# OPERADS WITH GENERAL GROUPS OF EQUIVARIANCE, AND SOME 2-CATEGORICAL ASPECTS OF OPERADS IN Cat

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ABSTRACT. We give a definition of an operad with general groups of equivariance suitable for use in any symmetric monoidal category with appropriate colimits. We then apply this notion to study the 2-category of algebras over an operad in **Cat**. We show that any operad is finitary, that an operad is cartesian if and only if the group actions are nearly free in a precise fashion, and that the existence of a pseudo-commutative structure largely depends on the groups of equivariance. We conclude by showing that the operad for braided strict monoidal categories has two canonical pseudo-commutative structures.

We show that every action operad gives rise to a notion of monoidal category via the categorical version of the Borel construction, embedding action operads into the category of 2-monads on **Cat**. We characterize those 2-monads in the image of this embedding, and as an example show that the theory of coboundary categories corresponds precisely to the operad of $n$-fruit cactus groups. We finally define $\mathbf{\Lambda}$-multicategories for an action operad $\mathbf{\Lambda}$, and show that they arise as monads in a Kleisli bicategory.

CONTENTS



Corrections left to do: 16, 20, 24, onwards

## INTRODUCTION

Operads were defined by May [31] in the early 70's to provide a convenient tool to approach problems in algebraic topology, notably the question of when a

1991 *Mathematics Subject Classification.* 18D05, 18D10, 18D50.
*Key words and phrases.* operad, 2-category, pseudo-commutativity.

space $X$ admits an $n$-fold delooping $Y$ so that $X \simeq \Omega^n Y$. An operad, like an algebraic theory [25], is a global description for an algebraic structure, with all operations given equal weight [1]. The theory of operads has seen great success, and we would like to highlight two reasons. First, operads can be defined in any suitable symmetric monoidal category, so that there are operads of topological spaces, of chain complexes, of simplicial sets, and of categories, to name a few examples. Moreover, symmetric (lax) monoidal functors carry operads to operads, so we can use operads in one category to understand objects in another via transport by such a functor. Second, operads in a fixed category are highly flexible tools. In particular, the categories listed above all have some inherent notion of "homotopy equivalence" which is weaker than that of isomorphism, so we can study operads which are equivalent but not isomorphic. These tend to have algebras which have similar features in an "up-to-homotopy" sense but very different combinatorial or geometric properties arising from the fact that different objects make up these equivalent but not isomorphic operads.

Operads in the category **Cat** of small categories have a unique flavor arising from the fact that **Cat** is not just a category but a 2-category. These 2-categorical aspects have not been widely treated in the literature, although a few examples can be found. Lack [24] mentions braided **Cat**-operads (the reader new to braided operads should refer to the work of Fiedorowicz [9]) in his work on coherence for 2-monads, and Batanin [3] uses lax morphisms of operads in **Cat** in order to define the notion of an internal operad. But aside from a few appearances, the basic theory of operads in **Cat** and their 2-categorical properties seems missing. This paper was partly motivated by the need for such a theory to be explained from the ground up.

There were two additional motivations for the work in this paper. In thinking about coherence for monoidal functors, the first author was led to a general study of algebras for multicategories internal to **Cat**. These give rise to 2-monads (or perhaps pseudomonads, depending on how the theory is set up), and checking abstract properties of these 2-monads prompts one to consider the simpler case of operads in **Cat** instead of multicategories. The other motivation was from the second author's attempt to understand the interplay between operads in **Cat**, operads in **Top**, and the passage from (bi)permutative categories to $E_\infty$ (ring) spaces. The first of these motivations raised the issue of when operads in **Cat** are cartesian, while the second led us to consider when an operad in **Cat** possesses a pseudo-commutative structure.

While considering how to best tackle a general discussion of operads in **Cat**, it became clear that restricting attention to the two most commonly used types of operads, symmetric and non-symmetric operads, was both short-sighted and unnecessary. Many theorems apply to both kinds of operads at once, with the difference in proofs being negligible; in fact, most of the arguments which applied to the symmetric case seemed to apply to the case of braided operads as well. This led us to the notion of an action operad $\Lambda$, and then to a definition of $\Lambda$-operads. In essence, this is merely the general notion of what it means for an operad $P = \{P(n)\}_{n\in\mathbb{N}}$ to

[1] We note that although similar, operads and Lawvere theories have slightly different flavors. We have applications in mind in which the algebra is defined using non-cartesian monoidal structures, like chain complexes, or the category is not locally presentable, and thus we are "only" interested in structures defined by operads.

have groups of equivariance $\Lambda = \{\Lambda(n)\}_{n\in\mathbb{N}}$ such that $\Lambda(n)$ acts on $P(n)$. Choosing different natural families of groups $\Lambda$, we recover known variants of the definition of operad.

| Groups $\Lambda$ | Type of operad |
|---|---|
| Terminal groups | Non-symmetric operad |
| Symmetric groups | Symmetric operad |
| Braid groups | Braided operad |

These definitions have appeared, with minor variations, in two sources of which we are aware. In Wahl's thesis [35], the essential definitions appear but not in complete generality as she requires a surjectivity condition. Zhang [36] also studies these notions[2], once again in the context of homotopy theory, but requires the superfluous condition that $e_1 = \mathrm{id}$ (see Lemma 1.23).

This paper consists of the following. In Section 1, we give the definition of an action operad $\Lambda$ and a $\Lambda$-operad. We give some basic results and perform some first calculations. We also explore the relationship between $\Lambda$-operads and symmetric operads, giving one explanation for the ubiquity of symmetric operads.

Section 2 gives a treatment of $\Lambda$-operads as monoids in a category of collections. This is standard material for symmetric operads, but in order to show that our $\Lambda$-operads fit into this philosophy, we must use the calculus of coends together with the Day convolution product [6]. The reader uninterested in these details can happily skip them, although we find the route taken here to be quite satisfactory in justifying the axioms for an action operad $\Lambda$ and the accompanying notion of $\Lambda$-operad. Many of our calculations are generalizations of those appearing in work of Kelly [19], although there are slight differences in flavor between the two treatments.

Section 3 works through the basic 2-categorical aspects of operads in **Cat**. We explain how every operad gives rise to a 2-monad, and show that all of the various 1-cells between algebras of the associated 2-monad correspond to the obvious sorts of 1-cells one might define between algebras over an operad in **Cat**. Similarly, we show that the algebra 2-cells, using the 2-monadic approach, correspond to the obvious notion of transformation one would define using the operad. In contrast to Section 2, we have given a very elementary treatment of this material since much of what we discuss here is relevant for topological applications.

Section 4 studies three basic 2-categorical properties of an operad, namely the property of being finitary, the coherence property, and the property of being 2-cartesian. The first of these always holds, as a simple calculation shows. The second is also easily shown to hold for any $\Lambda$-operad on **Cat** using a factorization system argument due to Power [32]. The third of these is the most complicated, and turns out to be equivalent to the action of $\Lambda(n)$ on $P(n)$ being free for all $n$, at least up to a certain kernel. In particular, our characterization clearly shows that every non-symmetric operad is 2-cartesian, and that a symmetric operad is 2-cartesian if and only if the symmetric group actions are all free. (It is useful to note that a 2-monad on **Cat** is 2-cartesian if and only if the underlying monad on the category of small categories is cartesian in the usual sense as the (strict) 2-pullback of a diagram is the same as its pullback.)

[2] Zhang calls our action operad a *group operad*. We dislike this terminology as it seems to imply that we are dealing with an operad in the category of groups, which is not the case unless all of the maps $\pi_n : \Lambda(n) \to \Sigma_n$ are zero maps.

Section 5 then goes on to study the question of when a $\Lambda$-operad $P$ admits a pseudo-commutative structure. Such a structure provides the 2-category of algebras with a richer structure that includes well-behaved notions of tensor product, internal hom, and multilinear map that fit together much as the analogous notions do in the category of vector spaces. When $P$ is contractible (i.e., each $P(n)$ is equivalent to the terminal category), this structure can be obtained from a collection of elements $t_{m,n} \in \Lambda(mn)$ satisfying certain properties. In particular, we show that every contractible symmetric operad is pseudo-commutative, and we prove that there exist such elements $t_{m,n} \in Br_{mn}$ so that every contractible braided operad is pseudo-commutative as well (in fact in two canonical ways). Thus Section 4 can be seen as a continuation, in the operadic context, of the work in [13], and in particular the "geometric" proof of the existence of a pseudo-commutative structure for braided strict monoidal categories demonstrates the power of being able to change the groups of equivariance.

Categories of interest are often monoidal: sets, topological spaces, and vector spaces are all symmetric monoidal, while the category of finite ordinals (under ordinal sum) is merely monoidal. But other categories have more exotic monoidal structures. The first such type of structure discovered was that of a braided monoidal category. These arise in categories whose morphisms have a geometric flavor like cobordisms embedded in some ambient space [14], in categories produced from double loop spaces [9], and categories of representations over objects like quasitriangular (or braided) bialgebras [34] . Another such exotic monoidal structure is that of a coboundary category, arising in examples from the representation theory of quantum groups [7].

Going back to the original work of May on iterated loop spaces [31], operads were defined in both symmetric and nonsymmetric varieties. But Fiedorowicz's work on double loop spaces [9] showed that there was utility in considering another kind of operad, this time with braid group actions instead of symmetric group actions. There is a clear parallel between these definitions of different types of operads and the definitions of different kinds of monoidal category, with each given by some general schema in which varying an $\mathbb{N}$-indexed collection of groups produced the types of operads or monoidal categories seen in nature. Building on the work in [**?**], the goal of this paper is to show that this parallel can be upgraded from an intuition to precise mathematics using the notion of action operad.

An action operad $\mathbf{\Lambda}$ is an operad which incorporates all of the essential features of the operad of symmetric groups. Thus $\Lambda(n)$ is no longer just a set, but instead also has a group structure together with a map $\pi_n : \Lambda(n) \to \Sigma_n$. Operadic composition then satisfies an additional equivariance condition using the maps $\pi_n$ and the group structures. Each action operad $\mathbf{\Lambda}$ produces a notion of $\mathbf{\Lambda}$-operad which encodes equivariance conditions using both the groups $\Lambda(n)$ and the maps $\pi_n$. Examples include the symmetric groups, the terminal groups (giving nonsymmetric operads), the braid groups (giving braided operads), and the $n$-fruit cactus groups [11] (giving a new notion of operad one might call cactus operads). Using a formula resembling the classical Borel construction for spaces with a group action, we can produce from any action operad $\mathbf{\Lambda}$ a notion of $\mathbf{\Lambda}$-monoidal category, in which the group $\Lambda(n)$ acts naturally on $n$-fold tensor powers of any object. Thus the categorical Borel construction embeds action operads into a category of monads on **Cat**, and we characterize the image of this embedding as those monads describing monoidal structures of a precise kind.

The paper is organized into the following sections. Section 1 reviews the definition of an action operad, and defines the categorical Borel construction on them. The key result, which reappears in proofs throughout the paper, is Theorem 5.8, characterizing action operads in terms of two new operations mimicking the block sum of permutations and the operation which takes a permutation of $n$ letters and produces a new permutation on $k_1 + k_2 + \cdots + k_n$ letters by permuting the blocks of $k_i$ letters. In Section 2, we use this characterization and Kelly's theory of clubs [15, 16, 17] to embed action operads into monads on **Cat** and determine the essential image of this embedding. Section 3 gives a construction of the free action operad from a suitable collection of data, and relates this to how clubs can be described using generators and relations. The results of Sections 2 and 3 show that the definitions of symmetric monoidal category or coboundary category, for example, correspond to the action operad constructed from the corresponding free symmetric monoidal or coboundary category on one object; these and other examples appear in detail in Section 4. Section 5 then extends the definition of $\mathbf{\Lambda}$-operad to that of $\mathbf{\Lambda}$-multicategory and shows that these arise abstractly via a Kleisli construction.

The author would like to thank Alex Corner and Ed Prior for conversations contributing to this research.

This research was supported by EPSRC 134023.

The authors would like to thank John Bourke, Martin Hyland, Tom Leinster, and Peter May for various conversations which led to this paper. While conducting this research, the second author was supported by an EPSRC Early Career Fellowship.

## 1. Action operads

In this section, we will explore the general definition of an operad $P$ which is equipped with groups of equivariance $\Lambda(n)$. The group $\Lambda(n)$ will act on the right on the object $P(n)$, and the operad structure of $P$ will be required to respect this action. For certain choices of the groups $\Lambda(n)$, we will recover standard notions of operads such as symmetric operads, non-symmetric operads, and braided operads. The definitions here will, unless otherwise stated, apply in any symmetric monoidal category $\mathcal{V}$ in which the functors $X \otimes -, - \otimes X$ preserve colimits for every object $X \in \mathcal{V}$.

**Conventions 1.1.** We adopt the following conventions throughout.

- $\Sigma_n$ is the symmetric group on $n$ letters, and $Br_n$ is the braid group on $n$ strands.
- For a group $G$, a right $G$-action on a set $X$ will be denoted $(x, g) \mapsto x \cdot g$. We will use both $\cdot$ and concatenation to represent multiplication in a group.
- The symbol $e$ will generically represent an identity element in a group. If we have a set of groups $\{\Lambda(n)\}_{n \in \mathbb{N}}$ indexed by the natural numbers, then $e_n$ is the identity element in $\Lambda(n)$.
- We will often be interested in elements of a product of the form
$$A \times B_1 \times \cdots \times B_n \times C$$
(or similar, for example without $C$). We will write elements of this set as $(a; b_1, \ldots, b_n; c)$, and in the case that we need equivalence classes of such elements they will be written as $[a; b_1, \ldots, b_n; c]$.

We begin with the basic definitions.

**Definition 1.2.** A *symmetric operad* $O$ (in the category of sets) consists of

- sets $O(n)$ for each natural number $n$,
- for each $n$, a right $\Sigma_n$-action on $O(n)$,
- an element $\mathrm{id} \in O(1)$, and
- functions

$$\mu : O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying the following three axioms.

(1) The element $\mathrm{id} \in O(1)$ is a two-sided unit for $\mu$ in the sense that

$$\begin{aligned} \mu(\mathrm{id}; x) &= x \\ \mu(x; \mathrm{id}, \ldots, \mathrm{id}) &= x \end{aligned}$$

for any $x \in O(n)$.

(2) The functions $\mu$ (called operadic multiplication or operadic composition) are associative in the sense that the diagram below commutes.

$$\begin{tikzcd}
O(n)\times O(k_1)\times\cdots\times O(k_n)\times O(l_{1,1})\times\cdots\times O(l_{1,k_1})\times\cdots\times O(l_{n,1})\times\cdots\times O(l_{n,k_n}) \arrow[dr, "\cong"] \arrow[dddd, "\mu\times 1"'] & \\
& O(n)\times\prod_{i=1}^{n} O(k_1)\times O(l_{i,1})\times\cdots\times O(l_{i,k_i}) \arrow[d, "1\times\prod\mu"] \\
& O(n)\times O(\sum l_{1,-})\times\cdots\times O(\sum l_{n,-}) \arrow[d, "\mu"] \\
& O(\sum l_{-,-}) \\
O(k_1+\cdots+k_n)\times O(l_{1,1})\times\cdots\times O(l_{1,k_1})\times\cdots\times O(l_{n,1})\times\cdots\times O(l_{n,k_n}) \arrow[ur, "\mu"] &
\end{tikzcd}$$

(3) The functions $\mu$ are equivariant with respect to the symmetric group actions, and so satisfies the following two equations.

$$\begin{aligned} \mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) &= \mu(x; y_1, \ldots, y_n) \cdot (\tau_1 \oplus \cdots \oplus \tau_n) \\ \mu(x \cdot \sigma; y_1, \ldots, y_n) &= \mu(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}) \cdot \sigma^{+} \end{aligned}$$

For the above equations to make sense, we must have

- $x \in O(n)$,
- $y_i \in O(k_i)$ for $i = 1, \ldots, n$,
- $\tau_i \in \Sigma_{k_i}$, and
- $\sigma \in \Sigma_n$, and then $\sigma^{+}$ is the permutation in Example 1.5 below.

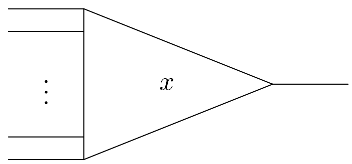
**Definition 1.3.** A *non-symmetric operad* $O$ consists of the same data as above but without any symmetric group actions, and only satisfying the first and second axioms.

**Remark 1.4.** (1) One can change from operads in **Sets** to operads in another symmetric monoidal category $\mathcal{V}$ by requiring each $O(n)$ to be an object of $\mathcal{V}$ and replacing all instances of cartesian product with the appropriate tensor product in $\mathcal{V}$. This includes replacing the element $\mathrm{id} \in O(1)$ with a map $I \to O(1)$ from the unit object of $\mathcal{V}$ to $O(1)$.

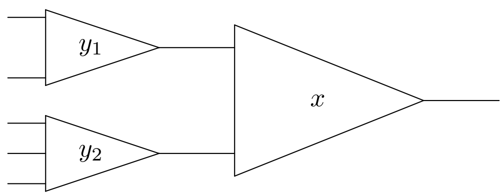
(2) Every symmetric operad has an underlying *symmetric collection* which consists of the natural number-indexed set $\{O(n)\}_{n\in\mathbb{N}}$ together with symmetric group actions, but without a chosen identity element or composition maps.

The category of symmetric collections is a presheaf category, and we will equip it with a monoidal structure in which monoids are precisely operads in Theorem 2.5. A similar construction, but without reference to group actions, shows that every non-symmetric operad has an underlying (non-symmetric) collection which is now merely a $\mathbb{N}$-indexed collection of sets.

One is intended to think that $x \in O(n)$ is a function with $n$ inputs and a single output, as below.




Operadic composition is then a generalization of function composition, with the pictorial representation below being $\mu(x; y_1, y_2)$ for $\mu : O(2) \times O(2) \times O(3) \to O(5)$.




**Example 1.5.** The canonical example of an operad is the symmetric operad which we write as $\Sigma$. The set $\Sigma(n)$ is the set of elements of the symmetric group $\Sigma_n$, and the group action is just multiplication on the right. The identity element $\mathrm{id} \in \Sigma(1)$ is just the identity permutation on a one-element set. Operadic composition in $\Sigma$ will then be given by a function

$$\Sigma(n) \times \Sigma(k_1) \times \cdots \times \Sigma(k_n) \to \Sigma(k_1 + \cdots + k_n)$$

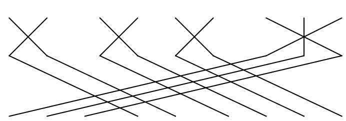
which takes permutations $\sigma \in \Sigma_n, \tau_i \in \Sigma_{k_i}$ and produces the following permutation in $\Sigma_{k_1+\cdots+k_n}$. First we form the block sum permutation $\tau_1 \oplus \cdots \oplus \tau_n$ which permutes the first $k_1$ elements according to $\tau_1$, the next $k_2$ elements according to $\tau_2$ and so on; this is an element of $\Sigma_{k_1+\cdots+k_n}$. Then we take the permutation $\sigma^+ \in \Sigma_{k_1+\cdots+k_n}$ which permutes the $n$ different blocks 1 through $k_1$, $k_1+1$ through $k_1+k_2$, and so on, according to the permutation $\sigma \in \Sigma_n$. Operadic composition in $\Sigma$ is then given by the formula

$$\mu(\sigma; \tau_1, \ldots, \tau_n) = \sigma^+ \cdot (\tau_1 \oplus \cdots \oplus \tau_n).$$

Below we have drawn the permutation for the composition

$$\mu : \Sigma(3) \times \Sigma(2) \times \Sigma(4) \times \Sigma(3) \to \Sigma(9)$$

evaluated on the element $\Big((123); (12), (12)(34), (13)\Big)$.



Note that $(12)(34) \in \Sigma(4)$ is actually $\mu(e_2; (12), (12))$, where $e_2 \in \Sigma_2$ is the identity permutation. Using this and operad associativity, one can easily check that

$$\mu\Big((123); (12), (12)(34), (13)\Big) = \mu\Big((1234); (12), (12), (12), (13)\Big),$$

where now the composition on the right side uses the function

$$\mu : \Sigma(4) \times \Sigma(2) \times \Sigma(2) \times \Sigma(2) \times \Sigma(3) \to \Sigma(9).$$

This equality is obvious using the picture above, but verifiable directly using only the algebra of the symmetric operad.

In the original topological applications [31], symmetric operads were the central figures. A further kind of operad was studied by Fiedorowicz in [9]; we give the definition below in analogy with that for symmetric operads, with interpretation to follow afterwards to make it entirely rigorous. We do this to emphasize the key features that we will generalize in Definition 1.10.

**Definition 1.6.** A *braided operad* consists of

- a non-symmetric operad $O$ and
- for each $n$, a right action of the $n$th braid group $Br_n$ on $O(n)$,

satisfying the following axioms.

$$\begin{aligned} \mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) &= \mu(x; y_1, \ldots, y_n) \cdot (\tau_1 \oplus \cdots \oplus \tau_n) \\ \mu(x \cdot \sigma; y_1, \ldots, y_n) &= \mu(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}) \cdot \sigma^+ \end{aligned}$$

For the above equations to make sense, we must have

- $x \in O(n)$,
- $y_i \in O(k_i)$ for $i = 1, \ldots, n$,
- $\tau_i \in Br_{k_i}$, and
- $\sigma \in Br_n$.

In order to make sense of this definition, we must define $\tau_1 \oplus \cdots \oplus \tau_n$ and $\sigma^+$ in the context of braids. The first is the block sum in the obvious sense: given $n$ different braids on $k_1, \ldots, k_n$ strands, respectively, we form a new braid on $k_1 + \cdots + k_n$ strands by taking a disjoint union where the braid $\tau_i$ is to the left of $\tau_j$ if $i < j$. The braid $\sigma^+$ is obtained by replacing the $i$th strand with $k_i$ consecutive strands, all of which are braided together according to $\sigma$. Finally, the notation $\sigma^{-1}(i)$ should be read as $\pi(\sigma)^{-1}(i)$, where $\pi : Br_n \to \Sigma_n$ is the underlying permutation map.

We require one final preparatory definition.

**Definition 1.7.** Let $O, O'$ be operads. Then an *operad map* $f : O \to O'$ consists of functions $f_n : O(n) \to O'(n)$ for each natural number such that the following axioms hold.

$$\begin{aligned} f(\mathrm{id}_O) &= \mathrm{id}_{O'} \\ f(\mu^O(x; y_1, \ldots, y_n)) &= \mu^{O'}(f(x); f(y_1), \ldots, f(y_n)) \end{aligned}$$

The maps $f_n$ are required to be equivariant with respect to the symmetric group actions if $f$ is to be a map of symmetric operads, or braid actions if $f$ is to be a map of braided operads.

**Conventions 1.8.** In the above definition and below, we adopt the convention that if an equation requires using operadic composition in more than one operad, we will indicate this by a superscript on each instance of $\mu$ unless it is entirely clear from context.

**Example 1.9.** One can form an operad $Br$ where $Br(n)$ is the underlying set of the $n$th braid group, $Br_n$. This is done in much the same way as we did for the symmetric operad, and the collection of maps $\pi_n : Br_n \to \Sigma_n$ giving the underlying permutations constitutes an operad map (of non-symmetric or braided operads) operads$Br \to \Sigma$.

One should note that the axioms for symmetric and braided operads each use the fact that the groups of equivariance themselves form an operad. This is what we call an action operad.

**Definition 1.10.** An *action operad* $\mathbf{\Lambda}$ consists of

- an operad $\Lambda = \{\Lambda(n)\}$ in the category of sets such that each $\Lambda(n)$ is equipped with the structure of a group and
- a map $\pi : \Lambda \to \Sigma$ which is simultaneously a map of operads and a group homomorphism $\pi_n : \Lambda(n) \to \Sigma_n$ for each $n$

such that one additional axiom holds. Write

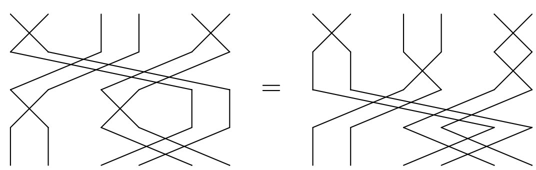
$$\mu : \Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n) \to \Lambda(k_1 + \cdots + k_n)$$

for the multiplication in the operad $\Lambda$. Let $(g; f_1, \ldots, f_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n)$ and let $(g'; f'_1, \ldots f'_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_{\pi(g)^{-1}(1)}) \times \cdots \times \Lambda(k_{\pi(g)^{-1}(1)})$. We require that

$$\mu(g'; f'_1, \ldots f'_n)\mu(g; f_1, \ldots, f_n) = \mu(g'g; f'_{\pi(g)(1)}f_1, \ldots, f'_{\pi(g)(n)}f_n) \tag{1}$$

in the group $\Lambda(k_1 + \cdots + k_n)$.

**Remark 1.11.**
- The final axiom is best explained using the operad $\Sigma$ of symmetric groups. Reading symmetric group elements as permutations from top to bottom, below is a pictorial representation of the final axiom for the map $\mu : \Sigma_3 \times \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_6$.



$\mu\big((23);(12),(12),\mathrm{id}\big)\cdot\mu\big((132);(12),\mathrm{id},(12)\big)$ $\qquad$ $\mu\big((23)\cdot(132);\mathrm{id}\cdot(12),(12)\cdot\mathrm{id},(12)\cdot(12)\big)$

- Our definition of an action operad is the same as that appearing in Wahl's thesis [35], but without the condition that each $\pi_n$ is surjective. It is also the same as that appearing in work of Zhang [36], although we prove later (see Lemma 1.23) that the condition $e_1 = \mathrm{id}$ in Zhang's definition follows from the rest of the axioms.

**Example 1.12.**
(1) The terminal operad $T$ in the category of sets has a unique action operad structure, $\mathbf{T}$. Since $T(n)$ is a singleton for each $n$, the group structure is unique as is the map $\pi$. The single action operad axiom is then automatic as both sides of Equation (1) are the unique element which happens to be the identity. This is the initial object in the category of action operads (see Definition 5.9 the definition of morphisms in the category of action operads).
(2) The symmetric operad $\Sigma$ has a canonical action operad structure. It is given by taking $\pi$ to be the identity map, and this action operad will be denoted $\mathbf{\Sigma}$. This is the terminal object in the category of action operads.
(3) Two less trivial examples are given by the braid groups, $\Lambda = \mathbf{Br}$, and the ribbon braid groups, $\Lambda = \mathbf{RBr}$. (A ribbon braid is given, geometrically, as a braid with strands replaced by ribbons in which we allow full twists. The actual definition of the ribbon braid groups is as the fundamental group

of a configuration space in which points have labels in the circle, $S^1$; see [33].) In each case, the homomorphism $\pi$ is given by taking underlying permutations, and the operad structure is given geometrically by using the procedure explained after Definition 1.6. We refer the reader to [9] for more information about braided operads, and to [33, 35] for information about the ribbon case.

(4) The operad of $n$-fruit cactus groups defined by Henriques and Kamnitzer in [11] has an action operad structure that we will discuss in Section 4.

**Example 1.13.** Every abelian group $A$ gives rise to action operad $A^\bullet$ as follows. The group $A^\bullet(n)$ is the direct sum of $n$ copies of $A$, $A^n$. The identity element is required to be $e \in A^1$, and the multiplication is defined by

$$\mu((a_1, \ldots, a_n); \mathbf{b_1}, \ldots, \mathbf{b_n}) = (a_1 + \mathbf{b_1}, a_2 + \mathbf{b_2}, \ldots, a_n + \mathbf{b_n})$$

where $\mathbf{b_i}$ is the string $b_{i1}, \ldots, b_{in_i}$, and $a_i + \mathbf{b_i}$ is

$$a_i + b_{i1}, a_i + b_{i2}, \ldots, a_i + b_{in_i}.$$

Action operads are themselves the objects of a category, **AOp**. The morphisms of this category are defined below.

**Definition 1.14.** A *map of action operads* $f : \Lambda \to \Lambda'$ consists of a map $f : \Lambda \to \Lambda'$ of the underlying operads such that

(1) $\pi^{\Lambda'} \circ f = \pi^{\Lambda}$ (i.e., $f$ is a map of operads over $\Sigma$) and
(2) each $f_n : \Lambda(n) \to \Lambda'(n)$ is a group homomorphism.

Another class of examples is that of operads *in* the category of groups, of which $A^\bullet$ above is an example.

**Proposition 1.15.** *There is a functor $Z : \mathbf{Op}(\mathbf{Grp}) \to \mathbf{AOp}$ from the category of operads in groups (using the cartesian monoidal structure) to the category of action operads. This functor is full, faithful, and its image is precisely the collection of action operads $\Lambda$ with $\pi_n(g) = e_n$ for all $n, g \in \Lambda(n)$.*

*Proof.* Given an operad in groups $P$, $Z(P)$ is the action operad with the same underlying operad and each $\pi_n$ the zero map. It is easy to verify that $\mu$ is a group homomorphism if and only if $\pi_n$ is zero for all $n$, and further that maps between such action operads are precisely the same thing as maps between the corresponding operads in the category of groups. □

We have the following corollary.

**Corollary 1.16.** *For an action operad $\Lambda$, the sets*

$$Ker\pi(n) = \{g \in \Lambda(n) : \pi_n(g) = e_n\}$$

*form an action operad for which the inclusion $Ker\pi \hookrightarrow \Lambda$ is a map of action operads. In particular, we have a sequence of action operad maps*

$$Ker\pi \hookrightarrow \Lambda \xrightarrow{\pi} \Sigma$$

*which is exact for each $n$.*

*Proof.* For $\mathrm{Ker}\,\pi \hookrightarrow \Lambda$ to be a map of action operads, we must define the map $\mathrm{Ker}\,\pi \to \Sigma$ to be zero, and we must check that the operadic multiplication of elements in the kernel is also in the kernel. This last fact is a trivial consequence of $\pi$ being an operad map. □

**Remark 1.17.** Zhang shows in Corollary 2.17 of [36] that the maps $\pi$ are either all zero or all surjective. In particular, we now see that every action operad is either an operad in the category groups or an extension of $\Sigma$ by such an operad. This gives a simple proof that the operads of pure braids and pure ribbon braids are both operads in groups.

**Remark 1.18.** The crossed simplicial groups of Fiedoriwicz and Loday [**?**] are related to action operads in the following way. There is a functor $C\colon \mathbf{AOp} \to \mathbf{CSGrp}$ from the category of action operads just described into the category of crossed simplicial groups. This functor takes an action operad $\Lambda$ and defines $C(\Lambda)(n) = \Lambda_{n+1}$. The face and degeneracy maps of the underlying simplicial structure are defined using the operadic composition inherent of $\Lambda$ - the description of the maps via the operad structure can be viewed in a similar way to Construction 1.1 of [**?**]. This functor is, however, not faithful, nor is it conservative.

**Example 1.19.** One can easily form new action operads from old ones by taking limits. To take a limit of a diagram in $\mathbf{AOp}$, one forgets down to the category of operads over $\Sigma$ and takes the limit there. Concretely, products in $\mathbf{AOp}$ are computed as products in $\mathbf{Op}/\Sigma$ which themselves are (possibly wide) pullbacks in the category of operads. This pullback will then be computed levelwise, showing that at each dimension we have a group structure with a group homomorphism to the appropriate $\Sigma_n$ and that the final action operad axiom holds since it does in each component. The equalizer of a pair of maps will then just be the levelwise equalizer. This shows that the pointwise product of an action operad $P$ with an action operad of the form $Z(Q)$ is again an action operad, but the pointwise product of two arbitrary action operads might not be.

Just as we had the definitions of operad, symmetric operad, and braided operad, we now come to the general definition of a $\Lambda$-operad, where $\Lambda$ is an action operad.

**Definition 1.20.** Let $\Lambda$ be an action operad. A $\Lambda$-*operad* $P$ (in **Sets**) consists of

- a non-symmetric operad $P$ in **Sets** and
- for each $n$, an action $P(n) \times \Lambda(n) \to P(n)$ of $\Lambda(n)$ on $P(n)$

such that the following two equivariance axioms hold.

$$\mu^P(x; y_1 \cdot g_1, \ldots, y_n \cdot g_n) = \mu^P(x; y_1, \ldots, y_n) \cdot \mu^\Lambda(e; g_1, \ldots, g_n)$$
$$\mu^P(x \cdot g; y_1, \ldots, y_n) = \mu^P(x; y_{\pi(g)^{-1}(1)}, \ldots, y_{\pi(g)^{-1}(n)}) \cdot \mu^\Lambda(g; e_1, \ldots, e_n)$$

**Definition 1.21.** The category $\Lambda$-$\mathbf{Op}$ of $\Lambda$-operads has objects the $\Lambda$-operads and morphisms those maps of operads which are levelwise equivariant with respect to the $\Lambda(n)$-actions.

**Example 1.22.**

1. Let $\mathbf{T}$ denote the terminal operad in **Sets** equipped with its unique action operad structure. Then a $\mathbf{T}$-operad is just a non-symmetric operad in **Sets**.
2. Let $\mathbf{\Sigma}$ denote the operad of symmetric groups with $\pi : \Sigma \to \Sigma$ the identity map. Then a $\mathbf{\Sigma}$-operad is a symmetric operad in the category of sets.
3. Let $\mathbf{Br}$ denote the operad of braid groups with $\pi_n : Br_n \to \Sigma_n$ the canonical projection of a braid onto its underlying permutation. Then a $\mathbf{Br}$-operad is a braided operad in the sense of Fiedorowicz [9].

The following lemma contains some useful basic calculations that we will require later for our proof that $\Lambda$-operads are the monoids in a category of collections.

**Lemma 1.23.** *Let $\Lambda$ be an action operad, and write $e_n$ for the unit element in the group $\Lambda(n)$.*

(1) *In $\Lambda(1)$, the unit element $e_1$ for the group structure is equal to the identity element for the operad structure, id.*
(2) *The equation*

$$\mu(e_n; e_{i_1}, \ldots, e_{i_n}) = e_I$$

*holds for any natural numbers $n, i_j, I = \sum i_j$.*
(3) *The group $\Lambda(1)$ is abelian.*

*Proof.* For the first claim, let $g \in \Lambda(1)$. Then

$$\begin{array}{rcl} g & = & g \cdot e_1 \\ & = & \mu(g; \mathrm{id}) \cdot \mu(\mathrm{id}; e_1) \\ & = & \mu(g \cdot \mathrm{id}; \mathrm{id} \cdot e_1) \\ & = & \mu(g \cdot \mathrm{id}; \mathrm{id}) \\ & = & g \cdot \mathrm{id} \end{array}$$

using that $e_1$ is the unit element for the group structure, that id is a two-sided unit for operad multiplication, and the final axiom for an action operad together with the fact that the only element of the symmetric group $\Sigma_1$ is the identity permutation. Thus $g = g \cdot \mathrm{id}$, so $\mathrm{id} = e_1$.

For the second claim, write the operadic product as $\mu(e; \underline{e})$, and consider the square of this element. We have

$$\begin{array}{rcl} \mu(e; \underline{e}) \cdot \mu(e; \underline{e}) & = & \mu(e \cdot e; \underline{e} \cdot \underline{e}) \\ & = & \mu(e; \underline{e}) \end{array}$$

where the first equality follows from the last action operad axiom together with the fact that $e$ gets mapped to the identity permutation; here $\underline{e} \cdot \underline{e}$ is the sequence $e_{i_1} \cdot e_{i_1}, \ldots, e_{i_n} \cdot e_{i_n}$. Thus $\mu(e; \underline{e})$ is an idempotent element of the group $\Lambda(I)$, so must be the identity element $e_I$.

For the final claim, note that operadic multiplication $\mu : \Lambda(1) \times \Lambda(1) \to \Lambda(1)$ is a group homomorphism by the action operad axioms, and $\mathrm{id} = e_1$ is a two-sided unit, so the Eckmann-Hilton argument shows that $\mu$ is actually group multiplication and that $\Lambda_1$ is abelian. □

An operad is intended to be an abstract description of a certain type of algebraic structure, and the particular instances of that structure are the algebras for that operad. We begin with the definition of an algebra over a non-symmetric operad and add in the group actions later.

**Definition 1.24.** Let $O$ be a non-symmetric operad. An *algebra* for $O$ consists of a set $X$ together with maps $\alpha_n : O(n) \times X^n \to X$ such that the following axioms hold.

(1) The element $\mathrm{id} \in O(1)$ is a unit in the sense that

$$\alpha_1(\mathrm{id}; x) = x$$

for all $x \in X$.

(2) The maps $\alpha_n$ are associative in the sense that the diagram

$$\begin{tikzcd}
O(n) \times O(k_1) \times X^{k_1} \times \cdots \times O(k_n) \times X^{k_n} \arrow[r, "1\times\alpha_{k_1}\times\cdots\alpha_{k_n}"] \arrow[d, "\cong"'] & O(n) \times X^n \arrow[dd, "\alpha_n"] \\
O(n) \times O(k_1) \times \cdots \times O(k_n) \times X^{k_1} \times \cdots \times X^{k_n} \arrow[d, "\mu\times 1"'] & \\
O(\sum k_i) \times X^{\sum k_i} \arrow[r, "\alpha_{\sum k_i}"'] & X
\end{tikzcd}$$

commutes.

Moving on to algebras for a $\Lambda$-operad, let $P$ be a $\Lambda$-operad and let $X$ be any set. Write $P(n) \times_{\Lambda(n)} X^n$ for the coequalizer of the pair of maps

$$P(n) \times \Lambda(n) \times X^n \rightrightarrows P(n) \times X^n$$

of which the first map is the action of $\Lambda(n)$ on $P(n)$ and the second map is

$$P(n) \times \Lambda(n) \times X^n \to P(n) \times \Sigma_n \times X^n \to P(n) \times X^n$$

using $\pi_n : \Lambda(n) \to \Sigma_n$ together with the canonical action of $\Sigma_n$ on $X^n$ by permutation of coordinates: $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. By the universal property of the coequalizer, a function $f : P(n) \times_{\Lambda(n)} X^n \to Y$ can be identified with a function $\tilde{f} : P(n) \times X^n \to Y$ such that

$$\tilde{f}(p \cdot g; x_1, \ldots, x_n) = \tilde{f}(p; x_{\pi(g)^{-1}(1)}, \ldots, x_{\pi(g)^{-1}(n)}).$$

**Definition 1.25.** Let $P$ be a $\Lambda$-operad. An *algebra* for $P$ consists of a set $X$ together with maps $\alpha_n : P(n) \times_{\Lambda(n)} X^n \to X$ such that the maps $\tilde{\alpha}_n$ satisfy the operad algebra axioms given in Definition 1.24.

**Remark 1.26.** It is worth noting that the equivariance required for a $P$-algebra is built into the definition above by requiring the existence of the maps $\alpha_n$ to be defined on coequalizers, even though the algebra axioms then only use the maps $\tilde{\alpha}_n$. Since every $\Lambda$-operad has an underlying plain operad (see 1.29, applied to the unique map $\mathbf{T} \to \Lambda$), this reflects the fact that the algebras for the $\Lambda$-equivariant version are always algebras for the plain version, but not conversely.

**Definition 1.27.** The category of algebras for $P$, $P$-**Alg**, has objects the $P$-algebras $(X, \alpha)$ and morphisms $f : (X, \alpha) \to (Y, \beta)$ those functions $f : X \to Y$ such that the following diagram commutes for every $n$.

$$\begin{tikzcd}
P(n) \times X^n \arrow[r, "1\times f^n"] \arrow[d, "\tilde{\alpha}_n"'] & P(n) \times Y^n \arrow[d, "\tilde{\beta}_n"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let $X$ be a set. Then the endomorphism operad of $X$, denoted $\mathcal{E}_X$, is given by the sets $\mathcal{E}_X(n) = \mathbf{Sets}(X^n, X)$ with the identity function in $\mathcal{E}_X(1)$ giving the unit element and composition of functions giving the composition operation. Concretely, composition is given by the formula

$$\mu(f; g_1, \ldots, g_n) = f \circ (g_1 \times \cdots \times g_n).$$

**Lemma 1.28.** *Let $\Lambda$ be an action operad, and let $X$ be a set. Then $\mathcal{E}_X$ carries a canonical $\Lambda$-operad structure.*

*Proof.* $\mathcal{E}_X$ is a symmetric operad, so we define the actions by

$$\mathcal{E}_X(n) \times \Lambda(n) \xrightarrow{1\times\pi_n} \mathcal{E}_X(n) \times \Sigma_n \to \mathcal{E}_X.$$

$\square$

The previous result is really a change-of-structure-groups result. We record the general result as the following proposition.

**Proposition 1.29.** *Let $f : \Lambda \to \Lambda'$ be a map of action operads. Then $f$ induces a functor $f^* : \Lambda'$-$\mathbf{Op} \to \Lambda$-$\mathbf{Op}$.*

We can now use endomorphism operads to characterize algebra structures.

**Proposition 1.30.** *Let $X$ be a set, and $P$ a $\Lambda$-operad. Then algebra structures on $X$ are in 1-to-1 correspondence with $\Lambda$-operad maps $P \to \mathcal{E}_X$.*

*Proof.* A map $P(k) \to \mathcal{E}_X(k)$ corresponds, using the closed structure on **Sets**, to a map $P(k) \times X^k \to X$. The monoid homomorphism axioms give the unit and associativity axioms, and the requirement that $P \to \mathcal{E}_X$ be a map of $\Lambda$-operads gives the equivariance condition. $\square$

**Remark 1.31.** The proposition above holds for $P$-algebras in any closed symmetric monoidal category. Having a closed structure (in addition to all small colimits) is a stronger condition than the tensor preserving colimits in each variable, but it is a natural one that arises in many examples.

**Definition 1.32.** Let $P$ be a $\Lambda$-operad. Then $P$ induces an endofunctor of **Sets**, denoted $\underline{P}$, by the following formula.

$$\underline{P}(X) = \coprod_n P(n) \times_{\Lambda(n)} X^n$$

We now have the following proposition; its proof is standard [31], and we leave it to the reader.

**Proposition 1.33.** *Let $P$ be a $\Lambda$-operad.*

1. *The $\Lambda$-operad structure on $P$ induces a monad structure on $\underline{P}$.*
2. *The category of algebras for the operad $P$ is isomorphic to the category of algebras for the monad $\underline{P}$.*

We end this section with a discussion of the relationship between symmetric operads and $\Lambda$-operads for an arbitrary action operad $\Lambda$.

**Theorem 1.34.** *Let $\Lambda$ be an action operad.*

1. *There is an adjunction between the category of $\Lambda$-operads and the category of symmetric operads, with right adjoint $\pi^* : \Lambda$-$\mathbf{Op} \to \Sigma$-$\mathbf{Op}$ and left adjoint denoted $S$.*
2. *The counit of this adjunction is an isomorphism, but the unit is not, thus this adjunction is not an equivalence of categories*
3. *There is a natural isomorphism of monads between $\underline{P}$ and $\underline{S(P)}$ for a $\Lambda$-operad $P$. In particular, these monads (and hence operads) have isomorphic categories of algebras*

*Proof.* Given any map of monoids $f : M \to N$ in a monoidal category, there is an adjunction between right $M$-modules and right $N$-modules given by $f^*$ as the right adjoint and $A \mapsto A \otimes_M N$ as the left adjoint. Thus we define

$$S(P)(n) = P(n) \times_{\Lambda(n)} \Sigma_n,$$

and this inherits a right $\Sigma_n$-action by multiplication. The unit of $S(P)$ is

$$* \xrightarrow{\eta} P(1) \longrightarrow P(1) \times_{\Lambda(1)} \Sigma_1 \cong P(1)/\Lambda(1).$$

For the multiplication, let $\underline{k} = k_1 + \cdots + k_n$, so we must define

$$\mu : (P(n) \times_{\Lambda(n)} \Sigma_n) \times (P(k_1) \times_{\Lambda(k_1)} \Sigma_{k_1}) \times \cdots \times (P(k_n) \times_{\Lambda(k_n)} \Sigma_{k_n}) \to P(\underline{k}) \times_{\Lambda(\underline{k})} \Sigma_{\underline{k}}.$$

Using the universal property of the coequalizer, this is induced by

$$\begin{array}{rcl} (P(n) \times \Sigma_n) \times (P(k_1) \times \Sigma_{k_1}) \times \cdots \times (P(k_n) \times \Sigma_{k_n}) & \cong & \\ \Big(P(n) \times \prod P(k_i)\Big) \times \Big(\Sigma_n \times \prod \Sigma_{k_i}\Big) & \to & \\ P(\underline{k}) \times \Sigma_{\underline{k}} & \to & P(\underline{k}) \times_{\Lambda(\underline{k})} \Sigma_{\underline{k}}. \end{array}$$

We leave verification of the associativity, unit, and equivariance axioms to the reader, they are simple applications of the same axioms for $P$ and $\Sigma$ together with some colimit universal properties and the $\Lambda$-operad axioms for $P$. It is then straightforward to check the bijection between $\Lambda$-operad maps $P \to \pi^* Q$ and symmetric operad maps $S(P) \to Q$, thus establishing the adjunction.

The second claim is a simple calculation using the coequalizer that defines $S(\pi^* Q)$, using that $Q(n)$ is itself the coequalizer of the obvious pair of maps $Q(n) \times \Sigma_n \times \Sigma_n$.

For the third claim, we get a natural isomorphism

$$P(n) \times_{\Lambda(n)} X^n \cong (P(n) \times_{\Lambda(n)} \Sigma_n) \times_{\Sigma_n} X^n$$

by the universal property of the colimits involved, so as functors $\underline{P} \cong \underline{S(P)}$. One can then easily verify that this isomorphism commutes with the unit and multiplication of the two monads involved using calculations similar to those used to establish the adjunction. ∎

**Remark 1.35.** (1) The adjunction alone is enough to establish that $P$ and $S(P)$ have isomorphic categories of algebras using Proposition 1.30.

(2) This theorem shows that semantically, one need never consider any kind of operad aside from symmetric operads: any other kind of operad can be symmetrized without altering the algebras. But as the operad should be considered a finer level of detail than the monad, restricting to symmetric operads misses the structure present in the more nuanced group actions.

Furthermore, there is clearly an artifact left from only considering the algebras themselves as objects in a symmetric monoidal category. It is well-known that a braided structure is all that is required for non-symmetric operads, and so one is left to consider that the natural home for algebras over a $\Lambda$-operad might be a type of monoidal structure other than symmetric in which case the theorem above gives no insight.

## 2. Operads as monoids

In this section, we will show that $\Lambda$-operads are the monoids in the category of $\Lambda$-collections equipped with an appropriate substitution product. Such a result is fairly standard [30], and in both the symmetric and non-symmetric cases can easily be proven directly. Since we work with an arbitrary action operad, however, it will be more economical to take the abstract approach using coends and Day convolution.

**Remark 2.1.** It is possible to consider $\Lambda$-operads in categories other than the category of sets. In this case we still use the notion of an action operad given above, but then take the operad $P$ to have objects $P(n)$ which are the objects of some closed symmetric monoidal category $\mathcal{V}$. We will rarely use anything that might require the closed structure as such, only the fact that the tensor product distributes over colimits in each variable. This is a consequence of the fact that both $X \otimes -$ and $- \otimes X$ are left adjoints in the case of a closed symmetric monoidal category. Thus while we set up the foundations using only operads in **Sets**, the diligent reader can easily modify this theory for their closed symmetric monoidal category of choice. In fact, we will use the same theory in **Cat** with its cartesian structure, noting only that the same arguments work in **Cat** with essentially no modification.

**Definition 2.2.** Let $\Lambda$ be an action operad. The category $\Lambda$-**Coll** of $\Lambda$-collections has objects $X = \{X(n)\}_{n \in \mathbb{N}}$ which consist of a set $X(n)$ for each natural number $n$ together with an action $X(n) \times \Lambda(n) \to X(n)$ of $\Lambda(n)$ on $X(n)$. A morphism $f : X \to Y$ in $\Lambda$-**Coll** consists of a $\Lambda(n)$-equivariant map $f_n : X(n) \to Y(n)$ for each natural number $n$.

**Remark 2.3.** The definition of $\Lambda$-**Coll** does not require that $\Lambda$ be an action operad, only that one has a natural number-indexed set of groups. Given any such collection of groups $\{\Lambda(n)\}_{n \in \mathbb{N}}$, we can form the category $\mathbb{\Lambda}$ whose objects are natural numbers and whose hom-sets are given by $\mathbb{\Lambda}(m, n) = \emptyset$ if $m \neq n$ and $\mathbb{\Lambda}(n, n) = \Lambda(n)$ (where composition and units are given by group multiplication and identity elements, respectively). Then $\Lambda$-**Coll** is the presheaf category

$$\hat{\mathbb{\Lambda}} = [\mathbb{\Lambda}^{\mathrm{op}}, \mathbf{Sets}],$$

with the opposite category arising from our choice of right actions. A key step in explaining how $\Lambda$-operads arise as monoids in the category of $\Lambda$-collections is to show that being an action operad endows $\mathbb{\Lambda}$ with a monoidal structure.

**Definition 2.4.** Let $\Lambda$ be an action operad, and let $X, Y$ be $\Lambda$-collections. We define the $\Lambda$-collection $X \circ Y$ to be

$$X \circ Y(n) = \Big( \coprod_{k_1 + \cdots + k_r = n} X(r) \times Y(k_1) \times \cdots \times Y(k_r) \Big) \times \Lambda(n) / \sim$$

where the equivalence relation is generated by

$$\begin{array}{rcl} (xh; y_1, \ldots, y_r; g) & \sim & (x; y_{\pi(h)^{-1}(1)}, \ldots, y_{\pi(h)^{-1}(r)}; \mu(h; e, \ldots, e)g), \\ (xe; y_1 g_1, \ldots, y_r g_r; g) & \sim & (x; y_1, \ldots, y_r; \mu(e; g_1, \ldots, g_r)g). \end{array}$$

For the first relation above, we must have that the lefthand side is an element of

$$X(r) \times Y(k_1) \times \cdots \times Y(k_r) \times \Lambda(n)$$

while the righthand side is an element of

$$X(r) \times Y(k_{\pi(h)^{-1}(1)}) \times \cdots \times Y(k_{\pi(h)^{-1}(r)}) \times \Lambda(n);$$

for the second relation, we must have $x \in X(r)$, $y_i \in Y(k_i)$, $f \in \Lambda(r)$, $g_i \in \Lambda(k_i)$, and $g \in \Lambda(n)$. The right $\Lambda(n)$-action on $X \circ Y(n)$ is given by multiplication on the final coordinate.

We will now develop the tools to prove that the category $\Lambda$-**Coll** has a monoidal structure given by $\circ$, and that operads are the monoids therein.

**Theorem 2.5.** *Let $\Lambda$ be an action operad.*

(1) *The category $\Lambda$-**Coll** has a monoidal structure with tensor product given by $\circ$ and unit given by the collection $I$ with $I(n) = \emptyset$ when $n \neq 1$, and $I(1) = \Lambda(1)$ with the $\Lambda$-action given by multiplication on the right.*
(2) *The category $\mathbf{Mon}(\Lambda\text{-}\mathbf{Coll})$ of monoids in $\Lambda$-**Coll** is equivalent to the category of $\Lambda$-operads.*

While this theorem can be proven by direct calculation using the equivalence relation given above, such a proof is unenlightening. Furthermore, we want to consider $\Lambda$-operads in categories other than sets, so an element-wise proof might not apply. Instead we now develop some general machinery that will apply to $\Lambda$-operads in any cocomplete symmetric monoidal category in which each of the functors $X \otimes -, - \otimes X$ preserve colimits (as is the case if the monoidal structure is closed). This theory also demonstrates the importance of the final axiom in the definition of an action operad. Our construction of the monoidal structure on the category of $\Lambda$-collections will require the Day convolution product [6]. This is a general construction which produces a monoidal structure on the category of presheaves $[\mathcal{V}^{\mathrm{op}}, \mathbf{Sets}]$ from a monoidal structure on the category $\mathcal{V}$. Since the category of $\Lambda$-collections is the presheaf category $[\mathbb{\Lambda}^{\mathrm{op}}, \mathbf{Sets}]$, we need to show that $\mathbb{\Lambda}$ has a monoidal structure.

**Proposition 2.6.** *The action operad structure of $\Lambda$ gives $\mathbb{\Lambda}$ a strict monoidal structure.*

*Proof.* The tensor product on $\mathbb{\Lambda}$ is given by addition on objects, with unit object 0. The only thing to do is define the tensor product on morphisms and check naturality for the associativity and unit isomorphisms, which will both be identities. On morphisms, $+$ must be given by a group homomorphism

$$+ : \Lambda(n) \times \Lambda(m) \to \Lambda(n+m),$$

and this is given by the formula

$$+(g,h) = \mu(e_2; g, h).$$

We need that $+$ is a group homomorphism, and the second part of Lemma 1.23 shows that it preserves identity elements. The final action operad axiom shows that it also preserves group multiplication since $\pi_2(e_2) = e_2$ (each $\pi_n$ is a group homomorphism) and therefore

$$\begin{aligned} \Big( + (g,h) \Big) \cdot \Big( + (g',h') \Big) &= \mu(e_2; g, h) \cdot \mu(e_2; g', h') \\ &= \mu(e_2 e_2; gg', hh') \\ &= +(gg', hh'). \end{aligned}$$

We now write $+(g,h)$ as $g + h$.

For naturality of the associator, we must have $(f+g)+h = f+(g+h)$. By the operad axioms for both units and associativity, the lefthand side is given by

$$\begin{aligned} \mu(e_2;\mu(e_2;f,g),h) &= \mu(e_2;\mu(e_2;f,g),\mu(\mathrm{id};h)) \\ &= \mu(\mu(e_2;e_2,\mathrm{id});f,g,h), \end{aligned}$$

while the righthand side is then

$$\mu(e_2;f,\mu(e_2;g,h)) = \mu(\mu(e_2;\mathrm{id},e_2);f,g,h).$$

By Lemma 1.23, both of these are equal to $\mu(e_3;f,g,h)$, proving associativity. Naturality of the unit isomorphisms follows similarly, using $e_0$. $\square$

Now that $\mathbb{\Lambda}$ has a monoidal structure, we get a monoidal structure on the category of $\mathbb{\Lambda}$-collections

$$[\mathbb{\Lambda}^{\mathrm{op}}, \mathbf{Sets}] = \hat{\mathbb{\Lambda}}$$

using Day convolution, denoted $\star$. Given collections $X, Y$, their convolution product $X \star Y$ is given by the coend formula

$$X \star Y(k) = \int^{m,n\in\mathbb{\Lambda}} X(m) \times Y(n) \times \mathbb{\Lambda}(k, m+n)$$

We refer the reader to [6] for further details. We do note, however, that the $n$-fold Day convolution product of a presheaf $Y$ with itself is given by the following coend formula.

$$Y^{\star n}(k) = \int^{(k_1,\ldots,k_n)\in\mathbb{\Lambda}^n} Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n)$$

Computations with Day convolution will necessarily involve heavy use of the calculus of coends, and we refer the reader in need of a refresher course on coends to [28]. Our goal is to express the substitution tensor product as a coend just as in [19], and to do that we need one final result about the Day convolution product.

**Lemma 2.7.** *Let $\Lambda$ be an action operad, let $Y \in \hat{\mathbb{\Lambda}}$, and let $k$ be a fixed natural number. Then the assignment*

$$n \mapsto Y^{\star n}(k)$$

*can be given the structure of a functor $\mathbb{\Lambda} \to \mathbf{Sets}$.*

*Proof.* Since the convolution product is given by a coend, it is the universal object with maps

$$Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \to Y^{\star n}(k)$$

such that the following diagram commutes for every $g_1 \in \Lambda(k_1), \ldots, g_n \in \Lambda(k_n)$.

$$\begin{tikzcd}
& Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \arrow[dr] & \\
Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \arrow[ur, "(-\cdot g_1,\ldots,-\cdot g_n)\times 1"] \arrow[dr, "1\times\big((g_1+\cdots+g_n)\cdot -\big)"'] & & Y^{\star n}(k) \\
& Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \arrow[ur] &
\end{tikzcd}$$

The first map along the top acts using the $g_i$'s, while the first map along the bottom is given by

$$h \mapsto \mu(e_n; g_1, \ldots, g_n) \cdot h$$

in the final coordinate.

Let $f \in \Lambda(n)$, considered as a morphism $n \to n$ in $\mathbb{\Lambda}$. We induce a map $f \bullet - : Y^{\star n}(k) \to Y^{\star n}(k)$ using the collection of maps

$$\prod_{i=1}^{n} Y(k_i) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \to \prod_{i=1}^{n} Y(k_{\pi(f)^{-1}(i)}) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n)$$

by using the symmetry $\pi(f)$ on the first $n$ factors and left multiplication by the element $\mu(f; e_{k_1}, \ldots, e_{k_n})$ on $\mathbb{\Lambda}(k, k_1 + \cdots + k_n)$. To induce a map between the coends, we must show that these maps commute with the two lefthand maps in the diagram above. For the top map, this is merely functoriality of the product together with naturality of the symmetry. For the bottom map, this is the equation

$$\mu(f; \overline{e}) \cdot \mu(e; g_1, \ldots, g_n) = \mu(e; g_{\pi(f)^{-1}1}, \ldots, g_{\pi(f)^{-1}n}) \cdot \mu(f; \overline{e}).$$

Both of these are equal to $\mu(f; g_1, \ldots, g_n)$ by the action operad axiom. Functoriality is then easy to check using that the maps inducing $(f_1 f_2) \bullet -$ are given by the composite of the maps inducing $f_1 \bullet (f_2 \bullet -)$. $\square$

We are now ready for the abstract description of the substitution tensor product. The following proposition is easily checked directly using the definition of the coend; in fact, the righthand side below should be taken as the definition of $X \circ Y$ as both sides are really the result of some colimiting process.

**Proposition 2.8.** *Let $X, Y \in \hat{\mathbb{\Lambda}}$. Then*

$$(X \circ Y)(k) \cong \int^{n} X(n) \times Y^{\star n}(k).$$

Finally we are in a position to prove Theorem 2.5. We make heavy use of the following consequence of the Yoneda lemma: given any functor $F : \mathbb{\Lambda} \to \mathbf{Sets}$ and a fixed object $a \in \mathbb{\Lambda}$, we have a natural isomorphism

$$\int^{n \in \mathbb{\Lambda}} \mathbb{\Lambda}(n, a) \times F(n) \cong F(a);$$

there is a corresponding result for $F : \mathbb{\Lambda}^{\mathrm{op}} \to \mathbf{Sets}$ using representables of the form $\mathbb{\Lambda}(a, n)$ instead.

*Proof of 2.5.* First we must show that $\Lambda$-**Coll** has a monoidal structure using $\circ$. To prove this, we must give the unit and associativity isomorphisms and then check the monoidal category axioms. First, note that the unit object is given as $I = \mathbb{\Lambda}(-, 1)$. Then for the left unit isomorphism, we have

$$\begin{aligned} I \circ Y(k) &\cong \int^{n} \mathbb{\Lambda}(n, 1) \times Y^{\star n}(k) \\ &\cong Y^{\star 1}(k) \\ &\cong Y(k) \end{aligned}$$

using only the properties of the coend. For the right unit isomorphism, we have

$$\begin{aligned} X \circ I(k) &\cong \int^{n} X(n) \times I^{\star n}(k) \\ &\cong \int^{n} X(n) \times \int^{k_1, \ldots, k_n} \mathbb{\Lambda}(k_1, 1) \times \cdots \times \mathbb{\Lambda}(k_n, 1) \times \mathbb{\Lambda}(k, k_1 + \cdots + k_n) \\ &\cong \int^{n} X(n) \times \mathbb{\Lambda}(k, 1 + \cdots + 1) \\ &= \int^{n} X(n) \times \mathbb{\Lambda}(k, n) \\ &\cong X(k) \end{aligned}$$

using the same methods.

For associativity, we compute $(X \circ Y) \circ Z$ and $X \circ (Y \circ Z)$.

$$\begin{array}{rcl} (X \circ Y) \circ Z(k) & = & \int^m X \circ Y(m) \times Z^{\star m}(k) \\ & = & \int^m \left( \int^l X(l) \times Y^{\star l}(m) \right) \times Z^{\star m}(k) \\ & \cong & \int^{m,l} X(l) \times Y^{\star l}(m) \times Z^{\star m}(k) \\ & \cong & \int^l X(l) \times \int^m Y^{\star l}(m) \times Z^{\star m}(k) \end{array}$$

The first isomorphism is from products distributing over colimits and hence coends, and the second is that fact plus the Fubini Theorem for coends [28]. A similar calculation shows

$$X \circ (Y \circ Z)(k) \cong \int^l X(l) \times (Y \circ Z)^{\star l}(k).$$

Thus the associativity isomorphism will be induced once we construct an isomorphism $\int^m Y^{\star l}(m) \times Z^{\star m} \cong (Y \circ Z)^{\star l}$. We do this by induction, with the $l = 1$ case being the isomorphism $Y^{\star 1} \cong Y$ together with the definition of $\circ$. Assuming true for $l$, we prove the case for $l + 1$ by the calculations below.

$$\begin{array}{rcl} (Y \circ Z)^{\star l+1} & \cong & (Y \circ Z) \star (Y \circ Z)^{\star l} \\ & \cong & (Y \circ Z) \star \left( \int^m Y^{\star l}(m) \times Z^{\star m} \right) \\ & = & \left( \int^n Y(n) \times Z^{\star n} \right) \star \left( \int^m Y^{\star l}(m) \times Z^{\star m} \right) \\ & = & \int^{a,b} \left( \int^n Y(n) \times Z^{\star n}(a) \right) \times \left( \int^m Y^{\star l}(m) \times Z^{\star m}(b) \right) \times \mathbb{A}(-, a+b) \\ & \cong & \int^{a,b,n,m} Y(n) \times Y^{\star l}(m) \times Z^{\star n}(a) \times Z^{\star m}(b) \times \mathbb{A}(-, a+b) \\ & \cong & \int^{n,m} Y(n) \times Y^{\star l}(m) \times Z^{\star (n+m)} \\ & \cong & \int^j \int^{n,m} Y(n) \times Y^{\star l}(m) \times \mathbb{A}(j, n+m) \times Z^{\star j} \\ & \cong & \int^j Y^{\star (l+1)}(j) \times Z^{\star j} \end{array}$$

Each isomorphism above arises from the symmetric monoidal structure on **Sets** using products, the monoidal structure on presheaves using $\star$, the properties of the coend, or the fact that products distribute over colimits.

For the monoidal category axioms on $\hat{\mathbb{A}}$, we only need to note that the unit and associativity isomorphisms arise, using the universal properties of the coend, from the unit and associativity isomorphisms on the category of sets together with the interaction between products and colimits. Hence the monoidal category axioms follow by those same axioms in **Sets** together with the universal property of the coend.

Now we must show that monoids in $(\hat{\mathbb{A}}, \circ)$ are operads. By the Yoneda lemma, a map of $\Lambda$-collections $\eta : I \to X$ corresponds to an element $\mathrm{id} \in X(1)$ since $I = \mathbb{A}(-, 1)$. A map $\mu : X \circ X \to X$ is given by a collection of $\Lambda(k)$-equivariant maps $X \circ X(k) \to X(k)$. By the universal property of the coend, this is equivalent to giving maps

$$\mu_{n,\underline{k}} : X(n) \times X(k_1) \times \cdots \times X(k_n) \times \mathbb{A}(k, k_1 + \cdots + k_n) \to X(k)$$

which are compatible with the actions of $\Lambda(k)$ (using the hom-set in the source, and the standard right action in the target) as well as each of $\Lambda(n), \Lambda(k_1), \ldots, \Lambda(k_n)$. The hom-set in $\mathbb{A}$ is nonempty precisely when $k = k_1 + \cdots + k_n$, so we define the operad multiplication $\mu$ for $X$ to be

$$\mu(x; y_1, \ldots, y_n) = \mu_{n,\underline{k}}(x; y_1, \ldots, y_n; e_k).$$

Compatibility with the actions of the $\Lambda(n), \Lambda(k_1), \ldots, \Lambda(k_n)$ give the equivariance axioms, and the unit and associativity for the monoid structure give the unit and associativity axioms for the operad structure. Finally, it is easy to check that a map of monoids is nothing more than an operad map which is appropriately equivariant for each $n$. $\square$

**Remark 2.9.** The above result can be interpreted for $\Lambda$-operads in an arbitrary cocomplete symmetric monoidal category $\mathcal{V}$ in which tensor distributes over colimits in each variable. In order to do so, the following changes must be made. First, cartesian products of objects $X(k)$ must be replaced by the tensor product in $\mathcal{V}$ of the same objects. Second, any product with a hom-set from $\Lambda$ must be replaced by a copower with the same set (recall that the copower of a set $S$ with an object $X$ is given by the formula $S \odot X = \coprod_S X$). The same changes also allow one to interpret the results below about algebras in such a category, unless noted otherwise.

## 3. Operads in Cat

This section will study those $\Lambda$-operads for which each $P(n)$ is a category, and from here onwards any operad denoted $P$ is in **Cat**. The extra structure that this 2-categorical setting provides allows us to consider notions of pseudoalgebras for an operad, as well as pseudomorphisms of operads. The induced monad associated to an operad of this sort can be shown to be a 2-monad (see [20] for background on 2-monads) and we will proceed to show that the notions of pseudoalgebra for both the operad and the associated 2-monad correspond precisely, i.e., there is an isomorphism of 2-categories between the 2-category with either strict or pseudo-level cells defined operadically and the 2-category with either strict or pseudo-level cells defined 2-monadically.

The associated monad $\underline{P}$ acquires the structure of a 2-functor as follows. We define $\underline{P}$ on categories much like before as the coproduct

$$\underline{P}(X) = \coprod_n P(n) \times_{\Lambda(n)} X^n,$$

whose objects will be written as equivalence classes $[p; x_1, \ldots, x_n]$ where $p \in P(n)$ and each $x_i \in X$, sometimes written as $[p; \underline{x}]$ when there is no confusion. On functors we define $\underline{P}$ in a similar way, exactly as with functions of sets. Given a natural transformation $\alpha \colon f \Rightarrow g$ we define a new natural transformation $\underline{P}(\alpha)$ as follows. The component of $\underline{P}(\alpha)$ at the object

$$[p; x_1, \ldots, x_n]$$

is given by the morphism

$$[1_p; \alpha_{x_1}, \ldots, \alpha_{x_n}]$$

in $\underline{P}(X)$. It is a simple observation that this constitutes a 2-functor, and that the components of the unit and multiplication are functors and are 2-natural.

**Remark 3.1.** The material in this section can be given a rather more abstract interpretation, in the sense of [**?**]. The idea here is that the category of $\Lambda$-collections acts on the category **Cat** via a functor $\diamond \colon \Lambda\textbf{-Coll} \times \textbf{Cat} \to \textbf{Cat}$ which sends $(P, X)$ to $\underline{P}(X)$ as described above. Fixing a $\Lambda$-collection $P$ produces an endofunctor $\underline{P} \colon \textbf{Cat} \to \textbf{Cat}$ which is then a monad when $P$ is a $\Lambda$-operad, just as monoids in $\Lambda$-**Coll** are precisely $\Lambda$-operads.

First we will set out some conventions and definitions.

**Conventions 3.2.** We will identify maps $\alpha_n\colon P(n) \times_{\Lambda(n)} X^n \to X$ with maps $\tilde{\alpha}_n\colon P(n) \times X^n \to X$ which are equivariant with respect to the $\Lambda$-actions via the universal property of the coequalizer. The coequalizer in **Cat** also has a 2-dimensional aspect to its universal property, so that a natural transformation $\Gamma\colon \alpha_n \Rightarrow \beta_n$ between functors as above determines and is determined by a transformation $\tilde{\Gamma}\colon \tilde{\alpha}_n \Rightarrow \tilde{\beta}_n$ with the property that the two possible whiskerings of $\tilde{\Gamma}$ with the two functors $P(n) \times \Lambda(n) \times X^n \to P(n) \times X^n$ are equal.

Note also that in the following definitions we will often write the composite

$$P(n) \times \prod (P(k_i) \times X^{k_i}) \to P(n) \times \prod P(k_i) \times X^{\Sigma k_i} \xrightarrow{\mu^P \times 1} P(\Sigma_{k_i}) \times X^{\Sigma k_i}$$

simply abbreviated as $\mu^P \times 1$. Furthermore, instead of using an element $\mathrm{id} \in P(1)$ as the operadic unit, we will now denote this as $\eta^P : 1 \to P(1)$.

We begin with the definitions of the pseudo-level cells in the operadic context, and after each specialize to the strict version.

**Definition 3.3.** Let $P$ be a $\Lambda$-operad. A *pseudoalgebra* for $P$ consists of:

- a category $X$,
- a family of functors

$$\left(\alpha_n : P(n) \times_{\Lambda(n)} X^n \to X\right)_{n\in\mathbb{N}},$$

- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism $\phi_{k_1,\ldots,k_n}$ (corresponding, via Conventions 3.2) to a natural isomorphism

$$\begin{tikzcd}
P_n \times \prod_{i=1}^n \left(P_{k_i} \times X^{k_i}\right) \arrow[rr, "1\times\prod\tilde{\alpha}_{k_i}"] \arrow[d] & & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_n \times \prod_{i=1}^n P_{k_i} \times X^{\Sigma k_i} \arrow[d, "\mu^P\times 1"'] \arrow[rr, phantom, "\Downarrow\tilde{\phi}_{k_1,\ldots,k_n}"] & & \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[rr, "\tilde{\alpha}_{\Sigma k_i}"'] & & X
\end{tikzcd}$$

- and a natural isomorphism $\phi_\eta$ corresponding to an isomorphism

$$\begin{tikzcd}
X \arrow[d, "\cong"'] \arrow[ddr, "1"] & \\
1 \times X \arrow[d, "\eta^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\phi}_\eta"] & \\
P(1) \times X \arrow[r, "\tilde{\alpha}_1"'] & X
\end{tikzcd}$$

satisfying the following axioms.

- For all $n, k_i, m_{ij} \in \mathbb{N}$, the following equality of pasting diagrams holds.

$$
\begin{tikzcd}
P_n \times \prod_i \big(P_{k_i} \times \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big)\big) \arrow[rr, "1\times\prod\left(1\times\prod\tilde{\alpha}_{m_ij}\right)"] \arrow[dd, "\mu^P\times 1"'] \arrow[dr, "\mu^P\times 1"] & & P_n \times \prod_i \big(P_{k_i} \times X^{k_i}\big) \arrow[d, "1\times\prod\tilde{\alpha}_{k_i}"] \\
& P_n \times \prod_i \big(P_{\Sigma m_{ij}} \times X^{\Sigma m_{ij}}\big) \arrow[ur, phantom, "\Downarrow 1\times\prod_i \tilde{\phi}_{m_{i1},\dots,m_{ik_i}}"] \arrow[r, "1\times\prod\tilde{\alpha}_{\Sigma m_{ij}}"] \arrow[dd, "\mu^P\times 1"] \arrow[ddr, phantom, "\Downarrow\tilde{\phi}_{\Sigma m_{1j},\dots,\Sigma m_{nj}}"] & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_{\Sigma k_i} \times \prod_i \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big) \arrow[dr, "\mu^P\times 1"'] & & \\
& P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \arrow[r, "\tilde{\alpha}_{\Sigma\Sigma m_{ij}}"'] & X
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
P_n \times \prod_i \big(P_{k_i} \times \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big)\big) \arrow[rr, "1\times\prod\left(1\times\prod\tilde{\alpha}_{m_ij}\right)"] \arrow[dd, "\mu^P\times 1"'] & & P_n \times \prod_i \big(P_{k_i} \times X^{k_i}\big) \arrow[dr, "1\times\prod\tilde{\alpha}_{k_i}"] \arrow[dd, "\mu^P\times 1"'] & \\
& & & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_{\Sigma k_i} \times \prod_i \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big) \arrow[rr, "1\times\prod\prod\tilde{\alpha}_{m_{ij}}"] \arrow[dr, "\mu^P\times 1"'] & & P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[ur, phantom, "\Downarrow\tilde{\phi}_{k_1,\dots,k_n}"] \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"] \arrow[dl, phantom, "\Downarrow\tilde{\phi}_{m_{11},\dots,m_{nk_n}}"] & \\
& P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \arrow[rr, "\tilde{\alpha}_{\Sigma\Sigma m_{ij}}"'] & & X
\end{tikzcd}
$$

- Each pasting diagram of the following form is an identity.

$$
\begin{tikzcd}
P_n \times X^n \arrow[dr, "\cong"] \arrow[ddrrr, bend left, "1"] \arrow[ddddrr, bend right, "1"'] & & & \\
& P_n \times (1 \times X)^n \arrow[dr, "1\times\left(\eta^P\times 1\right)^n"] \arrow[drr, phantom, "\Downarrow 1\times\tilde{\phi}_\eta^n"] & & \\
& & P_n \times (P_1 \times X)^n \arrow[r, "1\times\tilde{\alpha}_1^n"] \arrow[d, "\cong"'] & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
& & P_n \times P_1^n \times X^n \arrow[d, "\mu^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\phi}_{1,\dots,1}"] & {} \\
& & P_n \times X^n \arrow[r, "\tilde{\alpha}_n"'] & X
\end{tikzcd}
$$

**Definition 3.4.** Let $P$ be a $\Lambda$-operad. A *strict algebra* for $P$ consists of a pseudoalgebra in which all of the isomorphisms $\phi$ are identities.

**Definition 3.5.** Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a $\Lambda$-operad $P$. A *pseudomorphism* of $P$-pseudoalgebras consists of:

- a functor $f \colon X \to Y$
- and a family of natural isomorphisms (once again using 3.2)

$$\begin{tikzcd}
P_n \times X^n \arrow[r, "\tilde{\alpha}_n"] \arrow[d, "1 \times f^n"'] \arrow[dr, phantom, "\Downarrow \overline{f}_n"] & X \arrow[d, "f"] \\
P_n \times Y^n \arrow[r, "\tilde{\beta}_n"'] & Y
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$\begin{tikzcd}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) \arrow[rr, "1 \times \prod(1 \times f^{k_i})"] \arrow[dd, "\mu^P \times 1"'] & & P_n \times \prod_i (P_{k_i} \times Y^{k_i}) \arrow[dr, "1 \times \prod \tilde{\beta}_{k_i}"] \arrow[dd, "\mu^P \times 1"'] & \\
& & & P_n \times Y^n \arrow[dd, "\tilde{\beta}_n"] \arrow[dl, phantom, "\Downarrow \tilde{\psi}_{k_1, \ldots, k_n}"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[rr, "1 \times f^{\Sigma k_i}"] \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"'] & & P_{\Sigma k_i} \times Y^{\Sigma k_i} \arrow[dr, "\tilde{\beta}_{\Sigma k_i}"'] \arrow[dl, phantom, "\Downarrow \overline{f}_n"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) \arrow[rr, "1 \times \prod(1 \times f^{k_i})"] \arrow[dr, "1 \times \prod \tilde{\alpha}_{k_i}"] \arrow[dd, "\mu^P \times 1"'] & & P_n \times \prod_i (P_{k_i} \times Y^{k_i}) \arrow[dr, "1 \times \prod \tilde{\beta}_{k_i}"] \arrow[d, phantom, "\Downarrow 1 \times \prod \overline{f}_{k_i}"] & \\
& P_n \times X^n \arrow[rr, "1 \times f^n"] \arrow[dd, "\tilde{\alpha}_n"] \arrow[dl, phantom, "\Downarrow \tilde{\phi}_{k_1, \ldots, k_n}"] & {} & P_n \times Y^n \arrow[dd, "\tilde{\beta}_n"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"'] & & \Downarrow \overline{f}_n & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"'] \arrow[dddr, bend left, "1"] & \\
1 \times X \arrow[r, "1\times f"] \arrow[d, "\eta^P\times 1"'] & 1 \times Y \arrow[d, "\eta^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\psi}_\eta"] & {} \\
P_1 \times X \arrow[r, "1\times f"] \arrow[dr, "\tilde{\alpha}_1"'] & P_1 \times Y \arrow[dr, "\tilde{\beta}_1"] \arrow[d, phantom, "\Downarrow\overline{f}_1"] & \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] \arrow[dddr, bend left, "1"] & Y \arrow[dddr, bend left, "1"] & \\
1 \times X \arrow[d, "\eta^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\phi}_\eta"] & {} & \\
P_1 \times X \arrow[dr, "\tilde{\alpha}_1"'] & & \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

**Definition 3.6.** Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a $\Lambda$-operad $P$. A *strict morphism* of $P$-pseudoalgebras consists of a pseudomorphism in which all of the isomorphisms $\overline{f}_n$ are identities.

**Remark 3.7.** A strict algebra for a $\Lambda$-operad $P$ in **Cat** is precisely the same thing as an algebra for $P$ considered as an operad on the *category* of small categories and functors. A strict morphism between strict algebras is then just a map of $P$-algebras in the standard sense. We could also consider the notion of a lax algebra for an operad, or a lax morphism of algebras, simply by considering natural transformations in place of isomorphisms in the definitions.

**Definition 3.8.** Let $P$ be a $\Lambda$-operad and let $f, g\colon (X, \alpha, \phi, \phi_\eta) \to (Y, \beta, \psi, \psi_\eta)$ be pseudomorphisms of $P$-pseudoalgebras. A *$P$-transformation* is then a natural transformation $\gamma\colon f \Rightarrow g$ such that the following equality of pasting diagrams holds, for all $n$.

$$
\begin{tikzcd}
P_n \times X^n \arrow[r, bend left=40, "1\times f^n"] \arrow[r, "1\times g^n"'] \arrow[r, phantom, bend left=20, "\Downarrow 1\times\gamma^n"] \arrow[d, "\tilde{\alpha}_n"'] & P_n \times Y^n \arrow[d, "\tilde{\beta}_n"] \\
X \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\overline{g}_n"] & Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
P_n \times X^n \arrow[r, "1\times f^n"] \arrow[d, "\tilde{\alpha}_n"'] & P_n \times Y^n \arrow[d, "\tilde{\beta}_n"] \\
X \arrow[r, "f"] \arrow[r, bend right=40, "g"'] \arrow[r, phantom, bend right=20, "\Downarrow\gamma"] \arrow[ur, phantom, "\Downarrow\overline{f}_n"] & Y
\end{tikzcd}
$$

We can form various 2-categories using these cells.

**Definition 3.9.** Let $P$ be a $\Lambda$-operad.

- The 2-category $P$-$\mathbf{Alg}_s$ consists of strict $P$-algebras, strict morphisms, and $P$-transformations.
- The 2-category **Ps**-$P$-**Alg** consists of $P$-pseudoalgebras, pseudomorphisms, and $P$-transformations.

We also have the corresponding 2-monadic definitions, which we give for completeness. We state these for any 2-category $\mathcal{K}$, as specializing to **Cat** does not simplify them in any way.

**Definition 3.10.** Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A *$T$-pseudoalgebra* consists of an object $X$, a 1-cell $\alpha\colon TX \to X$, and invertible 2-cells

$$\begin{tikzcd}
T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "\eta_X"'] \arrow[drr, bend left, "1_X"] & \arrow[d, phantom, "\Downarrow\Phi_\eta"] & \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$\begin{tikzcd}
T^3X \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TX}"'] \arrow[dr, "T\mu_X"] & T^2X \arrow[d, phantom, "T\Phi\Downarrow"] \arrow[dr, "T\alpha"] & \\
T^2X \arrow[dr, "\mu_X"'] & T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
=
\begin{tikzcd}
T^3X \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_X"'] \arrow[dr, "T\alpha"] & \\
T^2X \arrow[r, "T\alpha"'] \arrow[dr, "\mu_X"'] & TX \arrow[d, phantom, "\Downarrow\Phi"] \arrow[r, phantom, "\Phi\Downarrow"] \arrow[dr, "\alpha"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

- The following pasting diagram is an identity.

$$\begin{tikzcd}
TX \arrow[dr, "T\eta_X"'] \arrow[drr, bend left, "1_{TX}"] \arrow[ddr, bend right, "1_{TX}"'] & \arrow[d, phantom, "\Downarrow T\Phi_\eta"] & \\
& T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

**Definition 3.11.** Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A *strict $T$-algebra* consists of a pseudoalgebra in which all of the isomorphisms $\Phi$ are identities.

**Definition 3.12.** Let $T$ be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$, $(Y, \beta, \Psi, \Psi_\eta)$ be $T$-pseudoalgebras. A *pseudomorphism* $(f, \bar{f})$ between these pseudoalgebras consists of a 1-cell $f\colon X \to Y$ along with an invertible 2-cell

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
T^2X \arrow[r, "T^2f"] \arrow[d, "\mu_X"'] \arrow[dr, "T\alpha"] \arrow[dr, phantom, "\Downarrow\Phi", shift right=4ex] & T^2Y \arrow[dr, "T\beta"] \arrow[d, phantom, "\Downarrow T\bar{f}"] & \\
TX \arrow[dr, "\alpha"'] & TX \arrow[r, "Tf"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TY \arrow[d, "\beta"] \\
& TX \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
T^2X \arrow[r, "T^2f"] \arrow[d, "\mu_X"'] & T^2Y \arrow[d, "\mu_Y"'] \arrow[dr, "T\beta"] & \\
TX \arrow[r, "Tf"'] \arrow[dr, "\alpha"'] \arrow[dr, phantom, "\Downarrow\bar{f}", shift left=4ex] & TX \arrow[dr, "\beta"'] \arrow[r, phantom, "\Psi\Downarrow"] & TY \arrow[d, "\beta"] \\
& TX \arrow[r, "f"'] & Y
\end{tikzcd}
$$

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\eta_X"'] & & Y \arrow[d, "\eta_Y"'] \arrow[ddr, bend left, "1_Y"] \arrow[d, phantom, "\Downarrow\Psi_\eta", shift left=4ex] & \\
TX \arrow[rr, "Tf"'] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\Downarrow\bar{f}"] & TY \arrow[dr, "\beta"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\eta_X"'] \arrow[ddr, bend left, "1_X"] & & Y \arrow[ddr, bend left, "1_Y"] & \\
TX \arrow[dr, "\alpha"'] \arrow[r, phantom, "\Downarrow\Phi_\eta"] & & & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

**Definition 3.13.** Let $T$ be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$, $(Y, \beta, \Psi, \Psi_\eta)$ be $T$-pseudoalgebras. A *strict morphism* $(f, \bar{f})$ consists of a pseudomorphism in which $\bar{f}$ is an identity.

**Remark 3.14.** Once again, the strict algebras and strict morphisms are exactly the same as algebras and morphisms for the underlying monad on the underlying category of $\mathcal{K}$.

**Definition 3.15.** Let $(f, \overline{f}), (g, \overline{g}) : X \to Y$ be pseudomorphisms of $T$-algebras. A $T$-*transformation* consists of a 2-cell $\gamma : f \Rightarrow g$ such that the following equality of pasting diagrams holds.

$$
\begin{tikzcd}
TX \arrow[r, bend left=40, "Tf"] \arrow[r, "Tg"'] \arrow[r, phantom, "\Downarrow T\gamma", shift left=2ex] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "f"] \arrow[r, bend right=40, "g"'] \arrow[r, phantom, "\Downarrow\gamma", shift right=2ex] & Y
\end{tikzcd}
$$

Once again, we have 2-categories defined using the different kinds of cells.

**Definition 3.16.** Let $T$ be a 2-monad.

- The 2-category $T$-$\mathbf{Alg}_s$ consists of strict $T$-algebras, strict morphisms, and $T$-transformations.

- The 2-category **Ps**-$T$-**Alg** consists of $T$-pseudoalgebras, pseudomorphisms, and $T$-transformations.

Our main result in this section is the following, showing that one can consider algebras and higher cells, in either strict or pseudo strength, using either the operadic or 2-monadic incarnation of a $\Lambda$-operad $P$. This extends Proposition 1.33.

**Theorem 3.17.** *Let $P$ be a $\Lambda$-operad in* **Cat**.

- *There is an isomorphism of 2-categories*
$$P\text{-}\mathbf{Alg}_s \cong \underline{P}\text{-}\mathbf{Alg}_s.$$
- *There is an isomorphism of 2-categories*
$$\mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$$
*extending the one above.*

*Proof.* We begin by noting that we suppress the difference between 2-cells $\Gamma$ and those $\tilde{\Gamma}$ as in 3.2, implicitly always using 2-cells defined on a coequalizer which are appropriately equivariant with respect to the group actions involved.

A proof of the first statement follows from our proof of the second by inserting identities where appropriate. Thus we begin by constructing a 2-functor $R\colon \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}$. We map a $\underline{P}$-pseudoalgebra $(X, \alpha, \Phi, \Phi_\eta)$ to the following $P$-pseudoalgebra on the same object $X$. First we define the functor $\alpha_n$ to be the composite

$$\alpha_n\colon P(n) \times_{\Lambda(n)} X^n \hookrightarrow \underline{P}(X) \xrightarrow{\alpha} X.$$

The isomorphisms $\phi_{k_1,\dots,k_n}$ are defined using $\Phi$ as in the following diagram

$$\begin{tikzcd}
P_n \times \prod_{i=1}^n \left(P_{k_i} \times X^{k_i}\right) \arrow[r] \arrow[dd] & P_n \times \prod_i \left(P_{k_i} \times_{\Lambda_{k_i}} X^{k_i}\right) \arrow[r, hook] & P_n \times \underline{P}(X)^n \arrow[r, "1\times\alpha^n"] \arrow[d] & P_n \times X^n \arrow[d] \\
 & & P_n \times_{\Lambda_n} \underline{P}(X)^n \arrow[d, hook] & P_n \times_{\Lambda_n} X^n \arrow[d, hook] \\
P_n \times \prod_i P_{k_i} \times X^{\Sigma k_I} \arrow[d, "\mu^P \times 1"'] & & \underline{P}^2(X) \arrow[r, "\underline{P}\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & \underline{P}(X) \arrow[d, "\alpha"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[r] & P_{\Sigma k_i} \times_{\Lambda_{\Sigma k_i}} X^{\Sigma k_i} \arrow[r, hook] & \underline{P}(X) \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

whilst $\Phi_\eta$ is simply sent to itself, since the composition of $\alpha$ with the composite of the coequalizer and inclusion map from $P(1) \times X$ into $\underline{P}(X)$ is just $\tilde{\alpha_1}$. Checking the axioms here is most easily done on components and it is easily seen that the axioms required of this data to be a $P$-pseudoalgebra are precisely those that they satisfy by virtue of $X$ being a $\underline{P}$-pseudoalgebra.

For a 1-cell $(f, \overline{f})\colon (X, \alpha) \to (Y, \beta)$, we send $f$ to itself whilst sending $\overline{f}$ to the obvious family of isomorphisms, as follows.

$$\begin{tikzcd}
P(n) \times X^n \arrow[r] \arrow[d, "1\times f^n"'] & P(n) \times_{\Lambda(n)} X^n \arrow[r, hook] \arrow[d, "1\times f^n"'] & \underline{P}(X) \arrow[r, "\alpha"] \arrow[d, "\underline{P}f"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & X \arrow[d, "f"] \\
P(n) \times Y^n \arrow[r] & P(n) \times_{\Lambda(n)} Y^n \arrow[r, hook] & \underline{P}(Y) \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

It is easy to check that the above data satisfy the axioms for being a pseudomorphism of $P$-pseudoalgebras, following from the axioms for $(f, \overline{f})$ being a pseudomorphism of $\underline{P}$-pseudoalgebras. A $\underline{P}$-transformation $\gamma\colon (f, \bar{f}) \Rightarrow (g, \bar{g})$ immediately gives a $P$-transformation $\bar{\gamma}$ between the families of isomorphisms we previously defined, with the components of $\bar{\gamma}$ being precisely those of $\gamma$. It is then obvious that $R$ is a 2-functor.

For there to be an isomorphism of 2-categories, we require an inverse to $R$, namely a 2-functor $S\colon \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$. Now assume that $(X, \alpha_n, \phi_{\underline{k}_i}, \phi_\eta)$ is a $P$-pseudoalgebra. We will give the same object $X$ a $\underline{P}$-pseudoalgebra structure. We can induce a functor $\alpha\colon \underline{P}(X) \to X$ by using the universal property of the coproduct.

$$\begin{tikzcd}
P(n) \times X^n \arrow[r] \arrow[drr, "\tilde{\alpha}_n"'] & P(n) \times_{\Lambda(n)} X^n \arrow[r, hook] \arrow[dr, "\alpha_n"] & \underline{P}(X) \arrow[d, "\exists!\alpha"] \\
& & X
\end{tikzcd}$$

Of course, this can be induced using either $\alpha_n$ or $\tilde{\alpha}_n$, each giving the same functor $\alpha$ by uniqueness. The components of the isomorphism $\Phi\colon \alpha \circ \underline{P}(\alpha) \Rightarrow \alpha \circ \mu_X$ can be given as follows. Let $|\underline{x}_i|$ denote the number of objects in the list $\underline{x}_i$. Then define the component of $\Phi$ at the object

$$[p; [q_1; \underline{x}_1], \ldots, [q_n; \underline{x}_n]]$$

to be component of $\phi_{|\underline{x}_1|, \ldots, |\underline{x}_n|}$ at the same object. To make this clearer, consider the object $[p; [q_1; x_{11}], [q_2; x_{21}, x_{22}], [q_3; x_{31}]]$. The component of $\Phi$ at this object is given by the component of $\phi_{1,2,1}$ at the same object. The isomorphism $\phi_\eta$ is again sent to itself.

Now given a 1-cell $f$ with structure 2-cells $\overline{f}_n$ we define a 1-cell $(F, \overline{F})$ with underlying 1-cell $f$ and structure 2-cell $\overline{F}$ with components

$$\overline{F}_{[p; x_1, \ldots, x_n]} := \left(\overline{f}_n\right)_{(p; x_1, \ldots, x_n)}.$$

For example, the component of $\overline{F}$ at the object $[p; x_1, x_2, x_3]$ would be the component of $f_3$ at the object $(p; x_1, x_2, x_3)$.

The mapping for 2-cells is just the identity as before. These mappings again constitute a 2-functor in the obvious way and from how they are defined it is also clear that this is an inverse to $R$. $\square$

**Remark 3.18.** Another interpretation of pseudoalgebras can be given in terms of pseudomorphisms of operads. Algebras for an operad $P$ can be identified with a morphism of operads $F\colon P \to \mathcal{E}_X$, where $\mathcal{E}_X$ is the endomorphism operad (Proposition 1.30). We can similarly define pseudomorphisms for a **Cat**-enriched $\Lambda$-operad and identify pseudoalgebras with pseudomorphisms into the endomorphism operad.

If $P$, $Q$ are $\Lambda$-operads then a *pseudomorphism* of $\Lambda$-operads $F\colon P \to Q$ consists of a family of $\Lambda$-equivariant functors

$$(F_n\colon P(n) \to Q(n))_{n \in \mathbb{N}}$$

together with isomorphisms instead of the standard algebra axioms. For example, the associativity isomorphism has the following form.

$$\begin{tikzcd} P(n)\times\prod_i P(k_i) \arrow[r, "F_n\times\prod_i F_{k_i}"] \arrow[d, "\mu^P"'] \arrow[dr, phantom, "\Downarrow\psi_{k_1,\ldots,k_n}"] & Q(n)\times\prod_i Q(k_i) \arrow[d, "\mu^Q"] \\ P(\Sigma k_i) \arrow[r, "F_{\Sigma k_i}"'] & Q(\Sigma k_i) \end{tikzcd}$$

These isomorphisms are then required to satisfy their own axioms, and these ensure that we have a weak map of 2-monads $\underline{P} \Rightarrow \underline{Q}$. In particular, one can show that a pseudomorphism from $P$ into the endomorphism operad $\mathcal{E}_X$ produces pseudoalgebras for the operad $P$ using the closed structure on **Cat**. While abstractly pleasing, we do not pursue this argument any further here.

## 4. Basic properties

This section will be concerned with characterizing various properties of those 2-monads induced by $\Lambda$-operads in **Cat**. We first show that these 2-monads are finitary. Second, we show that the coherence theorem in [24] applies to all such 2-monads and allows us to show that each pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra (and so similarly, by our previous results, to the pseudoalgebras for a $\Lambda$-operad $P$). Both of these results are simple extensions of well-known results about operads. Finally, we give conditions for these 2-monads to be 2-cartesian, describing how they interact with certain limits, namely 2-pullbacks. Operads do not always yield 2-cartesian 2-monads, and giving a complete characterization of when they do is more involved than our results on accessibility or coherence.

The 2-categories $\mathbf{Ps}$-$T$-$\mathbf{Alg}$ (of pseudoalgebras and weak morphisms) and $T$-$\mathbf{Alg}_s$ (of strict algebras and strict morphisms) are of particular interest. The behavior of colimits in both of these 2-categories can often be deduced from properties of the 2-monad $T$, the most common being that $T$ is finitary. In practice, one thinks of a finitary monad as one in which all operations take finitely many inputs as variables. If $T$ is finitary, then $T$-$\mathbf{Alg}_s$ will be cocomplete by standard results given in [5]. There are additional results of a purely 2-dimensional nature concerning finitary 2-monads, detailed in [24] and extending those in [5], namely the existence of a left adjoint

$$\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}_s$$

to the forgetful 2-functor which regards a strict algebra as a pseudoalgebra with identity structure isomorphisms.

We begin by showing each associated 2-monad is finitary.

**Proposition 4.1.** *Let $P$ be a $\Lambda$-operad. Then $\underline{P}$ is finitary.*

*Proof.* To show that $\underline{P}$ is finitary we must show that it preserves filtered colimits or, equivalently, that it preserves directed colimits (see [1]). Consider some directed colimit, colim$X_i$ say, in **Cat**. Then consider the following sequence of isomorphisms:

$$\begin{aligned}\underline{P}(\mathrm{colim}X_i) &= \coprod_n P(n) \times_{\Lambda(n)} (\mathrm{colim}X_i)^n \\ &\cong \coprod_n P(n) \times_{\Lambda(n)} \mathrm{colim}(X_i^n) \\ &\cong \coprod_n \mathrm{colim}(P(n) \times_{\Lambda(n)} X_i^n) \\ &\cong \mathrm{colim} \coprod_n P(n) \times_{\Lambda(n)} X_i^n = \mathrm{colim}\underline{P}(X_i).\end{aligned}$$

Since **Cat** is locally finitely presentable then directed colimits commute with finite limits, giving the first isomorphism. The second isomorphism follows from this fact as well as that colimits commute with coequalizers. The third isomorphism is simply coproducts commuting with other colimits. □

The next part of this section is motivated by the issue of coherence. At its most basic, a coherence theorem is a way of describing when a notion of weaker structure is in some way equivalent to a stricter structure. The prototypical case here is the coherence theorem for monoidal categories. In a monoidal category we require associator isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

for all objects in the category. The coherence theorem tells us that, for any monoidal category $M$, there is a strict monoidal category which is equivalent to $M$. In other words, we can treat the associators in $M$ as identities, and similarly for the unit isomorphisms.

The abstract approach to coherence considers when the pseudoalgebras for a 2-monad $T$ are equivalent to strict $T$-algebras, with the most comprehensive account appearing in [24]. Lack gives a general theorem which provides sufficient conditions for the existence of a left adjoint to the forgetful 2-functor

$$U \colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$$

for which the components of the unit of the adjunction are equivalences. We focus on one version of this general result which has hypotheses that are quite easy to check in practice. First we require that the base 2-category $\mathcal{K}$ has an enhanced factorization system. This is much like an orthogonal factorization system on a 2-category, consisting of two classes of maps $(\mathcal{L}, \mathcal{R})$, satisfying the lifting properties on 1-cells and 2-cells as follows. Given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

where $l \in \mathcal{L}$ and $r \in R$, there exists a unique morphism $m \colon B \to C$ such that $rm = g$ and $ml = f$. Similarly, given two commuting squares for which $rf = gl$ and $rf' = f'l$, along with 2-cells $\delta \colon f \Rightarrow f'$ and $\gamma \colon g \Rightarrow g'$ for which $\gamma * 1_l = 1_r * \delta$, there exists a unique 2-cell $\mu \colon m \Rightarrow m'$, where $m$ and $m'$ are induced by the 1-cell lifting property, satisfying $\mu * 1_l = \delta$ and $1_r * \mu = \gamma$. However, there is an additional

2-dimensional property of the factorization system which says that given maps $l \in \mathcal{L}$, $r \in \mathcal{R}$ and an invertible 2-cell $\alpha\colon rf \Rightarrow gl$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\swArrow\alpha"] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] \arrow[ur, "m"] \arrow[ur, phantom, shift right=3, "\swArrow\beta"] & D \end{tikzcd}$$

there is a unique pair $(m, \beta)$ where $m\colon C \to B$ is a 1-cell and $\beta\colon rm \Rightarrow g$ is an invertible 2-cell such that $ml = f$ and $\beta * 1_l = \alpha$.

Further conditions require that $T$ preserve $\mathcal{L}$ maps and that whenever $r \in \mathcal{R}$ and $rk \cong 1$, then $kr \cong 1$. In our case we are considering 2-monads on the 2-category **Cat**, which has the enhanced factorization system where $\mathcal{L}$ consists of bijective-on-objects functors and $\mathcal{R}$ is given by the full and faithful functors. This, along with the 2-dimensional property making it an enhanced factorization system, is described in [32]. The last stated condition, involving isomorphisms and maps in $\mathcal{R}$, is then clearly satisfied and so the only thing we need to check in order to satisfy the conditions of the coherence result are that the induced 2-monads $\underline{P}$ preserve bijective-on-objects functors, which follows simply from the fact that the set of objects functor preserves colimits, being left adjoint to the indiscrete category functor.

**Proposition 4.2.** *For any $\Lambda$-operad $P$, the 2-monad $\underline{P}$ preserves bijective-on-objects functors.*

**Corollary 4.3.** *Every pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra.*

We finally turn to a discussion of the interaction between operads and pullbacks. The monads arising from a non-symmetric operad are always cartesian, as described in [26]. The monads that arise from symmetric operads, however, are not always cartesian and so it is useful to be able to characterize exactly when they are. An example of where this fails is the symmetric operad for which the algebras are commutative monoids. In the case of 2-monads we can consider the strict 2-limit analogous to the pullback, the 2-pullback, and characterize when the induced 2-monad from a $\Lambda$-operad is 2-cartesian, as we now describe.

**Definition 4.4.** A 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ is said to be 2-*cartesian* if

- the 2-category $\mathcal{K}$ has 2-pullbacks,
- the functor $T$ preserves 2-pullbacks, and
- the naturality squares for the unit and multiplication of the 2-monad are 2-pullbacks.

It is important to note that the 2-pullback of a diagram is actually the same as the ordinary pullback in **Cat**, see [18]. Since we will be computing with coequalizers of the form $A \times_\Lambda B$ repeatedly, we give the following useful lemma.

**Lemma 4.5.** *Let $G$ be a group and let $A$, $B$ be categories for which $A$ has a right action by $G$ and $B$ has a left action by $G$. There is then an action of $G$ on the product $A \times B$ given by*

$$(a, b) \cdot g \colon = (a \cdot g, g^{-1} \cdot b).$$

*If this action of $G$ on $A \times B$ is free, then the category $A \times B/G$, consisting of the equivalence classes of this action, is isomorphic to the coequalizer $A \times_G B$.*

We begin our study of the cartesian property in the context of symmetric operads.

**Proposition 4.6.** *Let $P$ be a symmetric operad. Then the unit $\eta : id \Rightarrow \underline{P}$ for the associated monad is a cartesian transformation.*

*Proof.* In order to show that $\eta$ is cartesian, we must prove that for a functor $f : X \to Y$, the pullback of the following diagram is the category $X$.

$$\begin{tikzcd} & Y \arrow[d, "\eta_Y"] \\ \coprod P(n) \times_{\Sigma_n} X^n \arrow[r, "\underline{P}(f)"'] & \coprod P(n) \times_{\Sigma_n} Y^n \end{tikzcd}$$

The pullback of this diagram is isomorphic to the coproduct of the pullbacks of diagrams of the following form.

$$\begin{tikzcd} & Y \arrow[d] \\ P(1) \times X \arrow[r, "1\times f"'] & P(1) \times Y \end{tikzcd} \quad , \quad \begin{tikzcd} & \emptyset \arrow[d] \\ P(n) \times_{\Sigma_n} X^n \arrow[r, "1\times f^n"'] & P(n) \times_{\Sigma_n} Y^n \end{tikzcd}$$
$$n \neq 1$$

It is easy then to see that $X$ is the coproduct of the $n = 1$ cospan, and that the empty category is the pullback of each of the other cospans, making $X$ the pullback of the original diagram and verifying that $\eta$ is cartesian. $\square$

**Proposition 4.7.** *Let $P$ be a symmetric operad. Then the 2-monad $\underline{P}$ preserves pullbacks if and only if $\Sigma_n$ acts freely on $P(n)$ for all $n$.*

*Proof.* Consider the following pullback of discrete categories.

$$\begin{tikzcd} \{(x,y),(x,y'),(x',y),(x',y')\} \arrow[r] \arrow[d] & \{y,y'\} \arrow[d] \\ \{x,x'\} \arrow[r] & \{z\} \end{tikzcd}$$

Letting $\mathbf{4}$ denote the pullback and similarly writing $\mathbf{2}_X = \{x, x'\}$ and $\mathbf{2}_Y = \{y, y'\}$, we get the following diagram as the image of this pullback square under $\underline{P}$.

$$\begin{tikzcd} \coprod P(n) \times_{\Sigma_n} \mathbf{4}^n \arrow[r] \arrow[d] & \coprod P(n) \times_{\Sigma_n} \mathbf{2}_Y^n \arrow[d] \\ \coprod P(n) \times_{\Sigma_n} \mathbf{2}_X^n \arrow[r] & \coprod P(n)/\Sigma_n \end{tikzcd}$$

The projection map $\underline{P}(\mathbf{4}) \to \underline{P}(\mathbf{2}_Y)$ maps an element

$$[p; (x_1, y_1), \ldots, (x_n, y_n)]$$

to the element

$$[p; y_1, \ldots, y_n]$$

and likewise for the projection to $\underline{P}(\mathbf{2}_X)$.

Now assume that, for some $n$, the action of $\Sigma_n$ on $P(n)$ is not free. Then find some $p \in P(n)$ along with a nonidentity element $g \in \Sigma_n$ such that $p \cdot g = p$. We will show that the existence of $g$ proves that $\underline{P}$ is not cartesian.

Now $g \neq e$, so there exists an $i$ such that $g(i) \neq i$; without loss of generality, we may take $i = 1$. Using this $g$ we can find two distinct elements

$$[p; (x', y), (x, y), \ldots, (x, y), (x, y'), (x, y), \ldots, (x, y)]$$

and

$$[p; (x, y), \ldots, (x, y), (x', y'), (x, y), \ldots, (x, y)]$$

in $\underline{P}(\mathbf{4})$. In the first element we put $(x', y)$ in the first position and $(x, y')$ in position $g(1)$, whilst in the second element we put $(x', y')$ in position $g(1)$. Both of these elements, however, are mapped to the same elements in $\underline{P}(\mathbf{2}_X)$, since

$$\begin{aligned} [p; x', x, \ldots, x] &= [p \cdot g; (x', x, \ldots, x)] \\ &= [p; g \cdot (x', x, \ldots, x)] \\ &= [p; x, x, \ldots, x', \ldots, x]. \end{aligned}$$

Similarly, both of the elements are mapped to the same element in $\underline{P}(\mathbf{2}_Y)$, simply

$$[p; y, \ldots, y', \ldots, y].$$

The pullback of this diagram, however, has a unique element which is projected to the ones we have considered, so $\underline{P}(\mathbf{4})$ is not a pullback. Hence $\underline{P}$ does not preserve pullbacks if for some $n$ the action of $\Sigma_n$ on $P(n)$ is not free.

Now assume that each $\Sigma_n$ acts freely on $P(n)$. Given a pullback

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "R"'] & B \arrow[d, "S"] \\ C \arrow[r, "H"'] & D \end{tikzcd}$$

we must show that the image of the diagram under $\underline{P}$ is also a pullback. Now this will be true if and only if each individual diagram

$$\begin{tikzcd} P(n) \times_{\Sigma_n} A^n \arrow[r, "1 \times F^n"] \arrow[d, "1 \times R^n"'] & P(n) \times_{\Sigma_n} B^n \arrow[d, "1 \times S^n"] \\ P(n) \times_{\Sigma_n} C^n \arrow[r, "1 \times H^n"'] & P(n) \times_{\Sigma_n} D^n \end{tikzcd}$$

is also a pullback. The pullback of the functors $1 \times H^n$ and $1 \times S^n$ is a category consisting of pairs of objects $[p; \underline{c}]$ and $[q; \underline{b}]$, where $\underline{b}$ and $\underline{c}$ represent lists of elements in $B$ and $C$, respectively. These pairs are then required to satisfy the property that

$$[p; \underline{H(c)}] = [q; \underline{S(b)}].$$

Using the previous lemma, we know that a pair

$$([p; \underline{c}], [q; \underline{b}])$$

is in the pullback if and only if there exists an element $g \in \Sigma_n$ such that $p \cdot g = q$ and $Hc_i = (Sb_{\pi(g)^{-1}(i)})$. Using this we can define mutual inverses between $P(n) \times_{\Sigma_n} A^n$

and the pullback $Q'$. Considering the category $A$ as the pullback of the diagram we started with, we can consider objects of $P(n) \times_{\Sigma_n} A^n$ as being equivalence classes

$$[p; (b_1, c_1), \ldots, (b_n, c_n)]$$

where $p \in P(n)$ and $Hc_i = Sb_i$ for all $i$.

Taking such an object, we send it to the pair

$$([p; c_1, \ldots, c_n], [p; b_1, \ldots, b_n])$$

which lies in the pullback since the identity in $\Sigma_n$ satisfies the condition given earlier. An inverse to this sends a pair of equivalence classes in $Q'$ to the single equivalence class

$$[p; (c_1, b_{\pi(g)^{-1}(1)}), \ldots, (c_n, b_{\pi(g)^{-1}(n)})]$$

in $P(n) \times_{\Sigma_n} A^n$. If we apply the map into $Q'$ we get the pair

$$\big([p; c_1, \ldots, c_n], [p; b_{\pi(g)^{-1}(1)}, \ldots, b_{\pi(g)^{-1}(n)}]\big)$$

which is equal to the original pair since $p \cdot g = q$; the other composite is trivially an identity. A similar calculation on morphisms finishes the proof that $P(n) \times_{\Sigma_n} A^n$ is the pullback as required. □

**Proposition 4.8.** *Let $P$ be a symmetric operad. If the $\Sigma_n$-actions are all free, then the multiplication $\mu : \underline{P}^2 \Rightarrow \underline{P}$ of the associated monad is a cartesian transformation.*

*Proof.* Note that if all of the diagrams

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_X"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

are pullbacks then the outside of the diagram

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(f)"] \arrow[d, "\mu_X"'] & \underline{P}^2(Y) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_Y"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(f)"'] & \underline{P}(Y) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

is also a pullback and so each of the naturality squares for $\mu$ must therefore be a pullback. Now we can split up the square above, much like we did for $\eta$, and prove that each of the squares

$$\begin{tikzcd}
\coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i) \times_{\Sigma_{k_i}} X^{k_i}\right) \arrow[r] \arrow[d] & \coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i)/\Sigma_{k_i}\right) \arrow[d] \\
P(n) \times_{\Sigma_n} X^n \arrow[r] & P(n)/\Sigma_n
\end{tikzcd}$$

is a pullback. The map along the bottom is the obvious one, sending $[p; x_1, \ldots, x_n]$ simply to the equivalence class $[p]$. Along the right hand side the map is the one corresponding to operadic composition, sending $[q; [p_1], \ldots, [p_m]]$ to $[\mu^P(q; p_1, \ldots, p_n)]$. The pullback of these maps would be the category consisting of pairs

$$([p; x_1, \ldots, x_{\Sigma k_i}], [q; [p_1], \ldots, [p_n]]) ,$$

where $q \in P(n)$, $p_i \in P(k_i)$, $p \in P(\Sigma k_i)$, and for which $[p] = [\mu^P(q; p_1, \ldots, p_n)]$. The upper left category in the diagram, which we will refer to here as $Q$, has objects

$$[q; [p_1; \underline{x}_1], \ldots, [p_n; \underline{x}_n]].$$

There are obvious maps out of $Q$ making the diagram commute and as such inducing a functor from $Q$ into the pullback via the universal property. This functor sends an object such as the one just described to the pair

$$\left([\mu^P(q; p_1, \ldots, p_n); \underline{x}], [q; [p_1], \ldots, [p_n]]\right) .$$

Given an object in the pullback, we then have a pair, as described above, which has $[p] = [\mu^P(q; p_1, \ldots, p_n)]$ meaning that we can find an element $g \in \Sigma_{\Sigma k_i}$ such that $p = \mu^P(q; p_1, \ldots, p_n) \cdot g$. Thus we can describe an inverse to the induced functor by sending a pair in the pullback to the object

$$[q; [p_1; \pi(g)(\underline{x})_1], \ldots, [p_n; \pi(g)(\underline{x})_n]],$$

where $\pi(g)(\underline{x})_i$ denotes the $i$th block of $\underline{x}$ after applying the permutation $\pi(g)$. For example, if $\underline{x} = (x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31})$ and $\pi(g) = (1\,3\,5)$, then $\pi(g)(\underline{x}) = (x_{23}, x_{12}, x_{11}, x_{22}, x_{21}, x_{31})$. Thus $\pi(g)(\underline{x})_1 = (x_{23}, x_{12})$, $\pi(g)(\underline{x})_2 = (x_{11}, x_{22}, x_{21})$ and $\pi(g)(\underline{x})_3 = (x_{31})$.

Now applying the induced functor we find that we get back an object in the pullback for which the first entry is $[q; [p_1], \ldots, [p_n]]$ and whose second entry is

$$[\mu^P(q; p_1, \ldots, p_n); \pi(g)(\underline{x})] = [\mu^P(q; p_1, \ldots, p_n) \cdot g; \underline{x}] = [p; \underline{x}],$$

which is what we started with. Showing the other composite is an identity is similar, here using the fact that the identity acts trivially on $\mu^P(q; p_1, \ldots, p_n)$. Taking the coproduct of these squares then gives us the original diagram that we wanted to show was a pullback and, since each individual square is a pullback, so is the original. $\square$

Collecting these results together gives the following corollary.

**Corollary 4.9.** *The 2-monad associated to a symmetric operad $P$ is 2-cartesian if and only if the action of $\Sigma_n$ is free on each $P(n)$.*

We require one simple technical lemma before giving a complete characterization of $\Lambda$-operads which induce cartesian 2-monads.

**Lemma 4.10.** *Let $C$ be a category with a right action of some group $\Lambda$, and let $\pi : \Lambda \to \Sigma$ be a group homomorphism to any other group $\Sigma$. Then the right $\Sigma$-action on $C \times_\Lambda \Sigma$ is free if and only if the only elements of $\Lambda$ which fix an object of $C$ lie in the kernel of $\pi$.*

*Proof.* First, note that a group action on a category is free if and only if it is free on objects as fixing a morphism requires fixing its source and target. Thus our arguments need only concern the objects involved.

Since the set of objects functor preserves colimits, the objects of $C \times_\Lambda \Sigma$ are equivalence classes $[c; g]$ where $c \in C$ and $g \in \Sigma$, with $[c \cdot r; g] = [c; \pi(r)g]$. First assume the $\Sigma$-action is free. Then noting that $[c;e]\cdot g = [c;g]$, we have if $[c;g] = [c;e]$ then $g = e$. Let $r \in \Lambda$ be an element such that $c \cdot r = c$. Then

$$[c;e] = [c \cdot r;e] = [c;\pi(r)],$$

so $\pi(r) = e$.

Now assume that every element of $\Lambda$ fixing an object lies in the kernel of $\pi$. Let $\tau \in \Sigma$, and assume it fixes $[p;\sigma]$. Without loss of generality, we can take $\sigma = e$, so that

$$[p;\tau] = [p;e] \cdot \tau = [p;e].$$

Since the objects of $C \times_\Lambda \Sigma$ are equivalence classes as above, there is an element $r \in \Lambda$ such that $p \cdot r^{-1} = p$ and $\tau = \pi(r)$. But by assumption, we must have $r^{-1}$, and hence $r$, in the kernel, so $\tau = e$ and the $\Sigma$-action is free. $\square$

**Theorem 4.11.** *The 2-monad associated to a $\Lambda$-operad $P$ is 2-cartesian if and only if whenever $p \cdot g = p$ for an object $p \in P(n)$, $g \in Ker\,\pi(n)$.*

*Proof.* Since the monad $\underline{P}$ is isomorphic to $\underline{S(P)}$, we need only verify when $\underline{S(P)}$ is 2-cartesian. Thus the theorem is a direct consequence of Lemma 4.10 and Corollary 4.9. $\square$

## 5. The Borel construction for action operads

The clasical Borel construction is a functor from $G$-spaces to spaces, sending a $G$-space $X$ to $EG \times_G X$. Our goal in this section is to use the formal description of the Borel construction to construct some special operads in **Cat**. We start by reviewing the analogues of the functors $E, B : \mathbf{Grp} \to \mathbf{Top}$ now taking values in **Cat**.

**Definition 5.1.**
(1) Let $X$ be a set. We define the *translation category* $EX$ to have objects the elements of $X$ and morphisms consisting of a unique isomorphism between any two objects.
(2) Let $G$ be a group. The category $BG$ has a single object $*$, and hom-set $BG(*,*) = G$ with composition and identity given by multiplication and the unit element in the group, respectively.

**Definition 5.2.** A functor $F : X \to Y$ is an *isofibration* if given $x \in X$ and an isomorphism $f : y \cong F(x)$ in $Y$, there is an isomorphism $g$ in $X$ such that $F(g) = f$.

**Proposition 5.3.** *There is a natural transformation $p : EU \Rightarrow B$, where $U$ is the underlying set of a group, which is pointwise an isofibration. Applying the classifying space functor to the component $p_G$ gives a universal principal $G$-bundle.*

*Proof.* Given a group $G$, $p_G : EUG \to BG$ sends every object of $EUG$ to the unique object of $BG$. The unique isomorphism $g \to h$ in $EUG$ is mapped to $hg^{-1} : * \to *$. It is easy to directly check that this is an isofibration, as well as to see that the classifying spaces of $EUG$ and $BG$ are the spaces classically known as $EG, BG$, with $|p_G|$ being the standard universal principal $G$-bundle. $\square$

We will also need the functors $E, B$ defined for more than just a single set or group, in particular for the sets or groups which make up an operad and are indexed by the natural numbers.

**Notation 5.4.** Let $S$ be a set which we view as a discrete category.

(1) For any functor $F : S \to \mathbf{Sets}$, let $EF$ denote the composite $E \circ F : S \to \mathbf{Cat}$; we often view $F$ as an indexed set $\{F(s)\}$, in which case $EF$ is the indexed category $\{EF(s)\}$.
(2) For any functor $F : S \to \mathbf{Grp}$, let $BF$ denote the composite $B \circ F : S \to \mathbf{Cat}$; we often view $F$ as an indexed group $\{F(s)\}$, in which case $BF$ is the indexed category $\{BF(s)\}$.

The following lemma is a straightforward verification.

**Lemma 5.5.** *The functor $E$ : **Sets** $\to$ **Cat** is right adjoint to the set of objects functor. Therefore E preserves all limits, and in particular is a symmetric monoidal functor when both categories are equipped with their cartesian monoidal structures.*

Note that the operad of symmetric groups $\Sigma$ has its action operad structure determined by two auxiliary operations. The first is the block sum of permutations which we denote by

$$\beta : \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_{\underline{k}},$$

where $\underline{k} = \sum k_i$. The second is a kind of diagonal map which is defined for any natural number $n$ together with natural numbers $k_1, \ldots, k_n$. Then

$$\delta = \delta_{n;k_1,\ldots,k_n} : \Sigma_n \to \Sigma_{\underline{k}},$$

is defined on $\sigma \in \Sigma_n$ by permuting the elements $1, 2, \ldots, k_1$ together in a block according to the action of $\sigma \in \Sigma_n$ on 1, then $k_1 + 1, \ldots, k_1 + k_2$ together in a block according to the action of $\sigma$ on 2, and so on. The first of these, $\beta$, is a group homomorphism, while $\delta$ is a sort of twisted homomorphism, and taken together they define operadic multiplication in $\Sigma$. We now use these ideas to give the following algebraic characterization of action operads via the following definition.

**Definition 5.6.** For an action operad $\mathbf{\Lambda}$, define

$$\begin{array}{rcl} \beta(h_1, \ldots, h_n) & = & \mu(e; h_1, \ldots, h_n) \\ \delta(g) & = & \mu(g; e_1, \ldots, e_n). \end{array}$$

**Notation 5.7.** We will denote our identity elements in groups generically as $e$. If $\{G_i\}_{i \in I}$ are groups indexed by a set $I$, then $e_i$ is the identity element in $G_i$.

**Theorem 5.8.** *An action operad $\mathbf{\Lambda}$ determines, and is determined by, the following:*

- *groups $\Lambda(n)$ together with group homomorphisms $\pi_n : \Lambda(n) \to \Sigma_n$,*
- *a group homomorphism*

$$\Lambda(k_1) \times \cdots \times \Lambda(k_n) \xrightarrow{\beta} \Lambda(k_1 + \cdots + k_n),$$

  *for each $k_1, \ldots, k_n$ together with the degenerate case of $n = 0$ which then is a group homomorphism $1 \to \Lambda(0)$, and*
- *a function of sets*

$$\Lambda(n) \xrightarrow{\delta_{n;k_1,\ldots,k_n}} \Lambda(k_1 + \cdots + k_n)$$

  *for each $n, k_1, \ldots, k_n$,*

*subject to the axioms below. In what we write below, we use the following notational conventions.*

- *The symbols $f, g, h$, with or without subscripts, always refer to an element of some group $\Lambda(n)$.*

- *The symbols $j, k, m, n, p$ are all natural numbers, and $i$ is a natural number between 1 and $n$.*

*Axioms:*

(1) *The homomorphisms $\beta$ are natural with respect to the maps $\pi_n$, where $\underline{k} = k_1 + \cdots + k_n$.*

$$\begin{tikzcd} \Lambda(k_1) \times \cdots \times \Lambda(k_n) \arrow[r, "\beta"] \arrow[d, "\pi_1 \times \cdots \times \pi_n"'] & \Lambda(\underline{k}) \arrow[d, "\pi"] \\ \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \arrow[r, "\beta"'] & \Sigma_{\underline{k}} \end{tikzcd}$$

(2) *The homomorphism $\beta : \Lambda(k) \to \Lambda(k)$ is the identity.*

(3) *The homomorphisms $\beta$ are associative in the sense that*

$$\beta(h_{11}, \ldots, h_{1j_1}, h_{21}, \ldots, h_{2j_2}, \ldots, h_{nj_n}) = \beta\big(\beta(h_{11}, \ldots, h_{1j_1}), \ldots, \beta(h_{n1}, \ldots, h_{nj_n})\big)$$

*holds.*

(4) *The functions $\delta_{n;k_1,\ldots,k_n}$ are natural with respect to the maps $\pi_n$.*

$$\begin{tikzcd} \Lambda(n) \arrow[r, "\delta"] \arrow[d, "\pi"'] & \Lambda(k_1 + \cdots + k_n) \arrow[d, "\pi"] \\ \Sigma_n \arrow[r, "\delta"'] & \Sigma_{k_1 + \cdots + k_n} \end{tikzcd}$$

(5) *The functions $\delta_{n;1,\ldots,1}$, $\delta_{1;n} : \Lambda(n) \to \Lambda(n)$ are the identity.*

(6) *The equation $\delta_{n;k_i}(g)\delta_{n;j_i}(h) = \delta_{n;j_i}(gh)$ holds when*

$$k_i = j_{\pi(h)^{-1}(i)}.$$

(7) *The functions $\delta$ are associative in the sense that*

$$\delta_{m_1+\cdots+m_n;p_{11},\ldots,p_{1m_1},p_{21},\ldots,p_{nm_n}}\big(\delta_{n;m_1,\ldots,m_n}(f)\big) = \delta_{n;P_1,\ldots,P_n}(f)$$

*where $P_i = p_{i1} + \cdots + p_{im_i}$.*

(8) $\delta(g)\beta(h_1, \ldots, h_n) = \beta(h_{\pi(g)^{-1}(1)}, \ldots, h_{\pi(g)^{-1}(n)})\delta(g)$, *where $h_i \in \Lambda(k_i)$ and $\delta : \Lambda(n) \to \Lambda(k_1 + \cdots + k_n)$.*

(9) *The equation*

$$\beta(\delta_1(g_1), \ldots, \delta_n(g_n)) = \delta_c(\beta(g_1, \ldots, g_n))$$

*holds, where $\delta_i(g_i)$ is shorthand for $\delta_{k_i;m_{i1},\ldots,m_{ik_i}}(g_i)$ and $\delta_c$ is shorthand for*

$$\delta_{k_1+\cdots+k_n;m_{11},m_{12},\ldots,m_{1k_1},m_{21},\ldots,m_{nk_n}}.$$

*Proof.* Let $\mathbf{\Lambda}$ be an action operad, and define $\beta, \delta$ as in Definition 5.6. Since $\pi : \Lambda \to \Sigma$ is an operad map, axioms (1) and (4) hold. Since $\Lambda$ is an operad of sets, axioms (2) and (5) follow from the operad unit axioms, and axioms (3), (7), and (9) follow from the operad associativity axiom. Axioms (6) and (8) are special cases of the additional action operad axiom, as is the fact that $\beta$ is a group homomorphism.

Given the data above, we need only define the operad multiplication, verify the operad unit and multiplication axioms, and finally check the action operad axiom. Multiplication is given by

$$\mu(g; h_1, \ldots, h_n) = \delta_{n;k_1,\ldots,k_n}(g)\beta(h_1, \ldots, h_n)$$

where $h_i \in \Lambda(k_i)$. The unit is $e \in \Lambda(1)$.

We now verify the operad unit axioms.

$$\begin{array}{rcl} \mu(e;g) & = & \delta(e)\beta(g) \\ & = & e \cdot g \\ & = & g \\ \mu(h;e,\ldots,e) & = & \delta(h)\beta(e,\ldots,e) \\ & = & h \cdot e \\ & = & h \end{array}$$

These follow from axioms (2) and (5), together with the fact that $\beta$ is a group homomorphism.

For the operad associativity axiom, let

- $f \in \Lambda(m)$,
- $g_i \in \Lambda(n_i)$ for $i = 1,\ldots,m$, and
- $h_{ij} \in \Lambda(p_{i,j})$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n_i$.

Further, let $P_i = p_{i1} + \cdots + p_{in_i}$ and $\underline{h_{i\bullet}}$ denote the list $h_{i1}, h_{i2}, \ldots, h_{in_i}$. We must then show that

$$\mu\big(f;\mu(g_1;\underline{h_{1\bullet}}),\ldots,\mu(g_m;\underline{h_{m\bullet}})\big) = \mu\big(\mu(f;g_1,\ldots,g_m);h_{11},\ldots,h_{1n_1},h_{21},\ldots,h_{mn_m}\big).$$

By definition, the left side of this equation is

$$\delta_{m;P_1,\ldots,P_m}(f)\beta\big(\mu(g_1;\underline{h_{1\bullet}}),\ldots,\mu(g_m;\underline{h_{m\bullet}})\big),$$

and

$$\mu(g_i;\underline{h_{i\bullet}}) = \delta_{n_i;p_{i1},\ldots,p_{in_i}}(g_i)\beta(h_{i1},\ldots,h_{in_i}).$$

Since $\beta$ is a group homomorphism, we can then rewrite the left side as

$$\delta(f)\beta\big(\delta(g_1),\ldots,\delta(g_m)\big)\beta\big(\beta(h_{1\bullet}),\ldots,\beta(h_{m\bullet})\big)$$

where we have suppressed the subscripts on the $\delta$'s. By axiom (3) , we have

$$\beta\big(\beta(h_{1\bullet}),\ldots,\beta(h_{m\bullet})\big) = \beta(h_{11},\ldots,h_{1n_1},h_{21},\ldots,h_{mn_m}).$$

Further, axiom (9) above shows that

$$\beta\big(\delta(g_1),\ldots,\delta(g_m)\big) = \delta\big(\beta(g_1,\ldots,g_m)\big).$$

Thus we have shown that the left side of the operad associativity axiom is equal to

$$\delta(f)\delta\big(\beta(g_1,\ldots,g_m)\big)\beta(h_{11},\ldots,h_{1n_1},h_{21},\ldots,h_{mn_m}).$$

Now the right side is

$$\mu\big(\mu(f;g_1,\ldots,g_m);h_{11},\ldots,h_{1n_1},h_{21},\ldots,h_{mn_m}\big)$$

which is by definition

$$\delta\big(\mu(f;g_1,\ldots,g_m)\big)\beta(h_{11},\ldots,h_{1n_1},h_{21},\ldots,h_{mn_m}).$$

Thus verifying the operad associativity axiom reduces to showing

$$\delta(f)\delta\big(\beta(g_1,\ldots,g_m)\big) = \delta\big(\mu(f;g_1,\ldots,g_m)\big). \tag{2}$$

By the definition of $\mu$, we have

$$\delta(\mu(f;g_1,\ldots,g_m)) = \delta\big(\delta(f)\beta(g_1,\ldots,g_m)\big)$$

which is itself equal to

$$\delta\big(\delta(f)\big)\delta\big(\beta(g_1,\ldots,g_m)\big) \tag{3}$$

by axiom (6) above. Now the $\delta(f)$ on the left side of Equation (2) uses $\delta_{n;P_1,\ldots,P_n}$, while the $\delta(\delta(f))$ in Equation (3) is actually

$$\delta_{m_1+\cdots+m_n;q_{ij}}(\delta_{n;m_1,\ldots,m_n}(f))$$

where the $q_{ij}$ are defined, by axiom (6), to be given by

$$q_{ij} = p_{i,\pi(g_i)^{-1}(j)}$$

using the compatibility of $\beta$ and $\pi$ in axiom (1). By axiom (7), this composite of $\delta$'s is then $\delta_{n;Q_1,\ldots,Q_n}$ where $Q_i = q_{i1} + \cdots + q_{im_i}$. But by the definition of the $q_{ij}$, we immediately see that $Q_i = P_i$, so the $\delta(f)$ in Equation (2) is equal to the $\delta(\delta(f))$ appearing in Equation (3), concluding the proof of the operad associativity axiom.

The action operad axiom is now the calculation below, and uses axioms (4) and (8).

$$\begin{aligned}
\mu(g; h_1, \ldots, h_n)\mu(g'; h'_1, \ldots, h'_n) &= \delta(g)\beta(h_1, \ldots, h_n)\delta(g')\beta(h'_1, \ldots, h'_n) \\
&= \delta(g)\delta(g')\beta(h_{\pi(g')(1)}, \ldots, h_{\pi(g')(n)})\beta(h'_1, \ldots, h'_n) \\
&= \delta(gg')\beta(h_{\pi(g')(1)}h'_1, \ldots, h_{\pi(g')(n)}h'_n) \\
&= \mu(gg'; h_{\pi(g')(1)}h'_1, \ldots, h_{\pi(g')(n)}h'_n)
\end{aligned}$$

□

**Definition 5.9.** The category **AOp** of action operads has

- objects which are action operads $\mathbf{\Lambda}$, and
- morphisms $\mathbf{\Lambda} \to \mathbf{\Lambda}'$ which are those operad maps $f : \Lambda \to \Lambda'$ such that each $f_n : \Lambda(n) \to \Lambda'(n)$ is a group homomorphism and $\pi_n^{\Lambda} = \pi_n^{\Lambda'} \circ f_n$.

**Definition 5.10.** Let $\mathbf{\Lambda}$ be an action operad. A $\mathbf{\Lambda}$-*operad* $P$ (in **Sets**) consists of

- an operad $P$ in **Sets** and
- for each $n$, an action $\alpha_n : P(n) \times \Lambda(n) \to P(n)$ of $\Lambda(n)$ on $P(n)$

such that the following two equivariance axioms hold for $P$.

$$\begin{aligned}
\mu^P(x; y_1 \cdot g_1, \ldots, y_n \cdot g_n) &= \mu^P(x; y_1, \ldots, y_n) \cdot \mu^{\Lambda}(e; g_1, \ldots, g_n) \\
\mu^P(x \cdot g; y_1, \ldots, y_n) &= \mu^P(x; y_{\pi(g)^{-1}(1)}, \ldots, y_{\pi(g)^{-1}(n)}) \cdot \mu^{\Lambda}(g; e_1, \ldots, e_n)
\end{aligned}$$

We are additionally interested in $\mathbf{\Lambda}$-operads in **Cat** (or other cocomplete symmetric monoidal categories in which tensoring with a fixed object preserves colimits). While the definition above gives the correct notion of a $\mathbf{\Lambda}$-operad in **Cat** if we interpret the two equivariance axioms to hold for both objects and morphisms, it is useful to give a purely diagrammatic expression of these axioms. In the diagrams below, expressions of the form $G \times C$ for a group $G$ and category $C$ mean that the group $G$ is to be treated as a discrete category. This follows that standard method of how one expresses group actions in categories other than **Sets** using a copower. Thus the diagrams below are the two equivariance axioms given in Definition 5.10 expressed diagrammatically.

$$\begin{tikzcd}
P(n)\times P(k_1)\times\cdots\times P(k_n)\times\Lambda(k_1)\times\cdots\times\Lambda(k_n) \arrow[rr, "\cong"] \arrow[d, "\mu^P\times\mu^\Lambda(e;-)"'] & & P(n)\times P(k_1)\times\Lambda(k_1)\times\cdots\times P(k_n)\times\Lambda(k_n) \arrow[d, "\alpha_{k_1}\times\cdots\times\alpha_{k_n}"] \\
P(\underline{k})\times\Lambda(\underline{k}) \arrow[dr, "\alpha_{\underline{k}}"'] & & P(n)\times P(k_1)\times\cdots\times P(k_n) \arrow[dl, "\mu^P"] \\
& P(\underline{k}) &
\end{tikzcd}$$

$$\begin{tikzcd}
P(n)\times\Lambda(n)\times P(k_1)\times\cdots\times P(k_n) \arrow[rr, "\alpha_n\times 1"] \arrow[d, "1\times\Delta\times 1"'] & & P(n)\times P(k_1)\times\cdots\times P(k_n) \arrow[dd, "\mu^P"] \\
P(n)\times\Lambda(n)\times\Lambda(n)\times P(k_1)\times\cdots\times P(k_n) \arrow[d, "\cong"] & & \\
P(n)\times\Lambda(n)\times P(k_1)\times\cdots\times P(k_n)\times\Lambda(n) \arrow[d, "1\times\pi_n\times 1"'] & & P(\underline{k}) \\
P(n)\times\Sigma_n\times P(k_1)\times\cdots\times P(k_n)\times\Lambda(n) \arrow[rr, "\tilde{\mu}^P\times\mu^\Lambda(-;\underline{e})"'] & & P(\underline{k})\times\Lambda(\underline{k}) \arrow[u, "\alpha_{\underline{k}}"']
\end{tikzcd}$$

In the second diagram, the morphism $\tilde{\mu}^P : P(n)\times\Sigma_n\times P(k_1)\times\cdots\times P(k_n)$ is first the left action of $\Sigma_n$ on the product followed by the operad multiplication, and $\underline{e}$ is $e_{k_1},\ldots,e_{k_n}$.

**Proposition 5.11.** *Let $\mathbf{\Lambda}$ be an action operad. Then the operad $\Lambda$ is itself a $\mathbf{\Lambda}$-operad.*

*Proof.* One can in fact easily verify that the two equivariance axioms in the definition of a $\mathbf{\Lambda}$-operad follow from the final axiom for $\mathbf{\Lambda}$ being an action operad. □

**Definition 5.12.** Let $\mathbf{\Lambda}$ be an action operad. Then $B\mathbf{\Lambda}$ (see Notation 5.4) is the category with objects the natural numbers and

$$B\mathbf{\Lambda}(m,n) = \begin{cases} \Lambda(n), & m = n \\ \varnothing, & m \neq n, \end{cases}$$

where composition is given by group multiplication and the identity morphism is the unit element $e_n \in \Lambda(n)$.

**Theorem 5.13.** *Let $M, N$ be cocomplete symmetric monoidal categories in which the tensor product preserves colimits in each variable, and let $F : M \to N$ be a symmetric lax monoidal functor with unit constraint $\varphi_0$ and tensor constraint $\varphi_2$. Let $\mathbf{\Lambda}$ be an action operad, and $P$ a $\mathbf{\Lambda}$-operad in $M$. Then $FP = \{F(P(n))\}$ has a canonical $\mathbf{\Lambda}$-operad structure, giving a functor*

$$F_* : \mathbf{\Lambda}\text{-}Op(M) \to \mathbf{\Lambda}\text{-}Op(N)$$

*from the category of $\mathbf{\Lambda}$-operads in $M$ to the category of $\mathbf{\Lambda}$-operads in $N$.*

*Proof.* The category of $\mathbf{\Lambda}$-operads in $M$ is the category of monoids for the composition product $\circ_M$ on $[B\mathbf{\Lambda}^{\mathrm{op}}, M]$ constructed in [**?**]. Composition with $F$ gives a functor

$$F_* : [B\mathbf{\Lambda}^{\mathrm{op}}, M] \to [B\mathbf{\Lambda}^{\mathrm{op}}, N],$$

and to show that it gives a functor between the categories of monoids we need only prove that $F_*$ is lax monoidal with respect to $\circ_M$ and $\circ_N$. In other words, we must construct natural transformations with components $F_*X \circ_N F_*Y \to F_*(X \circ_M Y)$ and $I_{Op(N)} \to F_*(I_{Op(M)})$ and then verify the lax monoidal functor axioms. We note that in the calculations below, we often write $F$ instead of $F_*$, but it should be clear from context when we are applying $F$ to objects and morphisms in $M$ and when we are applying $F_*$ to a functor $B\mathbf{\Lambda}^{\mathrm{op}} \to M$.

We first remind the reader about copowers in cocomplete categories. For an object $X$ and set $S$, the copower $X \odot S$ is the coproduct $\coprod_{s \in S} X$. We have natural isomorphisms $(X \odot S) \odot T \cong X \odot (S \times T)$ and $X \odot 1 \cong X$, and using these we can define an action of a group $G$ on an object $X$ using a map $X \odot G \to X$. Any functor $F$ between categories with coproducts is lax monoidal with respect to those coproducts: the natural map $FA \coprod FB \to F(A \coprod B)$ is just the map induced by the universal property of the coproduct using $F$ applied to the coproduct inclusions $A \hookrightarrow A \coprod B, B \hookrightarrow A \coprod B$. In particular, for any functor $F$ we get an induced map $FX \odot S \to F(X \odot S)$.

The unit object in $[B\mathbf{\Lambda}^{\mathrm{op}}, M]$ for $\circ_M$ is the copower $I_M \odot B\mathbf{\Lambda}(-, 1)$. Thus the unit constraint for $F_*$ is the composite

$$I_N \odot B\mathbf{\Lambda}(-, 1) \xrightarrow{\varphi_0 \odot 1} FI_M \odot B\mathbf{\Lambda}(-, 1) \to F(I_M \odot B\mathbf{\Lambda}(-, 1)).$$

For the tensor constraint, we will require a map

$$t : (FY)^{\star n}(k) \to F\big(Y^{\star n}(k)\big)$$

where $\star$ is the Day convolution product; having constructed one, the tensor constraint is then the composite

$$\begin{array}{rcl} (FX \circ FY)(k) & \cong & \int^n FX(n) \otimes (FY)^{\star n}(k) \\ & \xrightarrow{\int 1 \otimes t} & \int^n FX(n) \otimes F(Y^{\star n}(k)) \\ & \xrightarrow{\int \varphi_2} & \int^n F(X(n) \otimes Y^{\star n}(k)) \\ & \longrightarrow & F(\int^n X(n) \otimes Y^{\star n}(k)) \\ & \cong & F(X \circ Y)(k), \end{array}$$

where both isomorphisms are induced by universal properties (see Section 3 for more details) and the unlabeled arrow is induced by the same argument as that for coproducts above but this time using coends. The arrow $t$ is constructed in a similar fashion, and is the composite below.

$$\begin{array}{rcl} (FY)^{\star n}(k) & = & \int^{k_1, \ldots, k_n} FY(k_1) \otimes \cdots \otimes FY(k_n) \odot B\mathbf{\Lambda}(k, \sum k_i) \\ & \to & \int^{k_1, \ldots, k_n} F(Y(k_1) \otimes \cdots \otimes Y(k_n)) \odot B\mathbf{\Lambda}(k, \sum k_i) \\ & \to & \int^{k_1, \ldots, k_n} F(Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\mathbf{\Lambda}(k, \sum k_i)) \\ & \to & F \int^{k_1, \ldots, k_n} Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\mathbf{\Lambda}(k, \sum k_i) \\ & = & F(Y^{\star n}(k)) \end{array}$$

Checking the lax monoidal functor axioms is tedious but entirely routine using the lax monoidal functor axioms for $F$ together with various universal properties of colimits, and we leave the details to the reader. □

Combining Theorem 5.13 and Proposition 5.11 with Corollary 5.5, we immediately obtain the following.

**Corollary 5.14.** *Let $\mathbf{\Lambda}$ be an action operad. Then $E\Lambda = \{E\big(\Lambda(n)\big)\}$ (see Notation 5.4) is a $\mathbf{\Lambda}$-operad in* **Cat**.

Any $\mathbf{\Lambda}$-operad $P$ in **Cat** gives rise to a 2-monad on **Cat** which we will also denote $P$. In our case, that 2-monad (also denoted $E\Lambda$) is given by

$$X \mapsto \coprod_{n\geq 0} E\Lambda(n) \times_{\Lambda(n)} X^n$$

where the action of $\Lambda(n)$ on $E\Lambda(n)$ is given by the obvious multiplication action on the right, and the action of $\Lambda(n)$ on $X^n$ is given using $\pi_n : \Lambda(n) \to \Sigma_n$ together with the standard left action of $\Sigma_n$ on $X^n$ in any symmetric monoidal category. The 2-monad $E\Sigma$ is that for symmetric strict monoidal categories (see Section 4 for this and further examples).

It will be useful for our calculations later to give an explicit description of the categories $E\Lambda(n) \times_{\Lambda(n)} X^n$. Objects are equivalence classes of tuples $(g; x_1, \ldots, x_n)$ where $g \in \Lambda(n)$ and the $x_i$ are objects of $X$, with the equivalence relation given by

$$(gh; x_1, \ldots, x_n) \sim (g; x_{\pi(h)^{-1}(1)}, \ldots, x_{\pi(h)^{-1}(n)});$$

we write these classes as $[g; x_1, \ldots, x_n]$. Morphisms are then equivalence classes of morphisms

$$(!; f_1, \ldots, f_n) : (g; x_1, \ldots, x_n) \to (g'; x'_1, \ldots, x'_n).$$

We have two distinguished classes of morphisms, one for which the map $! : g \to h$ is the identity and one for which all the $f_i$'s are the identity. Every morphism in $E\Lambda(n) \times X^n$ is uniquely a composite of a morphism of the first type followed by one of the second type. Now $E\Lambda(n) \times_{\Lambda(n)} X^n$ is a quotient of $E\Lambda(n) \times X^n$ by a free group action, so every morphism of $E\Lambda(n) \times_{\Lambda(n)} X^n$ is in the image of the quotient map. Using this fact, we can prove the following useful lemma.

**Lemma 5.15.** *For an action operad $\mathbf{\Lambda}$ and any category $X$, the set of morphisms from $[e; x_1, \ldots, x_n]$ to $[e; y_1, \ldots, y_n]$ in $E\Lambda(n) \times_{\Lambda(n)} X^n$ is*

$$\coprod_{g\in\Lambda(n)} \prod_{i=1}^{n} X(x_i, y_{\pi(g)(i)}).$$

*Proof.* A morphism with source $(e; x_1, \ldots, x_n)$ in $E\Lambda(n) \times X^n$ is uniquely a composite

$$(e; x_1, \ldots, x_n) \stackrel{(\mathrm{id}; f_1,\ldots,f_n)}{\longrightarrow} (e; x'_1, \ldots, x'_n) \stackrel{(!;\mathrm{id},\ldots,\mathrm{id})}{\longrightarrow} (g; x'_1, \ldots, x'_n).$$

Descending to the quotient, this becomes a morphism

$$[e; x_1, \ldots, x_n] \to [g; x'_1, \ldots, x'_n] = [e; x'_{\pi(g)^{-1}(1)}, \ldots, x'_{\pi(g)^{-1}(n)}],$$

and therefore is a morphism $[e; x_1, \ldots, x_n] \to [e; y_1, \ldots, y_n]$ precisely when $y_i = x'_{\pi(g)^{-1}(i)}$, and so $f_i \in X(x_i, y_{\pi(g)(i)})$. ☐

The 2-monad $E\Lambda$ is both finitary and cartesian (see [**?**]). In fact we can characterize this operad uniquely (up to equivalence) using a standard argument.

**Definition 5.16.** Let $\mathbf{\Lambda}$ be an action operad. A $\mathbf{\Lambda}_\infty$ *operad* $P$ is a $\mathbf{\Lambda}$-operad in which each action $P(n) \times \Lambda(n) \to P(n)$ is free and each $P(n)$ is contractible.

**Remark 5.17.** One should also note that by Corollary 5.14 we get a canonical $\mathbf{\Lambda}_\infty$ operad in **Cat**, namely $E\mathbf{\Lambda}$ itself, and thus also in the category of simplicial sets by taking the nerve (the nerve functor is represented by a cosimplicial category, namely $\Delta \subseteq \mathbf{Cat}$, so preserves products) and in suitable categories of topological spaces by taking the geometric realization (once again, product-preserving with the correct category of spaces). Thus we have something like a Barratt-Eccles $\mathbf{\Lambda}_\infty$ operad for any action operad $\mathbf{\Lambda}$.

**Remark 5.18.** The above definition makes sense in a wide context, but needs interpretation. We can interpret the freeness condition in any complete category, as completeness allows one to compute fixed points using equalizers. Contractibility then requires a notion of equivalence or weak equivalence, such as in an $(\infty, 1)$-category or Quillen model category, and a terminal object. Our interest is in the above definition interpreted in **Cat**, in which case both conditions (free action and contractible $P(n)$'s) mean the obvious thing.

**Proposition 5.19.** *For any two $\mathbf{\Lambda}_\infty$ operads $P, Q$ in* **Cat***, there is a span $P \leftarrow R \rightarrow Q$ of pointwise equivalences of $\mathbf{\Lambda}$-operads.*

*Proof.* Given $P, Q$ $\mathbf{\Lambda}_\infty$ operads in **Cat**, the product $P \times Q$ with the diagonal action is also $\mathbf{\Lambda}_\infty$. Each of the projection maps is a pointwise equivalence of $\mathbf{\Lambda}$-operads. □

**Remark 5.20.** Once again, this proof holds in a wide context. We required that the product of free actions is again free, true in any complete category. We also required that the product of contractible objects is contractible; this condition will hold, for example, in any Quillen model category in which all objects are fibrant or in which the product of weak equivalences is again a weak equivalence.

## 6. Abstract properties of the Borel construction

Kelly's theory of clubs [15, 16, 17] was designed to simplify and explain certain aspects of coherence results, namely the fact that many coherence results rely on extrapolating information about general free objects for a 2-monad $T$ from information about the specific free object $T1$ where 1 denotes the terminal category. This occurs, for example, in the study of the many different flavors of monoidal category: plain monoidal category, braided monoidal category, symmetric monoidal category, and so on. This section will explain how every action operad gives rise to a club, as well as compute the clubs which arise as the image of this procedure.

We begin by reminding the reader of the notion of a club, or more specifically what Kelly [15, 17] calls a club over **P**. We will only be interested in clubs over **P**, and thusly shorten the terminology to club from this point onward. Defining clubs is accomplished most succinctly using Leinster's terminology of generalized operads [26].

**Definition 6.1.** Let $C$ be a category with finite limits.

(1) A monad $T : C \to C$ is *cartesian* if the functor $T$ preserves pullbacks, and the naturality squares for the unit $\eta$ and the multiplication $\mu$ for $T$ are all pullbacks.
(2) The category of $T$-*collections*, $T$-**Coll**, is the slice category $C/T1$, where 1 denotes the terminal object.
(3) Given a pair of $T$-collections $X \xrightarrow{x} T1, Y \xrightarrow{y} T1$, their *composition product* $X \circ Y$ is given by the pullback below together with the morphism along the top.

$$\begin{tikzcd}
X \circ Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TY \arrow[r, "Ty"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu"] & T1 \\
X \arrow[r, "x"'] & T1 & &
\end{tikzcd}$$

(4) The composition product, along with the unit of the adjunction $\eta : 1 \to T1$, give $T$-**Coll** a monoidal structure. A $T$-*operad* is a monoid in $T$-**Coll**.

**Remark 6.2.** Everything in the above definition can be **Cat**-enriched without any substantial modifications. Thus we require our ground 2-category to have finite limits in the enriched sense, and the slice and pullbacks are the 2-categorical (and not bicategorical) versions. If we take this 2-category to be **Cat**, then in each case the underlying category of the 2-categorical construction is given by the corresponding 1-categorical version. From this point, we will not distinguish between the 1-dimensional and 2-dimensional theory. Our interest, of course, is in the 2-dimensional version.

Let $\Sigma$ be the operad of symmetric groups. This is the terminal object of the category of action operads, with each $\pi_n$ the identity map. Then $E\Sigma$ is a 2-monad on **Cat**, and by [**?**] it is cartesian.

**Definition 6.3.** A *club* is a $T$-operad in **Cat** for $T = E\Sigma$.

**Remark 6.4.** The category $\mathbf{P}$ in Kelly's terminology is the result of applying $E\Sigma$ to 1, and can be identified with $B\Sigma = \coprod B\Sigma_n$.

It is useful to break down the definition of a club. A club consists of

(1) a category $K$ together with a functor $K \to B\Sigma$,
(2) a multiplication map $K \circ K \to K$, and
(3) a unit map $1 \to K$

satisfying the axioms to be a monoid in the monoidal category of $E\Sigma$-collections. By the definition of $K \circ K$ as a pullback, we see that objects are tuples of objects of $K$ $(x; y_1, \ldots, y_n)$ where $\pi(x) = n$. A morphism

$$(x; y_1, \ldots, y_n) \to (z; w_1, \ldots, w_m)$$

exists only when $n = m$ (since $B\Sigma$ only has endomorphisms) and then consists of a morphism $f : x \to z$ in $K$ together with morphisms $g_i : y_i \to z_{\pi(x)(i)}$ in $K$.

**Notation 6.5.** For a club $K$ and a morphism $(f; g_1, \ldots, g_n)$ in $K \circ K$, we write $f(g_1, \ldots, g_n)$ for the image morphism under the functor $K \circ K \to K$.

We will usually just refer to a club by its underlying category $K$.

**Theorem 6.6.** *Let $\mathbf{\Lambda}$ be an action operad. Then the map of operads $\pi : \Lambda \to \Sigma$ gives the category $B\mathbf{\Lambda} = \coprod B\Lambda(n)$ the structure of a club.*

*Proof.* To give the functor $B\pi \colon B\mathbf{\Lambda} \to B\mathbf{\Sigma}$ the structure of a club it suffices (see [26]) to show that

- the induced monad, which we will show to be $E\Lambda$, is a cartesian monad on **Cat**,
- the transformation $\tilde{\pi} : E\Lambda \Rightarrow E\Sigma$ induced by the functor $E\pi$ is cartesian, and
- $\tilde{\pi}$ commutes with the monad structures.

$E\Lambda$ is always cartesian by results of [**?**]. The transformation $\tilde{\pi}$ is the coproduct of the maps $\tilde{\pi}_n$ which are induced by the universal property of the coequalizer as shown below.

$$\begin{tikzcd}
E\Lambda(n) \times \Lambda(n) \times X^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "E\pi \times \pi \times 1"'] & E\Lambda(n) \times X^n \arrow[r] \arrow[d, "E\pi \times 1"'] & E\Lambda(n) \times_{\Lambda(n)} X^n \arrow[d, dotted, "\tilde{\pi}_n"] \\
E\Sigma_n \times \Sigma_n \times X^n \arrow[r, shift left] \arrow[r, shift right] & E\Sigma_n \times X^n \arrow[r] & E\Sigma_n \times_{\Sigma_n} X^n
\end{tikzcd}$$

Naturality is immediate, and since $\pi$ is a map of operads $\tilde{\pi}$ also commutes with the monad structures.

It only remains to show that $\tilde{\pi}$ is cartesian and that the induced monad is actually $E\Lambda$. Since the monads $E\Lambda$ and $E\Sigma$ both decompose into a disjoint union of functors, we only have to show that, for any $n$, the square below is a pullback.

$$\begin{tikzcd} E\Lambda(n) \times_{\Lambda(n)} X^n \arrow[r] \arrow[d] & E\Sigma_n \times_{\Sigma_n} X^n \arrow[d] \\ B\Lambda(n) \arrow[r] & B\Sigma_n \end{tikzcd}$$

By the explicit description of the coequalizer given in [**?**], this amounts to showing that the square below is a pullback.

$$\begin{tikzcd} E\Lambda(n) \times X^n/\Lambda(n) \arrow[r] \arrow[d] & E\Sigma_n \times X^n/\Sigma_n \arrow[d] \\ B\Lambda(n) \arrow[r] & B\Sigma_n \end{tikzcd}$$

Here, $A \times B/G$ is the category whose objects are equivalence classes of pairs $(a, b)$ where $(a, b) \sim (ag, g^{-1}b)$, and similarly for morphisms. Now the bottom map is clearly bijective on objects since these categories only have one object. An object in the top right is an equivalence class

$$[\sigma; x_1, \ldots, x_n] = [e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}].$$

A similar description holds for objects in the top left, with $g \in \Lambda(n)$ replacing $\sigma$ and $\pi(g)^{-1}$ replacing $\sigma^{-1}$ in the subscripts. The map along the top sends $[g; x_1, \ldots, x_n]$ to $[\pi(g); x_1, \ldots, x_n]$, and thus sends $[e; x_1, \ldots, x_n]$ to $[e; x_1, \ldots, x_n]$, giving a bijection on objects.

Now a morphism in $E\Lambda(n) \times X^n/\Lambda(n)$ can be given as

$$[e; x_1, \ldots, x_n] \xrightarrow{[!; f_i]} [g; y_1, \ldots, y_n].$$

Mapping down to $B\Lambda(n)$ gives $ge^{-1} = g$, while mapping over to $E\Sigma_n \times X^n/\Sigma_n$ gives $[!; f_i]$ where $! : e \to \pi(g)$ is now a morphism in $E\Sigma_n$. In other words, a morphism in the upper left corner of our putative pullback square is determined completely by its images along the top and lefthand functors. Furthermore, given $g \in \Lambda(n)$, $\tau = \pi(g)$, and morphisms $f_i : x_i \to y_i$ in $X$, the morphism $[! : e \to g; f_i]$ maps to the pair $(g, [! : e \to \tau; f_i])$, completing the proof that this square is indeed a pullback. □

The club, which we now denote $K_{\mathbf{\Lambda}}$, associated to $E\Lambda$ has the following properties. First, the functor $K_{\mathbf{\Lambda}} \to B\Sigma$ is a functor between groupoids. Second, the functor $K_{\mathbf{\Lambda}} \to B\Sigma$ is bijective-on-objects. We claim that these properties characterize those clubs which arise from action operads. Thus the clubs arising from action operads are very similar to PROPs [27, 29].

**Theorem 6.7.** *Let $K$ be a club such that*

- *the map $K \to B\mathbf{\Sigma}$ is bijective on objects and*
- *$K$ is a groupoid.*

*Then $K \cong K_{\mathbf{\Lambda}}$ for some action operad $\mathbf{\Lambda}$. The assignment $\mathbf{\Lambda} \mapsto K_{\mathbf{\Lambda}}$ is a full and faithful embedding of the category of action operads* **AOp** *into the category of clubs.*

*Proof.* Let $K$ be such a club. Our hypotheses immediately imply that $K$ is a groupoid with objects in bijection with the natural numbers; we will now assume the functor $K \to B\mathbf{\Sigma}$ is the identity on objects. Let $\Lambda(n) = K(n,n)$. Now $K$ comes equipped with a functor to $B\mathbf{\Sigma}$, in other words group homomorphisms $\pi_n : \Lambda(n) \to \Sigma_n$. We claim that the club structure on $K$ makes the collection of groups $\{\Lambda(n)\}$ an action operad. In order to do so, we will employ Theorem 5.8.

First, we give the group homomorphism $\beta$ using Notation 6.5. Define

$$\beta(g_1, \ldots, g_n) = e_n(g_1, \ldots, g_n)$$

(see 6.5) where $e_n$ is the identity morphism $n \to n$ in $K(n,n)$. Functoriality of the club multiplication map immediately implies that this is a group homomorphism. Second, we define the function $\delta$ in a similar fashion:

$$\delta_{n;k_1,\ldots,k_n}(f) = f(e_1, \ldots, e_n),$$

where here $e_i$ is the identity morphism of $k_i$ in $K$.

There are now nine axioms to verify in Theorem 5.8. The club multiplication functor is a map of collections, so a map over $B\mathbf{\Sigma}$; this fact immediately implies that axioms (1) (using morphisms in $K \circ K$ with only $g_i$ parts) and (4) (using morphisms in $K \circ K$ with only $f$ parts) hold. The mere fact that multiplication is a functor also implies axioms (6) (once again using morphisms with only $f$ parts) and (8) (by considering the composite of a morphism with only an $f$ with a morphism with only $g_i$'s). Axiom (2) is the equation $e_1(g) = g$ which is a direct consequence of the unit axiom for the club $K$; the same is true of axiom (5). Axioms (3), (7), and (9) all follow from the associativity of the club multiplication.

Finally, we would like to show that this gives a full and faithful embedding $K_{-} : \mathbf{AOp} \to \mathbf{Club}$ of the category of action operads into the category of clubs. Let $f, f' : \mathbf{\Lambda} \to \mathbf{\Lambda}'$ be maps between action operads. Then if $K_f = K_{f'}$ as maps between clubs, then they must be equal as functors $K_{\mathbf{\Lambda}} \to K_{\mathbf{\Lambda}'}$. But these functors are nothing more than the coproducts of the functors

$$B(f_n), B(f'_n) : B\Lambda(n) \to B\Lambda'(n),$$

and the functor $B$ from groups to categories is faithful, so $K_{-}$ is also faithful. Now let $f : K_{\mathbf{\Lambda}} \to K_{\mathbf{\Lambda}'}$ be a maps of clubs. We clearly get group homomorphisms $f_n : \Lambda(n) \to \Lambda'(n)$ such that $\pi_n^{\Lambda} = \pi_n^{\Lambda'} f_n$, so we must only show that the $f_n$ also constitute an operad map. Using the description of the club structure above in terms of the maps $\beta, \delta$, we see that commuting with the club multiplication implies commuting with both of these, which in turn is equivalent to commuting with operad multiplication. Thus $K_{-}$ is full as well. $\square$

**Remark 6.8.** First, one should note that being a club over $B\mathbf{\Sigma}$ means that every $K$-algebra has an underlying strict monoidal structure. Second, requiring that $K \to B\mathbf{\Sigma}$ be bijective on objects ensures that $K$ does not have operations other than $\otimes$, such as duals or internal hom-objects, from which to build new types of objects. Finally, $K$ being a groupoid ensures that all of the "constraint morphisms" that exist in algebras for $K$ are invertible.

These hypotheses could be relaxed somewhat. Instead of having a club over $B\mathbf{\Sigma}$, we could have a club over the free symmetric monoidal category on one object (note that the free symmetric monoidal category monad on $\mathbf{Cat}$ is still cartesian). This would produce $K$-algebras with underlying monoidal structures which are not necessarily strict. This change should have relatively little impact on how the theory is developed. Changing $K$ to be a category instead of a groupoid would likely have a larger impact, as the resulting action operads would have monoids instead of groups at each level. We have made repeated use of inverses throughout the proofs in the basic theory of action operads, and these would have to be revisited if groups were replaced by monoids in the definition of action operads.

## 7. Presentations of action operads

One of the most useful methods for constructing new examples of some given algebraic structure is through the use of presentations. A presentation consists of generating data together with relations between generators using the operations of the algebra involved. In categorical terms, the generators and relations are both given as free gadgets on some underlying data, and the presentation itself is a coequalizer. This section will establish the categorical structure necessary to give presentations for action operads, and then explain how such a presentation is reflected in the associated club and 2-monad. The most direct route to the desired results uses the theory of locally finitely presentable categories. We recall the main definitions briefly, but refer the reader to [1] for additional details.

**Definition 7.1.** A *filtered category* is a nonempty category $C$ such that

- if $a, b$ are objects of $C$, then there is another object $c \in C$ and morphisms $a \to c, b \to c$; and
- if $f, g \colon a \to b$ are parallel morphisms in $C$, then there exists a morphism $h \colon b \to c$ such that $hf = hg$.

**Definition 7.2.** A *filtered colimit* is a colimit over a filtered category.

**Definition 7.3.** Let $C$ be a category with all filtered colimits. An object $x \in C$ is *finitely presentable* if the representable functor $C(x, -) : C \to \mathbf{Sets}$ preserves filtered colimits.

**Definition 7.4.** A *locally finitely presentable category* is a category $C$ such that

- $C$ is cocomplete and
- there exists a small subcategory $C_{fp} \subseteq C$ of finitely presentable objects such that any object $x \in C$ is the filtered colimit of some diagram in $C_{fp}$.

The definition of a locally finitely presentable category has many equivalent variants, and our applications are quite straightforward so we have given what is probably the most common version of the definition. We refer the reader to [1] for more information. The following result will be clear to the experts, so we just sketch a proof.

**Theorem 7.5.** *The category* $\mathbf{AOp}$ *is locally finitely presentable.*

*Proof.* First, note that there is a category $\mathbf{Op}^g$ whose objects are operads $P$ in which each $P(n)$ also carries a group structure. This is an equational theory using equations with only finitely many elements, so $\mathbf{Op}^g$ is locally finitely presentable. The symmetric operad is an object of this category, so the slice category $\mathbf{Op}^g/\Sigma$ is also locally finitely presentable. There is an obvious inclusion functor $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$. Now $\mathbf{AOp}$ is a full subcategory of $\mathbf{Op}^g/\Sigma$ which is closed under products, subobjects, and any object of $\mathbf{Op}^g/\Sigma$ isomorphic to an action operad is in fact an action operad, so the inclusion $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$ is actually the inclusion of a reflective subcategory. One easily checks that $\mathbf{AOp}$ is in fact closed under all limits and filtered colimits in $\mathbf{Op}^g/\Sigma$, so by the Reflection Theorem (2.48 in [1]), $\mathbf{AOp}$ is locally finitely presentable. $\square$

**Definition 7.6.** Let $\mathcal{S}$ be the set which is the disjoint union of the underlying sets of all the symmetric groups. Then $\mathbf{Sets}/\mathcal{S}$ is the slice category over $\mathcal{S}$ with objects $(X, f)$ where $X$ is a set and $f : X \to \mathcal{S}$ and morphisms $(X_1, f_1) \to (X_2, f_2)$ are those functions $g : X_1 \to X_2$ such that $f_1 = f_2 g$. We call an object $(X, f)$ a *collection over* $\mathcal{S}$.

**Remark 7.7.** In standard presentations of the theory of operads (see, for example, [30]), a nonsymmetric operad will have an underlying collection (or $\mathbb{N}$-indexed collection of sets) while a symmetric operad will have an underlying symmetric collection (or $\mathbb{N}$-indexed collection of sets in which the $n$th set has an action of $\Sigma_n$). Our collections over $\mathcal{S}$ more closely resemble the former as there is no group action present.

**Theorem 7.8.** *There is a forgetful functor* $U : \mathbf{AOp} \to \mathbf{Sets}/\mathcal{S}$ *which preserves all limits and filtered colimits.*

*Proof.* The functor $U$ is obvious, and the preservation of filtered colimits follows from the fact that these are computed pointwise, together with the fact that every map between action operads preserves underlying permutations. For limits, recall that limits in $\mathbf{AOp}$ are computed as in the category of operads over $\Sigma$. This means that equalizers are computed levelwise, and the product $\mathbf{\Lambda} \times \mathbf{\Lambda}'$ has underlying operad the pullback $\Lambda \times_\Sigma \Lambda'$; this pullback is itself computed levelwise. Together, these imply that $U$ also preserves all limits. $\square$

**Corollary 7.9.** *$U$ has a left adjoint* $F : \mathbf{Sets}/\mathcal{S} \to \mathbf{AOp}$, *the free action operad functor.*

*Proof.* The category $\mathbf{Sets}/\mathcal{S}$ is locally finitely presentable as it is equivalent to the functor category $[\mathcal{S}, \mathbf{Sets}]$ (here $\mathcal{S}$ is treated as a discrete category) and any presheaf category is locally finitely presentable. The functor $U$ preserves limits and filtered colimits between locally finitely presentable categories, so has a left adjoint (see Theorem 1.66 in [1]). $\square$

**Definition 7.10.** A *presentation* for an action operad $\mathbf{\Lambda}$ consists of

- a pair of collections over $\mathcal{S}$ denoted $\mathbf{g}, \mathbf{r}$,
- a pair of maps $s_1, s_2 : F\mathbf{r} \to F\mathbf{g}$ between the associated free action operads, and
- a map $p : F\mathbf{g} \to \mathbf{\Lambda}$ of action operads exhibiting $\mathbf{\Lambda}$ as the coequalizer of $s_1, s_2$.

In [15], Kelly discusses clubs given by generators and relations. His generators include functorial operations more general than what we are interested in here, and the natural transformations are not required to be invertible. In our case, the only generating operations we require are those of a unit and tensor product, as the algebras for $E\Lambda$ are always strict monoidal categories with additional structure. Tracing through his discussion of generators and relations for a club gives the following theorem.

**Theorem 7.11.** *Let* $\mathbf{\Lambda}$ *be an action operad with presentation given by* $(\mathbf{g}, \mathbf{r}, s_i, p)$. *Then the club* $E\Lambda$ *is generated by*

- *functors giving the unit object and tensor product, and*
- *natural transformations given by the collection* $\mathbf{g}$*: each element* $x$ *of* $\mathbf{g}$ *with* $\pi(x) = \sigma_x \in \Sigma_{|x|}$ *gives a natural transformation from the* $n$*th tensor power functor to itself,*

*subject to relations such that the following axioms hold.*

- *The monoidal structure given by the unit and tensor product is strict.*
- *The transformations given by the elements of* $\mathbf{g}$ *are all natural isomorphisms.*
- *For each element* $y \in \mathbf{r}$*, the equation* $s_1(y) = s_2(y)$ *holds.*

Bringing this down to a concrete level we have the following corollary.

**Corollary 7.12.** *Assume we have a notion* $\mathcal{M}$ *of strict monoidal category which is given by a set natural isomorphisms*

$$\mathcal{G} = \{(f, \pi_f) \,|\, x_1 \otimes \cdots \otimes x_n \xrightarrow{f} x_{\pi_f^{-1}(1)} \otimes \cdots \otimes x_{\pi_f^{-1}(n)}\}$$

*subject to a set* $\mathcal{R}$ *of axioms. Each such axiom is given by the data*

- *two finite sets* $f_1, \ldots, f_n$ *and* $f'_1, \ldots, f'_m$ *of elements of* $\mathcal{G}$*; and*
- *two formal composites* $F, F'$ *using only composition and tensor product operations and the* $f_i$*, respectively* $f'_i$*,*

*such that the underlying permutation of* $F$ *equals the underlying permutation of* $F'$ *(we compute the underlying permutations using the functions* $\beta, \delta$ *of Theorem 5.8). The element* $(\underline{f}, \underline{f}', F, F')$ *of the set* $\mathcal{R}$ *of axioms corresponds to the requirement that the composite of the morphisms* $f_i$ *using* $F$ *equals the composite of the morphisms* $f'_j$ *using* $F'$ *in any strict monoidal category of type* $\mathcal{M}$*. Then strict monoidal categories of type* $\mathcal{M}$ *are given as the algebras for the club* $E\Lambda$ *where* $\mathbf{\Lambda}$ *is the action operad with*

- $\mathbf{g} = \mathcal{G}$*,*
- $\mathbf{r} = \mathcal{R}$*,*
- $s_1$ *given by mapping the generator* $(\underline{f}, \underline{f}', F, F')$ *to the operadic composition of the* $f_i$ *using* $F$ *via* $\beta, \delta$*, and*
- $s_2$ *given by mapping the generator* $(\underline{f}, \underline{f}', F, F')$ *to the operadic composition of the* $f'_i$ *using* $F'$ *via* $\beta, \delta$*.*

## 8. Examples

In this section we will discuss examples of the preceding theory. We have seen that there are three equivalent incarnations of the same algebraic structure:

- as an action operad $\mathbf{\Lambda}$,
- as a 2-monad $X \mapsto \coprod E\Lambda(n) \times_{\Lambda(n)} X^n$ on **Cat**, or
- as a club $B\Lambda \to B\Sigma$ satisfying certain properties.

In practice, something like the third of these is the most likely to arise from applications (even if the notion of a club is perhaps less well-known outside of the categorical literature than that of an operad or a 2-monad) as a club can be given by a presentation as we discussed in Section 3. We will go into more detail here, explaining how particular monoidal structures of interest are represented in this theory.

**Example 8.1.** The 2-monad for symmetric strict monoidal categories (or permutative categories, as they are known in the topological literature) is given by $E\mathbf{\Sigma}$, so the notion of symmetric strict monoidal categories corresponds to the symmetric operad. While this example is well-known, we go into further detail to set the stage for less common examples.

The 2-monad $E\Sigma$ on **Cat** is given by

$$E\Sigma(X) = \coprod E\Sigma_n \times_{\Sigma_n} X^n.$$

An object of $E\Sigma_n \times_{\Sigma_n} X^n$ is an equivalence class of the form $[\sigma; x_1, \ldots, x_n]$ where $\sigma \in \Sigma_n$ and $x_i \in X$. The equivalence relation gives

$$[\sigma; x_1, \ldots, x_n] = [e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}],$$

so objects can be identified with finite strings of objects of $X$. Morphisms are given by equivalence classes of the form

$$[\sigma; x_1, \ldots, x_n] \xrightarrow{[!; f_1, \ldots, f_n]} [\tau; y_1, \ldots, y_n].$$

Here $! : \sigma \cong \tau$ is the unique isomorphism in $E\Sigma_n$, and $f_i : x_i \to y_i$ in $X$. Using the equivalence relation, we get that morphisms between finite strings

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

are given by a permutation $\rho \in \Sigma_n$ together with maps $f_i : x_i \to y_{\rho(i)}$ in $X$ (note that there are no morphisms between strings of different length); this is a special case of the calculation in 5.15. Thus $E\Sigma(X)$ is easily seen to be the free permutative category generated by $X$, and therefore $E\Sigma$-algebras are permutative categories.

**Example 8.2.** The template above can be used to show that the braid operad $\mathbf{B}$ corresponds to the 2-monad for braided strict monoidal categories. The details are almost exactly the same, only we use braids instead of permutations. The equivalence relation on objects gives

$$[\gamma; x_1, \ldots, x_n] = [e; x_{\pi(\gamma)^{-1}(1)}, \ldots, x_{\pi(\gamma)^{-1}(n)}],$$

where $\gamma \in B_n$ and $\pi(\gamma)$ is its underlying permutation; thus objects of $EB(X)$ are once again finite strings of objects of $X$. A morphism

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

is then given by a braid $\gamma \in B_n$ together with maps $f_i : x_i \to y_{\pi(\gamma)(i)}$ in $X$. Thus one should view a morphism as given by

- a finite ordered set $x_1, \ldots, x_n$ of objects of $X$ as the source,
- another such finite ordered set (of the same cardinality) $y_1, \ldots, y_n$ of objects of $X$ as the target,
- a geometric braid $\gamma \in B_n$ on $n$ strands, and
- for each strand, a morphism in $X$ from the object labeling the source of that strand to the object labeling the target.

This is precisely Joyal and Street's [14] construction of the free braided strict monoidal category generated by a category $X$, and thus we see that $EB$-algebras are braided strict monoidal categories.

This example can be extended to include ribbon braided categories as well. A *ribbon braid* is given, geometrically, in much the same way as a braid except that instead of paths $[0, 1] \to \mathbb{R}^3$ making up each individual strand, we use ribbons $[0, 1] \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$. This introduces the possibility of performing a full twist on a ribbon, and one can describe ribbon braided categories using generators and relations by introducing a natural twist isomorphism $\tau_A : A \to A$ and imposing one relation between the twist and the braid $\gamma_{A,B} : A \otimes B \to B \otimes A$. In [33], the authors show that the ribbon braid groups give an action operad **RB**, and that (strict) ribbon braided categories are precisely the algebras for $ERB$.

We now turn to an example which is not as widely known in the categorical literature, that of coboundary categories [7]. These arise in the representation theory of quantum groups and in the theory of crystals [11, 12]. Our goal here is to refine the relationship between coboundary categories and the operad of $n$-fruit cactus groups in [11] by using the theory of action operads and our Borel construction. We begin by recalling the definition of a coboundary category.

**Definition 8.3.** A *coboundary category* is a monoidal category $C$ equipped with a natural isomorphism $\sigma_{x,y} : x \otimes y \to y \otimes x$ (called the *commutor*) such that

- $\sigma_{y,x} \circ \sigma_{x,y} = 1_{x \otimes y}$ and
- the diagram

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r] \arrow[d, "\sigma_{x,y}1"'] & x \otimes (y \otimes z) \arrow[r, "1\sigma_{y,z}"] & x \otimes (z \otimes y) \arrow[d, "\sigma_{x,zy}"] \\
(y \otimes x) \otimes z \arrow[r, "\sigma_{yx,z}"'] & z \otimes (y \otimes x) \arrow[r] & (z \otimes y) \otimes x
\end{tikzcd}$$

commutes (in which the unlabeled morphisms are an associator and an inverse associator).

**Example 8.4.** (1) From the definition above, it is clear that any symmetric monoidal category is also a coboundary category.

(2) The name coboundary category comes from the original work of Drinfeld [7] in which he shows that the category of representations of coboundary Hopf algebra has the structure of coboundary category.

(3) Henriques and Kamnitzer [11] show that the category of crystals for a finite dimensional complex reductive Lie algebra has the structure of a coboundary category.

Our interest is in strict coboundary categories by which we mean coboundary categories with strict underlying monoidal category. Under the assumption of strictness, the second axiom above does not include associations for the tensor product and reduces to a square. To show that every coboundary category is equivalent to a strict coboundary category, we must introduce the 2-category **CobCat** of coboundary categories.

**Definition 8.5.** Let $(C, \sigma), (C', \sigma')$ be coboundary categories. A *coboundary functor $F : C \to C'$* is a strong monoidal functor (with invertible constraints $\varphi_0$ for the unit and $\varphi_{x,y}$ for the tensor product) such that

$$F\sigma_{x,y} \circ \varphi_{x,y} = \varphi_{y,x} \circ \sigma'_{Fx,Fy}$$

holds.

Coboundary functors are composed just as strong monoidal functors are, giving the following.

**Lemma 8.6.** *There is a 2-category* **CobCat** *of coboundary categories, coboundary functors, and monoidal transformations.*

**Proposition 8.7.** *Let $(C, \sigma)$ be a coboundary category. Then there is a strict coboundary category $(C', \sigma')$ which is equivalent to $(C, \sigma)$ in* **CobCat**.

*Proof.* Consider the underlying monoidal category of $(C, \sigma)$, which we will just write as $C$. We can find a strict monoidal category $C'$ by coherence for monoidal categories together with an equivalence, as monoidal categories, between $C$ and $C'$. By standard methods, this can be improved to an adjoint equivalence between $C$ and $C'$ in the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. Let $F : C \to C', G : C' \to C$ be the functors in this adjoint equivalence, and $\eta : 1 \Rightarrow FG$ the unit (which we note for emphasis is invertible). For objects $x, y \in C'$, we define a commutor $\sigma'$ for $C'$ as the composite

$$\begin{array}{rcl} xy & \xrightarrow{\eta\otimes\eta} & FGxFGy \\ & \cong & F(GxGy) \\ & \xrightarrow{F\sigma} & F(GyGx) \\ & \cong & FGyFGx \\ & \xrightarrow{\eta^{-1}\otimes\eta^{-1}} & yx. \end{array}$$

We then leave to the reader the computations to show that $\sigma'$ is a commutor for $C'$ and that $F, G$ become coboundary functors using $\sigma'$. $\square$

We now turn to the operadic description of strict coboundary categories; we note from this point onwards, all our coboundary categories are assumed to be strict.

**Definition 8.8.** Fix $n > 1$, and let $1 \leq p < q \leq n$, $1 \leq k < l \leq n$.

(1) $p < q$ is *disjoint* from $k < l$ if $q < k$ or $l < p$.
(2) $p < q$ *contains* $k < l$ if $p \leq k < l \leq q$.

**Definition 8.9.** Let $1 \leq p < q \leq n$, and define $\hat{s}_{p,q} \in \Sigma_n$ to be the permutation defined below.

$$\begin{array}{r|cccccccccccc} i & 1 & 2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & q-1 & q & q+1 & \cdots & n \\ \hat{s}_{p,q}(i) & 1 & 2 & \cdots & p-1 & q & q-1 & q-2 & \cdots & p+1 & p & q+1 & \cdots & n \end{array}$$

**Definition 8.10.** Let $J_n$ be the group generated by symbols $s_{p,q}$ for $1 \leq p < q \leq n$ subject to the following relations.

(1) For all $p < q$, $s_{p,q}^2 = e$.
(2) If $p < q$ is disjoint from $k < l$, then $s_{p,q}s_{k,l} = s_{k,l}s_{p,q}$.
(3) If $p < q$ contains $k < l$, then $s_{p,q}s_{k,l} = s_{m,n}s_{p,q}$ where
   - $m = \hat{s}_{p,q}(l)$ and
   - $n = \hat{s}_{p,q}(k)$.

It is easy to check that the elements $\hat{s}_{p,q} \in \Sigma_n$ satisfy the three relations in Definition 8.10, so $s_{p,q} \mapsto \hat{s}_{p,q}$ extends to a group homomorphism $\pi_n : J_n \to \Sigma_n$. This is the first step in proving the following.

**Theorem 8.11.** *The collection of groups $J = \{J_n\}$ form an action operad.*

*Proof.* We will use Theorem 5.8 to determine the rest of the action operad structure. Thus we must give, for any collection of natural numbers $n, k_1, \ldots, k_n$ and $\underline{k} = \sum k_i$, group homomorphisms $\beta : J_{k_1} \times \cdots \times J_{k_n} \to J_{\underline{k}}$ and functions $\delta : J_n \to J_{\underline{k}}$ satisfying nine axioms. We define both of these on generators, starting with $\beta$.

Let $s_{p_i,q_i} \in J_{k_i}$. Let $r_i = k_1 + k_2 + \cdots + k_{i-1}$. Define $\beta$ by

$$\beta(s_{p_1,q_1}, \ldots, s_{p_n,q_n}) = s_{p_1,q_1}s_{p_2+r_2,q_2+r_2}\cdots s_{p_n+r_n,q_n+r_n}.$$

Note that $s_{p_i+r_i,q_i+r_i}$ and $s_{p_j+r_j,q_j+r_j}$ are disjoint when $i \neq j$. It is easy to check that this disjointness property ensures that $\beta$ gives a well-defined group homomorphism

$$J_{k_1} \times \cdots \times J_{k_n} \to J_{\underline{k}}.$$

To define $\delta : J_n \to J_{\underline{k}}$ for natural numbers $n, k_1, \ldots, k_n$ and $\underline{k} = \sum k_i$, let $m_k = s_{1,k} \in J_k$. Then we start by defining

$$\delta(m_n) = m_{\underline{k}} \cdot \beta(m_{k_1}, m_{k_2}, \ldots, m_{k_n}).$$

Note that, by the containment relation, this is equal to

$$\beta(m_{k_n}, m_{k_{n-1}}, \ldots, m_{k_1}) \cdot m_{\underline{k}}.$$

Now $s_{p,q} \in J_n$ is equal to $\beta(e_{p-1}, m_{q-p+1}, e_{n-q})$ (here $e_i$ is the identity element in $J_i$) by definition of the $m_i$ and $\beta$, so we can define $\delta$ on any generator $s_{p,q}$ by

$$\delta(s_{p,q}) = \beta(e_A, M, e_B)$$

with

- $A = k_1 + k_2 + \cdots + k_{p-1}$,
- $M = m_{k_p+\cdots+k_q} \cdot \beta(m_{k_p}, m_{k_{p+1}}, \ldots, m_{k_q})$, and
- $B = k_{q+1} + k_{q+2} + \cdots + k_n$.

Unpacking this yields the following formula:

$$\delta(s_{p,q}) = s_{k_1+\cdots+k_{p-1}+1, k_1+\cdots+k_q} \cdot \beta(e_{k_1+\cdots+k_{p-1}}, m_{k_p}, \ldots, m_{k_q}, e_{k_{q+1}+\cdots+k_n}).$$

We extend $\delta$ to products of generators using axiom 6 of Theorem 5.8. As before, we must check that this gives a well-defined function on products of two generators in each of the relations of the cactus groups, and we must also check that this is well-defined on products of three or more generators. Thus we define

$$\delta_{n;j_i}(gh) = \delta_{n;k_i}(g)\delta_{n;j_i}(h)$$

where $k_i = j_{\pi(h)^{-1}(i)}$. There are three relations we must verify for compatibility.

- We must show that $\delta_{n;j_i}(s_{p,q}^2) = e$. By definition, we have

  $$\delta_{n;j_i}(s_{p,q}^2) = \delta_{n;k_i}(s_{p,q})\delta_{n;j_i}(s_{p,q})$$

  which is

  $$m_{\underline{j}}\beta(m_{j_n}, \ldots, m_{j_1}) m_{\underline{j}}\beta(m_{j_1}, \ldots, m_{j_n}).$$

  By the remarks above in the definition of $\delta$ and the fact that $s_{p,q}^2 = e$, the element above is easily seen to be the identity.
- We must show that $\delta(s_{p,q}s_{k,l}) = \delta(s_{k,l}s_{p,q})$ when $(p,q)$ is disjoint from $(k,l)$. This is another simple calculation using the definition of $\delta$ and the disjointness of the terms involved.
- We must show that $\delta(s_{p,q}s_{k,l}) = \delta(s_{a,b}s_{p,q})$, where $a = \hat{s}_{p,q}(l), b = \hat{s}_{p,q}(k)$, if $p < k < l < q$. In this case, we use all of the relations in the cactus groups to show that each side is equal to

$$\beta(e_{j_1}, \ldots, e_{j_{p-1}}, m_{j_p+\cdots+j_q} \cdot \beta(m_{j_p}, \ldots m_{j_{k-1}}, m_{j_k+\cdots j_l}, m_{j_{l+1}}, \cdots, m_{j_q}), m_{j_{q+1}}, \ldots, m_{j_n}).$$

In order to show that this gives a well-defined function on products of three or more generators, one proceeds inductively to show that $\delta\big((fg)h\big) = \delta\big(f(gh)\big)$ using the formula above. This is simply a matter of keeping track of the permutations used to define the subscripts for the different $\delta$'s and we leave it to the reader. This concludes the definition of the family of functions $\delta_{n;j_i}$.

There are now nine axioms to check in Theorem 5.8. Axioms (1) - (3) all concern $\beta$, and are immediate from the defining formula. Axiom (4) is obvious for the elements $m_k$, from which it follows in general by the formulas defining $\delta$. For axiom (5), one can check easily that

$$\delta_{n;1,\ldots,1}(m_n) = m_n, \quad \delta_{1;n}(m_n) = m_n$$

and once again the general case follows from these. Axiom (6) holds by the construction of $\delta$. Axiom (8) can be verified with only one $h_i$ nontrivial at a time, and then it is a simple consequence of the second and third relations for $J_n$.

Axiom (9) is straightforward to check when only a single $g_i$ is a generator and the rest are identities using the defining formulas, and the general case then follows using axiom (6). Using (9), we can then prove axiom (7) as follows; we suppress the subscripts on different $\delta$'s for clarity. We must show

$$\delta_{m_1+\cdots+m_n;p_{11},\ldots,p_{1m_1},p_{21},\ldots,p_{nm_m}}\big(\delta_{n;m_1,\ldots,m_n}(f)\big) = \delta_{n;P_1,\ldots,P_n}(f),$$

and we do so on $m_n$. By definition, we have

$$\delta\big(\delta(m_n)\big) = \delta\big(m_{\underline{k}}\beta(m_{k_1},\ldots,m_{k_n})\big),$$

which by axiom (6) is equal to

$$m_{P_1+\cdots P_n}\cdot\beta(m_{p_{11}},\ldots,m_{p_{n,m_n}})\cdot\delta\big(\beta(m_{k_1},\ldots,m_{k_n})\big).$$

Now this last term is equal to $\beta\big(\delta(m_{k_1}),\ldots,\delta(m_{k_n})\big)$ by axiom (9), which is then equal to

$$\beta\big(m_{P_1}\cdot\beta(m_{p_{11}},\ldots,m_{p_{1,m_1}}),\ldots,m_{P_n}\cdot\beta(m_{p_{n1}},\ldots,m_{p_{1,m_n}})\big).$$

Taken all together, the left hand side of axiom (9) is then

$$m_{P_1+\cdots P_n}\cdot\beta(m_{p_{11}},\ldots,m_{p_{n,m_n}})\cdot\beta\big(m_{P_1}\cdot\beta(m_{p_{11}},\ldots,m_{p_{1,m_1}}),\ldots,m_{P_n}\cdot\beta(m_{p_{n1}},\ldots,m_{p_{1,m_n}})\big).$$

All of the terms coming from an $m_{p_{ij}}$ can be collected together, and since $s^2_{p,q} = e$ for all $p, q$, these cancel. This leaves

$$m_{P_1+\cdots P_n}\cdot\beta\big(m_{P_1},\ldots,m_{P_n}\big)$$

which is the right hand side of axiom (9) as desired.

□

**Lemma 8.12.** *The 2-monad $C$ for strict coboundary categories is a club.*

*Proof.* This is obvious by 7.12. □

**Theorem 8.13.** *The free coboundary category on one element, $C1$, is isomorphic to $B\mathbf{J} = \coprod BJ_n$.*

*Proof.* The universal property we desire is with respect to strict coboundary functors (i.e., coboundary functors whose underlying monoidal functor is strict), so we must give $B\mathbf{J}$ the structure of a strict coboundary category and then check that to give a strict coboundary functor $B\mathbf{J} \to X$ to any other strict coboundary category is the same as giving an object of $X$.

The category $B\mathbf{J}$ has natural numbers as objects, and addition as its tensor product. The tensor product of two morphisms is given by $\beta$ as in 8.11, and it is simple to check that this is a strict monoidal structure. The commutor $\sigma_{m,n}$ is $s_{1,m+n}s_{1,m}s_{m+1,m+n}$. Using the relations in $J_n$, it is clear that $\sigma_{m,n}\sigma_{n,m}$ is the identity, so we only have one more axiom to verify in order to give a coboundary structure. By definition, this axiom is equivalent to the equation

$$\sigma_{m,p+n} \cdot \beta(e_m, \sigma_{n,p}) = \sigma_{n+m,p} \cdot \beta(\sigma_{m,n}, e_p)$$

holding for all $m, n, p$. Each side has six terms when written out using the definitions of $\sigma$ and $\beta$, two terms on each side cancel using $s_{p,q}^2 = e$ and the disjointness relation, and the other four terms match after using the disjointness relation. This establishes the coboundary structure on $B\mathbf{J}$; note that $\sigma_{1,1} = s_{1,2}$, the nontrivial element of $J(2)$.

Every strict coboundary functor $F : B\mathbf{J} \to X$ determines an object of $X$ by evaluation at 1. Conversely, given an object $x$ of a strict coboundary category $X$, we have an action of $J_n$ on $X(x^n, x^n)$ by Theorem 7 of [11] and therefore a strict monoidal functor $\overline{x} : B\mathbf{J} \to X$ with $\overline{x}(1) = x$. By construction, this strict monoidal functor is in fact a strict coboundary functor since it sends the commutor $\sigma_{1,1}$ in $B\mathbf{J}$ to $\sigma_{x,x}$ in $X$. In fact, the calculations in [11] leading up to Theorem 7 show that every element of $J_n$ is given as an operadic composition of $\sigma$'s, so requiring $\overline{x}$ to be a strict coboundary functor with $\overline{x}(1) = x$ determines the rest of the functor uniquely. This establishes the bijection between strict coboundary functors $F : B\mathbf{J} \to X$ and objects of $X$ which proves that $B\mathbf{J}$ is the free strict coboundary category on one object. □

**Corollary 8.14.** *The 2-monad $C$ for coboundary categories corresponds, using Theorem 6.7, to the action operad* $\mathbf{J}$.

## 9. Pseudo-commutativity

This final section gives conditions sufficient to equip the 2-monad $\underline{P}$ induced by a $\Lambda$-operad $P$ in **Cat** with a pseudo-commutative structure. Such a pseudo-commutativity will then give the 2-category **Ps-**$\underline{P}$**-Alg** some additional structure that we briefly explain here. For a field $k$, the category **Vect** of vector spaces over $k$ has many nice features. Of particular interest to us are the following three structures. First, the category **Vect** is monoidal using the tensor product $\otimes_k$. Second, the set of linear maps $V \to W$ is itself a vector space which we denote $[V, W]$. Third, there is a notion of multilinear map $V_1 \times \cdots \times V_n \to W$, with linear maps being the 1-ary version. While these three structures are each useful in isolation, they are tied together by natural isomorphisms

$$\mathbf{Vect}(V_1 \otimes V_2, W) \cong \mathbf{Vect}(V_1, [V_2, W]) \cong \mathbf{Bilin}(V_1 \times V_2, W)$$

expressing that $\otimes$ gives a closed monoidal structure which represents the multicategory of multilinear maps. Moreover, the adjunction between **Vect** and **Sets** respects all of this structure in the appropriate way. This incredibly rich interplay between the tensor product, the internal mapping space, and the multicategory of multilinear maps all arises from the free vector space monad on **Sets** being a *commutative* monad [21, 22, 23]. The notion of a pseudo-commutative 2-monad [13] is then a generalization of this machinery to a 2-categorical context, and can be viewed as a starting point for importing tools from linear algebra into category theory.

The aim of this section is to give conditions that ensure that the 2-monad $\underline{P}$ associated to a $\Lambda$-operad $P$ has a pseudo-commutative structure. We give the definition of pseudo-commutativity as in [13] but before doing so we require the definition of a strength for a 2-monad.

**Definition 9.1.** A *strength* for an endo-2-functor $T\colon \mathcal{K} \to \mathcal{K}$ on a 2-category with products and terminal object 1 consists of a 2-natural transformation $d$ with components

$$d_{A,B}\colon A \times TB \to T(A \times B)$$

satisfying the following unit and associativity axioms [21].

$$\begin{tikzcd}
1 \times TA \arrow[r, "d_{1,A}"] \arrow[dr, "\cong"'] & T(1 \times A) \arrow[d, "\cong"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "1 \times \eta"] \arrow[dr, "\eta"'] & A \times TB \arrow[d, "d_{A,B}"] \\
& T(A \times B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times TC \arrow[rr, "d_{AB,C}"] \arrow[d, "a"'] & & T\Big((A \times B) \times C\Big) \arrow[d, "Ta"] \\
A \times (B \times TC) \arrow[r, "1 \times d_{B,C}"'] & A \times T(B \times C) \arrow[r, "d_{A,BC}"'] & T\Big(A \times (B \times C)\Big)
\end{tikzcd}$$

$$\begin{tikzcd}
A \times T^2 B \arrow[r, "d_{A,TB}"] \arrow[d, "1 \times \mu"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] & T^2(A \times B) \arrow[d, "\mu"] \\
A \times TB \arrow[rr, "d_{A,B}"'] & & T(A \times B)
\end{tikzcd}$$

Similarly, a *costrength* for $T$ consists of a 2-natural transformation $d^*$ with components

$$d^*_{A,B}\colon TA \times B \to T(A \times B)$$

again satisfying unit and associativity axioms.

The strength and costrength for the associated 2-monad $\underline{P}$ are quite simple to define. We define the strength $d$ for $\underline{P}$ as follows. The component $d_{A,B}$ is a functor

$$d_{A,B}\colon A \times \left(\amalg P(n) \times_{\Lambda(n)} B^n\right) \to \amalg P(n) \times_{\Lambda(n)} (A \times B)^n$$

which sends an object $(a, [p; b_1, \ldots, b_n])$ to the object $[p; (a, b_1), \ldots, (a, b_n)]$. We also define the costrength similarly, sending an object $([p; a_1, \ldots, a_n], b)$ to the object $[p; (a_1, b), \ldots, (a_n, b)]$. Both the strength and the costrength are defined in the obvious way on morphisms.

**Remark 9.2.** It is crucial to note that the strength $d$ and the costrength $d^*$ do not depend on the $\Lambda$-actions in the following sense. The $\Lambda$-operad $P$ has an underlying non-symmetric operad that we also denote $P$, and it has a strength

$$d_{A,B}\colon A \times (\amalg P(n) \times B^n) \to \amalg P(n) \times (A \times B)^n$$

given by essentially the same formula:

$$\big(a; (p; b_1, \ldots, b_n)\big) \mapsto \big(p; (a, b_1), \ldots, (a, b_n)\big).$$

The strength for the $\Lambda$-equivariant $P$ is just the induced functor between coequalizers.

**Definition 9.3.** Given a 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ with strength $d$ and costrength $d^*$, a *pseudo-commutativity* consists of an invertible modification $\gamma$ with components

$$\begin{tikzcd}
TA \times TB \arrow[r, "d^*_{A,TB}"] \arrow[d, "d_{TA,B}"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma_{A,B}"] & T^2(A \times B) \arrow[d, "\mu_{A\times B}"] \\
T(TA \times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A \times B) \arrow[r, "\mu_{A\times B}"'] & T(A \times B)
\end{tikzcd}$$

satisfying the following three strength axioms, two unit (or $\eta$) axioms, and two multiplication (or $\mu$) axioms for all $A$, $B$, and $C$.

(1) $\gamma_{A\times B,C} * (d_{A,B} \times 1_{TC}) = d_{A,B\times C} * (1_A \times \gamma_{B,C})$
(2) $\gamma_{A,B\times C} * (1_{TA} \times d_{B,C}) = \gamma_{A\times B,C} * (d^*_{A,B} \times 1_{TC})$
(3) $\gamma_{A,B\times C} * (1_{TA} \times d^*_{B,C}) = d^*_{A\times B,C} * (\gamma_{A,B} \times 1_C)$
(4) $\gamma_{A,B} * (\eta_A \times 1_{TB})$ is the identity on $d$.
(5) $\gamma_{A,B} * (1_{TA} \times \eta_B)$ is the identity on $d^*$.
(6) $\gamma_{A,B} * (\mu_A \times 1_{TB})$ is equal to the pasting below.

$$\begin{tikzcd}
T^2A\times TB \arrow[r, "d^*_{TA,TB}"] \arrow[d, "d_{T^2A,B}"'] & T(TA\times TB) \arrow[r, "Td^*_{A,TB}"] \arrow[d, "Td_{TA,B}"'] & T^2(A\times TB) \arrow[r, "T^2d_{A,B}"] \arrow[d, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A\times B) \arrow[d, "T\mu_{A\times B}"] \\
T(T^2A\times B) \arrow[r, phantom, "\Downarrow\gamma_{TA,B}"] \arrow[d, "Td^*_{TA,B}"'] & T^2(TA\times B) \arrow[r, "T^2d^*_{A,B}"'] \arrow[d, "\mu_{TA\times B}"'] & T^3(A\times B) \arrow[r, "T\mu_{A\times B}"'] \arrow[d, "\mu_{T(A\times B)}"'] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\
T^2(TA\times B) \arrow[r, "\mu_{TA\times B}"'] & T(TA\times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B)
\end{tikzcd}$$

(7) $\gamma_{A,B} * (1_{TA} \times \mu_B)$ is equal to the pasting below.

$$\begin{tikzcd}
TA\times T^2B \arrow[r, "d^*_{A,T^2B}"] \arrow[d, "d_{TA,TB}"'] & T(A\times T^2B) \arrow[r, "Td_{A,TB}"] \arrow[d, phantom, "\Downarrow\gamma_{A,TB}"] & T^2(A\times TB) \arrow[d, "\mu_{A\times TB}"] \\
T(TA\times TB) \arrow[r, "Td^*_{A,TB}"'] \arrow[d, "Td_{TA,B}"'] & T^2(A\times TB) \arrow[r, "\mu_{A\times TB}"'] \arrow[d, "T^2d_{A,B}"] & T(A\times TB) \arrow[d, "Td_{A,B}"] \\
T^2(TA\times B) \arrow[r, phantom, "\Downarrow T\gamma_{A,B}"] \arrow[d, "T^2d^*_{A,B}"'] & T^3(A\times B) \arrow[r, "\mu_{T(A\times B)}"] \arrow[d, "T\mu_{A\times B}"] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\
T^3(A\times B) \arrow[r, "T\mu_{A\times B}"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B)
\end{tikzcd}$$

**Remark 9.4.** It is noted in [13] that this definition has some redundancy and therein it is shown that any two of the strength axioms immediately implies the third. Furthermore, the three strength axioms are equivalent when the $\eta$ and $\mu$ axioms hold, as well as the following associativity axiom:

$$\gamma_{A,B\times C} \circ (1_{TA} \times \gamma_{B,C}) = \gamma_{A\times B,C} \times (\gamma_{A,B} \times 1_{TC}).$$

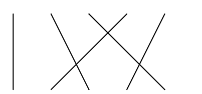
We need some notation before stating our main theorem. Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two lists. Then the set $\{(a_i, b_j)\}$ has $mn$ elements, and two natural lexicographic orderings. One of these we write as $\underline{(a, \underline{b})}$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} p < r, \text{ or} \\ p = r \text{ and } q < s. \end{cases}$$

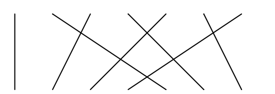
The other we write as $\underline{(\underline{a}, b)}$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} q < s, \text{ or} \\ q = s \text{ and } p < r. \end{cases}$$

The notation $(a, \underline{b})$ is meant to indicate that we have a single $a$ but a list of $b$'s, so then $\underline{(a, \underline{b})}$ would represent a list which itself consists of lists of that form. There is a unique permutation $\tau_{m,n} \in \Sigma_{mn}$ which has the property that $\tau_{m,n}(i) = j$ if the $i$th element of the ordered set $\underline{(a, \underline{b})}$ is equal to the $j$th element of the ordered set $\underline{(\underline{a}, b)}$. By construction, we have $\tau_{n,m} = \tau_{m,n}^{-1}$. We illustrate these permutations with a couple of examples.



$\tau_{2,3}$



$\tau_{4,2}$

Note then that $\tau_{m,n}$ is the permutation given by taking the transpose of the $m \times n$ matrix with entries $(a_i, b_j)$.

We now give sufficient conditions for equipping the 2-monad $\underline{P}$ associated to a $\Lambda$-operad $P$ with a pseudo-commutative structure. Let $\mathbb{N}_+$ denote the set of positive natural numbers.

**Theorem 9.5.** *Let $P$ be a $\Lambda$-operad. Then the following equip $\underline{P}$ with a pseudo-commutative structure.*

- *For each pair $(m, n) \in \mathbb{N}_+^2$, we are given an element $t_{m,n} \in \Lambda(mn)$ such that $\pi(t_{m,n}) = \tau_{m,n}$.*
- *For each $p \in P(n)$, $q \in P(m)$, we are given a natural isomorphism*

$$\lambda_{p,q} \colon \mu(p; q, \ldots, q) \cdot t_{m,n} \cong \mu(q; p, \ldots, p).$$

*We write this as $\lambda_{p,q} : \mu(p; \underline{q}) \cdot t_{m,n} \cong \mu(q; \underline{p})$.*

*These must satisfy the following:*

(1) *For all $n \in \mathbb{N}_+$,*

$$t_{1,n} = e_n = t_{n,1}$$

*and for all $p \in P(n)$, the isomorphism $\lambda_{p,id} : p \cdot e_n \cong p$ is the identity map.*

(2) *For all $l, m_1, \ldots, m_l, n \in \mathbb{N}_+$, with $M = \Sigma m_i$,*

$$\mu^\Lambda(e_l; t_{m_1,n}, \ldots, t_{m_l,n}) \cdot \mu^\Lambda(t_{l,n}; \underline{e_{m_1}, \ldots, e_{m_l}}) = t_{n,M}.$$

*Here $\underline{e_{m_1}, \ldots, e_{m_l}}$ is the list $e_{m_1}, \ldots, e_{m_l}$ repeated $n$ times.*

(3) *For all $l, m, n_1, \ldots, n_m \in \mathbb{N}_+$, with $N = \Sigma n_i$,*

$$\mu^\Lambda(t_{m,l}; \underline{e_{n_1}}, \ldots, \underline{e_{n_m}}) \cdot \mu^\Lambda(e_m; t_{n_1,l}, \ldots, t_{n_m,l}) = t_{N,l}.$$

*Here $\underline{e_{n_i}}$ indicates that each $e_{n_i}$ is repeated $l$ times.*

(4) *For any $l, m_i, n \in \mathbb{N}_+$, with $1 \leq i \leq n$, and $p \in P(l)$, $q_i \in P(m_i)$ and $r \in P(n)$, the following diagram commutes. (Note that we maintain the convention that anything underlined indicates a list, and double underlining indicates a list of lists. Each instance should have an obvious meaning from context and the equations appearing above.)*

$$\begin{tikzcd}
\mu\Big(p; \underline{\mu(q_i; \underline{r})}\Big) \cdot \mu(e_l; \underline{t_{n,m_i}})\mu(t_{n,l}; \underline{\underline{e_{m_i}}}) \arrow[r, equal] \arrow[d, equal] & \mu\Big(p; \underline{\mu(q_i; \underline{r})}\Big) \cdot t_{n,M} \arrow[d, equal] \\
\mu\Big(p; \underline{\mu(q_i; \underline{r}) \cdot t_{n,m_i}}\Big) \cdot \mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}) \arrow[d, "\mu(1;\underline{\lambda_{q_i,r}})\cdot 1"'] & \mu\Big(\mu(p; q_1, \ldots, q_n); \underline{\underline{r}}\Big) \cdot t_{n,M} \arrow[d, "\lambda_{\mu(p;q_1,\ldots,q_n),r}"] \\
\mu\Big(p; \underline{\mu(r; \underline{q_i})}\Big) \cdot \mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}) \arrow[d, equal] & \mu\Big(\mu(p; q_1, \ldots, q_n); \underline{\underline{r}}\Big) \arrow[dd, equal] \\
\mu\Big(\mu(p; \underline{r}) \cdot t_{n,l}; \underline{q_1, \ldots, q_n}\Big) \arrow[d, "\mu(\lambda_{p,r};1)"'] & \\
\mu\Big(\mu(r, \underline{p}); \underline{q_1, \ldots, q_n}\Big) \arrow[r, equal] & \mu\Big(r; \underline{\mu(p; q_1, \ldots, q_n)}\Big)
\end{tikzcd}$$

(5) *For any $l, m, n_i \in \mathbb{N}_+$, with $1 \leq i \leq m$, and $p \in P(l)$, $q \in P(m)$ and $r_i \in P(n_i)$, the following diagram commutes.*

$$\begin{tikzcd}
\mu\Big(\mu(p; \underline{q}) \cdot t_{m,l}; \underline{\underline{r_i}}\Big) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[r, equal] \arrow[dd, "\mu(\lambda_{p,q};1)\cdot 1"'] & \mu\Big(\mu(p; \underline{q}); \underline{\underline{r_i}}\Big) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
 & \mu\Big(p; \underline{\mu(q; \underline{r_i})}\Big) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\Big(\mu(q; \underline{p}); \underline{r_1}, \ldots, \underline{r_m}\Big) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] & \mu\Big(p; \underline{\mu(q; r_1, \ldots, r_m)}\Big) \cdot t_{N,l} \arrow[d, "\lambda_{p,\mu(q;r_1,\ldots,r_m)}"] \\
\mu\Big(q; \underline{\underline{\mu(p; r_i)}}\Big) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] & \mu\Big(\mu(q; r_1, \ldots, r_m); \underline{\underline{p}}\Big) \arrow[d, equal] \\
\mu\Big(q; \underline{\mu(p; \underline{r_i}) \cdot t_{n_i,l}}\Big) \arrow[r, "\mu(1;\underline{\lambda_{p,r_i}})"'] & \mu\Big(q; \underline{\mu(r_i; \underline{p})}\Big)
\end{tikzcd}$$

*Proof.* We begin the proof by defining an invertible modification $\gamma$ for the pseudo-commutativity for which the components are natural transformations $\gamma_{A,B}$. Such a transformation $\gamma_{A,B}$ has components with source

$$[\mu(p; \underline{q}); \underline{(x, \underline{y})}]$$

and target

$$[\mu(q; \underline{p}); \underline{(\underline{x}, y)}].$$

Now $\lambda_{p,q}\colon \mu(p;q,\ldots,q)\cdot t_{m,n} \cong \mu(q;p,\ldots,p)$ gives rise to another map by multiplication on the right by $t_{m,n}^{-1}$,

$$\lambda_{p,q}\cdot t_{m,n}^{-1}\colon \mu(p;q,\ldots,q) \cong \mu(q;p,\ldots,p)\cdot t_{m,n}^{-1},$$

so we define $(\gamma_{A,B})_{[p;a_1,\ldots,a_n],[q;b_1,\ldots,b_m]}$ to be the morphism which is the image of $(\lambda_{p,q}\cdot t_{m,n}^{-1}, 1)$ under the map

$$\coprod P(n)\times (A\times B)^n \to \coprod P(n)\times_{\Lambda(n)} (A\times B)^n.$$

We will write this morphism as $[\lambda_{p,q}t_{m,n}^{-1}, 1]$. In the case that either $p$ or $q$ is an identity then we choose the component of $\gamma$ to be the isomorphism involving the appropriate identity element using axiom 1 above.

There are two things to note about the definition above before we continue. First, it is easy to check that

$$t_{m,n}^{-1}\cdot \underline{(x,\underline{y})} = \underline{(\underline{x},y)}$$

since $\pi(t_{m,n}) = \tau_{m,n}$; this ensures that $\gamma$ has the correct target. Second, the morphism above has second component the identity. This is actually forced upon us by the requirement that $\gamma$ be a modification: in the case that $A,B$ are discrete categories, the only possible morphism is an identity, and the modification axiom then forces that statement to be true for general $A,B$ by considering the inclusion $A_0\times B_0 \hookrightarrow A\times B$ where $A_0,B_0$ are the discrete categories with the same objects as $A,B$.

We show that this is a modification by noting that it does not rely on objects in the lists $a_1,\ldots,a_n$ or $b_1,\ldots,b_m$, only on their lengths and the operations $p$ and $q$. As a result, if we have functors $f\colon X\to X'$ and $g\colon Y\to Y'$, then it is clear that

$$(\underline{P}(f\times g)\circ\gamma_{X,Y})_{[p;\underline{x}],[q;\underline{y}]} = [\lambda_{p,q},\underline{1}] = (\gamma_{X',Y'}\circ(\underline{P}f\times\underline{P}g))_{[p;\underline{x}],[q;\underline{y}]}.$$

As such we can simply write $(\gamma_{X,Y})_{[p;\underline{x}],[q;\underline{y}]}$ in shorthand as $\gamma_{p,q}$.

There are now seven axioms to check for a pseudo-commutativity: three strength axioms, two unit axioms, and two multiplication axioms. For the first strength axiom, we must verify that two different 2-cells of shape

$$\begin{tikzcd} A\times TB\times TC \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] & & T(A\times B\times C) \end{tikzcd}$$

are equal. The first of these is $\gamma$ precomposed with $d\times 1$, and so is the component of $\gamma$ at an object

$$\Big([p;(a,b_1),\ldots,(a,b_n)],[q;c_1,\ldots,c_m]\Big).$$

The second of these is $d$ applied to the component of $1\times\gamma$ at

$$\Big(a,([p;b_1,\ldots,b_n],[q;c_1,\ldots,c_m])\Big).$$

It is straightforward to compute that each of these maps is the image of $(\lambda_{p,q}\cdot t_{m,n}^{-1}, 1)$ under the functor

$$\coprod P(n)\times (A\times B)^n \to \coprod P(n)\times_{\Lambda(n)} (A\times B)^n.$$

The other two strength axioms follow by analogous calculations for other whiskerings of $\gamma$ with $d$ or $d^*$.

For the unit axioms, we must compute the components of $\gamma$ precomposed with $\eta\times 1$ for the first axiom and $1\times\eta$ for the second. Thus for the first unit axiom, we must compute the component of $\gamma$ at $\big([e;a],[p;b_1,\dots,b_m]\big)$. By definition, this is the image of $(\lambda_{e,p}\cdot t_{m,1}^{-1},1)$ under the map to the coequalizer, and by the first hypothesis of the theorem we know that $t_{m,1}^{-1}$ is the identity element and this isomorphism is the identity as well, so this component of $\gamma$ is also the identity. The second unit axiom follows similarly, using that $t_{1,n}^{-1}$ is the identity.

For the multiplication axioms, first note that hypothesis 2 in the statement of the theorem is necessary in order to ensure the existence of the top horizontal equality in the diagram of hypothesis 4; the same goes for hypotheses 3 and 5. We now explain how hypotheses 2 and 4 ensure that the first multiplication axiom holds, with the same reasoning showing that hypotheses 3 and 5 imply the second multiplication axiom.

We begin by studying the pasting diagram in the first multiplication axiom, but computing its values using the strength and costrength for the non-symmetric operad underlying $P$; this means that we evaluate on objects of the form $(p;a_1,\dots,a_n)$ rather than on their equivalence classes. Let $p\in P(l), q_i\in P(m_i)$ for $1\le i\le l$, and $r\in P(n)$. Computing the top and right leg around the pasting diagram gives the function on objects

$$\Big((p;(q_1;\underline{a_1}),\dots,(q_l;\underline{a_l})),(r;\underline{b})\Big)\mapsto\Big(\mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_l;\underline{r}));\underline{((a_{1\bullet},\underline{b}))},\dots,\underline{((a_{l\bullet},\underline{b}))}\Big),$$

where $\underline{((a_{i\bullet},\underline{b}))}$ is the list of pairs

$$(a_{i1},b_1),\dots,(a_{i1},b_m),(a_{i2},b_1),\dots,(a_{in_i},b_m).$$

Then $\underline{P}\gamma$ is the image of the morphism which is the identity on the $(a_{ij},b_k)$'s, and is

$$\mu\Big(p;\mu(q_1;\underline{r}),\dots,\mu(q_n;\underline{r})\Big)\xrightarrow{\mu\left(1;\lambda_{q_1,r}t_{n,m_1}^{-1},\dots,\lambda_{q_1,r}t_{n,m_l}^{-1}\right)}\mu\Big(p;\mu(r;\underline{q_1})t_{n,m_1}^{-1},\dots,\mu(r;\underline{q_l})t_{n,m_l}^{-1}\Big)$$

on the first component. By the $\Lambda$-operad axioms, the target of this morphism is equal to

$$\mu(p;\mu(r;\underline{q_1}),\dots,\mu(r;\underline{q_l}))\mu(e_l;t_{n,m_1}^{-1},\dots,t_{n,m_l}^{-1}).$$

Note that this is not the same object as one obtains by computing $T\mu\circ T^2d^*\circ Td\circ d^*$ using the underlying non-symmetric operad of $P$ as we are required to use the $\Lambda$-equivariance to ensure that the target of $\gamma$ is the correct one.

Next we compute the source of $(\mu\circ Td^*)*\gamma$, the other 2-cell in the pasting appearing in the first multiplication axiom. We compute this once again using the strength and costrength for the underlying non-symmetric operad, and note once again that this will not match our previous calculations precisely, but only up to an application of $\Lambda$-equivariance. This functor has its map on objects given by

$$\Big((p;(q_1;\underline{a_1}),\dots,(q_l;\underline{a_l})),(r;\underline{b})\Big)\mapsto\Big(\mu(\mu(p;\underline{r});\underline{q_1},\dots,\underline{q_l});\underline{(\underline{a_1},b_\bullet)},\dots,\underline{(\underline{a_l},b_\bullet)}\Big).$$

Note that if we apply $\Lambda$-equivariance, this matches the target computed above. Once again the component of $\gamma$ is the image of a morphism which is the identity on the $(a_{ij},b_k)$'s, and its first component is

$$\mu\big(\mu(p;\underline{r});\underline{q_1},\dots,\underline{q_l}\big)\xrightarrow{\mu\left(\lambda_{p,r}\cdot t_{n,l}^{-1};1,\dots,1\right)}\mu\big(\mu(r;\underline{p})\cdot t_{n,l}^{-1};\underline{q_1},\dots,\underline{q_l}\big).$$

We cannot compose these morphisms in $\coprod P(n) \times (A \times B)^n$ as they do not have matching source and target, but we can in $\coprod P(n) \times_\Lambda (A \times B)^n$. The resulting morphism has first component given by the image of

$$\mu\big(p;\mu(q_1;\underline{r}),\dots,\mu(q_n;\underline{r})\big) \xrightarrow{\mu\left(1;\lambda_{q_1,r}t^{-1}_{n,m_1},\dots,\lambda_{q_1,r}t^{-1}_{n,m_l}\right)} \mu\big(p;\mu(r;\underline{q_1})t^{-1}_{n,m_1},\dots,\mu(r;\underline{q_l})t^{-1}_{n,m_l}\big)$$
$$\xrightarrow{\mu\left(\lambda_{p,r}\cdot t^{-1}_{n,l};1,\dots,1\right)\cdot\mu(e_l;t^{-1}_{n,m_1},\dots,t^{-1}_{n,m_l})} \mu\big(\mu(r;\underline{p})\cdot t^{-1}_{n,l};\underline{q_1},\dots,\underline{q_l}\big)\cdot\mu(e_l;t^{-1}_{n,m_1},\dots,t^{-1}_{n,m_l}),$$

where we have made use of the operad axioms in identifying the target of the first map with the source of the second. Using the $\Lambda$-operad axioms again on the target, we get

$$\begin{aligned}&\mu\big(\mu(r;\underline{p})\cdot t^{-1}_{n,l};\underline{q_1},\dots,\underline{q_l}\big)\cdot\mu(e_l;t^{-1}_{n,m_1},\dots,t^{-1}_{n,m_l})\\ &= \mu\big(\mu(r;\underline{p});\underline{q_1,\dots,q_l}\big)\cdot\mu(t^{-1}_{n,l};\underline{e})\cdot\mu(e_l;t^{-1}_{n,m_1},\dots,t^{-1}_{n,m_l}).\end{aligned}$$

This composite of two morphisms, together with the necessary identities coming from operad axioms, is precisely the left and bottom leg of the diagram in hypothesis 4 in the statement of the theorem. Using the same method, one then verifies that $\gamma * (\mu \times 1)$ has its first component the image of the morphism appearing along the top and right leg of the diagram in hypothesis 4. The second component of these morphisms are all identities arising from $\Lambda$-equivariance, so the first multiplication axiom is a consequence of hypotheses 2 and 4. We leave the calculations for the second multiplication axiom to the reader as they are of the same nature. $\square$

**Corollary 9.6.** *Let $P$ be a non-symmetric operad. Then $\underline{P}$ is never pseudo-commutative.*

*Proof.* In the non-symmetric case, the 2-monad is just given using coproducts and products, there is no coequalizer. In order to define $\gamma$, we then need an isomorphism

$$\big(\mu(p;\underline{q});\underline{(x,\underline{y})}\big) \cong \big(\mu(q;\underline{p});\underline{(\underline{x},y)}\big).$$

When $A, B$ are discrete, there is no isomorphism $\underline{(x,\underline{y})} \cong \underline{(\underline{x},y)}$, and therefore no such $\gamma$ can exist. $\square$

A further property that a pseudo-commutativity can possess is that of symmetry. This symmetry is then reflected in the monoidal structure on the 2-category of algebras, which will then also have a symmetric tensor product (in a suitable, 2-categorical sense).

**Definition 9.7.** Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad on a symmetric monoidal 2-category $\mathcal{K}$ with symmetry $c$. We then say that a pseudo-commutativity $\gamma$ for $T$ is *symmetric* when the following is satisfied for all $A,\ B \in \mathcal{K}$:

$$Tc_{A,B} \circ \gamma_{A,B} \circ c_{TB,TA} = \gamma_{B,A}.$$

With the notion of symmetry at hand we are able to extend the above theorem, stating when $\underline{P}$ is symmetric.

**Theorem 9.8.** *The pseudo-commutative structure for $\underline{P}$ given by Theorem 9.5 is symmetric if for all $m, n \in \mathbb{N}_+$ the two conditions below hold.*

(1) $t_{m,n} = t^{-1}_{n,m}$.

(2) *The following diagram commutes:*

$$\begin{tikzcd}
\mu(p;\underline{q})\cdot t_{m,n}t_{n,m} \arrow[r, equal] \arrow[d, "\lambda_{p,q}\cdot 1"'] & \mu(p;\underline{q})\cdot e_{mn} \arrow[d, equal] \\
\mu(q;\underline{p})\cdot t_{n,m} \arrow[r, "\lambda_{q,p}"'] & \mu(p;\underline{q})
\end{tikzcd}$$

*Proof.* The commutativity of the diagram above ensures that the first component of the symmetry axiom commutes in $P(n)$ before taking equivalence classes in the coequalizer, just as in the proof of Theorem 9.5. $\square$

**Definition 9.9.** Let $P$ be a $\Lambda$-operad in **Cat**. We say that $P$ is *contractible* if each category $P(n)$ is equivalent to the terminal category.

**Corollary 9.10.** *If $P$ is contractible and there exist $t_{m,n}$ as in Theorem 9.5, then $\underline{P}$ acquires a pseudo-commutativity. Furthermore, it is symmetric if $t_{n,m} = t_{m,n}^{-1}$.*

*Proof.* The only thing left to define is the collection of natural isomorphisms $\lambda_{p,q}$. But since each $P(n)$ is contractible, $\lambda_{p,q}$ must be the unique isomorphism between its source and target, and furthermore the last two axioms hold since any pair of parallel arrows are equal in a contractible category. $\square$

**Corollary 9.11.** *If $P$ is a contractible symmetric operad then $\underline{P}$ has a symmetric pseudo-commutativity.*

*Proof.* We choose $t_{m,n} = \tau_{m,n}$. $\square$

**Remark 9.12.** If a $\Lambda$-operad $P$ is contractible, it is not the case that its symmetrization $S(P)$ (see Theorem 1.34) will also be contractible. The category $P(n)\times_{\Lambda(n)}\Sigma_n$ will necessarily be a groupoid as it is a colimit of groupoids: contractible categories are always groupoids, and both $\Lambda(n)$ and $\Sigma_n$ are discrete. Let $g \in \ker \pi_n$ be any non-identity element, and let $p \in P(n)$ be any object. Then

$$[p\cdot g, e] = [p, \pi(g)e] = [p, e],$$

but unless $p\cdot g = p$ in $P(n)$, there will be a unique isomorphism between them that will not be the identity, and hence will define a nontrivial automorphism of $[p, e]$ in $P(n)\times_{\Lambda(n)}\Sigma_n$. The existence of such ensures that $P(n)\times_{\Lambda(n)}\Sigma_n$ is not contractible.

We conclude with a computation using Theorem 9.5. This result (9.13 below) was only conjectured in [13], but we are able to prove it quite easily with the machinery developed thus far. Our strategy is to construct a $\Lambda$-operad which is contractible together with the group elements required in 9.5. Note that the symmetrized version of this operad will not be contractible, and we do not know of a proof using the structure of the symmetrized operad.

**Theorem 9.13.** *The 2-monad $\underline{B}$ for braided strict monoidal categories on* **Cat** *has two pseudo-commutative structures on it, neither of which are symmetric.*

In order to apply our theory, the 2-monad $\underline{B}$ must arise from a $\Lambda$-operad. The following proposition describes it as such, and can largely be found as Example 3.2 in the work of Fiedorowicz [9].

**Proposition 9.14.** *The 2-monad $\underline{B}$ is the 2-monad associated to the* $\mathbf{Br}$*-operad $B$ with the category $B(n)$ having objects the elements of the $n$th braid group $Br_n$ and a unique isomorphism between any pair of objects; the action of $Br_n$ on $B(n)$ is given by right multiplication on objects and is then uniquely determined on morphisms.*

The interested reader can easily verify that algebras for the $\mathbf{Br}$-operad $B$ are braided strict monoidal categories. The objects of $\underline{B}(X)$ can be identified with finite lists of objects of $X$, and morphisms are generated by the morphisms of $X$ together with new isomorphisms

$$x_1, \ldots, x_n \xrightarrow{\gamma} x_{\gamma^{-1}(1)}, \ldots, x_{\gamma^{-1}(n)}$$

where $\gamma \in Br_n$ and the notation $\gamma^{-1}(i)$ means, as before, that we take the preimage of $i$ under the permutation $\pi(\gamma)$ associated to $\gamma$. This shows that $\underline{B}(X)$ is the free braided strict monoidal category generated by $X$ according to [14], and it is easy to verify that the 2-monad structure on $\underline{B}$ arising from the $\mathbf{Br}$-operad structure on $B$ is the correct one to produce braided strict monoidal categories as algebras.

**Definition 9.15.** A braid $\gamma \in Br_n$ is *positive* if it is an element of the submonoid of $Br_n$ generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

**Definition 9.16.** A braid $\gamma \in Br_n$ is *minimal* if no pair of strands in $\gamma$ cross twice.

For our purposes, we are interested in braids which are both positive and minimal. A proof of the following lemma can be found in [8].

**Lemma 9.17.** *Let $PM_n$ be the subset of $Br_n$ consisting of positive, minimal braids. Then the function sending a braid to its underlying permutation is a bijection of sets $PM_n \to \Sigma_n$.*

**Remark 9.18.** It is worth noting that this bijection is not an isomorphism of groups, since $PM_n$ is not a group or even a monoid. The element $\sigma_1 \in Br_n$ is certainly in $PM_n$, but $\sigma_1^2$ is not as the first two strands cross twice. Thus we see that the product of two minimal braids is generally not minimal, but by definition the product of positive braids is positive.

*Proof of Theorem 9.13.* In order to use Theorem 9.5 with the action operad being the braid operad $\mathbf{Br}$, we must first construct elements $t_{m,n} \in Br_{mn}$ satisfying certain properties. Using Lemma 9.17, we define $t_{m,n}$ to be the unique positive minimal braid such that $\pi(t_{m,n}) = \tau_{m,n}$. Since $\tau_{1,n} = e_n = \tau_{n,1}$ in $\Sigma_n$ and the identity element $e_n \in Br_n$ is positive and minimal, we have that $t_{1,n} = e_n = t_{n,1}$ in $Br_n$. Thus in order to verify the remaining hypotheses, we must check two equations, each of which states that some element $t_{m,n}$ can be expressed as a product of operadic compositions of other elements.

Let $l, m_1, \ldots, m_l, n$ be natural numbers, and let $M = \sum m_i$. We must check that

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}) = t_{N,l}$$

in $Br_{lN}$. These braids have the same underlying permutations by construction, and both are positive since each operadic composition on the left is positive. The braid on the right is minimal by definition, so if we prove that the braid on the left is also minimal, they are necessarily equal. Now $\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$ is given by the braid for $t_{n,l}$ but with the first strand replaced by $m_1$ strands, the second strand replaced by $m_2$ strands, and so on for the first $l$ strands of $t_{n,l}$, and then repeating for each group of $l$ strands. In particular, since strands $i, i+l, i+2l, \ldots, i+(n-1)l$

never cross in $t_{n,l}$, none of the $m_i$ strands that each of these is replaced with cross. The braid $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ consists of the disjoint union of the braids for each $t_{n,m_i}$, so if two strands cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ then they must both cross in some $t_{n,m_i}$. The strands in $t_{n,m_i}$ are those numbered from $n(m_1 + \cdots + m_{i-1}) + 1$ to $n(m_1 + \cdots + m_{i-1} + m_i)$. This is a consecutive collection of $nm_i$ strands, and it is simple to check that these strands are precisely those connected (via the group operation in $Br_{Nl}$, concatenation) to the duplicated copies of strands $i, i+l, i+2l, \ldots, i+(n-1)l$ in $t_{n,l}$. Thus if a pair of strands were to cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$, that pair cannot also have crossed in $\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$, showing that the resulting product braid

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$$

is minimal. The calculation showing that

$$\mu(t_{m,l}; \underline{e_1}, \ldots, \underline{e_{n_m}})\mu(e_m; t_{n_1,l}, \ldots, t_{n_m,l})$$

is also minimal follows from the same argument, showing that it is equal to $t_{N,l}$ (here $N$ is the sum of the $n_i$, where once again $i$ ranges from 1 to $l$).

These calculations show, using Theorem 9.5, that the **Br**-operad $B$ induces a 2-monad which has a pseudo-commutative structure. As noted before, $B$-algebras are precisely braided strict monoidal categories. The second pseudo-commutative structure arises by using negative, minimal braids instead of positive ones, and proceeds using the same arguments. This finishes the first part of the proof of Theorem 9.13.

We will now show that neither of these pseudo-commutative structures is symmetric. The symmetry axiom in this case reduces to the fact that, in some category which is given as a coequalizer, the morphism with first component

$$f : \mu(p; \underline{q}) \cdot t_{n,m}t_{m,n} \to \mu(q; \underline{p}) \cdot t_{m,n} \to \mu(p; \underline{q})$$

is the identity. Now it is clear that $t_{n,m}$ is not equal to $t_{m,n}^{-1}$ in general: taking $m = n = 2$, we note that $t_{2,2} = \sigma_2$, and this element is certainly not of order two in $Br_4$. $B(4)$ is the category whose objects are the elements of $Br_4$ with a unique isomorphism between any two pair of objects, and $Br_4$ acts by multiplication on the right; this action is clearly free and transitive. We recall (see Lemma 4.5) that in a coequalizer of the form $A \times_G B$, we have that a morphism $[f_1, f_2]$ equals $[g_1, g_2]$ if and only if there exists an $x \in G$ such that

$$\begin{aligned} f_1 \cdot x &= g_1, \\ x^{-1} \cdot f_2 &= g_2. \end{aligned}$$

For the coequalizer in question, for $f$ to be the first component of an identity morphism would imply that $f \cdot x$ would be a genuine identity in $B(4)$ for some $x$. But $f \cdot x$ would have source $\mu(p; \underline{q})t_{n,m}t_{m,n}x$ and target $\mu(p; \underline{q})x$, which requires $t_{n,m}t_{m,n}$ to be the identity group element for all $n, m$. In particular, this would force $t_{2,2}$ to have order two, which we have noted above does not hold in $Br_4$, thus giving a contradiction. □

**Remark 9.19.** The pseudo-commutativities given above are not necessarily the only ones that exist for the **Br**-operad $B$, but we do not know a general strategy for producing others.

## 10. Profunctors and multicategories

In this section we generalize from operads to multicategories (or colored operads). The notions of plain and symmetric multicategories are standard [2], but in fact there is a corresponding notion of $\mathbf{\Lambda}$-multicategory for any action operad $\mathbf{\Lambda}$. We will give the basic definition and then show that it arises abstractly from a lifting of $E\Lambda$ as a 2-monad on **Cat** to a pseudomonad on **Prof**, the bicategory of categories, profunctors, and transformations. A quick treatment of similar material but restricted to the symmetric case can be found in [10].

**Definition 10.1.** Let $\mathbf{\Lambda}$ be an action operad. A $\mathbf{\Lambda}$-*multicategory* $M$ consists of the following data:

- a set of objects $M_0$;
- for any finite list $x_1, \ldots, x_n$ of objects and any object $y$, a set
$$M(x_1, \ldots, x_n; y)$$
of multi-arrows (or just arrows) from $x_1, \ldots, x_n$ to $y$;
- for each $\alpha \in \Lambda(n)$, an isomorphism
$$- \cdot \alpha : M(x_1, \ldots, x_n; y) \to M(x_{\pi(g)(1)}, \ldots, x_{\pi(g)(n)}; y);$$
- for each object $x$, an arrow $\mathrm{id}_x \in M(x; x)$; and
- a composition function
$$M(y_1, \ldots, y_k; z) \times M(x_{11}, \ldots, x_{1,n_1}; y_1) \times \cdots \times M(x_{k1}, \ldots, x_{k,n_k}; y_k) \to M(\underline{x}; z)$$
where $\underline{x} = x_{11}, \ldots, x_{1,n_1}, x_{21}, \ldots, x_{k,n_k}$, and which we write as
$$(g; f_1, \ldots, f_n) \mapsto g(f_1, \ldots, f_n).$$

These data are subject to the following axioms.

(1) id is a two-sided unit:
$$\begin{array}{rcl} \mathrm{id}(f) & = & f, \\ f(\mathrm{id}, \ldots, \mathrm{id}) & = & f. \end{array}$$

(2) Composition is associative:
$$f\Big(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n})\Big) = f(g_1, \ldots, g_n)(h_{11}, \ldots, h_{nm_n}).$$

(3) Composition respects the group actions:
$$\begin{array}{rcl} f(g_1 \cdot \alpha_1, \ldots, g_n \cdot \alpha_n) & = & f(g_1, \ldots, g_n) \cdot \mu^{\Lambda}(e; \alpha_1, \ldots, \alpha_n), \\ f \cdot \alpha(g_1, \ldots, g_n) & = & f(g_{\pi^{-1}(\alpha)(1)}, \ldots, g_{\pi^{-1}(\alpha)(n)}) \cdot \mu^{\Lambda}(\alpha; e_1, \ldots, e_n). \end{array}$$

**Definition 10.2.** Let $M, N$ be $\mathbf{\Lambda}$-multicategories. A $\mathbf{\Lambda}$-*multifunctor* $F$ consists of the following data:

- a function $F_0 : M_0 \to N_0$ on sets of objects and
- functions $F : M(x_1, \ldots, x_n; y) \to N(F_0(x_1), \ldots, F_0(x_n); F_0(y))$ which are $\Lambda(n)$-equivariant in that $F(f \cdot \alpha) = F(f) \cdot \alpha$.

These data are subject to the following axioms.

(1) $F$ preserves identites: $F(\mathrm{id}_x) = \mathrm{id}_{F_0(x)}$.
(2) $F$ preserves composition: $F\Big(f(g_1, \ldots, g_n)\Big) = F(f)\Big(F(g_1), \ldots, F(g_n)\Big)$.

Recall that the bicategory **Prof** has objects categories, 1-cells $F : X \nrightarrow Y$ profunctors from $X$ to $Y$ or equivalently functors

$$F : Y^{\mathrm{op}} \times X \to \mathbf{Sets},$$

and 2-cells transformations $F \Rightarrow G$. Composition of profunctors is given by the coend formula

$$G \circ F(z, x) = \int^{y \in Y} G(z, y) \times F(y, x)$$

and hence is only unital and associative up to coherent isomorphism. There is an embedding pseudofunctor $(-)^+ : \mathbf{Cat} \hookrightarrow \mathbf{Prof}$ which is the identity on objects and sends a functor $F : X \to Y$ to the profunctor $F^+$ defined by $F^+(y, x) = Y(y, Fx)$.

**Theorem 10.3.** *The 2-monad $E\Lambda$ on the 2-category* **Cat** *lifts to a pseudomonad $\widetilde{E\Lambda}$ on the bicategory* **Prof**.

*Proof.* On objects, we have $\widetilde{E\Lambda}(X) = E\Lambda(X)$. Let $F : X \nrightarrow Y$ be a profunctor given by the functor $F : Y^{\mathrm{op}} \times X \to \mathbf{Sets}$. We define $\widetilde{E\Lambda}F$ to be the functor

$$(E\Lambda(Y))^{\mathrm{op}} \times E\Lambda(X) \to \mathbf{Sets}$$

which is defined by the formulas

$$\widetilde{\Lambda}F\big([e; x_1, \ldots, x_n], [e; y_1, \ldots, y_m]\big) = \begin{cases} \varnothing & \text{if } n \neq m, \\ \coprod_{g \in \Lambda(n)} \prod_{i=1}^n F(y_i, x_{\pi(g)(i)}) & \text{if } n = m. \end{cases}$$

For a functor $G : X \to Y$, it is easy to check that

$$\widetilde{E\Lambda}(G^+) = \big(E\Lambda G\big)^+$$

using 5.15. The same formulas define the action of $\widetilde{E\Lambda}$ on 2-cells as well. The multiplication and unit of $\widetilde{E\Lambda}$ are just $\mu^+$ and $\eta^+$, where $\mu, \eta$ are the multiplication and unit, respectively, of $E\Lambda$. The remainder of the pseudomonad data comes from the pseudofunctoriality of $(-)^+$, and the axioms follow from the 2-monad axioms for $E\Lambda$ and the pseudofunctor axioms for $(-)^+$. $\square$

**Remark 10.4.** Since **Prof** is essentially the Kleisli bicategory for the free cocompletion pseudomonad, this lift corresponds to a pseudo-distributive law between $E\Lambda$ and the free cocompletion pseudomonad, but we do not pursue this perspective here.

Given a bicategory $B$ and a pseudomonad $T$ on $B$, we can form the Kleisli bicategory of $T$, $\mathbf{Kl}_T$. It has the same objects as $B$, but a 1-cell from $a$ to $b$ in $\mathbf{Kl}_T$ is a 1-cell $f : a \to Tb$ in $B$. In the case $B = \mathbf{Prof}, T = \widetilde{E\Lambda}$, the objects of $\mathbf{Kl}_T$ are categories, the 1-cells $X \nrightarrow Y$ are profunctors from $X$ to $E\Lambda Y$, or alternatively a functor $(E\Lambda Y)^{op} \times X \to \mathbf{Sets}$, and the 2-cells are natural transformation between such.

We now recall some standard definitions [4].

**Definition 10.5.** Let $B$ be a bicategory. A *monad* $(x, t, \mu, \eta)$ in $B$ consists of the following data:

- an object $x$,
- a 1-cell $t : x \to x$,
- a 2-cell $\mu : t^2 \Rightarrow t$, and
- a 2-cell $\eta : \mathrm{id}_x \Rightarrow t$.

These data are subject to the following axioms.

$$\begin{tikzcd}
(t \circ t) \circ t \arrow[r, "\cong"] \arrow[dd, "\mu * t"'] & t \circ (t \circ t) \arrow[d, "t * \mu"] \\
& t \circ t \arrow[d, "\mu"] \\
t \circ t \arrow[r, "\mu"'] & t
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_x \circ t \arrow[r, "\eta * t"] \arrow[dr, "\cong"'] & t \circ t \arrow[d, "\mu"] \\
& t
\end{tikzcd}
\quad
\begin{tikzcd}
t \circ \mathrm{id}_x \arrow[r, "t * \eta"] \arrow[dr, "\cong"'] & t \circ t \arrow[d, "\mu"] \\
& t
\end{tikzcd}$$

**Definition 10.6.** Let $(x, t, \mu, \eta), (x', t', \mu', \eta')$ be monads in $B$. An *oplax monad map* $(F, \alpha)$ from $t$ to $t'$ consists of the following data:

- a 1-cell $F : x \to x'$ and
- a 2-cell $\alpha : F \circ t \Rightarrow t' \circ F$.

These data are subject to the following axioms, in which we suppress the constraints of the bicategory $B$.

$$\begin{tikzcd}
Ft^2 \arrow[r, "\alpha * t"] \arrow[dd, "F * \mu"'] & t'Ft \arrow[d, "t' * \alpha"] \\
& t'^2 F \arrow[d, "\mu' * F"] \\
Ft \arrow[r, "\alpha"'] & t'F
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "F * \eta"] \arrow[dr, "\eta' * F"'] & Ft \arrow[d, "\alpha"] \\
& t'F
\end{tikzcd}$$

**Definition 10.7.** Let $(F, \alpha), (F', \alpha')$ be oplax monad maps from $t$ to $t'$. A *transformation of monad maps* $\Gamma : (F, \alpha) \Rightarrow (F', \alpha')$ is a 2-cell $\Gamma : F \Rightarrow F'$ such that

$$\begin{tikzcd}
Ft \arrow[r, "\alpha"] \arrow[d, "\Gamma * t"'] & t'F \arrow[d, "t' * \Gamma"] \\
F't \arrow[r, "\alpha'"'] & t'F'
\end{tikzcd}$$

commutes.

It is simple to check that monads, oplax monad maps, and transformations of monad maps form a bicategory.

**Theorem 10.8.** *There is a biequivalence between the category* $\mathbf{\Lambda}$-**Multicat** *of*

- $\mathbf{\Lambda}$*-multicategories and*
- $\mathbf{\Lambda}$*-multifunctors, and*

*the bicategory* $\mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\Lambda}})$ *of*

- *monads on sets (viewed as discrete categories) in* $\mathbf{Kl}_{\widetilde{E\Lambda}}$,
- *oplax monad maps* $(F, \alpha)$ *between them which are isomorphic to one of the form* $(f^+, \alpha)$ *for* $f : S \to T$ *for some function of the underlying sets, and*
- *transformations of monad maps.*

*Under this biequivalence, the category of* $\mathbf{\Lambda}$*-operads is equivalent to the bicategory of monads on the terminal set in* $\mathbf{Kl}_{\widetilde{E\Lambda}}$.

*Proof.* First, we note that $\mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\Lambda}})$ is a locally essentially discrete bicategory, by which we mean the hom-categories are all equivalent to discrete categories. We will show there is a unique isomorphism or no 2-cell at all between oplax monad maps of the form $(f^+, \alpha)$, from which the claim follows in general. A 2-cell between such has as its data a natural transformation $\gamma : f^+ \Rightarrow g^+$ which has components

$$\gamma_{[e;t_1,\ldots,t_n],s} : f^+([e; t_1, \ldots, t_n], s) \to g^+([e; t_1, \ldots, t_n], s).$$

Both of these sets are empty unless $n = 1$, and then the source is nonempty when $f(s) = t$ and the target is nonempty when $g(s) = t$; when nonempty, both of these sets are singletons. If both are nonempty for some $s$, then the functions $f, g$ agree on $s$. Assume the target is nonempty for some $([e; t], s)$ but that the source is empty, in other words that $g(s) = t$ but $f(s) \neq t$. Then consider $\gamma_{[e;f(s)],s}$. Its source is $f^+([e; f(s)], s)$ which is nonempty by construction, but its target is $g^+([e; f(s)], s)$. We know that $g(s) = t \neq f(s)$, so $g^+([e; f(s)], s)$ must be empty, giving a map from a nonempty set to an empty one, a contradiction. Thus there is a at most one 2-cell from an oplax monad map $(f^+, \alpha)$ to another $(g^+, \beta)$, such a map can only exist if $f = g$, and if it does exist then it is invertible. Thus the hom-categories of $\mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\Lambda}})$ are essentially discrete, and this bicategory is equivalent to a category.

We begin by describing an object of $\mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\Lambda}})$ which is a monad in $\mathbf{Kl}_{\widetilde{E\Lambda}}$ whose underlying category is a set $S$. A 1-cell $M : S \nrightarrow S$ is then a functor $(E\Lambda S)^{op} \times S \to \mathbf{Sets}$ which amounts to sets $M(s_1, \ldots, s_n; s)$ for $s_1, \ldots, s_n, s \in S$ together with a right action of $\Lambda(n)$ as in 10.1. A 2-cell $1_S \Rightarrow M$ consists of a $\Lambda(1)$-equivariant function $\Lambda(1) \to M(s; s)$ for each $s \in S$, in other words an element $\mathrm{id}_s \in M(s; s)$. A 2-cell $M \circ M \Rightarrow M$ then consists of a multicategorical composition function, as in 10.1, with appropriate equivariance built in by the coend used for composition of profunctors. Associativity and unit conditions are then seen to be the same as for $\mathbf{\Lambda}$-multicategories.

By definition, an oplax monad map $(f^+, \alpha) : (S, M) \to (S', M')$ consists of a function $f : S \to S'$ and a transformation $\alpha : M \circ f^+ \Rightarrow f^+ \circ M'$ satisfying two axioms. The transformation $\alpha$ amounts to giving $\Lambda(n)$-equivariant functions

$$M(s_1, \ldots, s_n; s) \to M'\big(f(s_1), \ldots, f(s_n); f(s)\big),$$

and the two axioms correspond to the unit and composition axioms for a $\mathbf{\Lambda}$-multifunctor.

These descriptions give the action on objects and morphisms of a pseudofunctor $\mathbf{\Lambda}$-**Multicat** $\to \mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\Lambda}})$ with local contractibility providing the pseudofunctoriality constraints as well as showing that the axioms for a pseudofunctor hold. It is also clear that this pseudofunctor is biessentially surjective and locally essentially surjective, so it is a biequivalence once again using local contractibility.

The final claim is then an immediate consequence of the definitions of $\mathbf{\Lambda}$-operad and $\mathbf{\Lambda}$-multicategory. □

## References

[1] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[2] J. Baez and J. Dolan. Higher-dimensional algebra. III. $n$-categories and the algebra of opetopes. *Adv. Math.*, 135(2):145–206, 1998.

[3] M. A. Batanin. The Eckmann-Hilton argument and higher operads. *Adv. Math.*, 217(1):334–385, 2008.

[4] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[5] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989.
[6] Brian Day. *Construction of biclosed categories*. 1970. Thesis (Ph.D.)–University of New South Wales.
[7] V. G. Drinfeld. Quasi-Hopf algebras. *Algebra i Analiz*, 1(6):114–148, 1989.
[8] Elsayed A. El-Rifai and H. R. Morton. Algorithms for positive braids. *Quart. J. Math. Oxford Ser. (2)*, 45(180):479–497, 1994.
[9] Zbigniew Fiedorowicz. The symmetric bar construction. preprint.
[10] R. Garner. Polycategories via pseudo-distributive laws. *Adv. Math.*, 218(3):781–827, 2008.
[11] A. Henriques and J. Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006.
[12] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
[13] Martin Hyland and John Power. Pseudo-commutative monads and pseudo-closed 2-categories. *J. Pure Appl. Algebra*, 175(1-3):141–185, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.
[14] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.
[15] G. M. Kelly. An abstract approach to coherence. In *Coherence in categories*, pages 106–147. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.
[16] G. M. Kelly. Many-variable functorial calculus. I. In *Coherence in categories*, pages 66–105. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.
[17] G. M. Kelly. On clubs and doctrines. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 181–256. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.
[18] G. M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39(2):301–317, 1989.
[19] G. M. Kelly. On the operads of J. P. May. *Repr. Theory Appl. Categ.*, (13):1–13, 2005.
[20] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 75–103. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.
[21] Anders Kock. Monads on symmetric monoidal closed categories. *Arch. Math. (Basel)*, 21:1–10, 1970.
[22] Anders Kock. Closed categories generated by commutative monads. *J. Austral. Math. Soc.*, 12:405–424, 1971.
[23] Anders Kock. Strong functors and monoidal monads. *Arch. Math. (Basel)*, 23:113–120, 1972.
[24] Stephen Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175(1-3):223–241, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.
[25] F. William Lawvere. *Functional semantics of algebraic theories*. ProQuest LLC, Ann Arbor, MI, 1963. Thesis (Ph.D.)–Columbia University.
[26] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
[27] S. Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965.
[28] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
[29] M. Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, volume 5 of *Handb. Algebr.*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008.
[30] M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
[31] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.
[32] A. J. Power. A general coherence result. *J. Pure Appl. Algebra*, 57(2):165–173, 1989.
[33] Paolo Salvatore and Nathalie Wahl. Framed discs operads and Batalin-Vilkovisky algebras. *Q. J. Math.*, 54(2):213–231, 2003.
[34] R. Street. *Quantum groups*, volume 19 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2007. A path to current algebra.
[35] Nathalie Wahl. *Ribbon braids and related operads*. 2001. Thesis (Ph.D.)–University of Oxford.
[36] Wenbin Zhang. Group operads and homotopy theory. preprint, arXiv:1111.7090v2.

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