The iso $(a, \underline{b}) \cong (\underline{a}, b)$ in $(A \times B)^{mn}$

uses a non-trivial permutation, so it is easiest to require that $\pi : G(n) \to \Sigma_n$ is surjective. Then $\exists\, g \in G(n)$ s.t.

$$[\mu(p;q,\dots,q),\ (a,\underline{b})] = [\mu(p;q,\dots,q)\cdot g^{-1} g,\ (a,\underline{b})]$$

$$= [\mu(p;q,\dots,q)g^{-1},\ \pi(g)\cdot(a,\underline{b})]$$

$$\pi(g)\cdot(a,\underline{b}) = (\underline{a}, b)$$

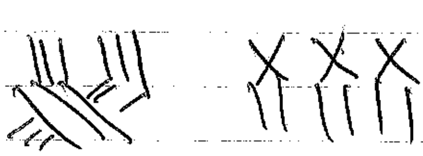
Correct thm #2: If $\pi : G(n) \to \Sigma_n$ is surjective & $P(n)$'s contractible, then P is ps-comm.

~~Correct thm #1:~~

$\Rightarrow$ Free strict monoidal – not ps-comm
Free strict sym/br – is ps-comm
Free strict G-sym – is ps-comm
$P(n) = [EG, E\Sigma_n]$

Then need an iso $\mu(p;q,\dots,q)\cdot g^{-1} \cong \mu(q;p,\dots,p)$.

Symmetries agree $p = \times$ $q = \times$ ✓
$p = \times$ $q = |||$



Natural iso

$$[\mu(q;p,\dots,p), \underline{(a,b)}] \cong [\mu(p;q,\dots,q), \underline{(a,b)}]$$

consists of
- $h \in G(mn)$ s.t. $\pi(h)\left(\underline{(a,b)}\right) = \underline{(a,b)}$
- plus $\mu(q;p,\dots,p)\cdot h \cong \mu(p;q,\dots,q)$.

What does naturality mean?

$$\begin{tikzcd}
{[\mu(q;p,\dots,p), \underline{(a,b)}]} \arrow[r, equal] \arrow[d, "{(F,G)}"] & {[\mu(q;p,\dots,p)\cdot h, \underline{(a,b)}]} \arrow[d, "{(\cong,1)}"] \\
{[\mu(q';p',\dots,p'), \underline{(a',b')}]} \arrow[d, equal] & {[\mu(p;q,\dots,q), \underline{(a,b)}]} \arrow[d, "{(F,G)}"] \\
{[\mu(q';p',\dots,p')\cdot h', \underline{(a',b')}]} \arrow[r, "{(\cong,1)}"'] & {[\mu(p';q',\dots,q'), \underline{(a',b')}]}
\end{tikzcd}$$

$F=1$: $\cancel{\ }\ \mu(q;p,\dots,p)\cdot h \cong \mu(p;q,\dots,q)$
$\qquad\qquad\qquad\qquad\qquad \cong \mu(q;p,\dots,p)\cdot h'$

$$\mu(q;p,\dots,p)\cdot h h'^{-1} \cong \mu(q;p,\dots,p)$$

$$\begin{tikzcd}
P(mn)\times \mathrm{Ker}\,\pi \arrow[r, hook] \arrow[rr, bend right, "\mathrm{proj}_1"', "\cong"] & P(mn)\times G(mn) \arrow[r] & P(mn)
\end{tikzcd}$$

not quite true, only true on $\mu(q;p,\dots,p)$

$G=1$: Does it follow that

$$\begin{tikzcd} \mu(q;p,\dots,p)\circ h h'^{-1} \arrow[r, phantom, "\cong"] \arrow[d, "F\circ 1"] & \mu(q;p,\dots,p) \arrow[d, "F"] \\ \mu(q';p',\dots,p')\circ h h'^{-1} \arrow[r, phantom, "\cong"] & \mu(q';p',\dots,p') \end{tikzcd} \quad ?$$

Probably not: * what is bottom iso?
- h, h' depended on $\underline{a}, \underline{b}, p, q$
- now have $\underline{a}, \underline{b}, p', q'$ along the bottom

---

1^{st} strength axiom:
- $h((\underline{a},\underline{b}),\underline{c},p,q) = h(\underline{b},\underline{c},p,q)$
- iso for $(\underline{a},\underline{b}),\underline{c},p,q$ = iso for $\underline{b},\underline{c},p,q$

3^{rd}:
- $h(\underline{a},(\underline{b},\underline{c}),p,q) = h(\underline{a},\underline{b},p,q)$
- same for iso

2^{nd}:
- $h((\underline{a},\underline{b}),\underline{c},p,q) = h(\underline{a},(\underline{b},\underline{c}),p,q)$
- same for iso

1^{st} unit axiom:
- $h \in \ker(\pi)$ $\quad h(\underline{a},\underline{b},1,q)$
- ~~$\mu(q;p,\dots,p)$~~ $\mu(q;1,\dots,1)\circ h = \mu(1;q)$

$\iff q\circ h = q$

If $G(n)$ acts freely, then $h = \mathrm{id} \in G(n)$

2^{nd} unit axiom is redundant

$$\gamma_{A\times B,C}\cdot(t_{AB}\times TC) = t_{A,B\times C}\cdot(A\times\gamma_{BC}).$$

$$\begin{tikzcd}
A\times \underline{P}B\times\underline{P}C \arrow[r, "t_{AB}\times TC"] & \underline{P}(A\times B)\times\underline{P}C \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\gamma_{A\times B,C}"] & \underline{P}((A\times B)\times\underline{P}C) \arrow[r] & \underline{P}^2((A\times B)\times C) \arrow[d] \\
& \underline{P}(\underline{P}(A\times B)\times C) \arrow[r] & \underline{P}^2((A\times B)\times C) \arrow[r] & \underline{P}((A\times B)\times C)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
A\times\underline{P}B\times\underline{P}C \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow A\times\gamma_{BC}"] & A\times\underline{P}(B\times\underline{P}C) \arrow[r] & A\times\underline{P}^2(B\times C) \arrow[d] & \\
A\times\underline{P}(\underline{P}B\times C) \arrow[r] & A\times\underline{P}^2(B\times C) \arrow[r] & A\times\underline{P}(B\times C) \arrow[r, "t_{A,B\times C}"'] & \underline{P}(A\times(B\times C))
\end{tikzcd}$$

$$\begin{tikzcd}
a,[p,\underline{b}],[q,\underline{c}] \arrow[r, mapsto] & {[p,(a,\underline{b})],[q,\underline{c}]} \arrow[r, mapsto] \arrow[d, mapsto] & {[p,((a,\underline{b}),[q,\underline{c}])]} \arrow[r, mapsto] & {[p,[q,((a,\underline{b}),\underline{c})]]} \arrow[d, mapsto] \\
& {[q,([p,(a,\underline{b})],\underline{c})]} \arrow[r, mapsto] & {[q,[p,((a,\underline{b}),\underline{c})]]} \arrow[dr, mapsto, bend right] & {[p(q,\dots,q),\underline{((a,\underline{b}),\underline{c})}]} \\
& & & {[q(p,\dots,p),\underline{((a,\underline{b}),\underline{c})}]}
\end{tikzcd}$$

$$\begin{tikzcd}
a,\underset{1,n}{\underline{b}},\underset{1,m}{\underline{c}} \arrow[r, mapsto] & (a,b_1),\dots,(a,b_n),\underline{c} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{array}{c}((a,b_1),c_1),\dots,((a,b_1),c_m),\\ \dots((a,b_n),c_1),\dots,((a,b_n),c_m)\end{array} \arrow[r, mapsto] & '' \\
& \begin{array}{c}((a,b_1),c_1),\dots,((a,b_n),c_1),\\ \dots,((a,b_1),c_m),\dots,((a,b_n),c_m)\end{array} \arrow[r, mapsto] & ''
\end{tikzcd}$$

$$a, \underline{b}, \underline{c} \longmapsto a, (b_1, c_1), \ldots, (b_1, c_m), \ldots, (b_n, c_1), \ldots, (b_n, c_m), \longmapsto$$

$$\downarrow$$

$$a, (b_1, c_1), \ldots, (b_n, c_1), \ldots, (b_1, c_m), \ldots, (b_n, c_m) \longmapsto$$

$$(a, (b_1, c_1)), \ldots, (a, (b_1, c_m)), \ldots, (a, (b_n, c_1)), \ldots, (a, (b_n, c_m))$$

$$(a, (b_1, c_1)), \ldots, (a, (b_1, c_m)), \ldots, (a, (b_n, c_1)), \ldots, (a, (b_n, c_m))$$

$$\downarrow \cong$$

$$(a, (b_1, c_1)), \ldots, (a,.$$

$$\begin{tikzcd}
\underline{P}A \times B \times \underline{P}C \arrow[r, "\underline{P}A \times t_{BC}"] & \underline{P}A \times \underline{P}(B \times C) \arrow[r] \arrow[d] & \underline{P}(A \times \underline{P}(B \times C)) \arrow[r] \arrow[d, phantom, "\Downarrow \gamma"] & \underline{P}^2(A \times (B \times C)) \arrow[d] \\
& \underline{P}(\underline{P}A \times (B \times C)) \arrow[r] & \underline{P}^2(A \times (B \times C)) \arrow[r] & \underline{P}(A \times (B \times C))
\end{tikzcd}$$

$$\parallel$$

$$\underline{P}A \times B \times \underline{P}C \longrightarrow$$

$$a \longmapsto [1, a]$$

γ.

$$A \times \underline{P}B \longrightarrow \underline{P}A \times \underline{P}B \longrightarrow$$

$$a, [q, \underline{b}]$$

19/09.

$$[p, [v_1, \underline{a}_1], \ldots, [v_n, \underline{a}_n]], [q, \underline{b}] \longmapsto [p(v_1, \ldots, v_n), \underbrace{(\underline{a}_1, \ldots, \underline{a}_n)}_{\underline{a}}], [q, \underline{b}].$$

$$[p(v_1, \ldots, v_n)(q, \ldots, q), \underbrace{(\underline{a}_1, b_1), \ldots, (\underline{a}_1, b_m), \ldots, (\underline{a}_n, b_1), \ldots, (\underline{a}_n, b_m)}_{x}]$$

$$[q(p(v_1, \ldots, v_n), \ldots, p(v_1, \ldots, v_n)), \underbrace{(\underline{a}_1, b_1), \ldots, (\underline{a}_n, b_1), \ldots, (\underline{a}_1, b_m), \ldots, (\underline{a}_n, b_m)}_{y}].$$

~~μ~~ $\cdot h$ s.t. $\pi_{(h)}(x) = y$.

- ~~$\mu(p(v_1, \ldots, v_n)(q, \ldots, q)) \cdot h \cong \mu(q(p(v_1, \ldots, v_n), \ldots$~~
- $p(v_1, \ldots, v_n)(q, \ldots, q) \cdot h \equiv q(p(v_1, \ldots, v_n), \ldots, p(v_1, \ldots, v_n))$.
- Figure out what the pasting diagram does.
  - then these are equal.