A NOTE ON GENERALISED EULER-LAGRANGE THEORIES IN DIFFERENTIAL COHESION

NICK GOLUB

Abstract. We use the axioms of differential cohesion for ∞-topoi so as to internalize the prequantum physics completely. We suggest the way how to substitute $U(1)$ phased field theories and the Euler-Lagrange theory by rather general $\mathbf{A}_\infty$-type field theories. This allows to construct prequantum field theories over general ∞-stacks, in particular on orbifolds. We give an example of how differetnial K-theory can be used to define prequantum field theories of a new type.

Contents

1. Introduction

In this note we come up with the way of defining the whole Euler-Lagrange theory using differential cohomology $\mathbf{A}_\infty$-rings in an abstract *differetnially-cohesive ∞-topos.* The classical Euler-Lagrange theory includes the notion of Lagrangian forms, sorcce forms (in particular the notion of equations of motion) and the Euler-Lagrange complex of forms on jet bundles of field theories. We shall review briefly the modern description of the Euler-Lagrange theory, add some details (such as the representation of field theories via coalgebras over some comonad 2.20). Recall the Euler-Lagrange theory is attached to the Deligne $U(1)$-differential cohomology whose cocycles are classified by the stack $\mathbf{B}^{p+1}U(1)_{conn}$.

Our abstract definition of prequantum field theories in *differentially-cohesive ∞-topoi* provides a generalisation from the paradigm of Deligne cohomology to an abstract differential cohomology theory (such as *differential K-theory* for instance). In particular we may extend construction of prequantum field theories to abstract contexts (for instance in the context of relative cohesion of $\mathbb{E}_\infty$-arithmetic geometry).

Moreover, it is very suggestive to study these new types of field theories since this approach somewhat resembles what happened with non-abelian gauge field theories. Yang and Mills proposed to generalise the gauge invariance of electrodynamics by using the $SU(2)$ group and constructing an action for the isospin doublet of protons and neutrons. Later after the discovery of the property of asymptotic freedom of non-abelian gauge field theories and this motivated the search of the theory of strong interactions.

Defining prequantum field theories is somewhat (see the characterization in terms of the coalgebras that we derived 2.0.8) the same as defining a $\mathbb{R}/\hbar\mathbb{Z}$ (p+1)-gerbe with connection on a bundle of a field theory. Our suggested theory of $\mathbf{A}_\infty$-type field theories can be used in the fashion of $U(1) \mapsto G$ generalization, where G is a non-abelian Lie group, so as to find new field theories which for some reason cannot be formulated within the classical Euler-Lagrange theory. In 2.1 we demonstrated that there are interesting examples of smooth spectra which possess the desired properties of the classical Euler-Lagrange theory.

On the other hand, we know that $\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ stacks classifying prequantum field theories are defined using the structure of smooth manifolds. In string theory for instance, compactifications on orbifolds, orientifolds are of a great importance and so in the direction of M-theory it becomes common to consider spacetime as a more general stack than a simple smooth manifold. Nonetheless, there are no accepted mathematical means in defining field theories on such singular spaces as orbifolds or rather general smooth homotopy types. Our work makes the construction of the Euler-Lagrange theories axiomatic which allows to define field theories on general smooth homotpy types (or even on differentially-cohesive) and in the case of orbifolds 2.26 we even construct an analog of $\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ stack.

2. Local prequantum field theories synthetically

In this section we shall review the basics of a modern point of view on variational theory - the cornerstone of field theory and prequantum geometry. Then, we apply the ideas to define prequantum field theories abstractly in arbitrary differentially-cohesive ∞-topoi.

From now on let Σ denote a smooth $p+1$-manifold. For the notations used for notable categories appearing especially in the variational diagram 2.0.6 see below 2.0.2.

2.0.1. In this section we exploit the notation of [16] and [26] (see there for notation of modalities, (co)monads on cohesive, differentially-cohesive ∞-topoi).

One of the most important aspects of fundamental physics is the notion of the *Lagrangian* and the action functional defining loci of sections (configurations of fields) satisfying the Euler-Lagrange equations.

Field theories are defined in terms of fiber bundles $E \to \Sigma$ with given systems of Lagrangians forms on jet bundles $(\mathcal{L}_i) : J^\infty E|_{U_i} \to \mathbf{\Omega}^{p+1}$

$$\mathcal{L}_i \equiv L_i(x^k, u^k, u^k_\alpha)dx^1 \wedge \cdots \wedge dx^{p+1} \tag{1}$$

, where $\{U_i\}$ is an atlas of Σ, u^k_α denote local jet coordinates ([4] Chapter One). The pullback along a section ϕ with a compact support defines locally on Σ a form, the integral of which gives the value of the action functional on the given configuration:

$$\begin{tikzcd}
J^\infty E \arrow[d] \\
E \arrow[d] \\
\Sigma \arrow[u, bend left, "\phi"] \arrow[uu, dashed, bend right=60, "j^\infty\phi"']
\end{tikzcd}$$

$$\mathbf{S} : \phi \mapsto \int_{supp\{\phi\}} j^\infty(\phi)^*\mathcal{L} \tag{2}$$

The Lagrangian is a fundamental concept and the formulation only through variational problem on action functionals has various technical issues:

(1) $\mathbf{S}$ correctly are defined only locally, whilst globally only the exponentiated action functionals $e^{i\mathbf{S}}$ are defined ([26], 1.1.1.3). The obstructions to globality (see below or [4]) are known and in general one works with refined local systems of Lagrangians to Cech-Deligne cocycles.

As indicated in [1] finding such refinements is a necessary procedure for preparation of a theory to quantization [26] and $\mathbf{S}$ are obtained via transgression ([26] 1.1.1, [11] 2.8, and for fiber integration used for transgression [13]).

(2) Despite giving the right Euler-Lagrange equations some of the action functionals turn out to be incorrect from the point of view of the variational theory. For instance, the famous Hilbert-Einstein action functiona is not correct ([3], pp 16) when we pose the problem of finding the field configurations (components of metric tensors) since it contains second derivatives of fields. The issue is known to be resolved by Gibbons-Hawking-York action or by the first-order formulations [3].

In its classical form, the formalism for pro-manifolds over smooth manifolds has been constructed and we will state its main concepts using the [26], [16], [12], [25], [28]. Our goal is to trace how classical aspects of variational theory fit into higher topos theory, which will lead us to a variational diagram 2.0.6 as a result.

2.0.2. In the following definition we remind the well established notation in literature [16]:

Definition 2.1.

- *CartSp* $\subset$ $SmMfd$ - sites of Cartesian spaces and smooth manifolds. These sites produce the same ∞-*topoi*, but when working with slices it is convenient to use smooth manifolds or formal smooth manifolds (**??**).
- *LocProMfd* - projective limits of finite dimensional smooth manifolds
- *InfThPoint* $\subset CALG^{op}_{\mathbb{R}}$ - full subcategory of the opposite of the category of commutative $\mathbb{R}$-algebras spanned by duals (denote duals by *Spec*) of Weil algebras (*Spec*($\mathbb{R} \oplus \mathcal{V}$), where $\mathcal{V}$ is a nilpotent ideal.

  As it is known that ∞-stacks over *InfThPoint* are identified with properly infinitesimal homotopy types in the ∞-topos of *formal smooth* ∞*-groupoids* (in particular the underlying homotopy types of ∞-*Lie algebras*).
- *FormalCartSp* $\subset CALG^{op}_{\mathbb{R}}$- spanned by duals of $\mathbb{R}$-algebras of smooth functions of Cartesian spaces infinitesimally thickened (*Spec*($\mathbb{C}^\infty(\mathbb{R}^n) \oplus_{\mathbb{R}} (\mathbb{R} \oplus \mathcal{V})$).

  It is endowed with a Grothendieck topology induced by good open covers of Cartesian spaces in *CartSp* site.
- *Sh*(*FormalCartSp*) $\equiv$ *FormalSmoothSets* - formal smooth sets. This 1-topos is known in literature as Cahiers topos [16] and serves as a 0-truncated base for the *synthetic differential geometry*.
- Sh_∞(*FormalCartSp*) $\equiv$ *FormalSmooth-*∞*-groupoids*
- *Diffeological*. It is known that diffeological spaces are precisely the concrete 0-truncated *smooth* ∞*-groupoids*.
- $LocProMfd_{\downarrow\Sigma}$ - category spanned by fibered pro-manifolds $\pi : E \to \Sigma$ where π is a surjective submersion ([16], 3.18).
- $DiffOp_{\downarrow\Sigma}(LocProMfd)$ - category of differential operators over Σ. Its objects are that of $LocProMfd_{\downarrow\Sigma}$ but morphisms are *differential operators*:

$$(D : E \to F) \equiv D : \Gamma_\Sigma(E) \to \Gamma_\Sigma(F) \tag{3}$$

  such that there is a *formal differential operator* $\hat{D} : J^\infty E \to F$ so that the following diagram commutes:

$$\begin{tikzcd}
J^\infty E \arrow[d] \arrow[drr, "\hat{D}"] & & \\
E \arrow[d] & & F \\
\Sigma \arrow[u, bend left, "\phi"] \arrow[uu, dashed, bend right, "j^\infty(\phi)"'] \arrow[urr, "D(\phi)"'] & &
\end{tikzcd}$$

2.0.3. By the classical works of Vinogradov [28], [29] it has become possible to treat local variational problems cohomologically ([4]). The main object to study has become the variational bicomplex (see in details [4], pp 22; [26] Proposition 1.3.22). Its construction procceds by considering the chain complex (this is actually the sheaf of chain complexes on the site $DiffOp_{\downarrow\Sigma}(LocProMfd)$ 2.0.2 (see [26], 6.5.10) of differential forms on jet bundles $\mathbf{\Omega}^*(J^\infty E)$.

It has a filtration realizing $\mathbf{\Omega}^*(J^\infty E)$ as a total complex of the so called variational bicomplex (see [4], Def. 1.7):

$$\mathbf{\Omega}^*(J^\infty E) = \bigoplus_{r+s=*} \Omega^{r,s}, d = d_H + d_V \tag{4}$$

There emphasize the following three types of differential forms defining this filtration:

- (Horizontal forms Ω_H) $J^\infty E$ is endowed with an atlas of a pro-manifold [4]. In this atlas horizonal forms are those that locally defined on $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ basis elements coming form the local basis on Σ:

$$L_{i_1,\ldots,i_k}(x^i, u^i, u^i_\alpha)dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{5}$$

- (Vertical Ω_V) Vertical forms are spanned by the basis of forms ([4]):

$$\theta^{\alpha_1}_{I_1} \wedge \cdots \wedge \theta^{\alpha_s}_{I_s} \wedge \eta \tag{6}$$

  where

$$\theta^\alpha_I = du^\alpha_I - u^\alpha_{I,i}dx^i \tag{7}$$

  η - any form.
- (Source forms) $\Omega^{p+1,r}_S$ defined via the Euler operator I (yielding the Euler-Lagrange equations for the case $r = 1$):

$$\Omega^{p+1,r}_S \equiv \mathfrak{I}(\Omega^{p+1,r})$$

  The Euler derivative is defined via

$$\delta_V \equiv \mathfrak{I} \circ d_V$$

Source forms are used to extend the complex of *horizontal* differential forms on $J^\infty E$ to the Euler-Lagrange complex $\mathbf{\Omega}^*_{EL_\Sigma}$ (see [26], Def. 6.5.93):

$$0 \longrightarrow \Omega^0_H \xrightarrow{d} \Omega^1_H \longrightarrow \ldots \longrightarrow \Omega^{p+1}_H \xrightarrow{\delta_V} \Omega^{p+1,1}_S \longrightarrow \Omega^{p+1,2}_S \longrightarrow \ldots$$

2.0.4. Vinogradov constructed ([27], [29]) the category 2.0.2 of partial differential equations. Marvan then realized in [22] that the functor:

$$J^\infty_\Sigma : LocProMfd_{\downarrow\Sigma} \to LocProMfd_{\downarrow\Sigma} \tag{8}$$

has a structure of *comonad* (which we refer to as jet comonad):

$$\triangle : J^\infty_\Sigma \to J^\infty_\Sigma J^\infty_\Sigma, \mu : J^\infty_\Sigma \to 1 \tag{9}$$

with $\triangle$ being comultiplication and a counit μ . In [22] it was shown that the Vinogradov's category PDE_Σ is equivalent to the *Eilenberg-Moore category of coalgebras over the jet comonad* J^∞_Σ (see 2.0.2).

From this equivalence the category of differential operators over Σ is realized as *co-Kleisli category of* J^∞_Σ (2.0.2, [16]):

$$\begin{tikzcd}
EM(J^\infty_\Sigma) \arrow[rr, leftrightarrows, "\sim"] & & PDE_\Sigma \\
Kl(J^\infty_\Sigma) \arrow[u, hook] \arrow[r, leftrightarrows, "\sim"] & DiffOp_{\downarrow\Sigma}(LocProMfd) \arrow[ur, hook]
\end{tikzcd}$$

2.0.5. The theory of partial differential equations, variational calculus belong to the framework of differential geometry. Following the *internalization* methodology one may ask in what system of axioms for toposes one may produce aspects of differential geometry abstractly.

Now we turn to the *Cahiers topos* 2.0.2 *FormalSmoothSets* which was initially constructed in [8] by Dubuc as a model for the Kock-Lawvere axioms. In [16] it was suggested that it is the property of *differential cohesion* of that may be used to extend the theory partial differential equations 2.0.4 to *formal smooth sets*.

Remark 2.2. In 2.1 below we shall use the non-truncated version of the results formulated further using only the axioms of *differential cohesion*. This allows us to study differential geometry in abstract *differentially-cohesive* ∞*-topoi*.

This is the point for us hinting that axioms of *differential cohesion* may be enough even to cover the prequantum field theory 2.1. We saw **??** how Cartan connections and thus gravity fields *internalize* within the *differentially-cohesive* ∞*-topoi*.

The following sequence of claims should convey the essence of the theory of *generalised PDE's*. The point for us is to follow up whether the main claims extend to a non-truncated case.

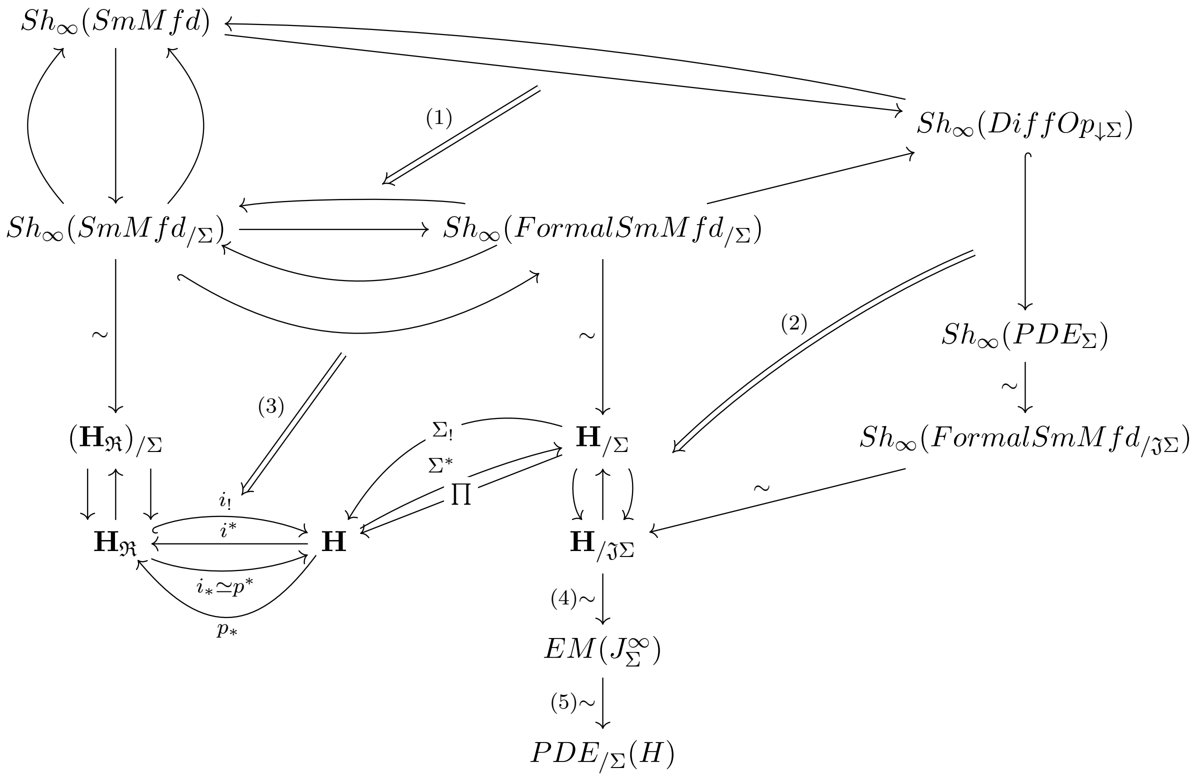
The key fact of the usage of V-manifolds (V is a *differentially-cohesive* ∞-group) is that they have a (see [16], **??**) a local trivialization of an *infinitesimal disk bundle*. In [16] this allowed to extend the *Cahiers topos* the approach by Marvan [22] to theory of *PDE's*:

(1) Axioms of differential cohesion on the *Cahiers topos* result in production of a comonad (**??**) J^∞_Σ
(2) If Σ is a V-manifold, where V is a group object in the *Cahiers topos*, then one may define the category $PDE_{/\Sigma}(\mathbf{H})$ of *formally integrable PDE's* over Σ ([16], Def.3.47)
(3) By [16] Th.3.52. an extension of the result by Marvan is given:

$$PDE_{/\Sigma}(\mathbf{H}) \simeq EM(J^\infty_\Sigma)$$

(4) For a smooth manifold Σ Vinogradov's category of *PDE's* is equivalent to $EM(J^\infty_\Sigma|_{LocaProMfd_{\downarrow\Sigma}})$ (Cor.3.55. [16])
(5) One may see that the results pertain to the non-truncated case in the ∞-topos of *formal smooth* ∞*-groupoids*

2.0.6. *Variational diagram.* The cornerstone of the prequantum geometry are constructions that are captured in the form of what we call *the variational diagram.* Most of the results may be found in [1], [26], [16], we summarise them and make some new observations concerning notable functors:



(1) The geometric morphism $Sh_\infty(FormalSmMfd_{/\Sigma}) \to Sh_\infty(FormalSmMfd_{/\mathfrak{J}\Sigma})$ descends to the adjunction:

$$FormalSmMfd_{/\Sigma} \underset{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}{\overset{(\eta_\Sigma)_!}{\rightleftarrows}} FormalSmMfd_{/\mathfrak{J}\Sigma}$$

(2) When dealing with bundles we shall need to calculate the functor $(\eta_\Sigma)_* : \mathbf{H}_{/\Sigma} \to \mathbf{H}_{/\mathfrak{J}\Sigma}$ which in general is difficult to understand. For instance consider 2.15 where the right adjoint to a pullback functor is $[G, -]//G$. Nonetheless, here we can simplify it:

Lemma 2.3. $(\eta_\Sigma)_* \simeq ((-) \circ (\eta_\Sigma)^*|_{FSmMfd_{/\mathfrak{J}\Sigma}})$.

Proof. Since both $(\eta_\Sigma)^*$ and $Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}$ preserve colimits and every ∞-stack is a homotopy colimit of representables ([9]) it suffices to verify whether functors coincide at representables.

For, let y denote the Yoneda embedding and consider evaluation of the left Kan extension

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}(y(K \to \mathfrak{J}\Sigma))(X \to \Sigma)$$

Further denote the restriction of the pullback functor to sites as $(\eta_{\Sigma)^*}$ Using the left Kan extension formula (see the "Pointwise Kan extensions" [24], [18]) this is equivalent to the colimit along the following category:

$$((X \to \Sigma) \downarrow (\eta_\Sigma)^*)^{op}$$

Its objects are $((H \to \mathfrak{J}\Sigma), (X \to \Sigma) \to (\eta_\Sigma)^*(H \to \mathfrak{J}\Sigma))$.

We notice that by the universal property of unit there is a final object:

$$(\eta_\Sigma)_!(X \to \Sigma), (X \to \Sigma) \underset{unit}{\to} (\eta_\Sigma)^*(\eta_\Sigma)_!(X \to \Sigma)$$

Thus, the left Kan extension has a value:

$$\simeq y(K \to \mathfrak{J}\Sigma)((\eta_\Sigma)_!(X \to \Sigma))$$

$$\simeq \mathbf{H}_{/\mathfrak{J}\Sigma}((\eta_{\Sigma !})(X \to \Sigma), K \to \mathfrak{J}\Sigma)$$

$$\simeq \mathbf{H}_{/\Sigma}((X \to \Sigma), (\eta_\Sigma)^*(K \to \mathfrak{J}\Sigma))$$

$$\simeq ((\eta_\Sigma)^* y(K \to \mathfrak{J}\Sigma))(X \to \Sigma)$$

On the other hand, we have:

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}} \simeq (\eta_\Sigma)^* \dashv (\eta_\Sigma)_*$$

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}} \dashv ((-) \circ (\eta_\Sigma)^*|_{FSmMfd_{/\mathfrak{J}\Sigma}})$$

which by essential uniqueness of adjoints proves the point. □

(3) Firstly consider $Sh_\infty(SmMfd_{/\Sigma}) \to \mathbf{H}_{\mathfrak{R}}$. By Lemma 6.1.3.13. [20] having that

$$\simeq Sh_\infty(SmMfd_{/\Sigma}) \simeq N(([SmMfd^{op}_{/\Sigma}, SSet]_{locproj})^\circ)$$

$$N((([SmMfd^{op}, SSet]_{locproj})_{/\Sigma})^\circ)$$

$$\simeq N((\dots)^\circ)_{/\Sigma} \simeq Sh_\infty(SmMfd)_{/\Sigma}$$

Another a bit nontrivial part of this rectangle is the commutativity of

$$\begin{array}{ccc} (\mathbf{H}_{\mathfrak{R}})_{/\Sigma} & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{H}_{\mathfrak{R}} \\ \downarrow & & \downarrow \\ \mathbf{H}_\Sigma & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{H} \end{array}$$

which follows from above note and that at level of sites, if we have a coreflection

$$C \begin{array}{c} \xrightarrow{\;i\;} \\ \xleftarrow[\;p\;]{} \end{array} D$$

then for any $\Sigma \in C$ it extends to a coreflection

$$C_{/\Sigma} \begin{array}{c} \xrightarrow{\;i_\Sigma\;} \\ \xleftarrow[\;p_\Sigma\;]{} \end{array} D_{/p\Sigma}$$

(4) This equivalence is notable. As indicated in the previous section this equivalence establishes the foundation of the theory of *formal generalised PDE's* and thus the prequantum field theory ([26], 6.3). Ideologically it says that bundles over an infinitesimal neighbourhood $\mathfrak{J}\Sigma$ of a smooth manifold encode equivalently as coalgebras of the *jet comonad* whose underlying objects are bundles over Σ.

When discussing examples of prequantum field theories we shall ussualy encounter *free coalgebras* which belong to the *Kleisli category* $KL(J^\infty_\Sigma)$ 2.0.4.

In fact that in $Sh_\infty(FormalCartSp)$ every homotopy type Σ is *formally smooth* and so the equivalence holds not only for smooth manifolds.

We shall discuss an extension of this equivalence to the tangent category $T_{\mathfrak{J}\Sigma}\mathbf{H}$ and the *Eilenberg-Moore category* of the *stable jet comonad.*

(5) This equivalence holds for V-manifolds Σ, where V is a framed object **??** (e.g. a type which has a group structure in the homotopy category level) ([16] Prop. 3.51., Th. 3.52).

2.0.7. *Cech-Deligne (or Cech-de Rham) cocycles.* Here we shall discuss some necessary constructions which we shall use further definitions in 2.1 leading us to an abstract definition of $\mathbf{A}_\infty$*-type prequantum field theories.* This shall allow us to build classical physics within an arbitrary differentially-cohesive ∞-topos.

Prequantum field theories 2.10 are cocycles in *twisted* cohomology in $\mathbf{H}$. Spectra representing such cohomology (and thus *hypercohomology*) are defined via the so-called *Dold-Kan correspondence.* In order to be able to proceed any further we need to briefly remind the necessary constructions which are discussed in ([10] 2.7., [11], [26] 6.4.15.2, [1]).

We shall further use the homological convention for chain complexes:

$$\ldots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

For definition of the Dold-Kan correspondence we address for example to ([19], 1.2.3). The Dold-Kan correspondence establishes an equivalence (which in fact extends to a simplicial Quillen equivalence between the model categories) of the categories of connective chain complexes and simplicial abelian groups. Simplicial abelian groups are included by U into *Kan* complexes ([23], [15]). The combination of these facts results in the right diagram:

$$sSet \underset{U}{\overset{F}{\rightleftarrows}} sAb \underset{\mathcal{DK}}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}(\mathbb{Z}) \qquad\qquad \infty Grpd \underset{\mathcal{DK}}{\overset{F}{\rightleftarrows}} N((Ch_{\geq 0}(\mathbb{Z})^{\circ})$$

Using

$$N((Ch_{\geq 0}(\mathbb{Z})^{\circ}) \xrightarrow{\mathcal{DK}} Spectra \qquad\qquad N([C^{op}; Ch_{\geq 0}(R)]_{locproj})^{\circ}) \xrightarrow{\mathcal{DK}} Sh_\infty(C; Spectra)$$

$Sp(\mathbf{H}_{\mathfrak{J}\Sigma}) \simeq Fun^{lex}(\mathbf{H}^{op}_{/\mathfrak{J}\Sigma}, Spectra)$ ([21] 1.Sheaves of spectra, [19] Rem. 1.4.2.9)

The first example is the *Deligne cohomology.* This hypercohomology is defined via the Deligne complex $\mathbf{B}^n U(1)_{conn,chn}$:

$$\begin{array}{llllllllll} X: & \mathbb{Z} & \hookrightarrow & \mathbb{R} & \longrightarrow & \Omega^1 & \longrightarrow & \ldots & \longrightarrow & \Omega^n \\ \mathbf{B}^n U(1)_{conn,chn}: & 0 & \longrightarrow & U(1) & \longrightarrow & \Omega^1 & \longrightarrow & \ldots & \longrightarrow & \Omega^n \end{array}$$

Where the evident morphism $X \to \mathbf{B}^n U(1)_{conn,chn}$ is quasi-equivalence and thus yields an equivalence of underlying homotopy types. This shall be particularly usefull when prequantizing the *higher U(1)-gauge field theories on a curved spacetime* (see below **??**).

It turns out ([26], [10]) that the *differential cohomology diagram* of the sheaf of spectra $\mathbf{B}^n U(1)_{conn}$ can be described at the level of chain complexes:

Lemma 2.4. *(*[26]*, 6.4.15.2)*

(1) $\mathbf{B}^n U(1)_{conn,chn}$ *is the homotopy pullback of the diagram of chain complexes:*

$$\begin{tikzcd} (U(1) \arrow[r] \arrow[d, "\frac{1}{2\pi i}dlog"'] & 0 \arrow[r] \arrow[d] & \ldots \arrow[r] & 0) \arrow[d] \\ (\Omega^1 \arrow[r] & \Omega^2 \arrow[r] & \ldots \arrow[r] & \Omega^{n+1}_{cl}) \\ (0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \ldots \arrow[r] & \Omega^{n+1}_{cl}) \arrow[u] \end{tikzcd}$$

(2) At the level of sheaves of spectra there is a homotopy pullback:

$$\begin{tikzcd} \mathbf{B}^n U(1)_{conn} \arrow[r] \arrow[d] & \Omega^{n+1}_{cl} \arrow[d] \\ \mathbf{B}^n U(1) \arrow[r] & \flat \mathbf{B}^{n+1}\mathbb{R} \simeq \flat_{dR}\mathbf{B}^{n+1}U(1) \end{tikzcd}$$

Proof. The model category of chain complexes with projective structure is right proper (since all objects are fibrant) so to construct the homotopy pullback it suffices to find a fibrant replacement of some map in our diagram ([15], 2). This fibrant replacement is constructed in [26], Definition 6.4.96. $\square$

In fact $\mathbf{B}^n U(1)_{conn}$ is a differential refinement of the $U(1)$-cohomology.
Now, we apply the functor :

$$\begin{tikzcd} T\mathbf{H} \arrow[r, "\Sigma^*"] \arrow[rr, bend left, "(-)_\Sigma:"] & T\mathbf{H}_{/\Sigma} \arrow[r, "(\eta_\Sigma)_*"] & T\mathbf{H}_{/\Im\Sigma} \\ Sp(\mathbf{H}) \arrow[r] \arrow[u, hook] \arrow[d, shift left, "\Omega^\infty_+"] & Sp(\mathbf{H}_{/\Sigma}) \arrow[r] \arrow[u, hook] \arrow[d, shift left] & Sp(\mathbf{H}_{/\Im\Sigma}) \arrow[u, hook] \arrow[d] \\ \mathbf{H} \arrow[r, "\Sigma^*"] \arrow[u, shift left, "\Sigma^\infty_+"] & \mathbf{H}_{/\Sigma} \arrow[r, "(\eta_\Sigma)_*"] \arrow[u, shift left] & \mathbf{H}_{/\Im\Sigma} \end{tikzcd}$$

to the spectrum $\mathbf{B}^n U(1)_{conn}$ and its defining diagram and obtain in $T_{\Im\Sigma}\mathbf{H}$:

$$\begin{tikzcd} & (\Omega^{p+1}_{cl})_\Sigma \arrow[dr] & \\ (\mathbf{B}^{p+1}(U(1))_{conn})_\Sigma \arrow[ur, "curv"] \arrow[dr] & & (\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma \\ & (\mathbf{B}^{p+1}(U(1)))_\Sigma \arrow[ur] & \end{tikzcd}$$

Further the so-called *variational Poincare lemma* comes into play. Its existence is the crucial property which shall be captured by us when fomralizing the *local prequantum field theories* within the *differentially-cohesive* ∞-topoi.

Lemma 2.5. *([26], [4]) Let $E \in LocProMfd_{\downarrow\Sigma}$ then the map $\Omega^*(J^\infty_\Sigma E) \to \mathbf{\Omega}^*_{EL_\Sigma}$ is a quasi-isomorphism. Moreover, this is the weak equivalence of sheaves of chain complexes on $DiffOp_{\downarrow\Sigma}$.*

([26] 6.5.10, or [1]) Applying the left Kan extension (2.0.6, (2)) to $E \mapsto \mathcal{DK}(\mathbf{\Omega}^*_{EL_\Sigma}(E))$ one constructs the spectrum which sits in the homotopy exact diagram in $Sp(PDE_\Sigma(\mathbf{H}_\Re)) \simeq Sp((\mathbf{H}_\Re)_{/\Sigma})$:

$$\begin{tikzcd} & \Omega^{\bullet \leq p+1}_H \arrow[rr, "\delta_V"] \arrow[dr] & & \Omega^{p+1,1}_{S,cl} \arrow[dr] & \\ (\flat\mathbf{B}^{p+1}\mathbb{R})_\Sigma \arrow[ur] \arrow[dr] & & \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn} \arrow[ur, "curv"] \arrow[dr] & & (\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma \\ & (\flat\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}))_\Sigma \arrow[rr, "(\beta)_\Sigma"] \arrow[ur] & & (\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}))_\Sigma \arrow[ur] & \end{tikzcd}$$

Corollary 2.6. *There is an equivalence of spectra* $(\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma \simeq \Omega^*_{EL_\Sigma, cl}$

A morphism from a bundle E in $DiffOp_{\downarrow\Sigma} \simeq KL(J^\infty_\Sigma)$: $(\eta_\Sigma)^*(E) \in \mathbf{H}_{/\Sigma} \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ is given equivalently by the

$$\mathbf{L} \equiv (\overset{(i)}{\mathcal{L}}_{p+1})_i, (\overset{(i_1 i_2)}{\kappa_p}{}_{i_1 i_2}), \ldots, (\overset{(i_0 \ldots i_{p+1})}{\kappa_0})_{i_0 \ldots i_{p+1}} \tag{10}$$

For an arbitrary bundle $E \in \mathbf{H}_{/\Sigma}$ with the help of the Lemma 2.3 one may realize that

Example 2.7. Let $\mathcal{G} \in \mathbf{H}$ be a classifying stack of fields, then the corresponding bundle $\Sigma^*(\mathcal{G})$ we further push-forward to $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G})) \in \mathbf{H}_{/\mathfrak{J}\Sigma}$. Using the formulas in Lemma 2.3 we conclude that at the sheaf level $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G}))(X \to \mathfrak{J}\Sigma) \simeq \mathcal{G}((\eta_\Sigma)^* X)$. In particular, probing $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G}))$ by $x : * \to \mathfrak{J}\Sigma$ is equivalent to giving the map from an infinitesimal disk:

$$\mathbb{D}^\Sigma_x \to \mathcal{G}$$

Usually we discuss $\mathcal{G} \in \mathbf{H}_\mathfrak{R} \subset \mathbf{H}$ and these probes correspond to giving a germ of field configurations and their derivatives at x (up to an order of infinitesimals exhibited by the differential cohesion of $\mathbf{H}$)

Example 2.8. A bundle for a G-gauge field theory reads as:

$$\mathcal{G}^\infty \times (\eta_X)_*(X^*(\mathbf{B}G_{conn})) \equiv \Xi \tag{11}$$

is the bundle for the Yang-mills theory on a spacetime, where G^∞ is the bundle for gravity.

Example 2.9. Sigma-model with brane Σ:

$$\Sigma \times G \to \Sigma \tag{12}$$

and thus, the classifying stack of fields is $\mathbb{R}$ which gives $(\eta_\Sigma)_*(\Sigma^*(G))(* \to \mathfrak{J}\Sigma) \simeq (\mathbb{D}^\Sigma_x \to G)$ which is equivalent to giving a germ of G-valued functions and their derivatives at a point x. **Notice:** in literature (and we follow that) we use the equivalence (2.0.6,(4)) and identify the bundles $\eta_{\Sigma *}(E)$ with $J^\infty_\Sigma E \in EM(J^\infty_\Sigma)$.

From [26], 6.5.11 we recall the Definition 2.10 and for the sake of generality (as it is done with the theory of *generalised PDE's* in [16]) extend the notion to *formal bundles*.

Definition 2.10. A prequantum field theory is the following data:

(1) A bundle $E \in \mathbf{H}_{/\Sigma}$ over a (p+1)-dimensional smooth manifold Σ
(2) A Cech-Deligne cocycle $L : (\eta_\Sigma)_* E \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ in $\mathbf{H}_{/\mathfrak{J}\Sigma} \simeq Sh_\infty(PDE_\Sigma)$, which is also called as *pre-quantum local Lagrangian*.

If we put any bundle in $E \in \mathbf{H}_{/\mathfrak{J}\Sigma}$ we extend the notion to what we suggest to call as *formal pre-quantum field theories*.

Remark 2.11. Notice that considering bundles over Σ and including them to $\mathbf{H}_{/\mathfrak{J}\Sigma}$ is equivalent to regarding them as free coalgebras under the equivalence (4) in 2.0.6. Thus, the *formal pre-quantum field theories* extend the notion to arbitrary coalgebras of the jet comonad J^∞_Σ.

2.0.8. *Prequantum field theories as coalgebras.* Now we shall focus on the view of *formal pre-quantum field theories* as coalgebras whose underlying objects are simply the twisted Deligne cocycles $E \to (\mathbf{B}^{p+1}U(1)_{conn})_\Sigma$. This shall be used by us so as to extend the theory to the full generality of *differentially-cohesive ∞-toposes.*

Recall from ([19], 4.7.3.) that a functor between ∞-categories $G : D \to C$ which admits a left adjoint F *exhibits D as monadic over C* provided the functor G' defining the factorization of G through $LMod_T$ is equivalence:

$$\begin{tikzcd} C \arrow[rr, bend left, "F"] & LMod_T(C) \arrow[l] & D \arrow[l, dashed, "G'"'] \arrow[ll, bend left, "G"] \end{tikzcd}$$

where $T \equiv G \circ F$ is a monad and $LMod_T$ is a category of algebras over T.

Dually all remains the same only that the category of *co-algebras* we shall further denote $EM(-)$ as we did with the jet comonad coalgebras (2.0.6, (4)).

Lemma 2.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor (preserves finite limits) with a right adjoint G between ∞-categories which admit finite limits. Assume that it exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$. Then,*

(1) F extends to comonadic F (which in fact is the derivative according [19]*, 6.2.), with the induced comonad $\partial(F \circ G) \simeq \partial(F) \circ \partial(G)$.*

(2) There is an equivalence between categories $EM(\partial T) \simeq Sp(EM(T))$

Proof. The first part is obvious from the dual version which is proven in [19], Example 4.7.3.10.: given $G : D \to C$ a functor between ∞-categories (both D and C admit finite limits) which exhibits D as monadic over C. The functor G is left exact, and therefore induces a functor $g : Sp(D) \to Sp(C)$. If g admits a left adjoint, then g exhibits $Sp(D)$ as monadic over $Sp(C)$. In our case we assumed F is left exact so as to extend it to the categories of spectra, and the existence of the right adjoint ∂G is evident which proves the first point.

□

Lemma 2.13. *If $f : X \to Y$ be an effective epimorphism in an ∞-topos $\mathbf{H}$, then the pullback functor f^* exhibits $\mathbf{H}_{/Y}$ as comonadic over $\mathbf{H}_{/X}$. Moreover, this all extends to the categories of spectra in that $\partial(f^*)$ is comonadic, monadic which moreover admits a left adjoint monadic functor $T \dashv \partial(f^* \circ f_*) \simeq \partial(f^*) \circ \partial(f_*)$:*

$$\begin{tikzcd} T_Y\mathbf{H} \arrow[r, "\sim"] \arrow[rr, bend right=10, "\partial f^*"'] & EM(\partial(f^* \circ f_*)) \arrow[r] & T_X\mathbf{H} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{H}_{/Y} \arrow[r, shift left, "f^*"] & \mathbf{H}_{/X} \arrow[l, shift left, "f_*"] \end{tikzcd} \qquad \begin{tikzcd} Sp(\mathbf{H}_{/Y}) \arrow[r, shift left, "\partial(f^*)"] & Sp(\mathbf{H}_{/X}) \arrow[l, shift left, "\partial(f_*)"] \end{tikzcd}$$

Proof. Indeed, the pullback functor F has both left $f_!$ and the right f_* adjoints since colimits are universal in ∞-topoi ([20]). Thus, we apply the above lemma and its dual version: f^* being right adjoint extends to $\partial f^* : Sp(\mathbf{H}_{/Y}) \to Sp(\mathbf{H}_{/X})$ which by the *Adjoint functor Theorem* and that ∞-topoi are presentable admits a left adjoint (notice that $f_!$ may admit no extension to the categories of spectra) which proves ∂f^* is monadic, with a monad T.

For the second part we evidently apply the (1) part of the previous lemma.

□

Remark 2.14. The above lemma allows us to describe twisted spectra as coalgebras in that we have the equivalence:

$$\begin{tikzcd} T_Y\mathbf{H} \arrow[r, "\sim"] \arrow[rr, bend right=10, "\partial f^*"'] & EM(\partial(f^* \circ f_*)) \arrow[r] & T_X\mathbf{H} \end{tikzcd}$$

$$\begin{tikzcd} T_X\mathbf{H} \arrow[r, "\sim"'] & LAlg_T(T_Y\mathbf{H}) \arrow[r] & T_Y\mathbf{H} \end{tikzcd}$$

Example 2.15. Let $G \in Grp(\mathbf{H})$ be a group in an ∞-topos $\mathbf{H}$, $* \to \mathbf{B}G$ be an essentially unique point of the delooping of G. It yields a known base change functor ([26], Double dimension reduction):

$$\begin{tikzcd} \mathbf{H}_{/\mathbf{B}G} \arrow[r, "pt^*"] & \mathbf{H} \arrow[l, bend right=40, "pt_!"'] \arrow[l, shift left=2, "{[G,-]//G}"] \end{tikzcd}$$

Now applying the above lemmas we may rewrite G-equivariant spectra as coalgebras over comonad $\partial(pt^* \circ [G, -]//G)$ whose underlying types are spectra in $\mathbf{H}$.

Theorem 2.16. *Let* $H : \mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}) \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ *be the projection from 2.5 then we have the following diagram where each category on the left is equivalent to the corresponding category of coalgebras whose objects belong to the category on the right:*

$$\begin{tikzcd}
T\mathbf{H}_{/\mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[r, "H^*" description] & T\mathbf{H}_{/(\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn})_\Sigma} \arrow[l, bend right, "H_!"'] \arrow[l, bend left, "H_*"] \\
Sp(\mathbf{H}_{/\Im\Sigma})_{/\mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook] \arrow[r] & Sp(\mathbf{H}_{/\Im\Sigma})_{/(\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn})_\Sigma} \arrow[u, hook] \arrow[l, bend right] \arrow[l, bend left] \\
(\mathbf{H}_{/\Im\Sigma})_{/\mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook, "\Sigma^\infty_+"] \arrow[r] & (\mathbf{H}_{/\Im\Sigma})_{/\mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook, "\Sigma^\infty_+"'] \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

Proof. Indeed, the projection H is an effective epimorphism ([26], 6.5.10). Thus, we apply the above lemmas to conclude (co)monadicity. □

Definition 2.17. Using the Lemma 2.13 we see that $\partial((\eta_\Sigma)_*)$ is comonadic and exhibits an extension of the jet comonad to tangent categories with the left adjoint $(T^\infty_\Sigma)^{sp}$ (Warning: this left adjoint is not a derivative of an infinitesimal disk bundle monad, since T^∞_Σ may not be left exact and in general does not possesses an extension to the category of spectra): $(T^\infty_\Sigma)^{sp} \dashv \partial J^\infty_\Sigma$.

$$\begin{tikzcd}
T_{\Im\Sigma}\mathbf{H} \arrow[r, "\sim"'] \arrow[rr, bend left, "\partial(\eta_\Sigma)^*"] & EM(\partial(J^\infty_\Sigma)) \arrow[r] & T_\Sigma\mathbf{H}
\end{tikzcd}$$

∂J^∞_Σ we simply call as a *stable jet comonad*, and its left adjoint $(T^\infty_\Sigma)^{sp}$ as a *stable infinitesimal disk comonad.*

Remark 2.18. Stable jet comonad allows us to work with field theories via the stable homotopy theory within the tangent category $T_{\Im\Sigma}\mathbf{H}$. This may be useful in further investigations, for example of localizations of field theories considered as morphisms of parameterized homotopy types.

For instance, using that via $Disc^{sp} : Spectra \to Sp(\mathbf{H})$ the ordinary spectra are embedded in $Sp(H) \simeq T_*\mathbf{H} \subset T\mathbf{H}$ the localizations (especially interesting would be to consider chromatic filtrations [6], or along the homology theories) [5]) in the ordinary stable homotopy theory shall induce localizations of *parameterized cohesive homotopy types.*

Now we unwind the (co)monadic data encoded in the Theorem 2.16:

Definition 2.19. Introduce the *topologization comonad*, with the structure map being the pullback of the effective epi. is an effecive epi.: $H_! \circ H^* \equiv \mathfrak{T}$

$$(\mathfrak{T}X \to X) \quad \sim \quad \begin{tikzcd}
 & & (\mathbf{B}^{p+1}(U(1))_{conn})_\Sigma \arrow[d, two heads] \\
H^*E \arrow[urr, dashed] \arrow[rr, "\mathfrak{T}(\mathcal{L})"] \arrow[dr, two heads, bend right] & & \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn} \\
 & E \arrow[ur, "\mathcal{L}"'] &
\end{tikzcd}$$

The Euler-Lagrange comonad $\mathfrak{E} \equiv H^* \circ H_*$ with the left adjoint $H^* \circ H_!$

Thus, we conclude this section the following:

Remark 2.20. Now, we see that defining a local prequantum field theory on Σ is equivalent to finding a bundle over Σ equipped $U(1)$ p-gerbe with connection and endowing it with a structure of a coalgebra of the *Euler-Lagrange comonad* $\mathfrak{E}$.

2.1. **Prequantum field theories in differential cohesion.** Here we are concerned with the problem of *internalization* of prequantum physics (2.1, 2.10) in arbitrary *differentially cohesive* ∞*-topoi.* Axioms of *differential cohesion* produce the same notions of *Cartan geometry, variational theory* ([16], [26]). We shall try to proceed even further and define abstractly the prequantum field theory using axioms of *differential cohesion*:

(1) In literature there only exist definitions of groups of synthetic phases $\mathbb{G}$ which play the role of $U(1)$ for defining field theories. Exponentials of action functionals are valued in $\mathbb{G}$ and prequantizations yield $\mathbb{G}$-bundles with connections on phase spaces. Nonetheless, this lacks the stable properties of $U(1)$, such structures as the Beilinson-Deligne cup product (which is very useful for defining prequantum field theories such as higher WZW models or Chern-Simons theories).

    Moreover, such an approach with $\mathbb{G}$ simply lacks the Euler-Lagrange theory. In 2.10 we see that that at the core of prequantum field theories lies $\mathbf{B}_H^{p+1}U(1)_{conn}$ stack which is the "horizontal part" of $\mathbf{B}^{p+1}U(1)_{conn}$. This horizontal part is abstractly defined via the exact hexagons for smooth spectra representing differential cohomology. Hence, we should combine these properties and yield an abstract definition of phases where $H\mathbb{Z}[1]$, and $\mathbf{B}^{p+1}U(1)_{conn}$ spectrum is substituted by some $\mathbf{A}_\infty$-ring.
(2) Proceed in 2.2 paragraph to the stable case so as to exploit the modalities of *cohesion* and work with spectra. As we shall see that stability will be very helpful in giving the abstract definition of prequantum field theories 2.23.
(3) Suggest how one can exploit *cohesive spectra* so as to produce the analogs of $\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn} \to \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ (2.23). It would be interesting to consider analogs of prequantum field theories but with Lagrangians (Cech-Deligne cocycles) valued in other differential cohomology (other than Deligne cohomology).

If we have a *formally smooth, framed* object V, then the generalised theory of PDE's applies. Thus, over V-manifolds we may construct and study analogs of prequantum field theories 2.23 which we had over smooth manifolds.

For, we need the following simple:

Lemma 2.21. *Let* $\mathbf{H}$ *be differentially-cohesive* ∞*-topos. If* V *is formally smooth framed object (e.g. group object at the homotopy category level of* $\mathbf{H}$*) and* X *is a* V*-manifold. Then,* X *is formally smooth anf locally framed.*

Proof. Let

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr, two heads] & \\ V & & X \end{tikzcd}$$

be an atlas of X. We have a pullback (by etaleness of $U \to V$):

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \Im U \arrow[d] \\ V \arrow[r] & \Im V \end{tikzcd}$$

where the bottom arrow is epi and the top arrow is epi. $\mathfrak{I}$ being left and right adjoint preserves epi [20] we have that in

$$\begin{tikzcd} U \arrow[r, two heads] \arrow[d, two heads, "et"] & \mathfrak{I}U \arrow[d, two heads] \\ X \arrow[r] & \mathfrak{I}X \end{tikzcd}$$

$\mathfrak{I}U \to \mathfrak{I}X$ is epi. Hence, $U \to \mathfrak{I}X$ being epi we conclude $X \to \mathfrak{I}X$ is epi and so X is formally smooth. It is locally framed. □

Thus, over V-manifolds we may construct and study analogs of prequantum field theories 2.23 which we had over smooth manifolds.

2.2. **A_∞-type field theories.** $\mathbf{A}_\infty$-rings are analogs of rings in stable homotopy theory ([19]).

Let E be a discrete spectrum and $\hat{E}[p+1]$ is a differential refinement (which in addition has an $\mathbf{A}_\infty$-ring structure; this may be used to define analogs of cup-product Chern-Simons theories abstractly, see [11], 2.7. for Beilinson-Deligne cup product in Deligne cohomology) of the $(p+1)$-th suspension of E (it may be written as $(p+1)$-th suspension of a spectrum $\hat{E}$, **??**). Then apply the functor as in 2.0.7 and produce:

This produces the homotopy exact hexagon:

$$\begin{tikzcd}[column sep=small] & (\int_{dR} \hat{E}[p])_\Sigma \arrow[rr, "\mathbf{d}"] \arrow[dr] & & (\flat_{dR}\hat{E}[p+2])_\Sigma \arrow[dr] & \\ (\int_{dR} \flat\hat{E}[p])_\Sigma \arrow[ur] \arrow[dr] & & (\hat{E}[p+1])_\Sigma \arrow[ur, "curv"] \arrow[dr] & & (\int \flat_{dR}\hat{E}[p+2])_\Sigma \\ & (\flat\hat{E}[p+1])_\Sigma \arrow[ur] \arrow[rr, "\beta_\Sigma"] & & (E[p+1])_\Sigma \arrow[ur, "ch"'] & \end{tikzcd}$$

Indeed, $(-)_\Sigma$ being the composition of right adjoint functors preserves homotopy pullbacks which by stability implies exactness.

Consider an effective epimorphism:

$$(\hat{E}[p+1])_\Sigma \xrightarrow{H} \mathbf{B}^{p+1}_H(E)_{conn}$$

This yields the base change essential geometric morphism:

$$\begin{tikzcd} (T\mathbf{H})_{/\mathbf{B}^{p+1}_H(E)_{conn}} \arrow[r, "H^*"] & (T\mathbf{H})_{/(\hat{E}[p+1])_\Sigma} \arrow[l, bend right, "H_!"'] \arrow[l, bend left, "H_*"] \end{tikzcd}$$

Then, define $, X_2, X_3 \to \mathbf{B}^{p+1}_H(E)_{conn}$ by applying H_* to the left hand side part of the diagram 2.2. Further, by taking the homotopy cofibers followed by the pushout on the right hand side part of the diagram we obtain:

$$\begin{tikzcd} & X_3 \arrow[rr] \arrow[dr] & & Y_3 \arrow[dr] & \\ X_1 \arrow[ur] \arrow[dr] & & \mathbf{B}^{p+1}_H(E)_{conn} \arrow[ur] \arrow[dr] & & Y_1 \\ & X_2 \arrow[ur] \arrow[rr, "\beta"] & & Y_2 \arrow[ur] & \end{tikzcd}$$

Proposition 2.22. *The diagram 2.2 is exact in that the diagonals are homotopy fiber sequences, the top and the bottom rows are long fiber sequences, rectangles are pullbacks.*

Proof. Firstly, notice that H_* being right adjoint descends to the category of spectra over $\mathfrak{J}\Sigma$ and preserves products in $(T\mathbf{H})_{/(\hat{E}[p+1])_\Sigma}$, whence the left hand side of the diag. 2.2 is pullback. By stability we obtain the same for the right hand side rectangle and for the diagonals.

To prove that the top and the bottom rows are exact consider the diag.:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "(1)"'] & 0 \arrow[d] \arrow[dl, Rightarrow, "(2)"'] \\
X_3 \arrow[r] \arrow[d] & \mathbf{B}_H^{p+1}(E)_{conn} \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "(3)"'] & Y_3 \arrow[d] \arrow[dl, Rightarrow, "(4)"'] \\
0 \arrow[r] & Y_2 \arrow[r] & Y_1
\end{tikzcd}$$

(1) is pullback, (2) is pullback since Y_1 is a homotopy cofiber, (3) is analogous, and (4) is pullback being a pushout. Thus, by pullback pasting lemma we conclude that the the left and the right parts of the diag. are pullbacks, whence the result. □

Definition 2.23. Let $\mathbf{H}$ be a differentialy-cohesive ∞-topos. Let $\hat{E} \in Sp(\mathbf{H}_{\mathfrak{R}})$ be a spectrum as before 2.2 with an effective epi. 2.2 such that

1. (Underlying spectra are the same) $Y_2 \simeq (E[p+1])_\Sigma$
2. (*Synthetic* variational Poincare lemma holds) $Y_1 \simeq (\int \flat_{dR}\hat{E}[p+2])_\Sigma$

Then, $\mathbf{B}_H^{p+1}(E)_{conn}$ is E-type Cech-Deligne stack.

If $\pi_0 T\mathbf{H}(\Sigma; \flat_{dR}\hat{E}[\geq p+2]) = 0, \pi_0 T\mathbf{H}(\Sigma; Y_3) = 0$ (which says that differential forms in which $\hat{E}$-curvatures take values are trivial on Σ), then given $(H \to \Sigma) \in (\mathbf{H}_{\mathfrak{R}})_{/\Sigma}$

$$\mathcal{E} \xhookrightarrow{fib} (\eta_\Sigma)_* H \xrightarrow{\mathcal{L}} \mathbf{B}_H^{p+1}(E)_{conn} \xrightarrow{curv} Y_3$$

$(\mathcal{L}, H)$ we suggest calling a *E-type prequantum field theory* ($\mathcal{L}$ as an E-type Cech-Deligne cocycle); fib is the *shell* of the theory and $curv(\mathcal{L})$ are the *E-type Euler-Lagrange equations.* Y_3 is the stack of E-source forms.

Remark 2.24. If V is formally smooth and framed, then the theory of generalised PDE's applies in that we have (4), (5) equivalences in the Variational diagram 2.0.6 (see Prop. 2.21 and [16] Th. 3.60.). Thus, the *shell* $\mathcal{E}$ spanned by E-type Euler-Lagrange equations indeed defines an object in $EM(J^\infty_\Sigma) \simeq PDE_{/\Sigma}(\mathbf{H})$ i.e. a generalised PDE.

Remark 2.25. How does the usual definition of prequantum field theories 2.10 fit into our? Let $E \equiv U(1) = H\mathbb{Z}[1] \in Sp(Sh_\infty(SmMfd))$ (2.0.2) be the (smooth) Eilenberg-Maclane spectrum corresponding to $\mathcal{DK}(0 \to \cdots \to C^\infty(-, U(1)))$.

Then, $\mathbf{B}^{p+1}U(1)_{conn}$ is the spectrum defined in 2.4 and being a spectrum for Deligne cohomology it differentially refines $\mathbf{B}^{p+1}U(1)$ (see 2.4). Having the *Variational Poincare lemma* 2.5 at hand one constructs $Y_3 \equiv \Omega^{p+1,1}_{S,cl}$ and so the $\mathbf{B}_H^{p+1}U(1)_{conn}$ stack. The dimension condition is evident.

Example 2.26. Let X be a smooth manifold endowed with an action of a group G, form $p: X \to X//G$ an effective epi. Any quotient by any action of an ∞-group G then satisfies our construction.

Then, we have:

$$(-)_{X//G} : \mathbf{H} \to \mathbf{H}_{/X//G} \to \mathbf{H}_{/\mathfrak{J}(\mathbf{X}//\mathbf{G})}$$

which yields $(\mathbf{B}^{p+1}U(1)_{conn})_{X//G}$. One may define prequantum field theories over $X//G$ as follows:

(1) Consider the fol. epi. of stacks $(\mathbf{B}^{p+1}U(1)_{conn})_X \to \mathbf{B}_H^{p+1}U(1)_{conn}$
(2) Apply $(\Im p)_*$ and obtain an effective epi. again
(3) Notice that $(\Im p)_*((\mathbf{B}^{p+1}U(1)_{conn})_X) \simeq (\mathbf{B}^{p+1}U(1)_{conn})_{X//G}$
(4) This yields $(\mathbf{B}^{p+1}U(1)_{conn})_{X//G} \to (\Im p)_*(\mathbf{B}_H^{p+1}U(1)_{conn})$

In the case, for instance when $X//G$ is an orbifold [2], the description of prequantum field theories simplifies:

Proposition 2.27. *Assume $X//G$ as a groupoid in $\mathbf{H}$ is etale, in that p is etale (local diffeomorphism). A prequantum field theory with a bundle $E \to X//G$ is equivalently described as a prequantum field theory 2.10 with a bundle $p^*(E)$*

Proof. We shall show that there is an equivalence:

$$(\Im p)^*(\eta_{X//G})_* \simeq (\eta_X)_*(p)^*$$

For, we show that they have equivalent left adjoints.

$$1 \to (\eta_X)_*(p^*) \sim (\eta_X)^* \to p^*$$

$$\sim (p_!)(\eta_X)^* \to 1$$

It suffices to prove $(\eta_{X//G})^*(\Im p)_! \simeq (p_!)(\eta_X)^*$ and this is easily seen from:

$$\begin{tikzcd}
(\eta_X)^*E \arrow[r] \arrow[d] \arrow[dd, bend right] & E \arrow[d] \arrow[dd, bend left] \\
X \arrow[r] \arrow[d] & \Im X \arrow[d] \\
X//G \arrow[r] & \Im(X//G)
\end{tikzcd}$$

The top rectangle is a pullback. $(\eta_{X//G})^*(\Im p)_!(E)$ is a pullback of the composition $E \to \Im(X//G)$ along $\eta_{X//G}$ on the other hand $(p_!)(\eta_X)^*(E)$ is a composition on the left. Since the bottom rectangle is a pullback (p is etale) (the word "formally" is omitted) then by pasting of pullbacks this implies the equivalence of these bundles.

This proves that

$$(\eta_{X//G})_*(E) \to (\Im p)_*(\mathbf{B}_H^{p+1}U(1)_{conn})$$

corresponds to $(\Im p)^*(\eta_{X//G})_*(E) \to \mathbf{B}_H^{p+1}U(1)_{conn}$ which is equivalent to $(\eta_X)_*(p^*)(E) \to \mathbf{B}_H^{p+1}U(1)_{conn}$. □

It would be interesting to consider the following examples of incarnations of $\mathbf{A}_\infty$*-type Cech-Deligne stacks*:

Example 2.28. (Hopkins-Singer coefficients) Given a chain complex C [14]

$$\begin{tikzcd}
E_{conn,-n} \arrow[r] \arrow[d] & (\Omega \otimes_{\mathbb{Z}} C)^{\bullet \geq -n} \arrow[d] \\
E \arrow[r] & C \otimes_{\mathbb{Z}} \mathbb{R}
\end{tikzcd}$$

this is the homotopy differential cohomology pullback. Given a morphism of chain complexes:

$$C \to C^{'}$$

it yields then a morphism of spectra $E_{conn,-n} \to E^{'}_{conn,-n}$. Then, by the above pullback we have that the geometric realizations of them coincide. The synthetic variational Poincare lemma holds as well (the bottom right of the pullback above).

This example is different when compared to Deligne coefficients:

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{conn} \simeq (\mathbb{Z}[n+1]^\infty_D)_{conn,0} & \longrightarrow & (\Omega \otimes \mathbb{Z}[n+1]^\infty_D)^{\geq 0} \simeq \Omega^{n+1}_{cl} \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) \simeq \mathbb{Z}[n+1]^\infty_D & \longrightarrow & \mathbb{Z}[n+1]^\infty_D \otimes \mathbb{R} \simeq \flat \mathbf{B}^{n+1}\mathbb{R} \end{array}$$

and the stack $\mathbf{B}^{p+1}_H U(1)_{conn}$ is defined via the Euler-Lagrange complex, which sits in the *slice topos* instead whilst our suggestion of Hopkins-Singer coefficients

Example 2.29. (Algebraic K-theory) This is construction is via (Bunke-Nikolaus-Volkl) [7]. Let $\mathcal{K}$ be Algebraic K-theory functor from a category of symmetric monoidal ∞-categories. $\mathbf{Vect}^{\oplus}$, $\mathbf{Vect}^{\oplus}_{conn}$ stacks of categories complex vector bundles (with connections respectively) with a direct sum exhibitting the monoidal structure. Then,

$$\mathcal{K}(\mathbf{Vect}^{\oplus}) : SmMfd^{op} \longrightarrow CMon_\infty \xrightarrow{\mathcal{K}} Spectra$$

$$\mathcal{K}(\mathbf{Vect}^{\oplus}_{conn}) : SmMfd^{op} \longrightarrow CMon_\infty \xrightarrow{\mathcal{K}} Spectra$$

yield differential refinements of ku spectrum. We have $\mathcal{K}(\mathbf{Vect}_{conn}) \to \mathcal{K}(\mathbf{Vect})$.

$\flat(-) \simeq K\mathcal{C}$ which by a long homotopy fiber sequence (bottom row in differential homotopy hexagon 2.2) implies the variational Poincare lemma in that $\int \flat_{dR}\mathcal{K}(\mathbf{Vect}) \simeq \int \flat_{dR}\mathcal{K}(\mathbf{Vect}_{conn})$.

Remark 2.30. One of the main features of prequantum field theories with phases in Deligne coefficients is the *transgression*, which comes from the fiber integration in Deligne cohomology ([11], [17], [1]) which produces action functionals, prequantization of covariant phase spaces and *extended QFT* after quantization. Thus, in general we would be more interested in differential cohomology theories endowed with fiber integration as in the case with differential K-theory.

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St. Petersburg State University
Email address: `st055904@student.spbu.ru`