MATH 3310 Convergence of Alternating Harmonic to log(2)

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Claim . The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n}$ converges to $\log 2 = 0.693147181...$

This claim will be proven using Exercise 7.5.8 (d) and (e) in [A]. I will begin with part (d), first proving the convergence of the sequence:

$$\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) - \log n \tag{1}$$

After I have proven the convergence of γ_n, I will use the definitions of γ_n and γ_{2n} to derive a value for $\log 2$ by considering the sequence $\gamma_{2n} - \gamma_n$. Throughout this proof, the fact that $log(x) = \int_1^x \frac{1}{t} dt$ will be used.

Proof. Convergence of (1):

To prove that γ_n is bounded below (by zero), I will prove that $(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) \geq \log n \ \forall \ n \in \mathbf{N}$

$\log n = \int_1^n \frac{1}{t} dt = \int_1^2 \frac{1}{t} dt + \int_2^3 \frac{1}{t} dt + ... + \int_{n-1}^n \frac{1}{t} dt$

For each of these definite integrals, $\frac{1}{t}$ is at its maximum value when t is equal to its lower bound value, i.e. $\forall t \in [n, n+1]$, $\frac{1}{t} \leq \frac{1}{n}$. So, we can plug in these upper values for $\frac{1}{t}$ into our integrals to get an integral greater than or equal to our integrals evaluated with $\frac{1}{t}$:

$\int_1^n \frac{1}{t} dt = \int_1^2 \frac{1}{t} dt + \int_2^3 \frac{1}{t} dt + ... + \int_{n-1}^n \frac{1}{t} dt \leq \int_1^2 \frac{1}{1} dt + \int_2^3 \frac{1}{2} dt + ... + \int_{n-1}^n \frac{1}{n-1} dt$

$\int_1^2 \frac{1}{1} dt + \int_2^3 \frac{1}{2} dt + ... + \int_{n-1}^n \frac{1}{n-1} dt = (2 - 1) + (\frac{3}{2} - 1) + ... + (\frac{n}{n-1} - \frac{n-1}{n-1}) = 1 + \frac{1}{2} + ... + (\frac{1}{n-1})$

$\implies \int_1^n \frac{1}{t} dt \leq (1 + \frac{1}{2} + ... + \frac{1}{n-1})$

$\implies \gamma_n = (1 + \frac{1}{2} + ... + \frac{1}{n}) - \int_1^n \frac{1}{t} dt \geq (1 + \frac{1}{2} + ... + \frac{1}{n}) - (1 + \frac{1}{2} + ... + \frac{1}{n-1}) = \frac{1}{n}$

$\implies \gamma_n \geq \frac{1}{n}$

Thus we have that γ_n is greater than or equal to $\frac{1}{n} \ \forall \ n \in \mathbf{N}$, proving that γ_n is bounded below by zero since $\frac{1}{n} > 0 \ \forall n \in \mathbf{N}$.

Next, I will show that (γ_n) is decreasing.

$\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) - \log n = \sum_{k=1}^{n} \frac{1}{k} - \log{(n)}$

$\gamma_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} - \log{(n+1)}$

$\gamma_{n+1} \leq \gamma_n \iff \sum_{k=1}^{n+1} \frac{1}{k} - \log{(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k} - \log{(n)}$

$\iff \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{k} - \log{(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k} - \log{(n)}$

$\iff \frac{1}{n+1} - \log{(n+1)} \leq -\log{(n)}$

$\iff \frac{1}{n+1} \leq \log{(n+1)} - \log{(n)} = \int_1^{n+1} \frac{1}{t} dt - \int_1^{n} \frac{1}{t} dt = \int_n^{n+1} \frac{1}{t} dt$

$\int_n^{n+1} \frac{1}{t} dt$ is at its minimum when t = n+1, so we can plug in this t value into the integral to state that our given integral must be greater than or equal to this value:

$\int_n^{n+1} \frac{1}{n+1} dt \leq \int_n^{n+1} \frac{1}{t} dt$

$\int_n^{n+1} \frac{1}{n+1} dt = \frac{1}{n+1} \implies \frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{t} dt$

So we have that the statement is true for all $n \in \mathbf{N}$, thus this implication proves that $\gamma_{n+1} \leq \gamma_n$ which implies that the sequence is decreasing. Then, since (γ_n) is decreasing and bounded below, we have that the sequence must converge by the Monotone Convergence Theorem.

Knowing that (γ_n) converges, we must have that (γ_{2n}) also converges as $n \to \infty$ and to the same value as (γ_n). Thus, we must also have that the sequence $\lim_{n\to\infty}(\gamma_n - \gamma_{2n}) = 0$.

$\lim_{n\to\infty}(\gamma_n - \gamma_{2n}) = 0$

$\implies \lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \log(n)) - (\sum_{k=1}^{2n} \frac{1}{k} - \log(2n))) = 0$

$\implies \lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k} - \log(n) + \log(2n)) = 0$

Using log rules proven in HW 12, we can rewrite $\log(2n)$ as $\log(2) + \log(n)$, giving:

$\implies \lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k} - \log(n) + \log(n) + \log(2)) = 0$

$\implies \lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k} + \log(2)) = 0$

Since $\log(2)$ is a constant, we can rewrite this by the ALT as:

$\lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k}) + \log(2) = 0$

$\implies \lim_{n\to\infty}((\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k}) = -\log(2)$

$\implies \lim_{n\to\infty}(-\sum_{k=n+1}^{2n}\frac{1}{k}) = -\log(2)$

$\implies -\lim_{n\to\infty}(\sum_{k=n+1}^{2n}\frac{1}{k}) = -\log(2)$

$\implies \lim_{n\to\infty}(\sum_{k=n+1}^{2n}\frac{1}{k}) = \log(2)$

Next, I will prove that $\sum_{k=n+1}^{2n}\frac{1}{k} = \sum_{i=1}^{2n}\frac{(-1)^{i+1}}{i}$ using proof by induction.

Let n = 1: $\sum_{k=2}^{2}\frac{1}{k} = \sum_{i=1}^{2}\frac{(-1)^{i+1}}{i} \Leftrightarrow \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$

Let n = 2: $\sum_{k=3}^{4}\frac{1}{k} = \sum_{i=1}^{4}\frac{(-1)^{i+1}}{i} \Leftrightarrow \frac{1}{3} + \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

Adding $\frac{1}{4}$ to both sides yields: $\frac{1}{3} + \frac{1}{2} = 1 - \frac{1}{2} + \frac{1}{3} \Leftrightarrow \frac{1}{3} + \frac{1}{2} = \frac{1}{2} + \frac{1}{3}$.

So, the above statement is true for n=1 and n=2. Assume that the above is true for n=r. Now, I will prove this true for n=r+1:

$\sum_{k=r+1}^{2r}\frac{1}{k} = \sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} \implies \sum_{k=r+1}^{2r}\frac{1}{k} - \frac{1}{r+1} = \sum_{i=1}^{2n}\frac{(-1)^{i+1}}{i} - \frac{1}{r+1}$

$\sum_{k=r+1}^{2r}\frac{1}{k} - \frac{1}{r+1}$ can be rewritten as a single sum by removing the $(r+1)^{th}$ index to get: $\sum_{k=r+2}^{2r}\frac{1}{k}$. Next, we need our summation to go to $2(r+1) = 2r+2$, which we can do by adding $\frac{1}{2r+1} + \frac{1}{2r+2}$ on both sides.

$\sum_{k=r+2}^{2r}\frac{1}{k} + \frac{1}{2r+1} + \frac{1}{2r+2} = \sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} - \frac{1}{r+1} + \frac{1}{2r+1} + \frac{1}{2r+2}$

$\implies \sum_{k=r+2}^{2r+2}\frac{1}{k} = \sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} - \frac{1}{r+1} + \frac{1}{2r+1} + \frac{1}{2r+2}$

The left hand side of the equation is now correctly defined for $n = r+1$. Now, the right hand side can be further simplified:

$\sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} - \frac{1}{r+1} + \frac{1}{2r+1} + \frac{1}{2r+2} = \sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} - \frac{2}{2r+2} + \frac{1}{2r+1} + \frac{1}{2r+2}$

$= \sum_{i=1}^{2r}\frac{(-1)^{i+1}}{i} - \frac{1}{2r+2} + \frac{1}{2r+1}$

$= \sum_{i=1}^{2r+2}\frac{(-1)^{i+1}}{i}$.

Thus, we now have that $\sum_{k=r+2}^{2r+2}\frac{1}{k} = \sum_{i=1}^{2r+2}\frac{(-1)^{i+1}}{i}$, showing that our statement is true for n = r + 1 and $\forall\ n \in \mathbf{N}$.

So, we have:

$\log(2) = \lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k} = \lim_{n\to\infty}\sum_{i=1}^{2n}\frac{(-1)^{i+1}}{i}$

Now, by the alternating series test, we know that $\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}$ converges since we can let $(a_n) = \frac{1}{i}$ which is both decreasing and converges to zero as $i \to \infty$. Thus, we can set the partial sum of our series to $S_m = \sum_{i=1}^{2m}\frac{(-1)^{i+1}}{i}$. Then we have that $\lim_{m\to\infty} S_m = \log(2)$ which implies that the infinite alternating harmonic series must also converge to this value, finally giving us:

$\log(2) = \lim_{m\to\infty} S_m = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

$\square$