

Graduate Seminar  
An introduction to stochastic calculus with applications  
to finance and SPDEs

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March 29, 2022

# Stochastic Calculus

In Real Analysis we study measurable functions:

$$f : D \rightarrow R,$$

where both  $D$  and  $R$  are measurable spaces:

$$(D, \sigma(D), \mu_1) \quad \text{and} \quad (R, \sigma(R), \mu_2).$$

In undergraduate Calculus, we focus only on real valued measurable on  $\mathbb{R}^d$ :

$$f : D \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d,$$

and the measure space that it implies is the following:

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda),$$

where  $\lambda$  is Lebesgue measure.

# Stochastic Calculus

When studying Stochastic Calculus, and in general Probability Theory, we consider measurable functions,

$$X : \Omega \rightarrow \mathbb{R}^d,$$

and we consider the measure spaces

$$(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda),$$

where  $\Omega$  and  $\mathcal{F}$  are referred to as the sample space and set of possible events and

$$\mathbb{P}(\Omega) = 1.$$

## Definition (Random Variable)

A measurable function  $X : \Omega \rightarrow \mathbb{R}^d$  is called a  $\mathbb{R}^d$ -valued Random Variable.

So essentially, Stochastic Calculus and Probability is just Real Analysis on Random Variables.

# Stochastic Processes

## Definition (Stochastic Process)

A stochastic process is a random variable that is indexed by set,  $T$ , which usually represents time:

$$X_t(\omega), \quad t \in T, \omega \in \Omega.$$

One can think a stochastic processes as a two parameter function

$$X_t(\omega) = X(t, \omega) : T \times \Omega \rightarrow \mathbb{R}^d.$$

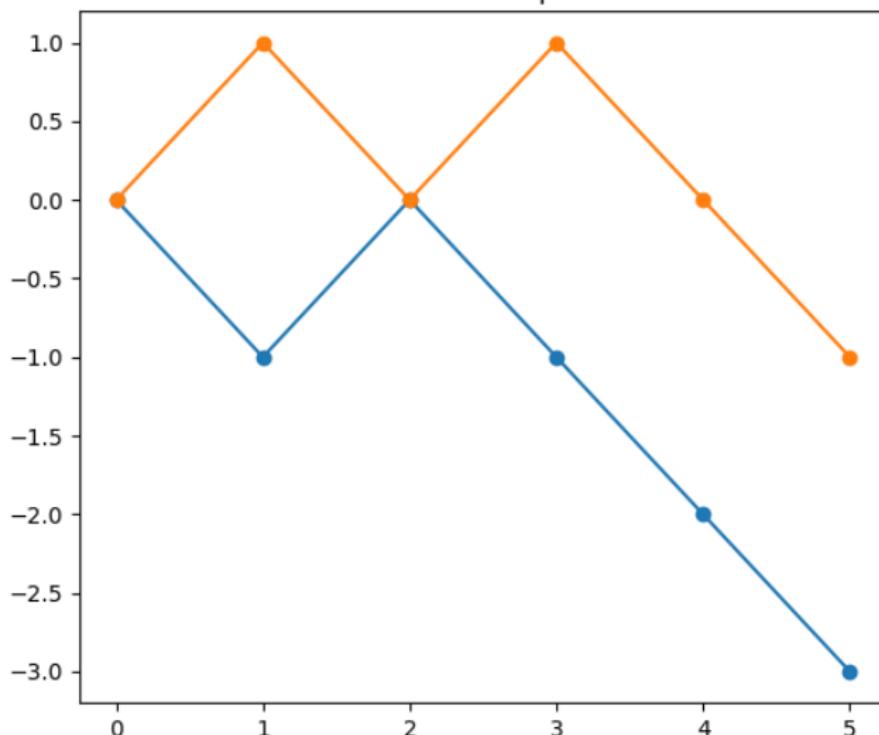
## Definition (Sample Path)

The variable  $\omega$  represents the idea of a single possible outcome. The following function is then referred to as a *sample path*:

$$t \mapsto X_t(\omega).$$

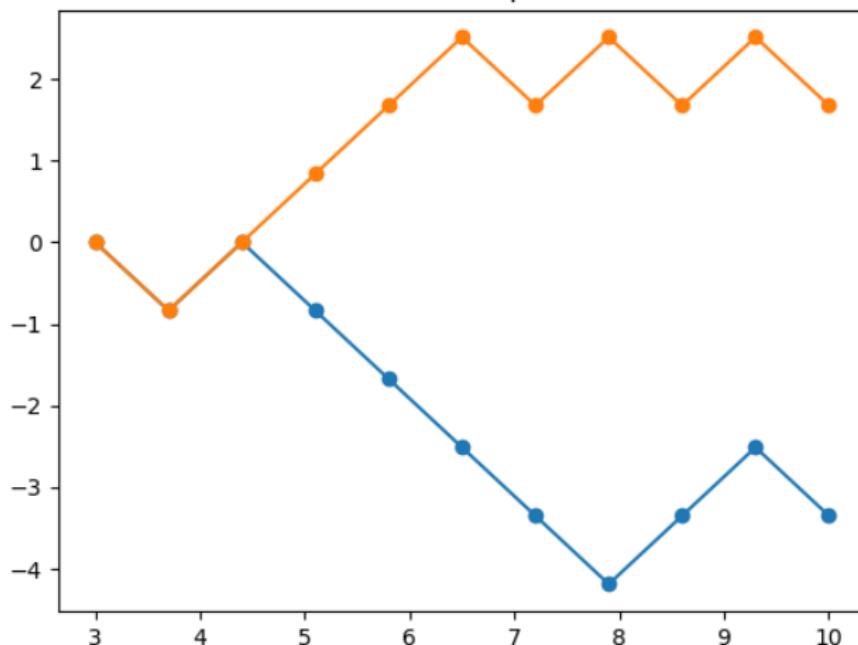
# Random Walk Example

2 sample paths of a random walk  
jump size = 1.0 and step size = 1.0  
number of steps = 5



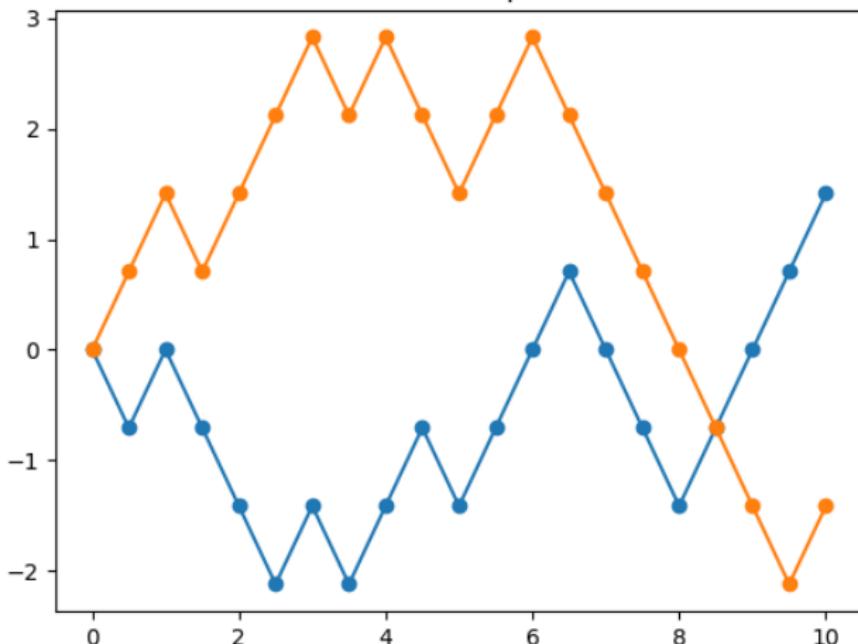
# Random Walk Example

↳ sample paths of a random walk  
jump size = 0.837 and step size = 0.7  
number of steps = 10



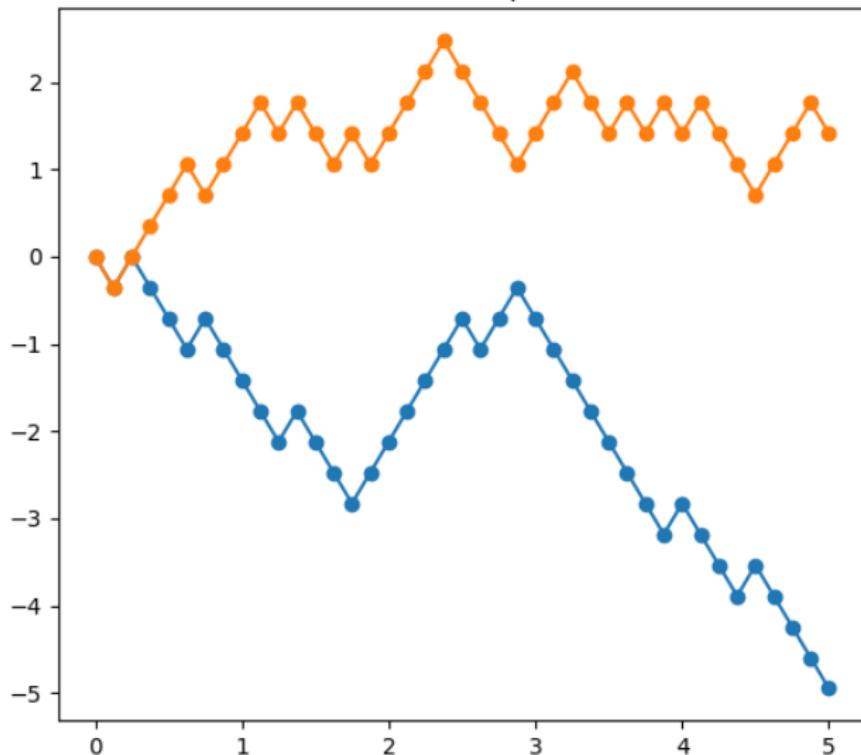
# Random Walk Example

2 sample paths of a random walk  
jump size = 0.707 and step size = 0.5  
number of steps = 20

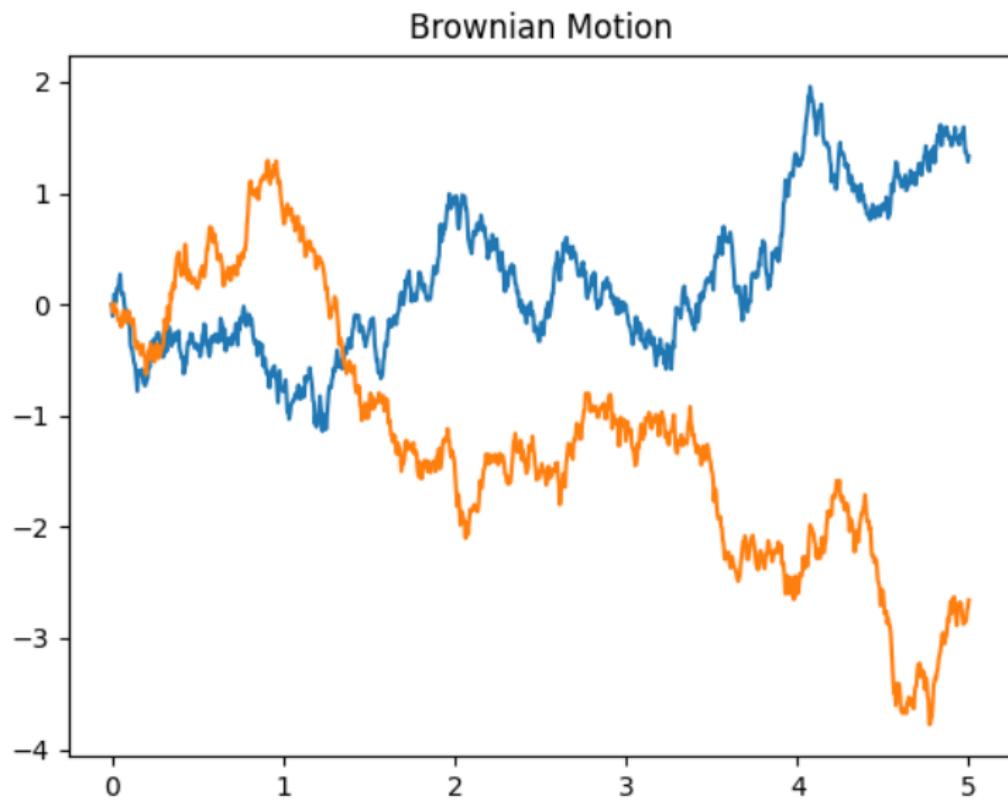


# Random Walk Example

2 sample paths of a random walk  
jump size = 0.354 and step size = 0.12  
number of steps = 40



## Random Walk Example



## Brownian Motion.

### Theorem

Consider a random walk  $X = \{X_0 = 0, X_{t_1}, \dots, X_{t_n}\}$ . Suppose that for  $k \in \{1, \dots, n\}$  that

$$\mathbb{P}[X_{t_k} = -h] = \mathbb{P}[X_{t_k} = h] = 1/2.$$

and suppose that the time increments are all constant:

$$t_k - t_{k-1} = d \quad \forall k \in \{1, \dots, n\}.$$

Suppose that  $h^2 = d$ . Now define

$$B_n(t_k) = \sum_{i=0}^k X_{t_i}.$$

Then  $B_n$  converges to a Brownian motion.

## The Ito integral (special case)

Suppose that  $f$  is a nice continuous function. Then we can define:

$$\int_0^t f(s)dB_s(\omega) := \lim_{\|P\| \rightarrow 0} \sum_{i \in P} f(t_{i-1})(B_t(\omega) - B_{t_{i-1}}(\omega)),$$

where  $P = t_0 = 0, t_1, t_2, \dots, t_n = t$  is a partition of  $[0, t]$ . In other words, the Ito integral in this case the left point Riemann sum.

## Theorem (Ito's Formula)

If  $f$  is twice continuously differentiable then we have the following "Fundamental Theorem of Ito Calculus"

$$\int_a^t \frac{d}{ds} f(B_s) dB_s = f(B_t) - f(B_a) - \frac{1}{2} \int_a^t \frac{d^2}{ds^2} (B_s) ds.$$

## Example

If we apply the above to  $f(x) = x^2/2$  with  $a = 0$  then we have

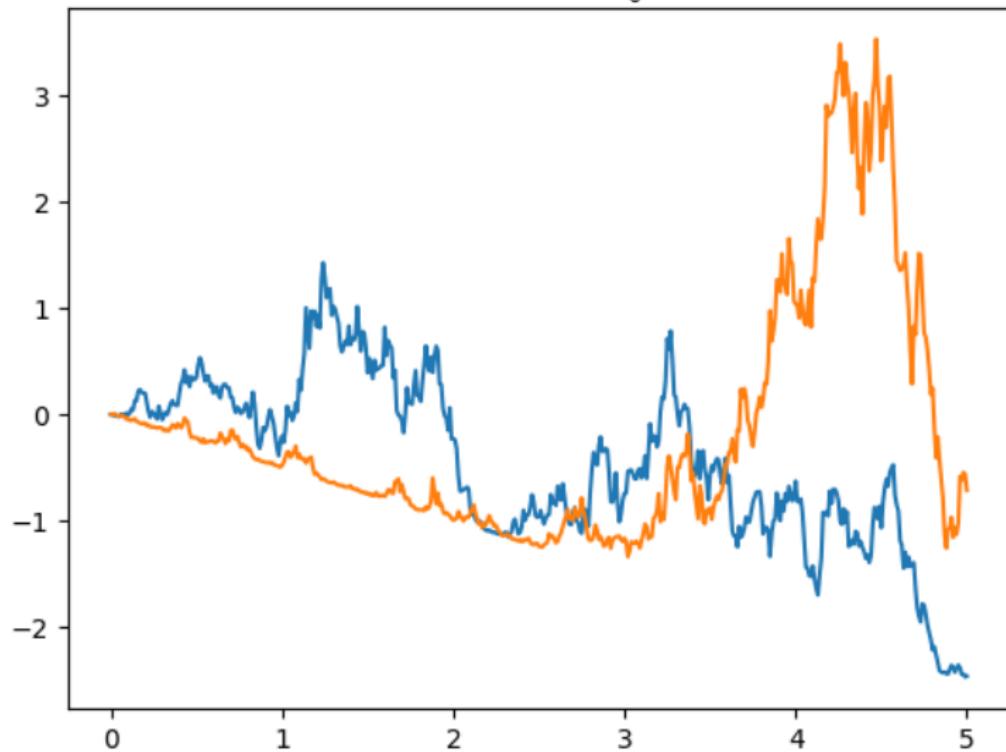
$$\int_0^t B_s \, dB_s = \frac{B_t^2}{2} - \frac{t}{2},$$

which is not quite the same thing as

$$\int_0^{B_t} x \, dx = \frac{B_t^2}{2}$$

## Two sample paths

Sample paths for  $\int_0^t B_s dB_s$



## One more example

For a nice function  $f(t, x)$  we can generalize Itos formula to

$$f(t, B_t) - f(a, B_a) = \int_a^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \int_a^t \frac{\partial f}{\partial t}(s, B_s) + \frac{\partial^2 f}{\partial x^2}(s, B_s) ds.$$

Now consider the function  $f(t, x) = tx$ . Then

$$\frac{\partial f}{\partial t}(t, x) = x, \quad \frac{\partial f}{\partial x}(t, x) = t, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = 0$$

Plugging this in above for  $a = 0$  gives us

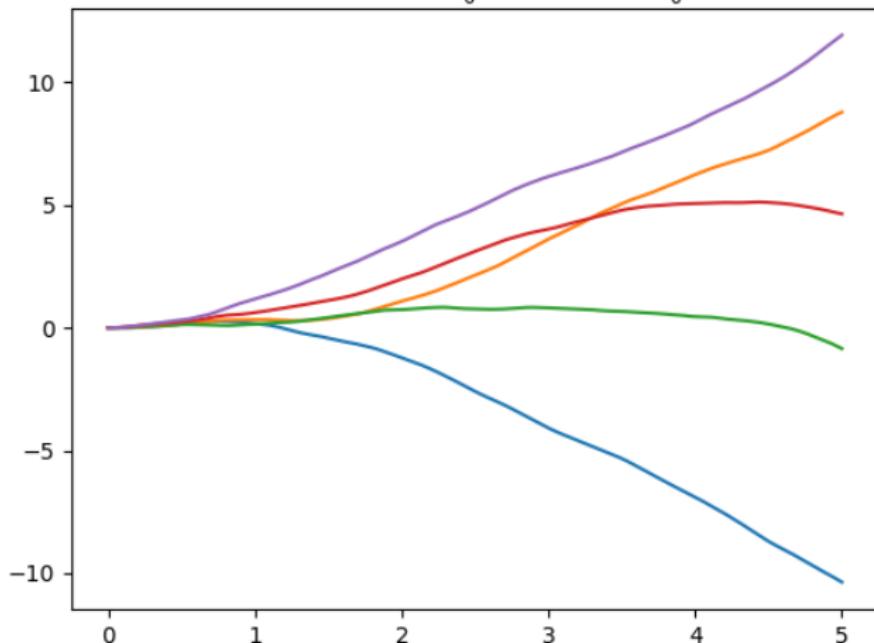
$$tB_t = \int_0^t s dB_s + \int_0^t B_s ds,$$

So we see that the Riemann integral of a Brownian motion is

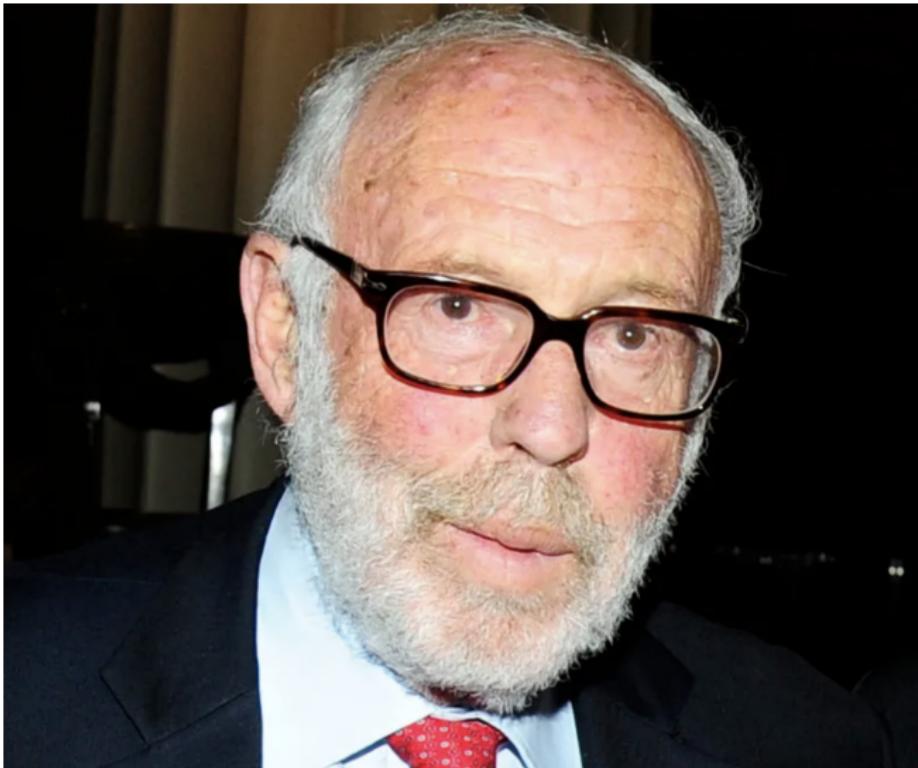
$$\int_0^t B_s ds = tB_t - \int_0^t s dB_s.$$

# Smoothness?!

5 sample paths of  $\int_0^t B_s \, ds = tB_t - \int_0^t s \, dB_s$



## Why do we care about Brownian Motion?



## An application to finance.

Consider a stock that has value  $S_t$  at time  $t$ . Consider  $t = 0$  to be the point of entry into the investment. We can define a return rate as follows:

$$S_t = S_0(1 + R_t), \quad R_t := \frac{S_t - S_0}{S_0} \text{ is the return rate over the period } [0, t].$$

We want to make predictions of our return. In a very simple model, we can assume that

$$R_t \sim \text{Normal}(\mu_t, \sigma^2), \quad \mu \in \mathbb{R}, \sigma \in (0, \infty).$$

In terms of Stochastic Calculus, we express this as the following stochastic differential equation:

$$R_t = \frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad \text{where } B_t \text{ is a Brownian motion.}$$

## Modeling a stock price with geometric brownian motion

The notation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad \text{where } B_t \text{ is a Brownian motion}$$

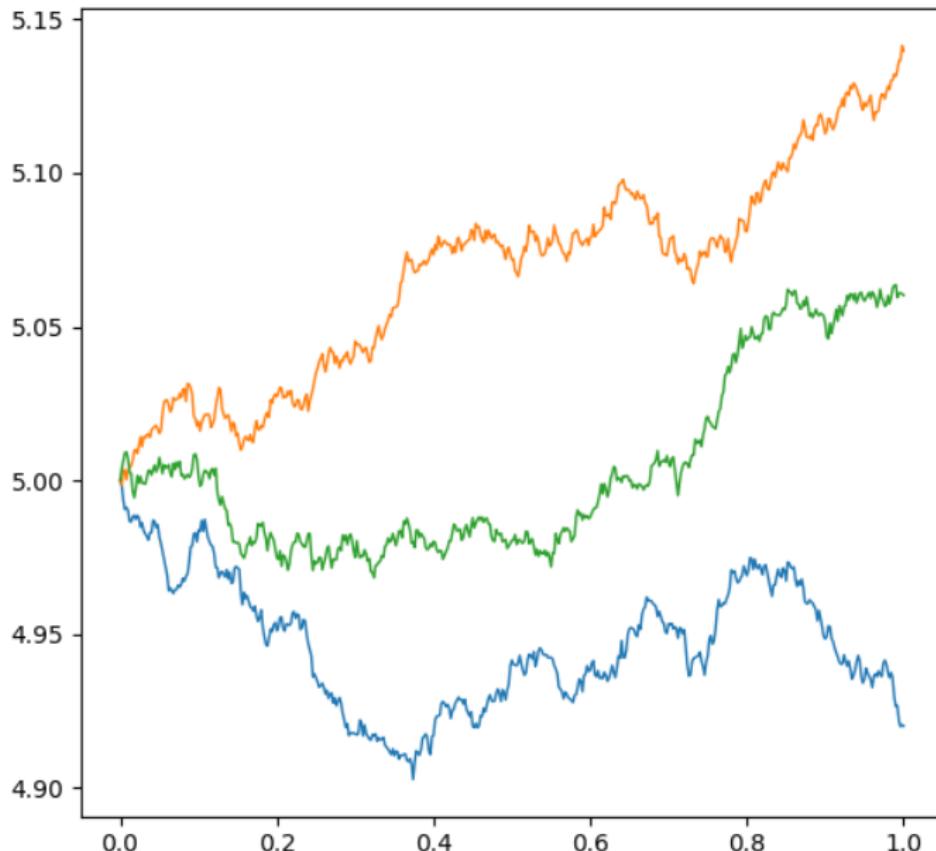
is only symbolic as  $dB_t$  does not exist. The above can be legitimately viewed as the following integral equation:

$$S_t - S_0 = \int_0^t \mu S_k dk + \int_0^t \sigma S_k dB_k.$$

The stochastic processes  $S_t$  that solves the above equation is called a Geometric Brownian Motion. Notice that we have came up with a random variable that models the stock price:

$$S_t(\omega) = S_0 + \int_0^t \mu S_k(\omega) dk + \int_0^t \sigma S_k(\omega) dB_k(\omega).$$

Geometric Brownian Motion:  $dX_t = \sigma X_t dW_t + \mu X_t dt$   
 $(a, b) = (0, 1)$ ,  $X_a = 5$ ,  $\sigma = 0.01$ , and  $\mu = 0.001$   
we subdivide the domain n = 500 times



## Black-Scholes option pricing model for European calls

Suppose at time  $t = 0$  you buy a contract from a seller, allowing you to purchase 1 share of a stock at time  $t = T$  and a set price of  $K$ , regardless of what the stock price at time  $T$  is.

$$\begin{aligned}\text{Payout and time } T &= \begin{cases} S_T - K & \text{if } S_T > k \\ 0 & \text{otherwise} \end{cases} \\ &= (S_T - K)^+\end{aligned}$$

What is a fair price for this contract?

For this we need a self-financing strategy.

We assume that the stocks price follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Let  $a_{t_i}$  be the amount of shares owned over the period  $t \in (t_{i-1}, t_i]$ . Then at any time  $t \in [0, T]$ , the amount of shares owned is

$$a_t = \sum_{i=1}^n a_{t_i} \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

and the realized over  $[0, T]$

$$\begin{aligned} a_T S_T - a_0 S_0 &= \sum_{i=1}^n a_{t_i} (S_{t_i} - S_{t_{i-1}}) = \int_0^T a_t dS_t \\ &= \mu \int_0^T a_t S_t dt + \sigma \int_0^T a_t S_t dB_t \end{aligned}$$

Summarizing the last slide...

$$\text{Our stock profit on time } [0, T] = a_T S_T - a_0 S_0 = \underbrace{\int_0^T a_t dS_t}_{\text{ }}$$

Now suppose we also invest in a bond with interest rate,  $r$ , that has price  $\beta_t$  at time  $t$ . Similarly...

$$\text{Our bond profit on time } [0, T] = b_T S_T - b_0 S_0 = \underbrace{\int_0^T b_t d\beta_t}_{\text{ }}$$

### Definition (Self Financing Strategy)

A pair of stocks and bonds,  $(a, b)$ , is self financing if

$$a_T S_T + b_T S_T = a_0 S_0 + b_0 S_0 + \int_0^T a_t dS_t + \int_0^T b_t d\beta_t$$

## Fair price for a European call option

A fair price for a European call option that allows the buyer to buy 1 share at price  $K$  at time  $T$  regardless of the price of the stock  $S_T$  at time  $T$  is the amount one would need to invest in a self financing strategy,

$$\underbrace{a_T S_T + b_T S_T}_{V_T} = \underbrace{a_0 S_0 + b_0 S_0}_{V_0} + \int_0^T a_t dS_t + \int_0^T b_t d\beta_t,$$

such that the value of the strategy at time  $T$  is equal to  $(S_T - K)^+$ :

$$V_T = (S_T - K)^+$$

Let  $f(S_t, T - t) = V_t$ . So we see that

$$f(S_T, 0) = V_T = (S_T - K)^+ \quad \text{and} \quad f(S_0, T) = V_0$$

So we need to find  $f$ . Moreover,  $f(S_0, T)$  is the fair price to pay.

It can be shown that  $f$  solves the following partial differential equation

$$\begin{cases} \frac{\partial f}{\partial s}(s, x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(s, x) + rx \frac{\partial f}{\partial x}(s, x) - rf(s, x) & (s, x) \in [0, T] \times (0, \infty) \\ f(0, x) = (x - K)^+ & x \in (0, \infty). \end{cases}$$

The solution is given by

$$f(t, x) = x\Phi(g(t, x)) - Ke^{-rt}\Phi(h(t, x)),$$

where

$$g(t, x) = \left[ \ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)t \right] / (\sigma\sqrt{t}),$$

and

$$h(t, x) = g(t, x) - \sigma/\sqrt{t},$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp\left(-\frac{|x|^2}{2}\right) dx$$

## Potential for unlimited profit

In practice, option contracts are priced by "the market". This means that the cost of a contract is literally what the buyer is willing to pay or what the seller is willing to sell at.

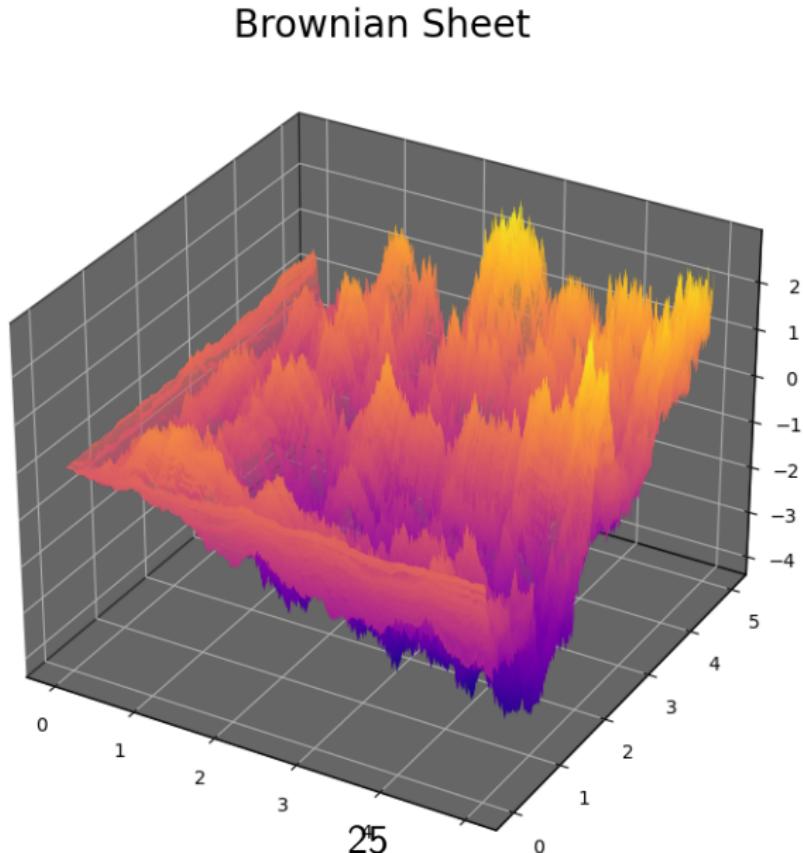
This gives the educated buyer (or seller) a chance to make profit.

Suppose the fair price for a call contract with strike  $K$  is  $\$q$  but we find a buyer, who does not know about the Black Scholes model, who wants to buy the contract for  $\$p > \$q$ . We can do the following:

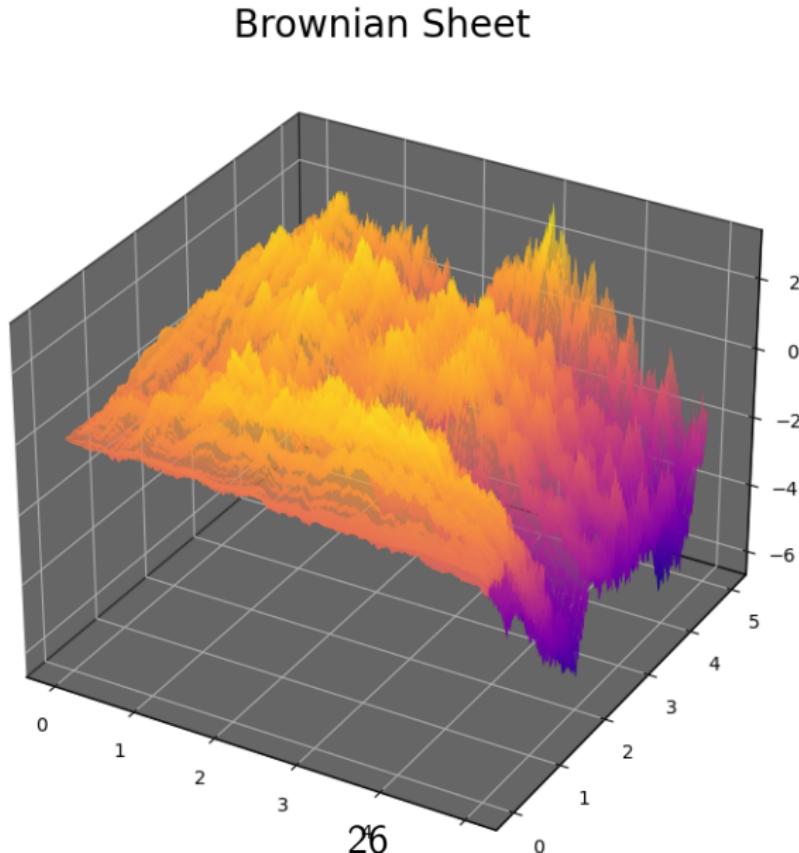
- ▶ sell a contract for  $\$p$ ,
- ▶ invest  $\$q$  in stock and bond at time  $t = 0$  in a self financing strategy (HARD TO DO),
- ▶ keep the remaining  $p - q$  dollars.
- ▶ pay the buyer  $(S_T - K)^+$  at expiration.

No matter what, we make  $p - q$  dollars. If we sell 100 contracts then we make  $100(p - q)\dots$

# Sample path of a Brownian Sheet, $W(t, x)$

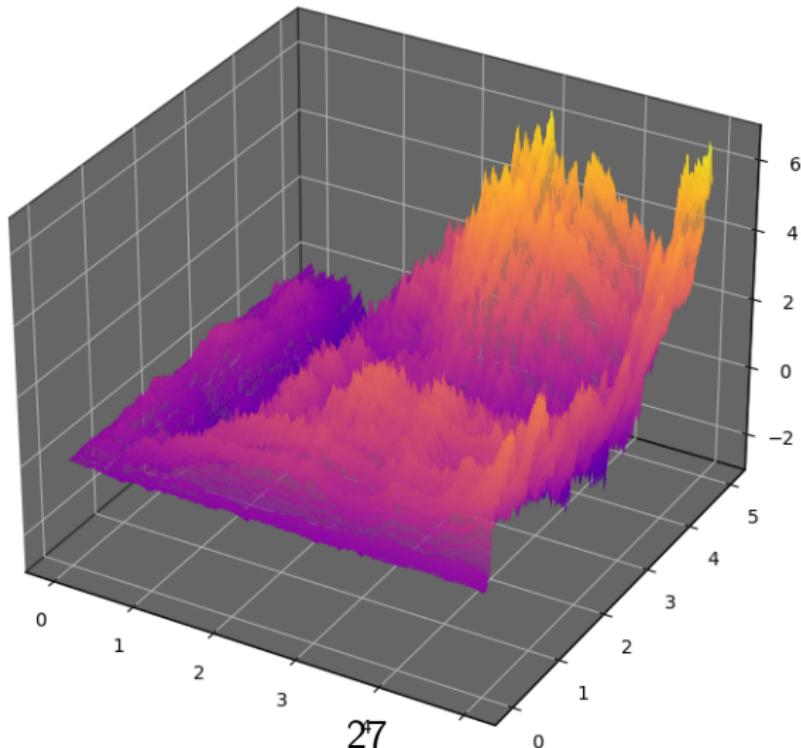


# Sample path of a Brownian Sheet, $W(t, x)$



# Sample path of a Brownian Sheet, $W(t, x)$

Brownian Sheet



## Stochastic Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = \lambda u(t, x) \frac{\partial^2}{\partial t \partial x} W(t, x) & t \geq 0, x \in \mathbb{R} \\ u(0, x) = f(x) & x \in \mathbb{R} \end{cases}$$

The solution to the stochastic heat equation is given through the following integral equation:

$$u(t, x) = \int_{\mathbb{R}} G(t, x - y) f(y) dy + \lambda \int_0^t \int_{\mathbb{R}} G(t - s, x - y) u(s, y) W(ds, dy),$$

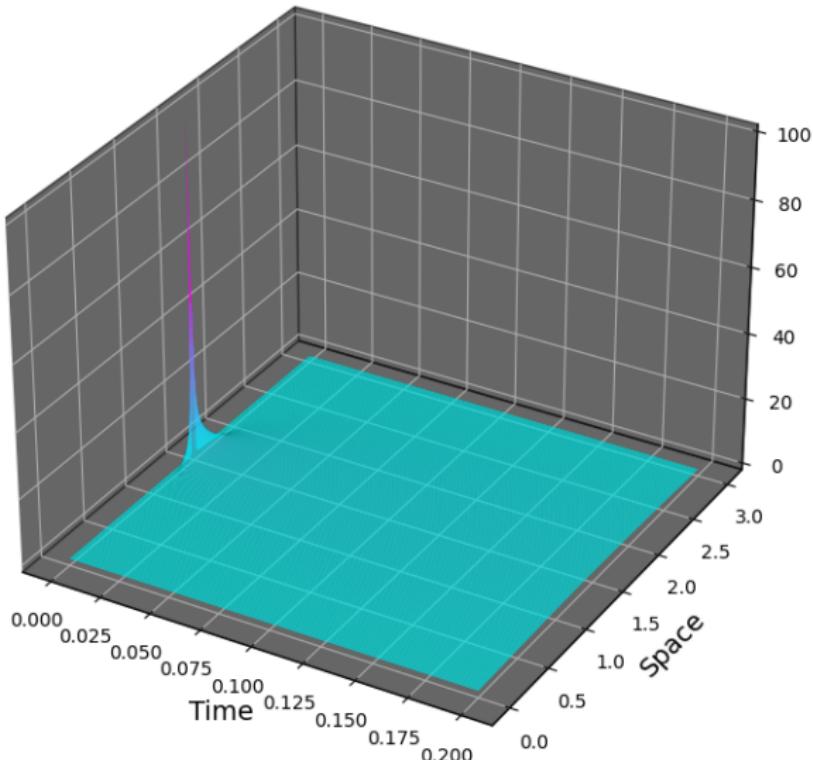
where

$$G(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right)$$

A finite difference method simulation of

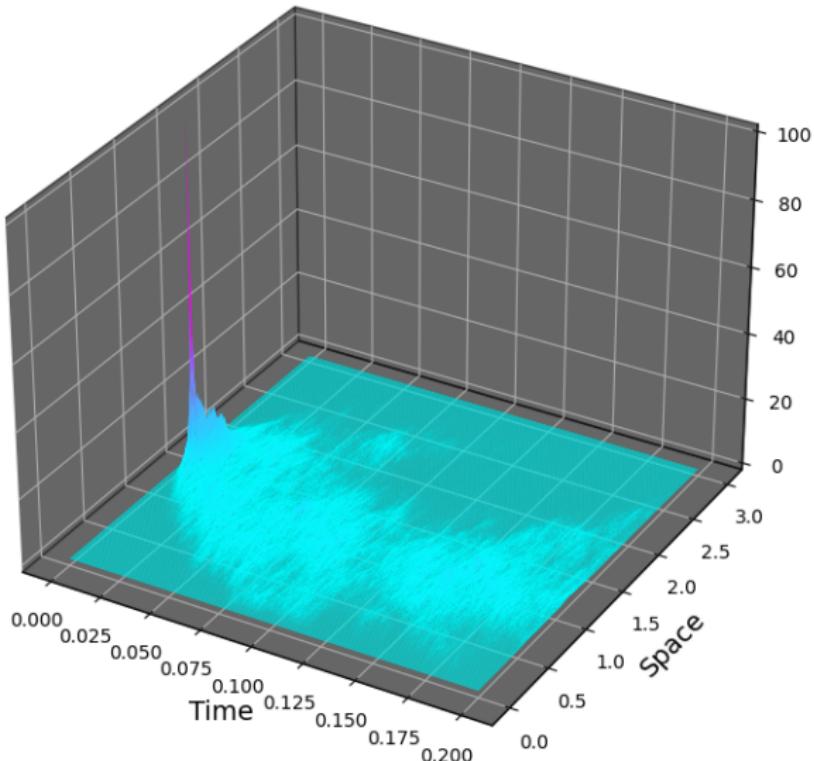
$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 0u(t, x)\dot{W}(t, x)$$

$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

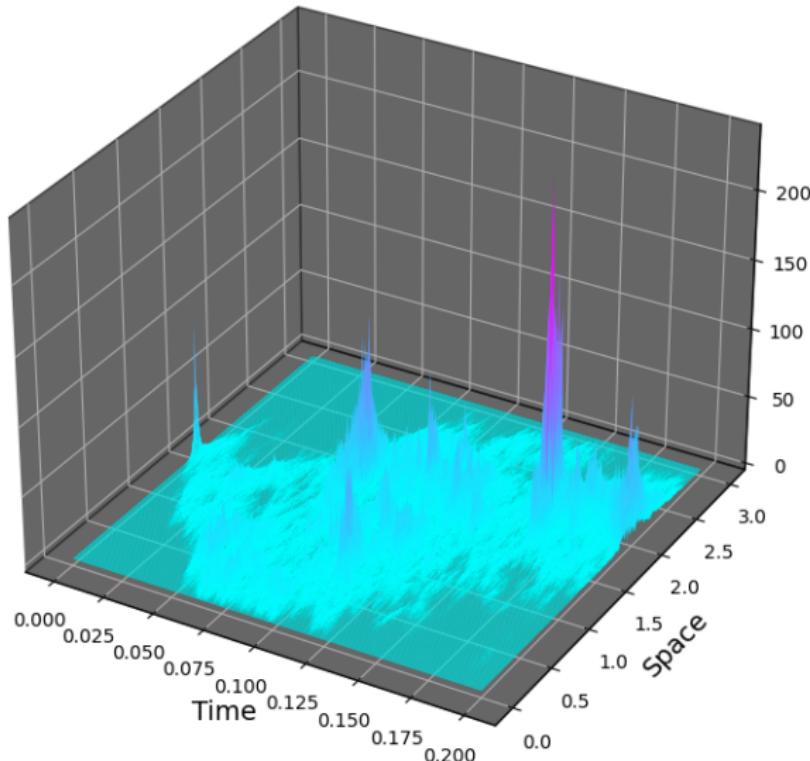
$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 1u(t, x)\dot{W}(t, x)$$
$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 2u(t, x)\dot{W}(t, x)$$

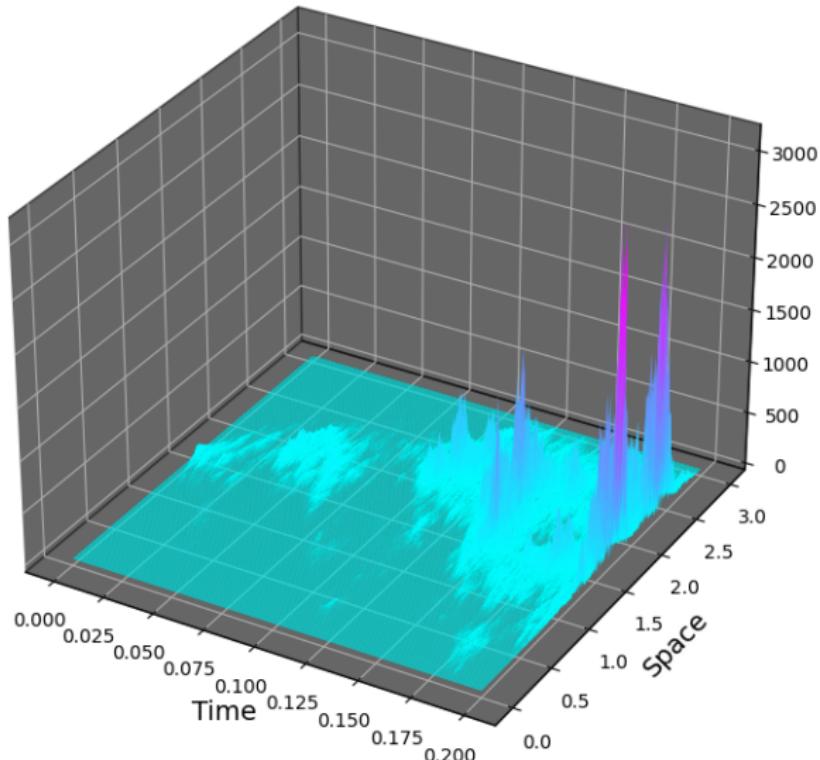
$$u(x, 0) = \delta_0(x)$$



A finite difference method simulation of

$$u_t(t, x) - \frac{1}{2}u_{xx}(t, x) = 2.15u(t, x)\dot{W}(t, x)$$

$$u(x, 0) = \delta_0(x)$$



## References

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