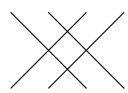
Theorem 0.1. *There is a functor* $C\colon \mathbf{AOp} \to \mathbf{CSGrp}$.

Proof. Let Λ be an action operad. Put $C(\Lambda)(n) = \Lambda_{n+1}$. We will show that this collection of groups constitutes a crossed simplicial group by following Proposition 1.6 of [**?**]. The group homomorphisms into the symmetric groups are given by π of the action operad. We define $s_i\colon \Lambda_n \to \Lambda_{n+1}$ by

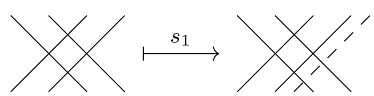
$$s_i(g) = \mu(g; e_1, \ldots, e_1, e_2, e_1, \ldots, e_1),$$

where e_2 is in the $\pi(g)^{-1}(i)^{th}$ input. Similarly for $d_i\colon \Lambda_n \to \Lambda_{n-1}$, using e_0 rather than e_2.

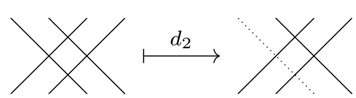
We will describe how to think of the elements $s_i(g)$ and $d_i(g)$ in terms of the diagrams that represent their permutations under π. For example, the permutation $(0\,2)(1\,3) \in \Sigma_3$ can be drawn as follows.



The diagram representing the permutation $s_i(g)$ is given by drawing the permutation for g and adding a new string running parallel and to the right of the i^{th} output string. E.g., $s_1((0\,2)(1\,3)) = (0\,3\,1\,4\,2)$ is represented by the following diagram.



Similarly the diagram representing the permutation $d_i(g)$ is given by drawing the permutation g and deleting the i^{th} output string. E.g., $d_2((0\,2)(1\,3)) = (0\,2\,1)$ is represented by the following diagram.



It is then simple to check that the simplicial identities hold for these maps. For example when $i < j$ we require that $d_i \cdot d_j = d_{j-1} \cdot d_i$. Following the method above we see that $d_i(d_j(g))$ is obtained by first writing out the diagram representing g, deleting the j^{th} output string, and renumbering the strings above this. We then follow this by deleting the i^{th} output string and renumbering the strings above this. If instead we consider $d_{j-1}(d_i(g))$ we see that the permutation is obtained by first writing out the diagram representing g, deleting the i^{th} output string, and renumbering the strings above this. Since we have renumbered the strings above the i^{th} output, the removal the $j-1^{th}$ output string in the resulting diagram is the same as removing the j^{th} output string as we did before. The identity follows.

We now require that these maps satisfy two further conditions as detailed in [**?**]. The first is that $s_i(gh) = s_i(g)s_{\pi(g)^{-1}(i)}(h)$ and similarly for the d_i. Now

$$s_i(gh) = \mu(gh; e_1, \ldots, e_1, e_2, e_1, \ldots, e_1)$$

with the e_2 in the $\pi(h^{-1}g^{-1})(i)^{th}$ position. Compare this with

$$s_i(g)s_{\pi(g)^{-1}(i)}(h) = \mu(g; e_1, \ldots, e_1, e_2, e_1, \ldots, e_2)\mu(h; e_1, \ldots, e_1, e_2, e_1, \ldots, e_1)$$

where the first composite has e_2 in the $\pi(g)^{-1}(i)^{th}$ position and the second has e_2 in the $\pi(h)^{-1}(\pi(g)^{-1}(i))^{th}$ position. The action operad axiom tells us that we can combine the two composites, where the element from the left composite that is multiplied with e_2 in the right composite is that which is in the $\pi(h)(\pi(h^{-1}g^{-1})(i))^{th}$ position, i.e., the e_2 from the left composite. Hence the identity is satisfied. Since it did not matter what the elements were we can repeat the argument for the d_i with e_0 in place of e_2.

The final condition of the proposition is that the following two diagrams commute.

$$\begin{tikzcd}
{[n+1]} \arrow[r, "\sigma_{\pi(g)^{-1}(i)}"] \arrow[d, "\pi(s_i(g))"'] & {[n]} \arrow[d, "\pi(g)"] \\
{[n+1]} \arrow[r, "\sigma_i"'] & {[n]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[n-1]} \arrow[r, "\delta_{\pi(g)^{-1}(i)}"] \arrow[d, "\pi(d_i(g))"'] & {[n]} \arrow[d, "\pi(g)"] \\
{[n-1]} \arrow[r, "\delta_i"'] & {[n]}
\end{tikzcd}$$

This can easily be seen as a consequence of how $s_i(g)$ and $d_i(g)$ are described in terms of the diagrams that represent their underlying permutations. Following the bottom path of the first diagram, the elements 1 to $g^{-1}(i)$ and $g^{-1}(i)+2$ to $n+1$ are permuted according to g, while $g^{-1}(i)+1$ is sent to $i+1$. This is followed by every $k \leq i$ being mapped to k, $i+1$ also being mapped to i, and every $k \geq i+1$ being mapped to $k-1$. The top path of the diagram maps each $k \leq g^{-1}(i)$ to k, maps $g^{-1}(i)+1$ to $g^{-1}(i)$ as well, and every $k \geq g^{-1}(i)+2$ to $k-1$. This is followed by permuting the elements according to g. A similar argument shows that the second diagram commutes as well.

On morphisms we simply send a map $F\colon \Lambda \to \Gamma$ of action operads to the collection of morphisms $F_n\colon \Lambda_n \to \Gamma_n$. That these interact appropriately with the face and degeneracy maps follows immediately from fact that F is a map of operads. To be a map of crossed simplicial groups we also require that $\pi^{\Gamma} \cdot F_n = \pi^{\Lambda}$ for all $n \in \mathbb{N}$ but this is simply part of the requirement of F being a map of action operads. $\square$