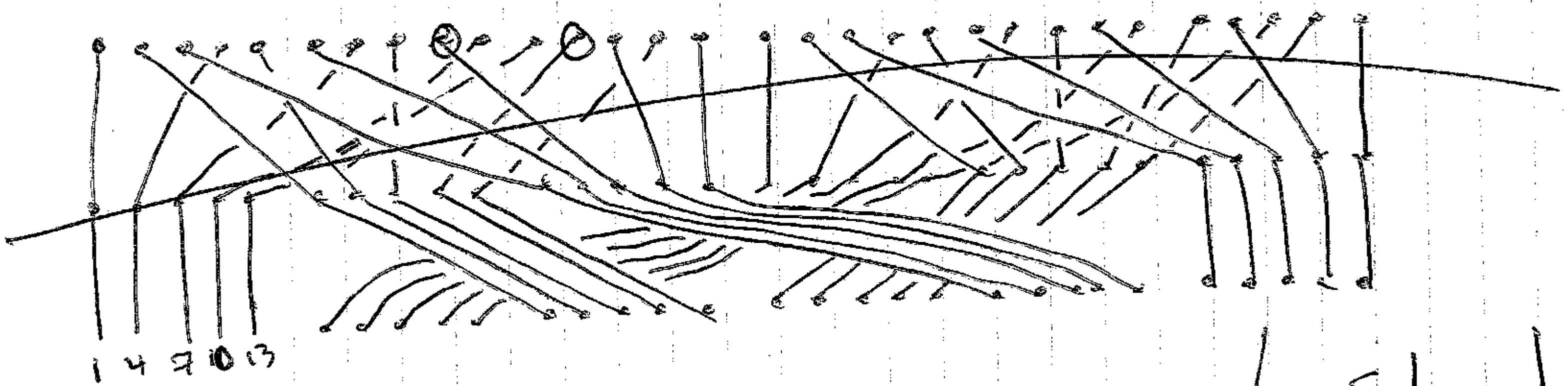
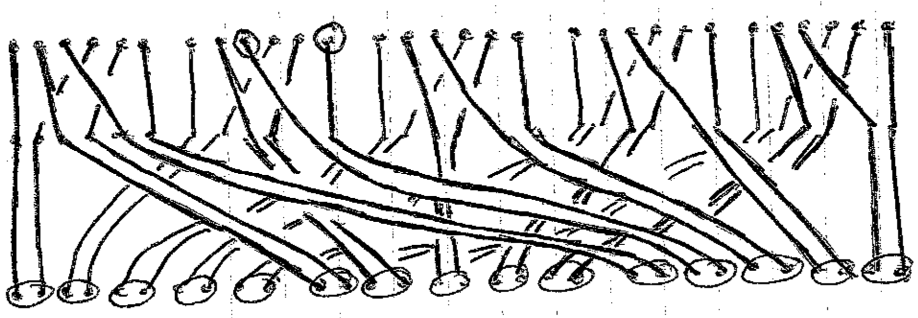
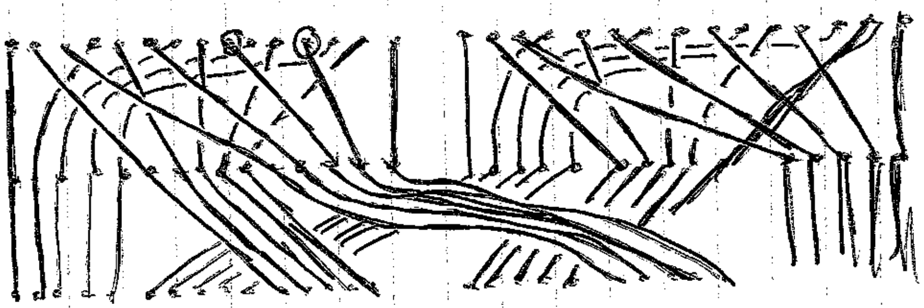
2,5|3)



(2,5|3)

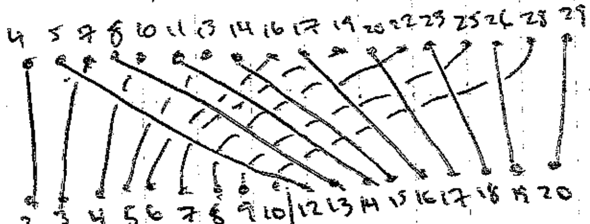
(5,2|3)



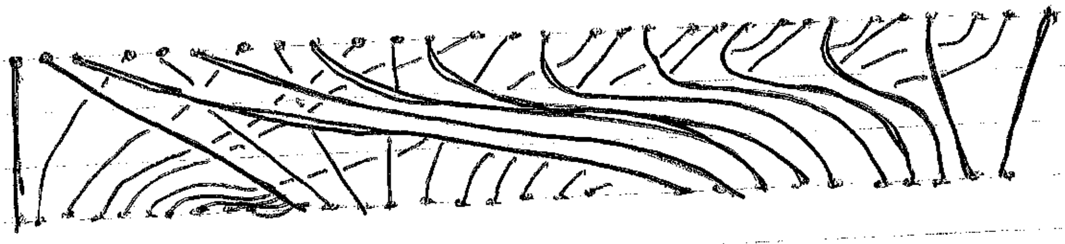


Top: 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30
4, 5, 8, 11, 14, 17, 20, 23, 26, 29
7, 10, 13, 16, 19, 22, 25, 28

1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30
4, 5, 8, 11, 14, 17, 20, 23, 26, 29
7, 10, 13, 16, 19, 22, 25, 28



(9|2)



Top: 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30
4, 5, 8, 11, 14

$(i, j, k) \longleftrightarrow (i-1)mn+(j-1)n+k$

$(l, m | n)$ $\quad (i, j, k) \mapsto (i, k, j)$

$(i-1)mn+(j-1)n+k \mapsto (i-1)mn+(k-1)m+j$

$(i_1, j_1, k_1) < (i_2, j_2, k_2)$ if:
- $i_1 < i_2$
- $i_1 = i_2$ & $j_1 < j_2$
- $i_1 = i_2, j_1 = j_2, k_1 < k_2$

$(i_1, k_1, j_1) < (i_2, k_2, j_2)$ if
- $i_1 < i_2$
- $i_1 = i_2$ & $k_1 < k_2$
- $i_1 = i_2, k_1 = k_2, j_1 < j_2$

$(i_1, j_1, k_1) \searrow' (i_2, j_2, k_2)$ if $(i_1, j_1, k_1) < (i_2, j_2, k_2)$ &
$(i_2, k_2, j_2) < (i_1, k_1, j_1)$

$\iff i_1 = i_2$ & $j_1 < j_2, k_2 < k_1$

$(k_1, i_1, j_1) < (k_2, i_2, j_2)$ if
- $k_1 < k_2$
- $k_1 = k_2, i_1 < i_2$
- $k_1 = k_2, i_1 = i_2, j_1 < j_2$

$(i_1, k_1, j_1) \searrow' (i_2, k_2, j_2)$ if $(i_1, k_1, j_1) < (i_2, k_2, j_2)$ &
$(k_2, i_2, j_2) < (k_1, i_1, j_1)$

$\iff i_1 < i_2$ & $k_2 < k_1$
or $i_1 = i_2, k_1 < k_2$ or $k_2 = k_1, i_2 < i_1$
or $i_1 = i_2, k_1 = k_2, j_1 < j_2$ or $k_2 = k_1, i_2 = i_1, j_2 < j_1$

$\iff i_1 < i_2, k_2 < k_1$

~~⇒ Not possible to have~~ $\cancel{(i_1, j_1, k_1) \searrow' (i_2, j_2, k_2)}$ ~~&~~

⇒ Not possible: $(i_1, j_1, k_1) \searrow' (i_2, j_2, k_2)$ & ⇒ minimal!
$(i_2, k_2, j_2) \searrow' (i_1, k_1, j_1)$

Write $(i_1, j_1, k_1) \downarrow (i_2, j_2, k_2)$ if

(1) • $(i_1, j_1, k_1) \searrow (i_2, j_2, k_2)$ or

(2) • $(i_1, k_1, j_1) \searrow (i_2, k_2, j_2)$

(1) means $i_1 = i_2$, $j_1 < j_2$ & $k_2 < k_1$

(2) means $i_1 < i_2$, $k_2 < k_1$

$$(lm \mid pn) \leftrightarrow \underset{1 \le \le lm \quad \le n}{\overset{lm \quad n}{(\alpha, \beta)}} \mapsto (\alpha-1)mn + \beta$$

$$\longmapsto (\beta, \alpha) = (\beta-1)lm + \alpha_1$$

$(\alpha_1, \beta_1) \searrow' (\alpha_2, \beta_2)$ if • $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ &

• $(\beta_2, \alpha_2) < (\beta_1, \alpha_1)$

$(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$: $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$, $\beta_1 < \beta_2$

$(\beta_2, \alpha_2) < (\beta_1, \alpha_1)$: $\beta_2 < \beta_1$, or $\beta_2 = \beta_1$, $\alpha_2 < \alpha_1$

$\Leftrightarrow \alpha_1 < \alpha_2$, $\beta_2 < \beta_1$

$r_1 \downarrow r_2 \Leftrightarrow r_1 = (i_1, j_1, k_1)$, $\boxed{r_2 = (i_2, j_2, k_2)}$

$= \boxed{(i_1-1)mn + (j_1-1)n + k_1} = (i_2-1)mn + (j_2-1)n + k_2$

• $i_1 = i_2$, $j_1 < j_2$ & $k_2 < k_1$

or • $i_1 < i_2$, $k_2 < k_1$

$r_1 \searrow' r_2 \Leftrightarrow r_1 = (\alpha_1, \beta_1)$, $r_2 = (\alpha_2, \beta_2)$

$= (\alpha_1 - 1)n + \beta_1$, $= (\alpha_2 - 1)n + \beta_2$

• $\alpha_1 < \alpha_2$, $\beta_2 < \beta_1$

Now $\alpha_1 - 1 = (i_1 - 1)m + (j_1 - 1)$

$\alpha_1 = (i_1 - 1)m + j_1$

$\alpha_1 < \alpha_2$ iff $(i_1-1)m + j_1 < (i_2 - 1)m + j_2$

Since $1 \le j \le m$, true iff

• $i_1 < i_2$, or

• $i_1 = i_2$ & $j_1 < j_2$

$k_2 < k_1$ iff $\beta_2 < \beta_1$

So $r_1 \searrow' r_2$ iff $r_1 \downarrow r_2$

$(i_1, j_1, k_1) \downarrow (i_2, j_2, k_2)$; $(i_2, j_2, k_2) \downarrow (i_3, j_3, k_3)$

$i_1 = i_2,\ j_1 < j_2$; $k_2 < k_1$ or $i_1 < i_2,\ k_2 < k_1$

$i_2 = i_3,\ j_2 < j_3$; $k_3 < k_2$ or $i_2 < i_3,\ k_3 < k_2$

Is $(i_1, j_1, k_1) \downarrow (i_3, j_3, k_3)$: $i_1 = i_3,\ j_1 < j_3,\ k_3 < k_1$ or $i_1 < i_3,\ k_3 < k_1$

$\Rightarrow k_3 < k_1$

1+1 $\Rightarrow$ $i_1 = i_2 = i_3,\ j_1 < j_2 < j_3$ ✓

1+2 $i_1 = i_2 < i_3,\ j_1 < j_2$ ✓

2+1 $i_1 < i_2 = i_3,\ j_2 < j_3$ ✓

2+2 $i_1 < i_2 < i_3$ ✓