$P(n) \times G(n) \longrightarrow P(n)$ $\qquad$ $G(n) \xrightarrow{\pi_n} \Sigma_n$

$x \cdot e = x$

$(x \cdot a) \cdot b = x \cdot (ab)$

<u>Prose</u>: $\underline{P}$ is 2-cartesian iff $P(n)$ is $G(n)$-free $\forall n$.

Free means: $x \cdot g_1 = x \cdot g_2 \Rightarrow g_1 = g_2$
$x \cdot g = x \Rightarrow g = e$

① Assume $P(n)$ is not $G(n)$-free for some n.
(a) $a \in \mathrm{ob}P(n)$, $g \neq e \in G(n)$ s.t. $a \cdot g = a$

Claim: $\{\mathrm{ob}\, P(n)\}$ is a G-operad in Sets.
True b/c $\mathrm{ob}: \mathcal{C}at \to Sets$ is monoidal

$\mathrm{ob}: \mathcal{C}at \to Sets$ also preserves limits so
$\{\mathrm{ob}\, P(n)\}$ not cartesian $\Rightarrow \{P(n)\}$ not 2-cartesian

- μ for $2 \to 1$:

$$\begin{array}{ccc} \underline{P}^2(2) & \longrightarrow & \underline{P}^2(1) \\ \downarrow & & \downarrow \\ \underline{P}(2) & \longrightarrow & \underline{P}(1) \end{array}$$

Too gross!

- $\underline{P}$ doesn't preserve pullbacks:

$$\begin{array}{ccc} \{(x,y),(x,y'),(x',y),(x',y')\} & \longrightarrow & \{y, y'\} \\ \downarrow & & \downarrow \\ \{x, x'\} & \longrightarrow & \{z\} \end{array}$$

is a pb in Sets.

$$\begin{array}{ccc} \coprod_{n \geq 0} P(n) \times_{G(n)} 4^n & \longrightarrow & \coprod P(n) \times_{G(n)} 2^n \\ \downarrow & & \downarrow \\ \coprod P(n) \times_{G(n)} 2^n & \longrightarrow & \coprod P(n)/G(n) \end{array}$$

$(a, y, \dots, y)$
$\downarrow$
$[a]$

$(a, x, x, \dots, x) = (a \cdot g, x, \dots, x) = (a, \pi_n(g)(x, \dots, x))$
$= (a, x, \dots, x)$

$\longmapsto [a]$

$$(a, (x',y'), (x,y), \dots, (x,y'), \dots, (x,y))$$
$$(a, (x,y), \dots, (x',y'), \dots (x,y'))$$

i^{th} if $\pi_n(g)(1) = i$

$$(a, y, \dots, y, y', y, \dots, y) \mapsto [a]$$

$$(a, x', x, \dots, x) \longmapsto [a]$$
$$(a, x, \dots, x, x', x, \dots, x)$$

(b) Assume $P(n)$ is $G(n)$-free $\forall n$.

* Naturality for η:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \eta_A & & \downarrow \eta_B \\ \coprod P(n) \times_{G(n)} A^n & \to & \coprod P(n) \times_{G(n)} B^n \end{array}$$

$\eta_A(a) = [1, a] \in P(1) \times_{G(1)} A$ (Remember: $1 \in P(1)$ is the operad unit)

Coproducts are disjoint in Cat: (or maybe stable under pb?)

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod Z_i & \xrightarrow{\coprod f_i} & \coprod Y_i \end{array}$$

$P \cong \coprod \text{pbs} \left(\begin{array}{c} X \\ \downarrow \\ Z_i \to Y_i \end{array} \right)$

$\Rightarrow$ Pb of
$$\begin{array}{ccc} & & B \\ & & \downarrow \eta_B \\ \underline{P}(A) & \to & \underline{P}(B) \end{array}$$
is $\coprod \text{pbs} \left(\begin{array}{ccc} & & B \\ & & \downarrow \\ P(n) \times_{G(n)} A^n & \to & P(n) \times_{G(n)} B^n \end{array} \right)$

but η_B lands in $P(1) \times_{G(1)} B$ so compute pb of

(*)
$$\begin{array}{ccc} Q & \longrightarrow & B \\ \downarrow & & \downarrow \\ P(1) \times_{G(1)} A & \longrightarrow & P(1) \times_{G(1)} B \end{array}$$

ob $Q = (b, [p, a])$ s.t. $[1, b] = [p, f(a)]$

Claim: $[\underline{x_1}, b] = [\underline{x_1}, b']$ in $P(1) \times_{G(1)} B$

$\Rightarrow b = b'$ True b/c eq. relation

Eg, $[p \cdot g, b] = [p, \pi_1(g) \cdot b] = [p, b]$

So need $G(1) = *$ (G is a reduced operad)

If $G(1) = *$, then (✱) becomes

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ P(1) \times A \arrow[r, "id \times f"'] & P(1) \times B \end{tikzcd} \Rightarrow Q \cong A$$

So naturality for η is a pb. $\square$

* Naturality for μ:

Use if all
$$\begin{tikzcd} T^2X \arrow[r] \arrow[d, "\mu"'] & T^2 1 \arrow[d, "\mu"] \\ TX \arrow[r] & T1 \end{tikzcd}$$
are pbs

then
$$\begin{tikzcd} T^2X \arrow[r, "T^2f"] \arrow[d] \arrow[dr, phantom, "LHS"] & T^2Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T^2 1 \arrow[d] \\ TX \arrow[r] & TY \arrow[r] & T1 \end{tikzcd}$$
and outside is pb so LHS is pb.

$$\begin{tikzcd} \underline{P}^2 X \arrow[r] \arrow[d] & \underline{P}^2 1 \arrow[d] \\ \underline{P} X \arrow[r] & \underline{P} 1 \end{tikzcd} \qquad \begin{tikzcd} \underline{P}^2 X \arrow[r] \arrow[d] & \underline{P}^2 1 \arrow[d] \\ \coprod P(n) \times_{G(n)} X^n \arrow[r] & \coprod P(n)/G(n) \end{tikzcd}$$

Split up the bottom.

$$\left[\omega_q, [v_1, \underline{x}_1], \dots, [v_j, \underline{x}_j]\right] \quad \coprod_k \coprod_{\sum_{i=1}^k m_i = n} P(k) \times_{G(k)} \left[P(m_1)/G(m_1) \times \cdots \times P(m_k)/G(m_k)\right]$$

$$[q, [r_1], \dots, [r_k]]$$

$$\begin{array}{ccc} P(n) \times_{G(n)} X^n & \longrightarrow & P(n)/G(n) \\ [p, \underline{x}] & \longmapsto & [p] = [q(r_1, \dots, r_k)] \end{array}$$

$$[\omega \cdot \alpha, [v_1, x_1], \dots, [v_j, x_j]] = [\omega, \pi_j(\alpha)([\ ], \dots, [\ ])]$$

P_b : $\left([q, [r_1], \dots, [r_k]], [p, \underline{x}]\right)$ where $r_i \in P(m_i)$ w/ $\sum m_i = n$

$[p] = [q(r_1, \dots, r_k)]$ in $P(n)/G(n)$

Equivariance for P:

$$\mu : P(n) \times P(k_1) \times \cdots \times P(k_n) \longrightarrow P(k_1 + \cdots + k_n)$$

$$\mu(q \cdot g, h_1, \dots, h_n) = \mu(q, \pi_n(g)(h_1, \dots, h_n))$$

$$\mu(q, h_1 \cdot g_1, \dots, h_n \cdot g_n) \overset{?}{=} \mu(q \cdot \mathrm{id}(g_1, \dots, g_n), h_1, \dots, h_n)$$

$$S_n \times S_m \overset{i}{\hookrightarrow} S_{n+m}$$

$$S_2 \times S_n \times S_m \xrightarrow{\mu} S_{n+m}$$

$$i(\alpha, \beta) = \mu(\mathrm{id}, \alpha, \beta)$$

$$\mu^P(q \cdot g, h_1 \cdot g_1, \dots, h_n \cdot g_n) = \mu^P(q, h_1, \dots, h_n) \cdot \mu^G(g, \pi(g)(g_1, \dots$$

$$\mu(q \cdot g, v_1, \dots, v_n) = \mu(q, v_1, \dots, v_n) \cdot \mu^G(g, 1, \dots, 1)$$

Want: $[\mu(q \cdot g, v_1, \dots, v_n), \underline{x}] = [\mu(q, v_1, \dots, v_n), \pi(g)(\underline{x})]$

$$[\mu(q \cdot g, v_i), \underline{x}] = [\mu(q, v_1, \dots, v_n) \cdot \mu^G(g, 1, \dots, 1), \underline{x}]$$

$$= [\mu(q, v), \mu^G(g, 1, \dots, 1) \cdot \underline{x}]$$

should be equal since π is an operad map

$\left[q, [v_1], \ldots, [v_n]\right], \; [p, \underline{x}]$ $\quad [p] = [q(v_1, \ldots, v_n)]$

$\exists!\, g$

$p = q(v_1, \ldots, v_n) \cdot g$

$[q(v_1, \ldots, v_n) \cdot g, \underline{x}] = [q(v_1, \ldots, v_n), \pi_g(\underline{x})]$

$\left[q, [v_1, \pi_g(\underline{x})_1], \ldots, [v_n, \pi_g(\underline{x})_n]\right]$

where $\pi_g(\underline{x}) = \pi_g(\underline{x})_1, \ldots, \pi_g(\underline{x})_n$

* Must still show that this is well-defined

If so, done! $\square$

* $\underline{P}$ preserves (2-)pbs:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "S"] \\ C \arrow[r, "G"'] & D \end{tikzcd} \quad \text{in } Cat$$

Must show
$$\begin{tikzcd} \underline{P}A \arrow[r] \arrow[d] & \underline{P}B \arrow[d] \\ \underline{P}C \arrow[r] & \underline{P}D \end{tikzcd}$$
is also pb.

$$\begin{tikzcd} \coprod P(n) \times_{G(n)} A^n \arrow[r] \arrow[d] & \coprod P(n) \times_{G(n)} B^n \arrow[d] \\ \coprod P(n) \times_{G(n)} C^n \arrow[r] & \coprod P(n) \times_{G(n)} D^n \end{tikzcd}$$
is pb iff each

$$\begin{tikzcd} P(n) \times_{G(n)} A^n \arrow[r] \arrow[d] & P(n) \times_{G(n)} B^n \arrow[d] \\ P(n) \times_{G(n)} C^n \arrow[r] & P(n) \times_{G(n)} D^n \end{tikzcd}$$

$$[q,[v_1],\dots,[v_n]],\ [p,\underline{x}] \quad \text{where } p = q(v_1,\dots,v_n)\circ g.$$

$$\downarrow$$

$$[q,[v_1,\pi_g(x)_1],\dots,[v_n,\pi_{(g)}(x)_n]]$$

$$\downarrow$$

$$[q,[v_1],\dots,[v_n]],\ [q(v_1,\dots,v_n),(\pi_{(g)}(x)_1,\dots,\pi_{(g)}(x)_n)]$$

$$\parallel$$

$$[q,[v_1],\dots,[v_n]],\ [q(v_1,\dots,v_n)\circ g,\ \underline{x}]$$

$$[q,[v_1,\underline{x}_1],\dots,[v_n,\underline{x}_n]]$$

$$\downarrow$$

$$[q,[v_1],\dots,[v_n]],\ [q(v_1,\dots,v_n),\underline{x}] \quad \text{where } \underline{x} = (\underline{x}_1,\dots,\underline{x}_n).$$

$$\downarrow$$

$$[q,[v_1,\pi_{(1)}(\underline{x})],\dots,[v_n,\pi_{(1)}(\underline{x})]]$$

$$\parallel$$

$$[q,[v_1,\underline{x}_1],\dots,[v_n,\underline{x}_n]].$$

2.7

Want $[p, x] = [q, y]$ iff $\exists g$ s.t.
1) $p = q \cdot g$
2) $[p, x] = [q \cdot g, x] = [q, \pi(g)(x)]$ $\quad y = \pi(g)(x)$

$$\begin{gathered}[p, b_i] \\ \downarrow \\ [p, S(b_i)]\end{gathered}$$

$$[q, c_i] \longmapsto [q, G(c_i)]$$

Is $ob : Cat \to Set$ a left adj?
If so, $Set(obC, X) \cong Cat(C, RX)$
Yes, RX = indiscrete cat on X
So, ob preserves coequalizers

$$ob\, P(n) \times G(n) \times (ob X)^n \rightrightarrows ob\, P(n) \times (ob X)^n \to \boxed{}$$

In Sets: $\underset{\text{group}}{A \times G} \times B^n \rightrightarrows A \times B^n \to A \times_G B^n$

$A \times_G B^n = A \times B^n / \sim \quad (a, \underline{b}) \sim (a', \underline{b}')$ if $\exists g \in G$ s.t. $a \cdot g = a'$, $\pi(g)(\underline{b}) = \underline{b}'$

use $P(n)$ is $G(n)$-free

$\exists! g$
$p = q \cdot g$ and $G(c_i) = \pi(g)(S(b_i))$

$G(c_i) = S(b_{\pi(g)(i)})$

$\cancel{[p, b_{\pi(g)(i)}]}$

$\cancel{[p, b_i]}$

$[q, c_i] \longmapsto [q, G(c_i)] = [q, S(b_{\pi(g)(i)})]$
$\Rightarrow P(n) \times_{G(n)} A^n$ is the pb $\square$