Age-structured and stage-structured population models

  

Today's topics

1. Introduction
2. Matrix Models
3. Reproductive value
4. Stage-structured models

Birth and death rates usually depend on age.

Motivation

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Growth rates will differ between populations with different age structures.

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Some age classes contribute more to population growth than others.

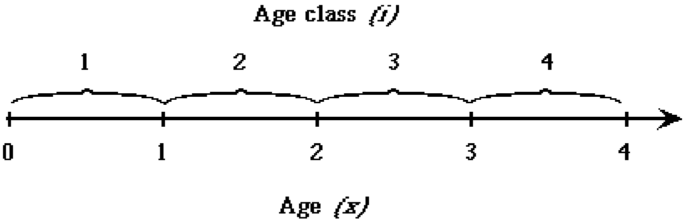
These facts have important management implications.

Matrix models

- Age is discrete

Life tables

- Age is continuous



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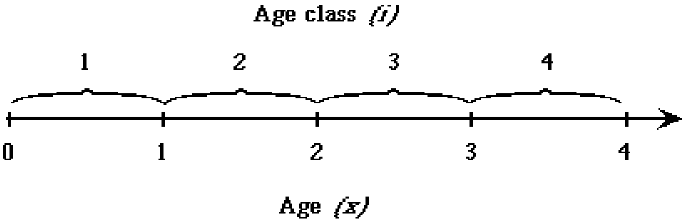

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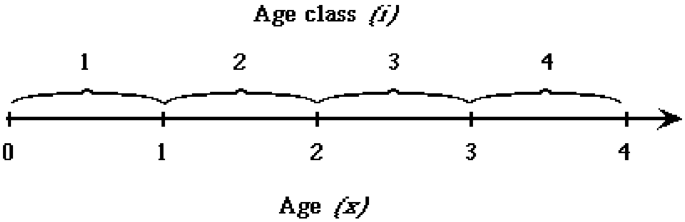

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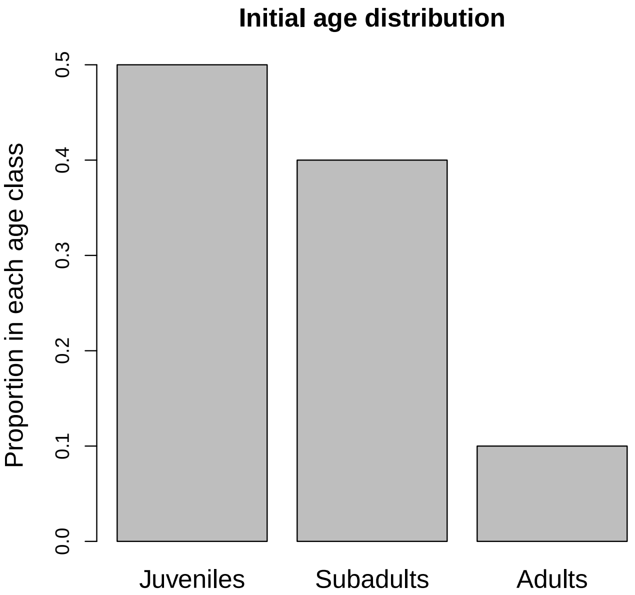
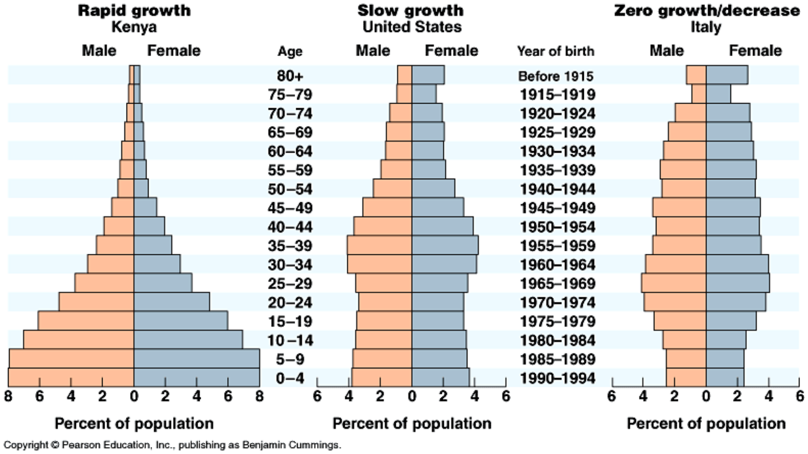
An age distribution describes the proportion of individuals in each age class


Initial age distribution
Proportion in each age class
0.0
0.1
0.2
0.3
0.4
0.5
Juveniles
Subadults
Adults

Declining populations have relatively more old individuals than growing populations



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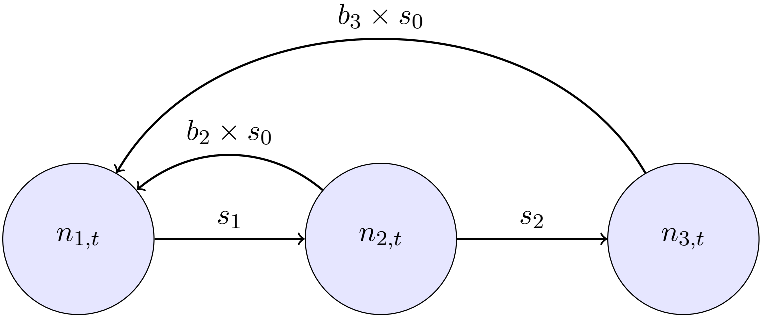

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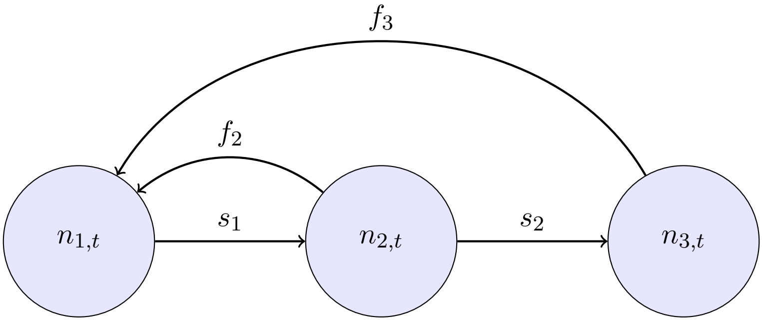
- Depends on age class survival rates s_i
- And age class birth rates b_i
- Fecundity is often defined as the product of birth rate and offspring survival, $f_i = b_i \times s_0$



Age class	Equation
1	$n_{1,t+1} = n_{1,t} \times f_1 + n_{2,t} \times f_2 + n_{3,t} \times f_3$

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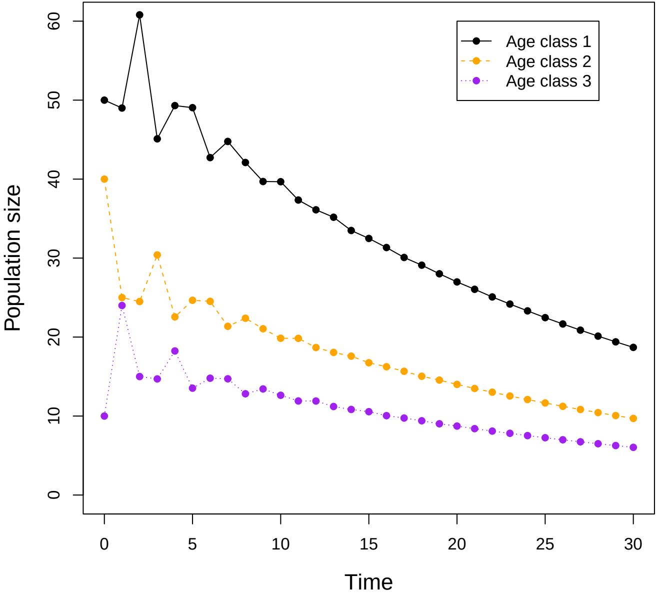
In stage-structured models, animals can stay in a class for more than one time interval.

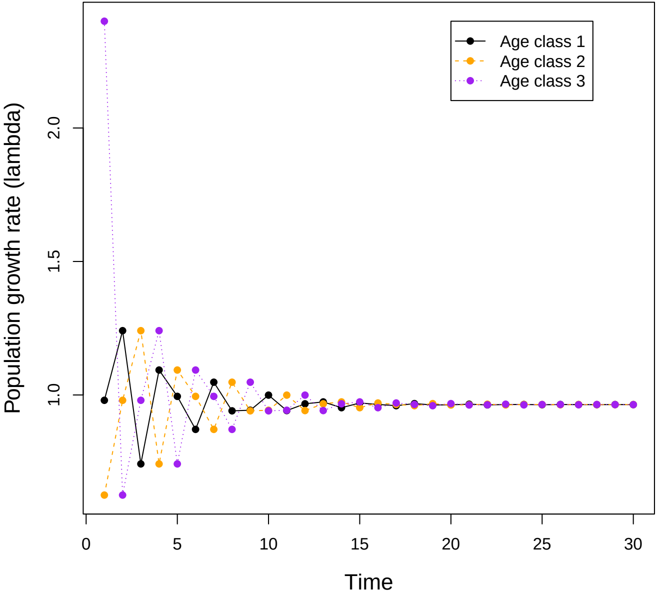
Let's choose values of s_i and f_i

Age class	Initial population size $n_{i,0}$	Survival probability s_i	Fecundity rate f_i
1	50	0.5	0.0
2	40	0.6	0.8
3	10	0.0	1.7

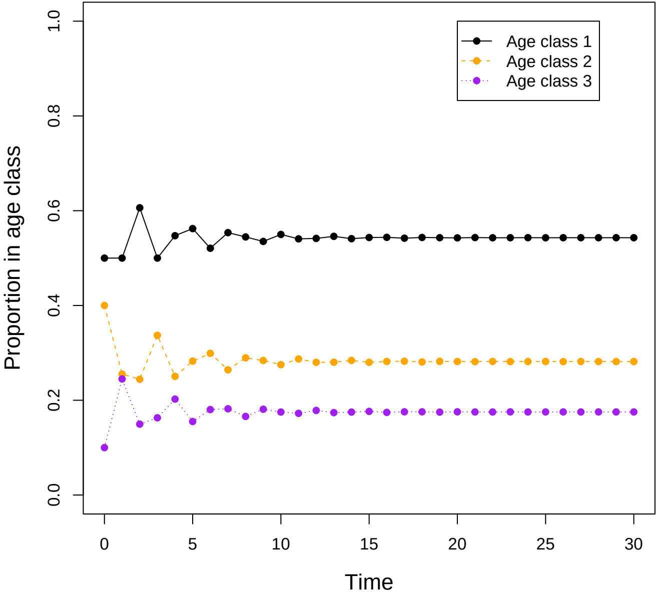
Population size, $n_{i,t}$



Growth rates, $n_{i,t+1}/n_{i,t} = \lambda_{i,t}$



Age distribution, $c_{i,t}$



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Three age class example – post-reproductive census

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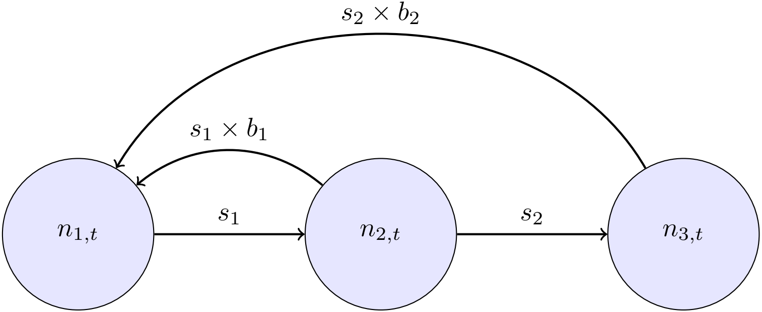

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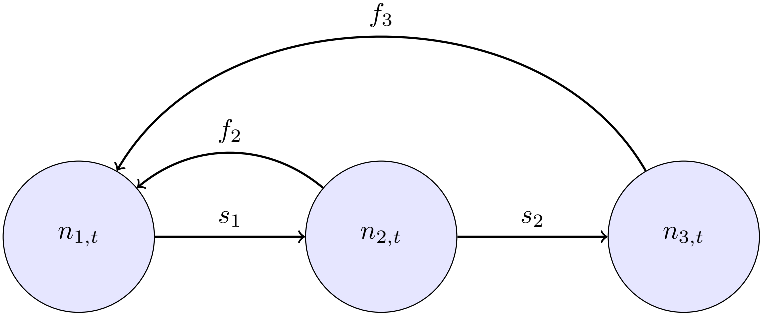

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Growth rate at the stable age distribution is the same for all age classes, and it is geometric!

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But, they make it much easier to compute important quantities like λ and *reproductive value*.

What is a matrix?

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Example of a 3×4 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{bmatrix}$$

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- Fertilities on first row
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Example:

$$\mathbf{A} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix}$$

These two expressions are equivalent:

Age class	Equation
1	$n_{1,t+1} = n_{1,t} \times f_1 + n_{2,t} \times f_2 + n_{3,t} \times f_3$
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AND

$$\mathbf{n}_{t+1} = \mathbf{A} \times \mathbf{n}_t$$

Matrix multiplication

$$= \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \times \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} aw + bx + cy + dz \\ ew + fx + gy + hz \\ iw + jx + ky + lz \\ mw + nx + oy + pz \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \times \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

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Reproductive Value

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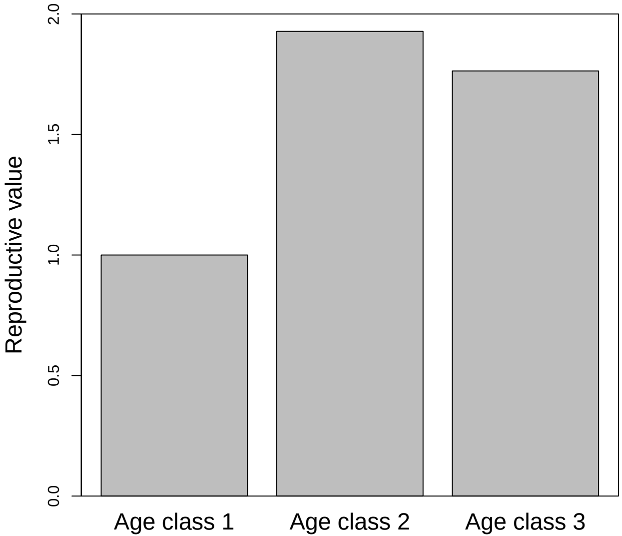
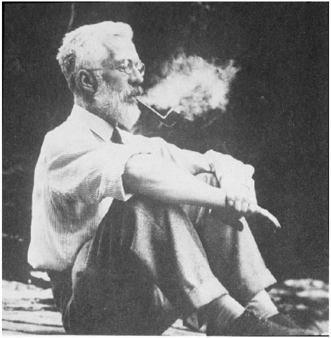
Depending on survival and fecundity rates, a first-year individual could have a higher or lower reproductive value than a second-year individual.

Reproductive value for previous example



Sir Ronald Aylmer Fisher was central to the development of the idea of reproductive value.



He was also a terrible person (eugenicist and Nazi sympathizer).

Net reproductive rate
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Generation time

The time required for a population to increase by a factor of R_0

Other properties of the Leslie matrix[1]

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Stage-structured population models

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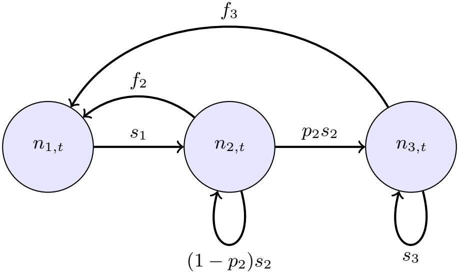

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This example allows for individual to remain in stages 2, 3, and 4 for more than one time period.

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Matrix models are a convenient method used to work with age-structured populations.