Mathematics problems

1 Elementary algebra

Problem 1.1. Simplify

$$\frac{x^{n+2}}{x^{n-2}}$$

1. $\frac{x^{n+2}}{x^{n-2}} = x^{n+2-(n-2)}$
2. Solution: $x^{n+2-(n-2)} = x^4$

Problem 1.2. Solve for x:

$$x^{-1} * 8 = 2$$

1. $\frac{1}{x^1} * 8 = 2$
2. $\frac{1}{x} = \frac{2}{8}$
3. $\frac{8}{2} = x$
4. Solution: $x = 4$

Problem 1.3. Calculate the missing value. If $a = 5$ and $b = 10$ then $(a^b)^0 = \ldots$

1. $(a^b)^0 = (5^{10})^0$
2. Solution: $(5^{10})^0 = 1$

Problem 1.4. Calculate

$$\frac{\sqrt{4x}}{\sqrt{x}}$$

1. $\frac{\sqrt{4x}}{\sqrt{x}} = \frac{\sqrt{4} * \sqrt{x}}{\sqrt{x}}$
2. $\frac{\sqrt{4} * \sqrt{x}}{\sqrt{x}} = \sqrt{4}$
3. Solution: $\sqrt{4} = 2$

Problem 1.5. Solve for x:

$$x^2 + (x+1)^2 = (x+2)^2$$

1. identities: $(x+1)^2 = x^2 + 2x + 1 and (x+2)^2 = x^2 + 4x + 4$
2. substituting: $x^2 + x^2 + 2x + 1 = x^2 + 4x + 4$
3. simplifying: $x^2 - 2x - 3 = 0$
4. reordering: $x(x-2) = 3$
5. Solutions: $x = -1, x = 3$

Problem 1.6. Find the solution set for the inequality below:

$$2^x > 1024$$

1. take the log of both sides: $\log_{10} 2^x > \log_{10} 1024$
2. use the log power identity: $x * log_{10}2 > \log_{10} 1024$
3. re-arrange: $x > \frac{\log_{10} 1024}{log_{10}2}$
4. Solution: $x > 10$

2 Functions of one variable

Problem 2.1 (Based on SYD 2.5.6). The relationship between temperatures measured in Celsius and Fahrenheit is linear. 0°C is equivalent to 32°F and 100°C is the same as 212°F. Which temperature is measured by the same number on both scales?

1. Given that when it is 0°C, it is 32°F and 100°C when it is 212°F, we know the relationship between Celsius and Fahrenheit can be described in the form of F°= m*C°+ b, with 'm' being the slope of the line and b 'the' y-intercept of the graphed line.
2. In this case, $m = \frac{(212-32)}{(100-0)} = \frac{18}{10}$ and $b = 32$. This gives us the formula $F^\circ = \frac{18}{10} * C^\circ + 32$
3. Solving the above equation for when C°= F°, yields the following $C^\circ = \frac{18}{10} * C^\circ + 32$.
4. Simplifying yields $-32 = \frac{8}{10} * C^\circ$
5. Solving for C°, C°= -40
6. Solution: F°and C°are equal at -40°

Problem 2.2. Take the following function $f(x) = 5x + 4$. Find y if $f(3) = y$.

1. $f(3)$ means the value of the function when $x = 3$. Therefore, $y = 5 * 3 + 4 = 19$
2. Solution: $y = 19$

Problem 2.3. Find all values of x that satisfy:

$$x^2 - 4x + 3 = 0$$

1. $x^2 - 4x + 3 = 0$ is equivalent to $(x - 3)(x - 1) = 0$
2. Solution: This equation holds when $x = 3$ or $x = 1$;

Problem 2.4. Assume that you invest 10 HUF for 90 years with a yearly compound interest of 2%. How much money do you receive 90 years later?

1. For this we use the compounding interest formula. $A = P(1 + \frac{r}{n})^{nt}$
2. Substituting, we get $10HUF * (1 + \frac{0.02}{1})^{1*90}$
3. Solution: This yields $10HUF * (1.02)^{90} = 59.43HUF$

Problem 2.5. Calculate the following value

$$e^{\ln 5}$$

1. If $e^x = a$ and by definition $x = ln(a)$, then $e^{ln(a)} = a$
2. Solution: Therefore $e^{ln(5)} = 5$

3 Calculus

Problem 3.1. Calculate the following sum

$$\sum_{i=1}^{\infty} \frac{12}{6^i}$$

1. The sum of an infinite geometric series follows the formula $\frac{a_1}{1-r}$ where a_1 is the first term of the series and r is the common ratio.

2. Solution: With $a = \frac{12}{6^1} = 2$ and $r = \frac{1}{6}$, $\sum\limits_{i=1}^{\infty} \frac{12}{6^i} = \frac{2}{1-\frac{1}{6}} = \frac{2}{\frac{5}{6}} = 2 * \frac{6}{5} = \frac{12}{5}$

Problem 3.2. Find the following limit

$$\lim_{x\to 1} \frac{6^{1-x}}{x}$$

1. By identity, $\lim\limits_{x\to 1} \frac{6^{1-x}}{x} = \frac{\lim\limits_{x\to 1} 6^{1-x}}{\lim\limits_{x\to 1} x}$

2. $\lim\limits_{x\to 1} 6^{1-x} = 1$ and $\lim\limits_{x\to 1} x = 1$

3. Solution: Therefore, $\frac{\lim\limits_{x\to 1} 6^{1-x}}{\lim\limits_{x\to 1} x} = \frac{1}{1} = 1$

Problem 3.3. Find the slope of the function $f(x) = x^5 - 8$ at $x = -3$.

1. The slope of the line at $x = -3$ can be found by taking the first derivative of the equation $f'(x)$ and finding $f'(-3)$

2. $f'(x) = \frac{\mathrm{d}}{\mathrm{d}\,x}(x^5 - 8) = \frac{\mathrm{d}}{\mathrm{d}\,x}(x^5) - \frac{\mathrm{d}}{\mathrm{d}\,x}8$

3. $\frac{\mathrm{d}}{\mathrm{d}\,x}(x^5) = 5x^4$ and $\frac{\mathrm{d}}{\mathrm{d}\,x}8 = 0$, meaning $f'(x) = 5x^4 - 0 = 5x^4$

4. Solution: $f'(-3) = 5(-3)^4 = 405$

Problem 3.4. Find the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}\,x} \frac{x^3 + 2x - 1}{x - 2}$$

1. Given the identity that $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ where $f = x^3 + 2x - 1$, $f' = 3x^2 + 2$, $g = x - 2$, $g' = 1$

2. Solution: The derivative equals $\frac{(3x^2+2)(x-2)-(x^3+2x-1)(1)}{(x-2)^2} = \frac{3x^3-6x^2+2x-4-x^3-2x+1}{(x-2)^2} = \frac{2x^3-6x^2-3}{(x-2)^2}$

Problem 3.5. Find the following second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} 4x^4 + 4x^2$$

1. $f'(x) = \frac{\mathrm{d}}{\mathrm{d}\,x} 4x^4 + 4x^2 = 16x^3 + 8x$

2. Solution: The second derivative is $\frac{\mathrm{d}}{\mathrm{d}\,x}(16x^3 + 8x) = 48x^2 + 8$

Problem 3.6. Find the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\,x} \frac{\ln x}{e^x}$$

1. Reorder the equation to the equivalent $e^{-x} * \ln x$.

2. Given the identity that $(fg)' = f'g + fg'$ where $f = \ln x$, $f' = \frac{1}{x}$, $g = e^{-x}$, $g' = -e^{-x}$

3. $f'(x) = \frac{e^{-x}}{x} + \ln x * -e^{-x} = \frac{e^{-x}}{x} - \ln x * e^{-x}$

4. Simplified, $\frac{e^{-x}(1-x\ln x)}{x}$

Problem 3.7. Consider the following function. Find all of its stationary points and classify them as local minima, local maxima or inflection points. Also decide whether it is convex or concave. If it has one or more inflection points then define where it is locally concave or locally convex. (You should create a table like we did in class)

$$f(x) = 3x^2 - 5x + 2$$

1. $3x^2 - 5x + 2 = (3x-2)(x-1)$

2. Inflection points occur where $f'(x) = 0$. $f'(x) = 6x - 5$ so there is one inflection point at $\frac{5}{6}$.

3. We look at the second derivative to establish concavity. $f''(x) = 6$, therefore the second derivative is always positive, making the graph "concave up".

x	$x < \frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6} < x$
f(x)	approaches $+\infty$	$\frac{-1}{12}$	approaches $+\infty$
f'(x)	-	0	+
slope	decreasing	global minimum	increasing
f"(x)	+	+	+
convexity	concave up	inflection point	concave up

Problem 3.8. Let $f(x,y) = x^2 + y^3$. Calculate $f(2,3)$

1. Solution: $f(2,3) = (2)^2 + (3)^3 = 31$

Problem 3.9. Consider the following function: $f(x,y) = \ln(x-y)$. For what combinations of x and y is this function defined?

1. Because the the limit of $\ln(a)$ is non-real for all a ¡ 0, $\ln(x-y)$ is defined for all $(x-y) > 0$.

Problem 3.10. Find the following partial derivative:

$$\frac{\partial}{\partial x} x^5 + xy^3$$

1. Solution: Using the power rule, $\frac{\partial}{\partial x}(x,y) = 5x^4 + y^3$

Problem 3.11. Find the local maxima or minima of the following function:

$$f(x,y) = x^2y^2 + 10$$

1. We can find the inflection points by looking for points where $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial y} = 0$

2. $\frac{\partial}{\partial x} = 2xy^2$ and $\frac{\partial}{\partial y} = 2x^2y$. Therefore there is a single inflection point for x at $x = 0$ and for y at $y = 0$.

3. The second partial derivative of x is $2y^2$ and the second partial derivative of y is $2x^2$. Since the second partial derivative is always positive, we know the graph is convex up.

4. Because the graph is convex up, we know the inflection points are also the global minimum at x=0, y=0.

Problem 3.12. Solve the following constrained optimization problem using Lagrange's method: $\max x^2y^2$ s.t. $x + y = 10$

1. $L(x, y, \lambda) = x^2y^2 - \lambda(x + y - 10)$
2. The partial derivative with respect to x is $0 = 2xy^2 - \lambda$
3. The partial derivative with respect to y is $0 = 2x^2y - \lambda$
4. The partial derivative with respect to λ is $x + y = 10$
5. Solving the system of equations, we get $2xy^2 = 2x^2y$ and $x + y = 10$. Simplifying gets $2y = 2x$. We then substitute $x = 10 - y$ and $y = 10 - x$.
6. Solution: $x = 5$ and $y = 5$, where $f(x, y) = 625$

4 Linear algebra

Problem 4.1. Take the following matrices:

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \\ 1 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 7 \\ 2 & 8 & 2 \end{bmatrix}$$

What is $A \cdot B$?

1. $A \cdot B = \begin{bmatrix} 14 & 50 & 26 \\ 7 & 13 & 37 \\ 19 & 73 & 25 \end{bmatrix}$

Problem 4.2. Take the following matrices:

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 9 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

What is $B \cdot A$?

1. $B \cdot A = \begin{bmatrix} 39 & 59 \\ 10 & 16 \end{bmatrix}$

Problem 4.3. What is the transpose of the following matrix?

$$\begin{bmatrix} 7.1 & 9.1 & 4.7 \\ 2 & 7.8 & 1.1 \\ 4 & 4.44 & 0 \end{bmatrix}$$

1. Solution: $\begin{bmatrix} 7.1 & 2 & 4 \\ 9.1 & 7.8 & 4.44 \\ 4.7 & 1.1 & 0 \end{bmatrix}$

Problem 4.4. Calculate the determinant of

$$\begin{bmatrix} 1 & 9 \\ 2 & 8 \end{bmatrix}$$

1. The determinant is $1 * 8 - 9 * 2 = 8 - 18 = -10$

5 Probability theory

Problem 5.1. You run an experiment where you throw a (regular, 6 sided) dice twice. The first number you get will be the first digit of a two-digit number, while the second number you get will be the second digit of the same two-digit number. What is the sample space of your experiment?

1. Because a die can have a value of 1-6, the sample space can be described as S = { 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 }

Problem 5.2. Assume that in a certain country 1% of the population uses a certain drug. You have a way to test drug use, which will give you a positive result in 99% of the cases where the individual is indeed a drug user and a negative result in 99.5% of the cases where the individual doesn't use the drug. What is the probability that a randomly selected citizen will have a positive drug test?

1. We want to find the probability that a random person tests positive. This can happen if they are a user and the test is positive and they aren't a user and get a false positive. To find the probability of a random citizen testing positive, we sum the probability of a citizen testing positive given they are a user and the probability of citizen falsely testing positive given they are not a user.
2. Solution: $.01 * .99 + .99 * .005 = 1.48\%$

Problem 5.3. Assume that in a certain country 1% of the population uses a certain drug. You have a way to test drug use, which will give you a positive result in 99% of the cases where the individual is indeed a drug user and a negative result in 99.5% of the cases where the individual doesn't use the drug. What is the probability that someone with a positive drug test is indeed a drug user?

1. We are looking for the probability that that a citizen tests positive minus the probability that it is a false positive. We use Bayes rule.
2. We take the probability of a correct positive and divide it by the probability of a positive.
3. Solution: This yields $\frac{.01*.99}{.01*.99+.99*.005} = 66.66\% acitizenwhotestspositiveisactuallyadruguser$