THE SYMMETRIC BAR CONSTRUCTION

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Abstract. We show that the cyclic bar construction, a contravariant functor on the cyclic category, extends to a covariant functor on the symmetric category, which we call the symmetric bar construction. This enables us to define symmetric homology for algebras, which we compute in the case of group algebras. Similar constructions and computations are possible for the hyperoctahedral and braid categories. The computations in the braid case yield as a fringe benefit an interesting relation between the theory of quantum groups and the homotopy theory of iterated loop spaces.

§0 Introduction.

In [5] Loday and the author developed a notion of crossed simplicial group and their associated homology theories, providing a natural generalization of Connes' cyclic homology. (Krausaskas [8] independently discovered the same notion which he called a skew-simplicial group.)

Briefly a crossed simplicial group is a simplicial set G_*, such that each G_n is a group, but with the faces and degeneracies not required to be group homomorphisms but rather "crossed homomorphisms", satisfying formulas of the form

$$d_i(g_1g_2) = d_i(g_1)d_{g_1^{-1}i}(g_2) \qquad s_i(g_1g_2) = s_i(g_1)s_{g_1^{-1}i}(g_2).$$

Associated to each crossed simplicial group is a category ΔG from which G_* can be reconstructed. The objects of ΔG are the natural numbers $[n]$, $n \geq 0$, while the morphisms factor uniquely in the form

$$[m] \xrightarrow{g} [m] \xrightarrow{\gamma} [n],$$

where $g \in G_m$ and γ is a morphism in the simplicial category Δ. An action of G_* on a simplicial object X_* in a category $\mathcal{C}$ corresponds to a functor $\Delta G^{op} \to \mathcal{C}$, whose restriction to Δ^{op} gives the simplicial object X_*.

The canonical case is the cyclic crossed simplicial group C_* of Connes. The associated category ΔC is Connes' cyclic category Λ and functors $\Delta C^{op} \to \mathcal{C}$ are precisely cyclic objects in the category $\mathcal{C}$. Given an algebra A over a commutative ring k, there is a natural functor

$$\begin{aligned} B_*^{cyc}A &\longrightarrow k\text{-}Mod \\ [n] &\mapsto A^{\otimes_k n+1} \end{aligned}$$

called the cyclic bar construction and the cyclic homology of A may be defined as

$$HC_*(A) = Tor_*^{\Delta C}(k, B_*^{cyc} A).$$

In the special case of $A = k[\Gamma]$, Γ a discrete group, $HC_*(A)$ may be identified with the Borel S^1-equivariant homology of the natural S^1 action on the free loop space of $B\Gamma$.

Much of this generalizes in a nice and straightforward way to the setting of crossed simplicial groups. The geometric realization of a crossed simplicial group G_* is a topological group $|G_*|$. For a functor $M : \Delta G^{op} \to k\text{-}Mod$ we can define the G_* homology as

$$HG_*(M) = Tor_*^{\Delta G}(k, M).$$

If $M = k[X]$ is the k-linearization of a G_*-set X_*, then we may identify $HG_*(M)$ with the Borel $|G_*|$-equivariant homology of the associated $|G_*|$ action on $|X_*|$.

Moreover there is an explicit classification of crossed simplicial groups. While we refer to [5] for details, we will only mention the existence of three interesting examples (in addition to the cyclic case and the closely related dihedral case): There are crossed simplicial groups $\{S_{n+1}\}_{n\geq 0}$, $\{H_{n+1}\}_{n\geq 0}$, and $\{B_{n+1}\}_{n\geq 0}$, where S_{n+1}, $H_{n+1} = S_{n+1} \wr \mathbb{Z}/2$, B_{n+1} denote the symmetric, hyperoctahedral, and braid groups respectively.

At this point the analogy with cyclic homology breaks down. For the symmetric crossed simplicial group we have (cf. [5])

(0) The geometric realization of $\{S_{n+1}\}_{n\geq 0}$ is a contractible topological group.
(1) For algebras A over k, $B_*^{cyc} A \to k\text{-}Mod$ does not extend in general to a functor $B_*^{sym} A : \Delta S^{op} \to k\text{-}Mod$.
(2) For any $M : \Delta S^{op} \to k\text{-}Mod$, $HS_*(M) = H_*(\text{underlying simplicial module } M_*)$.

What (1) seems to be implying is that the cyclic homology of algebras cannot be extended to a notion of symmetric homology of algebras. Point (2) (which follows from (0)) seems to be implying that the symmetric homology of k-modules in general is not interesting. Similar remarks hold for the hyperoctahedral and braid crossed simplicial groups.

In this paper we will show that contrary to the pessimistic indications above, there is a natural way to define symmetric homology of algebras. The key point is that while, as noted in (1) above, there is no natural way to extend the cyclic bar construction of an algebra to a functor $B_*^{sym} A : \Delta S^{op} \to k\text{-}Mod$, there is a very natural way to extend it to a functor of the opposite variance

$$B_*^{sym} A : \Delta S \to k\text{-}Mod. \qquad (*)$$

To make sense of this, one must realize that the cyclic category ΔC, unlike the symmetric category ΔS, is self-dual (cf. [3]). We will refer to $(*)$ as the symmetric bar construction and we define the symmetric homology of A by

$$HS_*(A) = Tor_*^{\Delta S}(B_*^{sym} A, k).$$

By composing with the natural projection $\Delta B \to \Delta S$, the symmetric bar construction gives rise to the braided bar construction $B_*^{braid} A : \Delta B \to k\text{-}Mod$ and a corresponding notion of braid homology. If A is an algebra with involution, then the dihedral structure on the cyclic bar construction (cf. [9]) extends to a functor $B^{oct} : \Delta H \to k\text{-}Mod$, which we will refer to as the hyperoctahedral bar construction, and which defines hyperoctahedral homology of algebras with involution (which we denote $HO_*(A)$).

Moreover, we will compute the symmetric, braid and hyperoctahedral homologies of group algebras. Our main result is

Theorem 1. *(i) Let Γ be a discrete group. Then*

$$HS_*(k[\Gamma]) \cong H_*(\Omega\Omega^\infty S^\infty(B\Gamma); k).$$

(ii) *Let Γ be a discrete group. Then*

$$HB_*(k[\Gamma]) \cong H_*(\Omega^2 S(B\Gamma); k).$$

(iii) *Let Γ be a discrete group with involution (satisfying $\overline{g_1 g_2} = \overline{g_2}\,\overline{g_1}$ as well as $\overline{\overline{g}} = g$). Then*

$$HO_*(k[\Gamma]) \cong H_*(\Omega^\infty \big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega\Omega^\infty S^\infty(B\Gamma)\big); k),$$

where $E\mathbb{Z}/2_+$ denotes $E\mathbb{Z}/2$ with a disjoint basepoint, the $\mathbb{Z}/2$ action on $B\Gamma$ is induced by the homomorphism $g \mapsto \overline{g}^{-1}$, and $\Omega^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} X)$ denotes the infinite loop space obtained from spacewise equivariant smashing of the BbbZ/2 equivariant spectrum associated to X with $E\mathbb{Z}/2_+$.

This paper is divided into three sections corresponding to symmetric, hyperoctahedral and braid homologies respectively. The symmetric case is discussed in detail, while the other cases are sketched in so much as they differ from the symmetric case. In the process of computing the braid homology of group rings we will prove the following result, related to recent developments in quantum groups, which is of independent interest.

Theorem 2. *The group completion of the nerve of a braided tensor category is a double loop space.*

This was noticed by Stasheff in [12], but as far as we know no proof has yet appeared in the literature.

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§1 Symmetric Homology.

Definition 1.1 Let A be an algebra over a ground ring k. Let ΔS denote the symmetric category. Define a functor $B_*^{sym}A : \Delta S \to k\text{-}Mod$ as follows. Let $B_n^{sym}A = A^{\otimes n+1}$ with tensor factors labelled by indices in $[n] = \{0, 1, \ldots, n\}$. Since morphisms $[m] \to [n]$ in ΔS have a unique factorization of the form $\gamma \cdot g$ with $g \in S_{m+1}$ and $\gamma : [m] \to [n]$ a morphism in Δ, it suffices to define the functor $B_*^{sym}A$ on the morphisms g and γ. We define

$$g(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = a_{g^{-1}0} \otimes a_{g^{-1}1} \otimes \cdots \otimes a_{g^{-1}m}.$$

As $\gamma : [m] \to [n]$ is a nondecreasing function, we define

$$\gamma(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = (\prod_{a_i \in \gamma^{-1}(0)} a_i) \otimes (\prod_{a_i \in \gamma^{-1}(1)} a_i) \otimes \cdots \otimes (\prod_{a_i \in \gamma^{-1}(n)} a_i).$$

Here we use the following conventions: the product $(\prod_{a_i \in \gamma^{-1}(j)} a_i)$ is arranged according to increasing order of the indices and an empty product is understood to be the multiplicative unit $1 \in A$. We will refer to $B_*^{sym}A$ as the *symmetric bar construction.*

Lemma 1.1. *$B_*^{sym}A : \Delta S \to k\text{-}Mod$ is well defined.*

Proof. It is clear that the restriction of $B_*^{sym}A$ to Δ is a well-defined functor. Hence it suffices to check that $B_*^{sym}A$ preserves the fundamental commutation relations

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta_{g^{-1}i}} & [n] \\ {\scriptstyle d_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n-1] & \xrightarrow{\delta_i} & [n] \end{array} \qquad \begin{array}{ccc} [n+1] & \xrightarrow{\sigma_{g^{-1}i}} & [n] \\ {\scriptstyle s_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n+1] & \xrightarrow{\sigma_i} & [n] \end{array}$$

It is obvious that $B_*^{sym}A$ preserves the first commutation relation. To check the second one, one needs merely to note that $s_i(g)$ send $g^{-1}i$ to i and $g^{-1}i+1$ to $i+1$. Hence one obtains an i-th tensor factor of $a_i a_{i+1}$ going around both sides of the diagram. □

Our next observation is that the symmetric bar construction can be regarded as an extension of the cyclic bar construction.

Lemma 1.2. *There is a duality isomorphism $\Delta C^{op} \cong \Delta C$ such that the composite*

$$\Delta C^{op} \cong \Delta C \subset \Delta S \xrightarrow{B_*^{sym}A} k\text{-}Mod$$

is the cyclic bar construction $B_*^{cyc}A$.

Proof. By definition we have

$$\begin{aligned} B_*^{sym}A(\sigma_i) &= B_*^{cyc}A(\delta_i) : A^{\otimes n+1} \to A^{\otimes n} \quad 0 \le i \le n-1 \\ B_*^{sym}A(\delta_{i+1}) &= B_*^{cyc}A(\sigma_i) : A^{\otimes n} \to A^{\otimes n+1} \quad 0 \le i \le n-1 \\ B_*^{sym}A(\tau) &= B_*^{cyc}A(\tau) : A^{\otimes n+1} \to A^{\otimes n+1} \\ B_*^{sym}A(s_0 \cdot \tau) &= B_*^{cyc}A(\delta_n) : A^{\otimes n+1} \to A^{\otimes n} \end{aligned}$$

Now define $D : \Delta C^{op} \to \Delta C$ on generators by the formulas

$$\begin{aligned} D(\delta_i) &= \sigma_i \quad 0 \le i \le n-1 \\ D(\sigma_i) &= \delta_{i+1} \quad 0 \le i \le n-1 \\ D(\tau) &= \tau \\ D(\delta_n) &= s_0 \cdot \tau \end{aligned}$$

It is straightforward to check that this defines an isomorphism $\Delta C^{op} \cong \Delta C$, and by definition $B_*^{sym} A \cdot D = B_*^{cyc} A$. □

Definition 1.2. Define the symmetric homology of an algebra A by

$$HS_*(A) = Tor_*^{\Delta S}(B_*^{sym} A, k).$$

We now proceed to the proof of Theorem 1(i). It will be useful to generalize a little bit and consider monoid algebras instead of group algebras. We have the following result.

Proposition 1.3. *Let M be a monoid. Then*

$$HS_*(k[M]) \cong H_*(B(C_\infty, C_1, M); k).$$

Here $B(C_\infty, C_1, M)$ denotes May's functorial 2-sided bar construction, with C_1, C_∞ being the monads associated to the little 1-cubes, respectively little ∞-cubes operads.

We will prove Proposition 1 and Theorem 1(i) by recasting them in topological form. First of all note that $B_*^{sym} k[M]$ can be decomposed as a composite of functors

$$\Delta S \xrightarrow{B_*^{sym} M} Sets \xrightarrow{k[\ ]} k\text{-}Mod,$$

where $B_n^{sym} M = M^{n+1}$ with the evident values on morphisms. It is then a general fact of homological algebra that

$$Tor_*^{\Delta S}(B_*^{sym} k[M], k) = H_*(hocolim_{\Delta S} B_*^{sym} M; k).$$

Hence Theorem 1(i) and Proposition 1.3 follow from

Proposition 1.4. *(i) Let M be a topological monoid. Then there is a natural equivalence*

$$hocolim_{\Delta S} B_*^{sym} M \simeq B(C_\infty, C_1, M).$$

(ii) Let G be a topological group. Then there is a natural equivalence

$$hocolim_{\Delta S} B_*^{sym} G \simeq \Omega\Omega^\infty S^\infty(BG).$$

A special case of this result is of particular importance, the case when $M = JX$ is a free monoid on a pointed space X. In this case, from the fact $C_1 X \simeq JX$ and general properties of the functorial 2-sided bar construction, we have a chain of equivalences

$$B(C_\infty, C_1, JX) \simeq B(C_\infty, C_1, C_1 X) \simeq C_\infty X,$$

whence we obtain

Corollary 1.5. *For any pointed space X, there is a natural equivalence*

$$hocolim_{\Delta S} B_*^{sym} JX \simeq C_\infty X.$$

The strategy of our proof is to derive Proposition 1.4 from Corollary 1.5 (actually a sharpened version of it).

We begin with some notation.

Definition 1.3. Let $\mathcal{J}$ denote the subcategory of ΔS generated by all permutations and injections. Then if X is any pointed space, there is an evident functor $TX : \mathcal{J} \to Top$ such that $TX([n]) = X^{n+1}$, $TX(g) : X^{n+1} \to X^{n+1}$ is given by permuting coordinates, and $TX(\delta_i) : X^n \to X^{n+1}$ is the inclusion given by inserting the basepoint into the i-th coordinate of X^{n+1}.

The main ingredient in the proof of Corollary 1.5 is the following observation.

Lemma 1.6. *Let X be a pointed space. Then there is an isomorphism of functors*

$$B_*^{sym} JX \cong TX \otimes_{\mathcal{J}} \Delta S,$$

i.e. $B_*^{sym} JX$ is the free extension of $TX : \mathcal{J} \to Top$ to ΔS.

Proof. First notice that there is an evident natural inclusion of functors $TX \subset B_*^{sym} JX|_{\mathcal{J}}$. By universality this then induces a natural transformation of ΔS-functors

$$\lambda : TX \otimes_{\mathcal{J}} \Delta S \to B_*^{sym} JX.$$

We show that this is an isomorphism by indicating how to construct an inverse.

Consider a typical element

$$\overline{x} = (x_{01} \dots x_{0i_0}, x_{11} \dots x_{1i_1}, \dots, x_{n1} \dots x_{ni_n}) \in (JX)^{n+1} = B_n^{sym} JX.$$

Then there is a surjection $\rho : [i_0 + i_1 + \cdots + i_n - 1] \to [n]$ such that

$$B_*^{sym} JX(\rho)(x_{01}, \dots, x_{0i_0}, x_{11}, \dots, x_{n1}, \dots, x_{ni_n}) = \overline{x}.$$

Define the inverse map to take $\overline{x}$ to the element

$$[(x_{01}, \dots, x_{0i_0}, x_{11}, \dots, x_{n1}, \dots, x_{ni_n}), \rho] \in TX \otimes_{\mathcal{J}} \Delta S([n]).$$

It is straightforward to verify that this defines an inverse to λ. $\square$

Proof of Corollary 1.5. First of all, for technical reasons discussed below, it is easier to work with the E_∞ operad $\mathcal{D}$ (with $\mathcal{D}(n) = ES_n$) and its associated monad D. Since the monads associated to different E_∞ operads are equivalent, we might as well show that $hocolim_{\Delta S} B_*^{sym} JX \simeq DX$.

We have by general nonsense

$$hocolim_{\Delta S} B_*^{sym} JX \cong B_*^{sym} JX \otimes_{\Delta S} B(-\backslash \Delta S)$$

where $-\backslash\Delta S$ is the functor $\Delta S^{op} \to Cat$ which assigns to $[n]$ the under category $[n]\backslash\Delta S$. But by Lemma 3

$$\begin{aligned} B_*^{sym} JX \otimes_{\Delta S} B(-\backslash\Delta S) &\cong TX \otimes_{\mathcal{J}} \Delta S \otimes_{\Delta S} B(-\backslash\Delta S) \\ &\cong TX \otimes_{\mathcal{J}} B(-\backslash\Delta S). \end{aligned}$$

There is also a functor $\mathcal{J}^{op} \to Top$ with $[n] \mapsto \mathcal{D}(n+1)$. Since $S_{n+1} = Aut_{\mathcal{J}}([n])^{op}$ acts freely on both $\mathcal{D}(n+1)$ and $B([n]\backslash\Delta S)$, both spaces are contractible, and $B\Delta S$ is contractible, the projection maps define a chain of equivalences

$$TX \otimes_{\mathcal{J}} B(-\backslash\Delta S) \xleftarrow{\simeq} TX \otimes_{\mathcal{J}} (B(-\backslash\Delta S) \times \mathcal{D}(-)) \xrightarrow{\simeq} TX \otimes_{\mathcal{J}} \mathcal{D}(-) \simeq \mathcal{D}X.$$

This follows from a variant of the usual argument as given in May [11], using an induction argument involving the pushout lemma for homotopy equivalences. The difference here is that the induction has to start with the part of the tensor product lying over $T(*)$, with the chain of equivalences

$$B\Delta S = colim_{n \in \mathcal{J}^{op}} B([n]\backslash\Delta S) \xleftarrow{\cong} colim_{n \in \mathcal{J}^{op}} B([n]\backslash\Delta S) \xrightarrow{\simeq} colim_{n \in \mathcal{J}^{op}} \mathcal{D}(n+1) = *.$$

This argument would not work with the operad $\mathcal{C}_\infty$, since $colim_{n \in \mathcal{J}^{op}} \mathcal{C}_\infty(n+1)$ is a two point space with a non-Hausdorff topology. $\square$

Proof of Proposition 1.4. As in the proof of Corollary 1.5, we prefer to work with the operad $\mathcal{D}$ and associated monad D, instead of $\mathcal{C}_\infty$ and C_∞. We resolve M by a simplicial monoid which is degreewise free, use the fact that homotopy colimits and the symmetric bar construction commute with geometric realization and apply Corollary 1.5.

$$\begin{aligned} hocolim_{\Delta S} B_*^{sym} M &\simeq hocolim_{\Delta S} B^{sym} B(J, J, M) \\ &\cong B(hocolim_{\Delta S} B_*^{sym} J, J, M) \\ &\simeq B(D, J, M) \end{aligned}$$

The last equivalence is a bit problematic because we are implicitly hypothesizing that the equivalence of Corollary 1.5 is compatible with the simplicial structure maps of the functorial 2-sided bar construction. This is clear for the degeneracy maps as well as for all the face maps except for the 0-faces, since the equivalence of Corollary 1.5 is evidently natural in X.

The 0-faces of $B(hocolim_{\Delta S} B_*^{sym} J, J, M)$ arise from the monad structure of J, whereas the 0-faces of $B(D, J, M)$ arise from the monad structure of D thus: $DJ \to DD \to D$. It is not at all clear that the equivalence of Corollary 4 converts one to the other. Proving this requires careful rechecking of the proof of Corollary 4.

By iterating Lemma 1.6 we see that

$$B_*^{sym} JJX \cong TX \otimes_{\mathcal{J}} \Delta S \otimes_{\mathcal{J}} \Delta S,$$

and it is not difficult to see that the monad structure map $B_*^{sym} JJX \to B_*^{sym} JX$ corresponds to the evident quotient map

$$(*) \qquad TX \otimes_{\mathcal{J}} \Delta S \otimes_{\mathcal{J}} \Delta S \to TX \otimes_{\mathcal{J}} \Delta S \otimes_{\Delta S} \Delta S \cong TX \otimes_{\mathcal{J}} \Delta S$$

The operad $\mathcal{D}$ has the advantage that the functor $\mathcal{J}^{op} \to Top$, $[n] \mapsto \mathcal{D}(n+1)$ extends to a functor $\Delta S^{op} \to Top$, $[n] \mapsto \mathcal{D}(n+1) = ES_{n+1}$. Under the equivalence of Corollary 1.5, $(*)$ evidently corresponds to

$$TX \otimes_{\mathcal{J}} \Delta S \otimes_{\mathcal{J}} \mathcal{D}(-) \to TX \otimes_{\mathcal{J}} \Delta S \otimes_{\Delta S} \mathcal{D}(-) \cong TX \otimes_{\mathcal{J}} \mathcal{D}(-)$$

and it is straightforward to check that this is the same map as

$$DJX \to DDX \to DX.$$

This proves part (i).

Next we can easily calculate that $\pi_0(hocolim_{\Delta S} B_*^{sym} M)$ is just the abelianization of $\pi_0 M$. In particular for G a group $hocolim_{\Delta S} B_*^{sym} G \simeq B(C_\infty, C_1, G)$ is grouplike, so

$$B(C_\infty, C_1, G) \to B(\Omega^\infty S^\infty, C_1, G)$$

is an equivalence. Part (ii) now follows from Proposition 1.7 below. □

Proposition 1.7. *For any topological monoid M, there is a natural equivalence*

$$B(\Omega^\infty S^\infty, C_1, M) \simeq \Omega\Omega^\infty S^\infty(BM).$$

Proof. There is a natural map of simplicial spaces

$$B_*(S, C_1, M) \to B_*(\Omega^\infty S^\infty S, C_1, M)$$

which induces a map

$$\Omega^\infty S^\infty B_*(S, C_1, M) \to \Omega^\infty S^\infty B_*(\Omega^\infty S^\infty S, C_1, M) \to B_*(\Omega^\infty S^\infty S, C_1, M),$$

where the last map arises from the monad structure map $\Omega^\infty S^\infty \Omega^\infty S^\infty \to \Omega^\infty S^\infty$. Looping gives a natural map

$$\Omega\Omega^\infty S^\infty B_*(S, C_1, M) \to B_*(\Omega\Omega^\infty S^\infty S, C_1, M) = B_*(\Omega^\infty S^\infty, C_1, M).$$

Geometric realization gives rise to a natural chain of maps

$$\Omega\Omega^\infty S^\infty B(S, C_1, M) \xleftarrow{\simeq} |\Omega\Omega^\infty S^\infty B_*(S, C_1, M)| \to B(\Omega^\infty S^\infty, C_1, M).$$

We claim that the composite of this chain is an equivalence. Arguing as in the proof of part (i) of Theorem 1.4, we can reduce to checking this assertion for the case $M = JX$ is a free monoid. On the right hand side we have $B(\Omega^\infty S^\infty, C_1, JX) \simeq \Omega^\infty S^\infty(X)$ by the same argument used to derive Corollary 1.5 from Theorem 1.4. A similar argument shows that $B(S, C_1, JX) \simeq SX$, so we get $\Omega^\infty S^\infty(X)$ on the left-hand side too.

The last ingredient in the proof is the result of Thomason [13] and [4], that the May "delooping" of M, $B(S, C_1, M)$, is naturally equivalent to the bar construction BM. □

Remark 1.4 By Lemma 1.2 there is a natural map

$$HC_*(A) \longrightarrow HS_*(A).$$

For group rings $k[\Gamma]$ this is induced by a map
(#)

$$ES^1 \times_{S^1} (B\Gamma)^{S^1} \simeq hocolim_{\Delta C^{op}} B^{cyc}_*\Gamma \longrightarrow hocolim_{\Delta S} B^{sym}_*\Gamma \simeq \Omega\Omega^\infty S^\infty(B\Gamma).$$

This map has been essentially analyzed by Carlsson and Cohen [2]. They consider the functor $B(X) = ES^1{}_+ \wedge_{S^1} (SX)^{S^1}$ and show that it has Waldhausen stabilization $\Omega^\infty BS^\infty(X) \simeq \Omega^\infty S^\infty(X)$. It follows that the functor $U(X) = ES^1{}_+ \wedge_{S^1} X^{S^1}$ has Waldhausen stabilization $\Omega^\infty US^\infty(X) \simeq \Omega\Omega^\infty S^\infty(X)$. The map (#) may be identified with the composite

$$ES^1 \times_{S^1} (B\Gamma)^{S^1} \xrightarrow{q} U(B\Gamma) \longrightarrow \Omega^\infty US^\infty(X).$$

§2 Hyperoctahedral Homology.

In this section we will discuss the modifications required to formulate and prove the corresponding results for hyperoctahedral homology.

Definition 2.1. Let A be an algebra over a commutative ground ring k equipped with an involution $a \mapsto \overline{a}$ satisfying

(i) $\overline{a+b} = \overline{a} + \overline{b}$

(ii) $\overline{ab} = \overline{b}\,\overline{a}$

(iii) $\overline{\overline{a}} = a$

Let ΔH denote the hyperoctahedral category. Then extend the symmetric bar construction of the preceding section to a functor $B^{oct}A : \Delta H \to k\text{-}Mod$ by defining

$$(1; \varepsilon_0, \varepsilon_1, \dots, \varepsilon_m)(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = (a_0^{\varepsilon_0} \otimes a_1^{\varepsilon_1} \otimes \cdots \otimes a_m^{\varepsilon_m}),$$

where $(1; \varepsilon_0, \varepsilon_1, \dots, \varepsilon_m) \in (\mathbb{Z}/2)^{m+1} \subseteq S_{m+1} \int \mathbb{Z}/2 = H_{m+1}$ and

$$a^{\varepsilon} = \begin{cases} a & \text{if } \varepsilon = 1 \\ \overline{a} & \text{if } \varepsilon = -1 \end{cases}$$

We shall refer to $B^{oct}A$ as the *hyperoctahedral bar construction.*

Lemma 2.1. *$B_*^{oct}A : \Delta H \to k\text{-}Mod$ is well defined.*

Proof. The proof is the same as in the symmetric case, with one additional ingredient. When checking the fundamental commutation relation

$$\begin{array}{ccc} [n+1] & \xrightarrow{\sigma_{g^{-1}i}} & [n] \\ {\scriptstyle s_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n+1] & \xrightarrow{\sigma_i} & [n] \end{array}$$

for g of the form $(1; \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$, one notes that

$$s_i(1; \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) = \begin{cases} (1; \varepsilon_0, \varepsilon_1, \dots, \varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n) & \text{if } \varepsilon_i = 1 \\ ((i\ i+1); \varepsilon_0, \varepsilon_1, \dots, \varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n) & \text{if } \varepsilon_i = -1 \end{cases}$$

If we apply the functor $B_*^{oct}A$ to the diagram and then chase an element $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$ around the resulting diagram along the top and right, we get an i-th tensor factor of $(a_i a_{i+1})^{\varepsilon_i}$. If we chase the element around the left and bottom of the diagram, we get either $a_i^{\varepsilon_i} a_{i+1}^{\varepsilon_i}$ or $a_{i+1}^{\varepsilon_i} a_i^{\varepsilon_i}$ depending on whether $\varepsilon_i = 1$ or $\varepsilon_i = -1$, which from the multiplicative property of the involution is the same thing. □

In [9] (cf. also [5]) Loday shows that for an algebra A with involution, the cyclic bar construction extends to a functor $B_*^{dih}A : \Delta D^{op} \to k\text{-}Mod$, where ΔD is the dihedral category corresponding to the dihedral crossed simplicial group. The proof of the following result is completely analogous to that of Lemma 1.2 and will be omitted.

Lemma 2.2. *There is a duality isomorphism $\Delta D^{op} \cong \Delta D$ such that the composite*

$$\Delta D^{op} \cong \Delta D \subset \Delta S \xrightarrow{B_*^{sym} A} k\text{-}Mod$$

is the dihedral bar construction $B_*^{dih} A$.

Definition 2.2. Define the hyperoctahedral homology of an algebra A with involution by

$$HO_*(A) = Tor_*^{\Delta H}(B_*^{oct} A, k).$$

We now proceed to outline the proof of Theorem 1(iii) in as much as it differs from the symmetric case. Again we begin by generalizing from group algebras to monoid algebras.

Proposition 2.3. *Let M be a monoid with involution. Then*

$$HO_*(k[M]) \cong H_*(B(C_\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} -), C_1, M); k).$$

Here we are implicitly using the obvious extension of the monad C_1 to the category of spaces with involution (ie., $\mathbb{Z}/2$ actions).

As in the symmetric case, we reformulate Proposition 2.3 in topological form:

Proposition 2.4. *(i) Let M be a topological monoid with involution. Then there is a natural equivalence*

$$hocolim_{\Delta H} B_*^{oct} M \simeq B(C_\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} -), C_1, M).$$

(ii) Let G be a topological group with involution. Then there is a natural equivalence

$$hocolim_{\Delta H} B_*^{oct} G \simeq \Omega^\infty\big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega\Omega^\infty S^\infty(BG)\big).$$

Again in the special case when $M = JX$ is a free monoid on a pointed space X with an involution (with the involution on JX being the evident extension of the one on X), this reduces to:

Corollary 2.5. *For any pointed space X with involution, there is a natural equivalence*

$$hocolim_{\Delta H} B_*^{oct} JX \simeq C_\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} X).$$

As in the symmetric case we derive Proposition 2.4 from a sharpened form of Corollary 2.5.

We begin with the hyperoctahedral analog of Definition 1.3

Definition 2.3. Let $\mathcal{J}_H$ denote the subcategory of ΔH generated by all automorphisms and injections. Then if X is any pointed space with involution, there is an evident functor $T_H X : \mathcal{J}_H \to Top$ such that $T_H X([n]) = X^{n+1}$, $T_H X(g) : X^{n+1} \to X^{n+1}$ is given by permuting coordinates and applying the involution to the appropriate coordinates, and $T_H X(\delta_i) : X^n \to X^{n+1}$ is the inclusion given by inserting the basepoint into the i-th coordinate of X^{n+1}.

The main ingredient in the proof of Corollary 2.5 is the following observation.

Lemma 2.6. *Let X be a pointed space with involution. Then there is an isomorphism of functors*
$$B_*^{oct}JX \cong T_H X \otimes_{\mathcal{J}_H} \Delta H,$$
i.e. $B_*^{oct}JX$ is the free extension of $T_H X : \mathcal{J}_H \to Top$ to ΔH.

The proof is identical to that of Lemma 1.6 and so will be omitted.

The proof of Corollary 2.5 parallels closely that of Corollary 1.5. Of course one is obliged to use an operad $\mathcal{H}$ with $\mathcal{H}(n) = EH_n$ instead of $\mathcal{D}$. The only additional argument required is to identify

$$T_H X \otimes_{\mathcal{J}_H} \mathcal{H}(-) \simeq H(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} X),$$

where H here denotes the monad associated with $\mathcal{H}$. If $X = \mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} Y$, this reduces to the obvious equivalence

$$T_H Y \otimes_{\mathcal{J}} \mathcal{H}(-) \simeq H(E\mathbb{Z}/2_+ \wedge Y),$$

and the general case can be reduced to this case by taking a simplicial resolution of X by spaces of the form $\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} Y$.

The proof of Proposition 2.4(i) follows closely that of Proposition 1.4 with the evident modifications. (Eg. $\pi_0(hocolim_{\Delta H} B_*^{oct} M)$ is the quotient monoid by the involution of the abelianization of $\pi_0(M)$.)

The proof of Proposition 2.4(ii) then utilizes

Proposition 2.7. *For any topological monoid M with involution, there is a natural equivalence*

$$B(\Omega^\infty S^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} -), C_1, M) \simeq \Omega^\infty\big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega\Omega^\infty S^\infty(BM)\big),$$

where the involution on BM is the one given by

$$(\#) \qquad [m, (t_0, t_1, \ldots, t_n)] \mapsto [\overline{m}, (t_n, \ldots, t_1, t_0)].$$

Proof Sketch. Evidently we can make the identification

$$B(\Omega^\infty S^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} -), C_1, M) \simeq \Omega^\infty\big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} B(\Omega^\infty S^\infty, , C_1, M)\big).$$

The proof now follows from the fact that the equivalence

$$B(\Omega^\infty S^\infty, C_1, M) \simeq \Omega\Omega^\infty S^\infty(BM)$$

constructed in Proposition 1.7 is a $\mathbb{Z}/2$ equivariant map. This fact follows from noting that the equivalence of [4] between the May delooping $B(S, C_1, M)$ and BM sends the natural involution on the former into the involution (#). □

The final ingredient in the proof of Proposition 2.4(ii) is that for a topological group G with involution, the involution (#) on BG is equivariantly homotopic to the involution induced by the homomorphism $g \mapsto \overline{g}^{\,-1}$. For a proof see [1].

§3 Braid Homology.

This section is devoted to the analogous results for the braid crossed simplicial group. The proof are entirely analogous to those for the symmetric case, except that certain constructions needed to formulate the proofs are not available in the existing literature and have to be defined. These constructions have some unexpected applications in another area of mathematics, namely the theory of quantum groups, giving a very easy proof of Theorem 2.

Definition 3.1. Let A be an algebra over a ground ring k. Let ΔB denote the braid crossed simplicial group. We define the braided bar construction $B^{braid}A : \Delta B \to k\text{-}Mod$ given by the composite

$$\Delta B \to \Delta S \xrightarrow{B^{sym}A} k\text{-}Mod.$$

We define the braid homology of an algebra as

$$HB_*(A) = Tor_*^{\Delta S}(B^{braid}A, k).$$

The analogues of Proposition 1.3, 1.4 and Corollary 1.5 are respectively

Proposition 3.1. *Let M be a monoid. Then*

$$HB_*(k[M]) \cong H_*(B(C_2, C_1, M); k).$$

Here C_2 denotes the monad associated to the little 2-cubes operad.

Proposition 3.2. *(i) Let M be a topological monoid. Then there is a natural equivalence*

$$hocolim_{\Delta B} B_*^{braid} M \simeq B(C_2, C_1, M).$$

(ii) Let G be a topological group. Then there is a natural equivalence

$$hocolim_{\Delta B} B_*^{braid} G \simeq \Omega^2 S(BG).$$

Corollary 3.3. *For any pointed space X, there is a natural equivalence*

$$hocolim_{\Delta B} B_*^{braid} JX \simeq C_2 X.$$

The proofs are entirely analogous to those in the symmetric case once we dispose of one technicality. Similarly to the little ∞-cubes operad in the symmetric case, the little 2-cubes operad has certain technical drawbacks. Thus we need to replace it by a braid analog of the E_∞ operad $\mathcal{D}$, and show that the resulting monad is equivalent to C_2. For E_∞ operads there is a simple trick to see that monads associated to different operads are equivalent: one compares both monads to the monad associated to the product of the two operads. The crucial point here is that the product of two E_∞ operads is an E_∞ operad, which depends on the fact that E_∞ operads are spacewise contractible. This is not the case for the little 2-cubes operad $\mathcal{C}_2$. Instead $\mathcal{C}_2(n) \simeq K(T_n, 1)$, where T_n is the pure braid group on n strings.

Fortunately there is a simple remedy: one notices that the universal covers $\widetilde{\mathcal{C}}_2(n)$ are contractible spaces. To make use of this observation we have to formulate a new notion of operad, where the braid groups play the role of the symmetric groups in the usual definition.

Definition 3.2. A braided operad $\mathcal{C}$ consists of spaces $\mathcal{C}(j)$ for $j \geq 0$ with $\mathcal{C}(0)$ a single point $*$, together with the following data:

(a) structure maps $\gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \to \mathcal{C}(j_1 + \cdots + j_k)$
(b) an identity element $1 \in \mathcal{C}(1)$
(c) right actions of the braid groups $\mathcal{C}(j) \times B_j \to \mathcal{C}(j)$.

These data are required to satisfy the usual associativity and identity conditions and the following equivariance conditions

$$\gamma(c\sigma; d_1, \ldots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)}\sigma(j_1, \ldots, j_k)$$
$$\gamma(c; d_1\tau_1, \ldots, d_k\tau_k) = \gamma(c; d_1, \ldots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k)$$

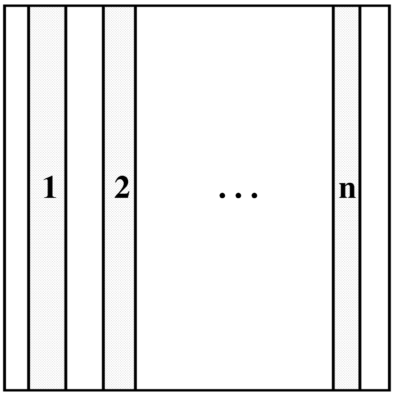
where $\sigma \in B_k$ acts on $\{1, \ldots, k\}$ via the natural projection $B_k \to S_k$, $\sigma(j_1, \ldots, j_k)$ is the braid obtained from σ by replacing the first strand of σ by j_1 parallel strands, $\ldots$, the k-th strand by j_k parallel strands, and $\tau_1 \oplus \cdots \oplus \tau_k$ denotes the block direct sum of the braids $\tau_1, \ldots, \tau_k$.

It is evident what is meant by a morphism of braided operads. Moreover an ordinary operad (with symmetric group actions), such as the endomorphism operad of a space, may be regarded as a braided operad via the natural projection $B_k \to S_k$. Thus we can define an action of a braided operad on a topological space as a morphism of braided operads into the endomorphism operad. We will be particularly interested in braided operads $\mathcal{C}$ such that each $\mathcal{C}(j)$ is contractible and all the braid group actions $\mathcal{C}(j) \times B_j \to \mathcal{C}(j)$ are free. We shall call such operads *B_∞ operads.*

Our first example of a B_∞ operad is constructed from the little 2-cubes operad $\mathcal{C}_2$.

Example 3.1. Let $\widetilde{\mathcal{C}}_2(n)$ denote the universal cover of $\mathcal{C}_2(n)$. We will define an operad structure map and braid group actions on $\widetilde{\mathcal{C}}_2$ by lifting them from the structure map, resp. symmetric group actions on $\mathcal{C}_2$. This might seem to present a difficulty— one might think that this would require choosing basepoints in the $\mathcal{C}_2(n)$'s, whereas there is no possible consistent choice. However one notices that the contractible A_∞ operad $\mathcal{C}_1$ (non-Σ version) can be naturally imbedded in $\mathcal{C}_2$ and this will serve in lieu of choosing basepoints.

Let $p : \widetilde{\mathcal{C}}_2(n) \to \mathcal{C}_2(n)$ denote the universal covering. Let $\mathcal{C}_1 \subset \mathcal{C}_2$ denote the "horizontal" imbedding of $\mathcal{C}_1$ as little cubes of the form



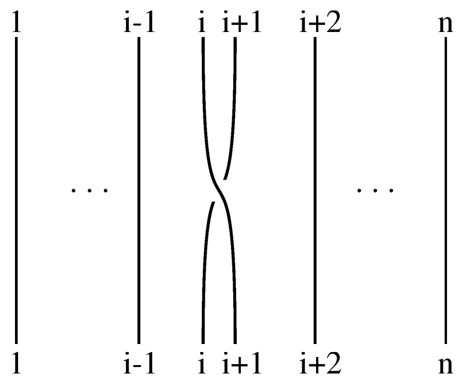
ordered from left to right as shown. Thus $p^{-1}(\mathcal{C}_1(n)) \subset \widetilde{\mathcal{C}}_2(n)$ is a disjoint union of components, each homeomorphic to $\mathcal{C}_1(n)$ via p. For each n, arbitrarily choose one of these components and denote it $\widetilde{\mathcal{C}}_1(n)$.

Now define the structure map $\widetilde{\gamma}$ for $\widetilde{\mathcal{C}}_2$ to be the unique lifting

$$\begin{array}{ccc}
\widetilde{\mathcal{C}}_2(k) \times \widetilde{\mathcal{C}}_2(j_1) \times \cdots \times \widetilde{\mathcal{C}}_2(j_k) & \dashrightarrow & \widetilde{\mathcal{C}}_2(j_1 + \cdots + j_k) \\
\big\downarrow p & & \big\downarrow p \\
\mathcal{C}_2(k) \times \mathcal{C}_2(j_1) \times \cdots \times \mathcal{C}_2(j_k) & \xrightarrow{\gamma} & \mathcal{C}_2(j_1 + \cdots + j_k)
\end{array}$$

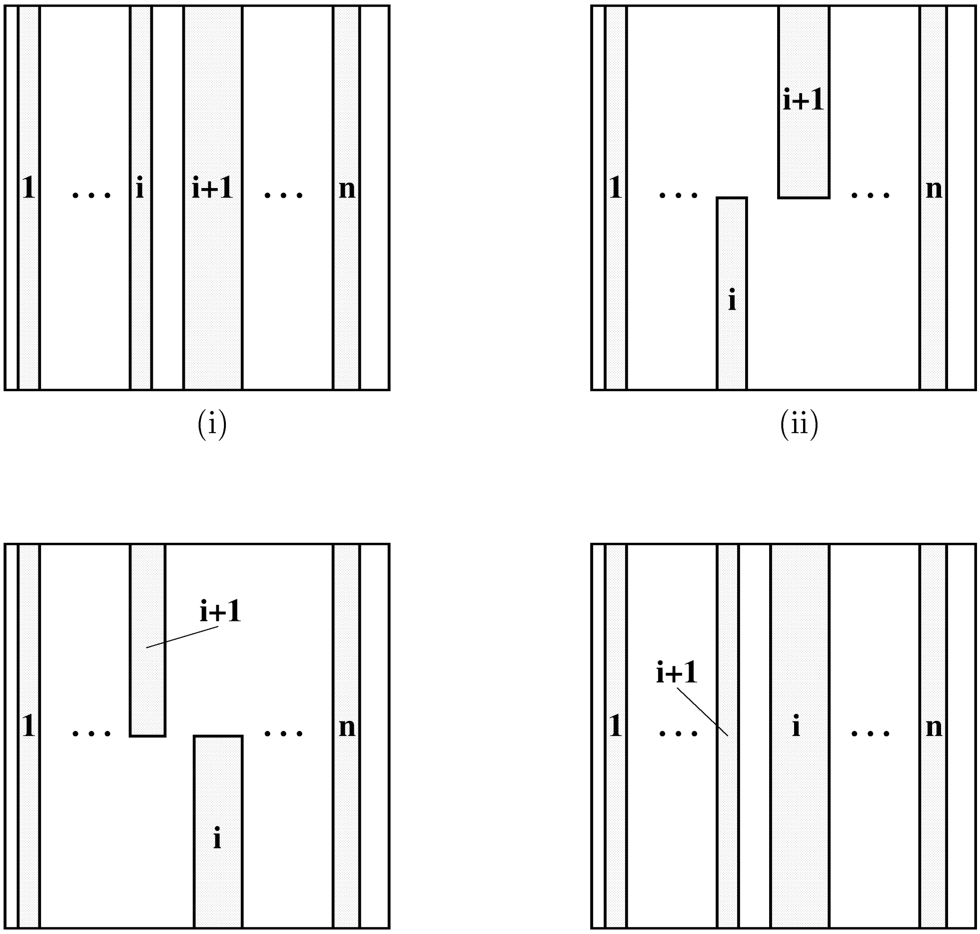
which takes $\widetilde{\mathcal{C}}_1(k) \times \widetilde{\mathcal{C}}_1(j_1) \times \cdots \times \widetilde{\mathcal{C}}_1(j_k)$ to $\widetilde{\mathcal{C}}_1(j_1 + \cdots + j_k)$. Also define the identity element of $\widetilde{\mathcal{C}}_2(1)$ to be the unique element $1 \in \widetilde{\mathcal{C}}_1(1)$ such that $p(1) = 1 \in \mathcal{C}_1(1) \subset \mathcal{C}_2(1)$.

Finally we have to define the braid group actions $\widetilde{\mathcal{C}}_2(n) \times B_n \to \widetilde{\mathcal{C}}_2(n)$. It suffices to specify actions by the standard generators $\widetilde{\sigma}_i$, $1 \le i \le n-1$, where $\widetilde{\sigma}_i$ denotes the braid



We will specify the action by this braid to be the lift of the action of $\sigma_i = (i\; i{+}1) \in S_n$ on $\mathcal{C}_2(n)$ subject to the following basepoint condition. Choose an arbitrary point

Let $\alpha : I \to \mathcal{C}_2(n)$ denote the path from x to $x\sigma_i$ specified by the following sequence of pictures



(i) (ii) (iii) (iv)

$x \in \mathcal{C}_1(n)$.
Let $\widetilde{\alpha}$ denote the lift of α which starts in $\widetilde{\mathcal{C}}_1$. Let $\widetilde{x} = \widetilde{\alpha}(0)$ and $\widetilde{y} = \widetilde{\alpha}(1)$, then specify that $\widetilde{x}\widetilde{\sigma} = \widetilde{y}$. It is straightforward to verify that this data specifies a B_∞ operad structure on $\widetilde{\mathcal{C}}_2$.

Our next example is the B_∞ analog of the E_∞ operad $\mathcal{D}$. Just as $\mathcal{D}$ naturally acts on classifying spaces of symmetric monoidal categories, the B_∞ operad $\mathcal{B}$ defined below naturally acts on classifying spaces of braided tensor categories. Moreover there is an analogous relation between $\mathcal{B}$ and ΔB, as there is between $\mathcal{D}$ and ΔS.

Example 3.2. Let $\mathcal{B}(n) = EB_n$. It will be convenient to regard $\mathcal{B}(n)$ as the translation category of the braid group. Thus the objects of the category are braids $\tau \in B_n$, with a unique morphism between any two objects. Thus the right action of B_n on itself specifies a right action of B_n on the translation category and hence also on the classifying space $\mathcal{B}(n)$. Clearly $\mathcal{B}(1)$ is a point, so it unquestionably has to be the identity element. The structure map

$$\gamma : \mathcal{B}(k) \times \mathcal{B}(j_1) \times \cdots \times \mathcal{B}(j_k) \to \mathcal{B}(j_1 + \cdots + j_k)$$

can be specified on the level of translation categories and hence on the level of braid groups. Given $\sigma \in B_k$, $\tau_1 \in B_{j_1}, \ldots, \tau_k \in B_{j_k}$, we define

$$\gamma(\sigma; \tau_1, \ldots, \tau_k) = \sigma(j_1, \ldots, j_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

in the notation of Definition 2. It is straightforward to verify that this specifies a structure of a B_∞ operad on $\mathcal{B}$.

We next discuss monads associated to B_∞ operads, and show that just as in the case of A_∞ and E_∞ operads, all B_∞ operads are essentially equivalent to each other.

Definition 3.3 Let $\mathcal{C}$ be a braided operad. Then the monad C associated to $\mathcal{C}$ is constructed as follows. Given a based space X denote

$$CX = \coprod_{n \geq 0} \mathcal{C}(n) \times_{B_n} X^n / \approx,$$

where $\approx$ is the same basepoint relation as in the case of ordinary operads, and as usual B_n acts on X^n via the projection $B_n \to S_n$. There is an obvious inclusion $X \to CX$, and the structure map of $\mathcal{C}$ defines a natural map $CCX \to CX$, which makes C into a monad. If $\mathcal{C}$ acts on X, the action maps define an evident "evaluation map" $CX \to X$. It is straightforward to check that CX satisfies the universal property of being free with respect to actions of $\mathcal{C}$.

If $\mathcal{C} \to \mathcal{V}$ is a map of B_∞ operads, the same argument as for E_∞ operads shows that the induced map of monads $CX \to VX$ is an equivalence. Thus for any two B_∞ operads $\mathcal{C}$ and $\mathcal{U}$, there is a natural chain of equivalences defined by projection maps $CX \xleftarrow{\simeq} (C \times U)X \xrightarrow{\simeq} UX$, where $(C \times U)X$ denotes the monad associated to the B_∞ operad $\mathcal{C} \times \mathcal{U}$. Thus given a $\mathcal{C}$-space X, the functorial 2-sided bar construction $B(U, C \times U, X)$ produces an equivalent $\mathcal{U}$-space. Thus all B_∞ operads specify essentially the same kind of structure on spaces they act on.

Since by construction the monad $\widetilde{C}_2$ associated to the B_∞ operad $\widetilde{\mathcal{C}}_2$ is *identical* to the monad C_2 associated to the little cubes operad $\mathcal{C}_2$, and because of the approximation theorem for 2-fold loop spaces, it follows that the group completion of CX is equivalent to $\Omega^2 S^2(X)$ for any space X and any B_∞ operad $\mathcal{C}$. It follows that

Proposition 3.4. *If a B_∞ operad acts on a space X, then its group completion is a double loop space.*

Definition 3.4 A braided tensor category (also known as braided monoidal category) is a category $\mathcal{C}$ together with a functor $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is strictly associative, has a strict 2-sided unit object E and with a natural commutativity isomorphism $c_{A,B} : A \square B \xrightarrow{\cong} B \square A$ satisfying the following properties:

(1) Unit Condition: $c_{A,E} = c_{E,A} = id_A$.

(2) Associativity Conditions: For any three objects A, B, C the following diagrams commute:

$$\begin{tikzcd}
A\Box B\Box C \arrow[rr, "c_{A\Box B,C}"] \arrow[dr, "id_A\Box c_{B,C}"'] & & C\Box A\Box B & A\Box B\Box C \arrow[rr, "c_{A,B\Box C}"] \arrow[dr, "c_{A,B}\Box id_C"'] & & B\Box C\Box A \\
& A\Box C\Box B \arrow[ur, "c_{A,C}\Box id_A"'] & & & B\Box A\Box C \arrow[ur, "id_B\Box c_{A,C}"'] &
\end{tikzcd}$$

Proof of Theorem 2. The coherence results of Joyal and Street [6] and [7] imply that the B_∞ operad $\mathcal{B}$ acts on the nerve of a braided tensor category. The derivation of this from the coherence results is entirely analogous to the proof that the E_∞ operad $\mathcal{D}$ acts on the nerve of a permutative category in [10]. Now apply Proposition 3.4. $\Box$

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