Thm: Let P be a G-operad. Then the following equip $\underline{P}$ w/ a ps-comm structure.

1) $\forall (m,n) \in \mathbb{N}_+^2$ (no 0 in $\mathbb{N}_+$), $t_{m,n} \in G(mn)$ s.t. $\pi(t_{m,n}) = \tau_{m,n}$.

2) $\forall\, p \in P(m),\ q \in P(n)$, a natural iso $\mu(p; q, \dots, q) \cdot t_{n,m} \cong \mu(q; p, \dots, p)$.

These must satisfy (I changed the order!) isos (plus other one)

- $t_{1,n} = e_n = t_{n,1}$ + $\mu(p; \underline{id}) \cdot t_{1,n} \cong \mu(id; p)$ ⟷ $p e e_m \cong p$ are identities
- $\mu(e_\ell; t_{n,m_1}, \dots, t_{n,m_\ell})\, \mu(t_{n,\ell}; \underbrace{e_{m_1}, \dots, e_{m_\ell}, e_{m_1}, \dots, e_{m_\ell}}_{\text{repeat n times}})$

$= t_{n,M}$ Plus axiom on μ #2 page

This holds for any $\ell, M, m_1, \dots, m_\ell, n$, $M = \sum m_i$

+ two μ axioms (another page).

(Lem:) this ps-comm is symmetric if $t_{m,n} = t_{n,m}^{-1}$.

Cor: If P is contractible (each $P(n) \simeq *$) and we have $t_{m,n}$ as above, then $\underline{P}$ acquires a ps-comm. which is symm if $t_{n,m} = t_{m,n}^{-1}$.

Cor: Any contractible symmetric operad is ps-comm. (+symm.)

Cor: Let P be a non-symmetric operad. Then $\underline{P}$ is never ps-comm.

Pf: There is no iso $\underline{(x,y)} \cong (\underline{x}, \underline{y})$. □

Section 4

1) Def on τ's
2) Def of ps-comm (copy HP)
3) Consequences: ps-comm ⇒ ps-closed ps-mon 2-cat
   +symm ⇒ +symm
   (explain from HP)
4) Main Thm + Cor's
5) Calculations for braided/ ~~ribbon braided cases~~

Need Lemma: $A \times_G B \cong A \times B/\sim$ where $(a,b) \sim (ag, g^{-1}b)$.

Symmetry: If $t_{n,m} = t_{m,n}^{-1}$;

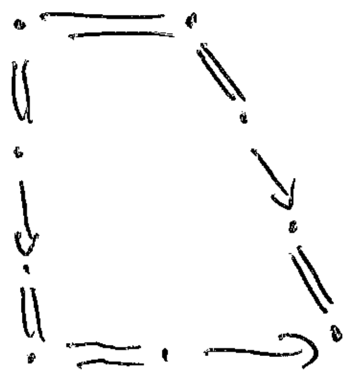
$$\begin{array}{ccc} \mu(p;q)\cdot t_{n,m} t_{m,n} & = & \mu(p;q)\cdot e_{mn} \\ {\scriptstyle iso\cdot 1}\downarrow & & \| \\ \mu(q;p)\cdot t_{m,n} & \longrightarrow & \mu(p;q) \end{array}$$

, then symmetric

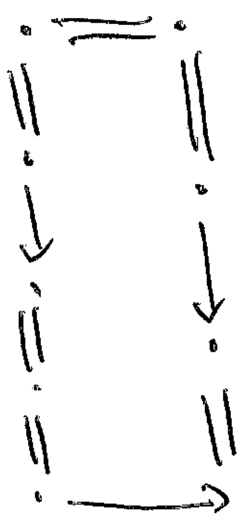
μ axiom #1

$$\begin{tikzcd}
\mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_\ell;\underline{r}))\cdot\mu(e_\ell;t_{n,m_1},\dots,t_{n,m_\ell})\,\mu(t_{n,\ell};\underbrace{e_{m_1},\dots,e_{m_\ell}}_{n\text{ times}}) \arrow[d, equal] \arrow[dr, equal] & \\
\mu(p;\mu(q_1;\underline{r})\cdot t_{n,m_1},\dots,\mu(q_\ell;\underline{r})\cdot t_{n,m_\ell})\,\mu(t_{n,\ell};\underline{e}) \arrow[d, "\mu(1;\,\text{iso's})\cdot 1"'] & \mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_\ell;\underline{r}))\cdot t_{n,m} \arrow[d, equal] \\
\mu(p;\mu(r;q_1),\dots,\mu(r;q_\ell))\cdot\mu(t_{n,\ell};\underline{e}) \arrow[d, equal] & \mu(\mu(p;q_1,\dots,q_\ell);\underline{r})\cdot t_{n,m} \arrow[dddddl, bend left, "\text{iso}", "\cancel{\mu(r;}"'] \\
\mu(\mu(p;\underline{r});q_1,\dots,q_\ell)\cdot\mu(t_{n,\ell};\underline{e}) \arrow[d, equal] & \\
\mu(\mu(p;\underline{r})\cdot t_{n,\ell};\underline{q_1,\dots,q_\ell}) \arrow[d, "\mu(\text{iso};1)"'] & \\
\mu(\mu(r;p);\underline{q_1,\dots,q_\ell}) \arrow[d, equal] & \\
\mu(r;\underline{\mu(p;q_1,\dots,q_\ell)}) &
\end{tikzcd}$$

Draw diagrams as



OR



(or close)

μ axiom #2 :

★ Additional equation

$$\mu(t_{m,e}; \underline{e_{n_1}}, \dots, \underline{e_{n_m}})\,\mu(e_m; t_{n_1,e}, \dots, t_{n_m,e}) = t_{N,e}$$

for any $e, m, n_1, \dots, n_m$ where $N = \sum n_i$

$$\mu(p; \underline{\mu(q; r_1, \dots, r_m)}) \cdot \mu(t_{m,e}; \underline{e_{n_1}}, \dots, \underline{e_{n_m}})\,\mu(e_m; t_{n_1,e}, \dots, t_{n_m,e})$$

$$= \mu(p; \underline{\mu(q; r_1, \dots, r_m)}) \cdot t_{N,e}$$

$$\mu(\mu(p;q); \underline{r_1, \dots, r_m}) \cdot \mu\mu$$

$$= \mu(\mu(p;q) \cdot t_{m,e}; \underline{r_1}, \dots, \underline{r_m}) \cdot \mu(e_m; t_{n_1,e}, \dots, t_{n_m,e})$$

$$\big\downarrow \mu(\mathrm{iso}; 1) \cdot 1$$

$$\mu(\mu(q;p); \underline{r_1}, \dots, \underline{r_m}) \cdot \mu$$

$$= \mu(q; \mu(p; \underline{r_1}), \dots, \mu(p; \underline{r_m})) \cdot \mu$$

$$= \mu(q; \mu(p; \underline{r_1}) \cdot t_{n_1,e}, \dots, \mu(p; \underline{r_m}) \cdot t_{n_m,e})$$

$$\big\downarrow \mu(1; \mathrm{iso})$$

$$\mu(q; \mu(r_1; f), \dots, \mu(r_m; f))$$

$$= \mu(\mu(q; r_1, \dots, r_m); f)$$

$$\mu(p; \underline{\mu(q; r_1, \dots, r_m)}) \cdot t_{N,e} \xrightarrow{\mathrm{iso}} \mu(\mu(q; r_1, \dots, r_m); f)$$

$$\underline{P}(X) = \coprod P(n) \times_{G(n)} X^n$$

$$X \times \underline{P}(Y) = X \times \coprod P(n) \times_{G(n)} Y^n \qquad (x, [p; y_1, \ldots, y_n])$$

$$\Big\downarrow \boxed{t}$$

$$\underline{P}(X \times Y) = \coprod P(n) \times_{G(n)} (X \times Y)^n \qquad [p; (x, y_1), \ldots, (x, y_n)]$$

$$t^* : \underline{P}(X) \times Y \longrightarrow \underline{P}(X \times Y)$$

$$([p; x_1, \ldots, x_n], y) \xmapsto{\boxed{t^*}} [p; (x_1, y), \ldots, (x_n, y)]$$

$$\begin{tikzcd}
\underline{P}(X) \times \underline{P}(Y) \arrow[r, "t^*"] \arrow[d, "t"'] & \underline{P}(X \times \underline{P}(Y)) \arrow[r, "\underline{P}(t)"] \arrow[d, phantom, "\swarrow \gamma_{X,Y}"] & \underline{P}^2(X \times Y) \arrow[d, "\mu"] \\
\underline{P}(\underline{P}(X) \times Y) \arrow[r, "\underline{P}(t^*)"'] & \underline{P}^2(X \times Y) \arrow[r, "\mu"'] & \underline{P}(X \times Y)
\end{tikzcd}$$

$$\begin{tikzcd}
([p; x_1, \ldots, x_n], [q; y_1, \ldots, y_m]) \arrow[r, mapsto, "t^*"] \arrow[d, mapsto, "t"'] & {[p; (x_1, [q; \underline{y}]), \ldots, (x_n, [q; \underline{y}])]} \arrow[d, mapsto, "\underline{P}(t)"] \\
{[q; ([p; \underline{x}], y_1), \ldots, ([p; \underline{x}], y_m)]} \arrow[d, mapsto, "\underline{P}(t^*)"'] & {[p; [q; (x_1, y_\cdot)], \ldots, [q; (x_n, y_\cdot)]]} \arrow[d, mapsto, "\mu"] \\
{[q; [p; (x_\cdot, y_1)], \ldots, [p; (x_\cdot, y_m)]]} \arrow[d, mapsto, "\mu"'] & {[\mu(p; q); \underline{(x, y_\cdot)}]} \\
{[\mu(q; p); \underline{(x, y)}]} &
\end{tikzcd}$$

Iso Morphisms: Swapped notation

$$[\mu(p; q); \underline{(x, y)}] \cong [\mu(q; p); \underline{(x, y)}]$$

Claim #1: A - right G-cat, B - left G-cat

$A \times_G B$ Any $[a, b] \longrightarrow [a', b']$ is of the form $[a, b] = [ag, g^{-1}b] \xrightarrow{[r, s]} [a', b']$

where $r: ag \to a'$, $s: g^{-1}b \to b'$

Pf: (coequalizer def'n (did on board). $\square$

Claim #2: Let X_0, Y_0 be the sets of obj of X, Y. Then

$$\underline{P}(X_0) \times \underline{P}(Y_0) \hookrightarrow \underline{P}(X) \times \underline{P}(Y) \xrightarrow{\Downarrow \gamma} \underline{P}(X \times Y)$$

$$\parallel$$

$$\underline{P}(X_0) \times \underline{P}(Y_0) \xrightarrow{\Downarrow \gamma} \underline{P}(X_0 \times Y_0) \hookrightarrow \underline{P}(X \times Y)$$

Pf. γ is a modification. $\square$

$G = 1$: Does it follow that

$$\begin{array}{ccc} \mu(q;p,\dots,p)\cdot h h'^{-1} & \cong & \mu(q;p,\dots,p) \\ \big\downarrow F\cdot 1 & & \big\downarrow F \\ \mu(q';p',\dots,p')\cdot h h'^{-1} & \cong & \mu(q';p',\dots,p') \end{array} \; ?$$

Probably not: * what is bottom iso?
- h, h' depended on $\underline{a}, \underline{b}, p, q$
- now have $\underline{a}, \underline{b}, p', q'$ along the bottom

---

1^{st} strength axiom:
- $h((a,\underline{b}), \underline{c}, p, q) = h(\underline{b}, \underline{c}, p, q)$
- iso for $(a,\underline{b}), \underline{c}, p, q$ = iso for $\underline{b}, \underline{c}, p, q$

3^{rd}:
- $h(\underline{a}, (\underline{b}, c), p, q) = h(\underline{a}, \underline{b}, p, q)$
- same for iso

2^{nd}:
- $h((\underline{a}, \underline{b}), \underline{c}, p, q) = h(\underline{a}, (\underline{b}, \underline{c}), p, q)$
- same for iso

1^{st} unit axiom:
- $h \in \ker(\pi)$ $\quad h(\underline{a}, \underline{b}, 1, q)$
- ~~$\mu(q;p,\dots,p)$~~ $\mu(q;1,\dots,1)\cdot h = \mu(1;q)$
  $\iff q\cdot h = q$

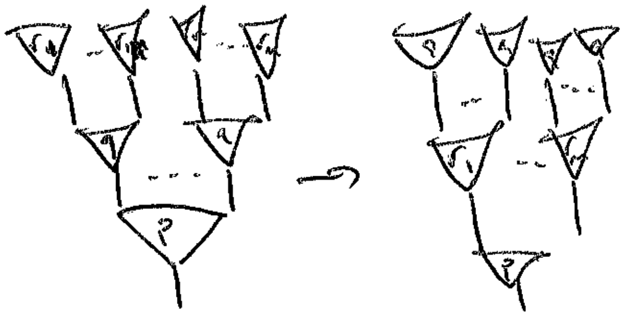
If $G(n)$ acts freely, then $h = id \in G(n)$

2^{nd} unit axiom is redundant

$p \in P(\ell)$, $q_i \in P(m_i)$ $i=1,\dots,\ell$, $r \in P(n)$ $M = \sum m_i$

$\mu(p;q) \cdot t_{\ell,n} \cong \mu(q;p)$

$\mu(q;r) \cdot t_{n,r} \cong \mu(r;q)$



$([p;[q_1;\underline{x_1}],\dots,[q_\ell;\underline{x_\ell}]],[r;\underline{y}])$

$\downarrow$

$[r;([p;[\underline{q};\underline{x}]],y_1),\dots,([p;[\underline{q};\underline{x}]],y_n)]$

$\downarrow$

$[r;[p;([q_1;x_{11}],y_1),\dots,([q_1;x_{1m_1}],y_1)],\dots[p;([q_\ell;x_{\ell 1}],$

$\downarrow$

$[\mu(r;p);[q\dots]]$

$\downarrow$

$\mu(\mu(r;p);\underline{q_1},\dots,\underline{q_\ell})$

$\downarrow$

$[p;([q_1;x_1],[r;y]),\dots,([q_\ell;x_\ell],[r;y])]$

$\downarrow$

~~$[p;(x_{11},[q_1;$~~

$[p;[q_1;(x_{11},[r;y]),\dots,(x_{1m_1},[r;y])],\dots[q_\ell;(\underline{x_{\ell 0},[r;y]})]]$

$\downarrow$

$[p;[q_1;[r;(x_{1\cdot},y)]$

$\downarrow$

$[p;[\mu(q_1;r)],\dots,[\mu(q_\ell;r)]]$

$\downarrow$

$\mu(p;\mu(q_1;r),\dots,\mu(q_\ell;r))$

$\mu^P(p;\mu(q_1;r)\cdot t_{m_1,n},\dots,\mu(q_\ell;r)\cdot t_{m_\ell,n}) = \mu^P(p;\mu(q_1;r),\dots,\mu(q_\ell;r))\cdot \mu^G(e;t_{m_1,n},\dots,t_{m_\ell,n})$

$\downarrow$

$\mu^P(p;\mu(r;q_1),\dots,\mu(r;q_\ell))$

$\shortparallel$

$\mu^P(\mu^P(p;r);q_1,\dots,q_\ell)$

$\mu(\mu(p;r)\cdot t_{\ell,n};q_1,\dots,q_\ell) =$

$\mu(\mu(p;r);\underbrace{q_{t^{-1}_{\ell,n}(1)},\dots,q_{t^{-1}_{\ell,n}(\ell)}}_{=q_1,\dots,q_\ell})\cdot\mu(t_{\ell,n};e)$

Thm: Let P be a G-operad. Then the following provide $\underline{P}$ w/ a pre-comm structure.

1) For each (m,n) an elt $t_{m,n} \in G(mn)$ s.t.

$$\pi(t_{m,n}) = \tau_{m,n}.$$

2) For each $p \in P(m)$, $q \in P(n)$, a natural iso

$$\mu^P(p; q, \dots, q) \cdot t_{mn} \cong \mu^P(q; p, \dots, p).$$

These satisfy:

- $t_{1,n} = e_n = t_{n,1}$
- axiom about operadic comp of t's
- axiom about associativity for iso's in (2)

t-axiom guess: $\mu(e_\ell; t_{m_1,n}, \dots, t_{m_\ell,n})\,\mu(t_{\ell,n}; \underbrace{e_{m_1}, \dots, e_{m_1}}_{n}, \dots, \underbrace{e_{m_\ell}, \dots, e_{m_\ell}}_{n})$

$$= t_{M,n} \qquad M = \sum m_i$$

Check w/ calculation in Σ:

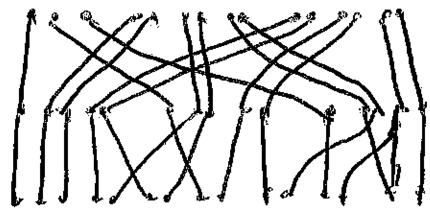
$\ell = 3$, $m_1 = 1$, $m_2 = 2$, $m_3 = 2$, $n = 3$

$\tau_{5,3} =$

Not equal

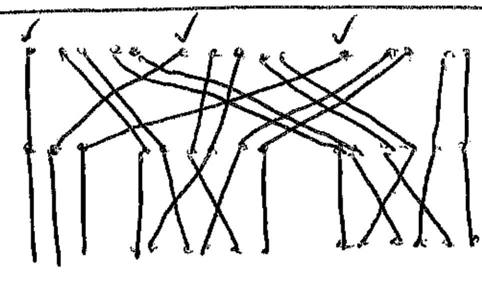
$\mu(\tau_{3,3}; e_1 e_1 e_1 e_2 e_2 e_2 e_2 e_2 e_2) =$

$\mu(e_3; \tau_{1,3}, \tau_{2,3}, \tau_{2,3}) =$



$\mu(\tau_{3,3}; e_1 e_2 e_2\, e_1 e_2 e_2\, e_1 e_2 e_2) =$

$\mu(e_3; \tau_{13}\, \tau_{23}\, \tau_{23}) =$



$$\mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_\ell;\underline{r})) = \mu(\mu(p;q_1,\dots,q_\ell);\underline{r})$$

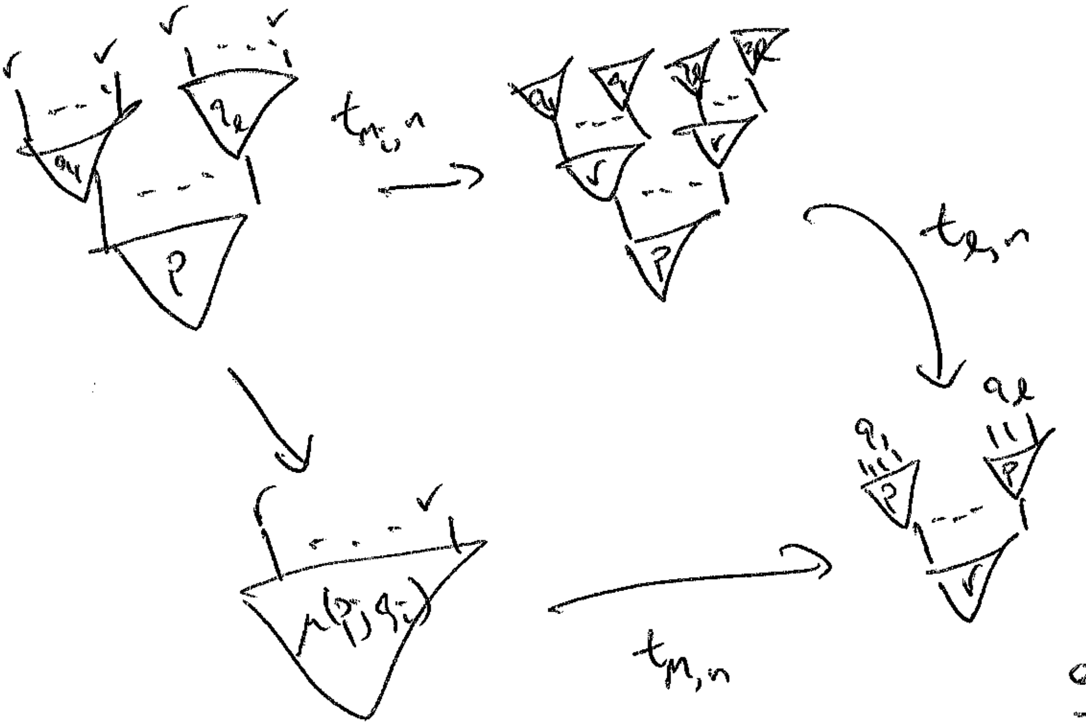
$$\mu(\mu(p;q);\underline{r})\cdot t_{\mu,n} \longrightarrow \mu(r;\underline{\mu(p;q)}) = \mu(\mu(r;\ell);\underline{q_1,\dots,q_\ell})$$

$$\cancel{\mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_\ell;\underline{r}))\cdot\mu(e_n;t_{\mu,n},\dots,t_{\mu_\ell,n})}$$

$$\mu(\mu(r;\ell);\underline{q_1,\dots,q_\ell}) \longleftarrow \mu(\mu(p;\underline{r})\cdot t_{\ell,n};\underline{q_1,\dots,q_\ell})$$

$$\|$$

$$\mu(\mu(p;\underline{r});q_1$$



$$\overset{e_{n_1},\dots,e_{n_\ell}}{}$$

$$\mu(p;\mu(q_1;\underline{r}),\dots,\mu(q_\ell;\underline{r}))\cdot\underbrace{\mu(e_\ell;t_{\mu,n},\dots,t_{\mu_\ell,n})\cdot\mu(t_{\ell,n};e_{n_1,q},\dots,e_{n_\ell,q})}$$

$$\|$$

$$\mu(p;\mu(q_1;\underline{r})\cdot t_{\mu,n},\dots,\mu(q_\ell;\underline{r})\cdot t_{\ell,n})\cdot\mu(t_{\ell,n};\underline{e}) \qquad \mu(p;\underline{\mu(q;\underline{r})})\cdot\mu(t_{\mu,n}$$

$$\downarrow \qquad\qquad \|$$

$$\mu(p;\mu(r;q_1),\dots,\mu(r;q_\ell))\cdot\mu(t_{\ell,n};\underline{e}) \qquad \mu(\mu(p;q);\underline{r})\cdot t_{\mu,n}$$

$$\| \qquad\qquad \downarrow$$

$$\mu(\mu(p;\underline{r});q_1,\dots,q_\ell)\cdot\mu(t_{\ell,n};\underline{e}) \qquad \mu(r;\underline{\mu(p;q)})$$

$$\|$$

$$\mu(\mu(p;\underline{r})\cdot t_{\ell,n};\underline{q_1,\dots,q_\ell})$$

$$\downarrow$$

$$\mu(\mu(r;\ell);\underline{q_1,\dots,q_\ell}) = \mu(r;\underline{\mu(p;q_1,\dots,q_\ell)})$$