2 Action operads

In this chapter, we will explore the general definition of an operad, P ,which is equipped with groups of equivariance, $\Lambda(n)$. The group $\Lambda(n)$ will act on the right on the object $P(n)$, and the operad structure of P will be required to respect this action. For certain choices of the groups $\Lambda(n)$, we will recover standard notions of operads such as symmetric operads, non-symmetric operads, and braided operads. The definitions here will, unless otherwise stated, apply in any symmetric monoidal category $\mathcal{V}$ in which the functors $X \otimes -, - \otimes X$ preserve colimits for every object $X \in \mathcal{V}$.

2.1 Introduction to operads

ventions

This section will provide an introduction to operads, including a general overview of the definitions of plain, symmetric, and braided operads. Remarks and examples are provided throughout and the section begins with conventions around notation.

Conventions 2.1. We adopt the following conventions throughout.

- *Σ_n is the symmetric group on n letters, and B_n is the braid group on n strands.*
- *For a group G, a right G-action on a set X will be denoted $(x, g) \mapsto x \cdot g$. We will use both $\cdot$ and concatenation to represent multiplication in a group.*
- *The symbol e will generically represent an identity element in a group. If we are considering a set of groups $\{\Lambda(n)\}_{n \in \mathbb{N}}$ indexed by the natural numbers, then e_n is the identity element in $\Lambda(n)$.*

hroughout

- *We will often be interested in elements of a product of the form*

$$A \times B_1 \times \cdots \times B_n \times C$$

(or similar, for example without C). We will write elements of this set as $(a; b_1, \ldots, b_n; c)$, and in the case that we need equivalence classes of such elements they will be written as $[a; b_1, \ldots, b_n; c]$.

We begin with the basic definitions. The diagrams for the axioms in some operadic definitions can look quite involved and we recommend that readers expand the products depicted, or write out a small example case, to better see what the axioms are describing.

Definition 2.2. A *symmetric operad* O (in the category of sets) consists of

- a set, $O(n)$, for each natural number n,
- for each n, a right Σ_n-action on $O(n)$,
- an element $\mathrm{id} \in O(1)$, and
- functions

$$\mu \colon O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying the following three axioms.

1. The element $\mathrm{id} \in O(1)$ is a two-sided unit for μ in the sense that

$$\mu(\mathrm{id}; x) = x$$
$$\mu(x; \mathrm{id}, \ldots, \mathrm{id}) = x$$

for any $x \in O(n)$.

2. The functions μ (called operadic multiplication or operadic composition) are associative in the sense that the diagram below commutes.

$$\begin{tikzcd}
O(n)\times(\prod_{i=1}^{n} O(k_i))\times(\prod_{i=1}^{n}\prod_{j=1}^{k_i} O(l_{i,j})) \arrow[r, "\cong"] \arrow[dd, "\mu\times 1"'] & O(n)\times\prod_{i=1}^{n}(O(k_i)\times\prod_{j=1}^{k_i} O(l_{i,j})) \arrow[d, "1\times\prod\mu"] \\
 & O(n)\times\prod_{i=1}^{n} O(\sum_{j=1}^{k_i} l_{i,j}) \arrow[d, "\mu"] \\
O(\sum_{i=1}^{n} k_i)\times\prod_{i=1}^{n}\prod_{j=1}^{k_i} O(l_{i,j}) \arrow[r, "\mu"'] & O(\sum_{i=1}^{n}\sum_{j=1}^{k_i} l_{i,j})
\end{tikzcd}$$

3. The functions μ are equivariant with respect to the symmetric group actions, and so satisfies the following two equations.

$$\mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) = \mu(x; y_1, \ldots, y_n) \cdot (\tau_1 \oplus \ldots \oplus \tau_n)$$
$$\mu(x \cdot \sigma; y_1, \ldots, y_n) = \mu\left(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \cdot \sigma^+$$

Remark 2.3. It is useful to write out in full what the sets in the diagram of the second axiom above mean. The use of numerous products and indices is to save space but the full picture becomes much clearer when these are expanded. For the equations in the third axiom above to make sense, we must have

- $x \in O(n)$,
- $y_i \in O(k_i)$ for $i = 1, \ldots, n$,
- $\tau_i \in \Sigma_{k_i}$,
- $\sigma \in \Sigma_n$, and
- $\tau_1 \oplus \ldots \oplus \tau_n, \sigma^+ \in \Sigma_{k_1 + \ldots + k_n}$ as in Example 2.6 below.

Definition 2.4. A *non-symmetric operad* O consists of the same data as above but without any symmetric group actions, and only satisfying the first and second axioms.

Remark 2.5.
1. One can change from operads in **Sets** to operads in another symmetric monoidal category $\mathcal{V}$ by requiring each $O(n)$ to be an object of $\mathcal{V}$ and replacing all instances of cartesian product with the appropriate tensor product in $\mathcal{V}$. This includes replacing the element $\mathrm{id} \in O(1)$ with a map $I \to O(1)$ from the unit object of $\mathcal{V}$ to $O(1)$.
2. Every symmetric operad has an underlying *symmetric collection* which consists of the natural number-indexed set $\{O(n)\}_{n \in \mathbb{N}}$ together with symmetric group actions, but without a chosen identity element or composition maps. The category of symmetric collections is a presheaf category, and we will equip it with a monoidal structure in which monoids are precisely operads in Theorem 2.70. A similar construction, but without reference to group actions, shows that every non-symmetric operad has an underlying (non-symmetric) collection which is now merely a $\mathbb{N}$-indexed collection of sets.

2.2 Action operads

One should note that the axioms for symmetric and braided operads each use the fact that the groups of equivariance themselves form an operad. This is what we call an action operad and we remark on similar structures defined and studied by others. After introducing the basic definition, we will consider familiar examples, such as those just noted, as well as less familiar examples such as those constituted by ribbon braid groups or so-called 'cactus' groups. We then proceed with some remarks about terminal and initial action operads, as well as maps between action operads.

Definition 2.11. An *action operad* $\mathbf{\Lambda}$ consists of

- an operad $\Lambda = \{\Lambda(n)\}$ in the category of sets such that each $\Lambda(n)$ is equipped with the structure of a group and
- a map $\pi\colon \Lambda \to \Sigma$ which is simultaneously a map of operads and a group homomorphism $\pi_n : \Lambda(n) \to \Sigma_n$ for each n

such that one additional axiom holds. Write

$$\mu\colon \Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n) \to \Lambda(k_1 + \cdots + k_n)$$

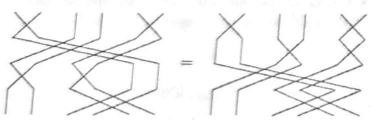
for the multiplication in the operad Λ. Let $(g; f_1, \ldots, f_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n)$ and let $(g'; f'_1, \ldots f'_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_{\pi(g)^{-1}(1)}) \times \cdots \times \Lambda(k_{\pi(g)^{-1}(1)})$. We require that

$$\mu\left(g'; f'_1, \ldots f'_n\right) \mu\left(g; f_1, \ldots, f_n\right) = \mu\left(g'g; f'_{\pi(g)(1)}f_1, \ldots, f'_{\pi(g)(n)}f_n\right) \tag{1}$$

in the group $\Lambda(k_1 + \cdots + k_n)$.

Remark 2.12.

- The final axiom is best explained using the operad Σ of symmetric groups. Reading symmetric group elements as permutations from top to bottom, below is a pictorial representation of the final axiom for the map $\mu\colon \Sigma_3 \times \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_6$.



$\mu((23);(12),(12),id)\cdot\mu((132);(12),id,(12))$ $\qquad$ $\mu((23)\cdot(132);id\cdot(12),(12)\cdot id,(12)\cdot(12))$

- Our definition of an action operad is the same as that appearing in Wahl's thesis [45], but without the condition that each π_n is surjective. It is also the same as that appearing in work of Zhang [47], although we prove later (see Lemma 2.27) that Zhang's condition of $e_1 \in \Lambda(1)$ being the identity element follows from the rest of the axioms.
- Yau [46] collects together a large number of results on the topic of action operads while also investigating the setting of infinity group operads.

take out? earlier?

Notation 2.13. F... group $\Lambda(n)$. We dr... clear from context ... Λ_n in place of $\Lambda(n)$...

Moved to original conventions

Example 2.14.

1. The terminal operad T in the category of sets has a unique action operad structure, $\mathbf{T}$. Since $T(n)$ is a singleton for each n, the group structure is unique, as is the map π. The single action operad axiom is then automatic as both sides of Equation (1) are the unique element which happens to be the identity. This is the initial object in the category of action operads (see Definition 2.18 for the definition of morphisms in the category of action operads).
2. The symmetric operad Σ has a canonical action operad structure. It is given by taking π to be the identity map, and this action operad will be denoted $\mathbf{\Sigma}$. This is the terminal object in the category of action operads.
3. Two less trivial examples are given by the braid groups, $\mathbf{\Lambda} = \mathbf{B}$, and the ribbon braid groups, $\mathbf{\Lambda} = \mathbf{RB}$. (A ribbon braid is given, geometrically, as a braid with strands replaced by ribbons in which we allow full twists. The actual definition of the ribbon braid groups is as the fundamental group of a configuration space in which points have labels in the circle, S^1; see [41].) In each case, the homomorphism π is given by taking underlying permutations, and the operad structure is given geometrically by using the procedure explained after Definition 2.7. We refer the reader to [9] for more information about braided operads, and to [41, 45] for information about the ribbon case. check
4. The operad of n-fruit cactus groups defined by Henriques and Kamnitzer in [14] has an action operad structure that we will discuss in Section 4.2.

Why two spots?

Definition 2.15. For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, \ldots, t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$b_i \cdot t_i = t_{i+1} \cdot b_i,$$
$$b_i \cdot t_{i+1} = t_i \cdot b_i.$$

rewrite

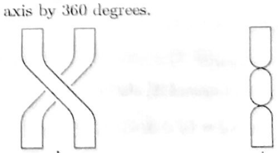
The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in Definition 2.7, but with some additional rules for dealing with twists. With regards to the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i = e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} = e_n, \qquad t_{(m+m')} = \left(t_{(m)} \otimes t_{(m')}\right) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Much as the symmetric groups can be represented by crossings of a collection of strings, and the braid groups by braidings of strings, the ribbon braid groups deal with the ways that one can braid together several flat ribbons, including the ability to twist a ribbon about its own axis by 360 degrees.



b $\qquad$ t

Now instead suppose that $\pi_2 \colon \Lambda(2) \to \Sigma_2$ is the zero map. Assume that $\pi_j \colon \Lambda(j) \to \Sigma_j$ is the zero map for some $j \geq 2$. Letting $\sigma \in \operatorname{Im}\pi_{j+1}$, there exists $g \in \Lambda(j+1)$ such that $\pi_{j+1}(g) = \sigma$. We can then consider the elements $\mu^{\Sigma}(\sigma; \underline{e_1}, e_0, \underline{e_1})$ where $\underline{e_1}$ means a sequence of e_1's and with e_0 in the kth position, with $1 \leq k \leq j+1$. Now

$$\begin{aligned}
\mu^{\Sigma}(\sigma; \underline{e_1}, e_0, \underline{e_1}) &= \mu^{\Sigma}(\pi_{j+1}(g); \underline{\pi_1(e_1)}, \pi_0(e_0), \underline{\pi_1(e_1)}) \\
&= \pi_j(\mu^{\Lambda}(g; \underline{e_1}, e_0, \underline{e_1})) \\
&= e_j.
\end{aligned}$$

We can think of this permutation as being σ with the kth string removed - in Remark 2.26 we comment on such 'face' and 'degeneracy' maps as used here, and see [5] for a more careful treatment of this idea. Now each of these is the identity e_j, as shown above. This means that σ must either have been the identity e_{j+1} or a transposition of the form $(a\ a+1)$, where $1 \leq a \leq j$.

If σ is the identity e_{j+1}, then we are done since this would give $\operatorname{Im}\pi_{j+1} = \{e_{j+1}\}$. Instead suppose that $\sigma = (a\ a+1) \in \Sigma_{j+1}$. Then if $1 < a \leq j$ we can use this to give

$$\begin{aligned}
(a-1\ a) &= \mu^{\Sigma}((a\ a+1); e_0, \underline{e_1}) \\
&= \mu^{\Sigma}(\pi_{j+1}(g); \pi_0(e_0), \underline{\pi_1(e_1)}) \\
&= \pi_j(\mu^{\Lambda}(g; e_0, \underline{e_1})) \\
&= e_j.
\end{aligned}$$

This gives a contradiction, hence $\sigma \neq (a\ a+1)$ and must be the identity e_{j+1}.

Similarly, if $\sigma = (1\ 2) \in \Sigma_{j+1}$, then in Σ_j we find that

$$\begin{aligned}
(1\ 2) &= \mu^{\Sigma}((1\ 2); \underline{e_1}, e_0) \\
&= \mu^{\Sigma}(\pi_{j+1}(g); \underline{\pi_1(e_1)}, \pi_0(e_0)) \\
&= \pi_j(\mu^{\Lambda}(g; \underline{e_1}, e_0)) \\
&= e_j.
\end{aligned}$$

Again, a contradiction, hence $\sigma = e_{j+1}$. □

Corollary 2.24. *Every action operad Λ with nontrivial π fits into a short exact sequence*

$$T \to Ker\pi \hookrightarrow \Lambda \xrightarrow{\pi} \Sigma \to T,$$

by which we mean this is a sequence of action operad maps which is an exact sequence of groups at each n.

Remark 2.25. Thus we see that an action operad is either an operad in groups, or is an extension of Σ by an operad in groups. This gives a simple proof that the operads of pure braids and pure ribbon braids are both operads in groups.

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?
prop.

Remark 2.26. The crossed simplicial groups of Fiedorowicz and Loday [10] are related to action operads in the following way. We can define a functor $C \colon \mathbf{AOp} \to \mathbf{CSGrp}$ from the category of action operads just described into the category of crossed simplicial groups. This functor takes an action operad Λ and defines $C(\Lambda)(n) = \Lambda(n+1)$. The face and degeneracy maps of the underlying simplicial structure are defined using the operadic composition inherent to Λ - the description of the maps via the operad structure can be viewed in a similar way to Construction 1.1 of [28]. This functor is, however, not faithful, nor is it conservative. (Recall that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is conservative if when Ff is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.)

We now study some of the structure on the groups $\Lambda(n)$ for small values of n. Recall that e_n is the identity element in the group $\Lambda(n)$.

Lemma 2.27. *Let Λ be an action operad.*

1. *In $\Lambda(1)$, the unit element e_1 for the group structure is equal to the identity element for the operad structure, id.*
2. *The equation*
$$\mu(e_n; e_{i_1}, \ldots, e_{i_n}) = e_I$$
holds for any natural numbers $n, i_j, I = \sum i_j$.
3. *The group $\Lambda(1)$ is abelian.*

Proof. For the first claim, let $g \in \Lambda(1)$. Then

$$\begin{aligned}
g &= g \cdot e_1 \\
&= \mu(g; \mathrm{id}) \cdot \mu(\mathrm{id}; e_1) \\
&= \mu(g \cdot \mathrm{id}; \mathrm{id} \cdot e_1) \\
&= \mu(g \cdot \mathrm{id}; \mathrm{id}) \\
&= g \cdot \mathrm{id}
\end{aligned}$$

using that e_1 is the unit element for the group structure, that id is a two-sided unit for operad multiplication, and the final axiom for an action operad together with the fact that the only element of the symmetric group Σ_1 is the identity permutation. Thus $g = g \cdot \mathrm{id}$, so $\mathrm{id} = e_1$.

For the second claim, write the operadic product as $\mu(e; \underline{e})$, and consider the square of this element. We find that

$$\begin{aligned}
\mu(e; \underline{e}) \cdot \mu(e; \underline{e}) &= \mu(e \cdot e; \underline{e} \cdot \underline{e}) \\
&= \mu(e; \underline{e}),
\end{aligned}$$

where the first equality follows from the last action operad axiom together with the fact that e gets mapped to the identity permutation; here $\underline{e} \cdot \underline{e}$ is the sequence $e_{i_1} \cdot e_{i_1}, \ldots, e_{i_n} \cdot e_{i_n}$. Thus $\mu(e; \underline{e})$ is an idempotent element of the group $\Lambda(I)$, so must be the identity element e_I.

For the final claim, note that the specific operadic multiplication map $\mu \colon \Lambda(1) \times \Lambda(1) \to \Lambda(1)$ is a group homomorphism following from the action operad axioms, and $\mathrm{id} = e_1$ is a two-sided unit, so the Eckmann-Hilton argument shows that μ is actually group multiplication and that $\Lambda(1)$ is abelian. □

Lemma 2.28. *Let Λ be an action operad. The group operation of $\Lambda(0)$ coincides with the operation*

$$g, h \in \Lambda(0) \mapsto \mu(e_2; g, h)$$

arising from the operad structure. Furthermore, $\Lambda(0)$ is abelian.

Proof. Note first that

$$\begin{aligned}
\mu(e_2; g, h) &= \mu(e_2 \cdot e_2; g \cdot e_0, e_0 \cdot h) \\
&= \mu(e_2; g, e_0) \cdot \mu(e_2; e_0, h),
\end{aligned}$$

so for the first claim that $\mu(e_2; g, h) = g \cdot h$ it will suffice to show that $\mu(e_2; g, e_0) = g$ and $\mu(e_2; e_0, h) = h$ for all $g, h \in \Lambda(0)$. We will use the fact that $\mu(e_0;) = e_0$, which follows below from a similar argument found in the second part of Lemma 2.27.

$$\begin{aligned}
\mu(e_0;)\mu(e_0;) &= \mu(e_0^2;) \\
&= \mu(e_0;).
\end{aligned}$$

Definition 2.32. A *filtered category* is a nonempty category C such that

- if a, b are objects of C, then there exists another object $c \in C$ and morphisms $a \to c, b \to c$; and
- if $f, g \colon a \to b$ are parallel morphisms in C, then there exists a morphism $h \colon b \to c$ such that $hf = hg$.

Definition 2.33. A *filtered colimit* is a colimit over a filtered category.

Definition 2.34. Let C be a category with all filtered colimits. An object $x \in C$ is *finitely presentable* if the representable functor $C(x, -) \colon C \to \mathbf{Sets}$ preserves filtered colimits.

Definition 2.35. A *locally finitely presentable category* is a category C such that

- C is cocomplete and
- there exists a small subcategory $C_{fp} \subseteq C$ of finitely presentable objects such that any object $x \in C$ is the filtered colimit of some diagram in C_{fp}.

The definition of a locally finitely presentable category has many equivalent variants, but we find this one most practicable to work with in this setting.

Theorem 2.36. *The category* $\mathbf{AOp}$ *is locally finitely presentable.*

Proof. First note that we can define a category $\mathbf{Op}^g$, whose objects are operads P in which each $P(n)$ also carries a group structure. This is an equational theory using equations with only finitely many elements, so $\mathbf{Op}^g$ is locally finitely presentable [1, Corollary 3.7]. A slice category of a locally finitely presentable category is itself locally finite presentable [1, Proposition 1.57] and since the symmetric operad is an object of $\mathbf{Op}^g$, the slice category $\mathbf{Op}^g/\Sigma$ is locally finitely presentable.

There is an obvious inclusion functor $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$. Now $\mathbf{AOp}$ is a full subcategory of $\mathbf{Op}^g/\Sigma$ which is closed under products, subobjects. Since any object of $\mathbf{Op}^g/\Sigma$ isomorphic to an action operad is in fact an action operad, the inclusion $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$ is actually the inclusion of a reflective subcategory. One can easily check that $\mathbf{AOp}$ is in fact closed under all limits and filtered colimits in $\mathbf{Op}^g/\Sigma$, so by the Reflection Theorem (2.48 in [1]), $\mathbf{AOp}$ is locally finitely presentable. □

Definition 2.37. Let $\mathcal{S}$ be the set which is the disjoint union of the underlying sets of all the symmetric groups. Then $\mathbf{Sets}/\mathcal{S}$ is the slice category over $\mathcal{S}$ with objects (X, f) where X is a set and $f \colon X \to \mathcal{S}$ and morphisms $(X_1, f_1) \to (X_2, f_2)$ are those functions $g \colon X_1 \to X_2$ such that $f_1 = f_2 g$. We call an object (X, f) a *collection over* $\mathcal{S}$.

Remark 2.38. In standard presentations of the theory of operads (see, for example, [36]), a nonsymmetric operad will have an underlying collection (or $\mathbb{N}$-indexed collection of sets) while a symmetric operad will have an underlying symmetric collection (or $\mathbb{N}$-indexed collection of sets in which the nth set has an action of Σ_n). Our collections over $\mathcal{S}$ more closely resemble the former as there is no group action present.

Example 2.39. One can easily form new action operads from old ones by taking limits. To take a limit of a diagram in $\mathbf{AOp}$, one forgets down to the category of operads over Σ and takes the limit there. Concretely, products in $\mathbf{AOp}$ are computed as products in $\mathbf{Op}/\Sigma$ which themselves are (possibly wide) pullbacks in the category of operads. This pullback will then be computed levelwise, showing that at each dimension there is a group structure with a group homomorphism to the appropriate Σ_n and that the final action operad axiom holds since it does in each component. The equalizer of a pair of maps will then just be the levelwise equalizer. This shows that the pointwise product of an action operad P with an action operad of the form $Z(Q)$ (as in Proposition 2.20) is again an action operad, but the pointwise product of two arbitrary action operads might not be.

Theorem 2.40. *There exists a forgetful functor* $U \colon \mathbf{AOp} \to \mathbf{Sets}/\mathcal{S}$ *which preserves all limits and filtered colimits.*

Proof. For a given action operad $\mathbf{\Lambda}$, we put $U(\mathbf{\Lambda}) = \left(\coprod_{\mathbb{N}} \Lambda(n), \coprod_{\mathbb{N}} \pi_n\right)$ and this easily extends to a mapping on morphisms using the universal property of the coproduct. The preservation of filtered colimits follows from the fact that these are computed pointwise, together with the fact that every map between action operads preserves underlying permutations. As equalizers are computed levelwise, and the product $\mathbf{\Lambda} \times \mathbf{\Lambda}'$ has underlying operad the pullback $\Lambda \times_\Sigma \Lambda'$; this pullback is itself computed levelwise. Together, these imply that U also preserves all limits. □

Corollary 2.41. *U has a left adjoint* $F \colon \mathbf{Sets}/\mathcal{S} \to \mathbf{AOp}$, *the free action operad functor.*

Proof. The category $\mathbf{Sets}/\mathcal{S}$ is locally finitely presentable as it is equivalent to the functor category $[\mathcal{S}, \mathbf{Sets}]$ (here $\mathcal{S}$ is treated as a discrete category) and any presheaf category is locally finitely presentable. The functor U preserves limits and filtered colimits between locally finitely presentable categories, so has a left adjoint (see Theorem 1.66 in [1]). □

Definition 2.42. A *presentation* for an action operad $\mathbf{\Lambda}$ consists of

- a pair of collections over $\mathcal{S}$ denoted $\mathbf{g}, \mathbf{r}$,
- a pair of maps $s_1, s_2 \colon F\mathbf{r} \to F\mathbf{g}$ between the associated free action operads, and
- a map $p \colon F\mathbf{g} \to \mathbf{\Lambda}$ of action operads exhibiting $\mathbf{\Lambda}$ as the coequalizer of s_1, s_2.

Give them for T, Σ, ?

2.5 Operads and algebras with general group actions

Just as we had the definitions of operad, symmetric operad, and braided operad, we now come to the general definition of a $\mathbf{\Lambda}$-operad, where $\mathbf{\Lambda}$ is an action operad. Thereafter we begin to delve into the theory around $\mathbf{\Lambda}$-operads. Beginning by recasting familiar operads as $\mathbf{\Lambda}$-operads for particular choices of $\mathbf{\Lambda}$, we then proceed to shadow standard operadic definitions for algebras and pseudoalgebras in this setting, characterising algebras using endomorphism $\mathbf{\Lambda}$-operads, and deriving monads from $\mathbf{\Lambda}$-operads. We show that there is an adjunction between $\mathbf{\Lambda}$-$\mathbf{Op}$ and Σ-$\mathbf{Op}$, and prove some technical results regarding maps between monads induced by $\mathbf{\Lambda}$-operads.

Definition 2.43. Let $\mathbf{\Lambda}$ be an action operad. A *$\mathbf{\Lambda}$-operad* P (in $\mathbf{Sets}$) consists of

- a non-symmetric operad P in $\mathbf{Sets}$ and
- for each n, an action $P(n) \times \Lambda(n) \to P(n)$ of $\Lambda(n)$ on $P(n)$

such that the following two equivariance axioms hold.

- For each $p \in P(n)$, $q_i \in P(k_i)$, and $g_i \in \Lambda(k_i)$:

$$\mu^P(p; q_1 \cdot g_1, \ldots, q_n \cdot g_n) = \mu^P(p; q_1, \ldots, y_n) \cdot \mu^\Lambda(e_n; g_1, \ldots, g_n).$$

- For each $p \in P(n)$, $g \in \Lambda(n)$, and $q_i \in \Lambda(k_i)$:

$$\mu^P(p \cdot g; y_1, \ldots, y_n) = \mu^P\left((p; y_{\pi(g)^{-1}(1)}, \ldots, y_{\pi(g)^{-1}(n)}\right) \cdot \mu^\Lambda(g; e_{k_1}, \ldots, e_{k_n}).$$

Remark 2.54. The proposition above holds for P-algebras in any closed symmetric monoidal category. Having a closed structure (in addition to all small colimits) is a stronger condition than the tensor preserving colimits in each variable, but it is a natural one that arises in many examples.

Definition 2.55. Let P be a Λ-operad. Then P induces an endofunctor of **Sets**, denoted $\underline{P}$, by the following formula.

$$\underline{P}(X) = \coprod_n P(n) \times_{\Lambda(n)} X^n$$

We now have the following proposition; its proof is standard [37], and we leave it to the reader should they be interested in the details.

Proposition 2.56. *Let P be a Λ-operad.*

1. *The Λ-operad structure on P induces a monad structure on $\underline{P}$.*
2. *The category of algebras for the operad P is isomorphic to the category of algebras for the monad $\underline{P}$.*

In the case that we take $P = \Lambda$, we do not get algebras more interesting than monoids.

Proposition 2.57. *Let Λ be an action operad. The category of algebras, Λ-**Alg**, for Λ taken as a Λ-operad, is equivalent to the category of monoids.*

Proof. The key observation here is that, for each $n \in \mathbb{N}$,

$$\Lambda(n) \times_{\Lambda(n)} X^n \cong X^n.$$

We will describe this bijection and leave the rest of the proof to the reader, which falls out of the various axioms either for being a monoid or for being a Λ-algebra.

Recall that the elements of $\Lambda(n) \times_{\Lambda(n)} X^n$ are equivalence classes of the form $[g; x_1, \ldots, x_n]$ for which

$$(gh; x_1, \ldots, x_n) \sim (g; x_{\pi(h)^{-1}(1)}, \ldots, x_{\pi(h)^{-1}(n)}).$$

There is an obvious map $X^n \to \Lambda(n) \times_{\Lambda(n)} X^n$ sending $(x_1, \ldots, x_n)$ to the equivalence class $[e; x_1, \ldots, x_n]$. The inverse to this map is given by the map $\Lambda(n) \times_{\Lambda(n)} X^n \to X^n$ sending $[g; x_1, \ldots, x_n]$ to the element $(x_{\pi(g)^{-1}(1)}, \ldots, x_{\pi(g)^{-1}(n)})$. It is then clear that these are inverses, relying on the equivalence relation to see that

$$[g; x_1, \ldots, x_n] = [e; x_{\pi(g)^{-1}(1)}, \ldots, x_{\pi(g)^{-1}(n)}].$$

That the second map is well-defined is simple to show. $\square$

We end this section with a discussion of the relationship between symmetric operads and Λ-operads for an arbitrary action operad Λ.

Theorem 2.58. *Let Λ be an action operad.*

redo more generally?

1. *An adjunction exists between the category of Λ-operads and the category of symmetric operads, with right adjoint $\pi^* \colon \Lambda\text{-Op} \to \Sigma\text{-Op}$ and left adjoint denoted S.*
2. *The counit of this adjunction is an isomorphism, but the unit is not, thus this adjunction is not an equivalence of categories*
3. *There exists a natural isomorphism of monads between $\underline{P}$ and $\underline{S(P)}$ for a Λ-operad P. In particular, these monads (and hence operads) have isomorphic categories of algebras.*

Proof. Given any map of monoids $f \colon M \to N$ in a monoidal category, there exists an adjunction between right M-modules and right N-modules given by f^* as the right adjoint and $A \mapsto A \otimes_M N$ as the left adjoint. Thus we define

$$S(P)(n) = P(n) \times_{\Lambda(n)} \Sigma_n,$$

and this inherits a right Σ_n-action by multiplication. The unit of $S(P)$ is

$$* \xrightarrow{\eta} P(1) \longrightarrow P(1) \times_{\Lambda(1)} \Sigma_1 \cong P(1)/\Lambda(1).$$

For the multiplication, let $K = k_1 + \cdots + k_n$, so we must define

$$\mu \colon (P(n) \times_{\Lambda(n)} \Sigma_n) \times \prod_{i=1}^{n} (P(k_i) \times_{\Lambda(k_i)} \Sigma_{k_i}) \to P(K) \times_{\Lambda(K)} \Sigma_K.$$

Using the universal property of the coequalizer, this is induced by the following composite.

$$\begin{aligned}(P(n) \times \Sigma_n) \times \prod_{i=1}^{n} (P(k_i) \times \Sigma_{k_i}) &\cong \left(P(n) \times \prod_{i=1}^{n} P(k_i)\right) \times \left(\Sigma_n \times \prod_{i=1}^{n} \Sigma_{k_i}\right) \\ &\xrightarrow{\mu^P \times \mu^\Sigma} P(K) \times \Sigma_K \\ &\longrightarrow P(K) \times_{\Lambda(K)} \Sigma_K\end{aligned}$$

We leave verification of the associativity, unit, and equivariance axioms to the reader; they are simple applications of the same axioms for P and Σ together with some colimit universal properties and the Λ-operad axioms for P. It is then straightforward to check the bijection between Λ-operad maps $P \to \pi^* Q$ and symmetric operad maps $S(P) \to Q$, thus establishing the adjunction.

The second claim is a simple calculation using the coequalizer that defines $S(\pi^* Q)$, using that $Q(n)$ is itself the coequalizer of the obvious pair of maps $Q(n) \times \Sigma_n \times \Sigma_n$.

For the third claim, we find a natural isomorphism

$$P(n) \times_{\Lambda(n)} X^n \cong (P(n) \times_{\Lambda(n)} \Sigma_n) \times_{\Sigma_n} X^n$$

by the universal property of the colimits involved, so as functors $\underline{P} \cong \underline{S(P)}$. One can then easily verify that this isomorphism commutes with the unit and multiplication of the two monads involved using calculations similar to those used to establish the adjunction. $\square$

Remark 2.59. 1. The adjunction alone is enough to establish that P and $S(P)$ have isomorphic categories of algebras using Proposition 2.53.

2. This theorem shows that semantically, one need never consider any kind of operad aside from symmetric operads: any other kind of operad can be symmetrized without altering the algebras. But as the operad should be considered a finer level of detail than the monad, restricting to symmetric operads misses the structure present in the more nuanced group actions.
3. Furthermore, there is clearly an artifact left from only considering the algebras themselves as objects in a symmetric monoidal category. It is well-known that a braided structure is all that is required for non-symmetric operads, and so one is left to consider that the natural home for algebras over a Λ-operad might be a type of monoidal structure other than symmetric in which case the theorem above gives no insight.

We end this section by presenting some results which allow us to transfer operad or algebra structures to other categories. We will use the following standard definitions of monad maps and transformations, as per [44].

Definition 2.60. Let S be a monad on a category C and T be a monad on a category D. A *monad map* of from S to T is a functor $F\colon C \to D$ together with a natural transformation $\alpha\colon TF \Rightarrow FS$ such that the following diagrams commute.

$$
\begin{tikzcd}
 & TFX \arrow[dd, "\alpha_X"] \\
FX \arrow[ur, "\eta^T_{FX}"] \arrow[dr, "F\eta^S_X"'] & \\
 & FSX
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TFSX \arrow[r, "\alpha_{SX}"] & FS^2X \arrow[dd, "F\mu^S_X"] \\
T^2FX \arrow[ur, "T\alpha_X"] \arrow[dr, "\mu^T_{FX}"'] & & \\
 & TFX \arrow[r, "\alpha_X"'] & FSX
\end{tikzcd}
$$

A *transformation* $\Gamma\colon (F,\alpha) \Rightarrow (G,\beta)$ between monad maps is a natural transformation $\Gamma\colon F \Rightarrow G$ such that the following diagram commutes.

Remark 2.61. Every monad map (F,α) induces a functor S-**Alg** $\to T$-**Alg** on the categories of algebras. An S-algebra (X,σ) is sent to the T-algebra $(FX, F\sigma\cdot\alpha_X)$, as we now describe. For $(FX, F\sigma\cdot\alpha_X)$ to be a T-algebra we require the usual diagrams to commute, shown as the outside of the diagrams below.

$$
\begin{tikzcd}
T^2FX \arrow[r, "T\alpha_X"] \arrow[dd, "\mu^T_{FX}"'] & TFSX \arrow[r, "TF\sigma"] \arrow[d, "\alpha_{SX}"] & TFX \arrow[d, "\alpha_X"] \\
 & FS^2X \arrow[r, "FS\sigma"] \arrow[d, "F\mu^S_X"] & FSX \arrow[d, "F\sigma"] \\
TFX \arrow[r, "\alpha_X"'] & FSX \arrow[r, "F\sigma"'] & FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\eta^T_{FX}"] \arrow[dr, "F\eta^S_X"] \arrow[ddr, "\mathrm{id}"'] & TFX \arrow[d, "\alpha_X"] \\
 & FSX \arrow[d, "F\sigma"] \\
 & FX
\end{tikzcd}
$$

The first diagram commutes since the left hand side is the second diagram required to commute for (F,α) to be a monad map, the square at the top right is an instance of naturality for α, while the bottom right square commutes since (X,σ) is an S-algebra. The second diagram commutes since the top triangle is again a requirement of α being a transformation, with the lower triangle commuting again as a result of (X,σ) being an S-algebra.

A morphism $f\colon (X,\sigma_X) \to (Y,\sigma_Y)$ of S-algebras is sent to the morphism $Ff\colon (FX, F\sigma_X\cdot\alpha_X) \to (FY, F\sigma_Y\cdot\alpha_Y)$, this being a map of T-algebras following from the naturality of α and of f being an S-algebra map. Functoriality follows from that of F.

Throughout the text we make reference to where results can be applied in a more general case where a symmetric monoidal category is cocomplete and for which the tensor product distributes over colimits in each variable. However, we include the following definition to be clear what is meant simply by a cocomplete symmetric monoidal category.

change

Definition 2.62. A cocomplete symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal category for which the underlying category is cocomplete.

revisit end of this section

The following three results tie in with material in the coming chapters but are of a general nature which fits the nature of this section.

Proposition 2.63. *Let C, D be cocomplete symmetric monoidal categories. Let Λ be an action operad, and P be a Λ-operad in C. Let $F\colon C \to D$ be a symmetric lax monoidal functor. Then FP is a Λ-operad in D, and there exists a monad map $(F,\psi)\colon (C, \underline{P}) \to (D, \underline{FP})$.*

Proof. Theorem 3.37 describes how the functor F can be used to describe a functor

$$\Lambda\text{-}Op(C) \to \Lambda\text{-}Op(D),$$

from which we see that FP is a Λ-operad in D.

The functor F constitutes the 1-cell of the monad map, while ψ is required to be a natural transformation as below.

$$
\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\underline{P}"'] \arrow[dr, phantom, "\swarrow\psi"] & D \arrow[d, "\underline{FP}"] \\
C \arrow[r, "F"'] & D
\end{tikzcd}
$$

We describe the components of this natural transformation at an object X of C below.

$$
\begin{aligned}
\underline{FP}(FX) &= \coprod_{n\in\mathbb{N}} FP(n) \otimes_{\Lambda(n)} (FX)^n \\
&\to \coprod_{n\in\mathbb{N}} F\left(P(n) \otimes_{\Lambda(n)} X^n\right) \\
&\to F\left(\coprod_{n\in\mathbb{N}} P(n) \otimes_{\Lambda(n)} X^n\right) \\
&= F(\underline{P}(X))
\end{aligned}
$$

The first morphism is a composite of the coherence cells of the type

$$FX \otimes FY \to F(X \otimes Y)$$

for the symmetric lax monoidal functor F, while the second morphism is the induced morphism out of the coproduct. Naturality follows from that of the component morphisms. It is then straightforward to see that the monad morphism diagrams commute since the diagrams involved consist of instances of the coherence axioms for F along with naturality of the coherence cells. □

Proposition 2.64. *Let C be a cocomplete symmetric monoidal category. Let Λ be an action operad, and P, Q be Λ-operads in C with a map $\sigma\colon P \to Q$ of Λ-operads between them. Then σ induces a monad map $(id, \sigma^*)\colon (C, \underline{Q}) \to (C, \underline{P})$ and hence a functor on categories of algebras.*

Proof. We will first describe the components of the natural transformation $\sigma^*\colon \underline{P} \Rightarrow \underline{Q}$. The component σ^*_X at an object X of C is a morphism between the coproducts

$$\sigma^*_X\colon \coprod_{n\in\mathbb{N}} P(n) \otimes_{\Lambda(n)} X^n \to \coprod_{n\in\mathbb{N}} Q(n) \otimes_{\Lambda(n)} X^n.$$

This is seen to be induced by the universal properties of the coequalizers and coproducts in the following diagram, where σ_n denotes the n-ary component of the Λ-operad map σ and λ^P, λ^Q, ρ^P, and ρ^Q denote the usual left and right actions.

revisit later

The upper square which includes λ^P and λ^Q commutes due to σ being a Λ-operad map, while the square with both ρ actions commutes because the σ_n and ρ do not interact. Since c_n^Q coequalizes λ^Q and ρ^Q, then this commutativity shows that $c_n^Q \cdot (\sigma_n \otimes \mathrm{id}) \cdot \lambda^P = c_n^Q \cdot (\sigma_n \otimes \mathrm{id}) \cdot \rho^P$, hence the morphism $\sigma^*_{n,X}$ exists. The morphism σ^*_X is then induced by the universal property of the coproduct $\underline{P}(X)$.

$$\begin{tikzcd}
P(n) \otimes \Lambda(n) \otimes X^n \arrow[r, "\sigma_n \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[d, shift right, "\lambda^P"'] \arrow[d, shift left, "\rho^P"] & Q(n) \otimes \Lambda(n) \otimes X^n \arrow[d, shift right, "\lambda^Q"'] \arrow[d, shift left, "\rho^Q"] \\
P(n) \otimes X^n \arrow[r, "\sigma_n \otimes \mathrm{id}"] \arrow[d, "c_n^P"] & Q(n) \otimes X^n \arrow[d, "c_n^Q"] \\
P(n) \otimes_{\Lambda(n)} X^n \arrow[r, "\exists ! \sigma^*_{n,X}"'] \arrow[d] & Q(n) \otimes_{\Lambda(n)} X^n \arrow[d] \\
\coprod_{n \in \mathbb{N}} P(n) \otimes_{\Lambda(n)} X^n \arrow[r, "\exists ! \sigma^*_X"] & \coprod_{n \in \mathbb{N}} Q(n) \otimes_{\Lambda(n)} X^n
\end{tikzcd}$$

It is then routine to check that these components are natural in X and constitute a monad map. That a functor is then induced on the category of algebras follows from Lemma 6.1.1 of [31]; the process is described above, following Definition 2.60. □

We can combine these two propositions.

Corollary 2.65. *If C, D, P, F are as in Proposition 2.63, and F is part of a monoidal adjunction (i.e., an adjunction in which both functors are symmetric lax monoidal, and the unit and counit are monoidal transformations) $F \dashv U$, then (F, id) and (U, id) are both monad maps. The unit $\eta\colon 1 \Rightarrow UF$ induces an operad map $\eta\colon P \Rightarrow UFP$, and a transformation between monad maps*

$$(id, id) \Rightarrow (id, \eta^*) \circ (U, \psi^U) \circ \left(F, \psi^F\right).$$

The counit $\varepsilon\colon FU \Rightarrow 1$ induces an operad map $\varepsilon\colon FUFP \Rightarrow FP$, and a transformation between monad maps

$$\left(F, \psi^F\right) \circ (id, \eta^*) \circ \left(U, \psi^U\right) \Rightarrow (id, id).$$

These constitute an adjunction $(F, \psi^F) \dashv (id, \eta^*) \circ (U, \psi^U)$ in the 2-category of monads, and hence induce an adjunction between $\underline{P}$-algebras in C and $\underline{FP}$-algebras in D.

2.6 Operads as monoids

In this section, we will show that Λ-operads are the monoids in the category of Λ-collections equipped with an appropriate substitution product. Such a result is fairly standard [36], and in both the symmetric and non-symmetric cases can easily be proven directly. Since we work with an arbitrary action operad, however, it will be more economical to take the abstract approach using coends and Day convolution.

Remark 2.66. It is possible to consider Λ-operads in categories other than the category of sets. In this case we still use the notion of an action operad given above, but then take the operad P to have objects $P(n)$ which are the objects of some closed symmetric monoidal category $\mathcal{V}$. We will rarely use anything that might require the closed structure as such, only the fact that the tensor product distributes over colimits in each variable. This is a consequence of the fact that both $X \otimes -$ and $- \otimes X$ are left adjoints in the case of a closed symmetric monoidal category. Thus while we set up the foundations using only operads in **Sets**, the diligent reader could easily modify this theory for their closed symmetric monoidal category of choice. In fact, we will use the same theory in **Cat** with its cartesian structure, noting only that the same arguments work in **Cat** with essentially no modification.

Definition 2.67. Let Λ be an action operad. The category Λ-**Coll** of Λ-collections has objects $X = \{X(n)\}_{n \in \mathbb{N}}$ which consist of a set $X(n)$ for each natural number n together with an action $X(n) \times \Lambda(n) \to X(n)$ of $\Lambda(n)$ on $X(n)$. A morphism $f\colon X \to Y$ in Λ-**Coll** consists of a $\Lambda(n)$-equivariant map $f_n\colon X(n) \to Y(n)$ for each natural number n.

Remark 2.68. The definition of Λ-**Coll** does not require that Λ be an action operad, only that one has a natural number-indexed set of groups. Given any such collection of groups $\{\Lambda(n)\}_{n \in \mathbb{N}}$, we can form the category $\boldsymbol{\Lambda}$ whose objects are natural numbers and whose hom-sets are given by $\boldsymbol{\Lambda}(m, n) = \emptyset$ if $m \neq n$ and $\boldsymbol{\Lambda}(n, n) = \Lambda(n)$ (where composition and units are given by group multiplication and identity elements, respectively). Then Λ-**Coll** is the presheaf category

$$\hat{\boldsymbol{\Lambda}} = [\boldsymbol{\Lambda}^{op}, \mathbf{Sets}],$$

with the opposite category arising from our choice of right actions. A key step in explaining how Λ-operads arise as monoids in the category of Λ-collections is to show that being an action operad endows $\boldsymbol{\Lambda}$ with a monoidal structure.

Definition 2.69. Let Λ be an action operad, and let X, Y be Λ-collections. We define the Λ-collection $X \circ Y$ to be

$$X \circ Y(n) = \left(\left(\coprod_{k_1 + \cdots + k_r = n} X(r) \times Y(k_1) \times \cdots \times Y(k_r) \right) \times \Lambda(n) \right) / \sim$$

where the equivalence relation is generated by

$$(xh; y_1, \ldots, y_r; g) \sim \left(x; y_{\pi(h)^{-1}(1)}, \ldots, y_{\pi(h)^{-1}(r)}; \mu(h; e, \ldots, e)g\right),$$
$$(xe; y_1 g_1, \ldots, y_r g_r; g) \sim \left(x; y_1, \ldots, y_r; \mu(e; g_1, \ldots, g_r)g\right).$$

For the first relation above, we must have that the lefthand side is an element of

$$X(r) \times Y(k_1) \times \cdots \times Y(k_r) \times \Lambda(n)$$

while the righthand side is an element of

$$X(r) \times Y\left(k_{\pi(h)^{-1}(1)}\right) \times \cdots \times Y\left(k_{\pi(h)^{-1}(r)}\right) \times \Lambda(n);$$

for the second relation, we must have $x \in X(r)$, $y_i \in Y(k_i)$, $f \in \Lambda(r)$, $g_i \in \Lambda(k_i)$, and $g \in \Lambda(n)$. The right $\Lambda(n)$-action on $X \circ Y(n)$ is given by multiplication on the final coordinate.

We will now develop the tools to prove that the category Λ-**Coll** has a monoidal structure given by $\circ$, and that operads are the monoids therein.

Theorem 2.70. *Let Λ be an action operad.*

1. *The category Λ-*Coll *has a monoidal structure with tensor product given by $\circ$ and unit given by the collection I with $I(n) = \emptyset$ when $n \neq 1$, and $I(1) = \Lambda(1)$ with the Λ-action given by multiplication on the right.*
2. *The category* Mon(Λ-Coll) *of monoids in* Λ-Coll *is equivalent to the category of Λ-operads.*

While this theorem can be proven by direct calculation using the equivalence relation given above, such a proof is unenlightening. Furthermore, we want to consider Λ-operads in categories other than sets, so an element-wise proof might not apply. Instead we now develop some general machinery that will apply to Λ-operads in any cocomplete symmetric monoidal category in which each of the functors $X \otimes -, - \otimes X$ preserve colimits (as is the case if the monoidal structure is closed). This theory also demonstrates the importance of the final axiom in the definition of an action operad. Our construction of the monoidal structure on the category of Λ-collections will require the Day convolution product [7]. This is a general construction which produces a monoidal structure on the category of presheaves $[\mathcal{V}^{op}, \mathbf{Sets}]$ from a monoidal structure on the category $\mathcal{V}$. Since the category of Λ-collections is the presheaf category $[\mathbb{\Lambda}^{op}, \mathbf{Sets}]$, we need to show that $\mathbb{\Lambda}$ has a monoidal structure.

Proposition 2.71. *The action operad structure of Λ gives $\mathbb{\Lambda}$ a strict monoidal structure.*

Proof. The tensor product on $\mathbb{\Lambda}$ is given by addition on objects, with unit object 0. The only thing to do is define the tensor product on morphisms and check naturality for the associativity and unit isomorphisms, which will both be identities. On morphisms, $+$ must be given by a group homomorphism

$$+ \colon \Lambda(n) \times \Lambda(m) \to \Lambda(n+m),$$

and this is given by the formula

$$+(g,h) = \mu(e_2; g, h).$$

6 notation?

We need that $+$ is a group homomorphism, and the second part of Lemma 2.27 shows that it preserves identity elements. The final action operad axiom shows that it also preserves group multiplication since $\pi_2(e_2) = e_2$ (each π_n is a group homomorphism) and therefore

$$\begin{aligned}(+(g,h)) \cdot (+(g',h')) &= \mu(e_2; g, h) \cdot \mu(e_2; g', h') \\ &= \mu(e_2 e_2; gg', hh') \\ &= +(gg', hh').\end{aligned}$$

We now write $+(g,h)$ as $g+h$.

For naturality of the associator, we must have $(f+g)+h = f+(g+h)$. By the operad axioms for both units and associativity, the lefthand side is given by

strictness

$$\begin{aligned}\mu(e_2; \mu(e_2; f, g), h) &= \mu(e_2; \mu(e_2; f, g), \mu(\mathrm{id}; h)) \\ &= \mu(\mu(e_2; e_2, \mathrm{id}); f, g, h),\end{aligned}$$

while the righthand side is then

$$\mu(e_2; f, \mu(e_2; g, h)) = \mu(\mu(e_2; \mathrm{id}, e_2); f, g, h).$$

By Lemma 2.27, both of these are equal to $\mu(e_3; f, g, h)$, proving associativity. Naturality of the unit isomorphisms follows similarly, using e_0. $\square$

Now that $\mathbb{\Lambda}$ has a monoidal structure, there is also a monoidal structure on the category of $\mathbb{\Lambda}$-collections

$$[\mathbb{\Lambda}^{op}, \mathbf{Sets}] = \hat{\mathbb{\Lambda}}$$

using Day convolution, denoted $\star$. Given collections X, Y, their convolution product $X \star Y$ is given by the coend formula

$$X \star Y(k) = \int^{m,n \in \mathbb{\Lambda}} X(m) \times Y(n) \times \mathbb{\Lambda}(k, m+n)$$

We refer the reader to [7] for further details. We do note, however, that the n-fold Day convolution product of a presheaf Y with itself is given by the following coend formula.

$$Y^{\star n}(k) = \int^{(k_1,\ldots,k_n) \in \mathbb{\Lambda}^n} Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n)$$

Computations with Day convolution will necessarily involve heavy use of the calculus of coends, and we refer the unfamiliar reader to [34] or [32]. Our goal is to express the substitution tensor product as a coend just as in [23], and to do that we need one final result about the Day convolution product.

Lemma 2.72. *Let Λ be an action operad, let $Y \in \hat{\mathbb{\Lambda}}$, and let k be a fixed natural number. Then the assignment*

$$n \mapsto Y^{\star n}(k)$$

can be given the structure of a functor $\mathbb{\Lambda} \to \mathbf{Sets}$.

Proof. Since the convolution product is given by a coend, it is the universal object with maps

$$Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \to Y^{\star n}(k)$$

such that the following diagram commutes for every $g_1 \in \Lambda(k_1), \ldots, g_n \in \Lambda(k_n)$, where $\underline{k} = k_1 + \cdots + k_n$.

$$\begin{tikzcd} \prod_{i=1}^n Y(k_i) \times \mathbb{\Lambda}(k, \underline{k}) \arrow[r, "(-\cdot g_1, \cdots, -\cdot g_n) \times 1"] \arrow[d, "1 \times ((g_1 + \cdots + g_n) \cdot -)"'] & \prod_{i=1}^n Y(k_i) \times \mathbb{\Lambda}(k, \underline{k}) \arrow[d] \\ \prod_{i=1}^n Y(k_i) \times \mathbb{\Lambda}(k, \underline{k}) \arrow[r] & Y^{\star n}(k) \end{tikzcd}$$

The first map along the top acts using the g_i's, while the first map along the bottom is given by

$$h \mapsto \mu(e_n; g_1, \ldots, g_n) \cdot h$$

in the final coordinate.

Let $f \in \Lambda(n)$, considered as a morphism $n \to n$ in $\mathbb{\Lambda}$. We induce a map $f \bullet - \colon Y^{\star n}(k) \to Y^{\star n}(k)$ using the collection of maps

$$\prod_{i=1}^n Y(k_i) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \to \prod_{i=1}^n Y(k_{\pi(f)^{-1}(i)}) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n)$$

by using the symmetry $\pi(f)$ on the first n factors and left multiplication by the element $\mu(f; e_{k_1}, \ldots, e_{k_n})$ on $\mathbb{\Lambda}(k, k_1 + \cdots + k_n)$. To induce a map between the coends, we must show that these maps commute with the two lefthand maps in the diagram above. For the top map, this is merely functoriality of the product together with naturality of the symmetry. For the bottom map, this is the equation

$$\mu(f;\overline{e})\cdot\mu(e;g_1,\dots,g_n) = \mu(e;g_{\pi(f)^{-1}1},\dots,g_{\pi(f)^{-1}n})\cdot\mu(f;\overline{e}).$$

Both of these are equal to $\mu(f;g_1,\dots,g_n)$ by the action operad axiom. Functoriality is then easy to check using that the maps inducing $(f_1 f_2)\bullet -$ are given by the composite of the maps inducing $f_1\bullet(f_2\bullet -)$. □

We are now ready for the abstract description of the substitution tensor product. The following proposition is easily checked directly using the definition of the coend; in fact, the righthand side below should be taken as the definition of $X\circ Y$ as both sides are really the result of some colimiting process.

Proposition 2.73. *Let $X, Y \in \hat{\Lambda}$. Then*

$$(X\circ Y)(k) \cong \int^n X(n)\times Y^{\star n}(k).$$

Finally we are in a position to prove Theorem 2.70. We make heavy use of the following consequence of the Yoneda lemma: given any functor $F\colon \Lambda \to$ **Sets** and a fixed object $a\in\Lambda$, we have a natural isomorphism

$$\int^{n\in\Lambda} \Lambda(n,a)\times F(n) \cong F(a);$$

there is a corresponding result for $F\colon \Lambda^{\mathrm{op}} \to$ **Sets** using representables of the form $\Lambda(a,n)$ instead.

Proof of Theorem 2.70. First we must show that Λ-**Coll** has a monoidal structure using $\circ$. To prove this, we must give the unit and associativity isomorphisms and then check the monoidal category axioms. First, note that the unit object is given as $I = \Lambda(-,1)$. Then for the left unit isomorphism, we find that

$$\begin{aligned} I\circ Y(k) &\cong \int^n \Lambda(n,1)\times Y^{\star n}(k)\\ &\cong Y^{\star 1}(k)\\ &\cong Y(k)\end{aligned}$$

using only the properties of the coend. For the right unit isomorphism, we have that

$$\begin{aligned} X\circ I(k) &\cong \int^n X(n)\times I^{\star n}(k)\\ &\cong \int^n X(n)\times \int^{k_1,\dots,k_n}\Lambda(k_1,1)\times\cdots\times\Lambda(k_n,1)\times\Lambda(\underline{k})\\ &\cong \int^n X(n)\times\Lambda(k,1+\cdots+1)\\ &= \int^n X(n)\times\Lambda(k,n)\\ &\cong X(k)\end{aligned}$$

using the same methods and where $\underline{k} = k_1+\cdots+k_n$.

For associativity, we compute $(X\circ Y)\circ Z$ and $X\circ(Y\circ Z)$.

$$\begin{aligned} ((X\circ Y)\circ Z)(k) &= \int^m (X\circ Y)(m)\times Z^{\star m}(k)\\ &= \int^m\left(\int^l X(l)\times Y^{\star l}(m)\right)\times Z^{\star m}(k)\\ &\cong \int^{m,l} X(l)\times Y^{\star l}(m)\times Z^{\star m}(k)\\ &\cong \int^l X(l)\times\int^m Y^{\star l}(m)\times Z^{\star m}(k)\end{aligned}$$

The first isomorphism is from products distributing over colimits and hence coends, and the second is that fact plus the Fubini Theorem for coends [34]. A similar calculation shows

$$(X\circ(Y\circ Z))(k) \cong \int^l X(l)\times(Y\circ Z)^{\star l}(k).$$

Thus the associativity isomorphism will be induced once we construct an isomorphism $\int^m Y^{\star l}(m)\times Z^{\star m}\cong (Y\circ Z)^{\star l}$. We do this by induction, with the $l=1$ case being the isomorphism $Y^{\star 1}\cong Y$ together with the definition of $\circ$. Assuming true for l, we prove the case for $l+1$ by the calculations below.

$$\begin{aligned} (Y\circ Z)^{\star(l+1)} &\cong (Y\circ Z)\star(Y\circ Z)^{\star l}\\ &\cong (Y\circ Z)\star\left(\int^m Y^{\star l}(m)\times Z^{\star m}\right)\\ &= \left(\int^n Y(n)\times Z^{\star n}\right)\star\left(\int^m Y^{\star l}(m)\times Z^{\star m}\right)\\ &= \int^{a,b}\left(\int^n Y(n)\times Z^{\star n}(a)\right)\times\left(\int^m Y^{\star l}(m)\times Z^{\star m}(b)\right)\times\Lambda(-,a+b)\\ &\cong \int^{a,b,n,m} Y(n)\times Y^{\star l}(m)\times Z^{\star n}(a)\times Z^{\star m}(b)\times\Lambda(-,a+b)\\ &\cong \int^{n,m} Y(n)\times Y^{\star l}(m)\times Z^{\star(n+m)}\\ &\cong \int^j\int^{n,m} Y(n)\times Y^{\star l}(m)\times\Lambda(j,n+m)\times Z^{\star j}\\ &\cong \int^j Y^{\star(l+1)}(j)\times Z^{\star j}\end{aligned}$$

Each isomorphism above arises from the symmetric monoidal structure on **Sets** using products, the monoidal structure on presheaves using $\star$, the properties of the coend, or the fact that products distribute over colimits.

For the monoidal category axioms on $\hat{\Lambda}$, we only need to note that the unit and associativity isomorphisms arise, using the universal properties of the coend, from the unit and associativity isomorphisms on the category of sets together with the interaction between products and colimits. Hence the monoidal category axioms follow by those same axioms in **Sets** together with the universal property of the coend.

Now we must show that monoids in $(\hat{\Lambda},\circ)$ are operads. By the Yoneda lemma, a map of Λ-collections $\eta\colon I\to X$ corresponds to an element $\mathrm{id}\in X(1)$ since $I=\Lambda(-,1)$. A map $\mu\colon X\circ X\to X$ is given by a collection of $\Lambda(k)$-equivariant maps $(X\circ X)(k)\to X(k)$. By the universal property of the coend, this is equivalent to giving maps

$$\mu_{n,\underline{k}}\colon X(n)\times X(k_1)\times\cdots\times X(k_n)\times\Lambda(k,k_1+\cdots+k_n)\to X(k)$$

For there to be an isomorphism of 2-categories, we require an inverse to R, namely a 2-functor S: **Ps**-P-**Alg** $\to$ **Ps**-$\underline{P}$-**Alg**. Now assume that $(X, \alpha_n, \phi_{k_i}, \phi_\eta)$ is a P-pseudoalgebra. We will give the same object X a $\underline{P}$-pseudoalgebra structure. We can induce a functor $\alpha\colon \underline{P}(X) \to X$ by using the universal property of the coproduct.

$$\begin{tikzcd} P(n)\times X^n \arrow[r] \arrow[drr, "\tilde{\alpha}_n"'] & P(n)\times_{\Lambda(n)} X^n \arrow[r, hook] \arrow[dr, "\alpha_n"] & \underline{P}(X) \arrow[d, "\exists!\alpha"] \\ & & X \end{tikzcd}$$

Of course, this can be induced using either α_n or $\tilde{\alpha}_n$, each giving the same functor α by uniqueness. The components of the isomorphism $\Phi\colon \alpha \circ \underline{P}(\alpha) \Rightarrow \alpha \circ \mu_X$ can be given as follows. Let $|\underline{x}_i|$ denote the number of objects in the list $\underline{x}_i$. Then define the component of Φ at the object

$$[p;[q_1;\underline{x}_1],\ldots,[q_n;\underline{x}_n]]$$

to be the component of $\phi_{|\underline{x}_1|,\ldots,|\underline{x}_n|}$ at the same object. To make this clearer, consider the object $[p;[q_1;x_{11}],[q_2;x_{21},x_{22}],[q_3;x_{31}]]$. The component of Φ at this object is given by the component of $\phi_{1,2,1}$ at the same object. The isomorphism ϕ_η is again sent to itself.

Now given a 1-cell f with structure 2-cells $\overline{f}_n$ we define a 1-cell $(F,\overline{F})$ with underlying 1-cell f and structure 2-cell $\overline{F}$ with components

$$\overline{F}_{[p;x_1,\ldots,x_n]} := \left(\overline{f}_n\right)_{(p;x_1,\ldots,x_n)}.$$

For example, the component of $\overline{F}$ at the object $[p;x_1,x_2,x_3]$ would be the component of $\overline{f}_3$ at the object $(p;x_1,x_2,x_3)$.

The mapping for 2-cells is just the identity as before. These mappings again constitute a 2-functor in the obvious way and from how they are defined it is also clear that this is an inverse to R. □

Remark 3.19. Another interpretation of pseudoalgebras can be given in terms of pseudomorphisms of operads. Algebras for an operad P can be identified with a morphism of operads $F\colon P \to \mathcal{E}_X$, where $\mathcal{E}_X$ is the endomorphism operad (Proposition 2.53). We can similarly define pseudomorphisms for a **Cat**-enriched Λ-operad and identify pseudoalgebras with pseudomorphisms into the endomorphism operad.

If P, Q are Λ-operads then a *pseudomorphism* of Λ-operads $F\colon P \to Q$ consists of a family of Λ-equivariant functors

$$(F_n\colon P(n) \to Q(n))_{n\in\mathbb{N}}$$

together with isomorphisms instead of the standard algebra axioms. For example, the associativity isomorphism has the following form.

$$\begin{tikzcd} P(n)\times\prod_i P(k_i) \arrow[r, "F_n\times\prod_i F_{k_i}"] \arrow[d, "\mu^P"'] \arrow[dr, phantom, "\Downarrow\psi_{k_1,\ldots,k_n}"] & Q(n)\times\prod_i Q(k_i) \arrow[d, "\mu^Q"] \\ P(\Sigma k_i) \arrow[r, "F_{\Sigma k_i}"'] & Q(\Sigma k_i) \end{tikzcd}$$

These isomorphisms are then required to satisfy their own axioms, and these ensure that we have a weak map of 2-monads $\underline{P} \Rightarrow \underline{Q}$. In particular, one can show that a pseudomorphism from P into the endomorphism operad $\mathcal{E}_X$ produces pseudoalgebras for the operad P using the closed structure on **Cat**. While abstractly pleasing, we do not pursue this argument any further here.

3.2 2-categorical properties of operads in Cat

This section will be concerned with characterizing various properties of those 2-monads induced by Λ-operads in **Cat**. We first show that these 2-monads are finitary. Second, we show that the coherence theorem in [29] applies to all such 2-monads and allows us to show that each pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra (and so similarly, by our previous results, to the pseudoalgebras for a Λ-operad P). Both of these results are simple extensions of well-known results about operads. Finally, we give conditions for these 2-monads to be 2-cartesian, describing how they interact with certain limits, namely 2-pullbacks. Operads do not always yield 2-cartesian 2-monads, and giving a complete characterization of when they do is more involved than our results on accessibility or coherence.

For a 2-monad T, the 2-categories **Ps**-T-**Alg** (of pseudoalgebras and weak morphisms) and T-**Alg**$_s$ (of strict algebras and strict morphisms) are of particular interest. The behavior of colimits in both of these 2-categories can often be deduced from properties of T, the most common being that T is finitary. In practice, one thinks of a finitary monad as one in which all operations take finitely many inputs as variables. If T is finitary, then T-**Alg**$_s$ will be cocomplete by standard results given in [6]. There are additional results of a purely 2-dimensional nature concerning finitary 2-monads, detailed in [29] and extending those in [6], namely the existence of a left adjoint

$$\textbf{Ps-}T\textbf{-Alg} \to T\textbf{-Alg}_s$$

to the forgetful 2-functor which regards a strict algebra as a pseudoalgebra with identity structure isomorphisms.

We begin by showing each associated 2-monad is finitary.

Proposition 3.20. *Let P be a Λ-operad. Then $\underline{P}$ is finitary.*

Proof. To show that $\underline{P}$ is finitary we must show that it preserves filtered colimits or, equivalently, that it preserves directed colimits (see [1]). Consider some directed colimit, $\mathrm{colim}X_i$ say, in **Cat**. Then consider the following sequence of isomorphisms:

Why directed shift at all?

$$\begin{aligned} \underline{P}(\mathrm{colim}X_i) &= \coprod_n P(n)\times_{\Lambda(n)} (\mathrm{colim}X_i)^n \\ &\cong \coprod_n P(n)\times_{\Lambda(n)} \mathrm{colim}(X_i^n) \\ &\cong \coprod_n \mathrm{colim}(P(n)\times_{\Lambda(n)} X_i^n) \\ &\cong \mathrm{colim}\coprod_n P(n)\times_{\Lambda(n)} X_i^n \\ &= \mathrm{colim}\underline{P}(X_i). \end{aligned}$$

Since **Cat** is locally finitely presentable then directed colimits commute with finite limits, giving the first isomorphism. The second isomorphism follows from this fact as well as that colimits commute with coequalizers. The third isomorphism is simply coproducts commuting with other colimits. □

The next part of this section is motivated by the issue of coherence. At its most basic, a coherence theorem is a way of describing when a notion of weaker structure is in some way equivalent to a stricter structure. The prototypical case here is the coherence theorem for monoidal categories. In a monoidal category we require associator isomorphisms

$$(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$$

for all objects in the category. The coherence theorem tells us that, for any monoidal category M, there exists a strict monoidal category which is equivalent to M. In other words, we can treat the associators in M as identities, and similarly for the unit isomorphisms.

The abstract approach to coherence considers when the pseudoalgebras for a 2-monad T are equivalent to strict T-algebras, with the most comprehensive account appearing in [29]. Lack gives a general theorem which provides sufficient conditions for the existence of a left adjoint to the forgetful 2-functor

$$U\colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$$

for which the components of the unit of the adjunction are equivalences. We focus on one version of ~~this~~ general result which has hypotheses that are quite easy to check in practice. First we require that the base 2-category $\mathcal{K}$ has an enhanced factorization system. This is much like an orthogonal factorization system on a 2-category, consisting of two classes of maps $(\mathcal{L}, \mathcal{R})$, satisfying the lifting properties on 1-cells and 2-cells as follows. Given a commutative square

Defs in cw's

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

where $l \in \mathcal{L}$ and $r \in \mathcal{R}$, there exists a unique morphism $m\colon B \to C$ such that $rm = g$ and $ml = f$. Similarly, given two commuting squares for which $rf = gl$ and $rf' = f'l$, along with 2-cells $\delta\colon f \Rightarrow f'$ and $\gamma\colon g \Rightarrow g'$ for which $\gamma * 1_l = 1_r * \delta$, there exists a unique 2-cell $\mu\colon m \Rightarrow m'$, where m and m' are induced by the 1-cell lifting property, satisfying $\mu * 1_l = \delta$ and $1_r * \mu = \gamma$. However, there is an additional 2-dimensional property of the factorization system which says that given maps $l \in \mathcal{L}$, $r \in \mathcal{R}$ and an invertible 2-cell $\alpha\colon rf \Rightarrow gl$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\swArrow\alpha"] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] \arrow[ur, "m"] & D \arrow[ul, phantom, "\swArrow\beta"] \end{tikzcd}$$

there exists a unique pair (m, β) where $m\colon B \to C$ is a 1-cell and $\beta\colon rm \Rightarrow g$ is an invertible 2-cell such that $ml = f$ and $\beta * 1_l = \alpha$.

Further conditions require that T preserve $\mathcal{L}$ maps and that whenever $r \in \mathcal{R}$ and $rk \cong 1$, then $kr \cong 1$. In our case we are considering 2-monads on the 2-category **Cat**, which has the enhanced factorization system where $\mathcal{L}$ consists of bijective-on-objects functors and $\mathcal{R}$ is given by the full and faithful functors. This, along with the 2-dimensional property making it an enhanced factorization system, is described in [38]. The last stated condition, involving isomorphisms and maps in $\mathcal{R}$, is then clearly satisfied and so the only thing we need to check in order to satisfy the conditions of the coherence result are that the induced 2-monads $\underline{P}$ preserve bijective-on-objects functors, which follows simply from the fact that the set of objects functor, Ob: **Cat** $\to$ **Set**, preserves colimits, being left adjoint to the indiscrete category functor, E: **Set** $\to$ **Cat**, as described in Lemma 3.35.

too long

why?

Proposition 3.21. *For any Λ-operad P, the 2-monad $\underline{P}$ preserves bijective-on-objects functors.*

State Power result clearly

Corollary 3.22. *Every pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra.*

We finally turn to a discussion of the interaction between operads and pullbacks. The monads arising from a non-symmetric operad are always cartesian, as described in [31]. The monads that arise from symmetric operads, however, are not always cartesian and so it is useful to be able to characterize exactly when they are. An example of where this fails is the symmetric operad for which the algebras are commutative monoids. In the case of 2-monads we can consider the strict 2-limit analogous to the pullback, the 2-pullback, and characterize when the induced 2-monad from a Λ-operad is 2-cartesian, as we now describe.

work out

Definition 3.23. A 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ is said to be *2-cartesian* if

- the 2-category $\mathcal{K}$ has 2-pullbacks,
- the functor T preserves 2-pullbacks, and
- the naturality squares for the unit and multiplication of the 2-monad are 2-pullbacks.

write out fully?

define cart trans

It is important to note that the 2-pullback of a diagram is actually the same as the ordinary pullback in **Cat**, see [20]. Since we will be computing with coequalizers of the form $A \times_\Lambda B$ repeatedly, we give the following useful lemma.

define notation ⓧ instead?

Lemma 3.24. *Let G be a group and let A, B be categories for which A has a right action by G and B has a left action by G. An action of G on the product $A \times B$ can then be defined by*

define these separately

$$(a, b) \cdot g\colon = (a \cdot g, g^{-1} \cdot b).$$

If this action of G on $A \times B$ is $\boxed{\text{free}}$ then the category $\boxed{(A \times B)/G,}$ consisting of the equivalence classes of this action, is isomorphic to the coequalizer $A \times_G B$.

Proof. The category $A \times_G B$ is defined as the coequalizer

$$\begin{tikzcd} A \times G \times B \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & A \times B \arrow[r, "\varepsilon"] & A \times_G B \end{tikzcd}$$

where $\lambda(a, g, b) = (a \cdot g, b)$ and $\rho(a, g, b) = (a, g \cdot b)$. However, the map $A \times B \to (A \times B)/G$, sending (a, b) to the equivalence class $[a, b] = [a \cdot g, g^{-1} \cdot b]$, also coequalizes λ and ρ since

$$[a \cdot g, b] = [(a \cdot g) \cdot g^{-1}, g \cdot b] = [a, g \cdot b].$$

Given any other category X and a functor $\chi\colon A \times B \to X$ which coequalizes λ and ρ, we define a functor $\phi\colon (A \times B)/G \to X$ by $\phi[a, b] = \chi(a, b)$. That this is well-defined is clear, since

$$\phi\left[a \cdot g, g^{-1} \cdot b\right] = \chi\left(a \cdot g, g^{-1} \cdot b\right) = \chi\left(a \cdot \left(gg^{-1}\right), b\right) = \chi(a, b) = \phi[a, b].$$

This is also unique and so we find that $(A \times B)/G$ satisfies the universal property of the coequalizer. □

We begin our study of the cartesian property in the context of symmetric operads.

Proposition 3.25. *Let P be a symmetric operad. Then the unit $\eta\colon id \Rightarrow \underline{P}$ for the associated monad is a cartesian transformation.*

Proof. In order to show that η is cartesian, we must prove that for a functor $f\colon X \to Y$, the pullback of the following diagram is the category X.

$$\begin{tikzcd} & Y \arrow[d, "\eta_Y"] \\ \coprod P(n) \times_{\Sigma_n} X^n \arrow[r, "\underline{P}(f)"'] & \coprod P(n) \times_{\Sigma_n} Y^n \end{tikzcd}$$

Make some stuff clear: coprods commute w/ pb's, ×'s pb's commute

The pullback of this diagram is isomorphic to the coproduct of the pullbacks of diagrams of the following form.

$$\begin{tikzcd} & Y \arrow[d] & & \emptyset \arrow[d] \\ P(1) \times X \arrow[r, "1\times f"'] & P(1) \times Y & P(n) \times_{\Sigma_n} X^n \arrow[r, "1\times f^n"'] & P(n) \times_{\Sigma_n} Y^n \end{tikzcd}$$

$$n \neq 1$$

It is easy then to see that X is the ~~coproduct~~ pullback of the $n = 1$ cospan, and that the empty category is the pullback of each of the other cospans, making X the pullback of the original diagram and verifying that η is cartesian. □

Proposition 3.26. *Let P be a symmetric operad. Then the 2-monad $\underline{P}$ preserves pullbacks if and only if Σ_n acts freely on $P(n)$ for all n.*

Proof. Consider the following pullback of discrete categories.

$$\begin{tikzcd} \{(x,y),(x,y'),(x',y),(x',y')\} \arrow[r] \arrow[d] & \{y,y'\} \arrow[d] \\ \{x,x'\} \arrow[r] & \{z\} \end{tikzcd}$$

Letting $\mathbf{4}$ denote the pullback and similarly writing $\mathbf{2}_X = \{x, x'\}$ and $\mathbf{2}_Y = \{y, y'\}$, the following diagram results as the image of this pullback square under $\underline{P}$.

$$\begin{tikzcd} \coprod P(n) \times_{\Sigma_n} \mathbf{4}^n \arrow[r] \arrow[d] & \coprod P(n) \times_{\Sigma_n} \mathbf{2}_Y^n \arrow[d] \\ \coprod P(n) \times_{\Sigma_n} \mathbf{2}_X^n \arrow[r] & \coprod P(n)/\Sigma_n \end{tikzcd}$$

The projection map $\underline{P}(\mathbf{4}) \to \underline{P}(\mathbf{2}_Y)$ maps an element

$$[p; (x_1, y_1), \ldots, (x_n, y_n)]$$

to the element

$$[p; y_1, \ldots, y_n]$$

and likewise for the projection to $\underline{P}(\mathbf{2}_X)$.

Now assume that, for some n, the action of Σ_n on $P(n)$ is not free. Then find some $p \in P(n)$ along with a nonidentity element $g \in \Sigma_n$ such that $p \cdot g = p$. We will show that the existence of g proves that $\underline{P}$ is not cartesian.

Now $g \neq e$, so there exists an i such that $g(i) \neq i$; without loss of generality, we may take $i = 1$. Using this g we can find two distinct elements

$$[p; (x', y), (x, y), \ldots, (x, y), (x, y'), (x, y), \ldots, (x, y)]$$

and

$$[p; (x, y), \ldots, (x, y), (x', y'), (x, y), \ldots, (x, y)]$$

in $\underline{P}(\mathbf{4})$. In the first element we put (x', y) in the first position and (x, y') in position $g(1)$, whilst in the second element we put (x', y') in position $g(1)$. Both of these elements, however, are mapped to the same elements in $\underline{P}(\mathbf{2}_X)$, since

$$\begin{aligned}[p; x', x, \ldots, x] &= [p \cdot g; (x', x, \ldots, x)] \\ &= [p; g \cdot (x', x, \ldots, x)] \\ &= [p; x, x, \ldots, x', \ldots, x].\end{aligned}$$

Similarly, both of the elements are mapped to the same element in $\underline{P}(\mathbf{2}_Y)$, simply

$$[p; y, \ldots, y', \ldots, y].$$

The pullback of this diagram, however, has a unique element which is projected to the ones we have considered, so $\underline{P}(\mathbf{4})$ is not a pullback. Hence $\underline{P}$ does not preserve pullbacks if for some n the action of Σ_n on $P(n)$ is not free.

Now assume that each Σ_n acts freely on $P(n)$. Given a pullback

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "R"'] & B \arrow[d, "S"] \\ C \arrow[r, "H"'] & D \end{tikzcd}$$

we must show that the image of the diagram under $\underline{P}$ is also a pullback. Now this will be true if and only if each individual diagram

$$\begin{tikzcd} P(n) \times_{\Sigma_n} A^n \arrow[r, "1\times F^n"] \arrow[d, "1\times R^n"'] & P(n) \times_{\Sigma_n} B^n \arrow[d, "1\times S^n"] \\ P(n) \times_{\Sigma_n} C^n \arrow[r, "1\times H^n"'] & P(n) \times_{\Sigma_n} D^n \end{tikzcd}$$

is also a pullback. The pullback of the functors $1 \times H^n$ and $1 \times S^n$ is a category consisting of pairs of objects $[p; \underline{c}]$ and $[q; \underline{b}]$, where $\underline{b}$ and $\underline{c}$ represent lists of elements in B and C, respectively. These pairs are then required to satisfy the property that

$$\left[p; \underline{H(c)}\right] = \left[q; \underline{S(b)}\right].$$

ref & explain

Using the previous lemma, we know that a pair

$$([p; \underline{c}], [q; \underline{b}])$$

is in the pullback if and only if there exists an element $g \in \Sigma_n$ such that $p \cdot g = q$ and $Hc_i = (Sb_{\pi(g)^{-1}(i)})$. Using this we can define mutual inverses between $P(n) \times_{\Sigma_n} A^n$ and the pullback Q'. Considering the category A as the pullback of the diagram we started with, we can consider objects of $P(n) \times_{\Sigma_n} A^n$ as being equivalence classes

$$[p; (b_1, c_1), \ldots, (b_n, c_n)]$$

where $p \in P(n)$ and $Hc_i = Sb_i$ for all i.

Taking such an object, we send it to the pair

$$([p; c_1, \ldots, c_n], [p; b_1, \ldots, b_n])$$

which lies in the pullback since the identity in Σ_n satisfies the condition given earlier. An inverse to this sends a pair of equivalence classes in Q' to the single equivalence class

$$[p; (c_1, b_{\pi(g)^{-1}(1)}), \ldots, (c_n, b_{\pi(g)^{-1}(n)})]$$

in $P(n) \times_{\Sigma_n} A^n$. If we apply the map into Q' we get the pair

$$([p; c_1, \ldots, c_n], [p; b_{\pi(g)^{-1}(1)}, \ldots, b_{\pi(g)^{-1}(n)}])$$

which is equal to the original pair since $p \cdot g = q$; the other composite is trivially an identity. A similar calculation on morphisms finishes the proof that $P(n) \times_{\Sigma_n} A^n$ is the pullback as required. □

reread!

Proposition 3.27. *Let P be a symmetric operad. If the Σ_n-actions are all free, then the multiplication $\mu\colon \underline{P}^2 \Rightarrow \underline{P}$ of the associated monad is a cartesian transformation.*

Proof. Note that if all of the diagrams

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_X"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

are pullbacks then the outside of the diagram

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(f)"] \arrow[d, "\mu_X"'] & \underline{P}^2(Y) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_Y"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(f)"'] & \underline{P}(Y) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

is also a pullback and so each of the naturality squares for μ must therefore be a pullback. Now we can split up the square above, much like we did for η, and prove that each of the squares below is a pullback.

$$\begin{tikzcd}
\coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i) \times_{\Sigma_{k_i}} X^{k_i}\right) \arrow[r] \arrow[d] & \coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i)/\Sigma_{k_i}\right) \arrow[d] \\
P(n) \times_{\Sigma_n} X^n \arrow[r] & P(n)/\Sigma_n
\end{tikzcd}$$

The map along the bottom is the obvious one, sending $[p; x_1, \ldots, x_n]$ simply to the equivalence class $[p]$. Along the right hand side the map is the one corresponding to operadic composition, sending $[q; [p_1], \ldots, [p_m]]$ to $[\mu^P(q; p_1, \ldots, p_n)]$. The pullback of these maps would be the category consisting of pairs

$$([p; x_1, \ldots, x_{\Sigma k_i}], [q; [p_1], \ldots, [p_n]]),$$

where $q \in P(n)$, $p_i \in P(k_i)$, $p \in P(\Sigma k_i)$, and for which $[p] = [\mu^P(q; p_1, \ldots, p_n)]$. The upper left category in the diagram, which we will refer to here as Q, has objects

$$[q; [p_1; \underline{x}_1], \ldots, [p_n; \underline{x}_n]].$$

There are obvious maps out of Q making the diagram commute and as such inducing a functor from Q into the pullback via the universal property. This functor sends an object such as the one just described to the pair

$$\left(\left[\mu^P(q; p_1, \ldots, p_n); \underline{x}\right], [q; [p_1], \ldots, [p_n]]\right).$$

Given an object in the pullback, we then have a pair, as described above, which has $[p] = [\mu^P(q; p_1, \ldots, p_n)]$ meaning that we can find an element $g \in \Sigma_{\Sigma k_i}$ such that $p = \mu^P(q; p_1, \ldots, p_n) \cdot g$. Thus we can describe an inverse to the induced functor by sending a pair in the pullback to the object

$$[q; [p_1; \pi(g)(\underline{x})_1], \ldots, [p_n; \pi(g)(\underline{x})_n]],$$

where $\pi(g)(\underline{x})_i$ denotes the ith block of $\underline{x}$ after applying the permutation $\pi(g)$. For example, if $\underline{x} = (x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31})$ and $\pi(g) = (1\ 3\ 5)$, then

$$\pi(g)(\underline{x}) = (x_{23}, x_{12}, x_{11}, x_{22}, x_{21}, x_{31}).$$

Thus $\pi(g)(\underline{x})_1 = (x_{23}, x_{12})$, $\pi(g)(\underline{x})_2 = (x_{11}, x_{22}, x_{21})$ and $\pi(g)(\underline{x})_3 = (x_{31})$.

Now applying the induced functor we find that we get back an object in the pullback for which the first entry is $[q; [p_1], \ldots, [p_n]]$ and whose second entry is

$$\left[\mu^P(q; p_1, \ldots, p_n); \pi(g)(\underline{x})\right] = \left[\mu^P(q; p_1, \ldots, p_n) \cdot g; \underline{x}\right] = [p; \underline{x}],$$

which is what we started with. Showing the other composite is an identity is similar, here using the fact that the identity acts trivially on $\mu^P(q; p_1, \ldots, p_n)$. Taking the coproduct of these squares then gives us the original diagram that we wanted to show was a pullback and, since each individual square is a pullback, so is the original. □

Collecting these results together gives the following corollary.

Corollary 3.28. *The 2-monad associated to a symmetric operad P is 2-cartesian if and only if the action of Σ_n is free on each $P(n)$.*

We require one simple technical lemma before giving a complete characterization of Λ-operads which induce cartesian 2-monads.

Lemma 3.29. *Let C be a category with a right action of some group Λ, and let $\pi\colon \Lambda \to \Sigma$ be a group homomorphism to any other group Σ. Then the right Σ-action on $C \times_\Lambda \Sigma$ is free if and only if the only elements of Λ which fix an object of C lie in the kernel of π.*

reread this too

Proof. First, note that a group action on a category is free if and only if it is free on objects as fixing a morphism requires fixing its source and target. Thus our arguments need only concern the objects involved.

Since the set of objects functor preserves colimits, the objects of $C \times_\Lambda \Sigma$ are equivalence classes $[c; g]$ where $c \in C$ and $g \in \Sigma$, with $[c \cdot r; g] = [c; \pi(r)g]$. First assume the Σ-action is free. Then noting that $[c; e] \cdot g = [c; g]$, we have if $[c; g] = [c; e]$ then $g = e$. Let $r \in \Lambda$ be an element such that $c \cdot r = c$. Then

$$[c; e] = [c \cdot r; e] = [c; \pi(r)],$$

so $\pi(\tau) = e$.

Now assume that every element of Λ fixing an object lies in the kernel of π. Let $\tau \in \Sigma$, and assume it fixes $[p;\sigma]$. Without loss of generality, we can take $\sigma = e$, so that

$$[p;\tau] = [p;e] \cdot \tau = [p;e].$$

Since the objects of $C \times_\Lambda \Sigma$ are equivalence classes as above, there exists an element $r \in \Lambda$ such that $p \cdot r^{-1} = p$ and $\tau = \pi(r)$. But by assumption, we must have r^{-1}, and hence r, in the kernel, so $\tau = e$ and the Σ-action is free. □

Theorem 3.30. *The 2-monad associated to a Λ-operad P is 2-cartesian if and only if whenever $p \cdot g = p$ for an object $p \in P(n)$, $g \in Ker\,\pi(n)$.*

Proof. Since the monad $\underline{P}$ is isomorphic to $\underline{S(P)}$, we need only verify when $\underline{S(P)}$ is 2-cartesian. Thus the theorem is a direct consequence of Lemma 3.29 and Corollary 3.28. □

3.3 The Borel construction for action operads

The classical Borel construction is a functor from G-spaces to spaces, sending a G-space X to $EG \times_G X$. Our goal in this section is to use the formal description of the Borel construction to construct some special operads in **Cat**. We start by reviewing the analogues of the functors E, B: **Grp** $\to$ **Top** now taking values in **Cat**.

Definition 3.31. 1. Let X be a set. We define the *translation category* EX to have objects the elements of X and morphisms consisting of a unique isomorphism between any two objects.

2. Let G be a group. The category BG has a single object $*$, and hom-set $BG(*,*) = G$ with composition and identity given by multiplication and the unit element in the group, respectively.

Definition 3.32. A functor $F\colon X \to Y$ is an *isofibration* if given $x \in X$ and an isomorphism $f\colon y \xrightarrow{\cong} F(x)$ in Y, then there exists an isomorphism g in X such that $F(g) = f$.

↳ $g: y' \cong x$

Proposition 3.33. *There exists a natural transformation $p\colon EU \Rightarrow B$, where U is the underlying set of a group, which is pointwise an isofibration. Applying the classifying space functor to the component p_G gives a universal principal G-bundle.*

Proof. Given a group G, $p_G\colon EUG \to BG$ sends every object of EUG to the unique object of BG. The unique isomorphism $g \to h$ in EUG is mapped to $hg^{-1}\colon * \to *$. It is easy to directly check that this is an isofibration, as well as to see that the classifying spaces of EUG and BG are the spaces classically known as EG, BG, with $|p_G|$ being the standard universal principal G-bundle. □

We will also need the functors E, B defined for more than just a single set or group, in particular for the sets or groups which make up an operad and are indexed by the natural numbers.

Notation 3.34. Let S be a set which we view as a discrete category.

1. For any functor $F\colon S \to$ **Sets**, let EF denote the composite $E \circ F\colon S \to$ **Cat**; we often view F as an indexed set $\{F(s)\}$, in which case EF is the indexed category $\{EF(s)\}$.

2. For any functor $F\colon S \to$ **Grp**, let BF denote the composite $B \circ F\colon S \to$ **Cat**; we often view F as an indexed group $\{F(s)\}$, in which case BF is the indexed category $\{BF(s)\}$.

The following lemma is a straightforward verification.

Lemma 3.35. *The functor E: **Sets** $\to$ **Cat** is right adjoint to the set of objects functor. Therefore E preserves all limits, and in particular is a symmetric monoidal functor when both categories are equipped with their cartesian monoidal structures.*

We are additionally interested in Λ-operads in **Cat** (or other cocomplete symmetric monoidal categories in which the tensor product preserves colimits in each variable). While the definition above gives the correct notion of a Λ-operad in **Cat** if we interpret the two equivariance axioms to hold for both objects and morphisms, it is useful to give a purely diagrammatic expression of these axioms. In the diagrams below, expressions of the form $G \times C$ for a group G and category C mean that the group G is to be treated as a discrete category. This follows the standard method of how one expresses group actions in categories other than **Sets** using a copower. Thus the diagrams below are the two equivariance axioms given in Definition 2.43 expressed diagrammatically.

$$\begin{tikzcd}
P(n)\times(\prod_{i=1}^n P(k_i))\times(\prod_{i=1}^n \Lambda(k_i)) \arrow[rr, "\cong"] \arrow[d, "\mu^P\times\mu^\Lambda(e;-)"'] & & P(n)\times(\prod_{i=1}^n(P(k_i)\times\Lambda(k_i))) \arrow[d, "1\times\prod_{i=1}^n\alpha_i"] \\
P(\underline{k})\times\Lambda(\underline{k}) \arrow[dr, "\alpha_{\underline{k}}"'] & & P(n)\times(\prod_{i=1}^n P(k_i)) \arrow[dl, "\mu^P"] \\
& P(\underline{k}) &
\end{tikzcd}$$

$$\begin{tikzcd}
P(n)\times\Lambda(n)\times\prod_{i=1}^n P(k_i) \arrow[rr, "\alpha_n\times 1"] \arrow[d, "1\times\Delta\times 1"'] & & P(n)\times\prod_{i=1}^n P(k_i) \arrow[dd, "\mu^P"] \\
P(n)\times\Lambda(n)\times\Lambda(n)\times\prod_{i=1}^n P(k_i) \arrow[d, "\cong"'] & & \\
P(n)\times\Lambda(n)\times\prod_{i=1}^n P(k_i)\times\Lambda(n) \arrow[d, "1\times\pi_n\times 1"'] & & P(\underline{k}) \\
P(n)\times\Sigma_n\times\prod_{i=1}^n P(k_i)\times\Lambda(n) \arrow[rr, "\tilde{\mu}^P\times\mu^\Lambda(-;\underline{e})"'] & & P(\underline{k})\times\Lambda(\underline{k}) \arrow[u, "\alpha_{\underline{k}}"']
\end{tikzcd}$$

Move this?

In the second diagram, the morphism

$$\tilde{\mu}^P\colon P(n)\times\Sigma_n\times\prod_{i=1}^n P(k_i) \to P(\underline{k}\times\Lambda(\underline{k}))$$

is first the left action of Σ_n on the product followed by the operad multiplication, and $\underline{e}$ is $e_{k_1},\ldots,e_{k_n}$.

Definition 3.36. Let Λ be an action operad. Then $B\Lambda$ (see Notation 3.34) is the category with objects the natural numbers and

$$B\Lambda(m,n) = \begin{cases} \Lambda(n), & m = n \\ \emptyset, & m \neq n, \end{cases}$$

where composition is given by group multiplication and the identity morphism is the unit element $e_n \in \Lambda(n)$.

Λ?

move this whole thing?

use changed def

Theorem 3.37. *Let M, N be cocomplete symmetric monoidal categories ~~in which the~~ tensor product preserves colimits in each variable, and let $F\colon M \to N$ be a symmetric ~~lax monoidal~~ functor with unit constraint φ_0 and tensor constraint φ_2. Let Λ be an action operad, and P a Λ-operad in M. Then $FP = \{F(P(n))\}$ has a canonical Λ-operad structure, giving a functor*

$$F_* \colon \Lambda\text{-}Op(M) \to \Lambda\text{-}Op(N)$$

from the category of Λ-operads in M to the category of Λ-operads in N.

Proof. The category of Λ-operads in M is the category of monoids for the composition product $\circ_M$ on $[B\Lambda^{op}, M]$ constructed in Section 2.6. Composition with F gives a functor

$$F_* \colon [B\Lambda^{op}, M] \to [B\Lambda^{op}, N],$$

and to show that it gives a functor between the categories of monoids we need only prove that F_* is lax monoidal with respect to $\circ_M$ and $\circ_N$. In other words, we must construct natural transformations with components $F_*X \circ_N F_*Y \to F_*(X \circ_M Y)$ and $I_{Op(N)} \to F_*(I_{Op(M)})$ and then verify the lax monoidal functor axioms. We note that in the calculations below, we often write F instead of F_*, but it should be clear from context when we are applying F to objects and morphisms in M and when we are applying F_* to a functor $B\Lambda^{op} \to M$.

We first remind the reader about copowers in cocomplete categories. For an object X and set S, the copower $X \odot S$ is the coproduct $\coprod_{s \in S} X$. We have natural isomorphisms $(X \odot S) \odot T \cong X \odot (S \times T)$ and $X \odot 1 \cong X$, and using these we can define an action of a group G on an object X using a map $X \odot G \to X$. Any functor F between categories with coproducts is lax monoidal with respect to those coproducts: the natural map $FA \coprod FB \to F(A \coprod B)$ is just the map induced by the universal property of the coproduct using F applied to the coproduct inclusions $A \hookrightarrow A \coprod B, B \hookrightarrow A \coprod B$. In particular, for any functor F there exists an induced map $FX \odot S \to F(X \odot S)$.

The unit object in $[B\Lambda^{op}, M]$ for $\circ_M$ is the copower $I_M \odot B\Lambda(-,1)$. Thus the unit constraint for F_* is the composite

$$I_N \odot B\Lambda(-,1) \xrightarrow{\varphi_0 \odot 1} FI_M \odot B\Lambda(-,1) \to F(I_M \odot B\Lambda(-,1)).$$

For the tensor constraint, we will require a map

$$t\colon (FY)^{*n}(k) \to F(Y^{*n}(k))$$

where $*$ is the Day convolution product; having constructed one, the tensor constraint is then the following composite.

$$\begin{aligned}
(FX \circ FY)(k) &\xrightarrow{\cong} \int^n FX(n) \otimes (FY)^{*n}(k) \\
&\xrightarrow{\int 1 \otimes t} \int^n FX(n) \otimes F(Y^{*n}(k)) \\
&\xrightarrow{\int \varphi_2} \int^n F(X(n) \otimes Y^{*n}(k)) \\
&\longrightarrow F\left(\int^n X(n) \otimes Y^{*n}(k)\right) \\
&\xrightarrow{\cong} F(X \circ Y)(k)
\end{aligned}$$

Both isomorphisms in the composite above are induced by universal properties (see Section 3.1 for more details) and the unlabeled arrow is induced by the same argument

as that for coproducts above but this time using coends. The arrow t is constructed in a similar fashion, and is the composite below.

$$\begin{aligned}
(FY)^{*n}(k) &= \int^{k_1,\ldots,k_n} FY(k_1) \otimes \cdots \otimes FY(k_n) \odot B\Lambda(k, \textstyle\sum k_i) \\
&\to \int^{k_1,\ldots,k_n} F(Y(k_1) \otimes \cdots \otimes Y(k_n)) \odot B\Lambda(k, \textstyle\sum k_i) \\
&\to \int^{k_1,\ldots,k_n} F(Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\Lambda(k, \textstyle\sum k_i)) \\
&\to F \int^{k_1,\ldots,k_n} Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\Lambda(k, \textstyle\sum k_i) \\
&= F(Y^{*n}(k))
\end{aligned}$$

Checking the lax monoidal functor axioms is tedious but entirely routine using the lax monoidal functor axioms for F together with various universal properties of colimits, and we leave the details to the reader. □

Combining Theorem 3.37 and Proposition 2.46 with Lemma 3.35, we immediately obtain the following.

Corollary 3.38. *Let Λ be an action operad. Then $E\Lambda = \{E(\Lambda(n))\}$ (see Notation 3.34) is a Λ-operad in* **Cat**.

Any Λ-operad P in **Cat** gives rise to a 2-monad on **Cat** which we will often also denote by P or, as in Section 3.1, by $\underline{P}$. In the context of Corollary 3.38, that 2-monad (also denoted $E\Lambda$) is given by

get one notation!

$$X \mapsto \coprod_{n \geq 0} E\Lambda(n) \times_{\Lambda(n)} X^n$$

where the action of $\Lambda(n)$ on $E\Lambda(n)$ is given by the obvious multiplication action on the right, and the action of $\Lambda(n)$ on X^n is given using $\pi_n \colon \Lambda(n) \to \Sigma_n$ together with the standard left action of Σ_n on X^n in any symmetric monoidal category. The 2-monad $E\Sigma$ is that for symmetric strict monoidal categories (see Section 4.2 for this and further examples).

Definition 3.39. A *Λ-monoidal category* is a strict algebra for the 2-monad $E\Lambda$. A *Λ-monoidal functor* is a strict morphism for this 2-monad $E\Lambda$. The 2-category **Λ-MonCat** is the 2-category $E\Lambda$-**Alg**$_s$ of strict algebras, strict morphisms, and algebra 2-cells for $E\Lambda$.

change conventio

Strict Λ-monoidal categories, in the sense of being strict algebras for the monad $E\Lambda$, can be characterised in more familiar terms by specifying a monoidal structure with appropriate equivariant interaction with the $\Lambda(n)$-actions. Explicit proofs of such can be found in Chapter 19 of [46], along with similar characterisations for strict Λ-monoidal functors and Λ-monoidal categories whose underlying monoidal structure is weak.

3.4 Free Λ-monoidal categories

It will be useful for our calculations later to give an explicit description of the categories $E\Lambda(n) \times_{\Lambda(n)} X^n$. Objects are equivalence classes of tuples $(g; x_1, \ldots, x_n)$ where $g \in \Lambda(n)$ and the x_i are objects of X, with the equivalence relation given by

$$(gh; x_1, \ldots, x_n) \sim (g; x_{\pi(h)^{-1}(1)}, \ldots, x_{\pi(h)^{-1}(n)});$$

we write these classes as $[g; x_1, \dots, x_n]$. Morphisms are then equivalence classes of morphisms

$$(!; f_1, \dots, f_n) \colon (g; x_1, \dots, x_n) \to (h; x'_1, \dots, x'_n)\,.$$

We have two distinguished classes of morphisms, one for which the map $! \colon g \to h$ is the identity and one for which all the f_i's are the identity. Every morphism in $E\Lambda(n) \times X^n$ is uniquely a composite of a morphism of the first type followed by one of the second type. Now $E\Lambda(n) \times_{\Lambda(n)} X^n$ is a quotient of $E\Lambda(n) \times X^n$ by a free group action, so every morphism of $E\Lambda(n) \times_{\Lambda(n)} X^n$ is in the image of the quotient map. Using this fact, we can prove the following useful lemma. why?

Lemma 3.40. *For an action operad Λ and any category X, the set of morphisms from $[e; x_1, \dots, x_n]$ to $[e; y_1, \dots, y_n]$ in $E\Lambda(n) \times_{\Lambda(n)} X^n$ is*

$$\coprod_{g \in \Lambda(n)} \prod_{i=1}^{n} X\left(x_i, y_{\pi(g)(i)}\right).$$

Proof. A morphism with source $(e; x_1, \dots, x_n)$ in $E\Lambda(n) \times X^n$ is uniquely a composite

$$(e; x_1, \dots, x_n) \xrightarrow{(\mathrm{id}; f_1, \dots, f_n)} (e; x'_1, \dots, x'_n) \xrightarrow{(!; \mathrm{id}, \dots, \mathrm{id})} (g; x'_1, \dots, x'_n)\,.$$

check

Descending to the quotient, this becomes a morphism

$$[e; x_1, \dots, x_n] \to [g; x'_1, \dots, x'_n] = \left[e; x'_{\pi(g)^{-1}(1)}, \dots, x'_{\pi(g)^{-1}(n)}\right],$$

and therefore is a morphism $[e; x_1, \dots, x_n] \to [e; y_1, \dots, y_n]$ precisely when $y_i = x'_{\pi(g)^{-1}(i)}$, and so $f_i \in X(x_i, y_{\pi(g)(i)})$. □

Notation 3.41. For $g \in \Lambda(n)$ and objects $x_1, \dots, x_n$ of a Λ-monoidal category X, we write

$$g^{\otimes} \colon x_1 \otimes \cdots x_n \to x_{\pi(g)^{-1}(1)} \otimes \cdots \otimes x_{\pi(g)^{-1}(n)}$$

for the image of the map

$$(!; \mathrm{id}, \dots, \mathrm{id}) \colon (e; x_1, \dots, x_n) \to (g; x_1, \dots, x_n)$$

in $E\Lambda(n) \times_{\Lambda(n)} X^n$.

Definition 3.42. We call the map

$$g^{\otimes} \colon x_1 \otimes \cdots \otimes x_n \to x_{\pi(g)^{-1}(1)} \otimes \cdots x_{\pi(g)^{-1}(n)}$$

the *action* of g on $x_1 \otimes \cdots x_n$.

+?

Remark 3.43. It is obvious that $g^{\otimes} h^{\otimes} = \mu(e_2; g, h)^{\otimes}$. The latter can also be written $\beta(g, h)^{\otimes}$ (using Theorem 2.30) or $(g \oplus h)^{\otimes}$ (using Notation 2.31). | just pick one

The 2-monad $E\Lambda$ is both finitary and cartesian (see Section 3.2). In fact we can characterize this operad uniquely (up to equivalence) using a standard argument.

Definition 3.44. Let Λ be an action operad. A Λ_∞ *operad* P is a Λ-operad in which each action $P(n) \times \Lambda(n) \to P(n)$ is free and each $P(n)$ is contractible.

Remark 3.45. One should also note that by Corollary 3.38 there exists a canonical Λ_∞ operad in **Cat**, namely $E\Lambda$ itself, and thus also in the category of simplicial sets by taking the nerve (the nerve functor is represented by a cosimplicial category, namely $\Delta \subseteq$ **Cat**, so preserves products) and in suitable categories of topological spaces by taking the geometric realization (once again, product-preserving with the correct category of spaces). Thus we have something like a Barratt-Eccles Λ_∞ operad for any action operad Λ.

Remark 3.46. The above definition makes sense in a wide context, but needs interpretation. We can interpret the freeness condition in any complete category, as completeness allows one to compute fixed points using equalizers. Contractibility then requires a notion of equivalence or weak equivalence, such as in an $(\infty, 1)$-category or Quillen model category, and a terminal object. Our interest is in the above definition interpreted in **Cat**, in which case both conditions (free action and contractible $P(n)$'s) mean the obvious thing.

Proposition 3.47. *For any two Λ_∞ operads P, Q in* **Cat**, *there exists a span $P \leftarrow R \to Q$ of pointwise equivalences of Λ-operads.*

Proof. Given Λ_∞ operads P, Q in **Cat**, the product $P \times Q$ with the diagonal action is also Λ_∞. Each of the projection maps is a pointwise equivalence of Λ-operads. □

Remark 3.48. Once again, this proof holds in a wide context. We required that the product of free actions is again free, true in any complete category. We also required that the product of contractible objects is contractible; this condition will hold, for example, in any Quillen model category in which all objects are fibrant or in which the product of weak equivalences is again a weak equivalence.

3.5 Abstract properties of the Borel construction

Kelly's theory of clubs [19, 21, 22] was designed to simplify and explain certain aspects of coherence results, namely the fact that many coherence results rely on extrapolating information about general free objects for a 2-monad T from information about the specific free object $T1$ where 1 denotes the terminal category. This occurs, for example, in the study of the many different flavors of monoidal category: plain monoidal category, braided monoidal category, symmetric monoidal category, and so on. This section will explain how every action operad gives rise to a club, as well as compute the clubs which arise as the image of this procedure.

We begin by reminding the reader of the notion of a club, or more specifically what Kelly [19, 22] calls a club over **P**. We will only be interested in clubs over **P**, and thusly shorten the terminology to club from this point onward. Defining clubs is accomplished most succinctly using Leinster's terminology of generalized operads [31].

Definition 3.49. Let C be a category with finite limits.

1. A monad $T \colon C \to C$ is *cartesian* if the functor T preserves pullbacks, and the naturality squares for the unit η and the multiplication μ for T are all pullbacks.
2. The category of T-*collections*, T-**Coll**, is the slice category $C/T1$, where 1 denotes the terminal object.
3. Given a pair of T-collections $X \xrightarrow{x} T1, Y \xrightarrow{y} T1$, their *composition product* $X \circ Y$ is given by the pullback below together with the morphism along the top.

$$\begin{tikzcd} X \circ Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TY \arrow[r, "Ty"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu"] & T1 \\ X \arrow[r, "x"'] & T1 & & \end{tikzcd}$$

4. The composition product, along with the unit of the adjunction $\eta \colon 1 \to T1$, give T-**Coll** a monoidal structure. A T-*operad* is a monoid in T-**Coll**.

Remark 3.50. Everything in the above definition can be **Cat**-enriched without any substantial modifications. Thus we require our ground 2-category to have finite limits in the enriched sense, and the slice and pullbacks are the 2-categorical (and not bicategorical) versions. If we take this 2-category to be **Cat**, then in each case the underlying category of the 2-categorical construction is given by the corresponding 1-categorical version. From this point, we will not distinguish between the 1-dimensional and 2-dimensional theory. Our interest, of course, is in the 2-dimensional version.

Let Σ be the operad of symmetric groups. This is the terminal object of the category of action operads, with each π_n the identity map. Then $E\Sigma$ is a 2-monad on **Cat**, and by results in Section 3.2 it is cartesian.

Definition 3.51. *A club is a T-operad in* **Cat** *for $T = E\Sigma$.*

Remark 3.52. The category **P** in Kelly's terminology is the result of applying $E\Sigma$ to 1, and can be identified with $B\Sigma = \coprod B\Sigma_n$.

It is useful to break down the definition of a club. A club consists of

1. a category K together with a functor $K \to B\Sigma$,
2. a multiplication map $K \circ K \to K$, and
3. a unit map $1 \to K$

satisfying the axioms to be a monoid in the monoidal category of $E\Sigma$-collections. By the definition of $K \circ K$ as a pullback, objects are tuples of objects of K $(x; y_1, \ldots, y_n)$ where $\pi(x) = n$. A morphism

$$(x; y_1, \ldots, y_n) \to (z; w_1, \ldots, w_m)$$

exists only when $n = m$ (since $B\Sigma$ only has endomorphisms) and then consists of a morphism $f\colon x \to z$ in K together with morphisms $g_i\colon y_i \to z_{\pi(x)(i)}$ in K.

Notation 3.53. For a club K and a morphism $(f; g_1, \ldots, g_n)$ in $K \circ K$, we write $f(g_1, \ldots, g_n)$ for the image of the morphism under the functor $K \circ K \to K$.

We will usually just refer to a club by its underlying category K.

Theorem 3.54. *Let Λ be an action operad. Then the map of operads $\pi\colon \Lambda \to \Sigma$ gives the category $B\Lambda = \coprod B\Lambda(n)$ the structure of a club.*

Proof. To give the functor $B\pi\colon B\Lambda \to B\Sigma$ the structure of a club it suffices (see [31]) to show that

- the induced monad, which we will show to be $E\Lambda$, is a cartesian monad on **Cat**,
- the transformation $\bar{\pi}\colon E\Lambda \Rightarrow E\Sigma$ induced by the functor $E\pi$ is cartesian, and
- $\bar{\pi}$ commutes with the monad structures.

The monad $E\Lambda$ is always cartesian by results of Section 3.2. The transformation $\bar{\pi}$ is the coproduct of the maps $\bar{\pi}_n$ which are induced by the universal property of the coequalizer as shown below.

$$\begin{tikzcd}
E\Lambda(n)\times\Lambda(n)\times X^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "E\pi\times\pi\times 1"'] & E\Lambda(n)\times X^n \arrow[r] \arrow[d, "E\pi\times 1"'] & E\Lambda(n)\times_{\Lambda(n)} X^n \arrow[d, dotted, "\bar{\pi}_n"] \\
E\Sigma_n\times\Sigma_n\times X^n \arrow[r, shift left] \arrow[r, shift right] & E\Sigma_n\times X^n \arrow[r] & E\Sigma_n\times_{\Sigma_n} X^n
\end{tikzcd}$$

Naturality is immediate, and since π is a map of operads $\bar{\pi}$ also commutes with the monad structures.

It only remains to show that $\bar{\pi}$ is cartesian and that the induced monad is actually $E\Lambda$. Since the monads $E\Lambda$ and $E\Sigma$ both decompose into a disjoint union of functors, we only have to show that, for any n, the square below is a pullback.

$$\begin{tikzcd}
E\Lambda(n) \times_{\Lambda(n)} X^n \arrow[r] \arrow[d] & E\Sigma_n \times_{\Sigma_n} X^n \arrow[d] \\
B\Lambda(n) \arrow[r] & B\Sigma_n
\end{tikzcd}$$

By Lemma 3.24, this amounts to showing that the square below is a pullback.

why?

$$\begin{tikzcd}
E\Lambda(n) \times X^n/\Lambda(n) \arrow[r] \arrow[d] & E\Sigma_n \times X^n/\Sigma_n \arrow[d] \\
B\Lambda(n) \arrow[r] & B\Sigma_n
\end{tikzcd}$$

Here, $(A \times B)/G$ is the category whose objects are equivalence classes of pairs (a, b) where $(a, b) \sim (ag, g^{-1}b)$, and similarly for morphisms. Now the bottom map is clearly bijective on objects since these categories only have one object. An object in the top right is an equivalence class

$$[\sigma; x_1, \ldots, x_n] = [e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}].$$

A similar description holds for objects in the top left, with $g \in \Lambda(n)$ replacing σ and $\pi(g)^{-1}$ replacing σ^{-1} in the subscripts. The map along the top sends $[g; x_1, \ldots, x_n]$ to $[\pi(g); x_1, \ldots, x_n]$, and thus sends $[e; x_1, \ldots, x_n]$ to $[e; x_1, \ldots, x_n]$, giving a bijection on objects.

Now a morphism in $(E\Lambda(n) \times X^n)/\Lambda(n)$ can be given as

$$[e; x_1, \ldots, x_n] \xrightarrow{[!; f_i]} [g; y_1, \ldots, y_n].$$

why?

Mapping down to $B\Lambda(n)$ gives $ge^{-1} = g$, while mapping over to $(E\Sigma_n \times X^n)/\Sigma_n$ gives $[!; f_i]$ where $!\colon e \to \pi(g)$ is now a morphism in $E\Sigma_n$. In other words, a morphism in the upper left corner of our putative pullback square is determined completely by its images along the top and lefthand functors. Furthermore, given $g \in \Lambda(n)$, $\tau = \pi(g)$, and morphisms $f_i\colon x_i \to y_i$ in X, the morphism $[!\colon e \to g; f_i]$ maps to the pair $(g, [!\colon e \to \tau; f_i])$, completing the proof that this square is indeed a pullback. □

The club, which we now denote K_Λ, associated to $E\Lambda$ has the following properties. First, the functor $K_\Lambda \to B\Sigma$ is a functor between groupoids. Second, the functor $K_\Lambda \to B\Sigma$ is bijective-on-objects. We claim that these properties characterize those clubs which arise from action operads. Thus the clubs arising from action operads are very similar to PROPs [33, 35].

Theorem 3.55. *Let K be a club such that*

- *the map $K \to B\Sigma$ is bijective on objects and*
- *K is a groupoid.*

Then $K \cong K_\Lambda$ for some action operad Λ. The assignment $\Lambda \mapsto K_\Lambda$ is a full and faithful embedding of the category of action operads **AOp** *into the category of clubs.*

Proof. Let K be such a club. Our hypotheses immediately imply that K is a groupoid with objects in bijection with the natural numbers; we will now assume the functor $K \to B\Sigma$ is the identity on objects. Let $\Lambda(n) = K(n, n)$. Now K comes equipped with a functor to $B\Sigma$, in other words group homomorphisms $\pi_n\colon \Lambda(n) \to \Sigma_n$. We claim

4 Monoidal structures and multicategories

4.1 Introduction to monoidal structures

QQQ

- need to start off by talking about monoidal structures and what this chapter will lead to
- starts off with Examples section
- monoidal categories via action operads
- pseudo-commutativity
- profunctors and multicategories

4.2 Examples

This should be moved earlier/combined

In this section we will discuss examples of the preceding theory. We have seen that there are three equivalent incarnations of the same algebraic structure:

- as an action operad Λ,
- as a 2-monad $X \mapsto \coprod E\Lambda(n) \times_{\Lambda(n)} X^n$ on **Cat**, or
- as a club $B\Lambda \to B\Sigma$ satisfying certain properties.

In practice, something like the third of these is the most likely to arise from applications (even if the notion of a club is perhaps less well-known outside of the categorical literature than that of an operad or a 2-monad) as a club can be given by a presentation as we discussed in Section 2.4. We will go into more detail here, explaining how particular monoidal structures of interest are represented in this theory.

Example 4.1. The 2-monad for symmetric strict monoidal categories (or permutative categories, as they are known in the topological literature) is given by $E\Sigma$, so the notion of symmetric strict monoidal categories corresponds to the symmetric operad. While this example is well-known, we go into further detail to set the stage for less common examples.

The 2-monad $E\Sigma$ on **Cat** is given by

$$E\Sigma(X) = \coprod E\Sigma_n \times_{\Sigma_n} X^n.$$

An object of $E\Sigma_n \times_{\Sigma_n} X^n$ is an equivalence class of the form $[\sigma; x_1, \ldots, x_n]$ where $\sigma \in \Sigma_n$ and $x_i \in X$. The equivalence relation gives

$$[\sigma; x_1, \ldots, x_n] = \left[e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right],$$

so objects can be identified with finite strings of objects of X. Morphisms are given by equivalence classes of the form

$$[\sigma; x_1, \ldots, x_n] \xrightarrow{[!; f_1, \ldots, f_n]} [\tau; y_1, \ldots, y_n].$$

Here $!\colon \sigma \cong \tau$ is the unique isomorphism in $E\Sigma_n$, and $f_i\colon x_i \to y_i$ in X. Using the equivalence relation, we find that morphisms between finite strings

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

are given by a permutation $\rho \in \Sigma_n$ together with maps $f_i\colon x_i \to y_{\rho(i)}$ in X (note that there are no morphisms between strings of different length); this is a special case of the calculation in Lemma 3.40. Thus $E\Sigma(X)$ is easily seen to be the free permutative category generated by X, and therefore Σ-monoidal categories are permutative categories.

Example 4.2. The template above can be used to show that the braid operad **B** corresponds to the 2-monad for braided strict monoidal categories. The details are almost exactly the same, only we use braids instead of permutations. The equivalence relation on objects gives

$$[\gamma; x_1, \ldots, x_n] = \left[e; x_{\pi(\gamma)^{-1}(1)}, \ldots, x_{\pi(\gamma)^{-1}(n)}\right],$$

where $\gamma \in B_n$ and $\pi(\gamma)$ is its underlying permutation; thus objects of $EB(X)$ are once again finite strings of objects of X. A morphism

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

is then given by a braid $\gamma \in B_n$ together with maps $f_i\colon x_i \to y_{\pi(\gamma)(i)}$ in X. Thus one should view a morphism as given by

- a finite ordered set $x_1, \ldots, x_n$ of objects of X as the source,
- another such finite ordered set (of the same cardinality) $y_1, \ldots, y_n$ of objects of X as the target,
- a geometric braid $\gamma \in B_n$ on n strands, and
- for each strand, a morphism in X from the object labeling the source of that strand to the object labeling the target.

This is precisely Joyal and Street's [17] construction of the free braided strict monoidal category generated by a category X, and thus **B**-monoidal categories are braided strict monoidal categories.

This example can be extended to include ribbon braided categories as well. A ***ribbon braid*** is given, geometrically, in much the same way as a braid except that instead of paths $[0,1] \to \mathbb{R}^3$ making up each individual strand, we use ribbons $[0,1] \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$. This introduces the possibility of performing a full twist on a ribbon, and one can describe ribbon braided categories using generators and relations by introducing a natural twist isomorphism $\tau_A\colon A \to A$ and imposing one relation between the twist and the braid $\gamma_{A,B}\colon A \otimes B \to B \otimes A$. In [41], the authors show that the ribbon braid groups give an action operad **RB**, and that (strict) ribbon braided categories are precisely the algebras for ERB.

We now turn to an example which is not as widely known in the categorical literature, that of coboundary categories [8]. These arise in the representation theory of quantum groups and in the theory of crystals [14, 15]. Our goal here is to refine the relationship between coboundary categories and the operad of n-fruit cactus groups in [14] by using the theory of action operads and our Borel construction. We begin by recalling the definition of a coboundary category.

Definition 4.3. A ***coboundary category*** is a monoidal category C equipped with a natural isomorphism $\sigma_{x,y}\colon x \otimes y \to y \otimes x$ (called the ***commutor***) such that

- $\sigma_{y,x} \circ \sigma_{x,y} = 1_{x \otimes y}$ and
- the diagram

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r] \arrow[d, "\sigma_{x,y} 1"'] & x \otimes (y \otimes z) \arrow[r, "1 \sigma_{y,z}"] & x \otimes (z \otimes y) \arrow[d, "\sigma_{x,zy}"] \\
(y \otimes x) \otimes z \arrow[r, "\sigma_{yx,z}"'] & z \otimes (y \otimes x) \arrow[r] & (z \otimes y) \otimes x
\end{tikzcd}$$

commutes (in which the unlabeled morphisms are an associator and an inverse associator).

br + co) = sym

Example 4.4. 1. As noted by Savage [42], any braiding automatically satisfies the cactus relation (the diagram in Definition 4.3). However, since braidings need not be involutions this does not mean that any braided monoidal category is a coboundary category. However, it should then be clear that any symmetric monoidal category is also a coboundary category.

2. The name coboundary category comes from the original work of Drinfeld [8] in which he shows that the category of representations of a coboundary Hopf algebra has the structure of coboundary category.

3. Henriques and Kamnitzer [14] show that the category of crystals for a finite dimensional complex reductive Lie algebra has the structure of a coboundary category.

Our interest is in strict coboundary categories by which we mean coboundary categories with strict underlying monoidal category. Under the assumption of strictness, the second axiom above does not include associations for the tensor product and reduces to a square. To show that every coboundary category is equivalent to a strict coboundary category, we must introduce the 2-category **CobCat** of coboundary categories.

Definition 4.5. Let $(C,\sigma),(C',\sigma')$ be coboundary categories. A *coboundary functor* $F\colon C \to C'$ is a strong monoidal functor (with invertible constraints φ_0 for the unit and $\varphi_{x,y}$ for the tensor product) such that

$$F\sigma_{x,y} \circ \varphi_{x,y} = \varphi_{y,x} \circ \sigma'_{Fx,Fy}$$

holds.

Coboundary functors are composed just as strong monoidal functors are, giving the following.

Lemma 4.6. *Coboundary categories, coboundary functors, and monoidal transformations form a 2-category, which we denote* **CobCat**.

Proposition 4.7. *Let (C,σ) be a coboundary category. Then there exists a strict coboundary category (C',σ') which is equivalent to (C,σ) in* **CobCat**.

Proof. Consider the underlying monoidal category of (C,σ), which we will just write as C. We can find a strict monoidal category C' by coherence for monoidal categories together with an equivalence, as monoidal categories, between C and C'. By standard methods [34], this can be improved to an adjoint equivalence between C and C' in the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. Let $F\colon C \to C', G\colon C' \to C$ be the functors in this adjoint equivalence, and $\eta\colon 1 \Rightarrow FG$ the unit (which we note for emphasis is invertible since it the unit of an adjoint equivalence). For objects $x,y \in C'$, we define a commutor σ' for C' as the following composite.

$$xy \xrightarrow{\eta\otimes\eta} FGxFGy \xrightarrow{\cong} F(GxGy) \xrightarrow{F\sigma} F(GyGx) \xrightarrow{\cong} FGyFGx \xrightarrow{\eta^{-1}\otimes\eta^{-1}} yx$$

We then leave to the reader, if they wish, the computations to show that σ' is a commutor for C' and that F,G become coboundary functors using σ'. □

We now turn to the operadic description of strict coboundary categories; we note from this point onwards, all our coboundary categories are assumed to be strict.

Definition 4.8. Fix $n > 1$, and let $1 \le p < q \le n$, $1 \le k < l \le n$.

1. $p < q$ *is disjoint from* $k < l$ if $q < k$ or $l < p$.
2. $p < q$ *contains* $k < l$ if $p \le k < l \le q$.

Definition 4.9. Let $1 \le p < q \le n$, and define $\hat{s}_{p,q} \in \Sigma_n$ to be the permutation defined below.

$$\begin{array}{c|cccccccccccccc} i & 1 & 2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & q-1 & q & q+1 & \cdots & n \\ \hat{s}_{p,q}(i) & 1 & 2 & \cdots & p-1 & q & q-1 & q-2 & \cdots & p+1 & p & q+1 & \cdots & n \end{array}$$

The n-fruit cactus group is then defined as follows.

Definition 4.10. Let J_n be the group generated by symbols $s_{p,q}$ for $1 \le p < q \le n$ subject to the following relations.

1. For all $p < q$, $s_{p,q}^2 = e$.
2. If $p < q$ is disjoint from $k < l$, then $s_{p,q}s_{k,l} = s_{k,l}s_{p,q}$.
3. If $p < q$ contains $k < l$, then $s_{p,q}s_{k,l} = s_{m,n}s_{p,q}$ where
   - $m = \hat{s}_{p,q}(l)$ and
   - $n = \hat{s}_{p,q}(k)$.

It is easy to check that the elements $\hat{s}_{p,q} \in \Sigma_n$ satisfy the three relations in Definition 4.10, so $s_{p,q} \mapsto \hat{s}_{p,q}$ extends to a group homomorphism $\pi_n\colon J_n \to \Sigma_n$. This is the first step in proving the following.

Theorem 4.11. *The collection of groups $J = \{J_n\}$ form an action operad.*

Proof. We will use Theorem 2.30 to determine the rest of the action operad structure. Thus we must give, for any collection of natural numbers $n, k_1, \ldots, k_n$ and $\underline{k} = \sum k_i$, group homomorphisms $\beta\colon J_{k_1} \times \cdots \times J_{k_n} \to J_{\underline{k}}$ and functions $\delta\colon J_n \to J_{\underline{k}}$ satisfying nine axioms. We define both of these on generators, starting with β.

Let $s_{p_i,q_i} \in J_{k_i}$. Let $r_i = k_1 + k_2 + \cdots + k_{i-1}$ for $i > 1$. Define β by

$$\beta(s_{p_1,q_1}, \ldots, s_{p_n,q_n}) = s_{p_1,q_1}s_{p_2+r_2,q_2+r_2}\cdots s_{p_n+r_n,q_n+r_n}.$$

Note that $s_{p_i+r_i,q_i+r_i}$ and $s_{p_j+r_j,q_j+r_j}$ are disjoint when $i \neq j$.

It is easy to check that this disjointness property ensures that β gives a well-defined group homomorphism

$$J_{k_1} \times \cdots \times J_{k_n} \to J_{\underline{k}}.$$

To define $\delta\colon J_n \to J_{\underline{k}}$ for natural numbers $n, k_1, \ldots, k_n$ and $\underline{k} = \sum k_i$, let $m_k = s_{1,k} \in J_k$. Then we start by defining

$$\delta(m_n) = m_{\underline{k}} \cdot \beta(m_{k_1}, m_{k_2}, \ldots, m_{k_n}).$$

Note that, by equation 8 of Theorem 2.30, this is equal to

$$\beta(m_{k_n}, m_{k_{n-1}}, \ldots, m_{k_1}) \cdot m_{\underline{k}}.$$

Now $s_{p,q} \in J_n$ is equal to $\beta(e_{p-1}, m_{q-p+1}, e_{n-q})$ (here e_i is the identity element in J_i) by definition of the m_i and β, so we can define δ on any generator $s_{p,q}$ by

$$\delta(s_{p,q}) = \beta(e_A, M, e_B)$$

with

- $A = k_1 + k_2 + \cdots + k_{p-1}$,
- $M = m_{k_p+\cdots+k_q} \cdot \beta(m_{k_p}, m_{k_{p+1}}, \ldots, m_{k_q})$, and

Sentence?

Corollary 4.14. *The 2-monad C for coboundary categories corresponds, using Theorem 3.55, to the action operad $\mathbf{J}$.*

4.3 Pseudo-commutativity

move w/ coherence, ~~cat~~ cart

This section gives conditions sufficient to equip the 2-monad $\underline{P}$ induced by a Λ-operad P in **Cat** with a pseudo-commutative structure. Such a pseudo-commutativity will then give the 2-category **Ps-$\underline{P}$-Alg** some additional structure that we briefly explain here. For a field k, the category **Vect** of vector spaces over k has many nice features. Of particular interest to us are the following three structures. First, the category **Vect** is monoidal using the tensor product $\otimes_k$. Second, the set of linear maps $V \to W$ is itself a vector space which we denote $[V, W]$. Third, there is a notion of multilinear map $V_1 \times \cdots \times V_n \to W$, with linear maps being the 1-ary version. While these three structures are each useful in isolation, they are tied together by natural isomorphisms

$$\mathbf{Vect}(V_1 \otimes V_2, W) \cong \mathbf{Vect}(V_1, [V_2, W]) \cong \mathbf{Bilin}(V_1 \times V_2, W)$$

expressing that $\otimes$ gives a closed monoidal structure which represents the multicategory of multilinear maps. Moreover, the adjunction between **Vect** and **Sets** respects all of this structure in the appropriate way. This incredibly rich interplay between the tensor product, the internal mapping space, and the multicategory of multilinear maps all arises from the free vector space monad on **Sets** being a *commutative* monad [26, 25, 27]. The notion of a pseudo-commutative 2-monad [16] is then a generalization of this machinery to a 2-categorical context, and can be viewed as a starting point for importing tools from linear algebra into category theory.

The aim of this section is to give conditions that ensure that the 2-monad $\underline{P}$ associated to a Λ-operad P has a pseudo-commutative structure. We give the definition of pseudo-commutativity as in [16] but before doing so we require the definition of a strength for a 2-monad.

Definition 4.15. A *strength* for an endo-2-functor $T \colon \mathcal{K} \to \mathcal{K}$ on a 2-category with products and terminal object 1 consists of a 2-natural transformation d with components

$$d_{A,B} \colon A \times TB \to T(A \times B)$$

satisfying the following unit and associativity axioms [26].

$$\begin{tikzcd} 1 \times TA \arrow[r, "d_{1,A}"] \arrow[dr, "\cong"'] & T(1 \times A) \arrow[d, "\cong"] \\ & TA \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[r, "1 \times \eta"] \arrow[dr, "\eta"'] & A \times TB \arrow[d, "d_{A,B}"] \\ & T(A \times B) \end{tikzcd}$$

$$\begin{tikzcd} (A \times B) \times TC \arrow[rr, "d_{AB,C}"] \arrow[d, "a"'] & & T((A \times B) \times C) \arrow[d, "Ta"] \\ A \times (B \times TC) \arrow[r, "1 \times d_{B,C}"'] & A \times T(B \times C) \arrow[r, "d_{A,BC}"'] & T(A \times (B \times C)) \end{tikzcd}$$

$$\begin{tikzcd} A \times T^2B \arrow[r, "d_{A,TB}"] \arrow[d, "1 \times \mu"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] & T^2(A \times B) \arrow[d, "\mu"] \\ A \times TB \arrow[rr, "d_{A,B}"'] & & T(A \times B) \end{tikzcd}$$

Similarly, a *costrength* for T consists of a 2-natural transformation d^* with components

$$d^*_{A,B} \colon TA \times B \to T(A \times B)$$

again satisfying unit and associativity axioms.

The strength and costrength for the associated 2-monad $\underline{P}$ are quite simple to define. We define the strength d for $\underline{P}$ as follows. The component $d_{A,B}$ is a functor

$$d_{A,B} \colon A \times (\amalg P(n) \times_{\Lambda(n)} B^n) \to \amalg P(n) \times_{\Lambda(n)} (A \times B)^n$$

which sends an object $(a, [p; b_1, \ldots, b_n])$ to the object $[p; (a, b_1), \ldots, (a, b_n)]$. We also define the costrength similarly, sending an object $([p; a_1, \ldots, a_n], b)$ to the object which is an equivalence class $[p; (a_1, b), \ldots, (a_n, b)]$. Both the strength and the costrength are defined in the obvious way on morphisms.

Remark 4.16. It is crucial to note that the strength d and the costrength d^* do not depend on the Λ-actions in the following sense. The Λ-operad P has an underlying non-symmetric operad that we also denote P, and it has a strength

$$d_{A,B} \colon A \times (\amalg P(n) \times B^n) \to \amalg P(n) \times (A \times B)^n$$

given by essentially the same formula:

$$(a; (p; b_1, \ldots, b_n)) \mapsto (p; (a, b_1), \ldots, (a, b_n)).$$

The strength for the Λ-equivariant P is just the induced functor between coequalizers.

Definition 4.17. Given a 2-monad $T \colon \mathcal{K} \to \mathcal{K}$ with strength d and costrength d^*, a *pseudo-commutativity* consists of an invertible modification γ with components

$$\begin{tikzcd} TA \times TB \arrow[r, "d^*_{A,TB}"] \arrow[d, "d_{TA,B}"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma_{A,B}"] & T^2(A \times B) \arrow[d, "\mu_{A \times B}"] \\ T(TA \times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A \times B) \arrow[r, "\mu_{A \times B}"'] & T(A \times B) \end{tikzcd}$$

satisfying the following three strength axioms, two unit (or η) axioms, and two multiplication (or μ) axioms for all A, B, and C.

1. $\gamma_{A \times B, C} * (d_{A,B} \times 1_{TC}) = d_{A, B \times C} * (1_A \times \gamma_{B,C})$.
2. $\gamma_{A, B \times C} * (1_{TA} \times d_{B,C}) = \gamma_{A \times B, C} * (d^*_{A,B} \times 1_{TC})$.
3. $\gamma_{A, B \times C} * (1_{TA} \times d^*_{B,C}) = d^*_{A \times B, C} * (\gamma_{A,B} \times 1_C)$.
4. $\gamma_{A,B} * (\eta_A \times 1_{TB})$ is the identity on d.
5. $\gamma_{A,B} * (1_{TA} \times \eta_B)$ is the identity on d^*.
6. $\gamma_{A,B} * (\mu_A \times 1_{TB})$ is equal to the pasting below.

$$\begin{tikzcd} T^2A \times TB \arrow[r, "d^*_{TA,TB}"] \arrow[d, "d_{T^2A,B}"'] & T(TA \times TB) \arrow[r, "Td^*_{A,TB}"] \arrow[d, "Td_{TA,B}"] & T^2(A \times TB) \arrow[r, "T^2 d_{A,B}"] \arrow[d, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A \times B) \arrow[d, "T\mu_{A \times B}"] \\ T(T^2A \times B) \arrow[r, phantom, "\Downarrow\gamma_{TA,B}"] \arrow[d, "Td^*_{TA,B}"'] & T^2(TA \times B) \arrow[r, "T^2 d^*_{A,B}"'] \arrow[d, "\mu_{TA \times B}"] & T^3(A \times B) \arrow[r, "T\mu_{A \times B}"'] \arrow[d, "\mu_{T(A \times B)}"] & T^2(A \times B) \arrow[d, "\mu_{A \times B}"] \\ T^2(TA \times B) \arrow[r, "\mu_{TA \times B}"'] & T(TA \times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A \times B) \arrow[r, "\mu_{A \times B}"'] & T(A \times B) \end{tikzcd}$$

7. $\gamma_{A,B} * (1_{TA} \times \mu_B)$ is equal to the pasting below.

$$\begin{tikzcd}
TA{\times}T^2B \arrow[r, "d^*_{A,T^2B}"] \arrow[d, "d_{TA,TB}"'] & T(A{\times}T^2B) \arrow[r, "Td_{A,TB}"] \arrow[d, phantom, "\Downarrow\gamma_{A,TB}"] & T^2(A{\times}TB) \arrow[d, "\mu_{A\times TB}"] \\
T(TA{\times}TB) \arrow[r, "Td^*_{A,TB}"'] \arrow[d, "Td_{TA,B}"'] & T^2(A{\times}TB) \arrow[r, "\mu_{A\times TB}"'] \arrow[d, "T^2d_{A,B}"] & T(A{\times}TB) \arrow[d, "Td_{A,B}"] \\
T^2(TA{\times}B) \arrow[r, phantom, "\Downarrow T\gamma_{A,B}"] \arrow[d, "T^2d^*_{A,B}"'] & T^3(A{\times}B) \arrow[r, "\mu_{T(A\times B)}"] \arrow[d, "T\mu_{A\times B}"] & T^2(A{\times}B) \arrow[d, "\mu_{A\times B}"] \\
T^3(A{\times}B) \arrow[r, "T\mu_{A\times B}"'] & T^2(A{\times}B) \arrow[r, "\mu_{A\times B}"'] & T(A{\times}B)
\end{tikzcd}$$

Remark 4.18. It is noted in [16] that this definition has some redundancy and therein it is shown that any two of the strength axioms immediately implies the third. Furthermore, the three strength axioms are equivalent when the η and μ axioms hold, as well as the following associativity axiom:

$$\gamma_{A,B\times C} \circ (1_{TA} \times \gamma_{B,C}) = \gamma_{A\times B,C} \times (\gamma_{A,B} \times 1_{TC}).$$

We need some further notation before stating our main theorem. Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two lists. Then the set $\{(a_i, b_j)\}$ has mn elements, and two natural lexicographic orderings. One of these we write as $(a, \underline{b})$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} p < r, \text{ or} \\ p = r \text{ and } q < s. \end{cases}$$

The other we write as $(\underline{a}, b)$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} q < s, \text{ or} \\ q = s \text{ and } p < r. \end{cases}$$

The notation $(a, \underline{b})$ is meant to indicate that there is a single a but a list of b's, so then $(\underline{a, \underline{b}})$ would represent a list which itself consists of lists of that form. There exists a unique permutation $\tau_{m,n} \in \Sigma_{mn}$ which has the property that $\tau_{m,n}(i) = j$ if the ith element of the ordered set $(a, \underline{b})$ is equal to the jth element of the ordered set $(\underline{a}, b)$. By construction, we have $\tau_{n,m} = \tau_{m,n}^{-1}$. We illustrate these permutations with a couple of examples.


$\tau_{2,3}$


$\tau_{4,2}$

Note then that $\tau_{m,n}$ is the permutation given by taking the transpose of the $m \times n$ matrix with entries (a_i, b_j).

We now define what it means for a Λ-operad to be pseudo-commutative, before then showing that such an operad yields a pseudo-commutative structure on the corresponding 2-monad $\underline{P}$. Let $\mathbb{N}_+$ denote the set of positive integers.

emph

Definition 4.19. Let P be a Λ-operad. A pseudo-commutative structure on P consists of:

- For each pair $(m, n) \in \mathbb{N}_+^2$, an element $t_{m,n} \in \Lambda(mn)$ such that $\pi(t_{m,n}) = \tau_{m,n}$.
- For each $p \in P(n)$, $q \in P(m)$, a natural isomorphism

$$\lambda_{p,q} \colon \mu(p; q, \ldots, q) \cdot t_{m,n} \cong \mu(q; p, \ldots, p).$$

We write this as $\lambda_{p,q} \colon \mu(p; \underline{q}) \cdot t_{m,n} \cong \mu(q; \underline{p})$.

These are required to satisfy the following axioms:

1. For all $n \in \mathbb{N}_+$,
$$t_{1,n} = e_n = t_{n,1}$$
and for all $p \in P(n)$, the isomorphism $\lambda_{p,\mathrm{id}} \colon p \cdot e_n \cong p$ is the identity map.
2. QQQ Equivariance axiom! (See Remark 11.2 in [13].) QQQ It's basically 'compose, act, switch' is the same as 'compose, switch, act', but can't actually see where it's used in the proof:
$$\lambda_{p \cdot g, q \cdot h} \circ \mu^P\left(\mathrm{id}_p \cdot g; \underline{\mathrm{id}_q \cdot h}\right) = \mu^P\left(\mathrm{id}_q \cdot h; \underline{\mathrm{id}_p \cdot g}\right) \circ \lambda_{p,q}.$$
For each $p \in P(n)$, $q \in P(m)$, $g \in \Lambda(n)$, and $h \in \Lambda(m)$, the following diagram commutes:
$$\begin{tikzcd}
\mu^P\left(p; \underline{q}\right) \cdot t_{m,n} \arrow[r, "\lambda_{p,q}"] \arrow[d, "\mu^P\left(\mathrm{id}_p \cdot g; \underline{\mathrm{id}_q \cdot h}\right)"'] & \mu^P\left(q; \underline{p}\right) \arrow[d, "\mu^P\left(\mathrm{id}_q \cdot h; \underline{\mathrm{id}_p \cdot g}\right)"] \\
\mu^P\left(p \cdot g; \underline{q \cdot h}\right) \cdot t_{m,n} \arrow[r, "\lambda_{p \cdot g, q \cdot h}"'] & \mu^P\left(q \cdot h; \underline{p \cdot g}\right)
\end{tikzcd}$$
3. For all $l, m_1, \ldots, m_l, n \in \mathbb{N}_+$, with $M = \Sigma m_i$,
$$\mu^\Lambda\left(e_l; t_{m_1,n}, \ldots, t_{m_l,n}\right) \cdot \mu^\Lambda\left(t_{l,n}; \underline{e_{m_1}, \ldots, e_{m_l}}\right) = t_{n,M}.$$
Here $\underline{e_{m_1}, \ldots, e_{m_l}}$ is the list $e_{m_1}, \ldots, e_{m_l}$ repeated n times.
4. For all $l, m, n_1, \ldots, n_m \in \mathbb{N}_+$, with $N = \Sigma n_i$,
$$\mu^\Lambda\left(t_{m,l}; \underline{e_{n_1}}, \ldots, \underline{e_{n_m}}\right) \cdot \mu^\Lambda\left(e_m; t_{n_1,l}, \ldots, t_{n_m,l}\right) = t_{N,l}.$$
Here $\underline{e_{n_i}}$ indicates that each e_{n_i} is repeated l times.
5. For any $l, m_i, n \in \mathbb{N}_+$, with $1 \leq i \leq n$, and $p \in P(l)$, $q_i \in P(m_i)$ and $r \in P(n)$, the following diagram commutes. (Note that we maintain the convention that anything underlined indicates a list, and double underlining indicates a list of lists. Each instance should have an obvious meaning from context and the equations

appearing above.)

$$
\begin{tikzcd}
\mu\left(p;\underline{\mu(q_i;\underline{r})}\right)\cdot\mu(e_i;t_{n,m_i})\mu(t_{n,l};\underline{\underline{e_{m_i}}}) \arrow[r, equal] \arrow[d, equal] & \mu\left(p;\underline{\mu(q_i;\underline{r})}\right)\cdot t_{n,M} \arrow[dd, equal] \\
\mu\left(p;\underline{\mu(q_i;\underline{r})\cdot t_{n,m_i}}\right)\cdot\mu(t_{n,l};\underline{e_{m_1},\ldots,e_{m_l}}) \arrow[d, "\mu(1;\lambda_{q_i,r})\cdot 1"'] & \\
\mu\left(p;\underline{\mu(r;\underline{q_i})}\right)\cdot\mu(t_{n,l};\underline{e_{m_1},\ldots,e_{m_l}}) \arrow[d, equal] & \mu\left(\mu(p;q_1,\ldots,q_n);\underline{r}\right)\cdot t_{n,M} \arrow[d, "\lambda_{\mu(p;q_1,\ldots,q_n),r}"] \\
\mu\left(\mu(p;\underline{r})\cdot t_{n,l};\underline{q_1,\ldots,q_n}\right) \arrow[d, "\mu(\lambda_{p,r};1)"'] & \mu\left(\mu(p;q_1,\ldots,q_n);\underline{r}\right) \arrow[d, equal] \\
\mu\left(\mu(r,\underline{p});\underline{q_1,\ldots,q_n}\right) \arrow[r, equal] & \mu\left(r;\underline{\mu(p;q_1,\ldots,q_n)}\right)
\end{tikzcd}
$$

6. For any $l, m, n_i \in \mathbb{N}_+$, with $1 \leq i \leq m$, and $p \in P(l)$, $q \in P(m)$ and $r_i \in P(n_i)$, the following diagram commutes.

$$
\begin{tikzcd}
\mu\left(\mu(p;\underline{q})\cdot t_{m,l};\underline{r_i}\right)\cdot\mu(e_m;\underline{t_{n_i,l}}) \arrow[r, equal] \arrow[d, "\mu(\lambda_{p,q};1)\cdot 1"'] & \mu\left(\mu(p;\underline{q});\underline{r_i}\right)\cdot\mu(t_{m,l};\underline{\underline{e_{n_i}}})\mu(e_m;\underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\left(\mu(q;\underline{p});\underline{r_1,\ldots,r_m}\right)\cdot\mu(e_m;\underline{t_{n_i,l}}) \arrow[dd, equal] & \mu\left(p;\underline{\mu(q;\underline{r_i})}\right)\cdot\mu(t_{m,l};\underline{\underline{e_{n_i}}})\mu(e_m;\underline{t_{n_i,l}}) \arrow[d, equal] \\
 & \mu\left(p;\underline{\mu(q;r_1,\ldots,r_m)}\right)\cdot t_{N,l} \arrow[d, "\lambda_{p,\mu(q;r_1,\ldots,r_m)}"] \\
\mu\left(q;\underline{\mu(p;\underline{r_i})}\right)\cdot\mu(e_m;\underline{t_{n_i,l}}) \arrow[d, equal] & \mu\left(\mu(q;r_1,\ldots,r_m);\underline{p}\right) \arrow[d, equal] \\
\mu\left(q;\underline{\mu(p;\underline{r_i})\cdot t_{n_i,l}}\right) \arrow[r, "\mu(1;\lambda_{p,r_i})"'] & \mu\left(q;\underline{\mu(r_i;\underline{p})}\right)
\end{tikzcd}
$$

Remark 4.20. Originally the definition presented above did not include the equivariance axiom (Axiom 2). That this axiom is required was identified by the authors of [12] with the detail described in Remark 11.2 of [13].

Theorem 4.21. *Let P be a Λ-operad equipped with a pseudo-commutative structure. Then $\underline{P}$ has a pseudo-commutativity.*

Proof. We begin the proof by defining an invertible modification γ for the pseudo-commutativity for which the components are natural transformations $\gamma_{A,B}$. Such a transformation $\gamma_{A,B}$ has components with source

$$\left[\mu\left(p;q\right);\underline{(a,b)}\right]$$

and target

$$\left[\mu\left(q;p\right);\underline{(a,b)}\right].$$

Now $\lambda_{p,q}\colon \mu(p;q,\ldots,q)\cdot t_{m,n} \cong \mu(q;p,\ldots,p)$ gives rise to another map by multiplication on the right by $t_{m,n}^{-1}$,

$$\lambda_{p,q}\cdot t_{m,n}^{-1}\colon \mu(p;q,\ldots,q) \cong \mu(q;p,\ldots,p)\cdot t_{m,n}^{-1},$$

so we define $\left(\gamma_{A,B}\right)_{[p;a_1,\ldots,a_n],[q;b_1,\ldots,b_m]}$ to be the morphism which is the image of $(\lambda_{p,q}\cdot t_{m,n}^{-1}, 1)$ under the map

$$\coprod P(n)\times(A\times B)^n \to \coprod P(n)\times_{\Lambda(n)}(A\times B)^n.$$

Naturality of $\gamma_{A,B}$ follows from that of each $\lambda_{p,q}$. We will write this morphism as $[\lambda_{p,q}t_{m,n}^{-1}, 1]$. In the case that either p or q is an identity then we choose the component of γ to be the isomorphism involving the appropriate identity element using axiom 1 above.

There are two things to note about the definition above before we continue. First, it is easy to check that

$$t_{m,n}^{-1}\cdot\underline{(a,b)} = \underline{(a,b)}$$

since $\pi(t_{m,n}) = \tau_{m,n}$; this ensures that γ has the correct target. Second, the morphism above has second component the identity. This is actually forced upon us by the requirement that γ be a modification: in the case that A, B are discrete categories, the only possible morphism is an identity, and the modification axiom then forces that statement to be true for general A, B by considering the inclusion $A_0 \times B_0 \hookrightarrow A \times B$ where A_0, B_0 are the discrete categories with the same objects as A, B.

We show that this is a modification by noting that it does not rely on objects in the lists $a_1,\ldots,a_n$ or $b_1,\ldots,b_m$, only on their lengths and the operations p and q. As a result, if there are functors $f\colon A\to A'$ and $g\colon B\to B'$, then it is clear that

$$(\underline{P}(f\times g)\circ\gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]} = [\lambda_{p,q},\underline{1}] = (\gamma_{A',B'}\circ(\underline{P}f\times\underline{P}g))_{[p;\underline{a}],[q;\underline{b}]}.$$

As such we can simply write $(\gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]}$ in shorthand as $\gamma_{p,q}$.

There are now seven axioms to check for a pseudo-commutativity: three strength axioms, two unit axioms, and two multiplication axioms. For the first strength axiom, we must verify that two different 2-cells of shape

$$
\begin{tikzcd}
A\times TB\times TC \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & T(A\times B\times C)
\end{tikzcd}
$$

are equal. The first of these is γ precomposed with $d\times 1$, and so is the component of γ at an object

$$([p;(a,b_1),\ldots,(a,b_n)],[q;c_1,\ldots,c_m]).$$

The second of these is d applied to the component of $1\times\gamma$ at

$$(a,([p;b_1,\ldots,b_n],[q;c_1,\ldots,c_m])).$$

It is straightforward to compute that each of these maps is the image of $(\lambda_{p,q}\cdot t_{m,n}^{-1}, 1)$ under the functor

$$\coprod P(n)\times(A\times B)^n \to \coprod P(n)\times_{\Lambda(n)}(A\times B)^n.$$

The other two strength axioms follow by analogous calculations for other whiskerings of γ with d or d^*.

are precisely those connected (via the group operation in B_{Nl}, concatenation) to the duplicated copies of strands $i, i+l, i+2l, \ldots, i+(n-1)l$ in $t_{n,l}$. Thus if a pair of strands were to cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$, that pair cannot also have crossed in $\mu\left(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}\right)$, showing that the resulting product braid

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu\left(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}\right)$$

is minimal. The calculation showing that

$$\mu\left(t_{m,l}; \underline{e_1, \ldots, e_{n_m}}\right)\mu\left(e_m; t_{n_1,l}, \ldots, t_{n_m,l}\right)$$

is also minimal follows from the same argument, showing that it is equal to $t_{N,l}$ (here N is the sum of the n_i, where once again i ranges from 1 to l).

These calculations show, using Theorem 4.21, that the B-operad B induces a 2-monad which has a pseudo-commutative structure. As noted before, B-algebras are precisely braided strict monoidal categories. The second pseudo-commutative structure arises by using negative, minimal braids instead of positive ones, and proceeds using the same arguments. This finishes the first part of the proof of Theorem 4.29.

We will now show that neither of these pseudo-commutative structures is symmetric. The symmetry axiom in this case reduces to the fact that, in some category which is given as a coequalizer, the morphism with first component

$$f\colon \mu\left(p; \underline{q}\right) \cdot t_{n,m}t_{m,n} \to \mu\left(q; \underline{p}\right) \cdot t_{m,n} \to \mu\left(p; \underline{q}\right)$$

is the identity. Now it is clear that $t_{n,m}$ is not equal to $t_{m,n}^{-1}$ in general: taking $m = n = 2$, we note that $t_{2,2} = \sigma_2$, and this element is certainly not of order two in B_4. $B(4)$ is the category whose objects are the elements of B_4 with a unique isomorphism between any two pair of objects, and B_4 acts by multiplication on the right; this action is easily shown to be free and transitive. We recall (see Lemma 3.24) that in a coequalizer of the form $A \times_G B$, a morphism $[f_1, f_2]$ equals $[g_1, g_2]$ if and only if there exists an $x \in G$ such that

$$f_1 \cdot x = g_1,$$
$$x^{-1} \cdot f_2 = g_2.$$

For the coequalizer in question, for f to be the first component of an identity morphism would imply that $f \cdot x$ would be a genuine identity in $B(4)$ for some x. But $f \cdot x$ would have source $\mu\left(p; \underline{q}\right) t_{n,m}t_{m,n}x$ and target $\mu\left(p; \underline{q}\right) x$, which requires $t_{n,m}t_{m,n}$ to be the identity group element for all n, m. In particular, this would force $t_{2,2}$ to have order two, which as noted above does not hold in B_4, thus giving a contradiction. □

Remark 4.35. The pseudo-commutativities given above are not necessarily the only ones that exist for the B-operad B, but we do not know a general strategy for producing others.

4.4 Profunctors and multicategories

In this section we generalize from operads to multicategories (or colored operads). The notions of plain and symmetric multicategories are standard [2], but in fact there is a corresponding notion of Λ-multicategory for any action operad Λ. We will give the basic definition and then show that it arises abstractly from a lifting of $E\Lambda$ as a 2-monad on **Cat** to a pseudomonad on **Prof**, the bicategory of categories, profunctors, and transformations. A quick treatment of similar material but restricted to the symmetric case can be found in [11].

mose this too

Definition 4.36. Let Λ be an action operad. A Λ-*multicategory* M consists of the following data:

- a set of objects M_0;
- for any finite list $x_1, \ldots, x_n$ of objects and any object y, a set

$$M(x_1, \ldots, x_n; y)$$

  of multi-arrows (or just arrows) from $x_1, \ldots, x_n$ to y;
- for each $\alpha \in \Lambda(n)$, an isomorphism

$$- \cdot \alpha\colon M(x_1, \ldots, x_n; y) \to M\left(x_{\pi(\alpha)(1)}, \ldots, x_{\pi(\alpha)(n)}; y\right);$$

- for each object x, an arrow $\mathrm{id}_x \in M(x; x)$; and
- a composition function

$$M(y_1, \ldots, y_k; z) \times \prod_{i=1}^{k} M(x_{i1}, \ldots, x_{in_i}; y_i) \to M(\underline{x}; z)$$

  where $\underline{x} = x_{11}, \ldots, x_{1,n_1}, x_{21}, \ldots, x_{k,n_k}$, and which we write as

$$(g; f_1, \ldots, f_k) \mapsto g(f_1, \ldots, f_k).$$

These data are subject to the following axioms.

1. id is a two-sided unit:

$$\mathrm{id}(f) = f,$$
$$f(\mathrm{id}, \ldots, \mathrm{id}) = f.$$

2. Composition is associative:

$$f\left(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n})\right) = f(g_1, \ldots, g_n)(h_{11}, \ldots, h_{nm_n}).$$

3. Composition respects the group actions:

$$f(g_1 \cdot \alpha_1, \ldots, g_n \cdot \alpha_n) = f(g_1, \ldots, g_n) \cdot \mu^{\Lambda}(e; \alpha_1, \ldots, \alpha_n),$$
$$(f \cdot \alpha)(g_1, \ldots, g_n) = f\left(g_{\pi^{-1}(\alpha)(1)}, \ldots, g_{\pi^{-1}(\alpha)(n)}\right) \cdot \mu^{\Lambda}(\alpha; e_1, \ldots, e_n).$$

Definition 4.37. Let M, N be Λ-multicategories. A Λ-*multifunctor* F consists of the following data:

- a function $F_0\colon M_0 \to N_0$ on sets of objects and
- functions $F\colon M(x_1, \ldots, x_n; y) \to N(F_0(x_1), \ldots, F_0(x_n); F_0(y))$ which are $\Lambda(n)$-equivariant in that $F(f \cdot \alpha) = F(f) \cdot \alpha$.

These data are subject to the following axioms.

1. F preserves identites: $F(\mathrm{id}_x) = \mathrm{id}_{F_0(x)}$.
2. F preserves composition: $F\left(f(g_1, \ldots, g_n)\right) = F(f)\left(F(g_1), \ldots, F(g_n)\right)$.