$$P(n)\times P(k_1)\times\cdots\times P(k_n)\xrightarrow{\mu}P(k_1+\cdots+k_n)$$

$$\begin{array}{ccc}
P(m)\times\left(\prod_{i=1}^{m}P(n_i)\times P(k_{i_1})\times\cdots\times P(k_{in_i})\right) & \xrightarrow{1\times\prod\mu} & P(m)\times P(\bar{k}_1)\times\cdots\times P(\bar{k}_m) \\
\downarrow\cong \ \leftarrow \text{need braidings} & & \downarrow\mu \\
P(m)\times P(n_1)\times\cdots\times P(n_m)\times\prod P(k_{i_j}) & & P(\Sigma k_{i_j}) \\
\searrow \mu\times 1 & & \nearrow \\
P(\Sigma n_j)\times\prod P(k_{i_j}) & \xrightarrow{\mu} &
\end{array}$$

T = free braided m.c. 2-monad on Cat

$T\xrightarrow{\alpha}S$: strict map of 2-monads

$$TA\xrightarrow{\alpha A}SA\to A$$

$$\begin{array}{ccc}
T^2 & \xrightarrow{\alpha^2} & S^2 \\
\mu\downarrow & & \downarrow\mu \\
T & \xrightarrow{\alpha} & S
\end{array}
,\qquad
\begin{array}{ccc}
1 & \xrightarrow{\eta} & T \\
& \searrow\eta & \downarrow\alpha \\
& & S
\end{array}
\qquad
\begin{array}{ccc}
TT & \xrightarrow{\alpha\alpha} & SS \\
T\alpha\searrow & & \nearrow\alpha S \\
& TS &
\end{array}$$

$S1$ gives an operad $\underline{S}$:

$$\underline{S}(n) = S1(n,n) \qquad \text{"}n = (\text{gen. obj.})^{\otimes n}\text{"}$$
$$= S1(\alpha_1(*^n),\alpha_1(*^n)) \qquad n=\alpha_1(*^n) \text{ where } * \text{ generates } T1$$

$$S1(\alpha_1 *^n,\alpha_1 *^n)\times S1(\alpha_1(*^{k_1}),\alpha_1 *^{k_1})\times\cdots\times S1(\alpha_1 *^{k_n},\alpha_1 *^{k_n})$$

$$\begin{array}{ccc}
T^2 1 & \xrightarrow{\alpha_1^2} & S^2 1 \\
\mu\downarrow & & \downarrow\mu_1 \\
T1 & \xrightarrow{\alpha_1} & S1
\end{array}
\qquad S1(\alpha_1 *,\alpha_1 *)\times S1(\alpha_1 *,\alpha_1 *)\to S1(\alpha_1 *^2,\alpha_1 *^2)$$

A - S-alg, $x\in A$

$$S1(n,n)\longrightarrow A(x^n,x^n)$$
$$S1\longrightarrow A$$

"$S1(n,n)$ acts on x^n"

G - group

$x\in A$

G acts on $x \Leftrightarrow G\to A(x,x)$

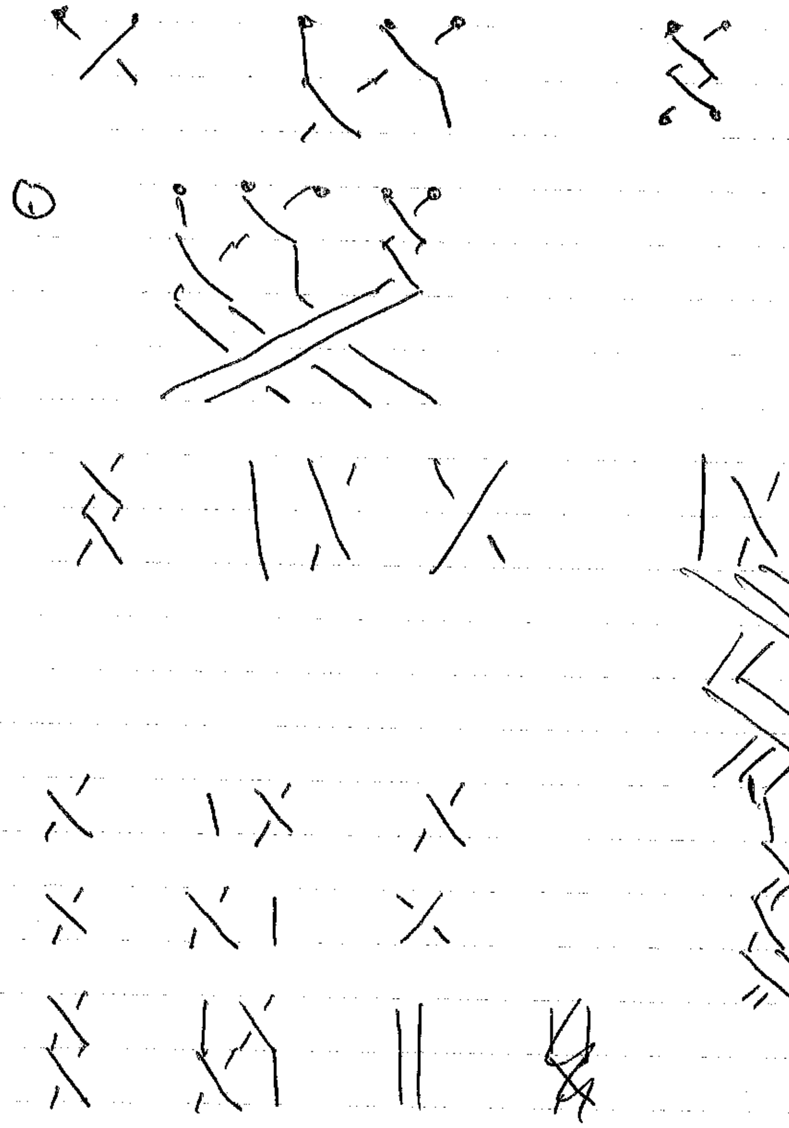
"$G\otimes x\longrightarrow x$"

$\bigoplus_{g\in G} 2\otimes x\longrightarrow x$

$$\begin{array}{ccc}
R\otimes M & \longrightarrow & M \\
1\otimes f\downarrow & & \downarrow f \\
R\otimes N & \longrightarrow & N
\end{array}$$

Failed attempt at "Br is an operad in groups"

$\pi_1(2,2) \times \pi_1(3,3) \times \pi_1(2,2) \longrightarrow \pi_1(5,5)$

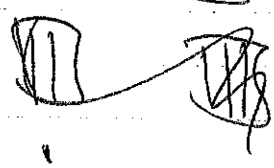


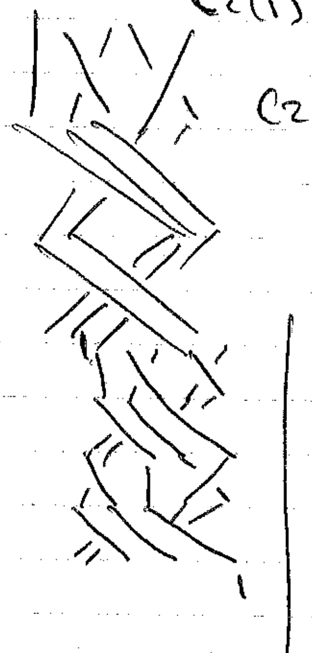
$C(2)(n) =$ 

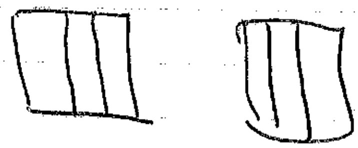
$\pi_1 C(2)(n) \cong Br_n$

$C_2(1) =$ 

$C_2(2)$ 







P – operad

X – P-alg $\qquad P(n) \otimes_{\Sigma(n)} X^n \longrightarrow X$

$$G \hookrightarrow [G^{op}, \mathrm{Sets}]$$

1.3

$$S1(\alpha, *, \alpha, *) \times S1(\alpha, *, \alpha, *) \longrightarrow TS1\left((\alpha, *, \alpha, *), (\alpha, *, \alpha, *)\right)$$

$$\downarrow \alpha$$

$$S^2 1 \left(\qquad \right)$$

$$\downarrow \mu$$

$$\left((*), (*)\right) \longmapsto \alpha_1(\alpha, *, \alpha, *) \qquad S1\left((\alpha, *)^2, (\alpha, *)^2\right)$$

S - symm mon. cat

$G \xrightarrow{\pi} \Sigma$ - map of operads in Sets

P - G-operad in S:

- $P(n) \in S$
- right action of $G(n)$ on $P(n)$ & $(x \cdot g) \cdot h = x \cdot (gh)$
  "$P(n) \times G(n) \longrightarrow P(n)$"
  1) $P(n) \otimes \coprod_{g \in G(n)} I \longrightarrow P(n)$
  2) $G(n)^{op} \longrightarrow S(P(n), P(n))$
- $\mu : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \longrightarrow P(k_1 + \cdots + k_n)$
  $\mu(q \cdot g, p_1 \cdot h_1, \ldots, p_n \cdot h_n) = \mu(q, p_1, \ldots, p_n) \cdot (g, h_1, \ldots, h_n)$
  $q \in P(n)$, $g \in G(n)$
  $p_i \in P(k_i)$, $g_i \in G(k_i)$

A P-algebra is an $X \in S$ plus
$$P(n) \underset{G(n)}{\otimes} X^n \longrightarrow X \quad \ldots$$

$$P(n) \otimes \coprod I \otimes X^n \underset{1 \otimes \text{act on } X^n}{\overset{\text{act on } P \otimes 1}{\rightrightarrows}} P(n) \otimes X^n \xrightarrow{\text{coeq}} P(n) \underset{G(n)}{\otimes} X^n$$

Symmetric operad: $G(n) = \Sigma_n$
Braided operad: $G(n) = Br_n$
Plain operad: $G(n) = *$

Alternating group?

$x \otimes y = -y \otimes x$

$G(n) = A_n \qquad P(n) = I \qquad X^n / A_n \longrightarrow X$

$G(n) = \Sigma_n \qquad P(n) = I \qquad X^n / \Sigma_n \longrightarrow X \qquad$ commutative

$$\underline{P}(X) = \coprod_{n \geq 0} P(n) \times_{G(n)} X^n$$

$$\underline{P}^2(X) = \coprod_{n \geq 0} P(n) \times_{G(n)} \underline{P}(X)^n$$

$$= \coprod_n P(n) \times_{G(n)} \Big(\coprod_m P(m) \times_{G(m)} X^m \Big)^n$$

$$\cancel{\cong \coprod_n \coprod_m P(n) \times_{G(n)} \big(P(m) \times_{G(m)} X^m \big)}$$

$$\cong \coprod_n P(n) \times_{G(n)} \coprod_{m_i} \Big[\big(P(m_1) \times_{G(m_1)} X^{m_1}\big) \times \cdots \times \big(P(m_n) \times_{G(m_n)} X^{m_n}\big) \Big]$$

① $$\cong \coprod_n \coprod_{m_i} P(n) \times_{G(n)} \Big[P(m_1) \times_{G(m_1)} X^{m_1} \times \cdots \times P(m_n) \times_{G(m_n)} X^{m_n} \Big]$$

$$\xrightarrow{\text{equivariant}} \coprod_n \coprod_{m_i} P(n) \times P(m_1) \times \cdots \times P(m_n) \times X^{m_1} \times \cdots \times X^{m_n}$$

$$\longrightarrow \coprod P(n) \times_{G(n)} X^n = \underline{P}(X)$$

① $$P(n) \times_{G(n)} \coprod A_i^n \cong \coprod P(n) \times_{G(n)} A_i^n$$

$$\begin{tikzcd}
P(n) \times G(n) \times \coprod A_i^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & P(n) \times \coprod A_i^n \arrow[r] \arrow[d, "\cong"] & P(n) \times_{G(n)} \coprod A_i^n \arrow[d, dashed, "\cong"] \\
\coprod P(n) \times G(n) \times A_i^n \arrow[r, shift left] \arrow[r, shift right] & \coprod P(n) \times A_i^n \arrow[r] & \coprod P(n) \times_{G(n)} A_i^n
\end{tikzcd}$$

$\underline{P}(X) \to X \quad \leftrightarrow$ maps $\quad P(n) \times_{G(n)} X^n \to X$

P-algs: $\quad P(n) \times_{G(n)} X^n \to X \quad$ + axioms

strict map: $\quad f: X \to Y \quad$ s.t. $\quad$ = map of P-algs (operad)

$$\begin{tikzcd} P(n) \times_{G(n)} X^n \arrow[r] \arrow[d, "1\times f^n"'] & X \arrow[d, "f"] \\ P(n) \times_{G(n)} Y^n \arrow[r] & Y \end{tikzcd}$$

alg 2-cell: $\alpha: f \Rightarrow g$ s.t. ....

ps map:
$$\begin{tikzcd} \underline{P}(X) \arrow[r, "x"] \arrow[d, "\underline{P}(f)"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "f"] \\ \underline{P}(Y) \arrow[r, "y"'] & Y \end{tikzcd}$$
s.t. ...

$\leftrightarrow$
$$\begin{tikzcd} P(n) \times_{G(n)} X^n \arrow[r] \arrow[d, "1\times f^n"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "f"] \\ P(n) \times_{G(n)} Y^n \arrow[r] & Y \end{tikzcd}$$
s.t. ...

Need to do: $\underline{P}$- ps algs, operad ps-maps

$\underline{P}$- ps algs $\leftrightarrow$ operad ps-maps $P \to \Sigma_X$

Higher cells? $\begin{tikzcd} P \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \Sigma_X \end{tikzcd}$

Finitary:

$$\underline{P}(\operatorname{colim} X_i) \cong \operatorname{colim} \underline{P}(X_i)$$

$$\coprod P(n) \times_{G(n)} (\operatorname{colim} X_i)^n \cong \coprod P(n) \times_{G(n)} \operatorname{colim}(X_i^n)$$

$$\cong \coprod \operatorname{colim} P(n) \times_{G(n)} X_i^n$$

$$\cong \operatorname{colim} \coprod P(n) \times_{G(n)} X_i^n = \operatorname{colim} \underline{P}(X_i)$$

Cartesian:

$G(n) = * \quad \forall n \quad \Rightarrow \underline{P}$ always cartesian (1-D)
1-D $\Rightarrow$ 2-D (elementary)

$G(n) = \Sigma_n$: $P(n) = *$ not cartesian (commu...
$P(n) = T(\Sigma_n)$ = translation category
$= E\Sigma_n$ contractible
obj = elts of Σ_n
$E\Sigma_n$ has a free action of Σ_n

$$E\Sigma_n / \Sigma_n \cong B\Sigma_n$$

$$g \xrightarrow{\sim} h \longmapsto hg^{-1}$$

P-algs "=" permutative cats
this is cartesian

Conjecture: P cartesian iff $P(n)$ is a free $G(n)$-category $\forall n$

Coherence:

$\mathbb{P}$ preserves bij-on-objs:

f, g b.o. $\Rightarrow$ $f \sqcup g$ is b.o.

f b.o. $\Rightarrow$ $P(n) \times_{G(n)} X^n \xrightarrow{1 \times f^n} P(n) \times_{G(n)} Y^n$ b.o.

$$\begin{tikzcd}
P(n) \times G(n) \times X^n \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "1\times 1\times f^n"', "\text{b.o.}"] & P(n) \times X^n \arrow[r] \arrow[d, "1\times f^n", "\text{b.o.}"'] & P(n) \times_{G(n)} X^n \arrow[r] \arrow[d, "1\times f^n"] & A \arrow[d] \\
P(n) \times G(n) \times Y^n \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & P(n) \times Y^n \arrow[r] \arrow[ur, dashed, "s"] \arrow[u, phantom, "\exists!"] & P(n) \times_{G(n)} Y^n \arrow[r] \arrow[ur, dashed, "r"] & B
\end{tikzcd}$$

Want: $r\lambda_1 = r\lambda_2$ $= s$ $\qquad ff r \lambda_1 = ff r \lambda_2$

$r \circ 1 \times f^n =$ $\qquad \alpha : F \Rightarrow G$ w/ $\alpha_x : Fx \to Gx \in L$ $\forall$

then $\operatorname{colim} \alpha = \operatorname{colim} F \to \operatorname{colim} G \in L$

$$\begin{tikzcd}
Fy \arrow[r, "Ff"] \arrow[d, "\alpha_y"'] \arrow[rr, bend left, "i_y"] & Fx \arrow[r] \arrow[d, "\alpha_x"] \arrow[ddrr, phantom, "L"] & \operatorname{colim} F \arrow[r, "f"] \arrow[d, "\operatorname{colim}\alpha"] & A \arrow[d, "R"] \\
Gy \arrow[r, "Gf"'] \arrow[rr, bend right, "j_y"'] & Gx \arrow[r, "j_x"'] \arrow[urr, dashed, "r_x"] & \operatorname{colim} G \arrow[r, "g"'] & B
\end{tikzcd}$$

★ $r_x Gf = r_y$

• $r_y \alpha_y = f i_y$

• $R r_y = g j_y$

① $r_x Gf \alpha_y = r_x \alpha_x Ff$
$= f\, i_x Ff$
$= f\, i_y$

② $R r_x Gf = g\, j_x Gf$
$= g\, j_y$

$\Rightarrow \operatorname{colim} \alpha \in L$