<u>Collection:</u>

Non-symmetric operads:

$\mathrm{Coll}(V) := [\mathbb{N}, V]$, $\mathbb{N}$ discrete

This is monoidal: $(F \circ G)(n) = \coprod_{k_1+\dots+k_m = n} F(m) \otimes G(k_1) \otimes \cdots \otimes G(k_m)$

Unit: $U(n) = \begin{cases} I & n = 1 \\ \emptyset & n \neq 1 \end{cases}$

A monoid in $\mathrm{Coll}(V)$ is an operad.

$F \circ F \longrightarrow F$ gives operadic composition.

$U \longrightarrow F$ gives unit in $F(1)$

Q: Convolution tensor product (Brian Day)?

Symmetric operads:

$\mathbb{N} \rightsquigarrow \Sigma$ $\quad (\Sigma^{op}?)$

$\Sigma \longrightarrow V$ is a symmetric collection.

$G$-collection: $G \longrightarrow V$

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Downarrow \, ?"] & V \\ \Sigma \arrow[ur, dashed, "\mathrm{Lan}_\pi F"'] & \end{tikzcd} \qquad \begin{tikzcd} {[G,V]} \arrow[r, shift left, "\mathrm{Lan}_\pi(-)"] \arrow[r, phantom, "\perp"] & {[\Sigma, V]} \arrow[l, shift left, "\pi^*"] \end{tikzcd}$$

<u>Conj</u>: $\pi$ operad map $\Rightarrow$ $\pi^*$ monoidal

<u>Q</u>: When is $\mathrm{Lan}_\pi(-)$ monoidal?

$\mathrm{Lan}_\pi(F)(n) \overset{?}{=} F(n)/\ker \pi_n$ need $\pi$ surj?

$G$-$\mathrm{Coll}(\mathrm{Cat})$ should be a monoidal 2-cat.

operads $\leftrightarrow$ monoids

pseudo-operads $\leftrightarrow$ pseudomonoids

$$\Sigma_X(n) \times G(n) \longrightarrow \Sigma_X(n)$$

$$[X^n, X] \times G(n) \xrightarrow{1 \times \pi_n} [X^n, X] \times \Sigma_n \longrightarrow [X^n, X]$$

Should prove that $\Sigma_X \cong \langle X, X \rangle$ (*)

- $\langle X, X \rangle$ uses some cotensors
- It is a (finitary?) 2-monad $Cat \to Cat$
- 2-monad maps $T \to \langle X, X \rangle$ correspond to strict alg str on $X$

Maybe (*) is false. But, we have

$$Mnd_?(Cat) \underset{L}{\overset{T}{\leftrightarrows}} Oper(Cat)$$ and maybe

left adj $\Sigma_X \cong L(\langle X, X \rangle)$.

A ps-comm on a G-operad $P$ consists of natural iso's

$$[\mu(q;p,\dots,p),(\underline{a},\underline{b})] \cong [\mu(p;q,\dots,q),(\underline{a},\underline{b})]$$

which correspond to
- $h \in G(mn)$ $(h = h(p,q,\underline{a},\underline{b}))$ s.t. $\pi(h)((\underline{a},\underline{b})) = (\underline{a},\underline{b})$
- $\mu(q;p,\dots,p)\cdot h \cong \mu(p;q,\dots,q)$

Swap direction makes nicer

We did strength/unit axioms on (4.3). Bis pasting diagram axioms:

HP #6 (1) $h = \mu(1;h'')\mu(h';1)$ (2) same w/ iso's } 5.3(c)

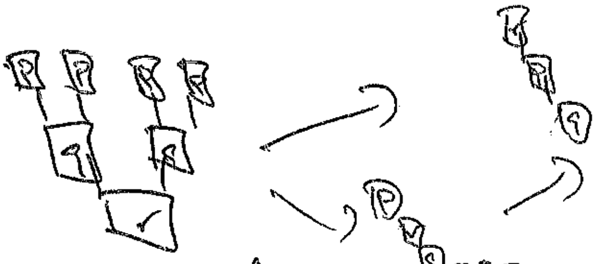
Conj: HP #7 is "$h = \mu(h';1)\mu(1;h'')$"

This equation only comes from a free action, otherwise we get $\_\cdot h = \_\cdot\mu(\ )\mu(\ )$. So (1) could follow just from the structure of $G$, then (2) must come from $P$.

<u>Conj</u>: Every $\pi: G \to \Sigma$ surjective has elts satisfying (1).

<u>Prop</u>: If $\pi$ is split by $\Sigma \hookrightarrow G$, then $G$ has elts satisfying (1).

$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[rr, bend right] & G \arrow[r, "\pi"] & \Sigma \end{tikzcd}$$



S.3(a)

<u>LHS</u>

- $h \in G\binom{lmn}{[illegible]}$ s.t. $\pi(h)(a_{ij}, b_k)\begin{bmatrix} i \\ j \\ k \end{bmatrix} = (a_{ij}, b_k)\begin{bmatrix} k \\ i \\ j \end{bmatrix}$
- $\mu(\mu(r; q, \dots, q); p, \dots, p) \cdot h \doteq \mu(p; \mu(r; q, \dots, q), \dots, \mu(r; q, \dots, q))$

<u>RHS</u>

- $h' \in G(mn)$ s.t. $\pi(h')(a_{\cdot j}, b_k)\begin{bmatrix} j \\ k \end{bmatrix} = (a_{\cdot j}, b_k)\begin{bmatrix} k \\ j \end{bmatrix}$
- $\mu(r; q, \dots, q) \cdot h' \doteq \mu(q; r, \dots, r)$

$+$

- $h_j \in G(ln)$ s.t. $\pi(h_j)(\quad)\begin{bmatrix} i \\ k \end{bmatrix} = \begin{bmatrix} k \\ i \end{bmatrix}$
- $\mu(r; p, \dots, p) \cdot h_j \doteq \mu(p; r, \dots, r)$

$h = \mu($

---

$$\mu(\mu(r; q, \dots, q); p, \dots, p) \cdot h \doteq \mu(p; \mu(r; q, \dots, q), \dots, \mu(r; q, \dots, q))$$

$$\shortparallel$$

$$\mu(r; \mu(q; p, \dots, p), \dots, \mu(q; p, \dots, p)) \cdot h$$

$$\mu(r; \mu(q; p, \dots, p), \dots, \mu(q; p, \dots, p)) \cdot h \doteq \mu(\mu(q; p, \dots, p); r, \dots, r)$$

$$\shortparallel$$

$$\mu(\mu(r; q, \dots, q); p, \dots, p) \cdot h$$

$$\cong$$

$$\mu(\mu(q; r, \dots, r) \cdot h'; p, \dots, p) \cdot h$$

$$\shortparallel$$

$$\mu(\mu(q; r, \dots, r); p, \dots, p) \cdot \mu(h'; \pi(h')(1, \dots, 1)) \cdot h$$

$$\begin{aligned}
\mu(r;\mu(q;p,\dots,p),\dots,\mu(q;p,\dots,p))\cdot h &\cong \mu(\mu(q;p);r) \\
&= \mu(\mu(r;q);p)\cdot h \\
&\cong \mu(\mu(q;r)\cdot h';p)\cdot h \\
&= \mu(\mu(q;r);p)\cdot\mu(h';1)\cdot h \\
&= \mu(q;\mu(r;p))\cdot\mu(h';1)\cdot h \\
&\cong \mu(q;\mu(p;r)\cdot h'')\cdot\mu(h';1)\cdot h \\
&= \mu(q;\mu(p;r))\cdot\mu(1;h'')\cdot\mu(h';1)\cdot h \\
&= \mu(\mu(q;p);r)\cdot\mu(1;h'')\cdot\mu(h';1)\cdot h
\end{aligned}$$

$$\mu(\overset{n}{r}; \mu(\overset{m}{q}; \overset{\ell}{p})) \cong \mu(\mu(q;p); r) \cdot h$$

$$\parallel$$

$$\mu(\mu(r;q); p)$$

$$\wr\wr$$

$$\mu(\mu(q;r)\cdot h'; p)$$

$$\parallel$$

$$\vdots$$

$$\cong$$

$$\mu(\mu(q;p); r) \cdot (\mu(\overset{m}{1}; \overset{n\ell}{h''}) \cdot \mu(\overset{mn}{h'}; \overset{\ell}{1}))$$

$h' \in G(mn)$
$h'' \in G(n\ell)$

---

<u>Conj. in 5.3</u>:

For each $\ell, m, n$, the functions

$$\Sigma_{mn} \times \Sigma_{n\ell} \longrightarrow \Sigma_{\ell mn}$$

$$(\sigma, \tau) \longmapsto \mu^{\Sigma}(1;\tau) \cdot \mu^{\Sigma}(\sigma, 1)$$

are jointly ~~i~~ injective on permutations arising from $\underline{(a, b)} \longmapsto \underline{(a, b)}$.

23/09.

$$\rho\, c_{A,B} \cdot \gamma_{A,B} \cdot c_{\rho B, \rho A} = \gamma_{B,A}^{-1}$$

$$\begin{array}{l}
[q,\underline{b}],[p,\underline{a}] \\
\quad \downarrow \\
[p,\underline{a}],[q,\underline{b}] \\
\quad \downarrow \\
[p(q,\dots,q),(\underline{a},\underline{b})] \\
\quad \cong \\
[q(p,\dots,p),(\underline{a},\underline{b})] \\
\quad \downarrow \\
[p(q,\dots,q),(\underline{b},\underline{a})] \\
\quad \cong \\
[q(p,\dots,p),(\underline{b},\underline{a})]
\end{array}
\qquad\qquad
\begin{array}{l}
[q,\underline{b}],[p,\underline{a}] \\
\quad \downarrow \\
[q(p,\dots,p),(\underline{b},\underline{a})] \\
\quad \cong \\
[p(q,\dots,q),(\underline{b},\underline{a})]
\end{array}$$

- $h \in G(mn)$ s.t. $\pi(h)(\underline{b},\underline{a}) = (\underline{b},\underline{a})$

; $\mathcal{M}(q;p,\dots,p) \cong \mathcal{M}(p;q,\dots,q)\cdot h$

$h$ s.t. $\pi(h)\,(\underline{b},\underline{a}) = (\underline{b},\underline{a})$

$k$ s.t. $\pi(k)\,(\underline{b},\underline{a}) = (\underline{b},\underline{a})$

Symmetric $\Rightarrow$ $h = k^{-1}$

Not possible for braids:

$m = n = 2$, pick $h, k \in B_4$ s.t.

$\pi(h) = \pi(k) = 1 \times 1 \in \Sigma_4$

Should show that the operad for strict braided mon. cats. has non-symm. ps-comm.