# ACTION OPERADS

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## 1. The hyperoctahedral groups do not form an action operad

This counterexample is due to Zhang (Example 2.28 of *Group operads and homotopy theory*).

Let $H_n = C_2 \wr \Sigma_n = C_2^N \rtimes z\Sigma_n$ be the $n^{th}$ hyperoctahedral group. In particular $H_1 = C_2$.

We can think of elements of the hyperoctahdral group as a permutation graph on the set $\{1, \ldots, n\}$ with labels from the group $C_2$ on the edges.

In order to define an action operad we need structure maps of the form

$$\gamma\colon H_1 \times H_k \to H_k.$$

Let $\tau$ be the generator of $H_1 = C_2$. Let $e_k$ denote the identity element of $H_k$. Let $r_k$ be the element of $H_k$ given by the order-reversing permutation with every edge labelled by $\tau$.

Let $a \in H_k$.

If we want to detect the order-reversing structure in the hyperoctahedral group, we have two options for the map $\gamma$.

- We could define $\gamma(\tau; a) = r_k \circ a$ or
- we could define $\gamma(\tau, a) = a \circ r_k$.

In other words, we either post-compose or pre-compose with the order-reversing permutation with all edges labelled by $\tau$.

These are not equal in general (for any $k \geqslant 3$ almost any choice of $a$ should yield an example of this. I've scribbled this out for $a = (\tau, e_1, e_1, \tau; (1\,3))$ for example).

Now, in $H_1$ we have $\tau \circ e_1 = e_1 \circ \tau$ and in $H_k$ we have $a \circ e_k = e_k \circ a$. Therefore we should have

$$\gamma(\tau; a) = \gamma(\tau \circ e_1; a \circ e_k) = \gamma(\tau \circ e_1; e_k \circ a) = \gamma(e_1 \circ \tau; a \circ e_k) = \gamma(e_1 \circ \tau; e_k \circ a).$$

The structure map $\gamma$ needs to be a crossed homomorphism, so we have a formula as part of the definition of action operad which tells us what each of these should be.

However, because we can commute $\tau$ and $e_1$, the definition of crossed homomorphism means that for some of these maps we are post-composing with $r_k$ and for others we are pre-composing, so the four maps are not equal in general, regardless of which convention we choose for $\gamma$. (As before, almost any choice for $k \geqslant 3$ should give an explicit counterexample.)