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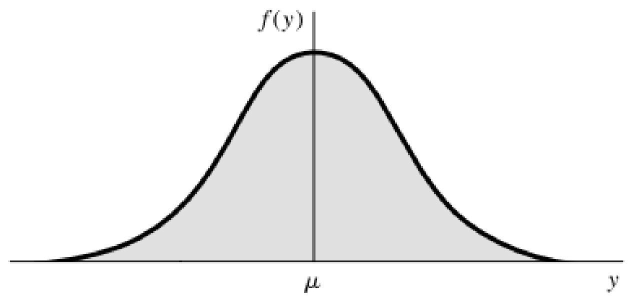
Normal, Gamma and Beta Distributions

Normal Distribution

The most frequently used continuous distribution, the Normal distribution, also known as Gaussian distribution, is a popular probability distribution that typically follows a symmetric bell curve. This distribution determines the probability that an event falls between two real numbers, as it approaches zero on each side. If the standard deviation is greater than zero and the mean is between negative infinity and positive infinity, then a normal probability distribution exists:



$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$$

Where y is between negative infinity and positive infinity.

The expected value and variance of Y are given by:

$$E(Y) = \mu$$

$$V(Y) = \sigma^2$$

The probability that an event exists on a certain interval is given by:

$$P(a \leq Y \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}\, dy$$

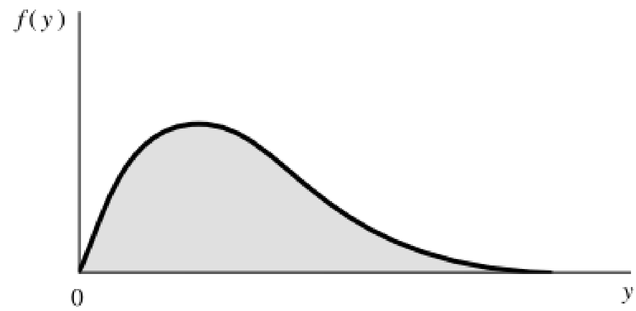
Exercise 4.73 The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm. What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?

$$P(947 \leq Y \leq 958) = \int_{947}^{958} \frac{1}{10\sqrt{2\pi}} e^{\frac{-(y-950)^2}{2(10)^2}}\, dy$$

$$= \frac{1}{10\sqrt{2\pi}} * 2.998941350960305 * 10^{1548}$$

Gamma Distribution

The gamma distribution deals with skewed bell curves if you will. As you go to the right, f(x) gradually decreases, deviating from what a bell curve represents. An example of this would be the rate at which customers queue at a checkout register or the rate at which aircraft engines malfunction. A random variable has a gamma distribution with parameters alpha greater than zero and beta greater than zero:



$$f(y) = \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\tau(\alpha)}$$

Where:

$$\tau(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}\,dy$$

And y is between zero and positive infinity and zero elsewhere.

The expected value and variance of Y with parameters α and β are given by:

$$E(Y) = \alpha\beta$$

$$V(Y) = \alpha\beta^2$$

Moving forward, if v is a positive integer, and if a random variable Y has parameters α = v/2 and β = 2, then Y has a chi-square distribution with v degrees of freedom. In this case, the expected value and variance of Y is:

$$E(Y) = v$$

$$V(Y) = 2v$$

A random variable Y has an exponential distribution where β > 0 if:

$$f(y) = \frac{1}{\beta}e^{-y/\beta}$$

Where y is between zero and positive infinity and zero elsewhere. In this case, the expected value and variance of Y is:

$$E(Y) = \beta$$

$$V(Y) = \beta^2$$

Exercise 4.88 The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will exceed 3.0 on the Richter scale.

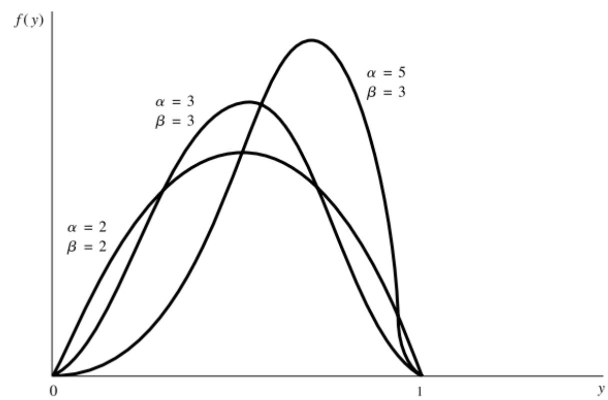
$$P(Y > 3) = 1 - P(Y \leq 3)$$

$$= 1 - (1 - e^{-\frac{3}{2.4}})$$

$$= 0.2865$$

Beta Distribution

The beta distribution contains two parameters and is defined by the bounds 0 ≤ y ≤ 1. This distribution is used for proportions. For example, impurities in a chemical product or proportion of time that a machine is under repair. It is sort of like a bell curve but skewed on both sides. A random variable Y has a beta distribution with parameters α > 0 and β > 0:



$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$$

Where:

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy$$

And y is between 0 and 1 and is 0 elsewhere.

The cumulative or incomplete beta function is given by:

$$F(y) = \int_0^y \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}\, dt$$

The expected value and variance of a beta function is given by:

$$E(Y) = \frac{\alpha}{\alpha+\beta}$$

$$V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Exercise 4.124 The percentage of impurities per batch in a chemical product is a random variable Y with density function. A batch with more than 40% impurities cannot be sold. Integrate the density directly to determine the probability that a randomly selected batch cannot be sold because of excessive impurities.

$$P(Y \geq 0.40) = \int_{0.40}^{1} 12y^2(1-y)\,dy$$

$$= 0.8208$$