★ EΛ-algs vs Λ-mon cat

CHAPTER 2

Invertible objects

The goal of this chapter will be to show that we can reconstruct all of the morphisms of L_n from the abelian group $\mathrm{M}(L_n)^{\mathrm{gp,ab}}$, and therefore that we can actually use the adjunction from Proposition 2.2.20 to help find a description of the free $E\Lambda$-algebra on n invertible objects.

The first step towards this goal will involve splitting $\mathrm{Mor}(L_n)$ up as the product of two other monoids. The first of these will encode all of the possible combinations of source and target data for morphisms in L_n, while the second will just be the endomorphisms of the unit object, $L_n(I, I)$. Once we have done this, we can then use the fact that $L_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L_n)$ in terms of its abelian group completion, $\mathrm{Mor}(L_n)^{\mathrm{gp,ab}}$. This is not quite the same thing as $\mathrm{M}(L_n)^{\mathrm{gp,ab}}$, but they are close enough that we can find a simple equation linking the two, which will in turn allow us to frame the former in terms of the quotient of $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$ we described last chapter. All together, this will constitute an expression for $\mathrm{Mor}(L_n)$ that is built up of pieces which we know how to calculate.

Current understanding is four steps:

(1) existence of L_n via adjoint functor theorem (2.1)
(2) L_n as a quotient of $E\Lambda(\underline{2n})$ (2.2–2.3)
(3) we split $L_n(I, I)$ (2.4–2.6)
(4) then we make a nasty quotient (2.7 until examples)

2.1. Introduction to invertibility

Our main focus will be on invertible objects, although not in the usual sense.

Definition 2.1.1. Let $(M, \otimes, I)$ be a monoidal category. An object $m \in M$ is *invertible* if there exists another object m^{-1} such that $m \otimes m^{-1} = I$ and $m^{-1} \otimes m = I$.

Most authors give the following as the definition of an invertible object, what we call weakly invertible.

Definition 2.1.2. Let $(M, \otimes, I)$ be a monoidal category. An object $m \in M$ is *weakly invertible* if there exists another object m^* such that $m \otimes m^* \cong I$ and $m^* \otimes m \cong I$.

We will later derive results about weakly invertible objects from those about invertible ones using standard 2-monadic techniques.

Definition 2.1.3. Given an $E\Lambda$-algebra X, we will denote by X_{inv} the sub-$E\Lambda$-algebra of X containing all objects which are invertible under tensor product, and all of the isomorphisms between them.

First note that this is indeed a well-defined $E\Lambda$-algebra since the tensor product of invertible objects is again invertible, the tensor product of isomorphisms

is again an isomorphism, and all of the morphisms giving the group actions are isomorphisms. The following proposition is then obvious.

Proposition 2.1.4. *The assignment $X \mapsto X_{\mathrm{inv}}$ can be extended to a 2-functor $(\text{-})_{\mathrm{inv}} : E\Lambda\text{-}\mathbf{Alg}_S \to E\Lambda\text{-}\mathbf{Alg}_S$.*

Proposition 2.1.5. *The 2-functor $(\text{-})_{\mathrm{inv}} : E\Lambda\text{-}\mathbf{Alg}_S \to E\Lambda\text{-}\mathbf{Alg}_S$ has a left adjoint. $L : E\Lambda\text{-}\mathbf{Alg}_S \to E\Lambda\text{-}\mathbf{Alg}_S$.*

Need 2-adj

PROOF. Since we already know that $E\Lambda\text{-}\mathbf{Alg}_S$ is locally finitely presentable, the conditions for ?? amount to showing that $(\text{-})_{\mathrm{inv}}$ preserves both limits and filtered colimits.

- Given an indexed collection of $E\Lambda$-algebras X_i, the $E\Lambda$-action of their product $\prod X_i$ is defined componentwise. In particular, this means that the tensor product of two objects in $\prod X_i$ is just the collection of the tensor products of their components in each of the X_i. An invertible object in $\prod X_i$ is thus simply a family of invertible objects from the X_i, so $(\prod X_i)_{\mathrm{inv}} = \prod (X_i)_{\mathrm{inv}}$.
- Given maps of $E\Lambda$-algebras $F : X \to Z$, $G : Y \to Z$. the $E\Lambda$-action of their pullback $X \times_Z Y$ is also defined component-wise. It follows that an invertible object in $X \times_Z Y$ is just a pair of invertible objects (x, y) from X and Y. such that $F(x) = G(y)$. But this is the same as asking for a pair of objects (x, y) from X_{inv} and Y_{inv} such that $F_{\mathrm{inv}}(x) = G_{\mathrm{inv}}(y)$, and hence $(X \times_Z Y)_{\mathrm{inv}} = X_{\mathrm{inv}} \times_{Z_{\mathrm{inv}}} Y_{\mathrm{inv}}$.
- Given a filtered diagram D of $E\Lambda$-algebras, the $E\Lambda$-action of its colimit $\mathrm{colim}(D)$ is defined in the following way: use filteredness to find an algebra which contains (representatives of the classes of) all the things you want to act on, then apply the action of that algebra. In the case of tensor products this means that $[x] \otimes [y] = [x \otimes y]$. and thus an invertible object in $\mathrm{colim}(D)$ is just (the class of) an invertible object in one of the algebras of D. In other words, $\mathrm{colim}(D)_{\mathrm{inv}} = \mathrm{colim}(D_{\mathrm{inv}})$.

Change to equalizers

Preservation of products and pullbacks give preservation of limits, and preservation of limits and filtered colimits together give the result. □

Definition 2.1.6. Let $L_n = L\Big(E\Lambda(\underline{n})\Big)$, where $\underline{n} = \{x_1, \ldots, x_n\}$ is a set with n elements.

Notation unit
$X \to (LX)_{\mathrm{inv}} = LX$

THEOREM 2.1.7. *The algebra L_n is the free $\mathbf{\Lambda}$-monoidal category on n invertible objects: for any other $E\Lambda$-algebra X, we have an isomorphism of categories*

$$E\Lambda\text{-}\mathbf{Alg}_S(L_n, X) \quad \cong \quad (X_{\mathrm{inv}})^n,$$

2-natural in X.

PROOF. We have the following 2-natural isomorphisms from the adjunctions for L and the free $\mathbf{\Lambda}$-monoidal category monad.

$$\begin{aligned} E\Lambda\text{-}\mathbf{Alg}_S\Big(L\Big(E\Lambda(\underline{n})\Big), X\Big) &\cong E\Lambda\text{-}\mathbf{Alg}_S(E\Lambda(\underline{n}), X_{inv}) \\ &\cong \mathbf{Cat}(\underline{n}, X_{inv}) \\ &\cong X^n_{inv} \end{aligned}$$

□

Proposition 2.1.8. *For any $E\Lambda$-algebra X, we have $\cancel{L(X)_{\mathrm{inv}}} = L(X)$.*

$(LX)_{\mathrm{inv}} = LX$

PROOF. Let $j : X_{inv} \to X$ denote the obvious inclusion functor. By construction, we have an equality of functors $(\text{-})_{\mathrm{inv}} \circ (\text{-})_{\mathrm{inv}} = (\text{-})_{\mathrm{inv}}$ and moreover $j_{inv} : (X_{inv})_{inv} \to X_{inv}$ is the identity for every X. Since $E\Lambda\text{-}\mathbf{Alg}_S(LX, Y) \cong E\Lambda\text{-}\mathbf{Alg}_S(X, Y_{\mathrm{inv}})$ is natural in Y, we have a commutative square

$$\begin{tikzcd} E\Lambda\text{-}\mathrm{Alg}_S(LX, LX_{\mathrm{inv}}) \arrow[r, "\cong"] \arrow[d, "j \circ -"'] & E\Lambda\text{-}\mathrm{Alg}_S(X, (LX_{\mathrm{inv}})_{\mathrm{inv}}) \arrow[d, "\mathrm{id}"] \\ E\Lambda\text{-}\mathrm{Alg}_S(LX, LX) \arrow[r, "\cong"'] & E\Lambda\text{-}\mathrm{Alg}_S(X, LX_{\mathrm{inv}}) \end{tikzcd}$$

Fix ( )'s

in which the identity on the right is really $j_{inv} \circ -$. Thus composition with j is an isomorphism in this case, so the identity map on LX factors as $j \circ g$ for some $g : LX \to (LX)_{inv}$. Since j is an inclusion, this factorization forces it to be the identity.

$$\begin{tikzcd} LX \arrow[rr, "1"] \arrow[dr, "g"'] & & LX \\ & (LX)_{\mathrm{inv}} \arrow[ur, "j"'] & \end{tikzcd}$$

□

Explain!

Corollary 2.1.9. *The component of the unit of the adjunction $L \dashv (-)_{inv}$ at $E\Lambda(\underline{n})$, $\eta_{E\Lambda(\underline{n})} : E\Lambda(\underline{n}) \to (L_n)_{inv}$, is an epimorphism in $E\Lambda\text{-}\mathrm{Alg}_S$.*

PROOF. By Proposition 2.1.8, we first have that $(L_n)_{inv} = L_n$. Let $F, G : L_n \to X$ be a pair of algebra maps for which $F\eta = G\eta$. Thus the algebra maps $E\Lambda(\underline{n}) \to X_{inv}$ corresponding to F, G are equal, so F, G are equal. □

QQQ Need to explain that the goal is to understand some group actions ← Move me

2.2. Adjunctions involving monoidal categories

This section will study three different adjunctions between $\mathbf{\Lambda}$-monoidal categories on the one hand and categories such as that of monoids or commutative monoids on the other hand. Two of these adjunctions are purely formal, and are induced by adjunctions between categories and sets, while the third requires a direct proof. We then investigate some simple consequences of these adjunctions for $\mathbf{\Lambda}$-monoidal categories of the form ~~$L(E\Lambda S)$ for a set S.~~ L_n?

Proposition 2.2.1. *The functor Ob from categories to sets, taking the set of objects, is left adjoint to the functor E of ?? QQQ. This adjunction is monoidal, hence induces an adjunction $Ob \dashv E$ between Λ-**MonCat** and **Mon**.*

PROOF. This is a straightforward application of ??; the only thing to note is that $\mathrm{Ob}(E\Lambda) = \Lambda$ as a $\mathbf{\Lambda}$-operad. □

Definition 2.2.2. Let $\mathbf{\Lambda}$ be an action operad. Let $T_{\mathbf{\Lambda}}$ denote the terminal $\mathbf{\Lambda}$-operad in sets, which is a singleton set in each dimension with the unique action of Λ_n.

Lemma 2.2.3. *The category of algebras for $T_{\mathbf{\Lambda}}$ is either the category of monoids, when $\mathbf{\Lambda}$ is not crossed, or the category of commutative monoids, when $\mathbf{\Lambda}$ is crossed.*

PROOF. In the case that $\mathbf{\Lambda}$ is not crossed, $\Lambda(n)$ acts on X^n trivially so the free $T_{\mathbf{\Lambda}}$-algebra monad coincides with the free monoid monad. When $\mathbf{\Lambda}$ is crossed, $\Lambda(n)$ acts on X^n via the surjection to Σ_n so the free $T_{\mathbf{\Lambda}}$-algebra monad coincides with the free commutative monoid monad. □

Notation 2.2.4. We write D for the discrete category functor $\mathrm{Set} \to \mathbf{Cat}$.

Proposition 2.2.5. *The functor π_0 from categories to sets, taking the set of path components, is left adjoint to D. This adjunction is monoidal. hence induces an adjunction between Λ-**MonCat** and $T_{\mathbf{\Lambda}}$-Alg.*

PROOF. This is now a simple application of ??, where we now only note that the unit $1 \Rightarrow \pi_0 \circ D$ is the identity transformation. □

Definition 2.2.6. For a monoid M, we write M^{gp} for its *group completion*, the universal group with a homomorphism $M \to M^{gp}$. We write the functor $M \mapsto M^{gp}$ as gp.

Fix gp supers

Remark 2.2.7. The category of groups is a reflective subcategory of the category of monoids, and gp is the reflection.

Proposition 2.2.8. *The functor $Ob \circ L \circ E\Lambda : Set \to$ **Mon** is naturally isomorphic to the composite of the free group functor and the inclusion of groups into monoids.*

Reword

PROOF. Using that the objects of $E\Lambda(M)_{inv}$ for a monoid M are the same as the objects of $E\Lambda(M^\times)$. where $M^\times$ is the subgroup of invertible elements of M, we see that both of these functors are left adjoints to the functor $M \mapsto M^\times$. □

There are several different ways to ~~actually~~ calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Notation 2.2.9. We write M^{*n} for the coproduct of n copies of the monoid M. We use the same notation for groups, although the n-fold coproduct of a group is different when considered as a monoid than as a group; it should be clear from context which we intend.

Corollary 2.2.10. *The object monoid of L_n is $\mathbb{Z}^{*n}$, the group completion of the object monoid of $E\Lambda(\underline{n})$. The restriction of η on objects, $\mathrm{Ob}(\eta)$, is then the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.*

What's n?

The core of Corollary 2.2.10 — that $\mathrm{Ob}(L_n)$ is the group completion of $\mathrm{Ob}(E\Lambda(\underline{n}))$ — makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it would seem reasonable to expect that $\pi_0(L_n)$ is also the group completion of $\pi_0(E\Lambda(\underline{n}))$. This is indeed the case, and the proof ~~proceeds in a way completely~~ is analogous ~~to Corollary 2.2.10~~.

Proposition 2.2.11. *The functor $\pi_0 \circ L : \Lambda$-**MonCat** $\to T_{\mathbf{\Lambda}}$-**Alg** is naturally isomorphic to the functor $gp \circ \pi_0$.*

~~PROOF. The proof is the same style as above, with both functors being left adjoints of the functor $M \mapsto (DM)_{inv} = D(M^\times)$.~~ □

Corollary 2.2.12. *The connected components of L_n are the group completion of the connected components of $E\Lambda(\underline{n})$. Also, the restriction of η onto connected components, $\pi_0(\eta)$, is the canonical map $\pi_0(E\Lambda(\underline{n})) \to \pi_0(E\Lambda(\underline{n}))^{\mathrm{gp}}$ associated with that group completion.*

We summarize these results below.

Corollary 2.2.13. *If G is a crossed action operad then*

- *the connected components of L_n are the monoid $\mathbb{Z}^n$,*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$, and*
- *the assignment of objects to their component is given by the quotient map of abelianization $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$.*

If instead G is non-crossed, then

- *the connected components of L_n are the monoid $\mathbb{Z}^{*n}$,*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$, and*
- *the assignment of objects to their component is $\mathrm{id}_{\mathbb{Z}^{*n}}$.*

We now turn to our third adjunction, which is of a less formal nature than the first two. We begin by showing that composition along invertible objects in X can always be restated in terms of the tensor product.

Lemma 2.2.14. *Let $f : x \to y$ and $f' : y \to z$ be morphisms in a strict monoidal category, and assume y is an invertible object with inverse y^{-1}. Then*

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y^{-1}} \otimes f$$

PROOF. By the interchange law for monoidal categories,

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y^{-1}} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y^{-1}} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y^{-1}} \circ \mathrm{id}_{y^{-1}}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y^{-1}} \otimes f. \end{aligned}$$

the case if

□

composition

Thus ~~in cases where~~ every object of X is invertible, the monoidal structure together with knowledge of each morphism's source and target will be enough to determine X uniquely. Since all objects in L_n are invertible, this means that we could choose to ignore composition of elements of $\mathrm{Mor}(L_n)$ QQQ this should have had a notation environment a long time ago QQQ for the time being, and focus on its status as a monoid under tensor product.

Definition 2.2.15. Let $\mathrm{M} : \mathbf{MonCat} \to \mathbf{Mon}$ be the functor which sends a monoidal category X to the quotient of its monoid of morphisms by the relation that sets $\otimes = \circ$. :

$$\mathrm{M}X \quad = \quad \mathrm{Mor}(X) \big/ f' \circ f \sim f' \otimes f$$

If $F : X \to Y$ is a strict monoidal functor, then MF is the monoid homomorphism sending the class of a morphism g to the class of Fg; the reader will easily check this is well-defined. We will call $\mathrm{M}X$ the *collapsed* morphisms of X.

Notation 2.2.16. If f is a morphism in X, we write its class in $\mathrm{M}X$ as $\mathrm{M}f$, and we write the single operation $\otimes$ rather than $\circ$. Note that the class $\mathrm{M}(f') \otimes \mathrm{M}(f)$ always contains $f' \otimes f$, but will only contain $f' \circ f$ if this latter morphism exists.

Now we need a candidate for the right adjoint to the functor M.

Definition 2.2.17. Let $B : \mathbf{Mon} \to \mathbf{Cat}$ denote the functor sending a monoid M to the one-object category whose single hom-set consists of the elements of M, with composition and identity being given by multiplication and the unit of

Explain strategy: what can M tell us? Forward refs.

M, respectively. We also denote by B the functor $\mathbf{CommMon} \to \mathbf{MonCat}$ the functor sending a commutative monoid A to the same category BA, now equipped with the monoidal structure which is trivial on the single object and $a \otimes b = a \circ b = ab$ on morphisms (writing the monoid operation in A as concatenation here).

Note that for any monoidal category M, the monoid $M(I, I)$ is always commutative by the Eckmann-Hilton argument [?, ?], hence the requirement of commutativity to lift B to a functor into monoidal categories.

Definition 2.2.18. For a monoid M, we write M^{ab} for its *abelianization*, the universal commutative monoid with a homomorphism $M \to M^{ab}$. We write the functor $M \mapsto M^{ab}$ as ab. We use the same notation for groups as well.

Fix ab supers

Remark 2.2.19. The categories of commutative monoids, resp. abelian groups, are reflective subcategories of the categories of monoids, resp. groups, and ab is the reflection.

Proposition 2.2.20. $\mathrm{B} : \mathbf{CommMon} \to \mathbf{MonCat}$ *is a right adjoint to the functor* $\mathrm{M}(-)^{\mathrm{ab}} : \mathbf{MonCat} \to \mathbf{CommMon}$.

PROOF. For a commutative monoid A, it is clear that $M(BA)^{ab} = A$, so we can define the counit to be the identity. For a monoidal category X, define $X \to B(MX^{ab})$ by sending every object of X to the unique object, and by sending the morphism f to the morphism given by the equivalence class of f. This is clearly natural in strict monoidal functors $X \to Y$, and the triangle identities make for a straightforward calculation.

□

Proposition 2.2.20 seems at first glance very similar to ????. However, our goal was to discover the relationship between the morphisms of $E\Lambda(\underline{n})$ and L_n, paralleling what we did in Corollaries 2.2.10 and 2.2.12, and in that regard M falls short in two very important ways.

(1) Our goal was to have an adjunction involving Λ-**MonCat**, not **MonCat**, since we want to apply it to strict $\mathbf{\Lambda}$-monoidal functors instead of arbitrary strict monoidal functors.
(2) Even if we do find a way to use this adjunction to extract information about L_n, it will not be the monoid $\mathrm{Mor}(L_n)$ we were originally after, only a strange abelianized version where tensor product and composition coincide.

Unfortunately, this adjunction seems to be the best that we can do.

Proposition 2.2.21. *The functor B :* $\mathbf{CommMon} \to \mathbf{MonCat}$ *lifts to a functor B :* $\mathbf{CommMon} \to \Lambda\text{-}\mathbf{MonCat}$. *This functor B has a left adjoint M' :* $\Lambda\text{-}\mathbf{MonCat} \to \mathbf{CommMon}$, *but both functors*

$$M' \circ E\Lambda,\ M' \circ L \circ E\Lambda : \mathcal{S}et \to \mathbf{CommMon}$$

are isomorphic to the functor sending every set to the trivial commutative monoid.

PROOF. To lift B to Λ-**MonCat**, we need only assign the trivial action of each Λ_n. The existence of M' is guaranteed by the adjoint functor theorem for locally finitely presentable categories since B clearly preserves limits and filtered colimits. For a set S and commutative monoid A, monoid homomorphisms $M' \circ E\Lambda(S) \to A$ are in bijection with strict $\mathbf{\Lambda}$-monoidal functors $E\Lambda(S) \to BA$ which are themselves

in bijection with functions $S \to \mathrm{Ob}(BA) = *$, so there is a unique such monoid homomorphism, and $M' \circ E\Lambda(S) \cong 0$ as commutative monoids. The same proof works for $M' \circ L \circ E\Lambda$ after we note that $(BA)_{inv} = BA$. □

We will eventually prove that the morphisms of L_n actually form a group under tensor product ??, so instead of working directly with the functor $\mathrm{M}(-)^{\mathrm{ab}}$: $\mathbf{MonCat} \to \mathbf{CommMon}$, we will focus on its composite with the group completion functor, gp : $\mathbf{CommMon} \to \mathbf{Ab}$. We end with a brief exploration of this new functor $\mathrm{M}(-)^{\mathrm{gp,ab}}$. It is well known that group completion and abelianization commute since their right adjoints commute, but we further note that group completion commutes with collapsing morphisms.

Lemma 2.2.22. *For any monoidal category X, define*

$$\begin{aligned} \mathrm{M}_{\mathrm{gp}}(X) &= \mathrm{Mor}(X)^{\mathrm{gp}} \big/ \mathrm{gp}(f' \circ f) \sim \mathrm{gp}(f' \otimes f) \\ \mathrm{M}_{\mathrm{ab}}(X) &= \mathrm{Mor}(X)^{\mathrm{ab}} \big/ \mathrm{ab}(f' \circ f) \sim \mathrm{ab}(f' \otimes f) \end{aligned}$$

Then

$$\mathrm{M}_{\mathrm{gp}}(X) \cong \mathrm{M}(X)^{\mathrm{gp}}, \qquad \mathrm{M}_{\mathrm{ab}}(X) \cong \mathrm{M}(X)^{\mathrm{ab}}$$

PROOF. This follows immediately from M being given explicitly as a quotient, and both functors ab and gp preserving colimits.. □

QQQ Institute the following where possible

Convention 2.2.23. We default to writing any abelian group, such as those in the image of $\mathrm{M}(-)^{\mathrm{gp,ab}}$, with additive notation, using $+$ for the group operation and 0 for the identity element.

2.3. The free object as a cokernel

This section will present a different approach to L_n. Consider for a moment the free $\mathbf{\Lambda}$-monoidal category on $2n$ objects $z_1, z_2, \ldots, z_{2n}$. If we were to take this category and then add the relations $z_{n+1} = z_1^{-1}, \ldots, z_{2n} = z_n^{-1}$, then we would be changing it from a structure with $2n$ independent generators into one with n independent generators and their inverses. Thus we would see L_n as a quotient of the larger monoidal category $E\Lambda(2n)$. We will now work towards making this idea precise, and then examine some of its consequences, the most important of which will be allowing us to describe the group $\mathrm{M}(L_n)^{\mathrm{gp,ab}}$.

Definition 2.3.1. Let δ be the map of $\mathbf{\Lambda}$-monoidal categories defined on generators by

$$\begin{array}{rcrcl} \delta & : & E\Lambda(\underline{2n}) & \to & E\Lambda(\underline{2n}) \\ & : & z_i & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. We will also denote by $q : E\Lambda(\underline{2n}) \to Q$ the cokernel of this map (i.e., the coequalizer of this functor and the functor sending all objects to the unit and all morphisms to the identity).

The first goal of this section is to show that $Q \cong L_n$.

Proposition 2.3.2. *The object monoid of Q is $\mathbb{Z}^{*n}$, and the restriction of q to objects $\mathrm{Ob}(q) : \mathrm{Ob}(E\Lambda(\underline{2n})) \to \mathrm{Ob}(Q)$ is the monoid homomorphism defined on generators as below.*

$$\begin{array}{rcrcl} \mathrm{Ob}(q) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

PROOF. Since Ob is a left adjoint, it preserves colimits and thus cokernels. It is clear, from the presentation, that the cokernel on objects is $\mathbb{Z}^{*n}$. □

Proposition 2.3.3. *Let $i : E\Lambda(\underline{n}) \to E\Lambda(\underline{2n})$ be the inclusion of $\mathbf{\Lambda}$-monoidal categories defined on generators by $i(z_j) = z_j$. Then $i \circ q$ exhibits Q as the initial $\mathbf{\Lambda}$-monoidal category on n invertible objects, so $Q \cong L_n$.*

qi

PROOF. Proposition 2.3.2 gives a unique map shown by the dashed arrow below.

$$\begin{tikzcd} E\Lambda(\underline{n}) \arrow[r, "i"] \arrow[dr] & E\Lambda(\underline{2n}) \arrow[r, "q"] & Q \\ & L_n \arrow[ur, dashed] & \end{tikzcd}$$

We also have a unique map $E\Lambda(\underline{2n}) \to L_n$ which agrees with the canonical map $E\Lambda(\underline{n}) \to L_n$ on $z_1, \ldots, z_n$ and sends z_{i+n} to the inverse of the image of z_i. This gives the diagram below, once again with the dashed arrow induced by the definition of δ and the universal property.

$$\begin{tikzcd} E\Lambda(\underline{n}) \arrow[r, "i"] \arrow[dr] & E\Lambda(\underline{2n}) \arrow[r, "q"] \arrow[d] & Q \arrow[dl, dashed] \\ & L_n & \end{tikzcd}$$

The two dashed arrows are then forced to be inverse to each other. □

Definition 2.3.4. We will say that a functor $F : C \to D$ is *surjective* if the underlying map of graphs is surjective. In particular F must be surjective on objects, but it need not be full.

Proposition 2.3.5. *Let $\phi, \phi' : X \to Y$ be a pair of $E\Lambda$-algebra maps, and $k : Y \to Z$ their coequalizer in $E\Lambda$-Alg_S. If the monoid $\mathrm{Ob}(Z)$ is also a group, then the functor k is surjective.*

PROOF. Since the functor $\mathrm{Ob} : E\Lambda\text{-}\mathrm{Alg}_S \to \mathbf{Mon}$ is a left adjoint, it preserves all colimits so the monoid homomorphism $\mathrm{Ob}(k) : \mathrm{Ob}(Y) \to \mathrm{Ob}(Z)$ is the coequalizer of the parallel pair $\mathrm{Ob}(\phi), \mathrm{Ob}(\phi')$ in the category of monoids. Every coequalizer is a regular epimorphism, and in the category of monoids regular epimorphisms coincide with surjective functions, so k is surjective on objects.

Next, let $f : v \to w$ and $f' : w' \to v'$ be any two morphisms of the category Y for which $k(f)$ and $k(f')$ are composable in Z. Since these maps are composable we know that $k(w)$ and $k(w')$ must be the same object of Z, and since $\mathrm{Ob}(Z)$ is a group we know this object has an inverse $k(w)^{-1} = k(w')^{-1}$. So by the surjectivity of k we can find another object y of Y for which $k(y) = k(w)^{-1}$. Using this, define the morphism $h : w' \otimes y \otimes v \to v' \otimes y \otimes w$ to be the tensor product $f' \otimes \mathrm{id}_y \otimes f$. Thus

$$\begin{aligned} k(h) &= k(f' \otimes \mathrm{id}_y \otimes f) \\ &= k(f') \otimes \mathrm{id}_{k(y)} \otimes k(f) \\ &= k(f') \otimes \mathrm{id}_{k(w)^{-1}} \otimes k(f) \end{aligned}$$

and so by Lemma 2.2.14, this is the composite $k(f') \circ k(f)$. Therefore the set of morphisms of Z which are images of morphisms of Y is closed under composition.

Now consider $k(Y)$, the subcategory of Z that contains every object x' for which there exists x in Y with $k(x) = x'$, and every morphism f' for which there exists f in Y with $q(f) = f'$. We know that the morphisms of $k(Y)$ are closed under composition, and so this is indeed a well-defined category. It is easy to see that $k(Y)$ is in fact a well-defined sub-$\mathbf{\Lambda}$-monoidal category, and that $k' : Y \to k(Y)$ is a surjective strict $\mathbf{\Lambda}$-monoidal functor. This shows that k' coequalizes ϕ, ϕ', so the inclusion $k(Y) \hookrightarrow Z$ must be an isomorphism and thus k is surjective. □

Remark 2.3.6. We note that, since we haven't used any feature of $\mathbf{\Lambda}$ in the above proofs, they all apply equally in the case of no group actions at all, i.e., strict monoidal categories or when $\mathbf{\Lambda} = \mathbf{T}$ is the terminal action operad consisting only of trivial groups.

Corollary 2.3.7. *The cokernel map $q : E\Lambda(\underline{2n}) \to L_n$ is surjective.*

Corollary 2.3.8. *We have an isomorphism of groups as below.*

$$\mathrm{M}(L_n)^{\mathrm{gp.ab}} \cong \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp.ab}} \big/ \ker\left(\mathrm{M}(q)^{\mathrm{gp.ab}}\right)$$

Move?
Remark?

One important consequence of the surjectivity of q is that it will allow us to use some results about the free algebra $E\Lambda(\underline{2n})$ to deduce information about the free invertible algebra L_n. In fact, we have done this once already: looking back at Proposition 2.3.2 with our current knowledge that $Q = L_n$, we can see that it is a direct analogue of ??, using the fact that q is surjective on objects.

QQQ Reference "Gnmapsaction" is missing.

In that same vein, one might ask if we can take ??, a statement about the morphisms $E\Lambda(\underline{2n})$, and extend it to an analagous result on L_n, using surjectivity of q on morphisms instead. ~~That is, since every morphism of $E\Lambda(\underline{2n})$ is an action morphism, and since strict $\mathbf{\Lambda}$-monoidal functors always send action morphisms to action morphisms, we should be able to use q to identify every morphism of L_n as an action morphism.~~

~~QQQ Get some notation for what Ed calls action maps, I don't like all the α's~~

Do this

Notation 2.3.9. Let's think about $g^{\otimes}$ for the iso induced by g.

Lemma 2.3.10. *Later I use: $(h \otimes g)^{\otimes} = h^{\otimes} \otimes g^{\otimes}$.*

Proposition 2.3.11. *Every morphism in L_n can be expressed as $g^{\otimes}$ for some $g \in \Lambda(m)$ and $x_i \in \{z_1, \ldots, z_n, z_1^*, \ldots, z_n^*\}$.*

PROOF. Let f be an arbitrary morphism in L_n. By surjectivity of q, there must exist at least one morphism f' in $E\Lambda(\underline{2n})$ such that $q(f') = f$, and from ?? we know that this f' can be expressed uniquely as $g^{\otimes}$ for some $g \in \Lambda(m)$. □

Note that while this result shows that every morphism of L_n is induced by the action of some $g \in \Lambda(m)$, it does not imply that this g is unique.

Convention 2.3.12. The monoid homomorphism $\mathrm{Ob}(q) : \mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ has a canonical section q^{-1} given as follows. An element $w \in \mathbb{Z}^{*n}$ can be written as a word in z_i, z_i^* in normal form, that is in the fewest number of symbols. We define $q^{-1}(z_i) = z_i$ and $q^{-1}(z_i^*) = z_{n+i}$, and then

$$q^{-1}(w) = q^{-1}(w_1) \cdots q^{-1}(w_k)$$

where $w = w_1 \cdots w_k$ is the normal form of w. We note that q^{-1} is merely a function, and not a monoid homomorphism.

Corollary 2.3.13. *For $g \in \Lambda(m)$ and $x_1, \ldots, x_m$ objects of L_n, the isomorphism*

$$g^{\otimes} : a_1 \otimes \cdots \otimes a_m \to a_{\pi(g)^{-1}(1)} \otimes \cdots \otimes a_{\pi(g)^{-1}(m)}$$

is

$$q\Big(g^{\otimes} : q^{-1}(a_1) \otimes \cdots \otimes q^{-1}(a_m) \to q^{-1}(a_{\pi(g)^{-1}(1)}) \otimes \cdots \otimes q^{-1}(a_{\pi(g)^{-1}(m)})\Big).$$

2.4. A second coequalizer argument

In this section, we will compare the coequalizer of a diagram of $\mathbf{\Lambda}$-monoidal categories as it is computed in $\mathbf{\Lambda MonCat}$, in $\mathbf{MonCat}$, and in $\mathbf{Cat}$. Our goal will be to show that the forgetful functors

$$\mathbf{\Lambda}\text{-}\mathbf{MonCat} \to \mathbf{MonCat} \to \mathbf{Cat}$$

all create reflexive coequalizers. This has the effect of allowing us to compute coequalizers in $\mathbf{\Lambda}$-$\mathbf{MonCat}$ by converting them into reflexive coequalizers and then computing them in either $\mathbf{MonCat}$ or $\mathbf{Cat}$. This technique will be necessary in our construction of the free $\mathbf{\Lambda}$-monoidal category on n invertible objects. The strategy here follows that in Section 4.1 of [?].

Lemma 2.4.1. *Let P be a $\mathbf{\Lambda}$-operad in $\mathbf{Cat}$, with $\underline{P}$ its associated 2-monad. Then $\underline{P}$ preserves reflexive coequalizers.*

PROOF. We have a functor $P : B\mathbf{\Lambda}^{op} \to \mathbf{Cat}$ which sends n to $P(n)$ and uses the right action $P(n) \times \mathbf{\Lambda}(n) \to P(n)$ to define the functor on morphisms. Given a category $X \in \mathbf{Cat}$, there is also a functor $R_X : B\mathbf{\Lambda} \to \mathbf{Cat}$ sending n to X^n and $g \in \Lambda(n)$ to the permutation given by $\pi(n)$. The weighted colimit $P \cdot R_X$ is then easily seen to be the free algebra $\underline{P}(X)$. The functor $P \cdot -$ is cocontinuous, so the free $\underline{P}$-algebra functor $\mathbf{Cat} \to \mathbf{Cat}$ will preserve whatever colimits the functor $X \mapsto R_X$ preserves. Colimits in the functor category $[B\mathbf{\Lambda}, \mathbf{Cat}]$ are computed pointwise, and the functors $X \mapsto X^n$ for $n \in \mathbb{N}$ preserve reflexive coequalizers, so $X \mapsto R_X$ does also and therefore the free $\underline{P}$-algebra functor preserves reflexive coequalizers. □

Lemma 2.4.2. *Let A, B, C be cocomplete categories, and let $A \xrightarrow{F} B \xrightarrow{G} C$ be functors such that G is conservative, and both G preserves and GF creates colimits of shape $\mathbb{D}$. Then F creates colimits of shape $\mathbb{D}$.*

For an monad T on a category C, the forgetful functor T-$\mathbf{Alg} \to C$ creates any colimit that T preserves. This fact and the previous lemmas prove the following proposition using the operad $P = E\mathbf{\Lambda}$.

Proposition 2.4.3. *For any action operad $\mathbf{\Lambda}$, the forgetful functor $\mathbf{\Lambda}$-$\mathbf{MonCat} \to \mathbf{Cat}$ is conservative and creates reflexive coequalizers. Consequently, $\mathbf{\Lambda}$-$\mathbf{MonCat} \to \mathbf{MonCat}$ is also conservative and creates reflexive coequalizers.*

Notation 2.4.4. Let $f : A \to C, g : B \to C$ be maps in a category with coproducts. We will write $f;g : A \coprod B \to C$ for the unique map induced by the universal property of the coproduct.

Lemma 2.4.5. *Let $f, g : A \to B$ be maps in a cocomplete category with coequalizer $c : B \to C$. Then c is also the coequalizer of $f; id_B$ and $g; id_B$.*

PROOF. A map $h : B \to X$ coequalizes $f; \mathrm{id}_B$ and $g; \mathrm{id}_B$ if and only if $hf = hg$ and $h\mathrm{id}_B = h\mathrm{id}_B$ by the universal property of the coproduct, so c is still the universal map which coequalizes. □

Definition 2.4.6. Let $\tilde{\delta} = \mathrm{id}_{E\Lambda(\underline{2n})}; \delta$. Explicitly, $\tilde{\delta}$ is the map of $E\Lambda$-algebras which acts on generators by

$$\begin{array}{rrcl} \tilde{\delta} : & E\Lambda(\underline{4n}) & \to & E\Lambda(\underline{2n}) \\ & z_i & \mapsto & z_i \\ & z_{n+i} & \mapsto & z_{n+i} \\ & z_{2n+i} & \mapsto & z_i \otimes z_{n+i} \\ & z_{3n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \le i \le n$. Similarly, let $\tilde{I} = \mathrm{id}_{E\Lambda(\underline{2n})}; I$. Explicitly, $\tilde{I}$ is the map of $E\Lambda$-algebras which acts on generators by

$$\begin{array}{rrcl} \tilde{I} : & E\Lambda(\underline{4n}) & \to & E\Lambda(\underline{2n}) \\ & z_i & \mapsto & z_i \\ & z_{n+i} & \mapsto & z_{n+i} \\ & z_{2n+i} & \mapsto & I \\ & z_{3n+i} & \mapsto & I \end{array}$$

for $1 \le i \le n$.

Corollary 2.4.7. *The functor q (Definition 2.3.1) is the coequalizer of $\tilde{\delta}$ and $\tilde{I}$ in $E\Lambda$-$\mathbf{Alg}_S$.*

By construction, $\tilde{\delta}$ and $\tilde{I}$ form a reflexive pair in $E\Lambda$-$\mathbf{Alg}_S$, so by Proposition 2.4.3 we have the further corollary below.

Where do we say this?

Corollary 2.4.8. *The functor q is the coequalizer of $\tilde{\delta}$ and $\tilde{I}$ in $\mathbf{MonCat}$ or in $\mathbf{Cat}$.*

Do this

2.5. From strictly to weakly invertible objects

We now turn to weakly invertible objects, and more specifically how we can apply our results about invertible objects to the weakly invertible ones. We begin with the analogue of Definition 2.1.3.

Definition 2.5.1. Given an $E\Lambda$-algebra X, we will denote by X_{winv} the sub-$E\Lambda$-algebra of X containing all objects which are weakly invertible under tensor product, and all of the isomorphisms between them.

We have analogues of the following propositions, the proofs of which are, *mutatis mutandis*, the same as for invertible objects.

Proposition 2.5.2. *The assignment $X \mapsto X_{\mathrm{winv}}$ can be extended to a 2-functor $(\text{-})_{\mathrm{winv}} : E\Lambda$-$\mathbf{Alg}_S \to E\Lambda$-$\mathbf{Alg}_S$.*

Proposition 2.5.3. *The 2-functor $(\text{-})_{\mathrm{inv}} : E\Lambda$-$\mathbf{Alg}_S \to E\Lambda$-$\mathbf{Alg}_S$ has a left adjoint, $K : E\Lambda$-$\mathbf{Alg}_S \to E\Lambda$-$\mathbf{Alg}_S$.*

Definition 2.5.4. Let K_n denote the free $\mathbf{\Lambda}$-monoidal category generated by n weakly invertible objects.

THEOREM 2.5.5. *The canonical $\mathbf{\Lambda}$-monoidal functor $\tau : K_n \to L_n$ is an equivalence of $\mathbf{\Lambda}$-monoidal categories.*

PROOF. Reduce to the case $n = 1$ using some coproducts

Prove a weakly invertible object is the same a strong $\mathbf{\Lambda}$-monoidal functor $L_1 \to X$

Use some coherence theorems □

CHAPTER 3

Invertibility and group actions

The goal of this chapter is to reduce coherence questions about invertible objects (be they strict or weak by Theorem 2.5.5) to the calculation of a group action. The group we are interested in is the group of automorphisms of the unit, $L_n(I, I)$, and it acts on $L_n(x, y)$ for any two objects x and y. In QQQ forward ref QQQ, we will explain how to use this group action to determine whether some pair of parallel morphisms are equal or not.

More intro here

3.1. Sources and targets in L_n

Definition 3.1.1. For any $E\Lambda$-algebra X, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ the monoid homomorphisms which send each morphism of X to its source and target, respectively. That is,

$$s(f : x \to y) = x, \qquad t(f : x \to y) = y.$$

Lemma 3.1.2. *Let X be an $E\Lambda$-algebra whose underlying category is a groupoid, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map induced from s and t using the universal property of products. Then the image of this map is*

This is a def

$$(s \times t)(X) = \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

where this pullback is taken over the canonical maps sending objects of X to their connected components:

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Ob}(X) \arrow[d, "{[-]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[-]}"] & \pi_0(X)
\end{tikzcd}$$

Recalling **????** and Corollaries 2.2.10 and 2.2.13, we can immediately conclude the following.

Corollary 3.1.3.

$$(s \times t)(E\Lambda(\underline{n})) \cong \begin{cases} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

$$(s \times t)(L_n) \cong \begin{cases} \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{Z}^{*n} & \textit{otherwise} \end{cases}$$

where the pullbacks are taken over the quotients of abelianization for $(\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$ and $(\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ respectively.

Now the group

Next, we want to show that this $(s \times t)(L_n)$ we have described is in fact a submonoid of $\mathrm{Mor}(L_n)$. We will accomplish this by first proving the analogous statement for all $E\Lambda(\underline{n})$, and then recovering the L_n version from it ~~later.~~ For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ such that the images of w and w' in the commutative monoid $\mathbb{N}^n$ are the same, there are generators $x_1, ..., x_m$ for which

after

$$w = x_1 \otimes ... \otimes x_m$$

and there exists at least one permutation $\sigma \in S_m$ such that

$$w' = x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}.$$

$\Lambda(n)$

S_m vs Σ_m

Since the underlying permutation maps $\pi : G(m) \to S_m$ of a crossed action operad G are all surjective, we can always find an element of $g \in G(m)$ for which $\pi(g) = \sigma$.

Notation 3.1.4. Let $\rho : \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to \bigcup E\Lambda(m)$ be an arbitrary, but fixed, function such that $\rho(w, w')$ satisfies $\pi(\rho(w, w')) = \sigma$ as defined above.

Think about this again

Lemma 3.1.5. *$\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is a free monoid.*

PROOF. Write the generators for the first copy of $\mathbb{N}^{*n}$ in $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ as $a_1, \ldots, a_n$, and the generators for the second copy of $\mathbb{N}^{*n}$ as $x_1, \ldots, x_n$. An element $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ can therefore be written uniquely as

$$(w, w') = (a_{i_1} a_{i_2} \cdots a_{i_k}, x_{i_1} x_{i_2} \cdots x_{i_j})$$

and being an element of the pullback immediately implies $k = j$. We then define an element $(a_{i_1} a_{i_2} \cdots a_{i_k}, x_{i_1} x_{i_2} \cdots x_{i_k})$ of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ to be indecomposable if there does not exist an $h < k$ such that $(a_{i_1} a_{i_2} \cdots a_{i_h}, x_{i_1} x_{i_2} \cdots x_{i_h})$ is also an element of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. It is clear that every element can be written as a product of indecomposables. If (w, w') can be written as a product of indecomposables in two ways, say

$$(w, w') = c_1 \cdots c_s = d_1 \cdots d_t,$$

then the length (QQQ we defined this kind of thing somewhere) of c_1 must be either strictly less than or strictly greater than the length of d_1; if they had equal length, then they would be equal. Without loss of generality assume c_1 has length l_c which is strictly less than the length l_d of d_1. But then the first l_c terms appearing in d_1 would agree with c_1, proving that d_1 is not indecomposable. Straightforward induction then finishes the proof that every element of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ can be written as a product of indecomposables in a unique way, making $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ free on the set of indecomposables. □

$l(c)$?

We can now construct the desired injection using the function ρ from Notation 3.1.4.

Proposition 3.1.6. $\mathrm{Mor}(E\Lambda(\underline{n})) \to (s \times t)(E\Lambda(\underline{n}))$ *is a split epi of monoids, so* $(s \times t)(E\Lambda(\underline{n}))$ *is (isomorphic to) a submonoid of* $\mathrm{Mor}(E\Lambda(\underline{n}))$. *Furthermore, we can choose the injection* $i : (s \times t)(E\Lambda(\underline{n})) \to \mathrm{Mor}(E\Lambda(\underline{n}))$ *such that the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Ob}(E\Lambda(\underline{n})) \arrow[r] \arrow[dr] & (s \times t)(E\Lambda(\underline{n})) \arrow[d, "i"] \\
& \mathrm{Mor}(E\Lambda(\underline{n}))
\end{tikzcd}$$

Need to make sure we say e_n is identity in $\Lambda(n)$

Why splitting?

PROOF. First, assume that the action operad G is non-crossed. Then there exists an obvious injective monoid homomorphism

$$\begin{array}{rrcl} i & : \ (s \times t)(E\Lambda(\underline{n})) & \to & \mathrm{Mor}(E\Lambda(\underline{n})) \\ & : \qquad \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & : \qquad w & \mapsto & (e_{|w|}, w) \end{array}$$

Notation!

The homomorphism property follows from the fact that the length $|w|$ defined in ?? is itself a homomorphism, so $|w \otimes w'| = |w| + |w'|$. Thus $(s \times t)(E\Lambda(\underline{n})) \subseteq \mathrm{Mor}(E\Lambda(\underline{n}))$ for non-crossed G.

Now assume that G is crossed. Fix a function ρ as in Notation 3.1.4. Because $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is a free monoid by Lemma 3.1.5, there is a unique monoid homomorphism

All G's to Λ's

$$\rho : \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \longrightarrow G$$

which agrees with the original function ρ on the generators in the source. By construction, we have

$$\pi(\rho(w, w'))(w) = w'$$

for any $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, not just the generators. Using ρ we define the homomorphism i to be

$$\begin{array}{rrcl} i & : \ (s \times t)(E\Lambda(\underline{n})) & \to & \mathrm{Mor}(E\Lambda(\underline{n})) \\ & \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & (w, w') & \mapsto & (\rho(w, w'), w). \end{array}$$

Moreover, for any two elements (v, v'), (w, w') of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ we have

$$\begin{array}{rcl} (\rho(v, v'), v) = (\rho(w, w'), w) & \Longrightarrow & \rho(v, v') = \rho(w, w'), \qquad v = w \\ & \Longrightarrow & v' = \pi(\rho(v, v'))(v) = \pi(\rho(w, w'))(w) = w' \end{array}$$

and thus i is injective. This injection splits $\mathrm{Mor}(E\Lambda(\underline{n})) \to (s \times t)(E\Lambda(\underline{n}))$ by construction.

Hmm...

For the final statement, note that the inclusion of $\mathrm{Ob}(E\Lambda(\underline{n}))$ into $(s \times t)(E\Lambda(\underline{n}))$ sends an object w to the pair (w, w). The only indecomposable such are (x_i, x_i) if we write the generating objects of $E\Lambda(\underline{n})$ as $x_1, x_2, \ldots, x_n$. Using the construction above, we can always choose $\rho(x_i, x_i) = e_1$. □

Ref

For the proof of Proposition 3.1.6, ρ can be any function satisfying the criteria in Notation 3.1.4, with the caveat that commuting with the inclusion of objects requires $\rho(x_i, x_i) = e_1$. The analogous proof for L_n, though, requires more properties of ρ which we establish now. Recall the functor δ from Definition 2.3.1.

Lemma 3.1.7. *If $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is indecomposable, then so is $\delta(w, w')$. For any indecomposable (w, w'), there exists a unique natural number k and indecomposable (v, v') such that*

$$(w, w') = \delta^k(v, v')$$

but (v, v') is not $\delta(u, u')$ for any indecomposable (u, u').

PROOF. The first statement follows immediately from the injectivity of δ on objects, and the second from the fact that δ doubles the length of any element. □

Proposition 3.1.8. $\mathrm{Mor}(L_n) \to (s \times t)(L_n)$ *is a split epi of monoids, so* $(s \times t)(L_n)$ *is (isomorphic to) a submonoid of* $\mathrm{Mor}(L_n)$. *Furthermore, we can choose the injection* $i : (s \times t)(L_n) \to \mathrm{Mor}(L_n)$ *such that the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Ob}(L_n) \arrow[r] \arrow[dr] & (s \times t)(L_n) \arrow[d, "i"] \\
& \mathrm{Mor}(L_n)
\end{tikzcd}$$

PROOF. Let $i : (s \times t)(E\Lambda(\underline{2n})) \to \mathrm{Mor}(E\Lambda(\underline{2n}))$ be a splitting as in Proposition 3.1.6. We will first show that the function ρ can be chosen to make the square

$$\begin{tikzcd}
(s \times t)(E\Lambda(\underline{2n})) \arrow[r, "i"] \arrow[d, "\delta"'] & \mathrm{Mor}(E\Lambda(\underline{2n})) \arrow[d, "\delta"] \\
(s \times t)(E\Lambda(\underline{2n})) \arrow[r, "i"'] & \mathrm{Mor}(E\Lambda(\underline{2n}))
\end{tikzcd}$$

commute. Recall that i is defined by first factoring an element into decomposables, and then applying ρ to each of those. Thus on an indecomposable (w, w'), the commutativity of this square is the claim that $\delta(\rho(w, w')) = \rho(\delta w, \delta w')$ using that $\delta(w, w') = (\delta w, \delta w')$ is also indecomposable by the first part of Lemma 3.1.7. By the second part of Lemma 3.1.7, write $(w, w') = \delta^k(v, v')$, so we need

$$\delta(\rho(\delta^k(v, v'))) = \delta^{k+1}\rho(v, v').$$

Furthermore, (v, v') cannot be written as $\delta(u, u')$ for any indecomposable (u, u'), so we may choose $\rho(v, v')$ arbitrarily and this will uniquely determine i in such a way that the square commutes.

for (v, v') not δ(something)

The functor $q : E\Lambda(\underline{2n}) \to L_n$ is the cokernel of δ, so the composite

$$\mathrm{Mor}(E\Lambda(\underline{2n})) \xrightarrow{\mathrm{Mor}\delta} \mathrm{Mor}(E\Lambda(\underline{2n})) \xrightarrow{\mathrm{Mor}q} \mathrm{Mor}(L_n)$$

is the zero map. From the definition of δ in Definition 2.3.1 and the description of the monoids $(s \times t)(E\Lambda(\underline{n}))$, $(s \times t)(L_n)$ in Corollary 3.1.3, the composite

Why?

$$(s \times t)(E\Lambda(\underline{2n})) \xrightarrow{(s\times t)(\delta)} (s \times t)(E\Lambda(\underline{2n})) \xrightarrow{(s\times t)(q)} (s \times t)(L_n)$$

This

exhibits $(s \times t)(L_n)$ as the cokernel of $(s \times t)(\delta)$ which induces a unique monoid homomorphism $i : (s \times t)(L_n) \to \mathrm{Mor}(L_n)$ making the square below commute.

$$\begin{tikzcd}
(s \times t)(E\Lambda(\underline{2n})) \arrow[r, "i"] \arrow[d, "q"'] & \mathrm{Mor}(E\Lambda(\underline{2n})) \arrow[d, "q"] \\
(s \times t)(L_n) \arrow[r, "i"'] & \mathrm{Mor}(L_n)
\end{tikzcd}$$

This homomorphism i splits $\mathrm{Mor}(L_n) \to (s \times t)(L_n)$ as desired.

The final claim about commuting with the inclusion of objects follows using the same argument as in Proposition 3.1.6, so we must also require $\rho(x_i, x_i) = e_1$. □

3.2. Unit endomorphisms of L_n

We now consider the monoid of unit endomorphisms, $L_n(I,I)$. This is a particularly important submonoid of the morphisms $\mathrm{Mor}(L_n)$, since it is the only submonoid which is also a homset of the category L_n. Moreover, because the maps in $L_n(I,I)$ all share the same source and target, what we have is not just a monoid under tensor product but also under composition as well. This fact leads to a series of special properties for $L_n(I,I)$, the first of which is just another instance of the classic Eckmann-Hilton argument. QQQ Reference here?

Proposition 3.2.1. *$L_n(I,I)$ is a commutative monoid under both tensor product and composition, with $f \otimes f' = f \circ f'$. Since L_n is a groupoid, this commutative monoid is actually an abelian group.*

Indeed, by using a slightly broader argument we can extend this result to every morphism of L_n.

Definition 3.2.2. A morphism $f : w \to v$ in a monoidal category X is *invertible under tensor product* or *has an inverse under tensor product* if there is another morphism $g : w^{-1} \to v^{-1}$ such that $f \otimes g = \mathrm{id}_I = g \otimes f$; note that this requires w, v to both be invertible with inverses w^{-1}, v^{-1}, respectively.

QQQ Change $y*$ to y^{-1}, but keep $f*$ as the tensor inverse of f

Proposition 3.2.3. *Every morphism $f : w \to v$ in L_n has an inverse under tensor product, $f^* : w^* \to v^*$. That is, the monoid $\mathrm{Mor}(L_n)$ is actually a group.*

PROOF. For any $f : w \to v$ in L_n, consider the map $\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$, where f^{-1} is the compositional inverse of f, as in the proof of ??. This morphism has source $w^* \otimes v \otimes v^* = w^*$ and target $w^* \otimes w \otimes v^* = v^*$, which allows us to apply the law of interchange to get

$$\begin{aligned} f \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) &= \big(f \circ \mathrm{id}_w\big) \otimes \big(\mathrm{id}_{v^*} \circ (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*})\big) \\ &= \big(f \otimes \mathrm{id}_{w^*}\big) \circ \big(\mathrm{id}_w \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*})\big) \\ &= (f \otimes \mathrm{id}_{v^*}) \circ (f^{-1} \otimes \mathrm{id}_{v^*}) \\ &= \mathrm{id}_I \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes f &= \big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \circ \mathrm{id}_{w^*}\big) \otimes \big(\mathrm{id}_v \circ f\big) \\ &= \big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes \mathrm{id}_v\big) \circ \big(\mathrm{id}_{w^*} \otimes f\big) \\ &= (\mathrm{id}_{w^*} \otimes f^{-1}) \circ (\mathrm{id}_{w^*} \otimes f) \\ &= \mathrm{id}_I, \end{aligned}$$

so $f^* := \mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$ is the inverse of f in the monoid $\mathrm{Mor}(L_n)$. □

Proposition 3.2.4. *$L_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L_n)$. Moreover, if Λ is a crossed action operad, then $L_n(I,I)$ is a subgroup of the center of $\mathrm{Mor}(L_n)$.*

PROOF. From ?? and Proposition 3.2.3, we know that $L_n(I,I)$ is a subgroup of $\mathrm{Mor}(L_n)$. For normality, we need to again consider both crossed and non-crossed action operads separately.

If G is non-crossed, then by Corollary 2.2.13 we know that the map assigning objects of L_n to their connected component is just the identity $\mathrm{id}_{\mathbf{Z}^{*n}}$. In other words, every object belongs to its own unique component, so that every morphism

of L_n is actually an endomorphism. It follows that the group $L_n(I,I)$ is the kernel of the source homomorphism s from Definition 3.1.1 and thus normal.

For crossed G, recall from ?? that all $E\Lambda$-algebras are spacial, and so in particular L_n is. This means that for any $h \in L_n(I,I)$ and $w \in \mathrm{Ob}(L_n)$ we will always have $h \otimes \mathrm{id}_w = \mathrm{id}_w \otimes h$. Thus for any $f : w \to v$ in $\mathrm{Mor}(L_n)$, we get

$$\begin{aligned} h \otimes f &= (\mathrm{id}_I \circ h) \otimes (f \circ \mathrm{id}_w) \\ &= (\mathrm{id}_I \otimes f) \circ (h \otimes \mathrm{id}_w) \\ &= (f \otimes \mathrm{id}_I) \circ (\mathrm{id}_w \otimes h) \\ &= (f \circ \mathrm{id}_w) \otimes (\mathrm{id}_I \circ h) \\ &= f \otimes h \end{aligned}$$

and so $L_n(I,I)$ is a subgroup of the center of $\mathrm{Mor}(L_n)$, thus normal. □

3.3. Splitting the monoid of morphisms

In this section we will collect together many of the arguments of this chapter to give the first closed form expression for the automorphism group of the unit object, I, in L_n.

Proposition 3.3.1. *For any action operad G,*

$$\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \ltimes L_n(I,I)$$

Moreover, if G is a crossed action operad, then

$$\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \times L_n(I,I)$$

PROOF. We just saw in Proposition 3.2.4 that $L_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L_n)$, so we can consider the quotient group

$$L_n(I,I) \hookrightarrow \mathrm{Mor}(L_n) \longrightarrow \mathrm{Mor}(L_n)/L_n(I,I)$$

By the universal property of quotients, the map $\mathrm{Mor}(L_n) \to \mathrm{Mor}(L_n)/L_n(I,I)$ will uniquely factor any homomorphism whose composite with the inclusion $L_n(I,I) \hookrightarrow \mathrm{Mor}(L_n)$ is the zero map. But our source/target map $s \times t : \mathrm{Mor}(L_n) \to (s \times t)(L_n)$ is one such homomorphism, since for any $h : I \to I$ clearly $(s \times t)(h) = (I,I)$, which is the identity element in $(s \times t)(L_n)$. Therefore there must exist a unique homomorphism u making the triangle below commute:

$$\begin{tikzcd} \mathrm{Mor}(L_n) \arrow[d] \arrow[dr, "s \times t"] & \\ \mathrm{Mor}(L_n)/L_n(I,I) \arrow[r, "u"] & (s \times t)(L_n) \end{tikzcd}$$

This map u will be surjective — because $s \times t$ is — but in fact it will also be injective. This is because if two morphisms f, f' of L_n have the same source and target, then the map $h = f^* \otimes f'$ is an element of $L_n(I,I)$ for which $f \otimes h = f'$, and so clearly f and f' are part of the same equivalence class in $\mathrm{Mor}(L_n)/L_n(I,I)$.

Thus u is bijective, so that

$$\mathrm{Mor}(L_n)/L_n(I,I) \cong (s \times t)(L_n)$$

and we have a group extension

$$0 \longrightarrow L_n(I,I) \hookrightarrow \mathrm{Mor}(L_n) \xrightarrow{s \times t} (s \times t)(L_n) \longrightarrow 0.$$

But recall from Proposition 3.1.8 that this extension is split, or equivalently $\mathrm{Mor}(L_n)$ is a semi direct product $(s \times t)(L_n) \ltimes L_n(I,I)$. However, if G is crossed then we also saw in Proposition 3.2.4 that $L_n(I,I)$ is a subgroup of the center of $\mathrm{Mor}(L_n)$, and so it will follow that $\mathrm{Mor}(L_n)$ is also a central extension of $(s \times t)(L_n)$. In that case $\mathrm{Mor}(L_n)$ is just the direct product $(s \times t)(L_n) \times L_n(I,I)$. □

G: $L_n(I,I) \cong \mathrm{Mor}/s \times t$

Convention 3.3.2. In what follows QQQ forward ref all the places we need this?, we fix an isomorphism $\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \times L_n(I,I)$; we note that this isomorphism (and in particular the projection homomorphism $\mathrm{Mor}(L_n) \to L_n(I,I)$) depends on the choices made in the proof of Proposition 3.1.8.

3.4. Splitting as a coherence theorem

Coherence theorems often take the form of a statement that all diagrams with a certain property commute. For example, coherence for ordinary monoidal categories states that every diagram in the free monoidal category generated by a set of objects commutes and coherence for symmetric monoidal categories states that a pair of parallel morphisms (i.e., the two legs around some diagram of interest) are equal if and only if their underlying permutations are equal.

Not totally happy here.

THEOREM 3.4.1. *Let Λ be an action operad, and L_n the free Λ-monoidal category on n invertible objects. Let x, y be objects of L_n, and $f, g : x \to y$ morphisms between them. Then the following are equivalent:*

1. *$f = g$,*
2. *$g^{-1}f = \mathrm{id}_x$, and*
3. *the automorphism of the unit object $x^{-1} \otimes (g^{-1}f)$ is the identity in $L_n(I,I)$.*

PROOF. 1 implies 2 and 2 implies 3 are trivial. For 3 implies 1, we need only note that the invertibility of x means that the function $x \otimes - : L_n(I,I) \to L_n(x,x)$ is a monoid isomorphism (with inverse $x^{-1} \otimes -$) so sends identities to identities. □

In practice, computing $x^{-1} \otimes (g^{-1}f)$ from $g^{-1}f$ is relatively simple, so we should view this theorem as affording the following strategy for resolving the potential equality of a pair of parallel morphisms $f, g : x \to y$.

1. Determine the group $L_n(I,I)$ in such a way as to make the equality of two elements decidable.
2. Compute $x^{-1} \otimes (g^{-1}f)$ as an element of this group.

practical?

The following chapters are devoted to the first step of this strategy.

Start

CHAPTER 4

Computing automorphisms of the unit

Now we turn to the matter of computing the group $L_n(I, I)$. This group completely controls the morphisms which arise from the presence of n different invertible objects as we saw in Theorem 3.4.1. Our computations are necessarily of a different flavor from the theory we have so far established as the goal has changed from establishing the existence of this group to determining it explicitly with generators and relations.

4.1. Nullary operations

We will begin our computational investigations by first studying the nullary operations, the elements $g \in \Lambda(0)$.

Lemma 4.1.1. *For any element $g \in \Lambda(m), m \in \mathbb{N}$ of an action operad Λ, the morphism*

$$g^{\otimes} : I^m \to I^m$$

in L_n is the identity. Equivalently, for any element $h \in \Lambda(0)$, h induces the identity morphism $I \to I$.

PROOF. First, let $g \in \Lambda(m)$. Then $g : I^m \to I^m$ is equal to $q(g) : q(I^m) \to q(I^m)$ which is equal to $q\delta(g) : q\delta(I^m) \to q\delta(I^m)$. Since q is the coequalizer of δ and the zero map, $q\delta(g) = \text{id}$ for all $g \in \Lambda(m)$. This clearly implies that every $h \in \Lambda(0)$ induces the identity map $I \to I$, but note that the morphism $g : I^m \to I^m$ is also the morphism induced by $\mu(g; e_0, \ldots, e_0) \in \Lambda(0)$, so these two claims are equivalent. □

This is a curious result. The morphisms $I \to I$ in $E\Lambda(\underline{n})$ are exactly the elements of the group $\Lambda(0)$, but all of these become the identity in L_n. In other words, L_n cannot detect any morphisms in $\Lambda(0)$, an idea we make precise ~~now.~~ in the next two results

Proposition 4.1.2. *Let Λ be a crossed action operad. Then there exists another crossed action operad Λ' given by $\Lambda'(m) = \Lambda(m)/\Lambda(0)$ for all $m \in \mathbb{N}$.*

PROOF. For any elements $g \in \Lambda(m)$ and $h \in \Lambda(0)$, their tensor product $h \otimes g := \mu(e_2; h, g)$ is also an element of $\Lambda(m)$. This defines a map $\Lambda(0) \times \Lambda(m) \to \Lambda(m)$, which is both a group homomorphism and a group action by $\boxed{??}$. This produces a homomorphism $\Lambda(0) \to \Lambda(m)$ for all m which lies in the center of $\Lambda(m)$ by $\boxed{QQQ}$??, hence is normal for all m. Furthermore, the induced map $\Lambda(0) \to \Lambda(m) \to \Sigma_m$ is the zero map, so we have an induced homomorphism $\Lambda(m)/\Lambda(0) \to \Sigma_m$. All that remains to be shown is that we the operadic multiplication for Λ induces one for the groups $\Lambda(n)/\Lambda(0)$, and that this multiplication satisfies the axioms for an action operad.

Let $h, h_1, \ldots, h_m \in \Lambda(0)$ and $k_1, \ldots, k_m \in \mathbb{N}$. We have the following calculation using the action operad axioms for Λ and the fact that $E\Lambda(\underline{1})$ is spacial (see ??) so the e_k commute with all elements of $\Lambda(0)$.

$$\begin{aligned}
\mu^{\Lambda}(h \otimes e_m ; h_1 \otimes e_{k_1}, \ldots, h_m \otimes e_{k_m}) &= \mu^{\Lambda}\big(\mu^{\Lambda}(e_2; h, e_m) ; h_1 \otimes e_{k_1}, \ldots, h_m \otimes e_{k_m}\big) \\
&= \mu^{\Lambda}\big(e_2 ; \mu^{\Lambda}(h; -), \mu^{\Lambda}(e_m; h_1 \otimes e_{k_1}, \ldots, h_m \otimes e_{k_m})\big) \\
&= \mu^{\Lambda}(h; -) \otimes \mu^{\Lambda}(e_m; h_1 \otimes e_{k_1}, \ldots, h_m \otimes e_{k_m}) \\
&= h \otimes h_1 \otimes e_{k_1} \otimes \ldots \otimes h_m \otimes e_{k_m} \\
&= e_{k_1} \otimes \ldots \otimes e_{k_m} \otimes h \otimes h_1 \ldots \otimes h_m \\
&= e_{k_1 + \ldots + k_m} \otimes h \otimes h_1 \ldots \otimes h_m
\end{aligned}$$

the above following square

~~This~~ shows that the ~~square below~~ commutes.

$$\begin{array}{ccc}
\Lambda(0) \times \Lambda(0) \times \ldots \times \ldots \Lambda(0) & \xrightarrow{\otimes} & \Lambda(0) \\
\downarrow & & \downarrow \\
\Lambda(m) \times \Lambda(k_1) \times \ldots \times \Lambda(k_m) & \xrightarrow[\mu]{} & \Lambda(k_1 + \ldots + k_m)
\end{array}$$

We write the image of $g \in \Lambda(m)$ under the quotient $\Lambda(m) \to \Lambda'(m)$ as $[g]$. We will show that the multiplication

$$\mu^{\Lambda'} : \Lambda'(m) \times \Lambda'(k_1) \times \cdots \times \Lambda'(k_m) \to \Lambda'(k_1 + \cdots + k_m)$$

given by

$$\mu^{\Lambda'}\big([g]; [g_1], \ldots, [g_m]\big) = [\mu^{\Lambda}\big(g; g_1, \ldots, g_m\big)]$$

is well-defined. With $h, h_1, \ldots, h_m \in \Lambda(0)$ as above, we have ?

$$\mu^{\Lambda}\big(g \cdot h \otimes e_m; g_1 \cdot h_1 \otimes e_{k_1}, \ldots, g_m \cdot h_m \otimes e_{k_m}\big) = \mu^{\Lambda}(g; g_1, \ldots, g_m) \cdot \mu^{\Lambda}(h \otimes e_m; \cdot h_1 \otimes e_{k_1}, \ldots, \cdot h_m \otimes e_{k_m})$$

using the fact that $\pi(h \otimes e_m)$ is the identity permutation. By the commutative square above, $\mu^{\Lambda}(h \otimes e_m; \cdot h_1 \otimes e_{k_1}, \ldots, \cdot h_m \otimes e_{k_m})$ is in the image of $\Lambda(0) \to \Lambda(k_1 + \cdots + k_m)$, so $\mu^{\Lambda'}$ is well-defined. It is now straightforward to check the rest of the action operad axioms for Λ' using those of Λ. □

We end this section by proving that the quotient action operad Λ' removes the unnecessary nullary operations without changing the invertible objects.

THEOREM 4.1.3. *Let Λ be a crossed action operad, and let Λ' be the action operad with $\Lambda'(m) = \Lambda(m)/\Lambda(0)$ constructed in Proposition 4.1.2. Then for any $n \in \mathbb{N}$,*

$$L\Lambda'_n \cong L\Lambda_n$$

both as $E\Lambda$-algebras and as $E\Lambda'$-algebras.

PROOF. The quotient map $[-] : \Lambda \to \Lambda'$ induces a Λ-algebra structure on $L\Lambda'_n$, and hence produces a unique strict Λ-monoidal functor $L\Lambda_n \to L\Lambda'_n$ by the universal property of $L\Lambda_n$. $L\Lambda_n$ is also a Λ'-monoidal category by defining $[g]^{\otimes} : x_1 \otimes \cdots \otimes x_n \to x_{[g]^{-1}1} \otimes \cdots \otimes x_{[g]^{-1}n}$ to be $g^{\otimes}$. This is well-defined, as whenever we have $[g] = [h]$ it is because there is some $x \in \Lambda(0)$ for which $h = x \otimes g = \mu_2(e_2; x, g)$. Then

$$h^{\otimes} = (x \otimes g)^{\otimes} = x^{\otimes} \otimes g^{\otimes} = \text{id}_I \otimes g^{\otimes} = g^{\otimes}$$

Check 4n vs 2n

using Lemma $\boxed{2.3.10}$. Thus we also have an induced map $L\Lambda'_n \to L\Lambda_n$ using the universal property of $L\Lambda'_n$. By inspection, both of these functors preserve the Λ- and Λ'-actions, hence must be inverse to each other. □

4.2. Reflexivity and a first calculation

say $\tilde{\delta}, \tilde{I}$

Recall that the functor $q : E\Lambda(\underline{2n}) \to L_n$ expresses L_n as a reflexive coequalizer of Λ-monoidal categories and hence also as a reflexive coequalizer of mere monoidal categories by Corollary 2.4.8. We can therefore use the adjunction from Proposition 2.2.20 to calculate $\mathrm{M}(L_n)^{\mathrm{gp,ab}}$.

Proposition 4.2.1. *Let Δ be the subgroup of $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$ generated by elements of the form*

$$\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) - \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f). \qquad f \in \mathrm{M}(E\Lambda(\underline{4n}))^{\mathrm{gp,ab}}$$

Then the abelianization of the group completion of the collapsed morphisms of L_n is computed as the quotient below.

$$\mathrm{M}(L_n)^{\mathrm{gp,ab}} \cong \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \Delta$$

PROOF. From Proposition 2.2.20, we know that $\mathrm{M}(\_)^{\mathrm{gp,ab}} : \mathbf{MonCat} \to \mathbf{Ab}$ is a left adjoint ~~functor~~. This means that it preserves all colimits in $\mathbf{MonCat}$, so

$$\mathrm{coeq}\big(\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}, \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}\big) \cong \mathrm{M}\big(\mathrm{coeq}(\tilde{\delta}, \tilde{I})\big)^{\mathrm{gp,ab}}.$$

The coequalizer of two abelian group homomorphisms is just the quotient of their common target by the image of their difference. Hence in this case we have

$$\begin{aligned}
\mathrm{M}(L_n)^{\mathrm{gp,ab}} &\cong \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \mathrm{im}\big(\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}} - \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}\big) \\
&\cong \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \Delta.
\end{aligned}$$

□

Purpose?

Remark 4.2.2. Note that $\mathrm{im}(\mathrm{M}(\delta)^{\mathrm{gp,ab}})$ is a subgroup of Δ because δ factors through $\tilde{\delta}$.

~~QQQ Rewrite with better notation~~

Our goal is to explicitly compute the group acting on a set of n invertible objects in a Λ-monoidal category, so we will need an explicit description of the elements of the subgroup Δ.

Improve

Lemma 4.2.3. *Δ is the subgroup of $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$ whose elements are tensor products of equivalence classes*

$$\big[\alpha^{E\Lambda(\underline{2n})}\big(\mu(g; e_{|\tilde{\delta}(x_1)|}, \ldots, e_{|\tilde{\delta}(x_m)|}); \mathrm{id}_{x'_1}, \ldots, \mathrm{id}_{x'_{m'}}\big)\big]$$
$$\otimes$$
$$\big[\alpha^{E\Lambda(\underline{2n})}\big(\mu(g; e_{|\tilde{I}(x_1)|}, \ldots, e_{|\tilde{I}(x_m)|}); \mathrm{id}_{x''_1}, \ldots, \mathrm{id}_{x''_{m''}}\big)\big]^*$$

where $g \in G(m)$, the x_i are generators of $\mathbb{N}^{*4n}$, the x'_i, x''_i are generators of $\mathbb{N}^{*2n}$, and

$$\begin{aligned}
\tilde{\delta}(x_1 \otimes \ldots \otimes x_m) &= x'_1 \otimes \ldots \otimes x'_{m'} \\
\tilde{I}(x_1 \otimes \ldots \otimes x_m) &= x''_1 \otimes \ldots \otimes x''_{m''}
\end{aligned}$$

PROOF. Let f be an element of $\mathrm{M}(\mathbb{G}_{4n})^{\mathrm{gp,ab}}$. By definition this means that f is an equivalence class of morphisms from $\mathbb{G}_{4n}$, and so by ?? there must exist $g \in G(m)$ and $x_1, \ldots, x_m \in \{z_1, \ldots, z_{4n}\}$ for which

$$f = [\alpha^{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \ldots, \mathrm{id}_{x_m})]$$

Thus

$$\begin{aligned}
\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) &= \mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}\big([\alpha^{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \ldots, \mathrm{id}_{x_m})]\big) \\
&= \big[\tilde{\delta}\big(\alpha^{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \ldots, \mathrm{id}_{x_m})\big)\big] \\
&= \big[\alpha^{E\Lambda(\underline{2n})}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, \ldots, \mathrm{id}_{\tilde{\delta}(x_m)})\big]
\end{aligned}$$

But again using ??, we know it must be possible to express the action morphism $\alpha^{E\Lambda(\underline{2n})}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, \ldots, \mathrm{id}_{\tilde{\delta}(x_m)})$ as an action morphism on the identities of generators. Since the source of this map is

$$\tilde{\delta}(x_1) \otimes \ldots \otimes \tilde{\delta}(x_m) = \tilde{\delta}(x_1 \otimes \ldots \otimes x_m) =: x'_1 \otimes \ldots \otimes x'_{m'}$$

clearly the x'_i are the generators we want, and so by expanding the $\tilde{\delta}(x_i)$ as tensor products of these we find that

$$\big[\alpha^{E\Lambda(\underline{2n})}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, \ldots, \mathrm{id}_{\tilde{\delta}(x_m)})\big] = \big[\alpha^{E\Lambda(\underline{2n})}\big(\mu(g; e_{|\tilde{\delta}(x_1)|}, \ldots, e_{|\tilde{\delta}(x_m)|}); \mathrm{id}_{x'_1}, \ldots, \mathrm{id}_{x'_{m'}}\big)\big]$$

For analogous reasons we also get

$$\begin{aligned}
\mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f) &= \big[\alpha^{E\Lambda(\underline{2n})}(g; \mathrm{id}_{\tilde{I}(x_1)}, \ldots, \mathrm{id}_{\tilde{I}(x_m)})\big] \\
&= \big[\alpha^{E\Lambda(\underline{2n})}\big(\mu(g; e_{|\tilde{I}(x_1)|}, \ldots, e_{|\tilde{I}(x_m)|}); \mathrm{id}_{x''_1}, \ldots, \mathrm{id}_{x''_{m''}}\big)\big]
\end{aligned}$$

and using these equations the lemma follows immediately from the definition of Δ. □

4.3. Consequences of the splitting

In this section we will combine our splitting result (Proposition 3.3.1) with the quotients in the previous section.

Corollary 4.3.1. *Let G be a crossed action operad. Then the endomorphisms of the unit object of L_n are*

$$L_n(I, I) \cong \mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ (s \times t)(L_n)^{\mathrm{ab}}$$

and therefore

$$\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \times \mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ (s \times t)(L_n)^{\mathrm{ab}}.$$

PROOF. Both statements follow from the simple fact that abelianization preserves products and quotients. □

Lemma 4.3.2. *Let X be any monoidal category whose objects and morphisms are all invertible under tensor product (see Definition 3.2.2 and Proposition 3.2.3). Then the monoid of collapsed morphisms $\mathrm{M}(X)$ is already a group, and its abelianization is isomorphic to $\mathrm{Mor}(X)^{\mathrm{ab}}/\mathrm{Ob}(X)^{\mathrm{ab}}$.*

PROOF. Let $f : x \to y$, $f' : y \to z$ be a composable pair of morphisms in X. Recall that in any monoidal category with invertible objects,

$$f' \circ f = f' \otimes \mathrm{id}_{y^{-1}} \otimes f$$

Justify

by Lemma 2.2.14, so both Mor(X) and M(X) are groups. Consider the canonical homomorphism $\psi : \mathrm{Mor}(X) \to \mathrm{M}(X) \to \mathrm{M}(X)^{\mathrm{ab}}$. Then we have

Hmm ...

See back 27/28

$$\begin{aligned}\psi(f' \otimes f) &= \psi(f' \circ f)\\ &= \psi(f' \otimes \mathrm{id}_{y^{-1}} \otimes f)\\ &= \psi(f') \otimes \psi(\mathrm{id}_{y^{-1}}) \otimes \psi(f)\\ &= \psi(f') \otimes \psi(f) \otimes \psi(\mathrm{id}_{y^{-1}})\\ &= \psi(f' \otimes f) \otimes \psi(\mathrm{id}_{y^{-1}})\end{aligned}$$

$\psi(\mathrm{id}_y) = \psi(\mathrm{id}_y \circ \mathrm{id}_y)$
$= \psi(\mathrm{id}_y \otimes \mathrm{id}_{y^{-1}} \otimes \mathrm{id}_y)$

so $\psi(\mathrm{id}_{y^{-1}}) = e$ for any object y, and Ob(X) is contained in the kernel of ψ.

Now let A be an abelian group and $\phi : \mathrm{Mor}(X)^{\mathrm{ab}} \to A$ any homomorphism of groups which satisfies the condition $\phi(\mathrm{ab}(\mathrm{id}_x)) = e$ 0 zero for all objects x. Then

$$\begin{aligned}\phi\big(\,\mathrm{ab}(f' \circ f)\big) &= \phi\big(\,\mathrm{ab}(f' \otimes \mathrm{id}_{y^{-1}} \otimes f)\big)\\ &= \phi\big(\,\mathrm{ab}(f')\big) \otimes \phi\big(\,\mathrm{ab}(\mathrm{id}_{y^{-1}})\big) \otimes \phi\big(\,\mathrm{ab}(f)\big)\\ &= \phi\big(\,\mathrm{ab}(f')\big) \otimes \phi\big(\,\mathrm{ab}(f)\big)\\ &= \phi\big(\,\mathrm{ab}(f' \otimes f)\big),\end{aligned}$$

but using Lemma 2.2.22 it is easy to see that this is the defining relation for the group $\mathrm{M}(X)^{\mathrm{ab}}$, so $\mathrm{M}(X)^{\mathrm{ab}}$ and $\mathrm{Mor}(X)^{\mathrm{ab}}/\mathrm{Ob}(X)^{\mathrm{ab}}$ have the same universal property and are therefore uniquely isomorphic under $\mathrm{Mor}(X)^{\mathrm{ab}}$. ☐

Here is the main theorem of this section, a first description of $L_n(I, I)$.

THEOREM 4.3.3. *For any crossed action operad* Λ,

$$L_n(I,I) \cong \frac{\left(\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)},$$

and the monoid of morphisms of L_n *is therefore*

$$\mathrm{Mor}(L_n) \quad\cong\quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \frac{\left(\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)}.$$

PROOF. We have the isomorphism

$$L_n(I,I) \quad\cong\quad \mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ (s \times t)(L_n)^{\mathrm{ab}}$$

from Corollary 4.3.1. Note that this quotient depends on the embedding of $(s \times t)(L_n)$ as a subgroup of the morphisms L_n in Proposition 3.1.8. The proof there ensured that the embedding we construct commutes with the inclusion of the monoid of objects, so

$$\mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ (s \times t)(L_n)^{\mathrm{ab}} \quad\cong\quad \frac{\left(\mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ \mathrm{Ob}(L_n)^{\mathrm{ab}}\right)}{\left((s \times t)(L_n)^{\mathrm{ab}} \big/ \mathrm{Ob}(L_n)^{\mathrm{ab}}\right)}.$$

Check:

$$\begin{array}{ccc} s{\times}t/\mathrm{ob} & \cong & (\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}/\mathbb{Z}^n \\ \downarrow & & \downarrow \\ \mathrm{Mor}/\mathrm{ob} & \cong & \mathrm{M}/\Delta \end{array}$$

(*)

Need

$$\begin{array}{ccccc} \mathrm{ob} & \hookrightarrow & s{\times}t & \longrightarrow & s{\times}t/\mathrm{ob} \\ \| & & \downarrow & & \downarrow \\ \mathrm{ob} & \hookrightarrow & \mathrm{Mor} & \longrightarrow & \mathrm{Mor}/\mathrm{ob} \\ & & & & \downarrow \\ & & & & \mathrm{Mor}/\mathrm{ob} \big/ s{\times}t/\mathrm{ob} \end{array}$$

Using Lemma 4.3.2 to change the numerator and Corollaries 2.2.10 and 3.1.3 to simplify the denominator, this quotient becomes

$$\mathrm{Mor}(L_n)^{\mathrm{ab}} \big/ (s \times t)(L_n)^{\mathrm{ab}} \cong \frac{\left(\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)} \quad (*)$$

as desired. ☐

It should be clear that there is further work to do in order to give a usable construction of the category L_n and the group $L_n(I, I)$. We will spend the next two sections performing ~~some~~ the necessary algebra to interpret the complicated parts of the formula above, namely $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$ and $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$.

4.4. Calculation: abelianizing sources and targets

Redo there!

Our first calculation will give a presentation for $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$. We were able to prove that $\mathbb{N}^{*n} \times_{\mathbb{Z}^N} \mathbb{N}^{*n}$ is a free monoid, but one cannot then conclude that $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ is a free group since group completion is a left adjoint and does not commute with pullbacks. Abelianization is also a left adjoint, and we cannot commute it past the pullback for the same reason. Thus we give a presentation directly. For the reader uninterested in the algebra involved, we record the main results of this section now:

- we prove
$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad\cong\quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$
in Proposition 4.4.2 and
- we prove
$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n \quad\cong\quad \mathbb{Z}^{\binom{n}{2}}$$
in Corollary 4.4.3.

Proposition 4.4.1. *The pullback group* $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ *is generated by two families of elements,*

$$\langle x \rangle \quad := \quad (x, x) \qquad \textit{and} \qquad \langle xy \rangle \quad := \quad (xy, yx)$$

where $x, y \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$ *are generators of the free group* $\mathbb{Z}^{*n}$ *or their inverses. These are subject to the relations*

$$\langle x \rangle^{-1} \quad=\quad \langle x^* \rangle, \qquad \langle xy \rangle^{-1} \quad=\quad \langle y^* x^* \rangle$$

$$\langle x x^* \rangle \quad=\quad e \quad=\quad \langle x^* x \rangle, \qquad \langle xx \rangle \quad=\quad \langle x \rangle^2$$

$$\langle xy \rangle \langle x^* \rangle \langle xy^* \rangle \quad=\quad \langle x \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle \quad=\quad \langle x \rangle \langle y \rangle \quad=\quad \langle yx \rangle \langle x^* \rangle \langle y^* \rangle \langle xy \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y \rangle \langle x^* \rangle \langle z^* \rangle^{-1} \langle zx \rangle \langle z^* \rangle \langle zy \rangle \quad=\quad \langle x \rangle \langle y \rangle \langle z \rangle$$

PROOF. We'll begin by constructing a certain monoidal category, which we'll call Z.

- The objects of Z are the elements of the group $\mathbb{Z}^{*n}$, with the usual multiplication as the tensor product.
- There is a unique morphisms between any two objects x and y for which $\mathrm{ab}(x) = \mathrm{ab}(y)$, where $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$ is the quotient map of abelianization. In other words, the morphisms of Z are the elements of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, with multiplication as the tensor product and composition given by
$$(x, y) \circ (y, z) \quad=\quad (x, z)$$
- The identity map on an object x is then the unique map $(x, x) : x \to x$.

Z is almost the subcategory of L_n whose morphisms are the subgroup isomorphic to $(s \times t)(L_n)$ that we chose in Proposition 3.1.8. However, we never required those representatives to be closed under composition, so Z is a strictly formal version of the subcategory on $(s \times t)(L_n)$, one that doesn't involve any specific choice of the map ρ. It is a well-defined monoidal category: the only thing that might not be immediately clear is the law of interchange, which is just given by

$$\begin{aligned}\big((x, y) \circ (y, z)\big) \otimes \big((x', y') \circ (y', z')\big) &= (x, z) \otimes (x', z')\\ &= (xx', zz')\\ &= (xx', yy') \circ (yy', zz')\\ &= \big((x, y) \otimes (x', y')\big) \circ \big((y, z) \otimes (y', z')\big)\end{aligned}$$

But now recall from Lemma 2.2.14 that in any monoidal category the composition of morphisms along an intertible object can be rewritten in terms of only the tensor product. In the case of Z, where all of the objects have inverses, we will have

$$(x, y) \circ (y, z) \quad=\quad (x, y) \otimes (y^*, y^*) \otimes (y, z)$$

Using this composition operation will make it easier to understand the structure of the group $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$.

Next, let $\mathbb{S}_{2n}$ be the free ES-algebra on $2n$ objects, where S is the symmetric action operad. Then there is an obvious monoidal functor $\psi : \mathbb{S}_{2n} \to Z$, given by

$$\begin{array}{rrcl} \psi : & \mathbb{S}_{2n} & \to & Z \\ : & z_i & \mapsto & z_i \\ : & z_{n+i} & \mapsto & z_i^* \\ : & \alpha(\sigma; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) & \mapsto & (x_1 \otimes ... \otimes x_m, x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}) \end{array}$$

A necessary condition for (y, y') to be an element of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ is that there exists some sequence of generators and their inverses $x_1, ..., x_m \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$ and some permutation $\sigma \in S_m$ for which

$$y \;=\; x_1 \otimes ... \otimes x_m, \qquad y' \;=\; x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}$$

Hence the functor ψ is clearly surjective. It follows from this that if we can find a collection of morphisms which generate $\mathrm{Mor}(\mathbb{S}_{2n})$ under composition and tensor product, their images under ψ will also generate $\mathrm{Mor}(Z) = \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ under composition and tensor product, and hence under just tensor product. To begin, we know that any permutation $\sigma \in S_m$ can be written as a product $\sigma_{i_k} \cdot ... \cdot \sigma_{i_1}$ of elementary transpositions, giving

$$\begin{aligned}\alpha(\,\sigma\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) &= \alpha(\,\sigma_{i_k} \cdot ... \cdot \sigma_{i_1}\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,)\\ &= \alpha(\,\sigma_{i_1}\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,)\\ &\quad \circ \alpha(\,\sigma_{i_2}\,;\,\mathrm{id}_{x_{\sigma_{i_1}(1)}}, ..., \mathrm{id}_{x_{\sigma_{i_1}(m)}}\,) \circ ...\\ &\quad \circ \alpha(\,\sigma_{i_k}\,;\,\mathrm{id}_{x_{\sigma_{i_{k-1}} \cdots \sigma_{i_1}(1)}}, ..., \mathrm{id}_{x_{\sigma_{i_{k-1}} \cdots \sigma_{i_1}(m)}}\,)\end{aligned}$$

Then if $\sigma_i = (i\ i+1) \in S_m$ is some elementary transposition we will have

$$\begin{aligned}\alpha(\,(i\ i+1)\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) &= \alpha(\,c_{i-1} \otimes (12) \otimes c_{m-i-1}\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,)\\ &= \mathrm{id}_{x_1 \otimes ... \otimes x_{i-1}} \otimes \alpha(\,(12)\,;\,\mathrm{id}_{x_i}, \mathrm{id}_{x_{i+1}}\,) \otimes \mathrm{id}_{x_{i+2} \otimes ... \otimes x_m}\end{aligned}$$

Therefore all of the morphisms of $\mathbb{S}_{2n}$ are generated by just the identities and the action maps $\alpha(\,(12)\,;\,\mathrm{id}_{x_1}, \mathrm{id}_{x_2}\,)$ for all pairs $x_1, x_2 \in \{z_1, ..., z_{2n}\}$. Passing through ψ, this means that elements of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ can always be expressed as a tensor product of elements of the form

$$(x, x) \qquad \text{or} \qquad (xy, yx), \qquad\qquad x, y \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$$

These are exactly the $\langle x \rangle$ and $\langle xy \rangle$ given in the statement of the proposition.

Now we need to consider what relations these generators will obey. Firstly, their definitions overlap in the following case:

$$\langle xx \rangle \quad=\quad (xx, xx) \quad=\quad (x, x) \otimes (x, x) \quad=\quad \langle x \rangle \langle x \rangle$$

Next we have to account for the law of interchange we discussed earlier. Using Lemma 2.2.14, we see that this condition will induce the following relation:

$$\begin{aligned}\langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle &= (xy, yx) \otimes (x^*, x^*) \otimes (y^*, y^*) \otimes (yx, xy)\\ &= (xy, yx) \otimes (yx, yx)^* \otimes (yx, xy)\\ &= (xy, yx) \circ (yx, xy)\\ &= (yx, xy) \otimes (yx, yx)^* \otimes (yx, xy)\\ &= (yx, xy) \otimes (x^*, x^*) \otimes (y^*, y^*) \otimes (xy, yx)\\ &= \langle yx \rangle \langle x^* \rangle \langle y^* \rangle \langle xy \rangle\end{aligned}$$

Also, by functoriality these generators will inherit any relations are obeyed to the corresponding morphisms of $\mathbb{S}_{2n}$, which in turn are just relations among different elementary transpositions. Each symmetric group S_m is subject to three families of these, namely

$$\begin{aligned}(\sigma_i)^2 &= e\\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \qquad j \neq i \pm 1\\ (\sigma_i \sigma_{i+1})^3 &= e\end{aligned}$$

The first one, the symmetry condition, corresponds to the relation

$$\begin{array}{lrcl} & (xy, yx) \circ (yx, xy) & = & (xy, xy) \\ \Longrightarrow & (xy, yx) \otimes (yx, yx)^* \otimes (yx, xy) & = & (x, x) \otimes (y, y) \\ \Longrightarrow & (xy, yx) \otimes (x^*, x^*) \otimes (y^*, y^*) \otimes (yx, xy) & = & (x, x) \otimes (y, y) \\ \Longrightarrow & \langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle & = & \langle x \rangle \langle y \rangle \end{array}$$

The second relation is just an example of interchange, which we have already looked at. The third yields

$$\begin{array}{r} (xy, yx)(x^*, x^*)(xz, zx)(x^*, x^*)(z^*, z^*)(y^*, y^*)(yz, zy) \\ (y^*, y^*)(yx, xy)(y^*, y^*)(x^*, x^*)(z^*, z^*)(zx, xz)(z^*, z^*)(zy, yz) \quad=\quad (x, x)(y, y)(z, z) \end{array}$$

or more simply,

$$\langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y^* \rangle \langle x^* \rangle \langle z^* \rangle \langle zx \rangle \langle z^* \rangle \langle zy \rangle \quad=\quad \langle x \rangle \langle y \rangle \langle z \rangle$$

Finally, we need to check how the invertibility of the objects of Z interacts with these generators. Most obviously, we have

$$\begin{aligned}\langle x \rangle^{-1} &= (x, x)^* &&= (x^*, x^*) &&= \langle x^* \rangle\\ \langle xy \rangle^{-1} &= (xy, yx)^* &&= (y^* x^*, x^* y^*) &&= \langle y^* x^* \rangle\end{aligned}$$

$$\begin{aligned} \langle xx^* \rangle &= (xx^*, x^*x) = (I, I) = e \\ \langle x^*x \rangle &= (x^*x, xx^*) = (I, I) = e \end{aligned}$$

But we can also insert an element and its inverse into different points of the source and target:

$$\begin{aligned} \langle x \rangle &= (x, x) \\ &= (xyy^*, yy^*x) \\ &= (xyy^*, yxy^*) \circ (yxy^*, yy^*x) \\ &= (xyy^*, yxy^*) \otimes (yxy^*, yxy^*)^* \otimes (yxy^*, yy^*x) \\ &= (xy, yx) \otimes (y^*, y^*) \otimes (y, y) \otimes (x, x)^*(y^*, y^*) \otimes (y, y) \otimes (xy^*, y^*x) \\ &= \langle xy \rangle \langle x^* \rangle \langle xy^* \rangle \end{aligned}$$

The relations $(xy, yx) = (zz^*xy, yzz^*x)$ and so forth are all composed of successive instance of the above, so these are all of the relations on our generators $\langle x \rangle$ and $\langle xy \rangle$. □

Proposition 4.4.2.

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \cong \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

PROOF. It follows immediately from Proposition 4.4.1 that the group $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$ has a presentation on generators

$$\langle x \rangle, \quad \langle xy \rangle, \quad x, y \in \{z_1, \dots, z_n, z_1^*, \dots, z_n^*\}$$

subject to the relations

$$\langle x \rangle^{-1} = \langle x^* \rangle, \qquad \langle xy \rangle^{-1} = \langle y^* x^* \rangle$$

$$\langle xx^* \rangle = e = \langle x^* x \rangle, \qquad \langle xx \rangle = \langle x \rangle^2$$

$$\langle xy \rangle \langle x^* \rangle \langle xy^* \rangle = \langle x \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle = \langle x \rangle \langle y \rangle = \langle yx \rangle \langle x^* \rangle \langle y^* \rangle \langle xy \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y^* \rangle \langle x^* \rangle \langle z^* \rangle \langle zx \rangle \langle z^* \rangle \langle zy \rangle = \langle x \rangle \langle y \rangle \langle z \rangle$$

but then also the commutativity conditions

$$\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle, \qquad \langle x \rangle \langle yz \rangle = \langle z \rangle \langle xy \rangle, \qquad \langle wx \rangle \langle yz \rangle = \langle yz \rangle \langle wx \rangle$$

Rearranging all of the former equations with the latter in mind, we get

$$\langle x \rangle^{-1} = \langle x^* \rangle, \qquad \langle xy \rangle^{-1} = \langle y^* x^* \rangle$$

$$\langle xx^* \rangle = e = \langle x^* x \rangle. \qquad \langle xx \rangle = \langle x \rangle^2 = \langle xy \rangle \langle xy^* \rangle$$

$$\langle xy \rangle \langle yx \rangle = \langle x \rangle^2 \langle y \rangle^2$$

$$\langle xy \rangle \langle yx \rangle \langle xz \rangle \langle zx \rangle \langle yz \rangle \langle zy \rangle = \langle x \rangle^4 \langle y \rangle^4 \langle z \rangle^4$$

The last of these relations is just a consequence of the one above that,

$$\begin{aligned} \langle xy \rangle \langle yx \rangle \langle xz \rangle \langle zx \rangle \langle yz \rangle \langle zy \rangle &= \big(\langle x \rangle^2 \langle y \rangle^2 \big) \big(\langle x \rangle^2 \langle z \rangle^2 \big) \big(\langle y \rangle^2 \langle y \rangle^2 \big) \\ &= \langle x \rangle^4 \langle y \rangle^4 \langle z \rangle^4 \end{aligned}$$

and in turn, the second-to-last follows from the relation above it.

$$\begin{aligned} \langle x \rangle^2 \langle y \rangle^2 &= \big(\langle xy \rangle \langle xy^* \rangle \big) \big(\langle yx \rangle \langle yx^* \rangle \big) \\ &= \langle xy \rangle \langle yx \rangle \langle xy^* \rangle \langle xy^* \rangle^{-1} \\ &= \langle xy \rangle \langle yx \rangle \end{aligned}$$

Without these, we are just left with equations in two or fewer variables. Then for any two $z_i, z_j \in \mathbb{Z}^{*n}$, $i < j$, the first three relations tell us that we only need to consider generators of the form

$$\langle z_i \rangle, \quad \langle z_j \rangle, \quad \langle z_i z_j \rangle, \quad \langle z_i^* z_j \rangle, \quad \langle z_i z_j^* \rangle, \quad \langle z_i^* z_j^* \rangle$$

Finally, the remaining relation $\langle x \rangle^2 = \langle xy \rangle \langle xy^* \rangle$ induces a system of four linear equations on these six generators, which can be solved to give

$$\begin{aligned} \langle z_i^* z_j \rangle &= \langle z_j \rangle^2 \langle z_i z_j \rangle^{-1} \\ \langle z_i z_j^* \rangle &= \langle z_i \rangle^2 \langle z_i z_j \rangle^{-1} \\ \langle z_i^* z_j^* \rangle &= \langle z_i \rangle^{-2} \langle z_j \rangle^{-2} \langle z_i z_j \rangle \end{aligned}$$

and three independent variables, $\langle z_i \rangle$, $\langle z_j \rangle$, and $\langle z_i z_j \rangle$. In other words, $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$ is the free abelian group generated by elements of this form, for $1 \le i < j \le n$, which means that

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

□

From this presentation, it should be immediately obvious how to calculate the denominator from ??.

Corollary 4.4.3.

$$\begin{aligned} (\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} / \mathbb{Z}^n &\cong \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}} / \mathbb{Z}^n \\ &\cong \mathbb{Z}^{\binom{n}{2}} \end{aligned}$$

PROOF. The $\mathbb{Z}^n$ term in the product of Proposition 4.4.2 represents the free abelian group generated by the morphisms

$$\langle x \rangle := (x, x) = \mathrm{id}_x$$

But this is exactly the same $\mathbb{Z}^n$ group that appears in the denominator of our quotient, $\mathrm{Ob}(L_n)^{\mathrm{ab}}$, so they cancel. □

QQQ I don't completely understand the following discussion.

Before moving on, we should be clear about exactly which $\mathbb{Z}^{\binom{n}{2}}$ subgroup of $\mathrm{M}(L_n)^{\mathrm{ab}}$ we have just identified — after all, we will eventually need to perform a quotient involving it. In Proposition 4.4.1 we defined the generators $\langle z_i z_j \rangle$ to be the elements $(z_i \otimes z_j, z_j \otimes z_i)$ of the monoid $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, which are the source/target combinations of morphisms of L_n. Using Proposition 3.1.8 we can identify this with a particular submonoid of the morphisms of L_n, specifically the image under q of the submonoid $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n} = (s \times t)(E\Lambda(\underline{2n})) \subseteq \mathrm{Mor}(E\Lambda(\underline{2n}))$ we chose in Proposition 3.1.6. In particular, since on objects we have $q(z_i) = z_i$ for all $1 \le i \le n$, the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ are clearly the image of the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$.

Thus, consider the following commutative diagram, whose top-left region comes from Proposition 3.1.8, bottom-left from the naturality of the adjoint functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$, and right-hand square from Lemma 4.3.2.

$$\begin{tikzcd}
& (s \times t)(E\Lambda(\underline{2n})) \arrow[rr, "q"] \arrow[dl] & & (s \times t)(L\mathbb{G}_n) \arrow[dl] \arrow[dr] & \\
\mathrm{Mor}(E\Lambda(\underline{2n})) \arrow[rr, "q"] \arrow[dr] & & \mathrm{Mor}(L_n) \arrow[dr] & & \dfrac{(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}{\mathrm{Ob}(L_n)^{\mathrm{ab}}} \arrow[dl] \\
& \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} \arrow[rr, "\mathrm{M}(q)^{\mathrm{gp,ab}}"] & & \mathrm{M}(L_n)^{\mathrm{gp,ab}} &
\end{tikzcd}$$

What we've just said that if we start with the element $(z_i \otimes z_j, z_j \otimes z_i)$ of $(s \times t)(E\Lambda(\underline{2n}))$, moving clockwise around the diagram will send it to the generator $\langle z_i z_j \rangle$ in $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} / \mathrm{Ob}(L_n)^{\mathrm{ab}} = \mathbb{Z}^{\binom{n}{2}}$. If we instead move anticlockwise, then we will first pass to our chosen representative $\alpha_{E\Lambda(\underline{2n})}(\rho(z_i \otimes z_j, z_j \otimes z_i); \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ in $\mathrm{Mor}(E\Lambda(\underline{2n}))$, then its equivalence class in $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$, then its equivalence class in $\mathrm{M}(L_n)^{\mathrm{gp,ab}}$, using the fact that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is the canonical map associated with the quotient

$$\mathrm{M}(L_n)^{\mathrm{gp,ab}} = \mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}} / \Delta$$

which we proved back in Proposition 4.2.1. Since the bottom-right inclusion completes this circuit, we see that the specific subgroup we are talking about in Corollary 4.4.3 is

$$\mathbb{Z}^{\binom{n}{2}} = \big\{ \big[\alpha_{E\Lambda(\underline{2n})} \big(\rho(z_i \otimes z_j, z_j \otimes z_i) : \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \big) \big] : 1 \le i < j \le n \big\} \subseteq \mathrm{M}(L_n)^{\mathrm{ab}}$$

Of course, ρ was an arbitrary permutation-preserving map $\mathbb{N}^{*n} \times_{\mathbb{N}} \mathbb{N}^{*n} \to G$, chosen using the freeness of its source monoid. Thus if we wanted to we could just pick the same element $\rho(2) \in \pi^{-1}((1\,2))$ to act as $\rho(z_i \otimes z_j, z_j \otimes z_i)$ for all i, j. For simplicity's sake, we will indeed be assuming this from now on.

4.5. Calculation: collapsed morphisms

The next group we are interested in understanding better is $\mathrm{M}(E\Lambda(\underline{2n}))^{\mathrm{gp,ab}}$. Per Lemma 2.2.22, the operations needed to produce this group out of $\mathrm{Mor}(E\Lambda(\underline{2n})) = \mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*2n}$ can be done in any order we choose, and so we will save the identification of $\otimes$ and $\circ$ until last. This will let us keep the tensor product as simple as possible while we are in the process of group completing and abelianizing it. We begin with group completion, using a method developed by Raouf Doss [?].

line break

Definition 4.5.1. We say that a monoid M is *left-cancellative* if for any $a, b, c \in M$, we have

$$ab = ac \implies b = c.$$

That is, common factors may be cancelled out on the left. Similarly, we call M *right-cancellative* if common factors can be cancelled on the right:

$$ac = bc \implies a = b.$$

A monoid that is both left- and right-cancellative is simply referred to as *cancellative*.

Definition 4.5.2. An element a of a monoid M is said to be *regular on the left* if it shares a common left-multiple with every other element of M. That is,

$$\forall b \in M, \quad \exists c, d \in M \quad : \quad ca = db.$$

The monoid as a whole is said to be *regular on the left* if all of its elements are. M is *quasi-regular on the left* if any two elements a, b of M share a common left-multiple ($ca = db$ as above) if and only if

$$\exists c', d' \in M \quad : \quad c'a = d'b, \qquad c' \text{ or } d' \text{ is regular in } M.$$

Again, we can define a similar condition for being quasi-regular on the right, and we say that a monoid is *quasi-regular* when it is both.

Proposition 4.5.3 ([?]). *If a monoid M is cancellative and quasi-regular on the left, then it can be embedded into a group.*

For a given action operad, both of these conditions will follow from the way that operadic multiplication interacts with the elements of the abelian group $\Lambda(0)$.

QQQ New notation: $(\mathbf{\Lambda}, \otimes)$ is the monoid associated to an action operad

Implement consistently Earlier??

Proposition 4.5.4. *For every action operad $\mathbf{\Lambda}$, the monoid $(\mathbf{\Lambda}, \otimes)$ is both cancellative and quasi-regular as a monoid under tensor product.*

PROOF. Let g and g' be any elements of Λ which share a left-multiple, so that there exists at least one pair h, h' in Λ for which

$$h \otimes g = h' \otimes g',$$

Make sure this notati appears earl is ref'ed

and without loss of generality assume that $\boxed{|g| \ge |g'|}$, so also $\boxed{|h| \le |h'|}$. The operadic product $\mu(h; e_0, \dots, e_0)$ is clearly an element of the group $\Lambda(0)$, and we know from ?? that this is an abelian group under tensor product, so also let $\mu(h; e_0, \dots, e_0)^*$ be its inverse. Then

• $(\Lambda, \otimes)$? ref

• $*$ = inv under $\otimes$, highlight earlier

$$\begin{aligned} g &= \mu(h; e_0, \dots, e_0)^* \otimes \mu(h; e_0, \dots, e_0) \otimes \mu(g; e_1, \dots, e_1) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(e_2 ; \mu(h; e_0, \dots, e_0), \mu(g; e_1, \dots, e_1) \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(\mu(e_2; h, g) : e_0, \dots, e_0, e_1, \dots, e_1 \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(h \otimes g ; e_0, \dots, e_0, e_1, \dots, e_1 \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(h' \otimes g' ; e_0, \dots, e_0, e_1, \dots, e_1 \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(\mu(e_2; h', g') ; e_0, \dots, e_0, e_1, \dots, e_1 \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu\big(e_2 ; \mu(h'; e_0, \dots, e_0, e_1, \dots, e_1), \mu(g'; e_1, \dots, e_1) \big) \\ &= \mu(h; e_0, \dots, e_0)^* \otimes \mu(h'; e_0, \dots, e_0, e_1, \dots, e_1) \otimes \mu(g'; e_1, \dots, e_1) \\ &= \big(\mu(h; e_0, \dots, e_0)^* \otimes \mu(h'; e_0, \dots, e_0, e_1, \dots, e_1) \big) \otimes g' \\ &=: k \otimes g' \end{aligned}$$

Put another way, if $h \otimes g = h' \otimes g'$, there exists $k \in \mathbf{\Lambda}$ such that $g = k \otimes g'$, or equivalently $e_0 \otimes g = k \otimes g'$. Since e_0 is obviously regular because it is the unit, the operad Λ viewed as a monoid is quasi-regular on the left; quasi-regularity on the right is analogous. Furthermore, following through the argument above when $h = h'$ ~~prove~~ shows that $h \otimes g = h \otimes g'$ implies $g = g'$, or left-cancellativity. □

Corollary 4.5.5. *The canonical map $\mathrm{gp} : (\mathbf{\Lambda}, \otimes) \to (\mathbf{\Lambda}, \otimes)^{\mathrm{gp}}$ associated with the group completion of $\mathbf{\Lambda}$ is an inclusion. Further, the canonical map $\mathrm{gp} : (\mathbf{\Lambda}, \otimes) \times_{\mathbb{N}} \mathbb{N}^{*n} \to ((\mathbf{\Lambda}, \otimes) \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp}}$ is also an inclusion.* line break

PROOF. The first statement is immediate. For the second, note that both $\mathbf{\Lambda}$ and $\mathbb{N}^{*n}$ embed into groups so their product and hence any submonoid of their product does too. ☐

As a consequence, we can identify the monoids $\mathbf{\Lambda}$ and $\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n}$ with their images in the group completion. Further, recall from Theorem 4.1.3 that for studying invertible objects, it suffices to study action operads $\mathbf{\Lambda}$ with $\Lambda(0)$ the trivial group.

Proposition 4.5.6. *Let M be a monoid with unit element 1, and assume M is equipped with a monoid homomorphism $\pi : M \to \mathbb{N}$ such that*

- *$\pi^{-1}(0) = \{1\}$, and*
- *if $hg = h'g'$ in M, then there exists a $k \in M$ such that $g = kg'$, $h' = hk$.*

Then M is a free monoid.

(*) QQQ Replace this whole proof with the one from Ed's email

PROOF. Let $\mathcal{G}$ be a subset of the monoid M. and $\mathcal{R}$ a collection of relations on the elements of $\mathcal{G}$, such that $(\mathcal{G}, \mathcal{R})$ is a presentation of M as a monoid. We assume that $1 \notin \mathcal{G}$ as it is the unit element. Notice that every relation in $\mathcal{R}$ can be written in the form $hg = h'g'$, where $g, g' \in \mathcal{G}$ are generators because the only other kind of relations are of the form $hg = 1$, and this is not possible if $\pi^{-1}(0) = \{1\}$. We can assume that $\pi(g) \geq \pi(g')$ and hence $\pi(h) \leq \pi(h')$ without loss of generality. By the second hypothesis, we can then find $k \in M$ for which $g = kg'$. $h' = hk$. We consider the following cases.

- $\pi(k) = \pi(g)$: This is not possible, as it would follow that $\pi(g') = 0$, and thus by assumption $g' = 1$. and we have assumed 1 is not a generator.
- $\pi(k) = 0$: This would mean that $k = 1$, and so we conclude that $g = g'$, $h = h'$. Thus we could simplify the presentation of M by replacing the relation $hg = h'g'$ in the set $\mathcal{R}$ with $h' = h$. If we define the length of a relation $hg = h'g'$ as $\pi(hg) = \pi(h'g')$, this substitution produces a relation of shorter length.
- $0 < \pi(k) < \pi(g|)$: In this case $\pi(g) > \pi(g')$ and thus $g \neq g'$. Therefore we could change our presentation of M by replacing g with k in the generator set $\mathcal{G}$. and also $h \otimes g = h' \otimes g'$ by $h' = h \otimes k$ in the relations $\mathcal{R}$.

Notice that in the latter two cases, we are always changing generators for ones that have strictly smaller image under π, and replacing relations with ones whose left- and right-hand sides have strictly smaller total length. But lengths are natural numbers, and therefore if we choose any relation in $\mathcal{R}$ and repeatedly apply this process to it, after a finite number of steps we will find that we have replaced it with $1 = 1$, the only relation whose sides have total length 0. Proceeding like this will let us eliminate all of the relations in $\mathcal{R}$, leaving us with a set $\mathcal{G}$ that freely generates the action operad G under tensor product. ☐

Corollary 4.5.7. *If $\mathbf{\Lambda}$ is an action operad with trivial $\Lambda(0)$, then $(\mathbf{\Lambda}, \otimes)$ is a free monoid under tensor product.*

Put short proof

QQQ Need to make sure we are using the correct notation: the operad, the monad, the monoid, etc

Check me!

We can now use these results to ~~express~~ describe the monoid $\mathrm{Mor}(E\Lambda(\underline{n}))$ using generators for the action operad $\mathbf{\Lambda}$.

Proposition 4.5.8. *Let $\mathcal{G}$ be a set that freely generates the action operad $\mathbf{\Lambda}$ under tensor product, and for each $m \in \mathbb{N}$ define $\mathcal{G}_m := \mathcal{G} \cap \Lambda(m)$, the subset of $\mathcal{G}$ containing all elements of length m. Then the monoid $\mathrm{Mor}(E\Lambda(\underline{n}))$ is*

$$\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n} \;\tilde{=}\; \mathbb{N}^{*(|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots)}$$

Just the
all = to

PROOF. Let (g, w) be an arbitrary element of $\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n}$. The monoid $\mathbf{\Lambda}$ is free of the generators $\mathcal{G}$, and $\mathbb{N}^{*n}$ is free on $\{z_1, \dots, z_n\}$, so we can find unique expansions of g and w as tensor products

$$\begin{aligned} g &= g_1 \otimes \dots \otimes g_k; & g_1, \dots, g_k &\in \mathcal{G} \\ w &= x_1 \otimes \dots \otimes x_m, & x_1, \dots, x_m &\in \{z_1, \dots, z_n\} \end{aligned}$$

But each of the generators $z_1, \dots, z_n$ has length 1, so the index m here is really just the length $|w|$, which by the definition of $\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n}$ is also the length $|g| = |g_1| + \dots + |g_k|$. Therefore we may write

$$\begin{aligned} (g, w) &= (g_1 \otimes \dots \otimes g_k, x_1 \otimes \dots \otimes x_{|w|}) \\ &= (g_1, x_1 \otimes \dots \otimes x_{|g_1|}) \otimes (g_2, x_{|g_1|+1} \otimes \dots \otimes x_{|g_1|+|g_2|}) \otimes \dots \\ &\quad \otimes (g_k, x_{|g_1|+\dots+|g_{k-1}|+1} \otimes \dots \otimes x_{|g_1|+\dots+|g_k|}) \end{aligned}$$

That is, every element in $\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n}$ may be expressed as a product of elements from the subset $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}$. Furthermore, the freeness of $\mathbf{\Lambda}$ and $\mathbb{N}^{*n}$ ~~make sure~~ ensures that this expansion is unique, since

$$\begin{aligned} & (g_1, x_1 \otimes \dots x_{|g_1|}) \otimes \dots \otimes (g_k, x_{|g_1|+\dots+|g_{k-1}|+1} \otimes \dots \otimes x_{|g_1|+\dots+|g_k|}) \\ = \; & (g'_1, x'_1 \otimes \dots \otimes x'_{|g'_1|}) \otimes \dots \otimes (g'_{k'}, x'_{|g'_1|+\dots+|g'_{k'-1}|+1} \otimes \dots \otimes x'_{|g'_1|+\dots+|g'_{k'}|}) \\ \implies \; & g_1 \otimes \dots \otimes g_k = g'_1 \otimes \dots \otimes g'_{k'}, \qquad x_1 \otimes \dots \otimes x_m = x'_1 \otimes \dots \otimes x'_{m'} \\ \implies \; & g_i = g'_i, \quad 1 \leq i \leq k = k', \qquad x_j = x'_j, \quad 1 \leq j \leq m = m' \end{aligned}$$

remove

Thus $\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n}$ is the free monoid on the set

$$\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n} = \mathcal{G}_0 \times \{z_1, \dots, z_n\}^0 \cup \mathcal{G}_1 \times \{z_1, \dots, z_n\}^1 \cup \mathcal{G}_2 \times \{z_1, \dots, z_n\}^2 \cup \dots$$

which is just the m-fold free product of $\mathbb{N}$ with itself, where m is the number of generators,

$$\begin{aligned} |\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}| &= |\mathcal{G}_0| \cdot |\{z_1, \dots, z_n\}^0| + |\mathcal{G}_1| \cdot |\{z_1, \dots, z_n\}^1| + |\mathcal{G}_2| \cdot |\{z_1, \dots, z_n\}^2| + \dots \\ &= |\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + \dots \end{aligned}$$

☐

This makes the group completion and abelianization ~~we want to do~~ trivial. to compute.

Corollary 4.5.9. *If $\mathcal{G}$ is a set that freely generates $\mathbf{\Lambda}$ under tensor product. and $\mathcal{G}_m := \mathcal{G} \cap \Lambda(m)$, then the abelian group $\mathrm{Mor}(E\Lambda(\underline{n}))^{\mathrm{gp,ab}}$ is*

$$(\mathbf{\Lambda} \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp,ab}} = \mathbb{Z}^{|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots}.$$

If we use a subset $\mathcal{G}' \subseteq \mathcal{G}$ which generates $\mathbf{\Lambda}$ using both the tensor product and group multiplication, then we can reduce the number of generators even further.

Corollary 4.5.10. *Let $\mathcal{G}$ be a subset of the action operad $\mathbf{\Lambda}$ that freely generates it under tensor product, and let $\mathcal{G}'$ be a subset of $\mathcal{G}$ which generates $\mathbf{\Lambda}$ under a combination of tensor product and group multiplication. Also let $\mathcal{G}_m := \mathcal{G} \cap G(m)$ and $\mathcal{G}'_m := \mathcal{G}' \cap G(m)$. Then*

$$\mathbb{Z}^{|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots} \Big/ {\otimes \sim \circ} \;\cong\; \mathbb{Z}^{|\mathcal{G}'_0|+n|\mathcal{G}'_1|+n^2|\mathcal{G}'_2|+\dots} \Big/ {\otimes \sim \circ}$$

(*)

PROOF. Composition in $E\Lambda(\underline{n})$ is given by group multiplication (when defined) by the formula

$$(g', \pi(g^{-1})(w)) \circ (g, w) = (g'g, w).$$

Therefore any generator $(g, w) \in \mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}$ can be expressed in terms of elements of $\mathcal{G}' \times_{\mathbb{N}} \mathbb{N}^{*n}$ by way of tensor product and group multiplication. After making the identification $\otimes \sim \circ$ we obtain the desired isomorphism. ☐

4.6. A full description of L_n

We now give our most complete description of the free $\mathbf{\Lambda}$-monoidal category generated by n invertible objects. We know that an arbitrary morphism of L_n can be described using its source, target, and an element of $L_n(I, I)$. We use this knowledge, together with a chosen isomorphism $\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \times L_n(I, I)$, to write down L_n as a $\mathbf{\Lambda}$-monoidal category.

Convention 4.6.1. Recall that we have fixed an isomorphism $\mathrm{Mor}(L_n) \cong (s \times t)(L_n) \times L_n(I, I)$ as per Proposition 3.3.1. Further, assume that $\Lambda(0)$ is the trivial group; if not, replace Λ with the action operad Λ' of Theorem 4.1.3.

linebreak

~~We begin by giving a recipe for constructing the group $L_n(I, I)$ from a presentation of the operad $\mathbf{\Lambda}$.~~

Definition 4.6.2. Let $\mathbf{\Lambda}$ be a crossed action operad. Choose a subset $\mathcal{G}$ that generates $\mathbf{\Lambda}$ under a combination of tensor product and group multiplication, which itself has subsets $\mathcal{G}_m := \mathcal{G} \cap \Lambda(m)$. Then for each $n \in \mathbb{N}$, we will denote by $A(\mathbf{\Lambda}, n)$ the abelian group obtained from the free abelian group

G' for consistent notation

$$\mathrm{F}(\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}) = \mathbb{Z}^{2n|\mathcal{G}_1|+(2n)^2|\mathcal{G}_2|+\dots}$$

via the following steps:

(1) For all $g, g' \in \mathcal{G}_m$ and $w \in \mathbb{N}^{*2n}$ with $|w| = m$, quotient out by the relation

$$(g, w) \otimes (g', \pi(g^{-1})(w)) \sim (g \cdot g', w)$$

(2) Quotient out by the subgroup Δ, which is generated by the equivalence classes of elements of the form

$$\begin{gathered} (\mu(g; e_{|\tilde{\delta}(x_1)|}, \dots, e_{|\tilde{\delta}(x_m)|}), \tilde{\delta}(x_1 \otimes \dots \otimes x_m)) \\ \otimes \\ (\mu(g; e_{|\tilde{I}(x_1)|}, \dots, e_{|\tilde{I}(x_m)|}), \tilde{I}(x_1 \otimes \dots \otimes x_m))^* \end{gathered}$$

where $g \in G(m)$, the x_i are generators of $\mathbb{N}^{*4n}$, and for all $1 \leq i \leq n$,

$$\begin{aligned} \tilde{\delta}(z_i) &= z_i, & \tilde{\delta}(z_{2n+i}) &= z_i \otimes z_{n+i}, & \tilde{I}(z_i) &= z_i, & \tilde{I}(z_{2n+i}) &= I, \\ \tilde{\delta}(z_{n+i}) &= z_{n+i}, & \tilde{\delta}(z_{3n+i}) &= z_{n+i} \otimes z_i & \tilde{I}(z_{n+i}) &= z_{n+i}, & \tilde{I}(z_{3n+i}) &= I \end{aligned}$$

(3) Choose any $\rho(2) \in \pi^{-1}((1\,2))$, and then quotient out by the $\mathbb{Z}^{\binom{n}{2}}$ subgroup generated by the equivalence classes of the elements

$$(\rho(2) : z_i, z_j), \qquad 1 \leq i < j \leq n$$

We write ~~Also, we will use~~ $\Psi : G \times_{\mathbb{N}} \mathbb{N}^{*2n} \to A(G, n)$ to represent the corresponding quotient map.

Λ

With this new notation, the main result of this thesis can be stated quite concisely.

THEOREM 4.6.3. *Let G be a crossed action operad. Then the free $E\Lambda$-algebra on n invertible objects is the category*

$$L_n \;\tilde{=}\; \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times BA(G, n)$$

equipped with ~~a component-wise tensor product,~~ tensor product given by componentwise

$$(x', y', h') \otimes (x, y, h) = (x' \otimes x, y' \otimes y, h'h)$$

and the $E\Lambda$-action given in Proposition 5.4.2: ?

$$\begin{gathered} \alpha_{L_n}(g : (x_1, y_1, h_1), \dots, (x_m, y_m, h_m)) \\ = \\ (\textstyle\bigotimes_i x_i, \; \bigotimes_i y_{\pi(g^{-1})(i)}, \; \Psi\alpha_{E\Lambda(\underline{2n})}(g; \mathrm{id}_{g^{-1}(y_1)}, \dots, \mathrm{id}_{g^{-1}(y_m)}) \otimes (\bigotimes_i h_i)) \end{gathered}$$

Rewrite

PROOF. The category $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times BA(G, n)$ is just the one which has objects $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, morphisms $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times A(G, n)$, and composition

$$(y, z, h') \circ (x, y, h) = (x, z, h'h)$$

We know that these objects and morphisms are correct by Corollaries 2.2.10 and 4.4.3, ??, Proposition 4.5.8, and Theorem 4.1.3, and those results also tell us that the monoidal structure is as given above. For composition, it follows from Lemma 2.2.14 that

$$\begin{aligned} (y, z, h') \circ (x, y, h) &= (y, z, h') \otimes \mathrm{id}_{y^*} \otimes (x, y, h) \\ &= (y, z, h') \otimes (y^*, y^*, \mathrm{id}_I) \otimes (x, y, h) \\ &= (y \otimes y^* \otimes x, z \otimes y^* \otimes y, h' \otimes \mathrm{id}_I \otimes h) \\ &= (x, z, h'h) \end{aligned}$$

The action we just found in ~~Proposition 5.4.2 then completes~~ this description of L_n. ☐

Check unit prop?

(*) For a simple example:

$$G'_m = \begin{cases} \emptyset & m \neq 2 \\ \{(12)\} & m = 2 \end{cases} \quad \text{for } \Sigma$$