MASTER THESIS: NEW TYPES OF FIELD THEORIES, GRAVITY IN COHESION AND TOPOLOGICAL INCOMPLETENESS OF GENERAL RELATIVITY

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ABSTRACT. We use the axioms of differential cohesion for ∞-topoi so as to internalize the prequantum physics completely. We suggest the way how to substitute $U(1)$ phased field theories and the Euler-Lagrange theory by rather general $\mathbf{A}_\infty$-type field theories. This allows to construct prequantum field theories over general ∞-stacks, in particular on orbifolds. Using the prequantum geometry we demonstrate that gravity theories (such as Einstein's general relativity) act incorrectly in the presence of prequantized field theories (*4d WZW over a Kahler manifold*). Namely, one encounters ambiguity when gluing geometries which locally satisfy gravitational equations. We introduce a topological coupling so as to cancel such ambiguities. We notice that this requires a deformation of gravity to the form of a *parameterized field theory.* In addition we construct the Whitehead towers of abstractly defined orthogonal groups in differentially-cohesive ∞-topoi which allows to study *spin-geometry, Riemann geometry*, and so on abstractly.

CONTENTS

1. INTRODUCTION

Our present work is motivated by a concrete task to figure out a deeper nature of gravity theory (Einstein's theory) so as to reduce the ideological distance between gravity and quantum theory. We exploit the recent advances of homotopy type theory towards the Hilbert's sixth problem [18]. Alongside we demonstrate how the physical concepts such as gravity fields, prequantum field theories, Euler-Lagrange theory may be formulated abstractly within any *differentially cohesive* ∞*-topoi.*

One of the main advantages of homotopy type theory is *internalization.* When making constructions in some particular contexts (in ∞-topoi) we often have an immediate abstract reformulation to general contexts (∞-topoi with additional structures, such as *cohesion*, *local* structure, *solid*...). Thus, for instance, the de Rham cohomology firstly realised as a cohomology theory for smooth manifolds may be formulated within any *locally* ∞*-connected* ([69], 5.2.10; [75]) ∞-topos. This is the slogan when applying the homotopy type theory to physics:

- Rewrite structures in physics via the *universal constructions* in a particular ∞-topos $\mathbf{H}$.
- Find a way to rewrite these constructions in abstract ∞-topoi with additional structures (like *cohesion*, or *differential cohesion* possessing such notions as *infinitesimality*, *frame bundles*, and we demonstrate *gravity fields, prequantum field theories, ...*)

Since ∞-topoi serve as some sort of *"substances"* (see for the intuition [69] names of paragraphs) that build the mathematics, fundamental physics being formalised *internalizes* and is built within the certain types of ∞-topoi ([69], 7. Physics).

We demonstrate that a large chunk of fundamental physics (prequantum geometry) *internalizes* in *differentially-cohesive* ∞*-topoi.* (In literature only the first step of the above slogan has been made completely so far [69], [3]). This allows us to define *new types of field theories* where Lagrangians are substituted by cocycles in abstract smooth spectra 4.27.

We use the *prequantum geometry* (which exploits the *higher topos theory*) so as to demonstrate that gravity theories (Einstein's gravity for instance) act incorrectly in the presence of prequantized (topologically nontrivial) field theories (5, 5). Cancellation of this incorrectness leads us to an idea of topological coupling.

1.1. **Gauge fields.** Let us remind how the main aspects of the fundamental physics such as gauge theory, gravity, quantization are formalized within the higher topos theory. All the fields that carry over fundamental interactions are gauge fields. Gauge field theory mathematically is what is now understood as ∞*-Chern-Weil theory*, where gauge fields are cocycles of differential cohomology. For the discussion of gauge field theories as aspects of differential cohomology we adress [73], [7]. One of the first realizations that electrodynamics is a gauge field theory was due the so called Dirac charge quantization (see for example [25], and concerning more recent work for the charge quantization of fluxes in M-theory see [24].

Higher topos theory allows us to describe gauge field theories as *sigma-models* of the form $X \to \mathbf{B}G_{conn}$, where the target space is now a stack of connections on G-bundles.

1.2. **Gravity.** Gravity fields are now recognized as cocycles but in *twisted* cohomology, where a *twist* exhibits a sort of geometry.

Example 1.1. Einstein realized that in order to combine the principles of equivalence, general and Lorentz covariance one needs to extend gravity to (pseudo)Riemanian geometry and gravity fields are

$$\begin{tikzcd} & \mathbf{B}O(1,3) \arrow[d] \\ X \arrow[ur, dashed, "o_X"] \arrow[r, "\tau_X"'] & \mathbf{B}GL(4) \arrow[ul, phantom, "\Leftarrow e" near start] \end{tikzcd}$$

(pseudo)Riemanian metric tensors which are equivalently depicted as a *reduction* (a homotopy known as a vielbein field e) of a tangent bundle to an orthogonal structure o_X. Here, the *twist* exhibits a tangent bundle on a smooth manifold X.

Example 1.2. It is of common occurrence in *generalised geometry* ([41], [16], [74], [31]) to consider maximal compact subgroups $O \subset G$ and if we have a *twist* exhibiting a generalised tangent bundle $\tau_X : X \to \mathbf{B}G$. Maximal compactness by the structure theorems of Lie groups ([35], [42]) implies that at the homotopy level the classifying spaces are equivalent in that any G structure reduces to O-structure. The incarnation of this is a well known fact that any smooth manifold admits a Riemannian metric ([83]).

For instance, the **typeII** superstring backgrounds are defined as

$$\begin{tikzcd} & \mathbf{B}(O(n)\times O(n)) \arrow[d, "\mathbf{typeII}"] \\ X \arrow[ur, dashed] \arrow[r, "\tau_X"'] \arrow[r, phantom, shift left=3, "\Rightarrow e"] & \mathbf{B}O(n,n) \end{tikzcd}$$

or in *exceptional generalised geometry*

$$\begin{tikzcd} & \mathbf{B}K_n \arrow[d, "\mathbf{ExcSugra}"] \\ X \arrow[ur, dashed] \arrow[r] \arrow[r, phantom, shift left=3, "\Rightarrow"] & \mathbf{B}E_{n(n)} \end{tikzcd}$$

See the review in ([69], 7.1.1.1.).

Remark 1.3. Moreover, gravity fields are recognized as *Cartan connections* and there are certain occasions when it becomes essential. For example, if we have a smooth manifold with $Spin(1,3)$-structure, meaning that we have:

$$\begin{tikzcd} & \mathbf{B}Spin(1,3) \simeq \mathbf{B}SL(2,\mathbb{C}) \arrow[d] \\ X \arrow[ur, dashed] \arrow[dr, "\tau_X"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}SO(1,3) \arrow[d] \\ & \mathbf{B}GL(4) \end{tikzcd}$$

we have a spinor (projective) representation of the Lorentz group $SL(2,\mathbb{C}) \to GL(4,\mathbb{C}) \equiv GL(\mathcal{V})$ which induces an associated vector bundle

$$\begin{tikzcd} P \times_{SL(2,\mathbb{C})} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}//SL(2,\mathbb{C}) \arrow[d] \\ X \arrow[r] & \mathbf{B}SL(2,\mathbb{C}) \end{tikzcd}$$

whose sections are Dirac (bi)spinor fields on X. Since this is a *projective* representation of the Lorentz group $SO(1,3)$ there is no way to introduce interaction of Dirac fields with gravity which exhibits only the *linear connection.* Thus, we need gravity to be defined via connections on the $Spin(1,3)$-bundle, which are called in literature as *spin-connections.* Mathematically this means that gravity fields in fact have a more refined version in a form of $(SO(1,3) \subset ISO(1,3))$-Cartan connections:

$$\begin{tikzcd} & \Omega^1(-, \mathfrak{iso}(1,3))//SO(1,3) \arrow[r] \arrow[d] & \mathbf{B}SO(1,3) \arrow[d] \\ X \arrow[ur, dashed, "\omega"] \arrow[r] \arrow[r, phantom, shift left=3, "\Rightarrow e"] & \mathbf{B}ISO(1,3)_{conn} \arrow[r] & \mathbf{B}ISO(1,3) \end{tikzcd}$$

such that locally the connection yields an isomorphism of the tangent space with the *internal* vector space on which the Lorentz group acts:

$$(e,\omega)_x : T_xX \to \mathfrak{iso}(1,3) \to \mathfrak{iso}(1,3)/\mathfrak{so}(1,3) \simeq \mathbb{R}^4$$

This says equivalently that we have a spin-conection together with a *vielbein* field e exhibiting the isomoprhism of tangent spaces T_xX with the "internal" space of the Lorentz group $\mathbb{R}^4$. (see [76], [46], [59] or see below 3.6 for the abstract treatment of Cartan-connections).

Moreover, there are not only the conceptual reasons to realize gravity field as Cartan connections but the empirical ones. For instance see [63], [64] where authors use the Einstein-Cartan field equations with the relativistic spin fluid as a source so as to demonstrate that gravitational repulsion of torsion prevents a singularity, replacing it with a nonsingular bounce, or see [9], [14] for bounces and the torsion-induced phase transitions in the early Universe. These approaches exploit high spin density fluids and thus using the Einstein-Cartan approach essential.

In our approach to defining the gravitational theories *internally* in differentially-cohesive ∞-topoi we are motivated by the significance of the approach of Cartan-connections.

1.3. **Quantization of gravity.** Modern Theoretical physics splits ideologically along a ubiquitous concept of interaction. Indeed, there are four fundamental interactions in nature: weak, strong, electromagnetic and gravitational. All except gravity are described within the Standard Model and are quantized. Recently, it was even suggested that in order to combine gravity with the quantum world it is not necessary to quantize gravity directly [60], so there are lots of conceptual contradictions and issues when we treat gravity in the context of quantum theory.

It is known that matter fields of nature are quantum and somehow must be compatible with gravity. Quantum field theory from a modern point of view should be considered as an *effective* theory and its problems in the form of divergencies are just the artefacts of its *effectiveness and incompleteness.* There is an understanding ([15], pp 3 and references therein) that it is *quantum gravity* that QFT is missing to be complete since gravity involvement provides certain cutoffs in QFT, ... Thus, one needs to understand better the true essence of gravity and somehow change the QFT so as to combine them (we suggest in 5 a homotopical point of view on a possible direction for it).

There are various conceptual and technical issues concerning the incompatibility of existing quantum theory and gravity (see for the review [15]), some of them we list below:

(1) (General covariance is in contradiction with locality) Observables in quantum gravity should seemingly respect the diffeomorphism invariance, but such observables are non-local.
(2) The problem of a time is a technical issue when quantizing canonically gravity. In general relativity no time is preferable due to because of the diffeomorphism invariance. The Hamiltonian of General relativity is a constraint and in order to capture any dynamics in quantum theory (so as to produce a nontrivial S-matrix) one needs to *choose a time and write it down in field terms.*
(3) Causality is violated since the fluctuations in metric (which is considered as a quantum field) might exchange past and future.
(4) Technical issues concerning the renormalizability [15].
(5) (**The lack of topological coupling mechanism in gravity**) Using the prequantum geometry in addition to the well-recognized quantization issues above we argue that there is a *topological issue* with gravity which extends the list of inconsistencies. We outline it briefly in the next section 1.4 and in 5 we discuss this phenomenon with its implications in details.

1.4. **Topological obstructions in gravity.** Let X be a smooth n-manifold further in this section.

Despite many similarities of gravitational theories with gauge field theories (and even when speaking dynamically, Einstein's gravity may be successfully rewritten in pure gauge form [5]) they are in fact unique and peculiar. Despite its simplicity this peculiarity of gravity theories that we are going to discuss seemingly has not yet been recognized. We shall demonstrate it in examples in 5 but here we shall briefly discuss it.

In Standard model all interactions are described via the gauge field theories which in fact all constitute the so-called Yang-Mills theories ([27]). Fields that constitute matter are fermion fields that take values in representations of the structure groups of gauge field theories.

Let G be a Lie group, $\mathcal{A} : X \to \mathbf{B}G_{conn}$ a gauge field configuration, and consider a representation $G \to GL(\mathcal{V})$. These fermion fields ψ interact with gauge fields via the so-called *minimal* coupling, which mathematically means that a gauge field being a connection on a principle bundle induces a connection on associated vector bundle and thus defining the *covariant derivative* acting on sections (fermions in a representation):

$$\partial \mapsto D = \partial - g\rho\mathcal{A}$$

where ρ is a matrix of an induced Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ and g is a coupling constant (measuring how strong are the effects of this interaction).

Further, one considers a Lagrangian for fermions interacting with gauge fields (not globally defined, but rather *local system of Lagrangians*, or the glued version - *Cech de Rham cocycle* [3], 4.0.7) (see the example 1.4):

$$\overset{(i)}{\mathcal{L}} = \overset{(i)}{\mathcal{L}}_{matter} + \mathcal{L}_{\mathcal{A}} \tag{1}$$

where $\mathcal{L}_{\mathcal{A}}$ in the case of Yang-Mills theories is $\sim tr(F^2)$ and thus globally well-defined, and $\overset{(i)}{\mathcal{L}}_{\psi\mathcal{A}} \sim \bar{\psi}^{(i)}\mathcal{A}\psi$. Then, the variation along $\mathcal{A}$ yields field equations

$$\cdots = \overset{(i)}{J} \equiv \frac{\delta}{\delta\mathcal{A}} \overset{(i)}{\mathcal{S}}_{\psi\mathcal{A}} \tag{2}$$

where on the right hand side we have a current, which is defined globally because of the presumed gauge invariance of interacting field theory for ψ.

When we work with representations, fields ψ are sections of vector bundles and topologically (in the sense of gluing Lagrangians globally on a bundle 5, 5.1, 4.0.7) this theories are trivial:

Indeed, there are obstructions to glue the Lagrangian forms globally (see [8], pp xii), which lie in $[EL(\overset{(i)}{\mathcal{L}_i})] \in H^{n+1}(\Omega_{EL_X}(E))$, using the *Variational Poincare lemma* (4.5) we find this obstruction in $H^{n+1}(J^\infty E; \mathbb{R})$ and since $J^\infty E$ is of the same homotopy type as E ([8]) we find this obstruction in $H^{n+1}(E; \mathbb{R})$. Since E is a vector bundle it has the same homotopy type as X, but X is n-dimensional, whence $H^{n+1}(\Omega_{EL_X}(E)) \simeq H^{n+1}(X; \mathbb{R}) \simeq 0$ and an obstruction vanishes. All this is quite trivial in the standard cases (see below 1.4)

Hence, we see that *we always may choose a globally defined Lagrangian for any field theory defined via a representation of the structure group of a considered gauge field theory.*

What is with gravity?

(1) Currents J are now substituted by the Energy-Momentum tensor (EMT). The difference is that there are more ways to interact with gravity (not only minimally via representations) for fields.
(2) Gravity fields exhibit geometry, and thus are present in definition of many other field theories and in particular in *non-linear sigma-models* (see 5.1))
(3) The bundles of such models can be topologically complicated and usually involve the consideration of a *bulk* or some higher dimensional manifold which bounds our 4-manifold.

This results in the need to prequantize (only locally-) defined Lagrangian systems as in the case of *4d Wess-Zumino-Witten model on a Kahler manifold* discussed below in 5. This model involves geometry (metric tensor is hidden in the Kahler form in the prequantized interaction term 51) and when we compute the

Energy-Momentum tensor of some field configurations we encounter **ambiguities** (5): *when we compute the Energy-Momentum tensor of field configurations, we see that it is defined correctly (as the Lagrangians) only locally and there are obstructions to glue them globally. On the other hand, if we assume that metric tensors satisfy graviational field equations (locally), then there is an obstruction to glue locally defined metrics to a global one.*

(4) Sigma-models arise in brane theories and are of a great importance in brane cosmology [56]. There are lots of models arising in M-theory contexts and they are nontrivial topologically (compactifications involve homotopically nontrivial manifolds) and so cannot be ignored.

Example 1.4. Let $\mathcal{A} : X \to \mathbf{B}SU(3)_{conn}$ be a gluon $SU(3)$-gauge field, whose underlying bundle we denote $[\mathcal{A}] : X \to \mathbf{B}SU(3)$. The quark fields are sections of a bundle associated with $[\mathcal{A}]$ under the fundamental representation of $SU(3)$ (or more precisely, since physicists assume it as self-explanatory under the representation of $SL(2,\mathbb{C}) \times SU(3)$) where a spin structure on a manifold X is chosen (in quantum theory one works usually perturbatively so that any topology of X is usually ignored whatsoever, so the underlying bundles $[\mathcal{A}]$ are all trivial). Thus, the quark fields are $(\psi_i)_{i=1,2,3}$ where ψ_i are Dirac spinors:

$$\begin{tikzcd} & \mathcal{V}//SU(3) \arrow[d] \\ X \arrow[ur, dashed, "(\psi_i)"] \arrow[r, "{[\mathcal{A}]}"'] & \mathbf{B}SU(3) \end{tikzcd}$$

The Lagrangian is given by:

$$\overset{(j)}{\mathcal{L}} = -tr(F^2) + \bar{\psi}_f(i\overset{(j)}{D} + m_f)\hat{\psi}_f$$

where f enumerates flavours [11] and the current reads $J^\mu = \bar{\psi}\gamma^\mu\psi$ which is evidently globally defined. This is the case for any current when we deal with interactions with gauge fields.

1.5. **Outline of the results.**

(1) Despite the realization of 2.2 of certain physical concepts inside *cohesive, differentially-cohesive ∞-topoi*, some of them remain only within the ∞-topos $Sh_\infty(FormalCartSp_{Sm})$ of *formal ∞-groupoids.*

One of them is the notion of a local prequantum field theory ([69], 6.5.11; or see below 4.10). Moreover, the local prequantum field theory relies on such notions as Euler-Lagrange complex which is constructed exploiting the structure of a smooth manifold. We shall see in 4.1 the construction of the theory of generalised PDEs and Euler-Lagrange theory extensively exploit formal smoothness and left Kan extensions along sites such as $DiffOp_{\downarrow\Sigma}$, PDE_Σ (4.0.2) - these are the essential structures of the ∞-topos of *formal ∞-groupoids* and of the smooth manifold structure of Σ.

Nonetheless, we show in 4.1 that local prequantum field theories may be rewritten internally in an arbitrary *differentially-cohesive ∞-topos.* We review the current state of the prequantum field theory and make some additional notes which we summarise in 3.1.5 4.0.6, 4.0.7, 4.10. In 4.0.8 we suggest a new approach to the Euler-Lagrange theory via coalgebras of (co)monads. This allows to speak of the prequantum field theories simply as *twisted* (by an infinitesimal neighbourhood $\mathfrak{I}\Sigma$; this yields the topology of a bundle of a prequantume field theory at hand) cocycles with some additional structure in the form of a coalgebra over the Euler-Lagrange comonad (this structure yields the dynamics of a theory)

In 4.1 and in 4.2, 4.27 we suggest an abstract definition of prequantum field theories in differentially-cohesive ∞-topoi. This allows, in particular, to define prequantum field theories not only over a smooth manifold, but over an arbitrary smooth ∞-groupoid. In 4.30 we show for instance how one can define Euler-Lagrange theory and prequantum field theory over a quatioent stack of a smooth manifold $X//G$. It turns out to simplify when $X//G$ is an orbifold. Moreover, in the context of *formally smooth* ∞*-groupoids* we suggest new analogs of the Euler-Lagrange theories, where the Deligne spectra $\mathbf{B}^{p+1}U(1)$ (and the coefficients for *Cech-de Rham* cohomology $\mathbf{B}^{p+1}_{H}U(1)_{conn}$, where cocycles are glued Lagrangian systems 4.1) are substituted by rather general smooth spectra such as Hopkins-Singer coefficients or differential refinements of algebraic K-theory.

(2) We know that gravity is realised as it may be seen from ([69], 7.1; or below in 1.2) as a *twisted* cohomology (twists exhibiting the frame bundles, or some other geometric data). Moreover, gravity fields rather take values in Cartan-connections (see 1.3). Cartan connections have been discussed synthetically only in the context of *formal smooth* ∞*-groupoids* [39] and we suggest a way to treat them abstractly in 3.2. In 3.3 we show how one can define such essential structures (for gravitational theories as well) such as *orthogonal, spin, string, FiveBrane* structures (see the Whitehead tower 3.3.1) abstractly on V-manifolds where V is an underlying homotopy type of a group with additional conditions in a *differentially-cohesive* ∞*-topos.*

(3) In 5 we found explicit formulas for prequantization of *4d WZW* field theory over a Kahler manifold. We use this model so as to demonstrate that the Energy-momentum tensor of gravity is not globally defined for all field configurations and glues over a manifold as a cocycle in differental cohomology. This is an implication of the fact that the *4d WZW* sigma-model has a nontrivial topologically bundle which yields obstructions to glue Lagrangians globally.

In 1.2 we argue that this is one of the difference of gravitational theories with gauge field theories. In addition to that, we notice that this topological inconsistency of gravity implies that gravity should be refined to a form *parameterized field theory* with a parameterization exhibited by a *topological coupling.*

2. Physics and Homotopy type theory

In this section we review some of the main defining constructions in the theory of *cohesive, differentially-cohesive* ∞*-topoi.* In 2.1 we remind the axioms of notable types of ∞*-topoi*, *modalities* and how some physical and mathematical constructions internalize higher topos theory.

2.1. **Modalities.**

2.1.1. *Cohesion.* Let $\mathbf{H}$ be an ∞ topos which is *local* in that there is an extra adjoint *CoDisc*, which in addition admits an adjoint π such that π preserves not only the terminal object but all finite products ([69], 4. Substance).

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=1, "\Gamma"'] & \infty Grpd \arrow[l, shift right=-1, "Disc"'] \arrow[l, shift left=3, "CoDisc"] \end{tikzcd}$$

- $\int \equiv Disc \circ \Pi$ - *shape modality*
- $\flat \equiv Disc \circ \Gamma$ - *flat modality*
- $\sharp \equiv CoDisc \circ \Gamma$ - *sharp modality*

Consider the following pullback and pushout squares which define the so called *de Rham modalities*:

$$\begin{tikzcd} \flat_{dR} \arrow[r] \arrow[d] & \flat \arrow[d] \\ * \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] & \int \arrow[d] \\ * \arrow[r] & \int_{dR} \end{tikzcd}$$

These yield the following adjunctions [69], Observation 5.2.64:

$$\mathbf{H}^{*/} \begin{array}{c} \xleftarrow{\quad \int_{dR} \quad} \\ \xrightarrow{\quad \flat_{dR} \quad} \\ \xleftarrow{\quad \Gamma_{dR} \quad} \end{array} \mathbf{H}$$

The de Rham modalities define the functor:

$$expLie \equiv \int_{dR} \circ \flat_{dR} : \mathbf{H}^{*/} \to \mathbf{H}^{*/} \tag{3}$$

2.1.2. *Stable cohomology and anti-Modal types.* Differential cohomology serves as a *refinement* of cohomology theories on smooth manifolds to that which requires no homotopy invariance [45]. In ([45], 5) one may see for examples of applications in Riemann geometry of differential cohomology, namely the theory of differential characters is useful in finding obstructions to existence of conformal immersions in $\mathbb{R}^{n+k}$.

One of the main features of differential cohomology is the so-called (homotopy) exact hexagon which decomposes the differential cohomology onto the topological and differential (connection) data. The differential cohomology hexagon was found in the context of ordinary differential cohomology and K-theory ([71], [72], [17], and differential K-theory is in fact characterised in terms of the exact hexagon [40], for the refinement to homotopy exact hexagons of classifying spaces [34], and for further discussions [7], [13], [69]).

For each presentable ∞-category C there is a tangent ∞-category TC ([51], 7.3., [69] Parameterized homotopy types, [52]) whose objects are the so-called parameterized homotopy types:

$$\begin{tikzcd} & \dots \arrow[d] & \\ \dots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & B \arrow[d] \\ & B \arrow[r] & E_n \arrow[r] \arrow[d] & \dots \\ & & \dots & \end{tikzcd}$$

$E_*, B \in \mathbf{H}$ and $E_n \to \Omega_B E_{n+1}$ is equivalence for each $n \in \mathbb{Z}$. the object 2.1.2 is in the tangent category $Sp(\mathbf{H}_{/B}) \simeq T_B\mathbf{H} \subset T\mathbf{H}$ to $\mathbf{H}$ at $B \in \mathbf{H}$.

The home for stable cohomology theories in a category $\mathbf{H}$ (good enough for Brown represantbility to apply, for instance in ∞-topoi, see [51] 1.4.1.) is the category of spectra $Sp(\mathbf{H}) = T_*\mathbf{H} \subset T\mathbf{H}$, whilst the paramterized homotopy types in $T\mathbf{H}$ represent the so-called *twisted cohomology* (see for the model category discussion [70], for the necessary and sufficient conditions for an ∞-category C to admit TC which is an ∞-topos and in particular for the resolution of the Joyal conjecture [38]).

One of the archetypical examples of twisted cohomology is the so-called Quillen cohomology based on the cotangent complexes (firstly appeared in the context of *Andre-Quillen cohomology of commutative algebras*, see for a brief exposition [80], Def.8.8.2.; then treated abstractly in ∞-categories [51], in combinatorial model categories [33], [32], [12]). We shall see in particular in 4.0.6 that prequantum field theories are defined as certain cocycles in *twisted cohomology* in the *differentially-cohesive* ∞*-topoi*, where the *twist* is exhibited by the infinitesimal neighbourhood $\Im\Sigma$ of a homotopy type $\Sigma \in \mathbf{H}$

Let $\hat{E}$ be a stable homotopy type in a *cohesive* ∞*-topos* in that $\hat{E} \to \Omega\Sigma(\hat{E})$ is equivalence. Given an ∞-topos $\mathbf{H}$ its tangent is again an ∞-topos $T\mathbf{H}$, and stable homotopy types in $T\mathbf{H}$ are precisely spectra in $\mathbf{H}$ ([69], 6.1.22.). Moreover, parameterized homotopy types extend the cohesion of $\mathbf{H}$, whose modalities are denoted similarly as those in $\mathbf{H}$:

$$\begin{tikzcd}
T\mathbf{H}_R \arrow[r, hook, shift left=3, "Ti_!"] \arrow[d, shift right, "\dashv"] & T\mathbf{H} \arrow[l, shift left=1, "Ti^*"] \arrow[l, shift right=1, "Ti_*"'] \arrow[d, shift right, "\dashv"] \\
T\mathbf{H}_R \arrow[r, shift right=3, "Ti^!"'] & {} \\
(\mathbf{H}_R)^{seq} \arrow[u, shift left, "spec"] \arrow[r, hook, shift left=3, "i_!"] \arrow[r, shift right=1, "i_*"'] & \mathbf{H}^{seq} \arrow[u, shift left] \arrow[l, shift left=1, "i^*"] \arrow[l, shift right=3, "i^!"']
\end{tikzcd}$$

In particular from this we may see how the axioms of cohesion yield the embedding of the ordinary spectra to the category of spectra $Sp(H) \equiv T_*\mathbf{H} \subset T\mathbf{H}$ 4.22.

For further results we address to [13], these may be summarised in:

Theorem 2.1. *For any $\hat{E} \in Sp(\mathbf{H}])$ there is natural homotopy exact hexagon.*

$$\begin{tikzcd}
 & \int_{dR} \hat{E}[-1] \arrow[rr, "\mathbf{d}"] \arrow[dr] & & \flat_{dR}\hat{E}[1] \arrow[dr] & \\
\int_{dR} \flat\hat{E}[-1] \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur, "curv"] \arrow[dr] & & \int \flat_{dR}\hat{E}[1] \\
 & \flat\hat{E} \arrow[rr, "\beta_\Sigma"] \arrow[ur] & & E \arrow[ur, "ch"'] &
\end{tikzcd}$$

Let $Ker(\flat)$ denote anti-$\flat$-modal types, meaning that $\flat$ vanishes on them. De Rham homotopy types (of the form $\flat_{dR}X$) are anti-$\flat$-modal. When constructing $\mathbf{A}_\infty$-type prequantum field theories we shall need the following converse to the Theorem above:

Proposition 2.2. *Let $\Omega^{\bullet\geq}$ be an anti-$\flat$-modal type, assume we have a E shape-modal type equipped with a morphism of spectra $E \to \int \Omega^{\bullet\geq}$. Then, the pullback:*

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & \Omega^{\bullet\geq} \arrow[d] \\
E \arrow[r] & \int \Omega^{\bullet\geq}
\end{tikzcd}$$

is equivalent to the differential cohomology pullback in the diagram 2.1.

Corollary 2.3. *There is a pullback of ∞-categories:*

$$\begin{tikzcd}
Sp(H) \arrow[r] \arrow[d] & Ker(\flat) \arrow[d] \\
Spectra^{\triangle[1]} \arrow[r] & Spectra
\end{tikzcd}$$

2.1.3. *Differential cohesion.* Let $\mathbf{H}$ be an ∞-topos, which has the following additional structure:

$$\begin{tikzcd}
\mathbf{H}_{\mathfrak{R}} \arrow[r, hook, shift left=3, "i_!"] \arrow[r, shift right=1, "i_*"'] & \mathbf{H} \arrow[l, shift left=1, "i^*"] \arrow[l, shift right=3, "i^!"']
\end{tikzcd}$$

where $i_!$ preserves finite products, $i^!$ is an additional right adjoint to i_*, which is necessarily full faithful together with $i_!$.

$\mathbf{H}$ is said to exhibit an infinitesimal neighbourhood of $\mathbf{H}_{\mathfrak{R}}$. Four adjoint functors yield the diagram of idempotent modalities:

$$\Re \dashv \Im \dashv \flat_{inf}$$

$$\int \dashv \flat \dashv \sharp$$

(1) *Reduction modality* $\Re \equiv i_! \circ i^*$
(2) *Infinitesimal shape modality* $\Im \equiv i_* \circ i^*$. Intuitively speaking if X is a smooth manifold, then $\Im X$ is a space in which infinitesimal points are glued whilst retaining the same *finite paths* as in X.
(3) $\flat_{inf} \equiv i_* \circ i^!$

See ([69], 5.2., 5.3.)

- *Points-to-pieces transform* $\flat \to \int$
- *Points-to-infinitesimal pieces* $\flat \to \Im$
- We have a factorization:

$$\begin{tikzcd} \flat \arrow[r] \arrow[dr] & \Im \arrow[d] \\ & \int \end{tikzcd}$$

- $\Im$ co-represents the *crystalline cohomology* ([69], 5.3.). It is known to produce *de Rham spaces*. Given an ∞-group G, in literature the mapping stack $[\Im X, \mathbf{B}G]$ is studied and known as a *moduli* ∞*-stack* of G*-local systems*, and quasicoherent sheaves on $\Im X$ are $\mathcal{D}$-modules (see [1], and this is related to [26] Langlands correspondence).
- $\flat$ represents the *flat cohomology* and holds in ∞*-connected* ∞*-topoi*
- $\flat_{dR}$ represents the *de Rham cohomology* and holds in ∞*-connected* ∞*-topoi*
- $\Sigma \in \mathbf{H}$, then there is a triple:

$$\begin{tikzcd} \mathbf{H}_{/\Im\Sigma} \arrow[r, "(\eta_\Sigma)^*"] & \mathbf{H}_{/\Sigma} \arrow[l, bend right, "(\eta_\Sigma)_!"'] \arrow[l, bend left, "(\eta_\Sigma)_*"] \end{tikzcd}$$

  which yields a jet comonad $J^\infty_\Sigma \equiv (\eta_\Sigma)^* \circ (\eta_\Sigma)_*$. In the case of Example 2.4 the jet comonad J^∞_Σ on the subcategory of locally pro-manifolds over Σ 4.0.2 coincides with the canonical jet comonad. This is used in [47] in internalization of the theory of jet bundles, *generalised PDE's* 4.0.5 in differentially-cohesive topoi.
- An *infinitesimal disk bundle monad* $T^\infty_\Sigma \equiv (\eta_\Sigma)^* \circ (\eta_\Sigma)_!$ which is used in the theory of framings, frame bundles (3.1, [69] 5.3.7.) and unwinding the definition:

$$\begin{tikzcd} T^\infty_\Sigma E \arrow[r] \arrow[d] & E \arrow[d] \\ T^\infty \Sigma \arrow[r] \arrow[d] & \Sigma \arrow[d] \\ \Sigma \arrow[r] & \Im\Sigma \end{tikzcd}$$

If we have orders k of infinitesimals in a differential cohesion, then we have $T^{(k)}_\Sigma \dashv J^{(k)}_\Sigma$. First order disk bundles are used in the theory of framings 3.1.

Example 2.4. The archetypical example of a *differentially-cohesive* ∞*-topos* is *FormalSmooth-*∞*-groupoids*:

$$\begin{tikzcd}
\mathbf{H}_{\mathfrak{R}} \equiv Sh_\infty(SmMfd) \arrow[r, hook, shift left=4.5, "i_!"] \arrow[r, leftarrow, shift left=1.5, "i^*" description] \arrow[r, shift right=1.5, "i_* \sim p^*" description] \arrow[r, leftarrow, shift right=4.5, "i^! = p_*" description] \arrow[d, leftarrow, shift right=4.5, "CoDisc"'] \arrow[d, dashed, shift right=1.5, "\Gamma" description] \arrow[d, leftarrow, shift left=1.5] \arrow[d, shift left=4.5, "\Pi"] & Sh_\infty(FormalSmMfd) \equiv \mathbf{H} \arrow[d] \\
\infty Grpd \arrow[r] & PSh_\infty(InfPoint) \equiv \mathbf{H}_{inf}
\end{tikzcd}$$

see (4.0.2, [47], [69]). The same theory holds for the site of complex analytic manifolds:

Example 2.5. Another example is arises in the so-called $\mathbb{E}_\infty$-arithmetic differential cohomology. It is defined within the ∞-topos of stacks over the ∞-site of formal duals to $\mathbb{E}_\infty$-rings with etale topology. It does not have a cohesive structure but it has a *relative differential cohesion* [54], [1].

One of the hallmarks of *differentially-cohesive topoi* is that they capture *the principle of locality* which states that configurations of physical fields can be reconstructed (in a neighbourhood of a point) using only the information (values, derivatives of fields) at a point.

Indeed, these ∞-topoi possess the notions of the theory of bundles, jets and thus makes it possible to redefine the main concepts of the field theory.

2.2. **Internalization.** Here we shall outline how certain constructions find their place abstractly in ∞-topoi. Higher geometric structures have a common appearance in physics and one cannot ignore the global topological structure of the manifold:

- Background fields of WZW models ([69], 7.5.1.; [81], 5, 5.1), B-fields are connections on $U(1)$-gerbes; C-fields in 11-dimensional supergravity [58] are connections on $U(1)$-2-gerbes.
- At the quantum level one encounters issues which can be only resolved by refining certain fields to higher geometric structures. For instance, a (super)membrane in M-theory which couples to the C-field. In supergravity, this locally is presented as an $\mathbb{R}$-valued differential 3-form. Nonetheless, when one studies of the path integral one realizes that this field is quantized in a rather nontrivial way [82] and must be presented as a rather a *shifted* version of degree 3 Cheeger-Simons differential character [19].
- Fields of superstring backgrounds 1.2, in exceptional geometry.
- The restrictions on the topology of a manifold at hand are demanded by the ability to define fields and their corresponding partition functions. There instances even in the experimentally verified physics such as fermions, which require a manifold to admit a *spin*-structure, and the description of theories with chiral fermions yields constraints on topology by an index theorem. More generally, dealing with quantum field theories one encounters *anomaly cancellations* which put restrictions on the topology of manifolds when we require quantum field theories to be consistently defined.
- Another instance in literature when higher geometric structures are involved are L_∞-algebras (sometimes implicitly as in [66], [69] BRST).

Further we briefly remind how the basic physical principles find their place in HoTT

(1) (The gauge principle)

Spaces of physical configurations now have a richer structure which one simply cannot quotient out. The objects are non-truncated homotopy types, so the *ambient theory is a HoTT* **H**. It is known and has been widely recognized in literature that when quantizing one simply *cannot ignore the homotopy types.* Even, for instance, an addition of G-symmetries to a classical field theory yields a deformation of a homotopy type of a configurational stack

$$[X.Fields] \mapsto [X, Fields]//G$$

which may be crucial at the quantum level.

Example 2.6. Consider a 3d Chern-Simons gauge field theory on Σ, it is known that gauge field configurations are gauge equivalent if they are joined by a diffeomorphism (connected to identity) and thus the theory is **implicitly** generally covariant. Nonetheless, the theory with **explicit** general covariance, meaning that the configurational 1-stack is now a quotient 2-stack:

$$[\Sigma, Fields]//Aut_0(\Sigma), \tag{4}$$

may be different at the quantum level ([69], General covariance).

(2) (Phases and local field theories)
Let $\mathbf{H}$ is *differentially-cohesive*, in which we have chosen an ∞-group $\mathbb{G}$ (usually *braided* (3.6, [69]) so that via some Hodge filtration on $\mathbb{G}$ one defines a notion of $\mathbb{G}$-connections).
Then, having a stack $\mathbf{B}\mathbb{G}_{conn}$ of $\mathbb{G}$-bundles with connections one defines a *context for the local action functionals* $(Fields \to \mathbf{B}\mathbb{G}_{conn}) \in \mathbf{H}_{/\mathbf{B}\mathbb{G}_{conn}}$, and its
We notice that even though in [2] one may see the discussion of phases and actions as simply the morphisms to the configurational stacks to $\mathbf{B}\mathbb{G}_{conn}$, actually yet no work has been done in the direction of the *internalization* of *prequantum field theory* in differentially-cohesive ∞-topoi in its complete form, namely the Euler-Lagrange theory 4.10, 4.1.
We introduce in 4.1 a general way to define prequantum field theories where *phases* are modulated by $\mathbf{A}_\infty$-rings.
(3) (Superposition and linearization) Group of phases $\mathbb{G}$ has a representation $\mathbb{G} \to GL_1(E)$ the group of units of a $\mathbb{E}_\infty$-ring. Then, each field theory yields:

$$Fields \overset{e^{iS}}{\to} \mathbf{B}\mathbb{G} \to GL_1(E) \to EMod$$

which linearizes spaces of states.
(4) Pull-push quantization ([69], Motivic quantization)

2.3. **Localization.** Localizations is one of the main tools in construction of presentable ∞-categories. Basics of the 0-truncated version of the theory may be found in the classical textbook [62], concerning the locally presentable categories [43]. The modern ∞-categorical treatment can be found in ([51]], 5.5.4., 5.5.5.)

From [51] recall that given a presentable ∞-category a localization $L : \mathcal{C} \to \mathcal{C}$ is uniquely determined by a family of morphisms $S \equiv \{f : Lf - equiv\}$, and L is accessible iff S is *strongly saturated* [51] and *is of small generation.* Moreover, given a set of morphisms S_0 there is a minimal strongly saturated family of morphisms S containing it (S_0 is said to generate S as a *strongly saturated* family).

Working with ∞-topoi we are more interested in producing *left exact localizations.* We shall discuss briefly how localizations interact with the notion of cohesiveness. In literature yet no results can be found [69], [68] concerning localizations of *cohesive* and differentially-cohesive ∞-topoi. Here we attach a few results concerning this matter and see for example an idea 4.22.

Proposition 2.7. *Let L be a left exact localization of a ∞-topos $\mathbf{H} \to S^{-1}\mathbf{H}$. Then,*
(1) If $\mathbf{H}$ is local then so is $S^{-1}\mathbf{H}$
(2) If $\mathbf{H}$ is (globally)∞-connected and for any $Y, X \in \mathbf{H}$ the following is equivalence

$$\mathbf{H}(Y, \flat X) \to \mathbf{H}(Y, L\flat X)$$

then so is $S^{-1}\mathbf{H}$.
(3) If $\mathbf{H}$ is cohesive and (2) holds, then $S^{-1}\mathbf{H}$ is cohesive as well.

Proof. Consider:

$$
\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=2, "\Pi"] \arrow[r, shift right=2, "\Gamma"'] \arrow[d, shift right, "L"'] & \infty Grpd \arrow[l, "Disc" description] \arrow[dl, shift right, "Disc'"'] \\
S^{-1}\mathbf{H} \arrow[u, shift right, "i"'] \arrow[ur, shift right, "\Gamma'"'] &
\end{tikzcd}
$$

by essential uniqueness of geometric morphisms and left exactness of L we conclude that $\Gamma i \simeq \Gamma'$ and $i \circ Disc \simeq Disc'$. Locality of $S^{-1}\mathbf{H}$ is implied by the *Adjoint functor theorem* and that $S^{-1}\mathbf{H}$ is presentable. Indeed, Γ' preserves small limits.

For (2)-(3) define a functor $\Pi' \equiv \Pi \circ i$ and construct the adjunction equivalences:

$$\infty Grpd((\Pi \circ i)(X), Y) \simeq \mathbf{H}(iX, Disc(Y)) \overset{L_{Disc(Y)}}{\to} S^{-1}\mathbf{H}(X, (L \circ Disc)(Y))$$

The last is an equivalence and whence the adjunction. Nowm we notice that $\prod'$ preserves all limits that $\prod$ does. $\square$

Proposition 2.8. *Let S_0 be a (small) set of morphisms in a cohesive ∞-topos $\mathbf{H}$ generating as a strongly saturated family of morphisms S. If any of the following equivalent conditions is satisfied and S_0 exhibits a left exact localization, then $S^{-1}\mathbf{H}$ is cohesive:*

(1) S_0 consists of $\int$-equivalences
(2) Every discrete type in $\mathbf{H}$ is S-local
(3) S consists of $\int$-equivalences

Proof. (1) $\Rightarrow$ (2): let $Y \in \mathbf{H}$ and $(f : X \to Z) \in S_0$ then we have:

$$
\begin{tikzcd}
\mathbf{H}(Z, \flat Y) \arrow[r] \arrow[d, "\sim"] & \mathbf{H}(X, \flat Y) \arrow[d, "\sim"'] \\
\mathbf{H}(\int Z, Y) \arrow[r] & \mathbf{H}(\int X, Y)
\end{tikzcd}
$$

where below the morphism is induced by $\int(f)$ which is equivalence by an assumption.

(2) $\Rightarrow$ (3): let $(f : X \to Z) \in S$, then for any $D \in S^{-1}\mathbf{H}$ and thus for any discrete homotopy type (i.e. of the form $\flat Y$ by the properties of cohesive ∞-topoi) we have $\mathbf{H}(Z, \flat Y) \to \mathbf{H}(X, \flat Y)$ and again using the adjunction $\int \dashv \flat$ as above we conclude $\int(f)$ is equivalence. (3) $\Rightarrow$ (1) is obvious. $\square$

Definition 2.9. In connection with the *Galois theory* ([69], 5.2.7.) we suggest the following: let $L : \mathbf{H} \to S^{-1}\mathbf{H}$ be a left exact accessible localization of a cohesive ∞-topos satisfying the condition of Proposition 2.8, then L is called a *Galois localization*, and a generating set S_0 is said to exhibit L as a Galois localization.

Example 2.10. Let $(X_i)_i$ be a set of objects in $\mathbf{H}$, then the set $(* \to \int_{dR} X_i)_i$ exhibits a Galois localization of $\mathbf{H}$ and the resulting localization is cohesive.

Remark 2.11. The argument of Proposition 2.8 extends to the parameterized homotopy types and the *relative cohesion*: let S_0 be a set of morphisms in $T\mathbf{H}$ exhibiting a Galois localization, then

$$
\begin{tikzcd}
T\mathbf{H} \arrow[r, shift left=3, "T\Pi"] \arrow[r, shift right=1, "T\Gamma" description] \arrow[d, shift right, "L"'] & T\infty Grpd \arrow[l, shift left=1, "TDisc" description] \arrow[l, shift left=5, "TCoDisc" description] \arrow[dl, shift right] \\
S^{-1}T\mathbf{H} \arrow[u, shift right, "i"'] \arrow[ur, shift right] &
\end{tikzcd}
$$

exhibits $S^{-1}T\mathbf{H}$ as relatively cohesive (and thus cohesive) to $T\infty Grpd$. This shall be useful when considering localizations induced by morhisms of cohesive spectra.

Example 2.12. As it is reminded above spectra in a cohesive ∞-topos sit in a homotopy pullback diagram 2.1. There are lots of various differential refinements of the same discrete spectra 4.2. The geometric realization map $f : \hat{E} \to E$ is an $\int$-equivalence ($\int$ is idempotent) which implies that $\{f\}$ exhibits a Galois localization of a cohesive ∞-topos of parameterized spectra.

Remark 2.13. In all the above examples in addition one must control whether a localization is left exact so as to produce a cohesive ∞-topos. Nonetheless, any such localization produces a presentable ∞-category and the *Galois condition* will be useful to control homotopy exact diagrams since the *fundamental path ∞-groupoid* functor adjunction $\Pi \dashv Disc$ descends to $S^{-1}\mathbf{H}$.

Example 2.14. From [53] it is known that the topological localizations of an ∞-category of presheaves $\mathcal{P}(C)$ (which turn out to be classified by Grothendieck topologies on C) are left exact accessible. Thus, it is interesting to ask when a topological localization (of a presheaf ∞-category) gives rise to a Galois localization.

Given an ∞-site its Grothendieck topology yields a localization of the category of presheaves:

$$\mathcal{P}(C) \mapsto S^{-1}\mathcal{P}(C) \equiv Sh(C)$$

where S consists of $\hat{C}\{U_i\} \to U$, $U \in C$, $\{U_i\}$ is a covering sieve on U and $\hat{C}\{U_i\}$ its Cech nerve (see for the bijective correspondence between monomorpisms to $U \in C$ and covering sieves 6.2.2. [53]). If C admits finite products, then the ∞-topos $\mathcal{P}(C)$ is evidently cohesive and Π is equivalent to $\varinjlim$ whence we conclude the following:

Proposition 2.15. *Let C be an ∞-site having finite products and such that for any $\{U_i\}$ covering sieve on $U \in C$ we have $\varinjlim(\hat{C}\{U_i\}) \to \varinjlim(U)$ is equivalence of ∞-groupoids. Then, $Sh(C)$ is a cohesive ∞-topos.*

Proof. Apply (3) of Proposition 2.7 and notice that morphisms at which we localize are $\int$-equivalences and so we have a left exact Galois localization (2.9, [53] Corollary 6.2.1.7., Lemma 6.2.2.7.). $\square$

Remark 2.16. Notice that our condition is a simplification of that given in ([69], 4.1.2.2. Presentation over ∞-cohesive sites) in that we do not require such conditions as

- $\hat{C}\{U_i\}$ is a degreewise a coproduct of representables. This condition is used to show that $\hat{C}\{U_i\} \to U$ is a cofibrant replacement of U and further when analyzing the localization one considers the homotopy function complexes $Map(\hat{C}\{U_i\}, -)$ where the cofibrantness of $\hat{C}\{U_i\}$ is crucial.
- $\varinjlim(\hat{C}\{U_i\}) \to \varinjlim(U) \simeq \Gamma(U)$ is a weak equivalence of simplicial sets. This condition is used in [69], Prop. 4.1.3.2. so as to prove that an additional right adjoint $\Gamma \dashv \nabla$ to global sections descends to the local projective structure. Clearly, this may not be a necessary (we have no counterexamples yet, but our ∞-categorical version has no such an assumption)

The thing is that in [69] one is concerned with the construction of such 1-sites which at the model categorical level would give a cohesion, i.e. $[C^{op}, SSet_{Quillen}]_{locproj}$ models a cohesive ∞-topos. Thus, one has such additional requirements as above, which turn out to be superfluous from the point of view of *direct* construction of cohesive ∞-topoi. For computational reasons it is very useful to have a model category of simplicial presheaves at hand which models the axioms of cohesion as in ([69], 4.1.2.2.).

3. Gravity fields via differential cohesion

In 1.2 we briefly discussed the mathematics of gravity theories. In this section we indicate how one can define Cartan connections (3.6) in an abstract *differentially-cohesive*

∞-*topos*. Yet this has been done only for Cartan geometries in ([69], 5.3.12.). Cartan connections are discussed synthetically in [39] but the definition relies on the structure of stacks $\mathbf{B}\mathbb{G}_{conn}$ for Lie groups $\mathbb{G}$. We show that $\Omega^1(-,\mathfrak{g})//H$ stacks can be defined abstractly for any $\mathbb{G}$ endowed with a Hodge filtration [39] and suggest an abstract way to discuss gravity fields.

Remark 3.1. Further we do not exploit the particular orders of infinitesimals in differential cohesion. Nonetheless, the terminology depends on whether we are working with infinite order infinitesimals (for instance, see [69] for the definition of ∞-*Lie algebras* or the *formal groups*) We shall use the convention:

- When using infinitesimal disks, for example when working with Cartan connections, we assume that these have the **first order**
- When we discuss ∞-*Lie algebras* or jet comonad we presume $\mathfrak{J}$ is of infinite order.

3.1. **Frame bundles, H-structures, Cartan geometries synthetically.** A V is a framed objects provided there is a trivialization of an infinitesimal disk bundle $T^1V \simeq V \times \mathbb{D}^V(1)$ for some $\mathbb{D}^V(1)$ is an infinitesimal disk in V:

$$\begin{tikzcd} \mathbb{D}^V \arrow[r] \arrow[d] & V \arrow[d] \\ * \arrow[r] & \mathfrak{J}V \end{tikzcd}$$

This holds for any group object V and this is naturally framed along the unit $e : * \to V$ ([47]).

The *twists* in cohomology defining gravity are exhibited by *geometries* 1.2. These geometries correspond to *frame bundles*, or some *generalised form* of such bundles 1.2. The theory of frame bundles internalizes in differential cohesion in the following way ([69], 5.3.11.):

- Consider a V-manifold Σ with an atlas:

$$\begin{tikzcd} & U \arrow[dl, "et"'] \arrow[dr, two heads, "et"] & \\ V & & \Sigma \end{tikzcd}$$

  V being framed Σ is locally framed in that its infinitesimal disk bundle trivializes ([69], Prop. 5.3.97.):

$$\begin{tikzcd} U \times \mathbb{D}^V(1) \arrow[r, two heads, "et"] \arrow[d] & T^1\Sigma \arrow[d] \\ U \arrow[r, two heads, "et"] & \Sigma \end{tikzcd}$$

- Define the general linear group $GL(V) \equiv Aut(\mathbb{D}^V(1))$
- From this diagram one sees that $T^1\Sigma \to \Sigma$ is an associated bundle with $GL(V)$-bundle ([69], Prop. 5.1.249., Def. 5.1.241.) and thus is classified by some:

$$\tau_\Sigma : \Sigma \to \mathbf{B}GL(V) \tag{5}$$

  which is called a *frame bundle* of Σ ([69], Def. 5.3.99.)

Another step is to define the Cartan connections. There are different methods to define stacks of differential forms with values in Lie algebras abstractly. One of them is via the Hodge filtrations:

Let $F^*\flat\mathbf{B}\mathbb{G}$ be a filtration of $\flat_{dR}\mathbf{B}^2\mathbb{G}$, then form pullbacks

$$\begin{tikzcd}
\flat\mathbf{B}\mathbb{G} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbf{B}\mathbb{G}_{conn} \arrow[r] \arrow[d] & \mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G})) \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] \\
\mathbf{B}\mathbb{G}_{conn^k} \arrow[r] \arrow[d] & F^{k+1}\flat\mathbf{B}^2\mathbb{G} \arrow[d] \\
\mathbf{B}\mathbb{G}_{conn^{k-1}} \arrow[r] \arrow[d] & F^{k}\flat\mathbf{B}^2\mathbb{G} \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] \\
\mathbf{B}\mathbb{G} \arrow[r] & \flat_{dR}\mathbf{B}^2\mathbb{G}
\end{tikzcd}$$

with a homotopy fiber defining stack of differential $Lie(\mathbb{G})$-valued 1-forms $\Omega^1(-;Lie(\mathbb{G})) \to \mathbf{B}\mathbb{G}_{conn} \to \mathbf{B}\mathbb{G}$.

3.2. **Cartan connections in differential cohesion.**

Definition 3.2. Let $\mathbf{G}$ be a $(n-1)$-truncated braided ([69], 1.3.4.3.) ∞-group equipped with Hodge filtration $F^*\flat\mathbf{B}^2\mathbf{G}$ with the corresponding stack of closed differential 2-forms $\mathbf{\Omega}^2_{cl}(-;Lie(\mathbf{G}) \equiv F^{n+1}\flat\mathbf{B}^2\mathbf{G}$. Assume we have a morphism of deloopings: $c : \mathbf{B}\mathbb{H} \to \mathbf{B}\mathbb{G}$. Form the pullback:

$$\begin{tikzcd}
\mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \arrow[r] \arrow[d] & \mathbf{B}\mathbb{H} \arrow[d] \\
\mathbf{B}\mathbb{G}_{conn} \arrow[r] & \mathbf{B}\mathbb{G}
\end{tikzcd}$$

which defines $\mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbf{H}$ stack.

We say that a Hodge filtration is $F^*\flat\mathbf{B}^2\mathbf{G}$ is *infinitesimally compatible* provided any $\mathbb{G}$-connection on an infinitesimal disk (of the first order) is equivalent to a trivial one:

$$\begin{tikzcd}
\mathbb{D}^X(1) \arrow[r, dashed, "\exists"'] \arrow[rr, bend left, "\forall"] & * \arrow[r] & \mathbf{B}\mathbb{G}_{conn}
\end{tikzcd}$$

Remark 3.3. Notice that in our definition of $\mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H}$ we do not assume that we have necessarily a homomorphism of ∞-groups $\mathbb{H} \to \mathbb{G}$. This may be very useful when discussing the characteristic classes.

Example 3.4. If $\mathbb{H} \simeq *$ is trivial, then the sheaf of differential 1-forms with values in Lie ∞-algebra of $\mathbb{G}$ is defined as a cofiber:

$$\mathbf{\Omega}^1(-;Lie(\mathbb{G})) \to \mathbf{B}\mathbb{G}_{conn} \to \mathbf{B}\mathbb{G}$$

Theorem 3.5. *(1) Let $\mathbb{H} \to \mathbb{G}$ be homomorphism between differentially cohesive ∞-groups. Then, there is a homotopy fiber sequence:*

$$\mathbb{G}/\mathbb{H} \longrightarrow \mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \longrightarrow \mathbf{B}\mathbb{G}_{conn}$$

(2) Assume that the Hodge filtration of $\mathbb{G}$ is infinitesimally compatible. Then, for any point $x : * \to X$ inclusions of infinitesimal disks factor through the quotient group:

$$\begin{tikzcd} & & \mathbb{G}//\mathbb{H} \arrow[d] \\ \mathbb{D}^X_x \arrow[r, dashed, "et"] \arrow[urr, dashed, bend left] & X \arrow[r] & \mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \end{tikzcd}$$

Proof.

$$\begin{tikzcd} \mathbb{G}/\mathbb{H} \arrow[r] \arrow[d] & \mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \arrow[r] \arrow[d] & \mathbf{B}\mathbb{H} \arrow[d] \\ * \arrow[r] & \mathbf{B}\mathbb{G}_{conn} \arrow[r] & \mathbf{B}\mathbb{G} \end{tikzcd}$$

the right rectangle is pullback by Definition 3.2. The outer rectangle is pullback by the long exact sequence [69] 5.1.178:

$$\mathbb{H} \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}//\mathbb{H} \longrightarrow \mathbf{B}\mathbb{H} \longrightarrow \mathbf{B}\mathbb{G}$$

For the second point, using (1) and that every $\mathbb{G}$ connection on an infinitesimal disk is trivial we conclude: $\mathbb{D}^X$

$$\begin{tikzcd} & & \mathbf{\Omega^1}(-;Lie(\mathbb{G}))//\mathbb{H} \arrow[d] \\ \mathbb{D} \arrow[r] \arrow[urr, dashed] & * \arrow[r] & \mathbf{B}\mathbb{G}_{conn} \end{tikzcd}$$

whence the factorization through the homotopy fiber $\mathbb{G}//\mathbb{H}$. $\square$

Definition 3.6. Let $\mathbb{H} \to \mathbb{G}$ be a homomorphism of differentially-cohesive ∞-groups (whose underlying homotopy types are reduced). Then, a *Cartan connection* on $X \in \mathbf{H}$ is a morphism:

$$\begin{tikzcd} & \mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \arrow[d] \\ X \arrow[r, "\nabla"'] \arrow[ur, dashed] \arrow[ur, phantom, shift right=3, "\Rightarrow"] & \mathbf{B}\mathbb{G}_{conn} \end{tikzcd}$$

such that for each $x : * \to X$ the factorization 3.5 is *etale*:

$$\mathbb{D}^X_x \to \mathbb{G}//\mathbb{H} \tag{6}$$

Remark 3.7. The Definition 3.6 recovers the synthetic definition given in [39] for Lie groups, where the stack $\mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H}$ is indeed the quotient stack of a sheaf of differential 1-forms with values in Lie algebra of a group $\mathbb{G}$.

Remark 3.8. Let $\mathbb{D}^X_x \to \mathbb{G}//\mathbb{H}$ be induced factorization 3.5. Denote the quotient *Lie ∞-algebra*: $\mathbb{D}^{\mathbb{G}//\mathbb{H}}_{e\mathbb{H}} \equiv \mathfrak{g}/\mathfrak{h}$. Then, using that $\mathfrak{J}(\mathbb{D}^X_x) \simeq *$ we have a factorization:

$$\begin{tikzcd} \mathbb{D}^X_x \arrow[dr, dashed] \arrow[drr, bend left, "et"] \arrow[ddr, bend right] & & \\ & \mathfrak{g}/\mathfrak{h} \arrow[r, "et"] \arrow[d] & \mathbb{G}//\mathbb{H} \arrow[d] \\ & * \arrow[r] & \mathfrak{J}(\mathbb{G}//\mathbb{H}) \end{tikzcd}$$

which by Lemma 3.11 implies that $\mathbb{D}^X_x \to \mathfrak{g}/\mathfrak{h}$ is etale and thus equivalence (since on $\mathfrak{J}$ they vanish).

The Definition 3.6 is equivalent to that the induced morphism

Ideologically, unwinding the definition above we see that giving a Cartan connection is equivalent to giving a $\mathbb{G}$-bundle with connection ∇ with the reduction of the underlying bundle of ∇ to $\mathbb{H}$:

$$\begin{tikzcd} & \mathbf{B}\mathbb{H} \arrow[dr] & \\ X \arrow[r, "\nabla"'] \arrow[ur, dashed, "red"] & \mathbf{B}\mathbb{G}_{conn} \arrow[r] \arrow[u, phantom, "\overset{e}{\Longrightarrow}"] & \mathbf{B}\mathbb{G} \end{tikzcd}$$

such that ∇ yields an equivalence infinitesimally of X with the quotient Lie ∞-algebra $\mathfrak{g}/\mathfrak{h}$.

A homotopy above e is usually called the *vielbein field*, or *tetrad field* when working with gravity.

Recall that in a *cohesive ∞-topos* $\mathbf{H}$ in an object X *pieces have points* provided the natural transformation (1) *points to pieces* 2.1 is effective epi on X ([69] Definition 4.1.16.):

$$\flat X \twoheadrightarrow \int X$$

In such a fashion we introduce:

Definition 3.9. In $X \in \mathbf{H}$ *infinitesimal pieces have points* provided

$$\flat X \twoheadrightarrow \Im X$$

is effective epi

Notice that evidently in *properly infinitesimal infinitesimal pieces have points* ($\flat X \to \Im X$ is equivalence [69], 5.3.6.) and the results below in this case become trivial. Nonetheless we expect some exotic examples of differentially cohesive ∞-topoi to come out where there are *infinitesimal stacks* with more than one point for example. One may skip the results till the next section 3.3

As it is indicated in [69] Definition 4.1.16., Remark 4.1.17. in *cohesive ∞-topoi* arising from *∞-cohesive sites*. Hence, for example the Sierpinski topos *does not admit a ∞-cohesive site* for its definition ([69] 6.1.1.)

Further when discussing the *Cartan geometry, Cartan connections* we shall be concerned with *differentially-cohesive* homotopy types which at *infinitesimal* level look the same around all points. For we introduce the following:

Definition 3.10. Let V be a *differentially-cohesive* homotopy type. V is said to be *infinitesimally homogeneous* provided there exists a type Ξ such that for each point x:

$$\begin{tikzcd} * \arrow[rr, "x"] \arrow[dr] & & X \\ & \mathbb{D}^X_x \arrow[ur, dashed, "et"'] & \end{tikzcd}$$

there is an equivalence $\Xi \simeq \mathbb{D}^X_x$.

Lemma 3.11. *If $f \circ g$ and f are etale morphisms then so is g.*

Proof. By pasting lemma on pullbacks and definition of etale morphisms **??**

$$\begin{tikzcd} U \arrow[r, "g"'] \arrow[rr, bend left, "f\circ g"] \arrow[d] & V \arrow[r, "f"'] \arrow[d] & W \arrow[d] \\ \Im U \arrow[r] & \Im V \arrow[r] & \Im W \end{tikzcd}$$

□

Example 3.12.
- If X has a unique global point [69] in that $\flat X \simeq *$, then since all morphisms $* \to X$ are homotopic, $\mathbb{D}^X_x$ are equivalent.
- X is a group object, then it is evidently *infinitesimally homogenous*

Theorem 3.13. *Assume that in $X \in \mathbf{H}$ infinitesimal pieces have points 3.9, X is infinitesimally homogenous 3.10, then X is a Ξ-manifold (and vice versa), where Ξ has a homotopy type of an infintesimal disk in X.*

Proof. We need to construct an atlas where the left leg is effective epi by [53] Corollary 6.2.3.11 (this makes the classes of Ξ-manifolds and X-manifolds coincide):

$$\begin{tikzcd} & U \arrow[dl, "et"', two heads] \arrow[dr, "et", two heads] & \\ \Xi & & X \end{tikzcd}$$

By [69] Example 4.1.17. if $\mathbf{H}$ has a an ∞-*cohesive site* then in all objects of $\mathbf{H}$ pieces have points. Now assume X satisfies the condition of our theorem.

$\flat^{rel} X \simeq \underset{\substack{\rightarrow \\ x:\flat X}}{lim}(\mathbb{D}^X_x)$ by Remark 5.3.59. and by infinitesimal homogeneity we have $\mathbb{D}^X_x \simeq \Xi$.

Thus, by Lemma 6.2.3.13 [53] we have that $\coprod_{x:\flat X} \Xi \to \flat^{rel} X$ is effective epi.

$$\begin{tikzcd} \coprod_{x:\flat X} \Xi \arrow[r, "(3)"'] \arrow[rr, bend left, "(1)"] & \flat^{rel} X \arrow[r, "(2)"'] & X \end{tikzcd}$$

(1) is etale morphism, since inclusions of infinitesimal disks are etale (consider the pullback from definition of etale morphisms and use that $\Im\mathbb{D} \simeq *$)
(2) etale by Proposition 5.3.65 [69]
(3) etale by 3.11

It suffice to show that (2) is effective epi. which by Corollary 6.2.3.12 [53] shall imply that (1) is effective epi. This defines immediately the atlas for X making it Ξ-manifold:

$$\begin{tikzcd} & \coprod_{x:\flat X} \Xi \arrow[dl, "et"', two heads] \arrow[dr, "et", two heads] & \\ \Xi & & X \end{tikzcd}$$

To prove that (1) is effective epi consider the following diagram:

$$\begin{tikzcd} \flat^{rel} X \arrow[r] \arrow[d] & \flat X \arrow[d] \\ X \arrow[r] & \Im X \end{tikzcd}$$

The rectangle is pullback and $\flat X \to \Im X$ is effective epi and so is $\flat^{rel} X \to X$. $\square$

Remark 3.14. In the above Theorem we have seen that for such homotopy types X the class of X-manifolds may be described using the infinitesimal homotopy types $\mathbb{D}$. Thus, given X it is locally framed 3.1 automatically which makes all the X-locally framed. This constitutes the main reason as to why in *generalised PDE* theory we have used V-*manifolds.*

Corollary 3.15. *Let X be an infinitesimally homogenous 3.10, formally smooth ([69]) and infinitesimal pieces have points. Then, the classes of $\mathbb{D}^X$-manifolds and X-manifolds coincide.*

Proof. Indeed, by Theorem 3.13 it suffice to notice that in X infinitesimal pieces have pieces. For, □

Remark 3.16. Notice that if we have an etale morphism $U \to X$ for each point $x : U \to X$ we have the equivalence $\mathbb{D}^U_x \simeq \mathbb{D}^X_x$. Indeed, consider the diagram:

$$\begin{tikzcd}
\mathbb{D}^U_x \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\
* \arrow[r] & \Im U \arrow[r] & \Im X
\end{tikzcd}$$

Corollary 3.17. *Let $H \to G$ be a homomorphism of differentially-cohesive ∞-groups, denote the quotient algebra $\mathfrak{g}/\mathfrak{h} \equiv \mathbb{D}^{G//H}_{eH}$. Let X be endowed with the $(H \to G)$-Cartan connection, then X is $\mathfrak{g}/\mathfrak{h}$-manifold (and thus a V-manifold if $\mathbb{G}/\mathbb{H}$ is a V-manifold) if infinitesimal pieces have points.*

Proof. Indeed, By above remark and by 3.8 we have that for each point x in X, there is $\mathbb{D}^X_x \simeq \mathfrak{g}/\mathfrak{h}$ and whence infinitesimal homogeneity of X 3.10. Then, applying the above theorem we get the $\mathfrak{g}/\mathfrak{h}$-manifold structure on X. □

3.3. **Gravity fields.**

Definition 3.18. Let $\mathbb{G}, \mathbb{H}$ be *braided differentially-cohesive ∞-groups* (n-1)-truncated equipped with Hodge filtrations. A homomorphism of groups $\mathbb{H} \to \mathbb{G}$ which extends to a morphism of filtrations:

$$\begin{tikzcd}
F^p \flat \mathbf{B}^2 \mathbb{H} \arrow[r] \arrow[d] & F^p \flat \mathbf{B}^2 \mathbb{G} \arrow[d] \\
\flat \mathbf{B}^2 \mathbb{H} \arrow[r] & \flat \mathbf{B}^2 \mathbb{G}
\end{tikzcd}$$

for any $p : 0 \leq p \leq n+1$
we call a Hodge homomorphism.

Proposition 3.19. *Let $\mathbb{G}$ be (n-1)-truncated differentially cohesive ∞-group equipped with a Hodge filtration a Maurer-Cartan homomorphism $\mathbb{H} \to \mathbb{G}$ which is (n-1)-truncated as a morphism in $\mathbf{H}$. Then, on $\mathbb{H}$ there is a Hodge filtration satisfying the following conditions:*

(1) $\mathbf{B}\mathbb{H}_{conn} \simeq \mathbf{\Omega}^1(-; Lie(\mathbb{G}))//\mathbb{H}$
(2) *There is a homotopy fiber sequence* $\flat \mathbf{B}\mathbb{H} \to \mathbf{B}\mathbb{H} \to \flat_{dR} \mathbf{B}^2 \mathbb{G}$

Proof. By induction we construct the following Hodge filtration on $\mathbb{H}$:

$$\begin{tikzcd}
\flat\mathbf{B}\mathbb{H} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbf{B}\mathbb{H}_{conn} \arrow[r] \arrow[d] & \mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{H})) \arrow[r] \arrow[d] & \mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G})) \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
\mathbf{B}\mathbb{H}_{conn^k} \arrow[r] \arrow[d] & F^{k+1}\flat\mathbf{B}^2\mathbb{H} \arrow[r] \arrow[d] & F^{k+1}\flat\mathbf{B}^2\mathbb{G} \arrow[d] \\
\mathbf{B}\mathbb{H}_{conn^{k-1}} \arrow[r] \arrow[d] & F^{k}\flat\mathbf{B}^2\mathbb{H} \arrow[r] \arrow[d] & F^{k}\flat\mathbf{B}^2\mathbb{G} \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
\mathbf{B}\mathbb{H} \arrow[r] & \flat_{dR}\mathbf{B}^2\mathbb{H} \arrow[r] & \flat_{dR}\mathbf{B}^2\mathbb{G}
\end{tikzcd}$$

Using where each $F^{k+1}\flat\mathbf{B}^2\mathbb{H}$ is defined by pullback for $k \geq 2$. Using that a homomorphism and applying the Lemma 5.5.6.14 [53] we conclude $\mathbb{H}$ is (n-1) truncated (as an underlying homotopy type). To prove $F^*\flat\mathbf{B}^2\mathbb{H}$ is indeed a Hodge filtration we need to show that the for the homotopy fiber:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{H})) \arrow[r] & \mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G}))
\end{tikzcd}$$

X is contractible. For, by pullback pasting lemma we conclude that

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
\flat_{dR}\mathbf{B}^2\mathbb{H} \arrow[r] & \flat_{dR}\mathbf{B}^2\mathbb{G}
\end{tikzcd}$$

is pullback, but since the homomorphism is Cartan-Maurer this has a trivial fiber and so is X.

Now, to prove (1) consider the following:

$$\begin{tikzcd}
\mathbf{B}\mathbb{G}_{conn} \arrow[r] \arrow[d] & \mathbf{B}\mathbb{G} \arrow[d] & \mathbf{B}\mathbb{H} \arrow[l] \arrow[d, "id"] \\
\mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G})) \arrow[r] & \flat_{dR}\mathbf{B}^2\mathbb{G} & \mathbf{B}\mathbb{H} \arrow[l]
\end{tikzcd}$$

The left hand side rectangle is pullback by definition of $\mathbf{B}\mathbb{G}_{conn}$. Then,

$$\mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \simeq (\mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G})) \underset{\flat_{dR}\mathbf{B}^2\mathbb{G}}{\times} \mathbf{B}\mathbb{G}) \underset{\mathbf{B}\mathbb{G}}{\times} \mathbf{B}\mathbb{H} \simeq$$

$$\simeq (\mathbf{\Omega}^2_{cl}(-;Lie(\mathbb{G})) \underset{\mathbf{B}\mathbb{G}}{\times} \mathbf{B}\mathbb{H}) \equiv \mathbf{B}\mathbb{H}_{conn}.$$

□

Proposition 3.20. *If $\mathbb{H} \to \mathbb{G}$ a Hodge homomorphism of $(n-1), (m-1)$ truncated braided differentially-cohesive ∞-groups ($n \geq m$). Then, it induces a morphism of stacks:*

$$\begin{tikzcd}
\mathbf{B}\mathbb{H}_{conn^n} \arrow[r, dashed] \arrow[rr, bend left] & \mathbf{\Omega}^1(-;Lie(\mathbb{G}))//\mathbb{H} \arrow[r] & \mathbf{B}\mathbb{G}_{conn}
\end{tikzcd}$$

Remark 3.21. Assume $\mathbb{H} \to \mathbb{G}$ is a Cartan-Maurer homorphism. Notice that by Propositions 3.19, 3.20 we have that on $\mathbb{H}$ (n-1)-truncated differentially-cohesive ∞-group we have a *universal* Hodge filtration F such that for any other Hodge filtration on $\mathbb{H}$ there is a factorization of a stack of $\mathbb{H}$-bundles with connections through the stack of bundles with connections associated with F.

3.3.1. *Whitehead towers, Riemanian geometry.* Now let us turn to discussion of gravity field in a differentially cohesive ∞-topos.

As it is indicated in 1.2 gravity fields are recognized as *twisted cohomology* where a *twist* exhibits some sort of geometry. Usually this geometry comes form the frame bundles 3.1 and the Einstein-Newton's principle of equivalence (which states that locally the effects of gravity can be suppressed by an appropriate choice of a non-inertial reference frame) relies on the fact that smooth manifolds "look the same".

In the previous section we discussed how to give an abstract definition of a *Cartan connection* but it is not enough for defining gravity fields. Indeed, gravity fields (in general relativity) modulate how the Poincare group of symmetries of an empty space breaks down to the group of (local) Lorentz symmetries (i.e. they are reductions of the frame bundle to $O(1,3)$) (this is actually how it was realized that gravity takes place in (pseudo-)Riemanian geometry). Moreover, the deloopings of orhogonal groups produce *Whitehead towers* ([68]) which gives the whole spectrum of structures in M-theory. In particular, it would be interesting to define the analogs of *spin*, *string*-structures in abstract *differentially-cohesive ∞-topoi.* Thus, our plan is:

(1) Given a geometrically cohesive ∞-group V in $\mathbf{H}$ we discuss how the orthogonal groups $O(V)$ can be defined and thus the Riemanian geometry abstractly.
(2) Construct isometry groups $Iso(V)$ under the assumption that V is *internalizable.*
(3) There difficulties with defining the analogs of Lorentz groups directly, since there are no evident universal characterizations of them (which is the case for the orthogonal groups). Nonetheless, we suggest another approach using the abstract $Spin(V)$ groups.
(4) Construct *geometric Whitehead towers* for $\mathbf{B}O(V)$ and using the Milnor's theorem we extract the abstract analogs of $Spin(V), String(V), \dots$ ∞-groups.

The classical result stating that *every smooth manifold admits a Riemann metric* should pertain generally. By classical structure theorems on Lie groups $O(n) \subset GL(n)$ is characterized (up to a conbjugation) by the property of being *maximal compact subgroup.* $O(n)$ inclusion is a homotopy equivalence (deformation retract $GL(n) \to O(n)$ is the Gramm-Schmidt process).

- Frame bundles are discussed in ([69], 5.3.11.), where $GL(V)$ groups are defined. Frame bundle construction discussed before is a functor

$$VMfd^{et}_{\mathbf{H}} \underset{\tau}{\to} \mathbf{H}_{\mathbf{B}GL(V)} \tag{7}$$

  (Prop. 5.3.101.).
- In 5.3.103. Schreiber suggests that in order to define *cohesive* $\mathbf{g}$-isometry groups, where $\mathbf{g}$ is a G-structure on a V-manifold, one needs to refine the above functor at $X \in VMfd^{et}_{\mathbf{H}}$ in that there is:

$$\tau^X_{(-)} : \mathbf{Aut}(X) \to \mathbf{Aut}_{\mathbf{H}}(\tau_X) \tag{8}$$

  a homomorphism of cohesive automorphism groups. Seemingly it can only be defined in concrete models so far. If for a V-manifold X this refinement exists X is called *internalizable*

- G-structures are def. in 5.3.12. and G-isometries in 5.3.15. Cohesive isometry groups are defined via pullback:

$$\begin{tikzcd} Iso(X,\mathbf{g}) \arrow[r] \arrow[d, "p"'] & Aut_{\mathbf{H}}(\mathbf{g}) \arrow[d] \\ Aut(X) \arrow[r, "\tau^X_{(-)}"'] & Aut_{\mathbf{H}}(\tau_X) \end{tikzcd}$$

Proposition 3.22. *There is a homomorphism of ∞-groups $\triangle : V \to GL(V)$ (which we a call a diagonal).*

Proof. $V \to Aut(V)$ is defined by the group structure on V. Then, using $GL(V) \equiv Aut(\mathbb{D}^V_e)$ and that $\mathbb{D}^V_e$ is a pullback we insert $Aut(V) \to Aut(\mathbb{D}^V_e)$. □

We suggest only some way to define orthogonal groups. Internalization process always accompanies some ambiguity. If one has a candidate for an orthogonal group then one may proceed further with the constructions of this section.

Definition 3.23. Let $i : O(V) \to GL(V)$ an (1-monomorphism) of ∞-groups is said to exhibit a V-orthogonal group if the following are satisfied:

(1) (Every V-manifold admits a V-orthogonal structure) i is an $\int$-equivalence
(2) V is *internalizable* (3.3.1)
(3) There is a canonical $O(V)$-structure on V (a euclidean metric):

$$\begin{tikzcd} & \mathbf{B}O(V) \arrow[d, "Orth"] \\ V \arrow[ur, "\tau_O"] \arrow[r, "\tau_{fr}"'] \arrow[ur, phantom, shift right=3, "\Longrightarrow"] & \mathbf{B}GL(V) \end{tikzcd}$$

(4) A homotopy fiber of $w_1 : \mathbf{B}O(V) \to \int_{(1)} \mathbf{B}O(V) \equiv \tau_{\leq 1} \int \mathbf{B}O(V) = \tau_{\leq 1} BO(V)$ is a delooping of some ∞-group which we denote as $SO(V)$ (see below 3.3.1).
(5) ...

Definition 3.24. If $i : O(V) \to GL(V)$ exhibits a V-orthogonal group. Then, by a pullback 3.3.1 we define $Iso(V) := Iso(V, \tau_O)$.

Notice, if V is geometrically contractible then $\int O(V) \to \int Iso(V), BO(V) \to BIso(V)$ are equivalences of ∞-groupoids.

Lemma 3.25. *The following is a homotopy fiber sequence:*

$$\Omega_{\tau_O}[\tau_{fr}, O]_{\mathbf{H}} \to Iso(V) \to Aut(V)$$

Proof. Apply 5.3.15. [69]. □

Now, recall from ([69],) the Whitehead tower (vertical tower) constructed for each X in an ∞-*connected* ∞-*topos* (every square is a pullback and $\int_{(n)} \equiv \tau_{\leq n} \int$):

$$\begin{tikzcd}
\cdots \arrow[d] & & & & & & & \\
X^{(n)} \arrow[rrr] \arrow[d] & & & * \arrow[d] & & & & \\
X^{(n-1)} \arrow[rrr] \arrow[d] & & & B^n \int_{(n)}(X) \arrow[rrr] \arrow[dddd] & & & * \arrow[d] & \\
\cdots \arrow[d] & & & & & & \cdots \arrow[ddd] & \\
X^{(1)} \arrow[d] & & & & & & & \\
X^{(0)} \arrow[rrrrrrrr] \arrow[d] & & & & & & & & & B \int_{(1)}(X) \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & \int(X) \arrow[r] & \cdots \arrow[r] & \int_{(n)}(X) \arrow[rrr] & & & \int_{(n-1)}(X) \arrow[r] & \cdots \arrow[rr] & & \int_{(1)}(X) \arrow[r] & \int_{(0)}(X)
\end{tikzcd}$$

Then apply to $\mathbf{B}O(V)$ and obtain:

$$\cdots \to \mathbf{B}Fivebrane(V) \to \mathbf{B}String(V) \to \mathbf{B}Spin(V) \to \mathbf{B}SO(V) \tag{9}$$

with abstractly defined characteristic classes:

$$\mathbf{B}SO(V) \to \mathbf{B}O(V) \stackrel{w_1}{\to} \int_{(1)} \mathbf{B}O(V) \tag{10}$$

$$\mathbf{B}Spin(V) \to \mathbf{B}SO(V) \stackrel{w_2}{\to} \int_{(2)} \mathbf{B}O(V) \tag{11}$$

and so on.

Remark 3.26. Notice that even if we do not construct $\ldots, Spin(V), SO(V)$ groups we may nonetheless define $\ldots, Spin(V), SO(V)$-structures.

4. Local prequantum field theories synthetically

In this section we shall review the basics of a modern point of view on variational theory - the cornerstone of field theory and prequantum geometry. Then, we apply the ideas to define prequantum field theories abstractly in arbitrary differentially-cohesive ∞-topoi.

From now on let Σ denote a smooth $p+1$-manifold. For the notations used for notable categories appearing especially in the variational diagram 4.0.6 see below 4.0.2.

4.0.1. One of the most important aspects of fundamental physics is the notion of the *Lagrangian* and the action functional defining loci of sections (configurations of fields) satisfying the Euler-Lagrange equations.

Field theories are defined in terms of fiber bundles $E \to \Sigma$ with given systems of Lagrangians forms on jet bundles $(\mathcal{L}_i) : J^\infty E|_{U_i} \to \mathbf{\Omega}^{p+1}$

$$\mathcal{L}_i \equiv L_i(x^k, u^k, u^k_\alpha) dx^1 \wedge \cdots \wedge dx^{p+1} \tag{12}$$

, where $\{U_i\}$ is an atlas of Σ, u^k_α denote local jet coordinates ([8] Chapter One). The pullback along a section ϕ with a compact support defines locally on Σ a form, the integral of which gives the value of the action functional on the given configuration:

$$\begin{tikzcd}
J^{\infty}E \arrow[d] \\
E \arrow[d] \\
\Sigma \arrow[u, bend left, "\phi"] \arrow[uu, dashed, bend right, "j^{\infty}\phi"']
\end{tikzcd}$$

$$\mathbf{S} : \phi \mapsto \int_{supp\{\phi\}} j^{\infty}(\phi)^{*}\mathcal{L} \tag{13}$$

The Lagrangian is a fundamental concept and the formulation only through variational problem on action functionals has various technical issues:

(1) $\mathbf{S}$ correctly are defined only locally, whilst globally only the exponentiated action functionals $e^{i\mathbf{S}}$ are defined ([69], 1.1.1.3). The obstructions to globality (see below or [8]) are known and in general one works with refined local systems of Lagrangians to Cech-Deligne cocycles.

As indicated in [3] finding such refinements is a necessary procedure for preparation of a theory to quantization [69] and $\mathbf{S}$ are obtained via transgression ([69] 1.1.1, [23] 2.8, and for fiber integration used for transgression [29]).

(2) Despite giving the right Euler-Lagrange equations some of the action functionals turn out to be incorrect from the point of view of the variational theory. For instance, the famous Hilbert-Einstein action functiona is not correct ([6], pp 16) when we pose the problem of finding the field configurations (components of metric tensors) since it contains second derivatives of fields. The issue is known to be resolved by Gibbons-Hawking-York action or by the first-order formulations [6].

In its classical form, the formalism for pro-manifolds over smooth manifolds has been constructed and we will state its main concepts using the [69], [47], [28], [67], [78]. Our goal is to trace how classical aspects of variational theory fit into higher topos theory, which will lead us to a variational diagram 4.0.6 as a result.

4.0.2. In the following definition we remind the well established notation in literature [47]:

Definition 4.1.
- $CartSp \subset SmMfd$ - sites of Cartesian spaces and smooth manifolds. These sites produce the same ∞-*topoi*, but when working with slices it is convenient to use smooth manifolds or formal smooth manifolds (**??**).
- *LocProMfd* - projective limits of finite dimensional smooth manifolds
- *InfThPoint* $\subset CALG_{\mathbb{R}}^{op}$ - full subcategory of the opposite of the category of commutative $\mathbb{R}$-algebras spanned by duals (denote duals by *Spec*) of Weil algebras (*Spec*($\mathbb{R} \oplus \mathcal{V}$), where $\mathcal{V}$ is a nilpotent ideal.

  As it is known that ∞-stacks over *InfThPoint* are identified with properly infinitesimal homotopy types in the ∞-topos of *formal smooth* ∞*-groupoids* (in particular the underlying homotopy types of ∞-*Lie algebras*).
- *FormalCartSp* $\subset CALG_{\mathbb{R}}^{op}$- spanned by duals of $\mathbb{R}$-algebras of smooth functions of Cartesian spaces infinitesimally thickened (*Spec*($\mathbb{C}^{\infty}(\mathbb{R}^n) \oplus_{\mathbb{R}} (\mathbb{R} \oplus \mathcal{V})$).

  It is endowed with a Grothendieck topology induced by good open covers of Cartesian spaces in *CartSp* site.
- *Sh*(*FormalCartSp*) $\equiv$ *FormalSmoothSets* - formal smooth sets. This 1-topos is known in literature as Cahiers topos [47] and serves as a 0-truncated base for the *synthetic differential geometry*.
- Sh_{∞}(*FormalCartSp*) $\equiv$ *FormalSmooth-*∞*-groupoids*
- *Diffeological*. It is known that diffeological spaces are precisely the concrete 0-truncated *smooth* ∞*-groupoids*.

- $LocProMfd_{\downarrow\Sigma}$ - category spanned by fibered pro-manifolds $\pi : E \to \Sigma$ where π is a surjective submersion ([47], 3.18).
- $DiffOp_{\downarrow\Sigma}(LocProMfd)$ - category of differential operators over Σ. Its objects are that of $LocProMfd_{\downarrow\Sigma}$ but morphisms are *differential operators*:

$$(D : E \to F) \equiv D : \Gamma_\Sigma(E) \to \Gamma_\Sigma(F) \tag{14}$$

  such that there is a *formal differential operator* $\hat{D} : J^\infty E \to F$ so that the following diagram commutes:

$$\begin{tikzcd}
J^\infty E \arrow[d] \arrow[drr, "\hat{D}"] & & \\
E \arrow[d] & & F \\
\Sigma \arrow[u, bend left, "\phi"] \arrow[uu, dashed, bend right, "j^\infty(\phi)"'] \arrow[urr, "D(\phi)"'] & &
\end{tikzcd}$$

4.0.3. By the classical works of Vinogradov [78], [79] it has become possible to treat local variational problems cohomologically ([8]). The main object to study has become the variational bicomplex (see in details [8], pp 22; [69] Proposition 1.3.22). Its construction procceds by considering the chain complex (this is actually the sheaf of chain complexes on the site $DiffOp_{\downarrow\Sigma}(LocProMfd)$ 4.0.2 (see [69], 6.5.10) of differential forms on jet bundles $\mathbf{\Omega}^*(J^\infty E)$.

It has a filtration realizing $\mathbf{\Omega}^*(J^\infty E)$ as a total complex of the so called variational bicomplex (see [8], Def. 1.7):

$$\mathbf{\Omega}^*(J^\infty E) = \bigoplus_{r+s=*} \Omega^{r,s}, d = d_H + d_V \tag{15}$$

There emphasize the following three types of differential forms defining this filtration:

- (Horizontal forms Ω_H) $J^\infty E$ is endowed with an atlas of a pro-manifold [8]. In this atlas horizonal forms are those that locally defined on $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ basis elements coming form the local basis on Σ:

$$L_{i_1,\ldots,i_k}(x^i, u^i, u^i_\alpha) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{16}$$

- (Vertical Ω_V) Vertical forms are spanned by the basis of forms ([8]):

$$\theta^{\alpha_1}_{I_1} \wedge \cdots \wedge \theta^{\alpha_s}_{I_s} \wedge \eta \tag{17}$$

  where

$$\theta^\alpha_I = du^\alpha_I - u^\alpha_{I,i} dx^i \tag{18}$$

  η - any form.
- (Source forms) $\Omega^{p+1,r}_S$ defined via the Euler operator I (yielding the Euler-Lagrange equations for the case $r = 1$):

$$\Omega^{p+1,r}_S \equiv \mathfrak{I}(\Omega^{p+1,r})$$

  The Euler derivative is defined via

$$\delta_V \equiv \mathfrak{I} \circ d_V$$

Source forms are used to extend the complex of *horizontal* differential forms on $J^\infty E$ to the Euler-Lagrange complex $\mathbf{\Omega}^*_{EL_\Sigma}$ (see [69], Def. 6.5.93):

$$0 \longrightarrow \Omega^0_H \xrightarrow{d} \Omega^1_H \longrightarrow \ldots \longrightarrow \Omega^{p+1}_H \xrightarrow{\delta_V} \Omega^{p+1,1}_S \longrightarrow \Omega^{p+1,2}_S \longrightarrow \ldots$$

4.0.4. Vinogradov constructed ([77], [79]) the category 4.0.2 of partial differential equations. Marvan then realized in [57] that the functor:

$$J^{\infty}_{\Sigma} : LocProMfd_{\downarrow\Sigma} \to LocProMfd_{\downarrow\Sigma} \tag{19}$$

has a structure of *comonad* (which we refer to as jet comonad):

$$\triangle : J^{\infty}_{\Sigma} \to J^{\infty}_{\Sigma} J^{\infty}_{\Sigma}, \mu : J^{\infty}_{\Sigma} \to 1 \tag{20}$$

with $\triangle$ being comultiplication and a counit μ . In [57] it was shown that the Vinogradov's category PDE_{Σ} is equivalent to the *Eilenberg-Moore category of coalgebras over the jet comonad* J^{∞}_{Σ} (see 4.0.2).

From this equivalence the category of differential operators over Σ is realized as *co-Kleisli category of* J^{∞}_{Σ} (4.0.2, [47]):

$$\begin{tikzcd}
EM(J^{\infty}_{\Sigma}) \arrow[rr, shift left, "\sim"] & & PDE_{\Sigma} \arrow[ll, shift left] \\
Kl(J^{\infty}_{\Sigma}) \arrow[u, hook] \arrow[r, shift left, "\sim"] & DiffOp_{\downarrow\Sigma}(LocProMfd) \arrow[l, shift left] \arrow[ur, hook]
\end{tikzcd}$$

4.0.5. The theory of partial differential equations, variational calculus belong to the framework of differential geometry. Following the *internalization* methodology one may ask in what system of axioms for toposes one may produce aspects of differential geometry abstractly.

Now we turn to the *Cahiers topos* 4.0.2 *FormalSmoothSets* which was initially constructed in [20] by Dubuc as a model for the Kock-Lawvere axioms. In [47] it was suggested that it is the property of *differential cohesion* of that may be used to extend the theory partial differential equations 4.0.4 to *formal smooth sets.*

Remark 4.2. In 4.1 below we shall use the non-truncated version of the results formulated further using only the axioms of *differential cohesion.* This allows us to study differential geometry in abstract *differentially-cohesive ∞-topoi.*

This is the point for us hinting that axioms of *differential cohesion* may be enough even to cover the prequantum field theory 4.1. We saw 3.6 how Cartan connections and thus gravity fields *internalize* within the *differentially-cohesive ∞-topoi.*

The following sequence of claims should convey the essence of the theory of *generalised PDE's.* The point for us is to follow up whether the main claims extend to a non-truncated case.

The key fact of the usage of V-manifolds (V is a *differentially-cohesive* ∞-group) is that they have a (see [47], 3.3) a local trivialization of an *infinitesimal disk bundle.* In [47] this allowed to extend the *Cahiers topos* the approach by Marvan [57] to theory of *PDE's*:

(1) Axioms of differential cohesion on the *Cahiers topos* result in production of a comonad (2.1.3) J^{∞}_{Σ}
(2) If Σ is a V-manifold, where V is a group object in the *Cahiers topos*, then one may define the category $PDE_{/\Sigma}(\mathbf{H})$ of *formally integrable PDE's* over Σ ([47], Def.3.47)
(3) By [47] Th.3.52. an extension of the result by Marvan is given:

$$PDE_{/\Sigma}(\mathbf{H}) \simeq EM(J^{\infty}_{\Sigma})$$

(4) For a smooth manifold Σ Vinogradov's category of *PDE's* is equivalent to $EM(J^{\infty}_{\Sigma}|_{LocaProMfd_{\downarrow\Sigma}})$ (Cor.3.55. [47])
(5) One may see that the results pertain to the non-truncated case in the ∞-topos of *formal smooth ∞-groupoids*

4.0.6. *Variational diagram.* The cornerstone of the prequantum geometry are constructions that are captured in the form of what we call *the variational diagram.* Most of the results may be found in [3], [69], [47], we summarise them and make some new observations concerning notable functors:

$$
\begin{tikzcd}
Sh_\infty(SmMfd) \arrow[d] \arrow[rrr, bend right=5] & & & Sh_\infty(DiffOp_{\downarrow\Sigma}) \arrow[lll, bend right=10] \arrow[d, hook] \\
Sh_\infty(SmMfd_{/\Sigma}) \arrow[u, bend left=40] \arrow[u, bend right=40] \arrow[rr] \arrow[d, "\sim"'] \arrow[rr, hook, bend right=30] & & Sh_\infty(FormalSmMfd_{/\Sigma}) \arrow[ll, bend right=10] \arrow[ll, bend left=20] \arrow[ru] \arrow[d, "\sim"'] \arrow[ul, phantom, "(1)"] & Sh_\infty(PDE_\Sigma) \arrow[d, "\sim"'] \arrow[dl, phantom, "(2)"] \\
(\mathbf{H}_\Re)_{/\Sigma} \arrow[dd, shift right] \arrow[dd, shift left=3] & & \mathbf{H}_{/\Sigma} \arrow[dl, bend right, "\Sigma_!"'] \arrow[dd, bend right] \arrow[dd, bend left] & Sh_\infty(FormalSmMfd_{/\mathfrak{J}\Sigma}) \arrow[ddl, "\sim"'] \\
& \Pi \arrow[ul, phantom, "(3)"] & & \\
\mathbf{H}_\Re \arrow[uu] \arrow[r, hook, bend left, "i_!"] \arrow[r, bend right, "i_*\simeq p^*"'] & \mathbf{H} \arrow[l, "i^*"'] \arrow[l, bend left=60, "p_*"] \arrow[uur, "\Sigma^*"] & \mathbf{H}_{/\mathfrak{J}\Sigma} \arrow[uu] \arrow[d, "(4)\sim"'] & \\
& & EM(J^\infty_\Sigma) \arrow[d, "(5)\sim"'] & \\
& & PDE_{/\Sigma}(H) &
\end{tikzcd}
$$

(1) The geometric morphism $Sh_\infty(FormalSmMfd_{/\Sigma}) \to Sh_\infty(FormalSmMfd_{/\mathfrak{J}\Sigma})$ descends to the adjunction:

$$FormalSmMfd_{/\Sigma} \underset{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}{\overset{(\eta_\Sigma)_!}{\rightleftarrows}} FormalSmMfd_{/\mathfrak{J}\Sigma}$$

(2) When dealing with bundles we shall need to calculate the functor $(\eta_\Sigma)_* : \mathbf{H}_{/\Sigma} \to \mathbf{H}_{/\mathfrak{J}\Sigma}$ which in general is difficult to understand. For instance consider 4.19 where the right adjoint to a pullback functor is $[G, -]//G$. Nonetheless, here we can simplify it:

Lemma 4.3. $(\eta_\Sigma)_* \simeq ((-) \circ (\eta_\Sigma)^*|_{FSmMfd_{/\mathfrak{J}\Sigma}})$.

Proof. Since both $(\eta_\Sigma)^*$ and $Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}$ preserve colimits and every ∞-stack is a homotopy colimit of representables ([21]) it suffices to verify whether functors coincide at representables.

For, let y denote the Yoneda embedding and consider evaluation of the left Kan extension

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}}(y(K \to \mathfrak{J}\Sigma))(X \to \Sigma)$$

Further denote the restriction of the pullback functor to sites as $(\eta_{\Sigma)^*}$ Using the left Kan extension formula (see the "Pointwise Kan extensions" [65], [49]) this is equivalent to the colimit along the following category:

$$((X \to \Sigma) \downarrow (\eta_\Sigma)^*)^{op}$$

Its objects are $((H \to \mathfrak{J}\Sigma), (X \to \Sigma) \to (\eta_\Sigma)^*(H \to \mathfrak{J}\Sigma))$.

We notice that by the universal property of unit there is a final object:

$$(\eta_\Sigma)_!(X \to \Sigma), (X \to \Sigma) \underset{unit}{\to} (\eta_\Sigma)^*(\eta_\Sigma)_!(X \to \Sigma)$$

Thus, the left Kan extension has a value:

$$\simeq y(K \to \mathfrak{J}\Sigma)((\eta_\Sigma)_!(X \to \Sigma))$$

$$\simeq \mathbf{H}_{/\mathfrak{J}\Sigma}((\eta_{\Sigma !})(X \to \Sigma), K \to \mathfrak{J}\Sigma)$$

$$\simeq \mathbf{H}_{/\Sigma}((X \to \Sigma), (\eta_\Sigma)^*(K \to \mathfrak{J}\Sigma))$$

$$\simeq ((\eta_\Sigma)^* y(K \to \mathfrak{J}\Sigma))(X \to \Sigma)$$

On the other hand, we have:

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}} \simeq (\eta_\Sigma)^* \dashv (\eta_\Sigma)_*$$

$$Lan_{((\eta_\Sigma)^*)|_{FSmMfd_{/\mathfrak{J}\Sigma}}} \dashv ((-) \circ (\eta_\Sigma)^*|_{FSmMfd_{/\mathfrak{J}\Sigma}})$$

which by essential uniqueness of adjoints proves the point. □

(3) Firstly consider $Sh_\infty(SmMfd_{/\Sigma}) \to \mathbf{H}_{\Re}$. By Lemma 6.1.3.13. [53] having that

$$\simeq Sh_\infty(SmMfd_{/\Sigma}) \simeq N(([SmMfd^{op}_{/\Sigma}, SSet]_{locproj})^\circ)$$

$$N((([SmMfd^{op}, SSet]_{locproj})_{/\Sigma})^\circ)$$

$$\simeq N((\dots)^\circ)_{/\Sigma} \simeq Sh_\infty(SmMfd)_{/\Sigma}$$

Another a bit nontrivial part of this rectangle is the commutativity of

$$\begin{array}{ccc} (\mathbf{H}_{\Re})_{/\Sigma} & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{H}_{\Re} \\ \downarrow & & \downarrow \\ \mathbf{H}_\Sigma & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & \mathbf{H} \end{array}$$

which follows from above note and that at level of sites, if we have a coreflection

$$C \underset{\leftarrow p -}{\overset{- i \to}{\rightleftarrows}} D$$

then for any $\Sigma \in C$ it extends to a coreflection

$$C_{/\Sigma} \underset{\leftarrow p_\Sigma -}{\overset{- i_\Sigma \to}{\rightleftarrows}} D_{/p\Sigma}$$

(4) This equivalence is notable. As indicated in the previous section this equivalence establishes the foundation of the theory of *formal generalised PDE's* and thus the prequantum field theory ([69], 6.3). Ideologically it says that bundles over an infinitesimal neighbourhood $\mathfrak{J}\Sigma$ of a smooth manifold encode equivalently as coalgebras of the *jet comonad* whose underlying objects are bundles over Σ.

When discussing examples of prequantum field theories we shall ussualy encounter *free coalgebras* which belong to the *Kleisli category* $KL(J^\infty_\Sigma)$ 4.0.4.

In fact that in $Sh_\infty(FormalCartSp)$ every homotopy type Σ is *formally smooth* and so the equivalence holds not only for smooth manifolds.

We shall discuss an extension of this equivalence to the tangent category $T_{\mathfrak{J}\Sigma}\mathbf{H}$ and the *Eilenberg-Moore category* of the *stable jet comonad*.

(5) This equivalence, as it is again noted in the previous section, holds for V-manifolds Σ, where V is a framed object 3.1 (e.g. a type which has a group structure in the homotopy category level) ([47] Prop. 3.51., Th. 3.52).

4.0.7. *Cech-Deligne (or Cech-de Rham) cocycles.* Here we shall discuss some necessary constructions which we shall use in 5 and motivate further definitions in 4.1 leading us to an abstract definition of $\mathbf{A}_\infty$*-type prequantum field theories* allowing us to build physics within arbitrary differentially-cohesive ∞-topoi.

Prequantum field theories 4.10 are cocycles in *twisted* cohomology in $\mathbf{H}$. Spectra representing such cohomology (and thus *hypercohomology*) are defined via the so-called *Dold-Kan correspondence*. In order to be able to proceed any further we need to briefly remind the necessary constructions which are discussed in ([22] 2.7., [23], [69] 6.4.15.2, [3]).

We shall further use the homological convention for chain complexes:

$$\ldots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

For definition of the Dold-Kan correspondence we address for example to ([51], 1.2.3). The Dold-Kan correspondence establishes an equivalence (which in fact extends to a simplicial Quillen equivalence between the model categories) of the categories of connective chain complexes and simplicial abelian groups. Simplicial abelian groups are included by U into *Kan* complexes ([61], [37]). The combination of these facts results in the right diagram:

$$sSet \underset{U}{\overset{F}{\rightleftarrows}} sAb \underset{\mathcal{DK}}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}(\mathbb{Z}) \qquad \infty Grpd \underset{\mathcal{DK}}{\overset{F}{\rightleftarrows}} N((Ch_{\geq 0}(\mathbb{Z})^\circ)$$

Using

$$N((Ch_{\geq 0}(\mathbb{Z})^\circ) \xrightarrow{\mathcal{DK}} Spectra \qquad N([C^{op}; Ch_{\geq 0}(R)]_{locproj})^\circ) \xrightarrow{\mathcal{DK}} Sh_\infty(C; Spectra)$$

$Sp(\mathbf{H}_{\mathfrak{I}\Sigma}) \simeq Fun^{lex}(\mathbf{H}^{op}_{/\mathfrak{I}\Sigma}, Spectra)$ ([55] 1.Sheaves of spectra, [51] Rem. 1.4.2.9)

The first example is the *Deligne cohomology.* This hypercohomology is defined via the Deligne complex $\mathbf{B}^n U(1)_{conn,chn}$:

$$\begin{array}{lccccccccc} X: & \mathbb{Z} & \hookrightarrow & \mathbb{R} & \longrightarrow & \Omega^1 & \longrightarrow & \ldots & \longrightarrow & \Omega^n \\ \mathbf{B}^n U(1)_{conn,chn}: & 0 & \longrightarrow & U(1) & \longrightarrow & \Omega^1 & \longrightarrow & \ldots & \longrightarrow & \Omega^n \end{array}$$

Where the evident morphism $X \to \mathbf{B}^n U(1)_{conn,chn}$ is quasi-equivalence and thus yields an equivalence of underlying homotopy types. This shall be particularly usefull when prequantizing the *higher U(1)-gauge field theories on a curved spacetime* (see below 5).

It turns out ([69], [22]) that the *differential cohomology diagram* of the sheaf of spectra $\mathbf{B}^n U(1)_{conn}$ can be described at the level of chain complexes:

Lemma 4.4. *(1) $\mathbf{B}^n U(1)_{conn,chn}$ is the homotopy pullback of the diagram of chain complexes:*

$$\begin{tikzcd} (U(1) \arrow[r] \arrow[d, "\frac{1}{2\pi i} dlog"'] & 0 \arrow[r] \arrow[d] & \ldots \arrow[r] & 0) \arrow[d] \\ (\Omega^1 \arrow[r] & \Omega^2 \arrow[r] & \ldots \arrow[r] & \Omega^{n+1}_{cl}) \\ (0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \ldots \arrow[r] & \Omega^{n+1}_{cl}) \arrow[u] \end{tikzcd}$$

(2) At the level of sheaves of spectra there is a homotopy pullback:

$$\begin{tikzcd}
\mathbf{B}^n U(1)_{conn} \arrow[r] \arrow[d] & \Omega^{n+1}_{cl} \arrow[d] \\
\mathbf{B}^n U(1) \arrow[r] & \flat \mathbf{B}^{n+1}\mathbb{R} \simeq \flat_{dR}\mathbf{B}^{n+1}U(1)
\end{tikzcd}$$

Proof. The model category of chain complexes with projective structure is right proper (since all objects are fibrant) so to construct the homotopy pullback it suffices to find a fibrant replacement of some map in our diagram ([37], 2). This fibrant replacement is constructed in [69], Definition 6.4.96. □

One immediately notices from a note on differential cohomology diagram 2.1 that in fact $\mathbf{B}^n U(1)_{conn}$ is a differential refinement of the $U(1)$-cohomology.

Now, we apply the functor :

$$\begin{tikzcd}
T\mathbf{H} \arrow[r, "\Sigma^*"] \arrow[rr, bend left, "(-)_\Sigma:"] & T\mathbf{H}_{/\Sigma} \arrow[r, "(\eta_\Sigma)_*"] & T\mathbf{H}_{/\mathfrak{J}\Sigma} \\
Sp(\mathbf{H}) \arrow[r] \arrow[u] \arrow[d, shift left, "\Omega^\infty_+"] & Sp(\mathbf{H}_{/\Sigma}) \arrow[r] \arrow[u] \arrow[d, shift left] & Sp(\mathbf{H}_{/\mathfrak{J}\Sigma}) \arrow[u] \arrow[d] \\
\mathbf{H} \arrow[r, "\Sigma^*"] \arrow[u, shift left, "\Sigma^\infty_+"] & \mathbf{H}_{/\Sigma} \arrow[r, "(\eta_\Sigma)_*"] \arrow[u, shift left] & \mathbf{H}_{/\mathfrak{J}\Sigma}
\end{tikzcd}$$

to the spectrum $\mathbf{B}^n U(1)_{conn}$ and its defining diagram and obtain in $T_{\mathfrak{J}\Sigma}\mathbf{H}$:

$$\begin{tikzcd}
 & (\Omega^{p+1}_{cl})_\Sigma \arrow[dr] & \\
(\mathbf{B}^{p+1}(U(1))_{conn})_\Sigma \arrow[ur, "curv"] \arrow[dr] & & (\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma \\
 & (\mathbf{B}^{p+1}(U(1)))_\Sigma \arrow[ur] &
\end{tikzcd}$$

Further the so-called *variational Poincare lemma* comes into play. Its existence is the crucial property which shall be captured by us when fomralizing the *local prequantum field theories* within the *differentially-cohesive* ∞-topoi.

Lemma 4.5. *([69], [8]) Let $E \in LocProMfd_{\downarrow\Sigma}$ then the map $\Omega^*(J^\infty_\Sigma E) \to \mathbf{\Omega}^*_{EL_\Sigma}$ is a quasi-isomorphism. Moreover, this is the weak equivalence of sheaves of chain complexes on $DiffOp_{\downarrow\Sigma}$.*

([69] 6.5.10, or [3]) Applying the left Kan extension (4.0.6, (2)) to $E \mapsto \mathcal{DK}(\mathbf{\Omega}^*_{EL_\Sigma}(E))$ one constructs the spectrum which sits in the homotopy exact diagram in $Sp(PDE_\Sigma(\mathbf{H}_\mathfrak{R})) \simeq Sp((\mathbf{H}_\mathfrak{R})_{/\Sigma})$:

$$\begin{tikzcd}
 & \Omega^{\bullet \leq p+1}_H \arrow[rr, "\delta_V"] \arrow[dr] & & \Omega^{p+1,1}_{S,cl} \arrow[dr] & \\
(\flat\mathbf{B}^{p+1}\mathbb{R})_\Sigma \arrow[ur] \arrow[dr] & & \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn} \arrow[ur, "curv"] \arrow[dr] & & (\flat\mathbf{B}^{p+2}\mathbb{R})_\Sigma \\
 & (\flat\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}))_\Sigma \arrow[rr, "(\beta)_\Sigma"] \arrow[ur] & & (\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}))_\Sigma \arrow[ur] &
\end{tikzcd}$$

Corollary 4.6. *There is an equivalence of spectra* $(\flat \mathbf{B}^{p+2}\mathbb{R})_\Sigma \simeq \Omega^*_{EL_\Sigma, cl}$

A morphism from a bundle E in $DiffOp_{\downarrow\Sigma} \simeq KL(J^\infty_\Sigma)$: $(\eta_\Sigma)^*(E) \in \mathbf{H}_{/\Sigma} \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ is given equivalently by the

$$\mathbf{L} \equiv (\overset{(i)}{\mathcal{L}}_{p+1})_i, (\overset{(i_1 i_2)}{\kappa_p}{}_{i_1 i_2}), \ldots, (\overset{(i_0 \ldots i_{p+1})}{\kappa_0})_{i_0 \ldots i_{p+1}} \tag{21}$$

For an arbitrary bundle $E \in \mathbf{H}_{/\Sigma}$ with the help of the Lemma 4.3 one may realize that

Example 4.7. Let $\mathcal{G} \in \mathbf{H}$ be a classifying stack of fields, then the corresponding bundle $\Sigma^*(\mathcal{G})$ we further push-forward to $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G})) \in \mathbf{H}_{/\mathfrak{J}\Sigma}$. Using the formulas in Lemma 4.3 we conclude that at the sheaf level $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G}))(X \to \mathfrak{J}\Sigma) \simeq \mathcal{G}((\eta_\Sigma)^* X)$. In particular, probing $(\eta_\Sigma)_*(\Sigma^*(\mathcal{G}))$ by $x : * \to \mathfrak{J}\Sigma$ is equivalent to giving the map from an infinitesimal disk:

$$\mathbb{D}^\Sigma_x \to \mathcal{G}$$

Usually we discuss $\mathcal{G} \in \mathbf{H}_{\mathfrak{R}} \subset \mathbf{H}$ and these probes correspond to giving a germ of field configurations and their derivatives at x (up to an order of infinitesimals exhibited by the differential cohesion of $\mathbf{H}$)

Example 4.8. A bundle for a G-gauge field theory reads as:

$$\mathcal{G}^\infty \times (\eta_X)_*(X^*(\mathbf{B}G_{conn})) \equiv \Xi \tag{22}$$

is the bundle for the Yang-mills theory on a spacetime, where G^∞ is the bundle for gravity.

Example 4.9. Sigma-model with brane Σ:

$$\Sigma \times G \to \Sigma \tag{23}$$

and thus, the classifying stack of fields is $\mathbb{R}$ which gives $(\eta_\Sigma)_*(\Sigma^*(G))(* \to \mathfrak{J}\Sigma) \simeq (\mathbb{D}^\Sigma_x \to G)$ which is equivalent to giving a germ of G-valued functions and their derivatives at a point x. This is used by us when we discuss the *4d WZW* prequantum theory 5.1. **Notice:** in literature (and we follow that) we use the equivalence (4.0.6,(4)) and identify the bundles $\eta_{\Sigma *}(E)$ with $J^\infty_\Sigma E \in EM(J^\infty_\Sigma)$.

Moreover, as we shall need those we introduce the following towers :

(1)

$$\begin{tikzcd}
 & \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn} \arrow[d] \\
 & \mathbf{B}(\mathbf{B}^{p}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}) \arrow[d] \\
 & \cdots \arrow[d] \\
 & (\mathbf{B}^{p}(\mathbf{B}(\mathbb{R}/\hbar\mathbb{Z})_{conn}) \arrow[d] \\
E \arrow[uuuur, dashed, bend left] \arrow[uuur, bend left, "L"] & (\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}))_\Sigma
\end{tikzcd}$$

where $\mathbf{B}^k(\mathbf{B}^{p+1-k}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn})$ is defined by the sheaf of chain complexes:

$$(\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \Omega^1_H \longrightarrow \ldots \longrightarrow \Omega^k_H \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0)$$

These correspond to choices of such L such that κ are constructed till the k-th elements: $((\overset{(i_0 \ldots i_{p+1-k})}{\kappa_k})_{i_1 \ldots i_{k+1}}, \ldots, (\overset{(i_1 \ldots i_{p+1})}{\kappa_0})_{i_0 \ldots i_{p+1}})$

(2) Another tower is a factorization of the atlas $\Omega^{p+1}_H \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$:

$$\begin{tikzcd}
 & \Omega^{p+1}_H \arrow[d] \\
 & \Omega^{p\leq p+1}_H \arrow[d] \\
 & \cdots \arrow[d] \\
 & \Omega^{q\leq p+1}_H \arrow[d] \\
 & \cdots \arrow[d] \\
 & \Omega^{1\leq p+1}_H \arrow[d] \\
X \arrow[r, "\mathbf{L}"'] \arrow[uuuuuur, dashed, bend left] \arrow[uuur, dashed, bend left] & \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}
\end{tikzcd}$$

A lift of $\mathbf{L}$ to $\Omega^{q\leq p+1}_H$ is equivalent to finding a Cech-Deligne cocycle 4.10 with vanishing $\kappa_{\leq q-1}$ gluing forms (see for such an example a prequantization of effective electrodynamics with a current 4.12, or the case of prequantized *4d-WZW* coupled to gravity via the Kahler form 5.1).

From [69], 6.5.11 we recall the Definition 4.10 and for the sake of generality (as it is done with the theory of *generalised PDE's* in [47]) extend the notion to *formal bundles*.

Definition 4.10. A prequantum field theory is the following data:

(1) A bundle $E \in \mathbf{H}_{/\Sigma}$ over a (p+1)-dimensional smooth manifold Σ
(2) A Cech-Deligne cocycle $L : (\eta_\Sigma)_* E \to \mathbf{B}^{p+1}_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ in $\mathbf{H}_{\mathfrak{J}\Sigma} \simeq Sh_\infty(PDE_\Sigma)$, which is also called as *pre-quantum local Lagrangian*.

If we put any bundle in $E \in \mathbf{H}_{/\mathfrak{J}\Sigma}$ we extend the notion to what we suggest to call as *formal pre-quantum field theories*.

Remark 4.11. Notice that considering bundles over Σ and including them to $\mathbf{H}_{/\mathfrak{J}\Sigma}$ is equivalent to regarding them as free coalgebras under the equivalence (4) in 4.0.6. Thus, the *formal pre-quantum field theories* extend the notion to arbitrary coalgebras of the jet comonad J^∞_Σ.

Example 4.12. In Theorem 4.15 we construct a Cech-Deligne cocycle $\mathcal{L}$ which lifts:

$$\begin{tikzcd}
 & \Omega^{2\leq 4}_H \arrow[d] \\
 & \Omega^{1\leq 4}_H \arrow[d] \\
X \arrow[r, "\mathbf{L}"'] \arrow[uur, dashed, "\mathcal{L}"] & \mathbf{B}^4_H(\mathbb{R}/\hbar\mathbb{Z})_{conn}
\end{tikzcd}$$

Prequantization is required for example in the *effective electrodynamics* (effectiveness means that the current J is prescribed and thus the theory is incorrect when is considered in gravity context, since the Energy-Momentum tensor is not gauge invariant and thus is not observable) Let $X \simeq \mathbb{R}^3 \setminus \{0\} \times \mathbb{R}$ be a spacetime which has a homotopy type of $\mathbb{S}^2$. For instance consider the following form on $\mathbb{R}^3 \setminus \{0\}$

$$\frac{1}{4\pi} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^3)^{\frac{3}{2}}} \equiv \omega \tag{24}$$

pullback this form to X. This form represents an integral cocycle in $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \simeq H^2_{dR}(X)$, and thus using the pullback for differential cohomology 4.4 we define a field configuration $\mathcal{A} \equiv ((\overset{(i)}{\mathcal{A}}), (\overset{(ij)}{\phi}), (\overset{(ijk)}{\lambda}))$ for electrodynamics such that the topology of the underlying bundle $[\mathcal{A}]$ is $c_1([\mathcal{A}]) = 1 \in H^2(X;\mathbb{Z})$ which, using the quasi-isomoprhism **??** again, is a cohomology class of the $\mathbb{Z}$ 2-cocycle $(\overset{(ijk)}{\lambda})$.

Electrodynamics, as is any Yang-Mills theory 22, is defined on a bundle mixed with gravity:

$$\mathcal{G}^{\infty} \times (\eta_X)_*(X^*(\mathbf{B}U(1)_{conn})) \equiv \Xi$$

For the Hodge star operation we use the following notation:

$$(*F)_{j_{k+1},\dots,j_n} = \frac{\sqrt{g}}{k!} F_{i_1,\dots,i_k} g^{i_1 j_1} \dots g^{i_k j_k} \epsilon_{j_1,\dots,j_n} \tag{25}$$

where g^{ij} are the components of the metric tensor Thus, for the curvature form we have

$$\sum_{i<j} (*F)_{ij} dx^i \wedge dx^j = \sum_{i<j} \sqrt{g} F^{km} \epsilon_{kmij} dx^i \wedge dx^j \tag{26}$$

and analogously one has the current 3-form:

$$j_{el} \equiv *j = j^0 dx^1 \wedge dx^2 \wedge dx^3 - j^1 dx^0 \wedge dx^2 \wedge dx^3 + \dots \tag{27}$$

The covariant Maxwell's equations satisfy:

$$d(*F) = j_{el} \tag{28}$$

Further we shall assume $\{U_i\}$ is a *good open cover* in that all finite intersections are contractible provided they are non-empty ([69], 3.)

Definition 4.13. Let X be a smooth manifold, $\mathfrak{X}(X)$ a vector space of vector fields on X and choose a good cover on X. Then,

$$H^p(X;U(1)) \times \mathfrak{X}(X) \to H^p(X;\mathbb{R})$$

$$(\alpha, J) \mapsto [(J^{\sigma} \partial_{\sigma}(\overset{(i_0,\dots,i_p)}{\phi})_{i_0,\dots,i_p})] \equiv \alpha j$$

where $((\overset{(i_0,\dots,i_p)}{\phi})_{i_0,\dots,i_p}, (\overset{i_0,\dots,i_{p+1}}{\lambda})_{i_0,\dots,i_{p+1}}) : X \to \mathcal{DK}(\mathbb{Z} \to \mathbb{R} \to \dots \to 0)$ is a representative of a cocycle $\alpha \in H^1(X;U(1))$ We call it the conjunction of J with α.

Lemma 4.14. *The conjunction operation in Definition 4.13 is correct and linear.*

Proof. Indeed, consider a transformation (for the sake of notation we shall prove the case $p = 1$) and choose a good open cover on X:

$$((\overset{(i_0,\dots,i_p)}{\phi})_{i_0,\dots,i_p}, (\overset{i_0,\dots,i_{p+1}}{\lambda})_{i_0,\dots,i_{p+1}}) \to ((\overset{(i_0,\dots,i_p)}{\phi'})_{i_0,\dots,i_p}, (\overset{i_0,\dots,i_{p+1}}{\lambda'})_{i_0,\dots,i_{p+1}})$$

Such transformations are defined by formulas as in ([3], pp 8) and in particular in conjunction we have $J^{\sigma}\partial_{\sigma}(\phi') = J^{\sigma}\partial_{\sigma}(\phi') + J^{\sigma}\partial_{\sigma}\gamma_i - J^{\sigma}\partial_{\sigma}\gamma_j$ and all the integer contributions cancel out under the derivative. This means that the cohomology classes coincide $[\mathrm{J}\partial(\phi)] = [J\partial(\phi')]$. □

Let

$$\omega_g = \sqrt{-g} dx^1 \wedge \dots dx^n$$

denote a volume form.

Theorem 4.15. *Restrict the bundle to $\mathcal{E} \subset \Xi$ to the one where electrodynamical part of sections is valued on-shell and metrics satisfy the constraint $\nabla(j) = 0$. Then, the following defines what we call the on shell prequantization of electrodynamics on curved spacetime:*

$$\overset{(i)}{\mathcal{L}} = \frac{1}{4} F \wedge *F + \overset{(i)}{\mathcal{A}}(j|_{U_i})\omega_g \tag{29}$$

$$\overset{(ij)}{\kappa_3} = \overset{(ij)}{\phi} j_{el}|_{U_{ij}} \tag{30}$$

$$\overset{(ijk)}{\kappa_2} = 2\pi \overset{(ijk)}{\lambda} (*F)|_{U_{ijk}} \tag{31}$$

$$\kappa_1 = 0 \tag{32}$$

$$\kappa_0 = 0 \tag{33}$$

Proof. We verify it directly.

$$\overset{(i)}{\mathcal{L}} - \overset{(j)}{\mathcal{L}} = (\overset{(i)}{\mathcal{A}}|_{U_{ij}}(j|_{U_{ij}}) - \overset{(j)}{\mathcal{A}}|_{U_{ij}}(j|_{U_{ij}}))\omega_g$$

On a good open cover the $U(1)$-cocycle of the underlying bundle is represented by $e^{i\phi_{ij}}$ and thus we have

$$= d\phi_{ij}(j|_{U_{ij}})\omega_g \equiv j^\sigma(\partial_\sigma \phi_{ij})\omega_g \tag{34}$$

This is precisely the conjunction of j with underlying cocycle $[\mathcal{A}] \in H^1(X; U(1))$ (4.13). Thus, an obstruction to existence of a trivialization ϕ such that 34 vanishes is in $[\mathcal{A}]j \in H^1(X; \mathbb{R})$.

We notice further that since j is covariantly constant, use the formula:

$$\frac{1}{\sqrt{-g}} \partial_\sigma (H^\sigma \sqrt{-g}) = \nabla_\sigma H^\sigma$$

to obtain

$$\overset{(i)}{\mathcal{L}} - \overset{(j)}{\mathcal{L}} = dx^1 \wedge \cdots \wedge dx^4 \partial_\sigma (j^\sigma \sqrt{-g}\phi_{ij}) \tag{35}$$

Then, using 27 we notice immediately that it is precisely

$$= d\overset{(ij)}{\kappa_3}$$

Further consider

$$\overset{(ij)}{\kappa_3} + \overset{(jk)}{\kappa_3} + \overset{(ki)}{\kappa_3} \tag{36}$$

$$= 2\pi \lambda_{ijk} j_{el}|_{U_{ijk}} \tag{37}$$

then using the Maxwell's equations 28 we have

$$= 2\pi \lambda_{ijk} d(*F)|_{U_{ijk}} \tag{38}$$

This proves that it is indeed a Cech-Deligne cocycle 4.0.7 and lifts as intended. ☐

4.0.8. *Prequantum field theories as coalgebras.* Now we shall focus on the view of *formal pre-quantum field theories* as coalgebras whose underlying objects are simply the twisted Deligne cocycles $E \to (\mathbf{B}^{p+1}U(1)_{conn})_{\Sigma}$. This shall be used by us so as to extend the theory to the full generality of *differentially-cohesive* ∞*-toposes.*

Recall from ([51], 4.7.3.) that a functor between ∞-categories $G : D \to C$ which admits a left adjoint F *exhibits* D *as monadic over* C provided the functor $G^{'}$ defining the factorization of G through $LMod_T$ is equivalence:

$$\begin{tikzcd} C \arrow[rr, bend left, "F"] & LMod_T(C) \arrow[l] & D \arrow[l, dashed, "G^{'}"'] \arrow[ll, bend left, "G"] \end{tikzcd}$$

where $T \equiv G \circ F$ is a monad and $LMod_T$ is a category of algebras over T.

Dually all remains the same only that the category of *co-algebras* we shall further denote $EM(-)$ as we did with the jet comonad coalgebras (4.0.6, (4)).

Lemma 4.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor (preserves finite limits) with a right adjoint G between ∞-categories which admit finite limits. Assume that it exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$. Then,*

(1) F extends to comonadic F (which in fact is the derivative according [51]*, 6.2.), with the induced comonad $\partial(F \circ G) \simeq \partial(F) \circ \partial(G)$.*
(2) There is an equivalence between categories $EM(\partial T) \simeq Sp(EM(T))$

Proof. The first part is obvious from the dual version which is proven in [51], Example 4.7.3.10.: given $G : D \to C$ a functor between ∞-categories (both D and C admit finite limits) which exhibits D as monadic over C. The functor G is left exact, and therefore induces a functor $g : Sp(D) \to Sp(C)$. If g admits a left adjoint, then g exhibits $Sp(D)$ as monadic over $Sp(C)$. In our case we assumed F is left exact so as to extend it to the categories of spectra, and the existence of the right adjoint ∂G is evident which proves the first point.

□

Lemma 4.17. *If $f : X \to Y$ be an effective epimorphism in an ∞-topos $\mathbf{H}$, then the pullback functor f^* exhibits $\mathbf{H}_{/Y}$ as comonadic over $\mathbf{H}_{/X}$. Moreover, this all extends to the categories of spectra in that $\partial(f^*)$ is comonadic, monadic which moreover admits a left adjoint monadic functor $T \dashv \partial(f^* \circ f_*) \simeq \partial(f^*) \circ \partial(f_*)$:*

$$\begin{tikzcd} T_Y\mathbf{H} \arrow[r, "\sim"] \arrow[rr, bend right, "\partial f^*"'] & EM(\partial(f^* \circ f_*)) \arrow[r] & T_X\mathbf{H} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{H}_{/Y} \arrow[r, shift left, "f^*"] & \mathbf{H}_{/X} \arrow[l, shift left, "f_*"] \end{tikzcd} \qquad \begin{tikzcd} Sp(\mathbf{H}_{/Y}) \arrow[r, shift left, "\partial(f^*)"] & Sp(\mathbf{H}_{/X}) \arrow[l, shift left, "\partial(f_*)"] \end{tikzcd}$$

Proof. Indeed, the pullback functor F has both left $f_!$ and the right f_* adjoints since colimits are universal in ∞-topoi ([53]). Thus, we apply the above lemma and its dual version: f^* being right adjoint extends to $\partial f^* : Sp(\mathbf{H}_{/Y}) \to Sp(\mathbf{H}_{/X})$ which by the *Adjoint functor Theorem* and that ∞-topoi are presentable admits a left adjoint (notice that $f_!$ may admit no extension to the categories of spectra) which proves ∂f^* is monadic, with a monad T.

For the second part we evidently apply the (1) part of the previous lemma.

□

Remark 4.18. The above lemma allows us to describe twisted spectra as coalgebras in that we have the equivalence:

$$\begin{tikzcd} T_Y\mathbf{H} \arrow[r, "\sim"] \arrow[rr, bend right, "\partial f^*"'] & EM(\partial(f^* \circ f_*)) \arrow[r] & T_X\mathbf{H} \end{tikzcd}$$

$$\begin{tikzcd} T_X\mathbf{H} \arrow[r, "\sim"'] & LAlg_T(T_Y\mathbf{H}) \arrow[r] & T_Y\mathbf{H} \end{tikzcd}$$

Example 4.19. Let $G \in Grp(\mathbf{H})$ be a group in an ∞-topos $\mathbf{H}$, $* \to \mathbf{B}G$ be an essentially unique point of the delooping of G. It yields a known base change functor ([69], Double dimension reduction):

$$\begin{tikzcd} \mathbf{H}_{/\mathbf{B}G} \arrow[r, "pt^*"] & \mathbf{H} \arrow[l, bend right, "pt_!"'] \arrow[l, bend left, "{[G,-]//G}"] \end{tikzcd}$$

Now applying the above lemmas we may rewrite G-equivariant spectra as coalgebras over comonad $\partial(pt^* \circ [G,-]//G)$ whose underlying types are spectra in $\mathbf{H}$.

Theorem 4.20. *Let $H : \mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z}) \to \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ be the projection from 4.5 then we have the following diagram where each category on the left is equivalent to the corresponding category of coalgebras whose objects belong to the category on the right:*

$$\begin{tikzcd} T\mathbf{H}_{/\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[r, "H^*" description] & T\mathbf{H}_{/(\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn})_\Sigma} \arrow[l, shift right=2, "H_!"'] \arrow[l, shift left=2, "H_*"] \\ Sp(\mathbf{H}_{/\mathfrak{J}\Sigma})_{/\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook] \arrow[r] & Sp(\mathbf{H}_{/\mathfrak{J}\Sigma})_{/(\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn})_\Sigma} \arrow[u, hook] \arrow[l, shift right=2] \arrow[l, shift left=2] \\ (\mathbf{H}_{/\mathfrak{J}\Sigma})_{/\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook, "\Sigma^\infty_+"'] \arrow[r] & (\mathbf{H}_{/\mathfrak{J}\Sigma})_{/\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}} \arrow[u, hook, "\Sigma^\infty_+"] \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

Proof. Indeed, the projection H is an effective epimorphism ([69], 6.5.10). Thus, we apply the above lemmas to conclude (co)monadicity. $\square$

Definition 4.21. Using the Lemma 4.17 we see that $\partial((\eta_\Sigma)_*)$ is comonadic and exhibits an extension of the jet comonad to tangent categories with the left adjoint $(T^\infty_\Sigma)^{sp}$ (Warning: this left adjoint is not a derivative of an infinitesimal disk bundle monad, since T^∞_Σ may not be left exact and in general does not possesses an extension to the category of spectra): $(T^\infty_\Sigma)^{sp} \dashv \partial J^\infty_\Sigma$.

$$\begin{tikzcd} T_{\mathfrak{J}\Sigma}\mathbf{H} \arrow[r, "\sim"'] \arrow[rr, bend left, "\partial(\eta_\Sigma)^*"] & EM(\partial(J^\infty_\Sigma)) \arrow[r] & T_\Sigma\mathbf{H} \end{tikzcd}$$

∂J^∞_Σ we simply call as a *stable jet comonad*, and its left adjoint $(T^\infty_\Sigma)^{sp}$ as a *stable infinitesimal disk comonad.*

Remark 4.22. Stable jet comonad allows us to work with field theories via the stable homotopy theory within the tangent category $T_{\mathfrak{J}\Sigma}\mathbf{H}$. This may be useful in further investigations, for example of localizations of field theories considered as morphisms of parameterized homotopy types.

For instance, using that via $Disc^{sp} : Spectra \to Sp(\mathbf{H})$ the ordinary spectra are embedded in $Sp(H) \simeq T_*\mathbf{H} \subset T\mathbf{H}$ the localizations (especially interesting would be to consider chromatic filtrations [12], or along the homology theories) [10]) in the ordinary stable homotopy theory shall induce localizations of *parameterized cohesive homotopy types.*

Now we unwind the (co)monadic data encoded in the Theorem 4.20:

Definition 4.23. Introduce the *topologization comonad*, with the structure map being the pullback of the effective epi. is an effecive epi.: $H_! \circ H^* \equiv \mathfrak{T}$

$$\begin{tikzcd}
 & & & (\mathbf{B}^{p+1}(U(1))_{conn})_{\Sigma} \arrow[d] \\
(\mathfrak{T}X \to X) & \sim & H^*E \arrow[ur, dashed] \arrow[r, "\mathfrak{T}(\mathcal{L})"] \arrow[dr, two heads, bend right] & \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn} \\
 & & & E \arrow[u, "\mathcal{L}"']
\end{tikzcd}$$

The Euler-Lagrange comonad $\mathfrak{E} \equiv H^* \circ H_*$ with the left adjoint $H^* \circ H_!$

Thus, we conclude this section the following:

Remark 4.24. Now, we see that defining a local prequantum field theory on Σ is equivalent to finding a bundle over Σ equipped $U(1)$ p-gerbe with connection and endowing it with a structure of a coalgebra of the *Euler-Lagrange comonad* $\mathfrak{E}$.

4.1. **Prequantum field theories in differential cohesion.** Here we are concerned with the problem of *internalization* of prequantum physics (4.1, 4.10) in arbitrary *differentially cohesive* ∞*-topoi*. We have seen that axioms of *differential cohesion* produce the same notions of *Cartan geometry, Cartan connections, variational theory* ([47], [69]). We shall try to proceed even further and define abstractly the prequantum field theory using axioms of *differential cohesion*:

(1) In literature there only exist definitions of groups of synthetic phases $\mathbb{G}$ which play the role of $U(1)$ for defining field theories. Exponentials of action functionals are valued in $\mathbb{G}$ and prequantizations yield $\mathbb{G}$-bundles with connections on phase spaces. Nonetheless, this lacks the stable properties of $U(1)$, such structures as the Beilinson-Deligne cup product (which is very useful for defining field theories, see for instance *4d WZW* 5.1).

    Moreover, such an approach with $\mathbb{G}$ simply lacks the Euler-Lagrange theory. In 4.10 we see that that at the core of prequantum field theories lies $\mathbf{B}_H^{p+1}U(1)_{conn}$ stack which is the "horizontal part" of $\mathbf{B}^{p+1}U(1)_{conn}$. This horizontal part is abstractly defined via the exact hexagons for smooth spectra representing differential cohomology. Hence, we should combine these properties and yield an abstract definition of phases where $H\mathbb{Z}[1]$, and $\mathbf{B}^{p+1}U(1)_{conn}$ spectrum is substituted by $\mathbf{A}_\infty$-ring.
(2) Proceed in 4.2 paragraph to the stable case so as to exploit the modalities 2.1 of *cohesion* and work with spectra. As we shall see that stability will be very helpful in giving the abstract definition of prequantum field theories 4.27.
(3) Suggest how one can exploit *cohesive spectra* so as to produce the analogs of $\mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn} \to \mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ (4.27). It is suggestive to consider analogs of prequantum field theories but with Lagrangians (Cech-Deligne cocycles) valued in other differential cohomology (other than Deligne cohomology).

We further notice that if we have a *formally smooth, framed* object V, then the generalised theory of PDE's applies. Thus, over V-manifolds we may construct and study analogs of prequantum field theories 4.27 which we had over smooth manifolds.

For, we need the following simple:

Lemma 4.25. *Let* $\mathbf{H}$ *be differentially-cohesive* ∞*-topos. If* V *is formally smooth framed object (e.g. group object at the homotopy category level of* $\mathbf{H}$*) and* X *is a* V*-manifold. Then,* X *is formally smooth anf locally framed.*

Proof. Let

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr, two heads] & \\ V & & X \end{tikzcd}$$

be an atlas of X. We have a pullback (by etaleness of $U \to V$):

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \Im U \arrow[d] \\ V \arrow[r] & \Im V \end{tikzcd}$$

where the bottom arrow is epi and the top arrow is epi. $\Im$ being left and right adjoint 2.1 preserves epi [53] we have that in

$$\begin{tikzcd} U \arrow[r, two heads] \arrow[d, two heads, "et"] & \Im U \arrow[d] \\ X \arrow[r] & \Im X \end{tikzcd}$$

$\Im U \to \Im X$ is epi. Hence, $U \to \Im X$ being epi we conclude $X \to \Im X$ is epi and so X is formally smooth. It is locally framed by 3.1. □

Thus, over V-manifolds we may construct and study analogs of prequantum field theories 4.27 which we had over smooth manifolds.

4.2. **A_∞-type field theories.** $\mathbf{A}_\infty$-rings are analogs of rings in stable homotopy theory ([51]).

Let E be a discrete spectrum and $\hat{E}[p+1]$ is a differential refinement (which in addition has an $\mathbf{A}_\infty$-ring structure; this may be used to define analogs of cup-product Chern-Simons theories abstractly, see [23], 2.7. for Beilinson-Deligne cup product in Deligne cohomology) of the $(p+1)$-th suspension of E (it may be written as $(p+1)$-th suspension of a spectrum $\hat{E}$, 2.1). Then apply the functor as in 4.0.7 and produce:

This produces the homotopy exact hexagon:

$$\begin{tikzcd} & (\int_{dR} \hat{E}[p])_\Sigma \arrow[rr, "\mathbf{d}"] \arrow[dr] & & (\flat_{dR}\hat{E}[p+2])_\Sigma \arrow[dr] & \\ (\int_{dR} \flat \hat{E}[p])_\Sigma \arrow[ur] \arrow[dr] & & (\hat{E}[p+1])_\Sigma \arrow[ur, "curv"] \arrow[dr] & & (\int \flat_{dR}\hat{E}[p+2])_\Sigma \\ & (\flat\hat{E}[p+1])_\Sigma \arrow[rr, "\beta_\Sigma"] \arrow[ur] & & (E[p+1])_\Sigma \arrow[ur, "ch"'] & \end{tikzcd}$$

Indeed, $(-)_\Sigma$ being the composition of right adjoint functors preserves homotopy pullbacks which by stability implies exactness.

Consider an effective epimorphism:

$$(\hat{E}[p+1])_\Sigma \xrightarrow{H} \mathbf{B}_H^{p+1}(E)_{conn}$$

This yields the base change essential geometric morphism:

$$\begin{tikzcd} (T\mathbf{H})_{/\mathbf{B}_H^{p+1}(E)_{conn}} \arrow[r, "H^*"] & (T\mathbf{H})_{/(\hat{E}[p+1])_\Sigma} \arrow[l, bend right, "H_!"'] \arrow[l, bend left, "H_*"] \end{tikzcd}$$

Then, define $, X_2, X_3 \to \mathbf{B}^{p+1}_H(E)_{conn}$ by applying H_* to the left hand side part of the diagram 4.2. Further, by taking the homotopy cofibers followed by the pushout on the right hand side part of the diagram we obtain:

$$\begin{tikzcd}
 & X_3 \arrow[rr] \arrow[dr] & & Y_3 \arrow[dr] & \\
X_1 \arrow[ur] \arrow[dr] & & \mathbf{B}^{p+1}_H(E)_{conn} \arrow[ur] \arrow[dr] & & Y_1 \\
 & X_2 \arrow[rr, "\beta"] \arrow[ur] & & Y_2 \arrow[ur] &
\end{tikzcd}$$

Proposition 4.26. *The diagram 4.2 is exact in that the diagonals are homotopy fiber sequences, the top and the bottom rows are long fiber sequences, rectangles are pullbacks.*

Proof. Firstly, notice that H_* being right adjoint descends to the category of spectra over $\mathfrak{J}\Sigma$ and preserves products in $(T\mathbf{H})_{/(\hat{E}[p+1])_\Sigma}$, whence the left hand side of the diag. 4.2 is pullback. By stability we obtain the same for the right hand side rectangle and for the diagonals.

To prove that the top and the bottom rows are exact consider the diag.:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & 0 \arrow[d] \\
X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & \mathbf{B}^{p+1}_H(E)_{conn} \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & Y_3 \arrow[d] \\
0 \arrow[r] & Y_2 \arrow[r] & Y_1
\end{tikzcd}$$

(1) is pullback, (2) is pullback since Y_1 is a homotopy cofiber, (3) is analogous, and (4) is pullback being a pushout. Thus, by pullback pasting lemma we conclude that the the left and the right parts of the diag. are pullbacks, whence the result. □

Definition 4.27. Let $\mathbf{H}$ be a differentialy-cohesive ∞-topos. Let $\hat{E} \in Sp(\mathbf{H}_{\mathfrak{R}})$ be a spectrum as before 4.2 with an effective epi. 4.2 such that

(1) (Underlying spectra are the same) $Y_2 \simeq (E[p+1])_\Sigma$
(2) (*Synthetic* variational Poincare lemma holds) $Y_1 \simeq (\int \flat_{dR}\hat{E}[p+2])_\Sigma$

Then, $\mathbf{B}^{p+1}_H(E)_{conn}$ is E-type Cech-Deligne stack.

If $\pi_0 T\mathbf{H}(\Sigma; \flat_{dR}\hat{E}[\geq p+2]) = 0, \pi_0 T\mathbf{H}(\Sigma; Y_3) = 0$ (which says that differential forms in which $\hat{E}$-curvatures take values are trivial on Σ), then given $(H \to \Sigma) \in (\mathbf{H}_{\mathfrak{R}})_{/\Sigma}$

$$\mathcal{E} \xhookrightarrow{fib} (\eta_\Sigma)_* H \xrightarrow{\mathcal{L}} \mathbf{B}^{p+1}_H(E)_{conn} \xrightarrow{curv} Y_3$$

$(\mathcal{L}, H)$ we suggest calling a *E-type prequantum field theory* ($\mathcal{L}$ as an E-type Cech-Deligne cocycle); fib is the *shell* of the theory and $curv(\mathcal{L})$ are the *E-type Euler-Lagrange equations.* Y_3 is the stack of E-source forms.

Remark 4.28. If V is formally smooth and framed, then the theory of generalised PDE's applies in that we have (4), (5) equivalences in the Variational diagram 4.0.6 (see Prop. 4.25 and [47] Th. 3.60.). Thus, the *shell* $\mathcal{E}$ spanned by E-type Euler-Lagrange equations indeed defines an object in $EM(J^\infty_\Sigma) \simeq PDE_{/\Sigma}(\mathbf{H})$ i.e. a generalised PDE.

Remark 4.29. How does the usual definition of prequantum field theories 4.10 fit into our? Let $E \equiv U(1) = H\mathbb{Z}[1] \in Sp(Sh_\infty(SmMfd))$ (4.0.2) be the (smooth) Eilenberg-Maclane spectrum corresponding to $\mathcal{DK}(0 \to \cdots \to C^\infty(-, U(1)))$.

Then, $\mathbf{B}^{p+1}U(1)_{conn}$ is the spectrum defined in 4.4 and being a spectrum for Deligne cohomology it differentially refines $\mathbf{B}^{p+1}U(1)$ (see 4.4). Having the *Variational Poincare lemma* 4.5 at hand one constructs $Y_3 \equiv \Omega^{p+1,1}_{S,cl}$ and so the $\mathbf{B}^{p+1}_H U(1)_{conn}$ stack. The dimension condition is evident.

Example 4.30. Let X be a smooth manifold endowed with an action of a group G, form $p : X \to X//G$ an effective epi. Any quotient by any action of an ∞-group G then satisfies our construction.

Then, we have:

$$(-)_{X//G} : \mathbf{H} \to \mathbf{H}_{/X//G} \to \mathbf{H}_{/\Im(\mathbf{X//G})}$$

which yields $(\mathbf{B}^{p+1}U(1)_{conn})_{X//G}$. One may define prequantum field theories over $X//G$ as follows:

(1) Consider the fol. epi. of stacks $(\mathbf{B}^{p+1}U(1)_{conn})_X \to \mathbf{B}^{p+1}_H U(1)_{conn}$
(2) Apply $(\Im p)_*$ and obtain an effective epi. again
(3) Notice that $(\Im p)_*((\mathbf{B}^{p+1}U(1)_{conn})_X) \simeq (\mathbf{B}^{p+1}U(1)_{conn})_{X//G}$
(4) This yields $(\mathbf{B}^{p+1}U(1)_{conn})_{X//G} \to (\Im p)_*(\mathbf{B}^{p+1}_H U(1)_{conn})$

In the case, for instance when $X//G$ is an orbifold [4], the description of prequantum field theories simplifies:

Proposition 4.31. *Assume $X//G$ as a groupoid in $\mathbf{H}$ is etale, in that p is etale (local diffeomorphism). A prequantum field theory with a bundle $E \to X//G$ is equivalently described as a prequantum field theory 4.10 with a bundle $p^*(E)$*

Proof. We shall show that there is an equivalence:

$$(\Im p)^*(\eta_{X//G})_* \simeq (\eta_X)_*(p)^*$$

For, we show that they have equivalent left adjoints.

$$1 \to (\eta_X)_*(p^*) \sim (\eta_X)^* \to p^*$$

$$\sim (p_!)(\eta_X)^* \to 1$$

It suffices to prove $(\eta_{X//G})^*(\Im p)_! \simeq (p_!)(\eta_X)^*$ and this is easily seen from:

$$\begin{tikzcd}
(\eta_X)^*E \arrow[r] \arrow[d] \arrow[dd, bend right] & E \arrow[d] \arrow[dd, bend left] \\
X \arrow[r] \arrow[d] & \Im X \arrow[d] \\
X//G \arrow[r] & \Im(X//G)
\end{tikzcd}$$

The top rectangle is a pullback. $(\eta_{X//G})^*(\Im p)_!(E)$ is a pullback of the composition $E \to \Im(X//G)$ along $\eta_{X//G}$ on the other hand $(p_!)(\eta_X)^*(E)$ is a composition on the left. Since the bottom rectangle is a pullback (p is etale) (the word "formally" is omitted) then by pasting of pullbacks this implies the equivalence of these bundles.

This proves that

$$(\eta_{X//G})_*(E) \to (\Im p)_*(\mathbf{B}^{p+1}_H U(1)_{conn})$$

corresponds to $(\Im p)^*(\eta_{X//G})_*(E) \to \mathbf{B}^{p+1}_H U(1)_{conn}$ which is equivalent to $(\eta_X)_*(p^*)(E) \to \mathbf{B}^{p+1}_H U(1)_{conn}$. □

It would be interesting to consider the following examples of incarnations of $\mathbf{A}_\infty$-*type Cech-Deligne stacks*:

Example 4.32. (Hopkins-Singer coefficients) Given a chain complex C [36]

$$\begin{tikzcd} E_{conn,-n} \arrow[r] \arrow[d] & (\Omega \otimes_{\mathbb{Z}} C)^{\bullet \geq -n} \arrow[d] \\ E \arrow[r] & C \otimes_{\mathbb{Z}} \mathbb{R} \end{tikzcd}$$

this is the homotopy differential cohomology pullback. Given a morphism of chain complexes:

$$C \to C^{'}$$

it yields then a morphism of spectra $E_{conn,-n} \to E^{'}_{conn,-n}$. Then, by the above pullback we have that the geometric realizations of them coincide. The synthetic variational Poincare lemma holds as well (the bottom right of the pullback above).

This example is different when compared to Deligne coefficients:

$$\begin{tikzcd} \mathbf{B}^n U(1)_{conn} \simeq (\mathbb{Z}[n+1]^{\infty}_{D})_{conn,0} \arrow[r] \arrow[d] & (\Omega \otimes \mathbb{Z}[n+1]^{\infty}_{D})^{\geq 0} \simeq \Omega^{n+1}_{cl} \arrow[d] \\ \mathbf{B}^n U(1) \simeq \mathbb{Z}[n+1]^{\infty}_{D} \arrow[r] & \mathbb{Z}[n+1]^{\infty}_{D} \otimes \mathbb{R} \simeq \flat \mathbf{B}^{n+1}\mathbb{R} \end{tikzcd}$$

and the stack $\mathbf{B}^{p+1}_H U(1)_{conn}$ is defined via the Euler-Lagrange complex, which sits in the *slice topos* instead whilst our suggestion of Hopkins-Singer coefficients

Example 4.33. (Algebraic K-theory) This is construction is via (Bunke-Nikolaus-Volkl) [13]. Let $\mathcal{K}$ be Algebraic K-theory functor from a category of symmetric monoidal ∞-categories. $\mathbf{Vect}^{\oplus}$, $\mathbf{Vect}^{\oplus}_{conn}$ stacks of categories complex vector bundles (with connections respectively) with a direct sum exhibitting the monoidal structure. Then,

$$\mathcal{K}(\mathbf{Vect}^{\oplus}) : SmMfd^{op} \longrightarrow CMon_{\infty} \xrightarrow{\mathcal{K}} Spectra$$

$$\mathcal{K}(\mathbf{Vect}^{\oplus}_{conn}) : SmMfd^{op} \longrightarrow CMon_{\infty} \xrightarrow{\mathcal{K}} Spectra$$

yield differential refinements of ku spectrum. We have $\mathcal{K}(\mathbf{Vect}_{conn}) \to \mathcal{K}(\mathbf{Vect})$.

$\flat(-) \simeq KC$ which by a long homotopy fiber sequence (bottom row in differential homotopy hexagon 4.2) implies the variational Poincare lemma in that $\int \flat_{dR}\mathcal{K}(\mathbf{Vect}) \simeq \int \flat_{dR}\mathcal{K}(\mathbf{Vect}_{conn})$.

Remark 4.34. One of the main features of prequantum field theories with phases in Deligne coefficients is the *transgression*, which comes from the fiber integration in Deligne cohomology ([23], [48], [3]) which produces action functionals, prequantization of covariant phase spaces and *extended QFT* after quantization.

5. Gravity as a parameterized field theory

There are lots of examples of theories naturally constructed as prequantum field theories. Examples include Chern-Simons theories ([3], [23]), WZW sigma-models [81], [44]. These theories are defined via the characteristic classes with values in the Deligne stack $\mathbf{B}^{p+1}U(1)_{conn}$.

Prequantum field theories are parallel (in the sense that we have a transgression 4.34) to extended quantum field theories (whose concordance classes are known to be classified [30]). **In this section we notice:** when one works with *nonlinear sigma-models* over a manifold Σ which in addition interact with gravity on Σ topological obstructions to glue their Lagrangians globally give rise to incorrectness of gravitational problem of gluing the geometries globally on Σ.

Further we remind an important model, a *4d WZW model over a (pseudo)Kahler manifold* which we further exploit so as to reveal a deeper structure of gravity. A nontrivial topology of a bundle over a manifold Σ of a field theory interacting with gravity yields restrictions on gluing the Energy-Momentum tensors on Σ along some sections.

We could not find an explicit *prequantization* of the *4d WZW action functional* in literature (**??**) and so we present formulas below. Even though the prequantization condition (conition making the $e^{i\mathbf{S}}$ well-defined).

Firstly we outline the main aspects of *2d WZW model* ([81], [69], [3] 2.4.) which shall be used in the explicit prequantization of the *4d WZW*:

(1) Given a semisimple, simply-connected compact Lie group G with Lie algera $\mathfrak{g}$, then the canonical *2d WZW sigma-model* is defined on a 2d-surface Σ by a bundle:

$$\Sigma \times G \to \Sigma$$

One chooses a 3-dimensional manifold $\mathbf{B}_3$ bounding Σ. Let $<-,->$ be Killing metric on $\mathfrak{g}$, with $\mu_3 \equiv <-,[-,-]>$ and the Cartan-Maurer form $\theta \in \Omega^1(G,\mathfrak{g})$. Firstly define the kinetic term [44]:

$$-\frac{1}{8\pi}\int_{\Sigma} d^2x < g^{-1}\partial g, g^{-1}\partial g > \tag{39}$$

(2) The second term was introduced by Witten in [81] so as for a theory to give a *conformal field theory* after quantization (via the Sugawara construction of embedding the Virasoro algebra into the universal envelopping of the Affine Lie algebra). This interaction term is defined on $\mathbf{B}_3$ and requires $\pi_2(G) = 0$ (in order to construct extensions of g, which holds for our G):

$$S_{int} = -\frac{1}{48\pi^2}\int_{\mathbf{B}_3} d^3x < g^{-1}\partial g, [g^{-1}\partial g \wedge g^{-1}\partial g] > \tag{40}$$

where $g : \Sigma \to G$ is now extended to $\mathbf{B}_3$

(3) To quantize the theory one needs to "descend" the term 40 to Σ and require $e^{i\mathbf{S}}$ to be well defined. The necessary and sufficient conditions for that is the *prequantization*. The procedure in its full form is better described in ([69], 7.5.1.):

(4) The term 40 is defined through a pullback along $g : \mathbf{B}_3 \to \mathbf{B}_3 \times G$ of the form $\mu_3(\theta)_H$ on $J^\infty(\mathbf{B}_3 \times G)$

(5) Using the ∞-*Chern-Weil theory* ([69], 7.5.1.) one refines a generator $BG \to K(\mathbb{Z},4)$ (for our G it is known that $H^4(BG;\mathbb{Z}) \simeq \mathbb{Z}$) to a 3-connection $\nabla_{CS} : \mathbf{B}G_{conn} \to \mathbf{B}^3U(1)_{conn}$ which descends (as indicated again in [69], 7.5.1.; this fact constitutes the *CS/WZW* correspondence) to:

$$\begin{tikzcd} & \mathbf{B}^2U(1)_{conn} \arrow[d] \\ G \arrow[ur, dashed, "\nabla_{WZW}"] \arrow[r, "\mu_3"'] & \Omega^3_{cl} \end{tikzcd}$$

which prequantizes (4.4) (in contrast with the *4d WZW* below this prequantization is constructed independently of Σ) the μ_3 form to a *Kalb-Ramon B-field.*

(6) Then, considering a descent for $G \to \mathbf{B}^2U(1)_{conn}$ by $(\{V_i\}_i, (\overset{(i)}{B})_i, \dots)$ (V_i is a cover of a group G) one exploits the Stokes theorem to rewrite (locally) the term 40:

$$S_{int} = -\frac{1}{48\pi^2}\int_{\mathbf{B}_3} d(g^*\overset{(i)}{B}) = -\frac{1}{48\pi^2}\int_{\Sigma} g^*\overset{(i)}{B} \tag{41}$$

Warning: the interaction term Lagrangians (and so the action functionals) are defined only locally in that B-fields (their horizontal parts of pullbacks to $J^\infty(\Sigma \times G)$) are defined on some cover of $J^\infty(\Sigma \times G)$.

(7) The approach works only perturbatively when we require $g : \Sigma \to G$ to factor through some contractible neighbourhood in G (for instance, g might be surjective then the above action term is not defined). In order to be able to define the field theory for arbitrary configurations one needs a global definition of $\mathcal{L}_{WZW}$, i.e. refining it to a *Cech-de Rham cocycle* by:

$$\mathcal{L}_{WZW} \equiv (\nabla_{WZW})_H : J^\infty(\Sigma \times G) \to \mathbf{B}^2_H U(1)_{conn} \tag{42}$$

(8) The transgression of it [3] yields a well-defined exponential $e^{iS_{WZW}}$ and so is the whole $e^{i\mathbf{S}}$.

5.1. **Prequantization of 4d WZW sigma model over a Kahler manifold.** Before we go further let us remind that a complex smooth manifold X equipped with a Hermitian metric h is Kahler, with $g(-,-) \equiv Re(h(-,-))$ Riemann metric is compatible with the complex structure:

$$g(Ju, Jv) = g(u, v), \forall u, v \tag{43}$$

A differential form $w \equiv Im(h(-,-))$ is symplectic (in this case is called Kahler). Equivalently, Kahler manifolds are complex manifolds with a metric compatible with the complex structure and such that $\omega(u, v) := g(u, Jv)$ is closed.

In holomorphic coordinates we have the (local) formulas:

$$\omega = \frac{i}{2} h_{j\bar{k}} dz^i \wedge d\bar{z}^k, g_{i\bar{j}} = \frac{1}{2}(h_{i\bar{j}} + h_{j\bar{i}}) \tag{44}$$

Now we turn to the *4d* case where gravity is present on our brane. *4d* analog of the *2d WZW* theory is studied in literature so as to construct solvable QFT's since the *2d WZW* turned out to be rational conformal field theory. This model has found its applications in non-perturbative string theory [50].

(1) Let X_4 be a 4-dimensional smooth manifold with a closed form $w \in \Omega^2_{cl}(X)$. In [50] for instance, authors considered a quantization of field theory with the following action functional:

$$S_\omega[g, g_0] = \frac{f^2}{8\pi} \int_{X_4} Tr(g^{-1}dg \wedge *(g^{-1}dg)) + \frac{i}{12\pi} \int_{X_4 \times I \equiv X_5} \omega \wedge Tr(g^{-1}dg)^3 \tag{45}$$

where $g : X_4 \to G$ is a field configuration. Field theory is defined on the space of such field configurations that are homotopic to a reference field g_0. The second term is defined via a homotopy from g to g_0. In [50] it is noted that here one uses X_5 in the form a cylinder (whilst in *2d WZW* we had Σ as a boundary) so as to not encounter such obstructions like X_4 might not be even cobordant to zero (so X_5 cannot be taken to be a cone).

(2) Let X_4 be a Kahler manifold with a Kahler form $\omega \equiv \frac{if^2}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then, one may rewrite the functional:

$$S_\omega[g, g_0] = -\frac{i}{4\pi} \int_{X_4} Tr((g^{-1}\partial g) \wedge (g^{-1}\bar{\partial} g)) + \frac{i}{12\pi} \int_{X_4 \times I \equiv X_5} \omega \wedge Tr(g^{-1}dg)^3 \tag{46}$$

which yields the following simple Euler-Lagrange equations:

$$\omega \wedge \bar{\partial}(g^{-1}\partial g) = 0 \tag{47}$$

(3) We could work with (pesudo)Kahler manifolds instead and nothing changes from that presceptive.

(4) The second term in 46 is quantized as it in the case of 5. For a measure $e^{i\mathbf{S}}$ to be well-defined (independently of a homotopy choices) one needs the following condition:

$$\omega \wedge \frac{1}{12\pi} Tr(g^{-1}dg)^3 \in H^5(X_5; \mathbb{Z}) \tag{48}$$

This says that the form ω must be Hodge in that the Kahler class quantizes $[\omega] \in H^2(X;\mathbb{Z})$. For the $Tr(g^{-1}dg)^3 = g^*\mu_3(\theta) \in \Omega^3(X_5, \mathfrak{g})$ (μ_3 is a cocycle on $\mathfrak{g}$ 5) we have the analogous condition as well.

(5) Recalling the pullback 4.4 for $\mathbf{B}^{p+1}U(1)_{conn}$ these conditions are equivalent to that ω is *prequantized* by fields (Kahler potentials):

$$((U_j), (\frac{i}{2}\bar{\partial}\rho_j), ...) : X \to \mathbf{B}U(1)_{conn} \tag{49}$$

with $d\frac{i}{2}\bar{\partial}\rho_j = \omega|_{U_j}$.

The same for the $g^*\mu_3(\theta)$ term, prequantization is already done in 5 and the explicit formula of this gerbe with connection on G:

$$(\{V_i\}_i, (\overset{(i)}{B})_i, (\overset{(ij)}{A})_{ij}, (\overset{(ijk)}{\phi})_{ijk}, (\lambda_{ijkm})_{ijkm}) : G \to \mathbf{B}^2U(1)_{conn} \tag{50}$$

(6) We see that the requirement of e^{iS} to be well-defined is equivalent to *prequantization* of the second (topological) term. Explicitly, this prequantization is constructed via the Beilinson-Deligne cup product [23]:

$$\mathcal{L}_{WZW} = \frac{i}{12\pi}(\nabla_{Hodge})_H \cup (\nabla_{WZW})_H \tag{51}$$

$$\nabla_{Hodge} \cup \nabla_{WZW} = ((\omega \wedge (\overset{(i)}{B}))_i, (\omega \wedge (\overset{(ij)}{A}))_i, (\omega \overset{(ijk)}{\phi})_{ijk}, (\frac{i}{2}\bar{\partial}\rho_i\lambda_{ijkm})_{ijkm}, \ldots, 0) \tag{52}$$

(7) Further we notice that if we have a field configuration $g|_{g^{-1}V_i} : g^{-1}V_i \to V_i$, we may then use the Stokes theorem (and that ω is closed) to write the topological term (the boundary is $X_4 \coprod X_4$ and the integral splits onto the sum where the first contribution is evaluated on g_0 which we may discard) on X_4 (locally):

$$\frac{i}{12\pi}\omega \wedge g^*\overset{(i)}{B} \equiv \overset{(i)}{\mathcal{L}} \in \Omega^4_H(X_4 \times G) \tag{53}$$

(8) Now we "descend" and de-transgress the action functional 46 to the Cech-de Rham cocycle on X_4:

$$-\frac{i}{4\pi}\omega\wedge < \theta_H \wedge *\theta_H > +\frac{i}{12\pi}\mathcal{L}_{WZW} \tag{54}$$

where $\theta_H \in \Omega^4_H(X_4 \times G) \subset \Omega^4(J^\infty_X(X \times G))$ is a horizontal part of the pullback of the Cartan-Maurer form to the jet bundle.

(9) This is the intuition of working with such brane field theories: having the theory defined on X_5 we "pull" the bulk information back onto X_4 in the form of a nontrivial topological term (WZW term in our case). We notice that the Kalb-Ramon field B is a *background field* on G and the field theory is independent of gauge transformations of B. As with the case of *2d WZW model* we needed prequantization to have a well defined e^{iS} and for all field configurations g, not only those which factor through a contractible neighbourhood in G. Consider for example $X_4 \equiv \mathbb{S}^4$ and $G \equiv SU(2)$ and a $g_0 : \mathbb{S}^4 \to SU(2) \simeq \mathbb{S}^3$ reference field configuration which has a homotopy type $[g_0] = 1 \in \pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2$. Then, this type of field theory does not admit any *local* formulation avoiding prequantization whatsoever.

(10) To the contrast with prequantized electrodynamics with currents 4.12, the *4d WZW* is a theory correct physically. **Firstly**, as it was indicated before, currents in electrodynamics J cannot be prescribed, since themselves are currents of spinor fields which transform under gauge transformations and thus, in this case Lagrangians of a *real* electrodynamics glue globally because of the gauge invariance. **Secondly**, the difference is that here B field is not a gauge field on X_4,

this field modulates how we *descend* the theory from X_5 to X_4. There is an arbitrariness in this, and this is known that descended action functionals 7 are defined only locally and not in a unique way. This arbitrariness has no implications on the dynamics on the theory 47 and its quantization.

(11) We now may compute the *Energy-Momentum tensor* (for the sake of brevity we shall only write the topological part): let g be a field configuration, restrict it on $g^{-1}V_i$ as in 7 and let $\mu, \bar{\nu}$ be indices on corresponding to a holomorphic chart on $g^{-1}V_i$, a indices of V_i chart, then

$$\frac{\delta \overset{(i)}{S}_{WZW}}{\delta g_{\mu\bar{\nu}}} = \overset{(i)}{T}{}^{\mu\bar{\nu}}[g] = \frac{2}{\sqrt{det(g_{\mu\bar{\nu}})}} \frac{\partial}{\partial g_{\mu\bar{\nu}}} g^* \overset{(i)}{\mathcal{L}} \tag{55}$$

$$g^* \overset{(i)}{\mathcal{L}} = -\frac{f^2}{24\pi} h_{j_1\bar{j}_2} \partial_{j_3} g^a \partial_{\bar{j}_4} g^b \epsilon^{j_1\bar{j}_2 j_3 \bar{j}_4} \overset{(i)}{B}_{ab} \tag{56}$$

then, applying 44:

$$\overset{(i)}{T}{}^{\mu\bar{\nu}}(g) = -\frac{f^2}{6\pi\sqrt{det(g_{\mu\bar{\nu}})}} (\epsilon^{\mu\bar{\nu}\sigma\bar{\gamma}} \overset{(i)}{B}_{ab} \partial_\sigma g^a \partial_{\bar{\gamma}} g^b + \epsilon^{\nu\bar{\mu}\sigma\bar{\gamma}} \overset{(i)}{B}_{ab} \partial_\sigma g^a \partial_{\bar{\gamma}} g^b) \tag{57}$$

We see that the Energy-Momentum tensor now acts as a cocycle with gluing conditions on $x \in g^{-1}V_i \cap g^{-1}V_j$ prescribed by ∇_{WZW} cocycle. If we had the Einstein's gravity on X_4 unchanged and solve the equations:

$$G_{\mu\nu} = T_{\mu\nu} \tag{58}$$

we encounter the following issue: Einstein's tensor is globally defined and thus is compatible on intersections $g^{-1}V_i \cap g^{-1}V_j$ whilst the Energy-Momentum tensor along a field configuration is a cocycle and we cannot glue the metric tensor globally by solving only the Einstein's equations.

(12) Notice that this prequantum field theory even if its Lagrangian 40 is defined using the metric tensors explicitly, the gluing condition of the ∇_{WZW} uses Kahler potentials ρ which in general do not possess an explicit description in terms of the metric.

This is important since if it was not the case we could define this prequantum field theory on the bundle $\mathcal{G}_{Kahler} \times X^*(G)$ where $\mathcal{G}$ is a bundle of (Kahler) metric tensors over X and then the *Energy-Momentum tensor* would naturally come out as Euler derivative $\delta_V \mathcal{L}_{WZW}$ which glues globally on any configuration $g : X \to G$. Thus the *4d WZW* as a *prequantum field theory* is defined only if we fix the Kahler metric on X. Despite that we may compute the Energy-Momentum tensor locally.

(13) Notice that we may consider this as a constraint on B field configurations. Indeed, there is for instance an analytical obstruction to glue geometries which follows from the Einstein's equations:

$$\nabla T = 0 \tag{59}$$

i.e. the Energy-Momentum tensor is covariantly constant since the covariant derivative of the Einstein's tensor vanishes. This is the analytical constraint (continuity condition) which is resolved when we "fall" *on-shell* of the theory. In our case we have that for a field configuration there must be a choice of $B, \mathcal{A}, \ldots$ fields so that the Energy-Momentum tensor trivializes as a cocycle. In particular this implies that:

$$g^* \nabla_{WZW} : X \to G \to \mathbf{B}^2 U(1)_{conn} \tag{60}$$

is a trivial $U(1)$-gerbe with connection.

This seems very restrictive to consider such constraints. Moreover, this would deform the theory at the quantum level and these deformations seem to be unnatural (it would be impossible to control them properly in the path integral).

Thus, *gravity needs to adjust to topological terms of field theories interacting with it so as to cancel such ambiguities.* This is the *parameterized nature of gravity.*

How do we do this? We may recall the related existing phenomena in the case of gravity:

Even for locally-defined field theories (topologically trivial) interacting with gravity there is an issue. Indeed, it is an obstruction to a proper nonminimal interaction (via Energy-Momentum tensor) of a field theory with gravity is that the Energy-Momentum tensor T must be divergence-free (this follows from the Einstein's equations). When we "deform" a given field theory we usually restrict ourselves to such field configurations that satisfy divergence-freeness or in particular to on-shell of the theory.

So we had: an obstruction to nonminimal interaction - divergence-freeness of EMT (analytical condition $\nabla_\mu T^{\mu\nu} = 0$). We deform a bundle of a theory (but Lagrangians remain the same): we take the bundle of a field theory at hand $(\mathcal{G}^\infty + E)$, where $\mathcal{G}^\infty$ is a bundle of gravity fields, E is a bundle of fields; then deform it to the shell $\mathcal{E} \subset (\mathcal{G}^\infty + E)$.

This deformation is "canonical" in that it can be done for all field theories at he same time.

Now in our case we have in addition: an obstruction to nonminimal interaction - EMT should be a globally defined tensor on a manifold (topological condition). We deform a field theory but here we change the Lagrangians themselves so that they could prequantize on a bundle of fields + metrics. Technically this means the following:

(1) We have a field theory defined on local charts of a total space of a bundle with local system of Lagrangians $(\mathcal{L}_i)_i$. Assuming this theory is nontrivial topologically in that given a gravity field g (consider WZW 4-dimensional in parallel to this argument) the pullback to E glues as a nontrivial Cech-de Rham cocycle $\mathcal{L}(G)$.

$$\begin{tikzcd}
(E|_{U_i})_i \arrow[r, hook, "g"'] \arrow[rr, bend left, "(g^*\mathcal{L}_i)_i"] \arrow[d, two heads] & ((\mathcal{G}^\infty + E)|_{U_i})_i \arrow[r, "(\mathcal{L}_i)_i"'] \arrow[d, two heads] & \Omega_H^{p+1} \arrow[d] & \forall g \in \Gamma(\mathcal{G}) \\
E \arrow[r, hook, "g"] \arrow[rr, bend right, "\mathcal{L}(g)"'] & ((\mathcal{G}^\infty + E)) \arrow[r, dashed, "\mathcal{L}_{tot}"] & B_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn} & \mathcal{L} + \mathcal{L}_{GR} \equiv \mathcal{L}
\end{tikzcd}$$

If the theory admits $\mathcal{L}_{tot}$ which lifts the Cech-de Rham cocycles $\mathcal{L}(g)$ to the Cech-de Rham cocycle then the Energy-Momentum tensor is correct globally-defined tensor since is a part of the variational derivative applied to $\mathcal{L}_{tot}$:

$$\delta_V(\mathcal{L}_{tot}) = G_{\mu\nu}\delta g^{\mu\nu} - \kappa T_{\mu\nu}\delta g^{\mu\nu} + EL(E)$$

(2) There are situations (WZW 4d for instance) where EMT is not correctly defined globally. This implies that there is no $\mathcal{L}_{tot}$ lift. For we need to deform a field theory (somewhat canonically) to $(\mathcal{L}_i^{'})_i$ which admits a lift $\mathcal{L}_{tot}^{'}$. This hypothetical mechanism we call as topological coupling. This is the problem worthy of future investigations.

6. Conclusions

(1) In our work we were driven by the methodology of *internalization.* Our goal was to extend such important aspects of fundamental physics as Cartan connections, orthogonal structures, *Riemann geometry spin-geometry*, *string-geometry* to an abstract *differentially-cohesive* ∞*-topos.* This is done in 3.3.

(2) In 4.1, 4.10 we defined what we call $\mathbf{A}_\infty$-type field theories. This leads to the following applications:

Firstly, our abstract definition of prequantum field theories in *differentially-cohesive* ∞*-topoi* provides us means to extend construction of fundamental physics to abstract contexts (for instance in the context of relative cohesion of $\mathbb{E}_\infty$-arithmetic geometry).

It is very suggestive to study these new types of field theories since this approach somewhat resembles what happened with non-abelian gauge field theories. Yang and Mills proposed to generalise the gauge invariance of electrodynamics by using the $SU(2)$ group and constructing an action for the isospin doublet of protons and neutrons. Later after the discovery of the property of asymptotic freedom of non-abelian gauge field theories and this motivated the search of the theory of strong interactions.

Defining prequantum field theories is somewhat (see the characterization in terms of the coalgebras that we derived 4.0.8) the same as defining a $\mathbb{R}/\hbar\mathbb{Z}$ (p+1)-gerbe with connection on a bundle of a field theory. Our suggested theory of $\mathbf{A}_\infty$-type field theories can be used in the fashion of $U(1) \mapsto G$ generalization, where G is a non-abelian Lie group, so as to find new field theories which for some reason cannot be formulated within the classical Euler-Lagrange theory. In 4.1 we demonstrated that there are interesting examples of smooth spectra which possess the desired properties of the classical Euler-Lagrange theory.

On the other hand, we know that $\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ stacks classifying prequantum field theories are defined using the structure of smooth manifolds. In string theory for instance, compactifications on orbifolds, orientifolds are of a great importance and so in the direction of M-theory it becomes common to consider spacetime as a more general stack than a simple smooth manifold. Nonetheless, there are no accepted mathematical means in defining field theories on such singular spaces as orbifolds or rather general smooth homotopy types. Our work makes the construction of the Euler-Lagrange theories axiomatic which allows to define field theories on general smooth homotpy types (or even on differentially-cohesive) and in the case of orbifolds 4.30 we even construct an analog of $\mathbf{B}_H^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{conn}$ stack.

(3) We were motivated to find new inconsistencies of gravity with quantum world so as to reveal a new direction where we should extend the existing gravity theories. In 5 we discussed the issue of coupling of gravity with a prequantized field theory (*4d WZW model on Kahler manifold*) and this resulted in the need of introduction of a topological coupling mechanism. This mechanism deserves further investigation in which firstly we are supposed to construct a functorial *topological term* $\mathbf{L}_{top}$. This approach to extension of gravity theory, in contrast with existing ones, reveals a deeper mathematical nature of gravity - it is a *parameterised field theory.*

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