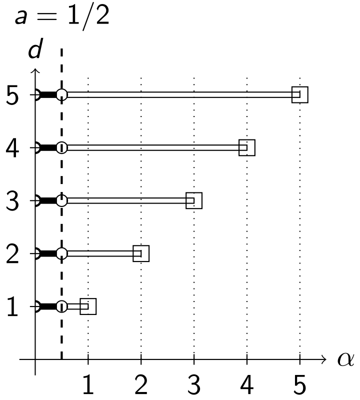
By setting $a = 2$, $b = 1$ and $r = 0$, we obtain the following condition for existence of a local solution: $\alpha = 2 \leq d$.



- When $\alpha = d = 2$, the critical time becomes $T_p = \dfrac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}$.
- Theorem 4.1 (Hu) proves that an $L^2(\Omega)$ solution exists for $t < 2$ but not for $t > 2\pi$.
- T_2 being precise implies $2 \leq T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \leq 2\pi$

Example: Solvability For The SHE With Fractional Laplace ($b = 1$, $r = 0$ and $0 < a \leq 2$)

A local solution exists when $\alpha = a \leq d$. The critical time is given by $T_p = \frac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}$.



The moment asymptotic

Theorem (Chen and Eisenberg)

Suppose a global solution to the SPDE in question exists and $\dot{W}$ is given through the generalized Riesz kernel defined above. Then,

$$\begin{aligned}\lim_{t_p\to\infty} & t_p^{-\beta}\log\|u(t,x)\|_p \\ & =\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\ & \qquad \times\left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right),\end{aligned}$$

where

$$\beta:=\frac{2(b+r)-\dfrac{b\alpha}{a}}{2(b+r)-\dfrac{b\alpha}{a}-1}\qquad \textit{and}\qquad t_p:=(p-1)^{1-1/\beta}\,t. \tag{0.3}$$

$\mathcal{M}$

We define

$$\begin{aligned}\mathcal{M}_{a,d}(\gamma,\theta) &:= \sup_{g\in\mathcal{F}_a}\left\{\left(\iint_{\mathbb{R}^{2d}} g^2(x)g^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y\right)^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\}\\ &= \sup_{g\in\mathcal{F}_a}\left\{\langle g^2 * g^2,\gamma\rangle^{1/2}_{L^2(\mathbb{R}^d)} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\},\end{aligned}$$

where

$$\mathcal{E}_a(g,g) := (2\pi)^{-d}\int_{\mathbb{R}^d}|\xi|^a|\mathcal{F}g(\xi)|^2\mathrm{d}\xi \quad \text{and} \tag{0.4}$$

$$\mathcal{F}_a := \left\{f\in L^2(\mathbb{R}^d):\ \|f\|_{L^2(\mathbb{R}^d)} = 1,\ \mathcal{E}_a(f,f) < \infty\right\}. \tag{0.5}$$

ρ

We define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x)$$

Theorem (Chen and Eisenberg)

When $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ where φ is the generalized Riesz kernel introduced above then,

$$\lim_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} \frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n} \xi_{\sigma(j)}|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}) \right]$$
$$= \log\left(\rho_{\nu,a}(\gamma)\right).$$

where γ is the spatial correlation function.

The Connection between ρ and $\mathcal{M}$

Theorem (Chen and Eisenberg)

When $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ where φ is the generalized Riesz kernel introduced above then,

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a}\mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty$$

ρ and the SWE

$$\zeta\left([0,t_1]\times[0,t_2]\right)=\int_0^{t_1}\int_0^{t_2}|X_1(s_1)+X_2(s_2)|^{-\sigma}\mathrm{d}s_1\mathrm{d}s_2$$

where $X_1(t)$ and $X_2(t)$ are i.i.d. d-dimensional symmetric stable processes of index $\beta=2$,

$$\mathbb{E}e^{i\lambda X_i(t)}=e^{-t|\lambda|^{\beta}}.$$

Theorem (X. Chen)

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\mathbb{E}\left[\zeta\big([0,\tau_1]\times[0,\tau_2]\big)^n\right]=\log\left[\rho_{2,2}(\gamma)\right]$$

ρ and the SWE

Theorem (X. Chen)

Suppose τ_1 and τ_2 are exponential random variables with mean 1. Then,

$$\mathbb{E}\left[\zeta\left([0,\tau_1]\times[0,\tau_2]\right)^n\right] = \int_{(\mathbb{R}^d)^n} \left[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\left|\sum_{j=k}^{n}\xi_{\sigma(j)}\right|^2}\right]^2 \mu(\mathrm{d}\vec{\xi})$$

It can be shown that

$$\frac{1}{(n!)^2}\mathbb{E}[\zeta([0,\tau_1]\times[0,\tau_2])^n] = \int_{(\mathbb{R}^d)^2} \mathcal{F}\mathcal{L}\left[\tilde{f}_n(\circ_1,\cdots\circ_n,0;\cdot)\right]^2(\xi_1,\cdots,\xi_n,1)\mu(\mathrm{d}\vec{\xi})$$

Exponential Growth of the Moments in Time

Theorem (Chen and Eisenberg)

For $p \geq 2$ fixed, we have that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) \\
&\quad = p(p-1)^{\frac{1}{2(b+r)-\frac{b\alpha}{a}-1}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\qquad \times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right).
\end{aligned}$$

From this we can deduce that $t \mapsto \mathbb{E}(|u(t,x)|^p)$ grows as fast as $\exp\left(K_1 t^{\beta}\right)$, in other words, the moments will grow exponentially in time.

Exponential Growth of the moments for Fixed Time

Theorem (Chen and Eisenberg)

For $t > 0$ fixed , we have that

$$\begin{aligned}
&\lim_{p\to\infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) \\
&\quad = t^{\beta}\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\qquad \times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).
\end{aligned}$$

From this we can deduce that $p \mapsto \mathbb{E}(|u(t,x)|^p)$ grows as fast as $\exp\left(K_2 p^{\beta}\right)$, in other words, the moments will grow exponentially in p as well.

Final Comments

Two possible future works.

1. Consider a time and spatially dependent noise. Ie, $W = \{W(\phi), \phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)\}$

$$\mathbb{E}(W(\phi)W(\psi)) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^{2d}} \gamma(t-s) f(x-y) \phi(t,x) \psi(s,y) \mathrm{d}x \mathrm{d}y \mathrm{d}t \mathrm{d}s$$

   where γ is the time correlation and f is the spatial correlation.

2. Under what type of convergence do we have as we vary the fractional parameters? How is the following limit taken:

$$\lim_{a \to a_0,\ b \to b_0,\ r \to r_0,\ \nu \to \nu_0,\ \theta \to \theta_0} u_{a,b,r,\nu,\theta}(t,x) \to u_{a_0,b_0,r_0,\nu_0,\theta_0}(t,x).$$