Operads and equivariance

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Contents

Abstract

1 Introduction

QQQ Needs an intro. Might be something salvageable from the original papers.

Original paper intro:

Operads were defined by May [39] in the early 70's to provide a convenient tool to approach problems in algebraic topology, notably the question of when a space X admits an n-fold delooping Y so that $X \simeq \Omega^n Y$. An operad, like an algebraic theory [30], is something like a presentation for a monad or algebraic structure. The theory of operads has seen great success, and we would like to highlight two reasons. First, operads can be defined in any suitable symmetric monoidal category, so that there are operads of topological spaces, of chain complexes, of simplicial sets, and of categories, to name a few examples. Moreover, symmetric (lax) monoidal functors carry operads to operads, so we can use operads in one category to understand objects in another via transport by such a functor. Second, operads in a fixed category are highly flexible tools. In particular, the categories listed above all have some inherent notion of "homotopy equivalence" which is weaker than that of isomorphism, so we can study operads which are equivalent but not isomorphic. These tend to have algebras which have similar features in an "up-to-homotopy" sense but very different combinatorial or geometric properties arising from the fact that different objects make up these equivalent but not isomorphic operads.

Operads in the category **Cat** of small categories have a unique flavor arising from the fact that **Cat** is not just a category but a 2-category. These 2-categorical aspects have not been widely treated in the literature, although a few examples can be found. Lack [29] mentions braided **Cat**-operads (the reader new to braided operads should refer to the work of Fiedorowicz [10]) in his work on coherence for 2-monads, and Batanin [2] uses lax morphisms of operads in **Cat** in order to define the notion of an internal operad. But aside from a few appearances, the basic theory of operads in **Cat** and their 2-categorical properties seems missing. This paper was partly motivated by the need for such a theory to be explained from the ground up.

There were two additional motivations for the work in this paper. In thinking about coherence for monoidal functors, the first author was led to a general study of algebras for multicategories internal to **Cat**. These give rise to 2-monads (or perhaps pseudomonads, depending on how the theory is set up), and checking abstract properties of these 2-monads prompts one to consider the simpler case of operads in **Cat** instead of multicategories. The other motivation was from the second author's attempt to understand the interplay between operads in **Cat**, operads in **Top**, and the passage from (bi)permutative categories to E_∞ (ring) spaces. The first of these motivations raised the issue of when operads in **Cat** are cartesian, while the second led us to consider when an operad in **Cat** possesses a pseudo-commutative structure.

While considering how to best tackle a general discussion of operads in **Cat**, it became clear that restricting attention to the two most commonly used types of operads, symmetric and non-symmetric operads, was both short-sighted and unnecessary. Many theorems apply to both kinds of operads at once, with the difference in proofs being negligible; in fact, most of the arguments which applied to the symmetric case seemed to apply to the case of braided operads as well. This led us to the notion of an action operad $\mathbf{G}$, and then to a definition of $\mathbf{G}$-operads. In essence, this is merely the general notion of what it means for an operad $P = \{P(n)\}_{n\in\mathbb{N}}$ to have groups of equivariance $\mathbf{G} = \{G(n)\}_{n\in\mathbb{N}}$ such that $G(n)$ acts on $P(n)$. Choosing different natural families of groups $\mathbf{G}$, we recover known variants of the definition of operad.

Groups $\mathbf{G}$	Type of operad
Terminal groups	Non-symmetric operad
Symmetric groups	Symmetric operad
Braid groups	Braided operad

These definitions have appeared, with minor variations, in two sources of which we are aware. In Wahl's thesis [47], the essential definitions appear but not in complete generality as she requires a surjectivity condition. Zhang [49] also studies these notions[1], once again in the context of homotopy theory, but requires the superfluous condition that $e_1 = \mathrm{id}$ (see Lemma 4.12).

This paper consists of the following. In Section 1, we give the definition of an action operad $\mathbf{G}$ and a $\mathbf{G}$-operad. We develop this definition abstractly so as to apply it in any suitable symmetric monoidal category. It is standard to express operads as monoids in a particular functor category using a composition tensor product. In order to show that our $\mathbf{G}$-operads fit into this philosophy, we must work abstractly and use the calculus of coends together with the Day convolution product [7]. The reader uninterested in these details can happily skip them, although we find the route taken here to be quite satisfactory in justifying the axioms for an action operad $\mathbf{G}$ and the accompanying notion of $\mathbf{G}$-operad. Many of our calculations are generalizations of those appearing in work of Kelly [22], although there are slight differences in flavor between the two treatments.

Section 2 works through the basic 2-categorical aspects of operads in **Cat**. We explain how every operad gives rise to a 2-monad, and show that all of the various 1-cells between algebras of the associated 2-monad correspond to the obvious sorts of 1-cells one might define between algebras over an operad in **Cat**. Similarly, we show that the algebra 2-cells, using the 2-monadic approach, correspond to the obvious notion of transformation one would define using the operad.

Section 3 studies three basic 2-categorical properties of an operad, namely the property of being finitary, the property of being 2-cartesian, and the coherence property. The first of these always holds, as a simple calculation shows. The second of these turns out to be equivalent to the action of $G(n)$ on $P(n)$ being free for all n, at least up to a certain kernel. In particular, our characterization clearly shows that every non-symmetric operad is 2-cartesian, and that a symmetric operad is 2-cartesian if and only if the symmetric group actions are all free. (It is useful to note that a 2-monad on **Cat** is 2-cartesian if and only if the underlying monad on the category of small categories is cartesian in the usual sense as the (strict) 2-pullback of a diagram is the same as its pullback.) The third property is also easily shown to hold for any $\mathbf{G}$-operad on **Cat** using a factorization system argument due to Power [40].

Section 4 then goes on to study the question of when a $\mathbf{G}$-operad P admits a pseudo-commutative structure. Such a structure provides the 2-category of algebras with a richer structure that includes well-behaved notions of tensor product, internal hom, and multilinear map that fit together much as the analogous notions do in the category of vector spaces. When P is contractible (i.e., each $P(n)$ is equivalent to the terminal category), this structure can be obtained from a collection of elements $t_{m,n} \in G(mn)$ satisfying certain properties. In particular, we show that every contractible symmetric operad is pseudo-commutative, and we prove that there exist such elements $t_{m,n} \in Br_{mn}$ so that every contractible braided operad is pseudo-commutative as well (in fact in two canonical ways). Thus Section 4 can be seen as a continuation, in the operadic context, of the work in [16], and in particular the "geometric" proof of the existence of a pseudo-commutative structure for braided strict monoidal categories

[1] Zhang calls our action operad a *group operad*. We dislike this terminology as it seems to imply that we are dealing with an operad in the category of groups, which is not the case unless all of the maps $\pi_n : G(n) \to \Sigma_n$ are zero maps.

demonstrates the power of being able to change the groups of equivariance.

The authors would like to thank John Bourke, Martin Hyland, Tom Leinster, and Peter May for various conversations which led to this paper. While conducting this research, the second author was supported by an EPSRC Early Career Fellowship.

Original Borel intro:

Categories of interest are often monoidal: sets, topological spaces, and vector spaces are all symmetric monoidal, while the category of finite ordinals (under ordinal sum) is merely monoidal. But other categories have more exotic monoidal structures. The first such type of structure discovered was that of a braided monoidal category. These arise in categories whose morphisms have a geometric flavor like cobordisms embedded in some ambient space [17], in categories produced from double loop spaces [10], and categories of representations over objects like quasitriangular (or braided) bialgebras [45] . Another such exotic monoidal structure is that of a coboundary category, arising in examples from the representation theory of quantum groups [9].

Going back to the original work of May on iterated loop spaces [39], operads were defined in both symmetric and nonsymmetric varieties. But Fiedorowicz's work on double loop spaces [10] showed that there was utility in considering another kind of operad, this time with braid group actions instead of symmetric group actions. There is a clear parallel between these definitions of different types of operads and the definitions of different kinds of monoidal category, with each given by some general schema in which varying an $\mathbb{N}$-indexed collection of groups produced the types of operads or monoidal categories seen in nature. Building on the work in [**cg**], the goal of this paper is to show that this parallel can be upgraded from an intuition to precise mathematics using the notion of action operad.

An action operad $\mathbf{\Lambda}$ is an operad which incorporates all of the essential features of the operad of symmetric groups. Thus $\Lambda(n)$ is no longer just a set, but instead also has a group structure together with a map $\pi_n : \Lambda(n) \to \Sigma_n$. Operadic composition then satisfies an additional equivariance condition using the maps π_n and the group structures. Each action operad $\mathbf{\Lambda}$ produces a notion of $\mathbf{\Lambda}$-operad which encodes equivariance conditions using both the groups $\Lambda(n)$ and the maps π_n. Examples include the symmetric groups, the terminal groups (giving nonsymmetric operads), the braid groups (giving braided operads), and the n-fruit cactus groups [14] (giving a new notion of operad one might call cactus operads). Using a formula resembling the classical Borel construction for spaces with a group action, we can produce from any action operad $\mathbf{\Lambda}$ a notion of $\mathbf{\Lambda}$-monoidal category, in which the group $\Lambda(n)$ acts naturally on n-fold tensor powers of any object. Thus the categorical Borel construction embeds action operads into a category of monads on **Cat**, and we characterize the image of this embedding as those monads describing monoidal structures of a precise kind.

The paper is organized into the following sections. Section 1 reviews the definition of an action operad, and defines the categorical Borel construction on them. The key result, which reappears in proofs throughout the paper, is Theorem 4.15, characterizing action operads in terms of two new operations mimicking the block sum of permutations and the operation which takes a permutation of n letters and produces a new permutation on $k_1 + k_2 + \cdots + k_n$ letters by permuting the blocks of k_i letters. In Section 2, we use this characterization and Kelly's theory of clubs [18, 20, 21] to embed action operads into monads on **Cat** and determine the essential image of this embedding. Section 3 gives a construction of the free action operad from a suitable collection of data, and relates this to how clubs can be described using generators and relations. The results of Sections 2 and 3 show that the definitions of symmetric monoidal category or coboundary category, for example, correspond to the action operad constructed from the corresponding free symmetric monoidal or coboundary category on one object; these and other examples appear in detail in Section 4. Section 5 then extends the definition of $\mathbf{\Lambda}$-operad to that of $\mathbf{\Lambda}$-multicategory and shows that these arise abstractly via a Kleisli construction.

Copied from text: Yau [48] collects together a large number of results on the topic of action operads while also investigating the setting of infinity group operads.

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2 Notation and Conventions

Notation 2.1 (Symmetric groups). We denote the symmetric group on the symbols $1, 2, \ldots, n$ by Σ_n. Elements of a symmetric group are usually denoted by lowercase Greek letters or written in cycle notation.

Notation 2.2 (Braid groups). We denote the braid group on n strands by B_n.

Notation 2.3 (Identity elements). The symbol e will generically represent an identity element in a group. If we are considering a set of groups $\{\Lambda(n)\}_{n\in\mathbb{N}}$ indexed by the natural numbers, then e_n is the identity element in $\Lambda(n)$. We will often drop the subscripts and just write e when the index n in $\Lambda(n)$ is either clear from context or unimportant to the argument at hand.

Convention 2.4 (Identity morphisms). We generically write an identity morphism $A \to A$ as either 1 or 1_A.

Notation 2.5 (Group actions). For a group G, a right G-action on a set X will be denoted $(x, g) \mapsto x \cdot g$ or $(x, g) \mapsto xg$. Similar notation will be used for left actions, and for multiplication in a group.

Convention 2.6 (Indexed objects). We generically write $\{\Lambda(n)\}_{n\in\mathbb{N}}$ for a $\mathbb{N}$-indexed family of objects $\Lambda(n)$. We will occasionally write Λ_n in place of $\Lambda(n)$, especially in diagrams or when the objects Λ_n have been independently defined, as in Notations 2.1 and 2.2.

Convention 2.7 (Products and quotients). We will often be interested in elements of a product of the form

$$A \times B(1) \times \cdots \times B(n) \times C$$

(or similar, for example without C). We will write elements of this set as $(a; b_1, \ldots, b_n; c)$, where $b_i \in B(i)$. In the case that we need to consider equivalence classes of such elements, these classes will be written as $[a; b_1, \ldots, b_n; c]$. The most common situation in which we consider such equivalence classes is that of a coequalizer of left and right group actions in the following sense. A coequalizer of maps

$$\begin{tikzcd} A \times G \times B \arrow[r, shift left, "1\times\lambda"] \arrow[r, shift right, "\rho\times 1"'] & A \times B \arrow[r, "\varepsilon"] & A \otimes_G B \end{tikzcd}$$

will be written as $A \otimes_G B$, where ρ is a right action of G on A and λ is a left action of G on B. This notation is meant to emphasize the analogy with tensor products of R-modules, even when the monoidal structure involved is cartesian. It also differentiates these coequalizers from pullbacks.

Convention 2.8 (Tilde for maps respecting equivariance). Suppose that $A \otimes_G B$ is a coequalizer as in Convention 2.7. By definition, maps $f\colon A \otimes_G B \to X$ are in bijection with maps $A \times B \to X$ that coequalize $1 \times \lambda$ and $\rho \times 1$. Given such a map f, we will always denote the corresponding map $A \times B \to X$ as $\tilde{f}$.

Convention 2.9 (Pullbacks). The pullback of the diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & A \end{tikzcd}$$

will be written as $X \times_A Y$.

Definition 2.10 (Underlying permutation). Suppose that $f\colon G \to \Sigma_n$ is a given group homomorphism, and $x \in G$. The *underlying permutation* of x is the element $f(x) \in \Sigma_n$. If there is likely to be some confusion as to which homomorphism f is being used, we will call $f(x)$ the *underlying permutation with respect to f*.

Notation 2.11 (Applying underlying permutations). Throughout we will be using maps $\pi_n\colon O(n) \to \Sigma_n$, where $O(n)$ is the set of n-ary operations of an operad O and Σ_n is the symmetric group on n elements. For any $\sigma \in O(n)$, we will write $\sigma(i)$ for $\pi_n(\sigma)(i)$, the image of i with respect to the underlying permutation of σ; the notation $\sigma^{-1}(i)$ will be used for the inverse image of i with respect to the underlying permutation of σ.

Remark 2.12 (Left action of symmetric groups on tuples). The most common group action we will encounter is the left action of the symmetric group Σ_n on a set of the form X^n. We write this action as $\sigma \cdot (x_1, \ldots, x_n)$, and emphasize that it is given by the formula

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

Definition 2.13 (Block sum). Let $k_1, \ldots, k_n$ be natural numbers and suppose that $\sigma_i \in \Sigma_{k_i}$ are permutations. The *block sum* of $\sigma_1, \ldots, \sigma_n$, written

$$\beta(\sigma_1, \ldots, \sigma_n),$$

is the permutation in Σ_K, where $K = \sum_{i=1}^n k_i$, given as described below. For $1 \leq j \leq K$, define c to be the unique integer such that

$$k_1 + \cdots + k_c < j \leq k_1 + \cdots + k_c + k_{c+1}.$$

Define

$$\beta(\sigma_1, \ldots, \sigma_n)(j) = k_1 + \cdots + k_c + \sigma_{c+1}\big(j - (\sum_{i=1}^{c} k_i)\big).$$

Remark 2.14. The formula above expresses the idea that $\beta(\sigma_1, \ldots, \sigma_n)$ permutes the first k_1 elements using σ_1, the next k_2 elements using σ_2, and so on.

Definition 2.15 (Duplication). Let $k_1, \ldots, k_n$ be natural numbers, and suppose that $\sigma \in \Sigma_n$ is a permutation. The *duplication* of σ with respect to $k_1, \ldots, k_n$, written

$$\delta_{n;k_1,\ldots,k_n}(\sigma),$$

is the permutation in Σ_K, where $K = \sum_{i=1}^n k_i$, given as described below. For $1 \leq j \leq K$, define c to be the unique integer such that

$$k_1 + \cdots + k_c < j \leq k_1 + \cdots + k_c + k_{c+1}.$$

Define

$$\delta_{n;k_1,\dots,k_n}(\sigma)(j) = \Big(\sum_{\sigma(k_i)<\sigma(k_{c+1})} k_i\Big) + j - \Big(\sum_{i=1}^{c} k_i\Big).$$

Remark 2.16. The formula above for $\delta_{n;k_1,\dots,k_n}(\sigma)$ is best explained by drawing the graph of σ as follows. The function σ can be represented by drawing two rows of n dots each, and connecting dot i in the top row to dot $\sigma(i)$ in the bottom row. Then $\delta_{n;k_1,\dots,k_n}(\sigma)$ is obtained by

- replacing dot i in the top row with k_i dots,
- replacing dot $\sigma(i)$ in the bottom row with k_i dots, and
- connecting these two sets of k_i dots in the unique way that preserves order.

Thus we see that the ith entry for σ is duplicated k_i times in $\delta_{n;k_1,\dots,k_n}(\sigma)$.

moved this remark here, fix and maybe shorten?

Remark 2.17. Permutations, as elements of Σ_n, can be considered as permutation *matrices* with exactly one 1 in each row and column. E.g., the permutation $(1\ 3\ 2) \in \Sigma_3$ can be considered as a matrix which permutes three elements $\begin{bmatrix} a & b & c \end{bmatrix}$ upon pre-multiplication:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \\ a \end{bmatrix}.$$

Then the block sum β (Definition 2.13) corresponds to the process of taking the block diagonal matrix of the original permutation matrices. So given elements $(1\ 2) \in \Sigma_2$, $e_1 \in \Sigma_1$, and $(1\ 2\ 3) \in \Sigma_3$, then

$$\beta((1\ 2), e_1, (1\ 2\ 3)) = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} 1 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which corresponds to the permutation $(1\ 2)(3)(4\ 5\ 6)$.

Similarly, we can describe the duplication δ (Definition 2.15) as an operation on permutation matrices. The idea here being that for $\sigma \in \Sigma_n$, $\delta_{n;k_1,\dots,k_n}(\sigma)$ takes a block diagonal of identity matrices $I_{k_1}, \dots, I_{k_n}$ (which corresponds to $\beta(e_{k_1}, \dots, e_{k_n}) \in \Sigma_{k_1+\dots+k_n}$), and permutes these according to the effect of the permutation σ. For example, given $\sigma = (1\ 2\ 3)$, then

$$\delta_{3;2,1,3}(\sigma) = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix} * \sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_3 \\ I_2 & 0 & 0 \\ 0 & I_1 & 0 \end{bmatrix}.$$

We make use of a similar interpretation of signed permutations and block diagonal matrices in a counterexample given in **??**.

Convention 2.18 (Superscripts). We generically use superscripts, when needed, to distinguish between operations of the same type associated to different structures. As an example, a monoid homomorphism $f\colon A \to B$ would have axioms written as

$$f(x \cdot^A y) = f(x) \cdot^B f(y),$$
$$f(1^A) = 1^B.$$

Part I: Operads and Action Operads

3 Background: Operads

This section will collect the basic background information on operads that we will later generalize in Section 8. We begin with the most common type of operad, a symmetric operad, before defining two more types of operad: plain and braided.

Definition 3.1 (Symmetric operad). A *symmetric operad* O (in the category of sets) consists of

- a set, $O(n)$, for each natural number n,
- for each n, a right Σ_n-action on $O(n)$,
- an element $\mathrm{id} \in O(1)$, and
- functions
$$\mu\colon O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying the following three axioms.

1. The element $\mathrm{id} \in O(1)$ is a two-sided unit for μ, meaning that
$$\mu(\mathrm{id}; x) = x,$$
$$\mu(x; \mathrm{id}, \ldots, \mathrm{id}) = x$$
for any $x \in O(n)$.
2. The functions μ are associative, meaning that the diagram below commutes.

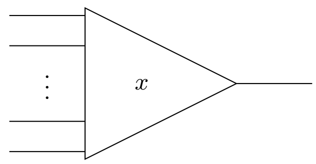
$$\begin{tikzcd}
O(n)\times\left(\prod_{i=1}^{n} O(k_i)\right)\times\left(\prod_{i=1}^{n}\prod_{j=1}^{k_i} O(l_{i,j})\right) \arrow[r, "\cong"] \arrow[dd, "\mu\times 1"'] & O(n)\times\prod_{i=1}^{n}\left(O(k_i)\times\prod_{j=1}^{k_i} O(l_{i,j})\right) \arrow[d, "1\times\prod\mu"] \\
 & O(n)\times\prod_{i=1}^{n} O\left(\sum_{j=1}^{k_i} l_{i,j}\right) \arrow[d, "\mu"] \\
O\left(\sum_{i=1}^{n} k_i\right)\times\prod_{i=1}^{n}\prod_{j=1}^{k_i} O(l_{i,j}) \arrow[r, "\mu"'] & O\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i} l_{i,j}\right)
\end{tikzcd}$$

3. The functions μ are equivariant with respect to the symmetric group actions, meaning that two equations hold.
   3.1 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\tau_i \in \Sigma_{k_i}$ for $i = 1, \ldots, n$. Then the first equivariance axiom is the requirement that
$$\mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) = \mu(x; y_1, \ldots, y_n) \cdot \beta(\tau_1, \ldots, \tau_n)$$
   holds, where β is the function from Definition 2.13.

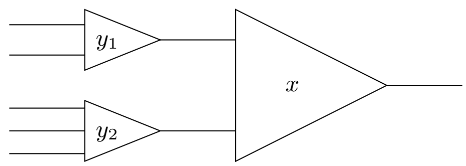
   3.2 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\sigma \in \Sigma_n$. Then the second equivariance axiom is the requirement that
$$\mu(x \cdot \sigma; y_1, \ldots, y_n) = \mu\left(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \cdot \delta_{n;k_1,\ldots,k_n}(\sigma)$$
   holds, where $\delta_{n;k_1,\ldots,k_n}$ is the function from Definition 2.15.

Terminology 3.2 (Operadic multiplication, composition). The functions μ in Definition 3.1 are called *operadic multiplication* or *operadic composition* maps.

Remark 3.3. One is intended to think that $x \in O(n)$ is a function with n inputs and a single output, as below.



Operadic composition is then a generalization of function composition, with the pictorial representation below being $\mu(x; y_1, y_2)$ for $\mu\colon O(2) \times O(2) \times O(3) \to O(5)$.



Terminology 3.4 (n-ary operations). The set $O(n)$ in Definition 3.1 is called the set of *n-ary operations* of O.

Remark 3.5. If O is an operad in a category other than **Sets** (see Remark 3.10), then we would call $O(n)$ the *object* of n-ary operations.

Here are two important examples of symmetric operads.

Example 3.6 (Symmetric operad of symmetric groups). The canonical example of *a* symmetric operad is *the* symmetric operad which we write as Σ. The set $\Sigma(n)$ is the set of elements of the symmetric group Σ_n, and the group action is just multiplication on the right. The identity element id $\in \Sigma(1)$ is just the identity permutation on a one-element set. Operadic composition in Σ will then be given by a function

$$\Sigma(n) \times \Sigma(k_1) \times \cdots \times \Sigma(k_n) \to \Sigma(k_1 + \cdots + k_n)$$

which takes permutations $\sigma \in \Sigma_n, \tau_i \in \Sigma_{k_i}$ and produces the following permutation in $\Sigma_{k_1+\cdots+k_n}$:

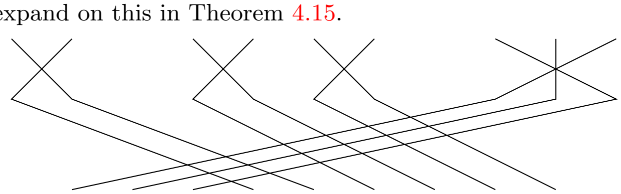
$$\mu(\sigma; \tau_1, \ldots, \tau_n) = \delta(\sigma) \cdot \beta(\tau_1, \ldots, \tau_n),$$

with β and δ as in Definitions 2.13 and 2.15.

Below we have drawn the permutation for the composition

$$\mu\colon \Sigma(3) \times \Sigma(2) \times \Sigma(4) \times \Sigma(3) \to \Sigma(9)$$

evaluated on the element $((1\ 2\ 3); (1\ 2), (1\ 2)(3\ 4), (1\ 3))$, in terms of β and δ. We expand on this in Theorem 4.15.



$\beta\big((1\ 2), (1\ 2)(3\ 4), (1\ 3)\big)$

$\delta_{3;2,4,3}\big((1\ 2\ 3)\big)$

End this example here, with labels, move the rest to after 2.26.

Example 3.7 (Endomorphism operad). Let X be a set. The *endomorphism operad* of X, denoted $\mathcal{E}_X$, consists of

- the sets
$$\mathcal{E}_X(n) = \mathbf{Sets}(X^n, X),$$
- the right group actions $\mathcal{E}_X(n) \times \Sigma_n \to \mathcal{E}_X(n)$ given by
$$(f \cdot \sigma)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$
- the element $\mathrm{id} \in \mathcal{E}_X(1)$ being the identity function $1\colon X \to X$, and
- operadic multiplication given by
$$\mu(g; f_1, \ldots, f_n) = g \circ (f_1 \times \cdots \times f_n).$$

We leave verification of the axioms to the reader.

Remark 3.8. The intuition in Remark 3.3 is connected with Example 3.7 through the concept of an algebra, see Section 8.

One can also drop the symmetric group actions entirely to obtain the notion of a non-symmetric or plain operad.

Definition 3.9 (Non-symmetric operad). A *non-symmetric operad* O consists of

- a set, $O(n)$, for each natural number n,
- an element $\mathrm{id} \in O(1)$, and
- functions
$$\mu\colon O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying axioms 1 and 2 from Definition 3.1.

Remark 3.10.

1. One can change from operads in **Sets** to operads in another (symmetric) monoidal category $\mathcal{V}$ by requiring each $O(n)$ to be an object of $\mathcal{V}$ and replacing all instances of cartesian product with the appropriate tensor product in $\mathcal{V}$. One would also the element $\mathrm{id} \in O(1)$ with a map $I \to O(1)$ from the unit object of $\mathcal{V}$ to $O(1)$. In the case of symmetric operads, one would also express the right group actions as homomorphisms
$$\Sigma_n^{op} \to \mathcal{V}\big(O(n), O(n)\big).$$
2. Every symmetric operad has an underlying *symmetric collection* which consists of the natural number-indexed set $\{O(n)\}_{n \in \mathbb{N}}$ together with symmetric group actions, but without a chosen identity element or composition maps. The category of symmetric collections is a presheaf category, and we will equip it with a monoidal structure in which monoids are precisely operads in **??**. A similar construction, but without reference to group actions, shows that every non-symmetric operad has an underlying (non-symmetric) collection which is now merely a $\mathbb{N}$-indexed collection of sets.

Example 3.11.

insert example of non-symmetric operad here, how about Trimble's thing in dimension 1?

Another example of non-symmetric operad is the operad of *pure* braid groups: see James Griffin's comment below the old blog post.

In seeking a definition of weak n-category which can be described through iterated enrichment, Trimble defines an operad E as follows:

- for $n \geq 0$, $E(n)$ is the space of continuous, endpoint preserving maps

$$[0,1] \to [0,n],$$

- the identity element $1 \in E(1)$ is the identity map

$$[0,1] \to [0,1],$$

- composition is described by 'reparameterisation'

this is harder to write down compactly than I remember, how much detail do we want about composition? and what am I writing about it being non-symmetric?

In the original topological applications [39], symmetric operads were the central figures. A further kind of operad was studied by Fiedorowicz in [10], that of a *braided* operad in which the braid groups take the place of the symmetric groups. We sketch that definition below.

cite Tom's survey [31] and Cheng/Lauda [6]? Or Cheng/Gurski [5] cobordisms paper?

Definition 3.12 (Braided operad, sketch). A *braided operad* consists of

- a non-symmetric operad O and
- for each n, a right action of the nth braid group B_n on $O(n)$,

such that the operadic multiplication functions μ are equivariant with respect to the braid group actions, meaning that two equations hold.

1 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\tau_i \in B_{k_i}$ for $i = 1, \ldots, n$. Then the first equivariance axiom is the requirement that

$$\mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) = \mu(x; y_1, \ldots, y_n) \cdot \beta(\tau_1, \ldots, \tau_n)$$

holds.

2 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\sigma \in B_n$. Then the second equivariance axiom is the requirement that

$$\mu(x \cdot \sigma; y_1, \ldots, y_n) = \mu\left(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \cdot \delta_{n;k_1,\ldots,k_n}(\sigma)$$

holds.

Remark 3.13. The above sketch omits the definitions of β, δ for braids. Formulas for these can be found in [48, Examples 5.1.11, 5.1.13], although the geometric interpretations are simple: β takes the disjoint union of braids, and $\delta_{n;k_1,\ldots,k_n}(\tau)$ is obtained by replacing the ith strand of τ by k_i parallel strands. These operations are sometimes referred to as 'cabling' operations for braids, as described in, for example, [8].

Example 3.14. Example 3.1 of [10] shows that there is a braided operad structure on the spaces $\tilde{C}_2(n)$, obtained as the universal covers of the spaces $C_2(n)$ appearing in the little 2-disks operad. The other canonical example of a braided operad is the operad of braid groups B_n, as obtained by applying Proposition 8.7.

We conclude this section by defining various categories of operads in **Sets**, although the reader can generalize these to categories of operads in any symmetric monoidal category. We focus on the case of symmetric operads, and explain after how to modify the definitions for non-symmetric or braided operads.

Definition 3.15 (Map of symmetric operads). Let O, O' be symmetric operads in **Sets**. Then a *map of symmetric operads* (or just *operad map* for short, when it is clear that the intent is to respect the symmetric group actions) $f : O \to O'$ consists of functions $f_n : O(n) \to O'(n)$ for each natural number such that the following axioms hold for all $x \in O(n), y_i \in O(k_i), \sigma \in \Sigma_n$.

$$f\left(\mathrm{id}_O\right) = \mathrm{id}_{O'}$$
$$f\left(\mu^O(x; y_1, \ldots, y_n)\right) = \mu^{O'}\left(f(x); f(y_1), \ldots, f(y_n)\right)$$
$$f(x \cdot \sigma) = f(x) \cdot \sigma$$

The next proposition states that symmetric operads and their maps form a category. We leave the proof to the reader.

let's use the following prop/notation pair as a template for how to define categories

Proposition 3.16. *There is a category with*

- *objects the symmetric operads O in* **Sets**,
- *morphisms the maps of symmetric operads between them,*
- *identities $1_O : O \to O$ given by*
$$(1_O)_n = 1_{O(n)} : O(n) \to O(n),$$
and
- *composition given by*
$$(g \circ f)_n = g_n \circ f_n.$$

Notation 3.17 (The category of symmetric operads). The category in Proposition 3.16 is called the *category of symmetric operads (in* **Sets***)*, and is denoted Σ-**Op**.

Remark 3.18 (The category of non-symmetric operads). Omitting symmetries entirely, we can also form the category of non-symmetric operads (in **Sets**), denoted **Op**. The objects are non-symmetric operads (Definition 3.9) and the morphisms have the same data as maps of symmetric operads (Definition 3.15) but only satisfy the first two axioms as there is no group action to preserve. Composition and identites are defined exactly as for symmetric operads.

Remark 3.19 (The category of braided operads). Replacing symmetries with braids, we can form the category of braided operads (in **Sets**), denoted B-**Op**. The objects are braided operads (Definition 3.12). The morphisms have the same data as maps of symmetric operads (Definition 3.15) and satisfy identical looking axioms so long as the equivariance axiom is interpretted using braids rather than symmetries. Composition and identites are defined exactly as for symmetric operads.

4 Action Operads

The axioms for both symmetric and braided operads use the following features.

1. For each n, we have a group Λ_n acting on the set $O(n)$ of n-ary operations of the operad. Each such group is equipped with a homomorphism $\pi_n : \Lambda_n \to \Sigma_n$, so that every element of Λ_n has an underlying permutation.

2. The first equivariance axiom requires the additional data of a family of functions

$$\beta\colon \Lambda_{k_1} \times \cdots \Lambda_{k_n} \to \Lambda_{k_1+\cdots+k_n}.$$

In order for this to be a well-defined function, the right group action axioms force these functions to be group homomorphisms.

3. The second equivariance axiom requires the additional data of a family of functions

$$\delta_{n;k_1,\ldots,k_n}\colon \Lambda_{k_1} \times \cdots \Lambda_{k_n} \to \Lambda_{k_1+\cdots+k_n}.$$

These functions are not forced to be group homomorphisms, but do satisfy some additional axioms.

In this section, we define *action operads* in Definition 4.1 in order to present a unified treatment of a family of groups satisfying the conditions above. In **??**, we define for each action operad Λ a notion of Λ-operad; symmetric operads will arise when $\Lambda = \Sigma$, non-symmetric operads will arise when Λ is the action operad of trivial groups, and braided operads will arise when $\Lambda = B$. Our definition of an action operad will not mention β or δ, but will instead use a single axiom relating the group structure, operadic multiplication, and underlying permutations. The main result of this section is Theorem 4.15 in which we prove that action operads can be described entirely in terms of the functions β, δ as above. We will give two examples of action operads (the symmetric groups and the trivial groups) in this section, and postpone the rest to Section 5.

Definition 4.1 (Action operad). An *action operad* (Λ, π) consists of

- an operad $\Lambda = \{\Lambda(n)\}$ in the category of sets such that each $\Lambda(n)$ is equipped with the structure of a group and
- a map $\pi\colon \Lambda \to \Sigma$ which is simultaneously a map of operads and a group homomorphism $\pi_n\colon \Lambda(n) \to \Sigma_n$ for each n

such that one additional axiom holds. Write

$$\mu\colon \Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n) \to \Lambda(k_1 + \cdots + k_n)$$

for the multiplication in the operad Λ. Let $(g; f_1, \ldots, f_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n)$ and let $(g'; f_1', \ldots f_n')$ be an element of the product $\Lambda(n) \times \Lambda(k_{g^{-1}(1)}) \times \cdots \times \Lambda(k_{g^{-1}(n)})$. We require that

$$\mu\left(g'; f_1', \ldots f_n'\right) \mu\left(g; f_1, \ldots, f_n\right) = \mu\left(g'g; f_{g(1)}' f_1, \ldots, f_{g(n)}' f_n\right) \tag{1}$$

in the group $\Lambda(k_1 + \cdots + k_n)$.

Notation 4.2. We write an action (Λ, π) as merely Λ. The maps π will be left implicit in the notation, as we will not have reason to study the case of a single operad Λ equipped with two different action operad structures via π and π'.

Remark 4.3. Our definition of an action operad is the same as the *operads from families of groups* appearing in Section 1.2 Wahl's thesis [47], but without the condition that each π_n is surjective. It is also the same as the *group operads* appearing in work of Zhang [49], although we prove later (see Lemma 4.12) that Zhang's condition of $e_1 \in \Lambda(1)$ being the identity element follows from the rest of the axioms.

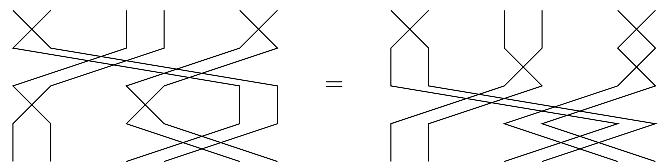
We now give the two examples of action operads that have already appeared in this paper: the symmetric groups and the trivial groups.

Example 4.4 (Action operad of symmetric groups). The symmetric operad Σ has a canonical action operad structure. It is given by taking π to be the identity map, and is the terminal object in the category of action operads.

Example 4.5 (Action operad of trivial groups). The terminal operad T in the category of sets has a unique action operad structure. Since $T(n)$ is a singleton for each n, the group structure is unique, as is the map π. The single action operad axiom is then automatic as both sides of Equation (1) are the identity. This is the initial object in the category of action operads.

Remark 4.6.

- As per Notation 2.11, we write $g(i)$ to mean $\pi(g)(i)$ and $g^{-1}(i)$ to mean $\pi(g)^{-1}(i)$.
- The final axiom is best explained using the operad Σ of symmetric groups. Reading symmetric group elements as permutations from top to bottom, below is a pictorial representation of the final axiom for the map $\mu\colon \Sigma_3 \times \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_6$.



$\mu((23);(12),(12),e_2)\cdot\mu((132);(12),e_2,(12))$ $\quad$ $\mu((23)\cdot(132);e_2\cdot(12),(12)\cdot e_2,(12)\cdot(12))$

Action operads are themselves the objects of a category, **AOp**. The morphisms of this category are defined below.

Definition 4.7 (Map of action operads). A *map of action operads* $f\colon \Lambda \to \Lambda'$ consists of a map $f\colon \Lambda \to \Lambda'$ of the underlying operads such that

1. $\pi^{\Lambda'} \circ f = \pi^{\Lambda}$ (i.e., f is a map of operads over Σ) and
2. each $f_n\colon \Lambda(n) \to \Lambda'(n)$ is a group homomorphism.

Proposition 4.8. *There is a category with*

- *objects the action operads O in* **Sets***,*
- *morphisms as defined in Definition 4.7,*
- *identities $1_\Lambda\colon \Lambda \to \Lambda$ given by the identity morphism of Λ as an operad, and*
- *composition given by composition of maps of operads.*

Notation 4.9 (The category of action operads). The category in Proposition 4.8 is called the *category of action operads (in* **Sets***)*, and is denoted **AOp**.

Proposition 4.10. *Let (Λ, π) be an action operad. The map $\pi\colon \Lambda \to \Sigma$ is a map of action operads.*

We now study some of the structure on the groups $\Lambda(n)$ for small values of n. Recall from Notation 2.3 that we write e_n for the identity element in the group $\Lambda(n)$. Many of our proofs rely on the following version of the Eckmann-Hilton argument.

Proposition 4.11 ((Eckmann-Hilton argument))**.** *Let G be a group with identity element e, and suppose $\varphi\colon G \times G \to G$ is a function. If φ is a homomorphism, meaning that*

$$\varphi(g', h') \cdot \varphi(g, h) = \varphi(g' \cdot g, h' \cdot h),$$

and $\varphi(g, e) = g = \varphi(e, g)$ for all elements $g \in G$, then

$$\varphi(g, h) = g \cdot h$$

and G is abelian.

Lemma 4.12. *Let Λ be an action operad.*

1. *In $\Lambda(1)$, the identity element for the group structure, e_1, is equal to the identity element for the operad structure, id.*
2. *The equation*
$$\mu(e_n; e_{i_1}, \ldots, e_{i_n}) = e_I$$
holds for any natural numbers $n, i_j, I = \sum_{j=1}^{n} i_j$.
3. *The group $\Lambda(1)$ is abelian.*

Proof. For the first claim, we will prove that $\mathrm{id} \cdot e_1 = \mathrm{id} \cdot \mathrm{id}$, so $e_1 = \mathrm{id}$ by cancellation. Note that since the only element of Σ_1 is the identity permutation, the action operad axiom Equation (1) is
$$\mu(g'; f') \cdot \mu(g; f) = \mu(g'g; f'f)$$
when $g, g' \in \Lambda(1)$. Thus we obtain

$$\begin{aligned}
\mathrm{id} \cdot e_1 &= \mu(\mathrm{id}; \mathrm{id}) \cdot \mu(\mathrm{id}; e_1) \\
&= \mu(\mathrm{id} \cdot \mathrm{id}; \mathrm{id} \cdot e_1) \\
&= \mu(\mathrm{id} \cdot \mathrm{id}; \mathrm{id}) \\
&= \mathrm{id} \cdot \mathrm{id}
\end{aligned}$$

using that id is the identity element for operadic multiplication, the $n = 1$ action operad axiom explained above, that e_1 is the identity for group multiplication, and that id is the identity for operadic multiplication again. Therefore $\mathrm{id} \cdot e_1 = \mathrm{id} \cdot \mathrm{id}$ as desired, and $e_1 = \mathrm{id}$.

For the second claim, we write $\mu(e_n; e_{i_1}, \ldots, e_{i_n})$ as $\mu(e; \underline{e})$, and consider the square of this element. We find that

$$\begin{aligned}
\mu(e; \underline{e}) \cdot \mu(e; \underline{e}) &= \mu(e \cdot e; \underline{e} \cdot \underline{e}) \\
&= \mu(e; \underline{e}),
\end{aligned}$$

where the first equality follows from the last action operad axiom together with the fact that e gets mapped to the identity permutation; here $\underline{e} \cdot \underline{e}$ is the sequence $e_{i_1} \cdot e_{i_1}, \ldots, e_{i_n} \cdot e_{i_n}$. Thus $\mu(e; \underline{e})$ is an idempotent element of the group $\Lambda(I)$, so must be the identity element e_I.

For the final claim, note that the specific operadic multiplication map $\mu \colon \Lambda(1) \times \Lambda(1) \to \Lambda(1)$ is a group homomorphism following from the action operad axioms, and $\mathrm{id} = e_1$ is a two-sided unit, so Proposition 4.11 shows that μ is actually group multiplication and that $\Lambda(1)$ is abelian. □

Lemma 4.13. *Let Λ be an action operad, and $g_i \in \Lambda(k_i)$ for $i = 2, \ldots, n$. Then*

$$\mu(e_n; e_0, g_2, \ldots, g_n) = \mu(e_{n-1}; g_2, \ldots, g_n).$$

Similarly, $\mu(e_n; h_1, \ldots, h_{n-1}, e_0) = \mu(e_{n-1}; h_1, \ldots, h_{n-1})$ for any $h_i \in \Lambda(k_i)$ for $i = 1, \ldots, n-1$.

Proof. We will only check the first claim, as the second follows by analogous calculations. The equalities

$$\begin{aligned}
\mu(e_n; e_0, g_2, \ldots, g_n) &= \mu\big(\mu(e_2; e_1, e_{n-1}); e_0, g_2, \ldots, g_n\big) \\
&= \mu\big(e_2; \mu(e_1; e_0), \mu(e_{n-1}; g_2, \ldots, g_n)\big) \\
&= \mu\big(e_2; e_0, \mu(e_{n-1}; g_2, \ldots, g_n)\big)
\end{aligned}$$

follow from the second part of Lemma 4.12, operadic associativity, and the first part of Lemma 4.12, respectively. Therefore the first equality in the lemma follows from the special case when $n = 2$, the equality

$$\mu(e_2; e_0, g) = g, \tag{2}$$

by subsituting $g = \mu(e_{n-1}; g_2, \ldots, g_n)$. In order to prove Equation (2), we use the same methods as above to obtain

$$\begin{aligned} g &= \mu(e_1; g) \\ &= \mu(\mu(e_2; e_0, e_1); g) \\ &= \mu(\mu(e_2; e_0, e_1); \mu(e_1; g)) \\ &= \mu(e_2; e_0, g). \end{aligned}$$

This calculation verifies Equation (2), and so completes the proof of the first equality in the statement of the lemma. ☐

Corollary 4.14. *Let Λ be an action operad. For any $g, h \in \Lambda(0)$, the equation*

$$g \cdot h = \mu(e_2; g, h)$$

holds. As a consequence, $\Lambda(0)$ is abelian.

Proof. The function $\Lambda(0) \times \Lambda(0) \to \Lambda(0)$ given by

$$g, h \mapsto \mu(e_2; g, h)$$

is a group homomorphism by the action operad axiom Equation (1) as we verify below.

$$\mu(e_2; g', h') \cdot \mu(e_2; g, h) = \mu(e_2 \cdot e_2; g' \cdot g, h' \cdot h) = \mu(e_2; g' \cdot g, h' \cdot h)$$

In order to apply Proposition 4.11 and conclude that $g \cdot h = \mu(e_2; g, h)$, we must verify that

$$\mu(e_2; e_0, g) = g = \mu(e_2; g, e_0)$$

for all $g \in \Lambda(0)$, but this follows immediately from Lemma 4.13. Thus the function $\mu(e_2; -, -)$ satisfies the hypotheses in Proposition 4.11. Therefore $g \cdot h = \mu(e_2; g, h)$ and $\Lambda(0)$ is abelian. ☐

The symmetric operad structure on the symmetric groups in Example 3.6 was constructed using the functions β, δ from Definition 2.13 and Definition 2.15, respectively. We are now ready to show that any action operad can be described in this way, as promised in the introductory remarks to this section.

Theorem 4.15. *An action operad Λ determines, and is uniquely determined by, the following:*

- *groups $\Lambda(n)$ together with group homomorphisms $\pi_n \colon \Lambda(n) \to \Sigma_n$,*
- *a group homomorphism*

$$\Lambda(k_1) \times \cdots \times \Lambda(k_n) \xrightarrow{\beta} \Lambda(k_1 + \cdots + k_n),$$

for each $n > 0$ and $k_1, \ldots, k_n$, and

Used to have: together with the degenerate case of $n = 0$ which then is a group homomorphism $1 \to \Lambda(0)$. Do we need this?

Think the $n = 0$ case just tells us that $e_0 \cdot e_0 = e_0$?

- *a function of sets*
$$\Lambda(n) \xrightarrow{\delta_{n;k_1,\ldots,k_n}} \Lambda(k_1 + \cdots + k_n)$$
for each $n, k_1, \ldots, k_n$*,*

subject to the axioms below. In what we write below, we use the following notational conventions.

- *The symbols* f, g, h*, with or without subscripts, always refer to an element of some group* $\Lambda(n)$*.*
- *The symbols* j, k, m, n, p *are all natural numbers, and* i *is a natural number between 1 and* n*.*

Axioms:

1. *The homomorphisms* β *are natural with respect to the maps* π_n*, where* $K = k_1 + \cdots + k_n$*.*
$$\begin{tikzcd} \Lambda(k_1) \times \cdots \times \Lambda(k_n) \arrow[r, "\beta"] \arrow[d, "\pi_1 \times \cdots \times \pi_n"'] & \Lambda(K) \arrow[d, "\pi"] \\ \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \arrow[r, "\beta"'] & \Sigma_K \end{tikzcd}$$
2. *The homomorphism* $\beta \colon \Lambda(k) \to \Lambda(k)$ *is the identity.*
3. *The homomorphisms* β *are associative in the sense that the equation*
$$\beta(\underline{h_1}, \ldots, \underline{h_n}) = \beta(\beta(\underline{h_1}), \ldots, \beta(\underline{h_n}))$$
holds, where $\underline{h_i} = h_{i1}, \ldots, h_{ij_i}$*.*
4. *The functions* $\delta_{n;k_1,\ldots,k_n}$ *are natural with respect to the maps* π_n*, where* $K = k_1 + \cdots + k_n$*.*
$$\begin{tikzcd} \Lambda(n) \arrow[r, "\delta"] \arrow[d, "\pi"'] & \Lambda(k_1 + \cdots + k_n) \arrow[d, "\pi"] \\ \Sigma_n \arrow[r, "\delta"'] & \Sigma_{k_1 + \cdots + k_n} \end{tikzcd}$$
5. *The function* $\delta_{n;1,\ldots,1} \colon \Lambda(n) \to \Lambda(n)$ *is the identity. The function* $\delta_{1;n} \colon \Lambda(1) \to \Lambda(n)$ *maps* e_1 *to* e_n*.*

   I think something more general is true: any δ maps e_n to e_K
6. *The equation* $\delta_{n;k_1,\ldots,k_n}(g)\delta_{n;j_1,\ldots,j_n}(h) = \delta_{n;j_1,\ldots,j_n}(gh)$ *holds when*
$$k_i = j_{h^{-1}(i)}.$$
7. *The functions* δ *are associative in the sense that the equation*
$$\delta_{m_1+\cdots+m_n;\underline{p_1},\ldots,\underline{p_n}}(\delta_{n;m_1,\ldots,m_n}(g)) = \delta_{n;P_1,\ldots,P_n}(g)$$
holds, where $P_i = p_{i1} + \cdots + p_{im_i}$ *and* $\underline{p_i} = p_{i1}, \ldots, p_{im_i}$*.*
8. *The equation*
$$\delta_{n;k_1,\ldots,k_n}(g)\beta(h_1, \ldots, h_n) = \beta(h_{g^{-1}(1)}, \ldots, h_{g^{-1}(n)})\delta_{n;k_{g^{-1}(1)},\ldots,k_{g^{-1}(n)}}(g)$$
holds, where $h_i \in \Lambda(k_i)$*.*

9. *The equation*

$$\beta(\delta_1(g_1), \ldots, \delta_n(g_n)) = \delta_c(\beta(g_1, \ldots, g_n))$$

holds, where $\delta_i(g_i)$ *is shorthand for* $\delta_{k_i;m_{i1},\ldots,m_{ik_i}}(g_i)$ *and* δ_c *is shorthand for*

$$\delta_{k_1+\cdots+k_n;m_{11},m_{12},\ldots,m_{1k_1},m_{21},\ldots,m_{nk_n}}.$$

Proof. Let Λ be an action operad, and define

$$\beta(g_1, \ldots, g_n) = \mu(e_n; g_1, \ldots, g_n),$$
$$\delta_{n;k_1,\ldots,k_n}(g) = \mu(g; e_{k_1}, \ldots, e_{k_n}).$$

Since $\pi\colon \Lambda \to \Sigma$ is an operad map, Axioms 1 and 4 hold by the definition of the operad structure on Σ in Example 3.6. Since Λ is an operad of sets, Axioms 2 and 5 follow from the operad unit axioms and the first part of Lemma 4.12, and Axioms 3, 7, and 9 follow from the operad associativity axiom and the second part of Lemma 4.12. Axioms 6 and 8 are special cases of the additional action operad axiom, as is the fact that β is a group homomorphism.

Conversely, given the data above, we need only define the operad multiplication, verify the operad unit and multiplication axioms, and finally check the action operad axiom. Multiplication is given by

$$\mu(g; h_1, \ldots, h_n) = \delta_{n;k_1,\ldots,k_n}(g)\beta(h_1, \ldots, h_n) \tag{3}$$

where $h_i \in \Lambda(k_i)$. The identity element id for the operad structure is $e_1 \in \Lambda(1)$.

We now verify the operad unit axioms. Let $g, h \in \Lambda(n)$. Then

$$\begin{aligned}
\mu(e_1; g) &= \delta(e_1)\beta(g) \\
&= e_1 \cdot g \\
&= g, \\
\mu(h; e_1, \ldots, e_1) &= \delta_{n;1,\ldots,1}(h)\beta(e_1, \ldots, e_1) \\
&= h \cdot e_n \\
&= h
\end{aligned}$$

by Axioms 2 and 5, together with the fact that β is a group homomorphism. Thus e_1 satisfies the identity axioms for operadic multiplication.

For the operad associativity axiom, let

- $f \in \Lambda(m)$,
- $g_i \in \Lambda(n_i)$ for $i = 1, \ldots, m$, and
- $h_{ij} \in \Lambda(p_{i,j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$.

Further, let $P_i = p_{i1} + \cdots + p_{in_i}$ and $\underline{h_i}$ denote the list $h_{i1}, h_{i2}, \ldots, h_{in_i}$. We must then show that

$$\mu\big(f; \mu(g_1; \underline{h_1}), \ldots, \mu(g_m; \underline{h_m})\big) = \mu\big(\mu(f; g_1, \ldots, g_m); \underline{h_1}, \ldots, \underline{h_m}\big).$$

By definition, the left side of this equation is

$$\delta_{m;P_1,\ldots,P_m}(f)\beta\big(\mu(g_1; \underline{h_1}), \ldots, \mu(g_m; \underline{h_m})\big),$$

and

$$\mu\left(g_i; \underline{h_i}\right) = \delta_{n_i;p_{i1},\ldots,p_{in_i}}(g_i)\beta\left(h_{i1}, \ldots, h_{in_i}\right).$$

From this point, we suppress subscripts on the δ's. Since β is a group homomorphism, we can then rewrite the left side as

$$\delta(f)\beta\big(\delta(g_1),\ldots,\delta(g_m)\big)\beta\big(\beta(\underline{h_1}),\ldots,\beta(\underline{h_m})\big)$$

where we have suppressed the subscripts on the δ's. By Axiom 3,

$$\beta\big(\beta(\underline{h_1}),\ldots,\beta(\underline{h_m})\big)=\beta\left(\underline{h_1},\ldots,\underline{h_m}\right).$$

Furthermore, Axiom 9 above shows that

$$\beta\big(\delta(g_1),\ldots,\delta(g_m)\big)=\delta\big(\beta(g_1,\ldots,g_m)\big).$$

Thus we have shown that the left side of the operad associativity axiom is equal to

$$\delta(f)\delta\big(\beta(g_1,\ldots,g_m)\big)\beta\left(\underline{h_1},\ldots,\underline{h_m}\right).$$

Now the right side is

$$\mu\big(\mu(f;g_1,\ldots,g_m);\underline{h_1},\ldots,\underline{h_m}\big),$$

which is by definition

$$\delta\big(\mu(f;g_1,\ldots,g_m)\big)\beta\left(\underline{h_1},\ldots,\underline{h_m}\right).$$

Cancelling the $\beta\left(\underline{h_1},\ldots,\underline{h_m}\right)$ terms, verifying the operad associativity axiom reduces to showing

$$\delta(f)\delta\big(\beta(g_1,\ldots,g_m)\big)=\delta\big(\mu(f;g_1,\ldots,g_m)\big). \tag{4}$$

By the definition of μ,

$$\delta\big(\mu(f;g_1,\ldots,g_m)\big)=\delta\big(\delta(f)\beta(g_1,\ldots,g_m)\big)$$

which is itself equal to

$$\delta\big(\delta(f)\big)\delta\big(\beta(g_1,\ldots,g_m)\big) \tag{5}$$

by Axiom 6 above. Now the $\delta(f)$ on the left side of Equation (4) uses $\delta_{n;P_1,\ldots,P_n}$, while the $\delta(\delta(f))$ in Equation (5) is actually

$$\delta_{m_1+\cdots+m_n;q_{ij}}(\delta_{n;m_1,\ldots,m_n}(f))$$

where the q_{ij} are defined, by Axiom 6, to be given by

$$q_{ij}=p_{i,g_i^{-1}(j)}$$

using the compatibility of β and π in Axiom 1. By Axiom 7, this composite of δ's is then $\delta_{n;Q_1,\ldots,Q_n}$ where $Q_i=q_{i1}+\cdots+q_{im_i}$. But by the definition of the q_{ij}, we immediately see that $Q_i=P_i$, so the $\delta(f)$ in Equation (4) is equal to the $\delta(\delta(f))$ appearing in Equation (5), concluding the proof of the operad associativity axiom.

Writing $\mu(g;\underline{h})=\mu\left(g;h_1,\ldots,h_n\right)=$ and $\mu(g';\underline{h'})=\mu\left(g';h_1',\ldots,h_n'\right)$, the action operad axiom is now the calculation below, and uses Axioms 4 and 8.

$$\begin{aligned}\mu(g;\underline{h})\mu(g';\underline{h'})&=\delta\left(g\right)\beta\left(h_1,\ldots,h_n\right)\delta\left(g'\right)\beta\left(h_1',\ldots,h_n'\right)\\&=\delta\left(g\right)\delta\left(g'\right)\beta\left(h_{g'(1)},\ldots,h_{g'(n)}\right)\beta\left(h_1',\ldots,h_n'\right)\\&=\delta\left(gg'\right)\beta\left(h_{g'(1)}h_1',\ldots,h_{g'(n)}h_n'\right)\\&=\mu\left(gg';h_{g'(1)}h_1',\ldots,h_{g'(n)}h_n'\right)\end{aligned}$$

□

Proposition 4.16 (Corollary 2.17, [49]). *Let Λ be an action operad. Then the homomorphisms $\pi_n\colon \Lambda(n) \to \Sigma_n$ are either all surjective or all the zero map.*

Proof. We will prove each case separately. The two cases coincide for $n = 0, 1$ as both Σ_0, Σ_1 are the trivial group and therefore any homomorphism with one of them as its codomain is both surjective and the zero map. Since Σ_2 only has one non-identity element, any homomorphism $G \to \Sigma_2$ must necessarily be surjective or the zero map.

Suppose that $\pi_2\colon \Lambda(2) \to \Sigma_2$ is surjective, so there exists $g \in \Lambda(2)$ such that $\pi_2(g) = (1\ 2)$. Let $n > 2$. Since Σ_n is generated by the adjacent transpositions $(a\ a+1)$, we will show that each such element is in the image of π_n. Write $\underline{x}^i$ for the i-tuple $x, x, \ldots, x$. Then $(a\ a+1) = \beta(\underline{e_1}^{a-1}, (1\ 2), \underline{e_1}^{n-a-1})$ in Σ, so

$$\begin{aligned}(a\ a+1) &= \beta(\underline{e_1}^{a-1}, (1\ 2), \underline{e_1}^{n-a-1})\\ &= \beta(\underline{\pi_1(e_1)}^{a-1}, \pi_2(g), \underline{\pi_1(e_1)}^{n-a-1})\\ &= \pi_n(\beta(\underline{e_1}^{a-1}, g, \underline{e_1}^{n-a-1}))\end{aligned}$$

by Axiom 1 of 4.15. Thus π_n is surjective for all $n > 2$ if π_2 is surjective.

Now we will consider the case where π_2 is the zero map. Suppose that there exists $g \in \Lambda(n)$ such that $\pi_n(g) = \sigma \neq e_n$ in Σ_n. Then we can find $1 \leq i < j \leq n$ such that $\sigma(j) < \sigma(i)$. Consider the element

$$h = \delta_{n;\underline{0}^{i-1},1,\underline{0}^{j-i-1},1,\underline{0}^{n-j}}(g) \in \Lambda(2).$$

By the assumption that π_2 is the zero map, we must have that $\pi_2(h) = e_2$, but by Axiom 4 of 4.15 we also compute

$$\pi_2(h) = \delta_{n;\underline{0}^{i-1},1,\underline{0}^{j-i-1},1,\underline{0}^{n-j}}\big(\pi_n(g)\big) = \delta_{n;\underline{0}^{i-1},1,\underline{0}^{j-i-1},1,\underline{0}^{n-j}}\big(\sigma\big).$$

The element $\delta_{n;\underline{0}^{i-1},1,\underline{0}^{j-i-1},1,\underline{0}^{n-j}}\big(\sigma\big)$ is equal to $(1\ 2)$ by the choice of i, j and Definition 2.15. These two computations of $\pi_2(h)$ are in contradiction, so there must be no such $g \in \Lambda(n)$. Thus if π_2 is the zero map, so is π_n for all $n > 2$.

□

5 Examples

In this section, we expand our collection of examples and non-examples of action operads. In all but one case, Example 5.4, the examples we provide have appeared elsewhere. The non-examples we provide were largely sourced from questions received after preliminary talks on this research by the authors.

Example 5.1 (Action operad of braid groups). One can form an operad B where $B(n)$ is the underlying set of the nth braid group, B_n. We define the operad structure using the functions β, δ from Remark 3.13. Yau checks that these groups and functions satisfy the axioms of an action operad in [48, Prop 5.2.5], but we note that each of the nine axioms in Theorem 4.15 follows immediately by using the geometric definitions of β, δ.

Example 5.2 (Action operad of ribbon braid groups).

new:

For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is defined to be the semidirect product $\mathbb{Z}^n \trianglelefteq B_n$, where the action of B_n on $\mathbb{Z}^n$ is given, using the underlying permutation of a braid γ, by the formula

Okay, I think there are some issues with inverses in all the literature, or maybe everyone is secretly using a right action. I am attempting to fix that, but maybe no one cares

$$\gamma \cdot (a_1, \ldots, a_n) = (a_{\gamma^{-1}(1)}, \ldots, a_{\gamma^{-1}(n)}).$$

Alternatively, RB_n can be described as the group of isotopy classes of braids equipped with a framing. A purely algebraic presentation of RB_n is given by generators

ref on the framed braid group papers on the nlab page

$$\sigma_1, \ldots, \sigma_{n-1}, t_1, \ldots, t_n$$

where

- the σ_i are the usual braid generators and satisfy the relations of the braid group, and
- the t_i are the *full twists* and satisfy the additional equations

$$\begin{aligned} t_i t_j &= t_j t_i, \\ \sigma_i t_j &= t_{\sigma_i^{-1}(j)} \sigma_i \end{aligned}$$

  for all i, j.

check these equations agree with yours below, and if they don't figure out if yours also have the issue about inverses from above

old:

For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, \ldots, t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$\begin{aligned} b_i \cdot t_i &= t_{i+1} \cdot b_i, \\ b_i \cdot t_{i+1} &= t_i \cdot b_i. \end{aligned}$$

The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in **??**, but with some additional rules for dealing with twists. With regards to the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i = e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} = e_n, \qquad t_{(m+m')} = \big(t_{(m)} \otimes t_{(m')}\big) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Example 5.3 (Action operad of cactus groups). The operad of n-fruit cactus groups defined by Henriques and Kamnitzer in [14] has an action operad structure that we will discuss in Section 16.

Example 5.4 (Action operad from an abelian group). Every abelian group A gives rise to an action operad $A^\bullet$ as follows. The group $A^\bullet(n)$ is the direct sum of n copies of A, A^n. The identity element is required to be $e \in A^1$, and the multiplication is defined by

$$\mu((a_1, \ldots, a_n); \underline{b_1}, \ldots, \underline{b_n}) = (a_1 + \underline{b_1}, a_2 + \underline{b_2}, \ldots, a_n + \underline{b_n})$$

where $\underline{b_i}$ is the string $b_{i1}, \ldots, b_{ik_i}$, and $a_i + \underline{b_i}$ is

$$a_i + b_{i1}, a_i + b_{i2}, \ldots, a_i + b_{ik_i}.$$

The map $\pi_n \colon A^\bullet(n) \to \Sigma_n$ is the zero map.

The characterisation of action operads in terms of maps π, β, and δ as in the above Theorem 4.15 allows us to more easily check for counterexamples, as we show in the latter examples below. Some of these, such as the cyclic groups, reflexive groups, and hyperoctahedral groups, do however form crossed simplicial groups as described in Remark 5.7.

Example 5.5 (Non-examples: subgroups of symmetric groups). By Proposition 4.16, the only action operad $\pi \colon \Lambda \to \Sigma$ for which the homomorphisms π_n are injective but not surjective is the action operad of trivial groups. Thus there is no family of proper, nontrivial subgroups of the symmetric groups that admits an action operad structure. In particular, the families of cyclic groups $\{C_n\}$, reflexive groups $\Lambda(n) = C_2$ of [28], and alternating groups $\{A_n\}$ do not admit action operad structures.

Example 5.6 (Non-example: hyperoctahedral groups). Copied:

In Example 2.28 of [49], Zhang describes one way in which the sequence of hyperoctahedral groups $H_n = C_2 \wr \Sigma_n$ do not form an action operad. We clarify that counterexample here, while also describing another. The group H_n can be described in many ways: as permutations π of the set $\{-n, 1-n, \ldots, -1, 1, \ldots, n-1, n\}$ such that $\pi(i) = -\pi(-i)$ for all i, as the subgroup of $O(n)$ consisting of those matrices with all integer coefficients, or as invertible $n \times n$-matrices whose entries consist of -1, 0, or 1 and in which each row and column has exactly one non-zero entry.

In order to describe the hyperoctahedral groups as an action operad, we could use Theorem 4.15 and define maps π, β, and δ. The obvious map $\pi_n \colon H_n \to \Sigma_n$ takes the 'absolute value' of a signed permutation matrix, giving back the underlying permutation. It seems obvious then to define β to be the block sum of signed permutation matrices in much the same way as for the symmetric groups.

For the maps δ there seem to be two sensible options to try. The first captures Zhang's counterexample by first taking r_n to be the order-reversing signed permutation matrix where all entries are -1, i.e., the $n \times n$ matrix with -1 in every entry of the anti-diagonal. Then we can define $\delta_{n;k_1,\ldots,k_n}(\sigma)$ to be the block sum $\beta(r_{k_1}, \ldots, r_{k_n})$ acted on by the product of r_n and σ. For example,

$$\begin{aligned}
\delta_{3;2,1,3}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) &= \left(r_3 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) * \begin{bmatrix} r_2 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} r_2 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -r_1 & 0 \\ 0 & 0 & r_3 \\ r_2 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

In particular, this gives $\delta_{1;n}([-1]) = r_n$ as in [49]. Taking $\sigma = ((2\ 3); -1, 1, -1)$, as above, we can show that Axiom 8 fails in Theorem 4.15. The left-hand side of the axiom would be $\delta_{1;3}([-1])\beta(\sigma) = r_n \cdot \sigma = ((1\ 3\ 2); 1, -1, 1)$. However, the right-hand side of the axiom would be $\beta(\sigma)\delta_{1;3}([-1]) = \sigma \cdot r_n = ((1\ 2\ 3); 1, -1, 1)$. Clearly defining $\delta_{n;k_1,\ldots,k_n}(\sigma) = (\sigma \cdot r_n) * \beta(r_{k_1}, \ldots, r_{k_n})$ would run into the same problem.

I got confused reading this, the calculations don't match what I expect

fixed, I think? and cut down a bit

An alternative way of defining δ is to take $\delta_{n;k_1,\ldots,k_n}(\sigma) = \sigma * \beta(e_{k_1}, \ldots, e_{k_n})$, without involving the order-reversing permutation r_n, having the effect of making $\delta_{1;n}([-1]) = -I_n$. This then does satisfy Axiom 8, but fails Axiom 6 instead; working through the following counterexample shows this to be the case:

$$\delta_{3;2,1,3}\left(\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\right)\delta_{3;3,1,2}\left(\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}\right)$$
$$\neq \delta_{3;3,1,2}\left(\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{bmatrix}\right).$$

Remark 5.7. The crossed simplicial groups of Krasauskas [28] and Fiedoriwicz and Loday [11] are related to action operads in the following way. On objects, we define a functor $C\colon \mathbf{AOp} \to \mathbf{CSGrp}$ from the category of action operads to the category of crossed simplicial groups by defining $C(\Lambda)(n) = \Lambda(n+1)$. The face and degeneracy maps of the underlying simplicial structure are defined using the operadic composition of Λ as in [27, Construction 1.1] or [3, Section 2]. On morphisms, something. This functor is neither faithful nor conservative.

Why do we shift the index here? Doesn't this cause us to lose informatino about $\Lambda(0)$?

is it just $C(f)(n) = f_{n+1}$? That seems conservative to me

6 Action Operads as Extensions

In this short section, we situate action operads between operads in the category of groups and symmetric operads. We prove two main results. In Proposition 6.3, we prove that operads in the category of groups are precisely the same as action operads for which the homomorphisms π_n are all the zero map; these are called "non-crossed group operads" in [49]. We then turn to studying kernels, images, and short exact sequences of action operads. We finally prove, in Corollary 6.8, that every action operad with surjective π_n's can be expressed as an extension of the action operad Σ by an operad in the category of groups.

Remark 6.1 (Operads in the category of groups). The category **Grp** of groups and group homomorphisms is symmetric monoidal using the cartesian product of groups. Thus we can form the category of operads in the category of groups, denoted $\mathbf{Op}(\mathbf{Grp})$, as in Remark 3.10. The objects of this category are operads P in **Sets** with the additional data of a group structure on each $P(n)$ such that operadic multiplication is a group homomorphism and $\mathrm{id} = e_1$ in $P(1)$; morphisms $f\colon P \to Q$ are those maps of operads such that $f_n\colon P(n) \to Q(n)$ is a group homomorphism for each n.

Proposition 6.2. *Let P be an operad in* **Grp***. Then there is an action operad, denoted $Z(P)$, with*

- *$Z(P)(n) = P(n)$,*
- *the same operadic multiplication as P, and*
- *each $\pi_n\colon P(n) \to \Sigma_n$ the zero map.*

Furthermore, if Λ is an action operad for which each π_n is the zero map, then the groups $\Lambda(n)$ define an operad in **Grp** *using the operadic multiplication of Λ.*

Proof. It is easy to verify, using Equation (1) of Definition 4.1, that the operadic multiplication μ of an action operad is a group homomorphism if and only if π_n is zero for all n. □

Proposition 6.3. *The assignment on objects $P \mapsto Z(P)$ extends to a functor*

$$Z\colon \mathbf{Op}(\mathbf{Grp}) \to \mathbf{AOp}.$$

This functor is full, faithful, and its image at the level of objects is precisely the collection of action operads Λ for which each π_n is the zero map.

Proof. Let $f\colon P \to Q$ be a morphism in $\mathbf{Op}(\mathbf{Grp})$, meaning that f consists of a family of group homomorphisms $f_n\colon P(n) \to Q(n)$ that define a map of operads. Define $Z(f)_n = f_n$. We must check that these functions define a map of action operads; functoriality will follow immediately, as composition and identities in both $\mathbf{Op}(\mathbf{Grp})$ and $\mathbf{AOp}$ are given levelwise. Since each action operad $Z(P)$ has π_n the zero map for all n, the first numbered axiom in Definition 4.7 is satisfied trivially. The second numbered axiom follows from the definition of a morphism in $\mathbf{Op}(\mathbf{Grp})$. This completes the proof that $Z(f)$ is a map of action operads, and the same reasoning shows that every map of action operads $g\colon Z(P) \to Z(Q)$ is $Z(g')$ for a unique $g'\colon P \to Q$ in $\mathbf{Op}(\mathbf{Grp})$, thus Z is full and faithful. $\square$

Proposition 6.4. *Let $f\colon \Lambda \to \Lambda'$ be a map of action operads.*

1. *The groups*
$$\operatorname{Ker} f_n = \{x \in \Lambda(n) \;:\; f(x) = e_n\}$$
form an action operad for which the inclusion $\operatorname{Ker} f \hookrightarrow \Lambda$ is a map of action operads.
2. *The groups*
$$\operatorname{Im} f_n = \{f(x) \;:\; x \in \Lambda(n)\}$$
form an action operad for which the inclusion $\operatorname{Im} \pi \hookrightarrow \Lambda'$ is a map of action operads.

Proof. For the first part, we start by defining $\pi_n^{\operatorname{Ker} f}\colon \operatorname{Ker} f_n \to \Sigma_n$ as the composite group homomorphism

$$\operatorname{Ker} f_n \hookrightarrow \Lambda(n) \xrightarrow{\pi_n^{\Lambda}} \Sigma_n.$$

Since $\pi^{\Lambda} = \pi^{\Lambda'} \circ f$, the composites $\pi_n^{\operatorname{Ker} f}$ are all the zero map. Next we verify that the subgroups $\operatorname{Ker} f_n$ are closed under operadic multiplication. Let $y \in \operatorname{Ker} f_n$ and $x_i \in \operatorname{Ker} f_{k_i}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} f\big(\mu(y; x_1, \ldots, x_n)\big) &= \mu\big(f(y); f(x_1), \ldots, f(x_n)\big) \\ &= \mu(e_n; e_{k_1}, \ldots, e_{k_n}) \\ &= e_{k_1 + \cdots + k_n} \end{aligned}$$

by the assumption that f is a map of operads, that y and each x_i is in the kernel, and Lemma 4.12, showing that the kernel subgroups are closed under operadic multiplication. The operadic identity $\mathrm{id} \in \Lambda(1)$ is an element of $\operatorname{Ker} f_1$ because it is equal to e_1 by Lemma 4.12. Thus the groups $\operatorname{Ker} f_n$ form a sub-operad of Λ, and the action operad axiom Equation (1) of Definition 4.1 for $\operatorname{Ker} f$ follows immediately from the same axiom for Λ. This completes the proof of the first claim, and in fact shows, via Proposition 6.2, that these groups constitute an operad in $\mathbf{Grp}$.

For the second part, we start by defining $\pi_n^{\operatorname{Im} f}\colon \operatorname{Im} f_n \to \Sigma_n$ as the composite group homomorphism

$$\operatorname{Im} f_n \hookrightarrow \Lambda'(n) \xrightarrow{\pi_n^{\Lambda'}} \Sigma_n.$$

These subgroups are closed under operadic multiplication in Λ' using that f is a map of action operads. The operadic identity $\mathrm{id} \in \Lambda'(1)$ is an element of $\operatorname{Im} f_1$ because it is equal to e_1 by Lemma 4.12. This completes the proof that the groups $\operatorname{Im} f_n$ form a sub-operad of Λ', and the action operad axiom Equation (1) of Definition 4.1 for $\operatorname{Im} f$ follows immediately from the same axiom for Λ', finishing the proof of the second claim. $\square$

Example 6.5 (Action operads of pure braids, pure ribbon braids). The nth pure braid group, PB_n, is defined as the kernel of the homomorphism $\pi_n \colon B_n \to \Sigma_n$, or equivalently as the subgroup of the nth braid group consisting of those braids with underlying permutation the identity. Proposition 6.4 gives a simple proof that the pure braid groups form an operad in the category of groups. Similarly, the pure ribbon braid group, PRB_n, is defined as the kernel of $\pi_n \colon RB_n \to \Sigma_n$, and these groups also constitute an operad in the category of groups.

Remark 6.6 (Kernels and images of π). We note that if (Λ, π) is an action operad, then we can apply the results of Proposition 6.4 to π by Proposition 4.10. The action operad $\operatorname{Ker} \pi_n$ will then be an operad in groups, and the action operad $\operatorname{Im} \pi_n$ will be a sub-action operad of Σ. By Proposition 4.16, this means that the action operad $\operatorname{Im} \pi_n$ is either Σ or the trivial action operad T (Example 4.5).

Definition 6.7. A *short exact sequence of action operads* consists of action operads $\Lambda_1, \Lambda_2, \Lambda_3$ and maps of action operads $f \colon \Lambda_1 \to \Lambda_2$, $g \colon \Lambda_2 \to \Lambda_3$ such that

- the action operad $\operatorname{Ker} f$ is the trivial action operad T,
- the action operad $\operatorname{Im} f$ is the action operad $\operatorname{Ker} g$, and
- the action operad $\operatorname{Im} g$ is the action operad Λ_3.

We denote such a short exact sequence as

$$T \to \Lambda_1 \xrightarrow{f} \Lambda_2 \xrightarrow{g} \Lambda_3 \to T,$$

and we say that a short exact sequence exhibits Λ_2 *as an extension of* Λ_3 *by* Λ_1.

The following corollary puts Remark 6.6 into the language of short exact sequences and extensions.

Corollary 6.8. *Let (Λ, π) be an action operad and assume that every homomorphism $\pi_n \colon \Lambda(n) \to \Sigma_n$ is surjective. Then there is a short exact sequence of action operads*

$$T \to \operatorname{Ker} \pi \hookrightarrow \Lambda \xrightarrow{\pi} \Sigma \to T.$$

In particular, every action operad Λ is either an operad in groups or an extension of Σ by an operad in groups.

7 Presentations for Action Operads

This section details how to provide presentations for action operads using the theory of locally finitely presentable (lfp) categories. We refer the reader to [1] for a full treatment of lfp categories. Our treatment here diverges slightly from how one might give presentations for symmetric operads because it is necessary to build in the underlying permutation map $\pi \colon \Lambda \to \Sigma$ from the beginning. In the theory of plain operads, the starting point is a collection: sets $\{P(n)\}$ indexed by the natural numbers. In the theory of symmetric operads, these are enhanced to symmetric collections: sets $\{P(n)\}$ indexed by the natural numbers, together with a right action $P(n) \times \Sigma_n \to P(n)$. Our analogue of collections (see Definition 7.1) are now sets $P(n)$, indexed by natural numbers, equipped with functions $\pi_n \colon P(n) \to \Sigma_n$. Thus the natural notion of the arity of an element in an action operad is not a natural number n, but rather a pair (n, σ) where $\sigma \in \Sigma_n$.

Definition 7.1 (Collections over $\mathcal{S}$). Let $\mathcal{S}$ be the set that is the disjoint union of the underlying sets of all the symmetric groups. Then $\mathbf{Sets}/\mathcal{S}$ is the slice category over $\mathcal{S}$ with objects (X, f) where X is a set and $f\colon X \to \mathcal{S}$ and morphisms $(X_1, f_1) \to (X_2, f_2)$ are those functions $g\colon X_1 \to X_2$ such that $f_1 = f_2 g$. We call an object (X, f) a *collection over* $\mathcal{S}$, and say that an element $x \in X$ has *underlying permutation* σ if $f(x) = \sigma$.

Notation 7.2. If (X, f) is a collection over $\mathcal{S}$, we write $X(\sigma)$ for $f^{-1}(\sigma)$. In other words, $X(\sigma)$ is the set of elements of X with underlying permutation $\sigma \in \Sigma_n \subseteq \mathcal{S}$.

Theorem 7.3. *The category* $\mathbf{AOp}$ *of action operads is a variety of* $\mathcal{S}$*-sorted finitary algebras, and therefore is a finitary monadic category over* $\mathbf{Sets}/\mathcal{S}$*. In particular,* $\mathbf{AOp}$ *is locally finitely presentable.*

Proof. In order to prove that $\mathbf{AOp}$ is a variety of $\mathcal{S}$-sorted finitary algebras (henceforth shortened to $\mathbf{AOp}$ *is a variety*), we must define a set $\mathcal{O}$ of operation symbols and a set of equations E such that action operads are the

- collections (X, f) over $\mathcal{S}$,
- equipped with functions

  $$X(\theta)\colon X(\sigma_1) \times \cdots \times X(\sigma_n) \to X(\sigma)$$

  for each operation symbol $\theta \in \mathcal{O}$ of type $\theta\colon \sigma_1, \ldots, \sigma_n \to \sigma$,
- satisfying the equations in E.

The set $\mathcal{O}$ of operation symbols is defined to have the elements given below.

1. For each $\sigma, \tau \in \Sigma_n$, we define an operation symbol

   $$\star[\sigma, \tau]\colon \sigma, \tau \to \sigma\tau,$$

   where the target $\sigma\tau$ is the product of these permutations in Σ_n.
2. For each natural number n, we define an operation symbol

   $$U_n\colon \ \to e_n,$$

   where the source is the empty list of permutations and the target is the identity element $e_n \in \Sigma_n$.
3. For each $\sigma \in \Sigma_n$, we define an operation symbol

   $$i[\sigma]\colon \sigma \to \sigma^{-1}.$$

4. Let μ denote the operadic multiplication in the operad of symmetric groups, Σ, from Example 3.6. For each $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$ for $i = 1, \ldots, n$, we define an operation symbol $\theta[\sigma; \tau_i]$ of type

   $$\theta[\sigma; \tau_i]\colon \sigma, \tau_1, \ldots, \tau_n \to \mu(\sigma; \tau_1, \ldots, \tau_n).$$

The set E of equations is defined to have the elements below.

1. We write $x \star y$ for $\star[\sigma, \tau](x, y)$, where x is a variable of type σ and y is a variable of type τ. For each triple $\rho, \sigma, \tau \in \Sigma_n$, there is an equation

   $$(x \star y) \star z = x \star (y \star z).$$

2. An $\mathcal{O}$-algebra X has, for each n, an element $u_n \in X(e_n)$ given by U_n. For each $\sigma \in \Sigma_n$, there are equations

$$u_n \star x = x,$$
$$x \star u_n = x.$$

3. We write x^{-1} for $i[\sigma](x)$, where x is a variable of type σ. For each $\sigma \in \Sigma_n$, there are equations

$$x^{-1} \star x = u_n,$$
$$x \star x^{-1} = u_n.$$

4. We write $\theta(x; y_1, \ldots, y_n)$ for $\theta[\sigma; \tau_i](x, y_1, \ldots, y_n)$, where x is a variable of type $\sigma \in \Sigma_n$ and for each $i = 1, \ldots, n$ y_i is a variable of type $\tau_i \in \Sigma_{k_i}$. Then for each
   - $\rho \in \Sigma_n$;
   - $\sigma_i \in \Sigma_{k_i}$, for $i = 1, \ldots, n$; and
   - $\tau_{i,j} \in \Sigma_{h_{i,j}}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$;

   there is an equation

$$\theta\big(\theta(x; y_1, \ldots, y_n); z_{1,1}, \ldots, z_{1,k_1}, \ldots, z_{n,k_n}\big) = \theta\big(x; \theta(y_1; z_{1,1}, \ldots, z_{1,k_1}), \ldots, \theta(y_n; z_{n,1}, \ldots, z_{n,k_n})\big).$$

5. For each $\sigma \in \Sigma_n$, there are equations

$$\theta(u_1; x) = x,$$
$$\theta(x; u_1, \ldots, u_1) = x.$$

6. Let x be a variable of type $\sigma \in \Sigma_i$, y_i be a variable of type $\tau_i \in \Sigma_{k_i}$ for $i = 1, \ldots, n$, x' be a variable of type $\sigma' \in \Sigma_n$, and y'_i be a variable of type $\tau'_i \in \Sigma_{\sigma^{-1}(i)}$. Then for each such choice of permutations, there is an equation

$$\theta(x'; y'_1, \ldots, y'_n) \star \theta(x; y_1, \ldots, y_n) = \theta(x' \star x; y'_{\sigma(1)} \star y_1, \ldots, y'_{\sigma(n)} \star y_n).$$

The category of $\mathcal{O}$-algebras satisfying the equations in E is isomorphic to **AOp** as follows. Given such an algebra (X, f), define an action operad (Λ^X, π) by defining

$$\Lambda^X(n) = \coprod_{\sigma \in \Sigma_n} X(\sigma)$$

and defining $\pi_n \colon X(n) \to \Sigma_n$ to be f restricted to $\Lambda^X(n) \subseteq X$. Each $X(n)$ is a group using $\star$ as its multiplication and u_n as its identity element, and π_n is a homomorphism since $\pi_n(x \star y) = \sigma\tau$ when $x \in X(\sigma), y \in X(\tau)$ by the definition of the source and target of $\star$. The operadic multiplication is given by the operation symbols $\theta[\sigma; \tau_i]$, and equations 4 and 5 in E are the operadic associativity and unit axioms, using the first part of Lemma 4.12. The additional action operad axiom is equation 6 in E. A morphism of $\mathcal{O}$-algebras is easily seen to define a map of action operads, and these assignments are an isomorphism between the category **AOp** and the category of $\mathcal{O}$-algebras satisfying the equations in E. This completes the proof that **AOp** is the variety defined by $\mathcal{O}$ and E. It is therefore a finitary monadic category over **Sets**$/\mathcal{S}$ by [1, Thm 3.18] and locally finitely presentable by [1, Cor 3.7]. □

For our purposes, the most important consequence of Theorem 7.3 is that we can freely generate an action operad from a collection over $\mathcal{S}$, as stated below.

Corollary 7.4. *The underlying collection functor* $U\colon \mathbf{AOp} \to \mathbf{Sets}/\mathcal{S}$ *has a left adjoint* $F\colon \mathbf{Sets}/\mathcal{S} \to \mathbf{AOp}$*, the free action operad functor.*

Definition 7.5 (Presentation for action operads). A *presentation* for an action operad Λ consists of

- a pair of collections over $\mathcal{S}$ denoted $\mathbf{g}, \mathbf{r}$,
- a pair of maps $s_1, s_2\colon F\mathbf{r} \to F\mathbf{g}$ between the associated free action operads, and
- a map $p\colon F\mathbf{g} \to \Lambda$ of action operads exhibiting Λ as the coequalizer of s_1, s_2.

Example 7.6 (The presentation for Σ). Here we explicitly give a presentation for the action operad of symmetric groups. Recall that the symmetric group Σ_n has a presentation, as a *group*, with

- generators $\sigma_{1;n}, \ldots, \sigma_{n-1;n}$ and
- relations
  1. $\sigma_{i;n}^2 = e_n$ for all i,
  2. $\sigma_{i;n}\sigma_{j;n} = \sigma_{j;n}\sigma_{i;n}$ for i, j satisfying $|i-j| \geq 2$, and
  3. $\sigma_{i;n}\sigma_{i+1;n}\sigma_{i;n} = \sigma_{i+1;n}\sigma_{i;n}\sigma_{i+1;n}$ for $1 \leq i < n-1$.

Note the nonstandard naming of the generators as $\sigma_{i;n}$ instead of merely σ_i. We have included this additional information in our generators as it is necessary to distinguish between the generator of Σ_2 traditionally denoted σ_1 and the generator of Σ_3 traditionally denoted with the same notation, for example.

Utilizing the operad structure, we notice that

$$\sigma_{i;n} = \beta(e_{i-1}, \sigma_{1;2}, e_{n-i-1}).$$

Furthermore, the second relation above is a consequence of this expression, the fact that β is a group homomorphism, and the second part of Lemma 4.12. Thus as an action operad, Σ is generated by the single element $\sigma_{1;2} \in \Sigma_2$.

We now define the relations for this presentation of Σ as an action operad. The first relation is

$$\sigma^2 = e_2. \tag{6}$$

The second relation is

$$\mu(\sigma; e_1, e_2) = \mu(e_2; e_1, \sigma) \cdot \mu(e_2; \sigma, e_1). \tag{7}$$

We therefore claim that Σ has a presentation given by

- $\mathbf{g} = \{\sigma\}$, defined as a collection over $\mathcal{S}$ by the function sending σ to $(1\ 2) \in \Sigma_2$;
- $\mathbf{r} = \{\rho_1, \rho_2\}$, defined as a collection over $\mathcal{S}$ by the function sending ρ_1 to $e_2 \in \Sigma_2$ and ρ_2 to $(1\ 3\ 2) \in \Sigma_3$;
- $s_1\colon F\mathbf{r} \to F\mathbf{g}$ defined uniquely by requiring

$$\begin{aligned} s_1(\rho_1) &= \sigma^2, \\ s_1(\rho_2) &= \mu(\sigma; e_1, e_2); \end{aligned}$$

- and $s_2\colon F\mathbf{r} \to F\mathbf{g}$ defined uniquely by requiring

$$\begin{aligned} s_2(\rho_1) &= e_2, \\ s_2(\rho_2) &= \mu(e_2; e_1, \sigma) \cdot \mu(e_2; \sigma, e_1). \end{aligned}$$

We note that both s_1, s_2 map ρ_1 to $(1\ 2)(1\ 2) = e_2 \in \Sigma_2$, and both s_1, s_2 map ρ_2 to $(1\ 3\ 2) = (2\ 3)(1\ 2)$, thus defining maps of collections over $\mathcal{S}$.

In order to prove that the above is a presentation for Σ, we must define a map of action operads $t\colon F\mathbf{g} \to \Sigma$ that exhibits Σ as the coequalizer of s_1, s_2. Define t by requiring $t(\sigma) = (1\ 2)$. The calculations above prove that $t \circ s_1 = t \circ s_2$, and now we must prove that t is the universal map of action operads coequalizing s_1, s_2. Let Λ be an action operad, and $f\colon F\mathbf{g} \to \Lambda$ a map of action operads such that $f \circ s_1 = f \circ s_2$. We construct a unique map of action operads $\tilde{f}\colon \Sigma \to \Lambda$ such that $f = \tilde{f} \circ t$. If such an $\tilde{f}$ exists, it must map the transposition $\sigma_{1;2} = (1\ 2) \in \Sigma_2$ to $f(\sigma)$. Since each other generator (of Σ_n as a group) $\sigma_{i;n}$ is the image

$$\sigma_{i;n} = \beta(e_{i-1}, \sigma_{1;2}, e_{n-i-1}) = \mu(e_3; e_{i-1}, \sigma_{1;2}, e_{n-i-1}) \tag{8}$$

of $\sigma_{1;2}$ under an operadic multiplication, any map of action operads $\tilde{f}$ satisfying $f = \tilde{f} \circ t$ is unique if it exists, and $\tilde{f}_n$ must be defined on the generators of Σ_n using Equation (8) by

$$\begin{aligned} \tilde{f}_n(\sigma_{i;n}) &= \tilde{f}\big(\mu(e_3; e_{i-1}, \sigma_{1;2}, e_{n-i-1})\big) \\ &= \mu\big(\tilde{f}_n(e_3); \tilde{f}_n(e_{i-1}), \tilde{f}_n(\sigma_{1;2}), \tilde{f}_n(e_{n-i-1})\big) \\ &= \mu(e_3; e_{i-1}, f(\sigma), e_{n-i-1}). \end{aligned}$$

In order to show that the formula

$$\tilde{f}_n(\sigma_{i;n}) = \mu(e_3; e_{i-1}, f(\sigma), e_{n-i-1})$$

defines a unique homomorphism $\tilde{f}_n\colon \Sigma_n \to \Lambda(n)$, we must check that it respects the relations in the presentation of Σ_n given above. We only check the third axiom, and only in the case $i = 1, n = 3$; the rest we leave as a simple exercise for the reader. In order to verify that $\tilde{f}_3$ respects this relation, we must show that

$$\tilde{f}_3(\sigma_{1;3})\ \tilde{f}_3(\sigma_{2;3})\ \tilde{f}_3(\sigma_{1;3}) = \tilde{f}_3(\sigma_{2;3})\ \tilde{f}_3(\sigma_{1;3})\ \tilde{f}_3(\sigma_{2;3}). \tag{9}$$

By Lemma 4.13, the left side of the above is

$$\mu\big(e_2; f(\sigma), e_1\big)\mu\big(e_2; e_1, f(\sigma)\big)\mu\big(e_2; f(\sigma), e_1\big) = f\big(\mu(e_2; \sigma, e_1)\mu(e_2; e_1, \sigma)\mu(e_2; \sigma, e_1)\big).$$

Since f is a map of action operads and coequalizes s_1, s_2, we obtain

$$f\big(\mu(e_2; \sigma, e_1)\mu(e_2; e_1, \sigma)\mu(e_2; \sigma, e_1)\big) = f\big(\mu(e_2; \sigma, e_1)\mu(\sigma; e_1, e_2)\big)$$

by the equality $fs_1(\rho_2) = fs_2(\rho_2)$. Finally, the action operad axiom shows that

$$f\big(\mu(e_2; \sigma, e_1)\mu(\sigma; e_1, e_2)\big) = f\big(\mu(\sigma; e_1, \sigma)\big).$$

A similar argument shows that the right side of Equation (9) is equal to

$$f\big(\mu(e_2; e_1, \sigma)\mu(e_2; \sigma, e_1)\mu(e_2; e_1, \sigma)\big),$$

and once by coequalizing s_1, s_2 is therefore $f\big(\mu(\sigma; e_1, \sigma)\big)$. We have now verified that $\tilde{f}_n$ respects the relations for the presentation of Σ_n, and therefore defines a unique group homomorphism $\tilde{f}_n\colon \Sigma_n \to \Lambda(n)$.

By Theorem 4.15, to show that the homomorphisms $\tilde{f}_n$ defined by

$$\tilde{f}_n(\sigma_{i;n}) = \mu(e_3; e_{i-1}, f(\sigma), e_{n-i-1})$$

give a map of action operads, it suffices to check that they commute with the operations δ, β and preserve underlying permutations. We sketch this proof below, and leave the routine details to the reader.

- First, check that the equality

$$\tilde{f}_N\big(\beta(\tau_1,\dots,\tau_k)\big) = \beta\big(\tilde{f}_{n_1}(\tau_1),\dots,\tilde{f}_{n_k}(\tau_k)\big)$$

follows from the special case when all the τ_i are identity elements except one, and that τ_i is σ; this reduction uses that β and the $\tilde{f}_{n_k}$'s are homomorphisms. Check that special case using Item 3.

- Second, check that the equality

$$\tilde{f}_N\big(\delta_{n;k_1,\dots,k_n}(\tau)\big) = \delta_{n;k_1,\dots,k_n}\big(\tilde{f}_n(\tau)\big)$$

follows once it is verified in the special cases that $\tau = \sigma_{i;n}$ for some i; this reduction uses that $\tilde{f}_n$ is a homomorphism and Item 6. Then show, using Item 9, that it suffices to check the case of $\sigma_{1;2}$ only.

- Third, we check

$$\tilde{f}_{i+j}\big(\delta_{2;i,j}(\sigma_{1;2})\big) = \delta_{2;i,j}\big(\tilde{f}_2(\sigma_{1;2})\big)$$

by induction. Fixing i and inducting on j, we start on the right and compute

$$\begin{aligned}
\delta_{2;i,j+1}\tilde{f}_2(\sigma_{1;2}) &= \mu\big(\tilde{f}_2(\sigma_{1;2}); e_i, e_{j+1}\big)\\
&= \mu\big(f(\sigma); e_i, e_{j+1}\big)\\
&= \mu\Big(\mu\big(f(\sigma); e_1, e_2\big); e_i, e_1, e_j\Big)\\
&= \mu\Big(f\big(\mu(\sigma; e_1, e_2)\big); e_i, e_1, e_j\Big)\\
&= \mu\Big(f\big(\mu(e_2; e_1, \sigma)\mu(e_2; \sigma, e_1)\big); e_i, e_1, e_j\Big)\\
&= \mu\Big(\mu\big(e_2; e_1, f(\sigma)\big)\mu\big(e_2; f(\sigma), e_1\big); e_i, e_1, e_j\Big)\\
&= \mu\Big(\mu\big(e_2; e_1, f(\sigma)\big); e_1, e_i, e_j\Big)\mu\Big(\mu\big(e_2; f(\sigma), e_1\big); e_i, e_1, e_j\Big)\\
&= \mu\Big(e_2; e_1, \mu\big(f(\sigma); e_i, e_j\big)\Big)\mu\Big(e_2; \mu\big(f(\sigma); e_i, e_1\big), e_j\Big)\\
&= \mu\Big(e_2; e_1, \tilde{f}_{i+j}\big(\mu(\sigma_{1;2}; e_i, e_j)\big)\Big)\mu\Big(e_2; \tilde{f}_{i+1}\big(\mu(\sigma_{1;2}; e_i, e_1)\big), e_j\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(e_2; e_1, \mu(\sigma_{1;2}; e_i, e_j)\big)\Big)\tilde{f}_{i+j+1}\Big(\mu\big(e_2; \mu(\sigma_{1;2}; e_i, e_1), e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(e_2; e_1, \mu(\sigma_{1;2}; e_i, e_j)\big)\mu\big(e_2; \mu(\sigma_{1;2}; e_i, e_1), e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(e_2; e_1, \mu(t(\sigma); e_i, e_j)\big)\mu\big(e_2; \mu(t(\sigma); e_i, e_1), e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(\mu(e_2; e_1, t(\sigma))\mu(e_2; t(\sigma), e_1); e_i, e_1, e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(t(\mu(\sigma; e_1, e_2)); e_i, e_1, e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}\Big(\mu\big(\mu(\sigma_{1;2}; e_1, e_2); e_i, e_1, e_j\big)\Big)\\
&= \tilde{f}_{i+j+1}(\sigma_{1;2}; e_i, e_{j+1})\\
&= \tilde{f}_{i+j+1}\delta_{2;i,j+1}(\sigma_{1;2}).
\end{aligned}$$

The equalities above are derived, in order, from the following:

we could number the equalities above if that would improve clarity

1. the definition of δ,

2. the definition of $\tilde{f}_2$,
3. operad associativity,
4. that f is a map of action operads,
5. that f coequalizes s_1 and s_2,
6. that f is a map of action operads,
7. the action operad axiom Equation (1),
8. operad associativity,
9. induction on j,
10. that $\tilde{f}$ commutes with β,
11. that $\tilde{f}$ is a group homomorphism,
12. the definition $\sigma_{1;2} = t(\sigma)$,
13. the action operad axiom Equation (1),
14. that t is an action operad map coequalizing s_1 and s_2,
15. that t is an action operad map and $\sigma_{1;2} = t(\sigma)$,
16. operad associativity, and
17. the definition of δ.

The argument for fixing j and inducting on i is similar.

- Fourth, we check the base case for induction. When $i = j = 0$, we note that $\delta_{2;0,0}(\sigma_{1;2}) = e_0$. Since $\tilde{f}_0$ is a group homomorphism, we therefore must check that the element $\delta_{2;0,0}\big(\tilde{f}_2(\sigma_{1;2})\big) = \delta_{2;0,0}\big(f(\sigma)\big)$ equals e_0. We do this by showing that $\delta_{2;0,0}\big(f(\sigma)\big) = \mu\big(f(\sigma); e_0, e_0\big)$ squares to itself as follows, using similar methods as above:

$$\begin{aligned}
\mu\big(f(\sigma); e_0, e_0\big) &= \mu\Big(\mu\big(f(\sigma); e_1, e_2\big); e_0, e_0, e_0\Big) \\
&= \mu\Big(\mu\big(e_2; e_1, f(\sigma)\big)\mu\big(e_2; f(\sigma), e_1\big); e_0, e_0, e_0\Big) \\
&= \mu\Big(\mu\big(e_2; e_1, f(\sigma)\big); e_0, e_0, e_0\Big)\mu\Big(\mu\big(e_2; f(\sigma), e_1\big); e_0, e_0, e_0\Big) \\
&= \mu\Big(e_2; e_0, \mu\big(f(\sigma); e_0, e_0\big)\Big)\mu\Big(e_2; \mu\big(f(\sigma); e_0, e_0\big), e_0\Big) \\
&= \mu\big(f(\sigma); e_0, e_0\big)\mu\big(f(\sigma); e_0, e_0\big).
\end{aligned}$$

This concludes the induction argument, and shows that the $\tilde{f}$ maps commute with the δ's. Therefore the $\tilde{f}$'s assemble to define a map of operads.

- Finally, we must check that $\tilde{f}$ is a map of operads over Σ, meaning that $\pi \circ \tilde{f} = 1_\Sigma$. Since every generator $\sigma_{i;n}$ is an operadic composition of $\sigma_{1;2}$ and identity elements, and $\tilde{f}$ preserves operadic composition and identities, it suffices to check the equality
$$\pi \circ \tilde{f}(\sigma_{1;2}) = \sigma_{1;2}.$$
By definition, $\tilde{f}(\sigma_{1;2}) = f(\sigma)$, and $f\colon F\mathbf{g} \to \Lambda$ is a map of action operads, so $\pi\big(f(\sigma)\big) = \sigma_{1;2}$ by the definition of $\mathbf{g}$ as a collection over $\mathcal{S}$. This step completes the proof that the homomorphisms $\tilde{f}_n$ define a map of action operads $\tilde{f}\colon \Sigma \to \Lambda$.

merging part 2 into part 1, move this intro?

In the second part of this paper, we study Λ-operads: these are operads P equipped with group actions of the form $P(n) \times \Lambda(n) \to P(n)$ and satisfying equivariance axioms similar to those for symmetric operads (Definition 3.1). Section 8 presents the definitions of Λ-operads and their algebras.**??** studies the monads associated to Λ-operads. Section 9 shows that Λ-operads are the monoids in a category of Λ-collections via the substitution product.

Remark 7.7.

maybe move this slightly for better visibility/numbering

The material in this part is presented in the case when the Λ-operads P have underlying *sets* $P(n)$, but it can be easily generalized. In fact, **??** II deals almost exclusively when the operad P is an operad in **Cat**. Here are the changes required to account for operads in another monoidal category $\mathcal{V}$.

- $\mathcal{V}$ should be a closed, symmetric monoidal category. It is possible to relax the closed condition to merely requiring that $X \otimes -$, and therefore also $- \otimes X$ by symmetry, preserves colimits.
- Any right group action $P(n) \times \Lambda(n) \to P(n)$ should be interpretted as a monoid homomorphism
$$\Lambda(n)^{op} \to \mathcal{V}\big(P(n), P(n)\big),$$
or equivalently repackaged as an action $P(n) \cdot \Lambda(n) \to P(n)$ using the copower $\cdot$.

8 Λ-Operads and their Algebras

This section presents the definition of a Λ-operad (Definition 8.1), where Λ is an action operad. This definition unifies the various types (non-symmetric, symmetric, and braided) of operads discussed in Section 3 under one umbrella term. The different group actions arise from different choices of Λ. We also define algebras over a Λ-operad in Definition 8.9, and prove a change-of-action operad result in Theorem 8.15.

Definition 8.1 (Λ-operads). Let Λ be an action operad. A Λ-*operad* P (in **Sets**) consists of

- a non-symmetric operad P in **Sets** and
- for each n, an action $P(n) \times \Lambda(n) \to P(n)$ of $\Lambda(n)$ on $P(n)$

such that the following two equivariance axioms hold.

- For each $p \in P(n)$, $q_i \in P(k_i)$, and $g_i \in \Lambda(k_i)$ for $i = 1, \ldots, n$:
$$\mu^P(p; q_1 \cdot g_1, \ldots, q_n \cdot g_n) = \mu^P(p; q_1, \ldots, y_n) \cdot \beta^\Lambda(g_1, \ldots, g_n).$$
- For each $p \in P(n)$, $g \in \Lambda(n)$, and $q_i \in \Lambda(k_i)$ for $i = 1, \ldots, n$:
$$\mu^P(p \cdot g; q_1, \ldots, q_n) = \mu^P\left(p; q_{g^{-1}(1)}, \ldots, q_{g^{-1}(n)}\right) \cdot \delta^\Lambda_{n;k_1,\ldots,k_n}(g).$$

Remark 8.2. Using Equation (3) from the proof of Theorem 4.15, the two equivariance axioms in Definition 8.1 can be combined into the single equality
$$\mu\left(p; p_1, \ldots p_n\right) \mu\left(\tau; \tau_1, \ldots, \tau_n\right) = \mu\left(p \cdot \tau; p_{\tau(1)} \cdot \tau_1, \ldots, p_{\tau(n)} \cdot \tau_n\right).$$

Definition 8.3 (Map of Λ-operads). Let P and Q be Λ-operads. A *map* $f\colon P \to Q$ *of* Λ*-operads* consists of an operad map (Definition 8.3) that is levelwise equivariant with respect to the $\Lambda(n)$-actions, i.e., for each $n \in \mathbb{N}$ the following diagram commutes.

$$\begin{tikzcd} P(n) \times \Lambda(n) \arrow[r, "f_n \times 1"] \arrow[d] & Q(n) \times \Lambda(n) \arrow[d] \\ P(n) \arrow[r, "f_n"'] & Q(n) \end{tikzcd}$$

Proposition 8.4. *There is a category with*

- *objects the* Λ*-operads* P *in* **Sets**,
- *morphisms the maps of* Λ*-operads between them,*
- *identities* $1_P\colon P \to P$ *given by*

$$(1_P)_n = 1_{P(n)}\colon P(n) \to P(n),$$

  and
- *composition given by*

$$(g \circ f)_n = g_n \circ f_n.$$

Notation 8.5 (The category of Λ-operads). The category in Proposition 3.16 is called the *category of* Λ*-operads (in* **Sets***)*, and is denoted Λ-**Op**.

Example 8.6 (Non-symmetric, symmetric, and braided operads expressed as Λ-operads). We can express the non-symmetric, symmetric, and braided operads of Section 3 as the Λ-operads for the appropriate choice of Λ.

1. Let T denote the terminal operad in **Sets** equipped with its unique action operad structure. Then a T-operad is just a non-symmetric operad in **Sets**.
2. Let Σ denote the operad of symmetric groups with $\pi\colon \Sigma \to \Sigma$ the identity map. Then a Σ-operad is a symmetric operad in the category of sets.
3. Let B denote the operad of braid groups with $\pi_n\colon B_n \to \Sigma_n$ the canonical projection of a braid onto its underlying permutation. Then a B-operad is a braided operad in the sense of Fiedorowicz [10].

A further example of a Λ-operad is given by the underlying operad, Λ, of Λ itself.

Proposition 8.7. *Let* Λ *be an action operad. Then the operad* Λ *is itself a* Λ*-operad.*

Proof. The underlying operad Λ is of course an operad in **Sets**. The action $\Lambda(n) \times \Lambda(n) \to \Lambda(n)$ is given simply by the group multiplication in $\Lambda(n)$. The two equivariance axioms are then both instances of the action operad axiom of Λ. $\square$

An operad is intended to be an abstract description of a certain type of algebraic structure, and the particular instances of that structure are the algebras for that operad. We give the general definition first in Definition 8.9, and then recover algebras over non-symmetric, symmetric, and braided operads in Example 8.11.

Remark 8.8. In preparation for the definition of an algebra over a Λ-operad, we make the following two remarks.

1. If Λ is an action operad and X is a set, then $\Lambda(n)$ acts on X^n by
$$\Lambda(n) \times X^n \xrightarrow{\pi_n \times 1} \Sigma_n \times X^n \xrightarrow{\kappa_n} X^n,$$
where κ_n is defined by the formula
$$\kappa_n(\sigma; x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$
Thus we would write
$$g \cdot (x_1, \ldots, x_n) = (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)})$$
using Notation 2.11.
2. Following the previous item, we define $P(n) \otimes_{\Lambda(n)} X^n$ as in Convention 2.7. As the definition of an algebra over P will involve maps with source object $P(n) \otimes_{\Lambda(n)} X^n$, we remind the reader of the tilde notation for maps respecting coequalizers, Convention 2.8.

Definition 8.9 (P-algebras). Let Λ be an action operad, and P be a Λ-operad. An *algebra* for P, or *P-algebra*, consists of a set X together with maps
$$\alpha_n \colon P(n) \otimes_{\Lambda(n)} X^n \to X$$
such that the maps $\tilde{\alpha}_n$ satisfy the following axioms.

1. The element $\mathrm{id} \in P(1)$ is a unit in the sense that
$$\tilde{\alpha}_1(\mathrm{id}; x) = x$$
for all $x \in X$.
2. The maps $\tilde{\alpha}_n$ are associative in the sense that the following diagram commutes.

$$\begin{tikzcd}
P(n) \times \prod_{i=1}^n \left(P(k_i) \times X^{k_i}\right) \arrow[rr, "1 \times \tilde{\alpha}_{k_1} \times \cdots \times \tilde{\alpha}_{k_n}"] \arrow[d, "\cong"'] & & P(n) \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P(n) \times \left(\prod_{i=1}^n P(k_i)\right) \times \prod_{i=1}^n \left(X^{k_i}\right) \arrow[d, "\mu \times 1"'] & & \\
P(\sum k_i) \times X^{\sum k_i} \arrow[rr, "\tilde{\alpha}_{\sum k_i}"'] & & X
\end{tikzcd}$$

Remark 8.10. It is worth reiterating that the equivariance required for a P-algebra is built into the definition above by requiring the existence of the maps α_n to be defined on coequalizers, even though the algebra axioms then only use the maps $\tilde{\alpha}_n$. Since every Λ-operad has an underlying non-symmetric operad (see Theorem 8.15, applied to the unique map $T \to \Lambda$), this reflects the fact that the algebras for the Λ-equivariant version are always algebras for the plain version, but not conversely.

Example 8.11 (Algebras over non-symmetric, symmetric, and braided operads as Λ-operads). We can recover standard notions of algebras over non-symmetric, symmetric, and braided operads as algebras over a Λ-operad.

1. For the action operad T of trivial groups, a T-operad is a non-symmetric operad. The coequalizer $P(n) \otimes_{T(n)} X^n$ is isomorphic to $P(n) \times X^n$, so without loss of generality we can assume that $\tilde{\alpha}_n = \alpha_n$. This recovers the usual notion of an algebra over a non-symmetric operad, see [37, Definition 1.20].

2. For the action operad Σ of symmetric groups, a Σ-operad is a symmetric operad. Definition 8.9 is equivalent to May's original definition [39, Definition 1.1] by Remark 8.8.

3. For the action operad B of braid groups, a B-operad is a braided operad in the sense of Fiedorowicz [10]. Once again, Definition 8.9 is equivalent to Fiedorowicz's definition [10, Definition 3.2] by Remark 8.8.

Definition 8.12 (Map of P-algebras). Let P be a Λ-operad, and let (X, α) and (Y, β) be P-algebras. Then a *map of P-algebras* $f\colon (X, \alpha) \to (Y, \beta)$ is a function $f\colon X \to Y$ such that the following diagram commutes for every n.

$$\begin{tikzcd} P(n) \times X^n \arrow[r, "1 \times f^n"] \arrow[d, "\tilde{\alpha}_n"'] & P(n) \times Y^n \arrow[d, "\tilde{\beta}_n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proposition 8.13. *Let Λ be an action operad and P be a Λ-operad. There is a category with*

- *objects the P-algebras (X, α),*
- *morphisms the maps of P-algebras between them,*
- *identities $1_{(X,\alpha)}\colon (X, \alpha) \to (X, \alpha)$ given by the identities 1_X, and*
- *composition given by composition of the underlying functions.*

Notation 8.14 (The category of P-algebras). The category in Proposition 8.13 is called the *category of P-algebras (in* **Sets***)*, and is denoted P-**Alg**.

The final goal of this section is to recast the category of algebras over a Λ-operad P using the endomorphism operad of Example 3.7. We begin by sketching a change-of-action operad result.

Theorem 8.15. *Let $f\colon \Lambda \to \Lambda'$ be a map of action operads.*

1. *The map f induces a functor $f^*\colon \Lambda'$-**Op** $\to \Lambda$-**Op** with the property that $(f^*Q)(n) = Q(n)$ for every Λ'-operad Q.*

2. *The map f induces a functor $f_!\colon \Lambda$-**Op** $\to \Lambda'$-**Op**, where $(f_!P)(n)$ is defined by the coequalizer below,*

$$P(n) \times \Lambda(n) \times \Lambda'(n) \underset{\rho \times 1}{\overset{1 \times (\star \circ f_n \times 1)}{\rightrightarrows}} P(n) \times \Lambda'(n) \xrightarrow{\varepsilon} P(n) \otimes_{\Lambda(n)} \Lambda'(n)$$

where $\star\colon \Lambda'(n) \times \Lambda'(n) \to \Lambda'(n)$ is group multiplication and $\rho\colon P(n) \times \Lambda(n) \to P(n)$ is the right action given by the Λ-operad structure.

3. *The functor $f_!$ is left adjoint to f^*.*

Proof. The right action of $\Lambda(n)$ on $(f^*Q)(n) = Q(n)$ is given as the composite

$$Q(n) \times \Lambda(n) \overset{1 \times f_n}{\to} Q(n) \times \Lambda'(n) \to Q(n),$$

where the second map is the action given by the Λ'-operad structure on Q. This group action, together with the operadic multiplication maps for Q as a Λ'-operad, give f^*Q a Λ-operad structure. Given a map $h\colon P \to Q$ of Λ'-operads, we must check that the maps $h_n\colon P(n) \to Q(n)$ also constitute a map of Λ-operads $f^*P \to f^*Q$. The functions h_n give a map of underlying operads by definition, so we need only verify the equivariance with respect to the $\Lambda(n)$-actions. This equivariance diagram commutes by the functoriality of products and the definition of h as a map of Λ'-operads. It is then straightforward to check the functoriality of these assignments, finishing the proof of the first claim.

For the second claim, we first observe that elements of this coequalizer are equivalence classes $[p,\tau]$ where $p \in P(n)$, $\tau \in \Lambda'(n)$, and the relation is given by $[p\cdot\sigma,\tau] = [p, f(\sigma)\tau]$ for $\sigma \in \Lambda(n)$. Then $(f_!P)(n)$ inherits a right $\Lambda'(n)$-action by multiplication in the second coordinate. The coequalizer displayed in the proposition statement is easily seen to be reflexive, with a common section given by mapping a pair $(p,\tau) \in P(n)\times\Lambda'(n)$ to $(p, e_n, \tau) \in P(n)\times\Lambda(n)\times\Lambda'(n)$. Since the product of reflexive coequalizers is again a coequalizer, we define operadic multiplication

$$m\colon f_!P(n) \times f_!P(k_1) \times \cdots \times f_!P(k_n) \to f_!P\left(\sum k_i\right)$$

to be uniquely determined by the universal property of its source as the coequalizer of a pair of maps

$$\big(P(n)\Lambda(n)\Lambda'(n)\big) \times \prod_{i=1}^{n} \big(P(k_i)\Lambda(n)\Lambda'(k_i)\big) \rightrightarrows \big(P(n)\times\Lambda'(n)\big) \times \prod_{i=1}^{n}\big(P(k_i)\times\Lambda'(k_i)\big). \tag{10}$$

Define a function

$$\tilde{m}\colon \big(P(n)\times\Lambda'(n)\big) \times \prod_{i=1}^{n}\big(P(k_i)\times\Lambda'(k_i)\big) \to f_!P\left(\sum k_i\right)$$

by

$$\tilde{m}\big((p,\tau);(p_1,\tau_1),\ldots,(p_n,\tau_n)\big) = \left[\mu^P(p;p_{\tau^{-1}(1)},\ldots,p_{\tau^{-1}(n)}), \mu^{\Lambda'}(\tau;\tau_1,\ldots,\tau_n)\right].$$

The function $\tilde{m}$ will induce the operadic multiplication on $f_!P$ once we verify that it coequalizes the two maps in Equation (10). In order to do so, we must check that

$$\tilde{m}\big((p\cdot\sigma,\tau);(p_1\cdot\sigma_1,\tau_1),\ldots,(p_n\cdot\sigma_n,\tau_n)\big) = \tilde{m}\big((p,f(\sigma)\tau);(p_1,f(\sigma_1)\tau_1),\ldots,(p_n,f(\sigma_n)\tau_n)\big).$$

By definition the left side is

$$\left[\mu^P(p\cdot\sigma;p_{\tau^{-1}(1)}\cdot\sigma_{\tau^{-1}(1)},\ldots,p_{\tau^{-1}(n)}\cdot\sigma_{\tau^{-1}(n)}),\ \mu^{\Lambda'}(\tau;\tau_1,\ldots,\tau_n)\right], \tag{11}$$

while the right side is

$$\left[\mu^P(p;p_{(f(\sigma)\tau)^{-1}(1)},\ldots,p_{(f(\sigma)\tau)^{-1}(n)}),\ \mu^{\Lambda'}\big(f(\sigma)\tau;f(\sigma_1)\tau_1,\ldots,f(\sigma_n)\tau_n\big)\right]. \tag{12}$$

By the action operad axioms and the fact that f is a map of action operads, we have the equalities

$$\begin{aligned}\mu\big(f(\sigma)\tau;f(\sigma_1)\tau_1,\ldots,f(\sigma_n)\tau_n\big) &= \mu\big(f(\sigma);f(\sigma_{\tau^{-1}(1)}),\ldots,f(\sigma_{\tau^{-1}(n)})\big)\mu\big(\tau;\tau_1,\ldots,\tau_n\big)\\ &= f\left(\mu\big(\sigma;\sigma_{\tau^{-1}(1)},\ldots,\sigma_{\tau^{-1}(n)}\big)\right)\mu\big(\tau;\tau_1,\ldots,\tau_n\big).\end{aligned}$$

By the equality $[p\cdot\sigma,\tau] = [p,f(\sigma)\tau]$, we therefore conclude that Equation (12) equals

$$\left[\mu^P(p;p_{(f(\sigma)\tau)^{-1}(1)},\ldots,p_{(f(\sigma)\tau)^{-1}(n)})\cdot\mu\big(\sigma;\sigma_{\tau^{-1}(1)},\ldots,\sigma_{\tau^{-1}(n)}\big),\mu\big(\tau;\tau_1,\ldots,\tau_n\big)\right].$$

By the single-equality version of the Λ-operad equivariance axioms from Remark 8.2, we obtain that the first coordinate can be rewritten as

$$\mu^P(p; p_{(f(\sigma)\tau)^{-1}(1)}, \ldots, p_{(f(\sigma)\tau)^{-1}(n)}) \cdot \mu\big(\sigma; \sigma_{\tau^{-1}(1)}, \ldots, \sigma_{\tau^{-1}(n)}\big) =$$
$$\mu^P(p \cdot \sigma; p_{(f(\sigma)\tau)^{-1}\left(\sigma(1)\right)} \cdot \sigma_{\tau^{-1}(1)}, \ldots, p_{(f(\sigma)\tau)^{-1}\left(\sigma(n)\right)} \cdot \sigma_{\tau^{-1}(n)}).$$

The indices on the terms $p_{(f(\sigma)\tau)^{-1}\left(\sigma(1)\right)}$ can be simplified using that f is a map of action operads, and hence preserves underlying permutations. Thus

$$\begin{aligned}(f(\sigma)\tau)^{-1}\big(\sigma(i)\big) &= \tau^{-1}\left(f(\sigma)^{-1}\left(\sigma(i)\right)\right) \\ &= \tau^{-1}(i)\end{aligned}$$

because the underlying permutation of σ is equal to that of $f(\sigma)$. After substituting these simplifications into the above, we obtain Equation (11), completing the proof that $\tilde{m}$ coequalizes the two maps in Equation (10) and therefore induces a unique operadic multiplication on $f_!P$. By definition, the operadic multiplication map $m\colon f_!P(n) \times f_!P(k_1) \times \cdots \times f_!P(k_n) \to f_!P\left(\sum k_i\right)$ is therefore defined to be

$$m\left([p,\tau]; [p_1,\tau_1], \ldots, [p_n,\tau_n]\right) = \Big[\mu^P(p; p_{\tau^{-1}(1)}, \ldots, p_{\tau^{-1}(n)}), \mu^{\Lambda'}(\tau; \tau_1, \ldots, \tau_n)\Big]. \quad (13)$$

The unit for $f_!P$ is $[\mathrm{id}, e_1]$ where $\mathrm{id} \in P(1)$ is the unit for P and $e_1 \in \Lambda'(1)$ is the unit for Λ'. It is straightforward to show that $[\mathrm{id}, e_1]$ acts as a unit for $f_!P$ using the unit axioms for P and Λ', and we leave these calculations to the reader.

Next we check the associativity axiom for $f_!P$. We do so by checking that the function

$$\widehat{m}\colon \big(P(n) \times \Lambda'(n)\big) \times \prod_{i=1}^{n} \big(P(k_i) \times \Lambda'(k_i)\big) \to P\left(\sum k_i\right) \times \Lambda'\left(\sum k_i\right)$$

given by the formula

$$\widehat{m}\left((p,\tau); (p_1,\tau_1), \ldots, (p_n,\tau_n)\right) = \Big(\mu^P(p; p_{\tau^{-1}(1)}, \ldots, p_{\tau^{-1}(n)}), \mu^{\Lambda'}(\tau; \tau_1, \ldots, \tau_n)\Big)$$

satisfies associativity with respect to operadic composition. The actual operadic composition is then obtained from $\widehat{m}$ by passing to equivalence classes via the coequalizer in Equation (10), so associativity of $\widehat{m}$ will ensure the associativity of m. First note that the second coordinate of the operadic composite $\widehat{m}\left((p,\tau); (p_1,\tau_1), \ldots, (p_n,\tau_n)\right)$ is $\mu^{\Lambda'}(\tau; \tau_1, \ldots, \tau_n)$, so associativity in the second coordinate follows immediately from associativity of Λ'.

In order to check associativity in the first coordinate, we introduce the following notation. We write a list $a_1, \ldots, a_n$ as $\underline{a_i}$. If $\sigma \in \Sigma_n$, we write $\sigma \bullet_i \underline{a_i}$ for the list $a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}$. Then the first coordinate of

$$\widehat{m}\left((p,\tau); \underline{\widehat{m}\left((p_i,\tau_i); \underline{(p_{i,j},\tau_{i,j})}\right)}\right)$$

is $\mu\left(p; \tau \bullet_i \underline{\mu(p_i; \tau_i \bullet_j \underline{p_{i,j}})}\right)$. By the associativity of P, the first coordinate is therefore equal to

$$\mu\left(\mu(p; \tau \bullet_i \underline{p_i}); \underline{\tau \bullet_i \tau_i \bullet_j \underline{p_{i,j}}}\right). \quad (14)$$

On the other hand, the first coordinate of

$$\widehat{m}\left(\widehat{m}\left((p,\tau); \underline{(p_i,\tau_i)}\right); \underline{(p_{i,j},\tau_{i,j})}\right)$$

is $\mu\left(\mu(p;\tau\bullet_i \underline{p_i});\mu(\tau;\underline{\tau_i})\bullet_{i,j}\underline{p_{i,j}}\right)$ where $\underline{p_{i,j}}$ is ordered lexicographically and $\bullet_{i,j}$ means that $\mu(\tau;\underline{\tau_i})$ acts upon this list. By Equation (3) from the proof of Theorem 4.15, we have

$$\begin{aligned}\mu(\tau;\underline{\tau_i})\bullet_{i,j}\underline{p_{i,j}} &= \delta(\tau)\beta(\underline{\tau_i})\bullet_{i,j}\underline{p_{i,j}}\\ &= \delta(\tau)\bullet_{i,j}\underline{\tau_i\bullet_j\underline{p_{i,j}}}\\ &= \tau\bullet_i\underline{\tau_i\bullet_j\underline{p_{i,j}}}.\end{aligned}$$

Therefore the first coordinate of $\widehat{m}\left(\widehat{m}\left((p,\tau);\underline{(p_i,\tau_i)}\right);\underline{(p_{i,j},\tau_{i,j})}\right)$ is

$$\mu\left(\mu(p;\tau\bullet_i\underline{p_i});\mu(\tau;\underline{\tau_i})\bullet_{i,j}\underline{p_{i,j}}\right)=\mu\left(\mu(p;\tau\bullet_i\underline{p_i});\tau\bullet_i\underline{\tau_i\bullet_j\underline{p_{i,j}}}\right),$$

matching Equation (14) and verifying operadic associativity.

Finally we check the equivariance axiom in the form of Remark 8.2. This axiom is the equality

$$m\left([p,\tau];\underline{[p_i,\tau_i]}\right)\cdot\mu(\omega;\underline{\omega_i})=m\left([p,\tau]\cdot\omega;[p_{\omega(1)},\tau_{\omega(1)}]\cdot\omega_1,\ldots,[p_{\omega(n)},\tau_{\omega(n)}]\cdot\omega_n\right).$$

Using the formula for m, it is straightforward to verify that both sides are equal to

$$\left[\mu(p;\tau\bullet_i\underline{p_i}),\mu(\tau\omega;\tau_{\omega(1)}\omega_1,\ldots,\tau_{\omega(n)}\omega_n)\right].$$

Thus the equivariance axiom for a Λ'-operad holds for $f_!P$, and completes the proof that $f_!P$ is a Λ'-operad.

Next we turn to the functoriality of $f_!$. Let $g\colon P\to Q$ be a map of Λ-operads. Define $f_!g\colon f_!P\to f_!Q$ by the formula

$$(f_!g)_n\left([p,\tau]\right)=[g(p),\tau], \tag{15}$$

for $[p,\tau]\in f_!P(n)$. Since g is a map of Λ-operads, it preserves the identity elements, so

$$(f_!g)\left([\mathrm{id},e_1]\right)=[g(\mathrm{id}),e_1]=[\mathrm{id},e_1],$$

showing that $f_!g$ also preserves identity elements. It is clear that $f_!g$ is levelwise equivariant with respect to the $\Lambda'(n)$-actions, so we need only check that it preserves operadic multiplication. We leave this calculation to the reader, as it is a straightforward application of the fact that g is a map of Λ-operads so preserves operadic multiplication. This completes the definition of $f_!$ on morphisms. It is immediate that $f_!$ preserves composition and identity maps, and is thus a functor

$$f_!\colon\Lambda\text{-}\mathbf{Op}\to\Lambda'\text{-}\mathbf{Op}$$

as desired, completing the proof of the second claim in the statement.

We now prove the third claim, that $f_!$ is left adjoint to f^*. The unit of this adjuntion has a component at the Λ-operad P given by a map $\eta^P\colon P\to f^*f_!P$, and is defined by the formula

$$\eta^P_n(p)=[p,e_n]$$

for all $p\in P(n)$. The map η^P preserves the operadic identity by definition, and it preserves operadic multiplication by a simple application of Equation (13) defining operadic multiplication in $f_!P$. Furthermore, η^P preserves the right Λ-actions because

$$\begin{aligned}\eta^P(p\cdot\tau)&=[p\cdot\tau,e_n]\\ &=[p,f(\tau)e_n]\\ &=[p,e_n]\cdot\tau\\ &=\eta^P(p)\cdot\tau.\end{aligned}$$

Thus η^P is a map of Λ-operads. It is then simple to check that these components define a natural transformation $1 \Rightarrow f^* f_!$.

The counit of this adjunction has a component at the Λ'-operad Q given by a map $\varepsilon^Q \colon f_! f^* Q \to Q$, and is defined by the formula

$$\varepsilon_n^Q \left([q, \tau]\right) = q \cdot \tau.$$

Since $f_! f^* Q(n)$ is defined to be $f^* Q(n) \otimes_{\Lambda(n)} \Lambda'(n)$ and the action of $\Lambda(n)$ on $f^* Q(n)$ is given by $q \cdot_{f^* Q} \sigma = q \cdot_Q f(\sigma)$, the equivalence relation defining $f_! f^* Q(n)$ is generated by the equalities

$$[q \cdot_Q f(\sigma), \tau] = [q, f(\sigma)\tau]$$

for all $\sigma \in \Lambda(n)$. In particular, the formula for ε_n^Q is well-defined on equivalence classes. It is obvious that ε^Q preserves the operadic identity and the right $\Lambda'(n)$-actions, so we only need to show that it preserves operadic multiplication. We therefore compute that

$$\begin{aligned}
\varepsilon^Q \left(m \left([q, \tau]; \underline{[q_i, \tau_i]} \right) \right) &= \varepsilon^Q \left([\mu(q; \tau \bullet_i \underline{q_i}), \mu(\tau; \underline{\tau_i})] \right) \\
&= \mu(q; \tau \bullet_i \underline{q_i}) \cdot \mu(\tau; \underline{\tau_i}) \\
&= \mu(q \cdot \tau; \underline{q_i \cdot \tau_i}) \\
&= m \left(\varepsilon^Q [q, \tau]; \underline{\varepsilon^Q [q_i, \tau_i]} \right)
\end{aligned}$$

by the definition of m, the definition of ε^Q, the equation in Remark 8.2, and the definition of ε^Q again. Thus ε^Q preserves operadic multiplication, and is a map of Λ'-operads. As with η, naturality in Q is simple to check.

For a Λ-operad P, the composite $\varepsilon^{f_! P} \circ f_! \eta^P$ is given by

$$\begin{aligned}
\varepsilon^{f_! P} \circ f_! \eta^P [p, \tau] &= \varepsilon^{f_! P} \left[[p, e_n], \tau \right] \\
&= [p, e_n \tau] \\
&= [p, \tau],
\end{aligned}$$

so is the identity. Likewise, for a Λ'-operad Q the composite $f^* \varepsilon^Q \circ \eta^{f^* Q}$ is given by

$$\begin{aligned}
f^* \varepsilon^Q \circ \eta^{f^* Q}(q) &= f^* \varepsilon^Q [q, e_n] \\
&= q \cdot e_n \\
&= q,
\end{aligned}$$

so is also the identity. These two calculations verify the triangle identities, and therefore prove that $f_! \dashv f^*$. $\square$

We will examine the adjunction from Theorem 8.15 further in the case that the map f is the underlying permutation map $\pi \colon \Lambda \to \Sigma$ for an action operad (Λ, π). Using the formulas in the proof above, we examine the unit and counit separately.

Proposition 8.16. *Let (Λ, π) be an action operad. The unit of the adjunction $\pi_! \dashv \pi^*$ from Theorem 8.15 is an isomorphism if and only if $\Lambda \cong \Sigma$ via π.*

Proof. The component $\eta^P \colon P \to \pi^* \pi_! P$ of the unit at a Λ-operad P is defined by the formula

$$\eta_n^P(p) = [p, e_n].$$

Taking Λ as a Λ-operad, we see that $\pi^* \pi_! \Lambda$ can be chosen to be Σ_n, in which case the map η_n^Λ is just π_n. The unit η can only be an isomorphism if each η^P is, so it is a necessary condition that $\Lambda \cong \Sigma$ via π. Sufficiency of this condition is obvious by the formulas defining $f^*, f_!$ applied to an isomorphism f, completing the proof. $\square$

Proposition 8.17. *Let (Λ, π) be an action operad. The counit of the adjunction $\pi_! \dashv \pi^*$ from Theorem 8.15 is an isomorphism if and only if $\pi\colon \Lambda \to \Sigma$ is surjective.*

Proof. For a symmetric operad Q, the counit $\varepsilon^Q\colon \pi_!\pi^*Q \to Q$ is defined on the n-ary operations by

$$\varepsilon_n^Q([q,\tau]) = q \cdot \tau,$$

where $q \in Q(n)$ and $\tau \in \Sigma_n$. This function is surjective for any action operad Λ and any Λ-operad Q, so we only check injectivity.

When $\pi\colon \Lambda \to \Sigma$ is not surjective, it is the zero map by Proposition 4.16. The equivalence relation defining $\pi_!\pi^*Q(n)$ is $[q \cdot_{\pi^*Q} \lambda, \tau] = [q, \pi(\lambda)\tau]$ for $q \in Q(n)$, $\lambda \in \Lambda(n)$, and $\tau \in \Sigma_n$, but $\pi(\lambda) = e_n \in \Sigma_n$. Furthermore, the action $q \cdot_{\pi^*Q} \lambda$ is computed in $\pi^*Q(n)$, so is actually

$$q \cdot_{\pi^*Q} \lambda = q \cdot_Q \pi(\lambda) = q.$$

These calculations show that the equivalence relation degenerates to $[q,\tau] = [q,\tau]$, so $\pi_!\pi^*Q(n) \cong Q(n) \times \Sigma_n$, and the counit map is just given by the group action of Σ_n on $Q(n)$, so in particular is not injective.

Now assume that $\pi\colon \Lambda \to \Sigma$ is surjective, and assume that $\varepsilon_n^Q([q_1,\tau_1]) = \varepsilon_n^Q([q_2,\tau_2])$, or equivalently that

$$q_1 \cdot \tau_1 = q_2 \cdot \tau_2.$$

We must check that $[q_1,\tau_1] = [q_2,\tau_2]$. Then for any $\lambda \in \Lambda(n)$ such that $\pi(\lambda) = \tau_1\tau_2^1$, we have

$$\begin{aligned}
[q_1,\tau_1] &= [q_1, \pi(\lambda)\pi(\lambda^{-1})\tau_1] \\
&= [q_1\pi(\lambda), \pi(\lambda^{-1})\tau_1] \\
&= [q_1\tau_1\tau_2^{-1}, \tau_2\tau_1^{-1}\tau_1] \\
&= [q_2,\tau_2],
\end{aligned}$$

proving that ε_n^Q is injective and completing the proof that if π is surjective then ε is an isomorphism. □

Lemma 8.18. *Let P be a symmetric operad and X be a set. P-algebra structures on X, given by $\{\alpha_n\colon P(n) \otimes_{\Sigma(n)} X^n \to X\}$ as in Definition 8.9, are in bijection with maps of symmetric operads $\alpha\colon P \to \mathcal{E}_X$.*

Proof.

This is almost in May GILS, but doesn't have enough details. We should provide a few.

□

Corollary 8.19. *Let Λ be an action operad with underlying permutation map $\pi\colon \Lambda \to \Sigma$. For any Λ-operad P and any set X, P-algebra structures on X are in bijection with*

- *maps of Λ-operads $\alpha\colon P \to \pi^*\mathcal{E}_X$ or*
- *maps of symmetric operads $\alpha'\colon \pi_!P \to \mathcal{E}_X$.*

Proof. By the definition of the $\Lambda(n)$-actions on $P(n)$ and X^n, the maps

$$P(n) \otimes_{\Lambda(n)} X^n \to X$$

defining a P-algebra structure in Definition 8.9 are in bijection with $\Lambda(n)$-equivariant maps $P(n) \to \pi^*\mathcal{E}_X(n)$. The correspondence between the axioms in Definition 8.9 and the map of Λ-operad axioms (Definition 8.3) follows just as in the proof of Lemma 8.18. The equivalence between the Λ- and symmetric operads versions follows immediately from adjointness, Theorem 8.15. □

Definition 8.20 (Endofunctor induced by a Λ-operad). Let P be a Λ-operad. Then P induces an endofunctor of **Sets**, denoted $\underline{P}$, by the following formula.

$$\underline{P}(X) = \coprod_n P(n) \otimes_{\Lambda(n)} X^n$$

We now have the following proposition; its proof is standard [39], and we leave it to the reader. But maybe there actually is no proof in GILS??

Proposition 8.21. *Let P be a Λ-operad.*

1. *The Λ-operad structure on P induces a monad structure on $\underline{P}$ via the operadic multiplication and operadic identities for P. We denote this monad $(\underline{P}, \mu, id)$, or just $\underline{P}$ when μ, id are understood.*
2. *The category of algebras for the Λ-operad P is isomorphic to the category of algebras for the monad $(\underline{P}, \mu, id)$.*

In the case that we take the operad P to also be Λ, we do not get algebras more interesting than monoids.

Proposition 8.22. *Let Λ be an action operad. The category of algebras for Λ taken as a Λ-operad, Λ-**Alg**, is isomorphic to the category of monoids.*

Proof. The category of monoids is $\underline{T}$-**Alg** where T is the terminal action operad, so we will produce an isomorphism of monads $R\colon \underline{T} \cong \underline{\Lambda}$. For a set X, $\underline{T}(X)$ is $\coprod_n T(n) \otimes_{T(n)} X^n \cong \coprod_n X^n$, while $\underline{\Lambda}(X)$ is

$$\coprod_n \Lambda(n) \otimes_{\Lambda(n)} X^n.$$

The elements of the coequalizer $\Lambda(n) \otimes_{\Lambda(n)} X^n$ are equivalence classes $[g; x_1, \ldots, x_n]$ under the equivalence relation

$$(gh; x_1, \ldots, x_n) \sim \big(g; x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n)}\big)\,.$$

The functions $R_{X;n}\colon \Lambda(n) \otimes_{\Lambda(n)} X^n \to X^n$ defined by

$$R_{X;n}\big(\ [g; x_1, \ldots, x_n]\ \big) = \big(x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}\big)$$

are bijections, and are easily seen to be natural in X. Define $R_X = \coprod_n R_{X;n}$. We leave it to the reader that these components also commute with the multiplication and unit of the monads $\underline{T}, \underline{\Lambda}$, so produce the desired isomorphism of monads. The isomorphism of monads R then induces an isomorphism between categories of algebras, proving the desired claim. $\square$

The monad-theoretic incarnation of Corollary 8.19 is then the following.

Corollary 8.23. *Let (Λ, π) be an action operad. For any Λ-operad P, there exists a natural isomorphism of monads between $\underline{P}$ and $\underline{\pi_! P}$. In particular, these monads (and hence operads) have isomorphic categories of algebras.*

Proof. The isomorphism is induced by the universal property of the coequalizer by noting that $P(n) \otimes_{\Lambda(n)} X^n$, the nth summand in $\underline{P}(X)$, and $\big(P(n) \otimes_{\Lambda(n)} \Sigma_n\big) \otimes_{\Sigma_n} X^n$, the nth summand in $\underline{\pi_! P}(X)$, both satisfy the same universal property. $\square$

9 The Substitution Product

In this section, we will show that Λ-operads are the monoids in the category of Λ-collections equipped with an appropriate substitution product. Such a result is fairly standard (see [37, Section 1.8]), and in both the symmetric and non-symmetric cases can easily be proven directly. Since we work with an arbitrary action operad, however, it will be more economical to take the abstract approach using coends and Day convolution.

Definition 9.1 (Λ-collections). Let Λ be an action operad.

1. The category $B\Lambda$ has
   - objects natural numbers $n \in \mathbb{N}$, and
   - morphism sets $B\Lambda(m,n)$ empty when $m \neq n$

     $$B\Lambda(n,n) = \Lambda(n),$$

     with composition given by group multiplication and identities given by the elements e_n.
2. The category Λ-**Coll** of Λ-*collections* is the presheaf category

   $$[B\Lambda^{\mathrm{op}}, \mathbf{Sets}].$$

Remark 9.2. The definition of Λ-**Coll** does not require that Λ be an action operad, only that one has a natural number-indexed set of groups.

Definition 9.3 (The substitution product $\circ$). Let Λ be an action operad, and let X, Y be Λ-collections. We define the Λ-collection $X \circ Y$ by

$$X \circ Y(k) = \left(\left(\coprod_{k_1 + \cdots + k_r = k} X(r) \times Y(k_1) \times \cdots \times Y(k_r) \right) \times \Lambda(k) \right) / \sim$$

where the equivalence relation is generated by the following.

1. For $x \in X(r)$, $h \in \Lambda(r)$, $y_i \in Y(k_i)$ for $i = 1, \ldots, r$, and $g \in \Lambda(k)$, we have

   $$(xh; y_1, \ldots, y_r; g) \sim \left(x; y_{h^{-1}(1)}, \ldots, y_{h^{-1}(r)}; \delta_{r;k_1,\ldots,k_r}(h)g\right).$$

2. For $x \in X(r)$, $y_i \in Y(k_i)$ for $i = 1, \ldots, r$, $g_i \in \Lambda(k_i)$ for $i = 1, \ldots, r$, and $g \in \Lambda(k)$, we have

   $$(x; y_1g_1, \ldots, y_rg_r; g) \sim (x; y_1, \ldots, y_r; \beta(g_1, \ldots, g_r)g).$$

We will now develop the tools to prove that the category Λ-**Coll** has a monoidal structure given by $\circ$, and that operads are the monoids with respect to this monoidal structure. We provide the statement here.

Theorem 9.4. *Let Λ be an action operad.*

1. *The category Λ-**Coll** has a monoidal structure with tensor product given by $\circ$ and unit given by the collection I with $I(n) = \emptyset$ when $n \neq 1$ and $I(1) = \Lambda(1)$.*
2. *The category **Mon**(Λ-**Coll**) of monoids in Λ-**Coll** is equivalent to the category of Λ-operads.*

While this theorem can be proven by direct calculation using the equivalence relation given above, such a proof is unenlightening. Furthermore, we want to consider Λ-operads in categories other than sets, so an element-wise proof might not apply. Instead we will develop general machinery that will apply to Λ-operads in any cocomplete symmetric monoidal category, by which we mean a category that is cocomplete, equipped with a symmetric monoidal structure, and the functors $X \otimes -, - \otimes X$ preserve colimits for every object X (as is the case if the monoidal structure is closed). Our construction of the monoidal structure on the category of Λ-collections will require the Day convolution product [7], and we begin by proving that $B\Lambda$ has a monoidal structure.

Proposition 9.5. *The action operad structure of Λ gives $B\Lambda$ a strict monoidal structure.*

Proof. The tensor product on $B\Lambda$ is given by addition on objects, with unit object 0; we denote tensor product by $+$. On morphisms, $+$ must be given by a group homomorphism

$$+ \colon \Lambda(n) \times \Lambda(m) \to \Lambda(n+m),$$

and is defined by the formula

$$+(g,h) = \beta(g,h).$$

By Theorem 4.15, β is a homomorphism as desired, and we now write $+(g,h)$ as $g+h$.

Addition of objects is strictly associative and unital. Strict associativity at the level of morphisms follows from Item 3, and strict unitality at the level of morphisms follows from Item 3 and Lemma 4.13. Thus $B\Lambda$ is a strict monoidal category as desired, completing the proof.

□

Now that $B\Lambda$ has a monoidal structure, there is also a monoidal structure on the category of $B\Lambda$-collections using Day convolution, denoted $\star$.

Definition 9.6 (Day convolution, [7]). Given collections X, Y, their *convolution product* $X \star Y$ is given by the coend formula

$$X \star Y(k) = \int^{m,n \in B\Lambda} X(m) \times Y(n) \times B\Lambda(k, m+n).$$

Remark 9.7. Given that $B\Lambda(k, m+n)$ is empty unless $k = m+n$, the coend in Definition 9.6 can be rewritten as

$$X \star Y(k) = \int^{m+n=k} X(m) \times Y(n) \times \Lambda(k).$$

In this formulation, $\Lambda(m) \times \Lambda(n)$ acts on $X(m) \times Y(n)$ by the product of their separate actions, and acts on $\Lambda(k)$ by $(g,h) \cdot t = \beta(g,h)t$.

Remark 9.8 (n-fold Day convolution). The n-fold Day convolution product of a Λ-collection Y with itself is given by the following coend formula.

$$Y^{\star n}(k) = \int^{k_1 + \cdots + k_n = k} Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k)$$

Computations with Day convolution will necessarily involve heavy use of the calculus of coends, and we refer the unfamiliar reader to [35] or [33]. Our goal is to express the substitution tensor product as a coend just as in [22], and to do that we need one final result about the Day convolution product.

Lemma 9.9. *Let Λ be an action operad, Y be a Λ-collection, and k be a fixed natural number. Then the assignment*

$$n \mapsto Y^{\star n}(k)$$

*can be given the structure of a functor $B\Lambda \to$ **Sets**.*

Proof. Since the convolution product is given by a coend, it is the universal object with maps

$$\theta_{k_1,\ldots,k_n;k} \colon Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k) \to Y^{\star n}(k),$$

for $k = k_1 + \cdots + k_n$, such that the following diagram commutes for every $g_1 \in \Lambda(k_1), \ldots, g_n \in \Lambda(k_n)$.

$$\begin{tikzcd}
Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k) \arrow[r, "(-\cdot g_1, \cdots, -\cdot g_n)\times 1"] \arrow[d, "1\times\left((g_1+\cdots+g_n)\cdot -\right)"'] & Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k) \arrow[d, "\theta_{k_1,\ldots,k_n;k}"] \\
Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k) \arrow[r, "\theta_{k_1,\ldots,k_n;k}"'] & Y^{\star n}(k)
\end{tikzcd}$$

Let $f \in \Lambda(n)$, considered as a morphism $n \to n$ in $B\Lambda$. We induce a map

$$f \bullet - \colon Y^{\star n}(k) \to Y^{\star n}(k)$$

using the universal property of the coend. For each k and $k_1, \ldots, k_n$ such that $k = k_1 + \cdots + k_n$, define

$$f[k_1, \ldots, k_n] \colon Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k) \to Y(k_{f^{-1}(1)}) \times \cdots \times Y(k_{f^{-1}(n)}) \times \Lambda(k)$$

by

$$f[k_1, \ldots, k_n](y_1, \ldots, y_n; g) = \big(y_{f^{-1}(1)}, \ldots, y_{f^{-1}(n)}; \delta_{n;k_1,\ldots,k_n}(f)g\big).$$

We now check that the following diagram commutes, where underlined elements represent lists of the indicated elements indexed from 1 to n, e.g.,

$$\beta(\underline{g_i}) = \beta(g_1, \cdots, g_n) = g_1 + \cdots + g_n.$$

$$\begin{tikzcd}
\prod_{i=1}^n Y(k_i) \times \Lambda(k) \arrow[rr, "(\underline{-\cdot g_i})\times 1"] \arrow[dr, "{f[k_1,\ldots,k_n]}"] \arrow[dd, "1\times(\beta(\underline{g_i})\cdot -)"'] & & \prod_{i=1}^n Y(k_i) \times \Lambda(k) \arrow[dr, "{f[k_1,\ldots,k_n]}"] \arrow[d, phantom, "①"] & \\
& \prod_{i=1}^n Y(k_{f^{-1}(i)}) \times \Lambda(k) \arrow[rr, "\left(\underline{-\cdot g_{f^{-1}(i)}}\right)\times 1"'] \arrow[dd, "1\times\left(\beta\left(\underline{g_{f^{-1}(i)}}\right)\cdot -\right)"] \arrow[d, phantom, "②"'] \arrow[ddrr, phantom, "③"] & & \prod_{i=1}^n Y(k_{f^{-1}(i)}) \times \Lambda(k) \arrow[dd, "\theta_{\underline{k_{f^{-1}(i)}};k}"] \\
\prod_{i=1}^n Y(k_i) \times \Lambda(k) \arrow[dr, "{f[k_1,\ldots,k_n]}"'] & & & \\
& \prod_{i=1}^n Y(k_{f^{-1}(i)}) \times \Lambda(k) \arrow[rr, "\theta_{\underline{k_{f^{-1}(i)}};k}"'] & & Y^{\star n}(k)
\end{tikzcd}$$

Square ① commutes by naturality of symmetries, square ② commutes by Item 8 from Theorem 4.15, and square ③ commutes by the definition of the coend. Therefore by the universality property, there is a unique map $f \bullet - \colon Y^{\star n}(k) \to Y^{\star n}(k)$ such that

$$\theta_{k_{f^{-1}(1)},\dots,k_{f^{-1}(n)};k} \circ f[k_1,\dots,k_n] = (f \bullet -) \circ \theta_{k_1,\dots,k_n;k} \tag{16}$$

for all k and $k_1,\dots,k_n$ such that $k = k_1 + \cdots + k_n$. Given $f_1, f_2 \in \Lambda(n)$, we have

$$\begin{aligned}(f_2 \bullet -) \circ (f_1 \bullet -) \circ \theta_{\underline{k_i};k} &= (f_2 \bullet -) \circ \theta_{\underline{k_{f_1^{-1}(i)}};k} \circ f_1 \left[\underline{k_i}\right] \\ &= \theta_{\underline{k_{f_1^{-1}(f_2^{-1}(i))}};k} \circ f_2 \left[k_{f_1^{-1}(1)},\dots,k_{f_1^{-1}(n)}\right] \circ f_1 \left[\underline{k_i}\right] \\ &= \theta_{\underline{k_{(f_2 f_1)^{-1}(i)}};k} \circ (f_2 f_1) \left[\underline{k_i}\right] \\ &= ((f_2 f_1) \bullet -) \circ \theta_{\underline{k_i};k}\end{aligned}$$

by Equation (16) twice, the left action of Σ_n on n-tuples as in Remark 2.12, and Item 6 from Theorem 4.15. By the universal property of the coend, we conclude that $(f_2 \bullet -) \circ (f_1 \bullet -) = \big((f_2 f_1) \bullet -\big)$, verifying functoriality and completing the proof. □

Did these use to come up as 'Item'? Or have I broken something?

We can easily change them to whatever you like. They are just items in a list, not sure the best name for that

Remark 9.10 (Yoneda via coends). We make heavy use of the following consequence of the Yoneda lemma: given any functor $F \colon B\Lambda \to \mathbf{Sets}$ and a fixed object $a \in B\Lambda$, there is a natural isomorphism

$$\int^{n \in B\Lambda} B\Lambda(n,a) \times F(n) \cong F(a)$$

given by sending the pair (g,x), for $g \in B\Lambda(n,a)$ and $x \in F(n)$, to $F(g)(x)$. There is a corresponding result for $F \colon B\Lambda^{\mathrm{op}} \to \mathbf{Sets}$, using representables of the form $B\Lambda(a,n)$ instead.

We are now ready for the abstract description of the substitution tensor product.

Lemma 9.11. *Let X, Y be Λ-collections. Then there is a natural isomorphism*

$$X \circ Y \cong \int^n X(n) \times Y^{\star n},$$

induced by the colimit structures.

Proof. The coend $\int^n X(n) \times Y^{\star n}(k)$ can be expanded as follows, using Remark 9.8, the fact that $A \times -$ preserves colimits for any A, and the Fubini theorem for coends [33, Theorem 1.3.1].

$$\begin{aligned}\int^n X(n) \times Y^{\star n}(k) &\cong \int^n X(n) \times \left(\int^{k_1+\cdots+k_n=k} Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k)\right) \\ &\cong \int^{n,k_1+\cdots+k_n=k} X(n) \times Y(k_1) \times \cdots \times Y(k_n) \times \Lambda(k).\end{aligned}$$

This final coend, when written out as a coequalizer, gives the formula in Definition 9.3. The two isomorphisms above are natural in both variables by the universal property of the colimits involved. □

Corollary 9.12. *Let Y be a Λ-collection.*

1. *The functor $- \circ Y \colon \Lambda$-$\mathbf{Coll} \to \Lambda$-$\mathbf{Coll}$ has a right adjoint $[Y, -]$.*

2. *For any other Λ-collection X, there is a natural isomorphism*

$$X^{\star n} \circ Y \cong (X \circ Y)^{\star n},$$

induced by the colimit structures.

Proof. We define the Λ-collection $[Y, Z]$ by

$$[Y, Z](k) = \Lambda\text{-}\mathbf{Coll}(Y^{\star k}, Z)$$

on objects and Lemma 9.9 on morphisms via precomposition. Then

$$\begin{aligned}
\Lambda\text{-}\mathbf{Coll}\,(X \circ Y, Z) &\cong \Lambda\text{-}\mathbf{Coll}\left(\int^{n} X(n) \times Y^{\star n}, Z\right) \\
&\cong \int_{n} \Lambda\text{-}\mathbf{Coll}\big(X(n) \times Y^{\star n}, Z\big) \\
&\cong \int_{n} \mathbf{Sets}\big(X(n), \Lambda\text{-}\mathbf{Coll}(Y^{\star n}, Z)\big) \\
&\cong \Lambda\text{-}\mathbf{Coll}\big(X, [Y, Z]\big)
\end{aligned}$$

by Lemma 9.11, the representable functor $\Lambda\text{-}\mathbf{Coll}\,(-, Z)$ mapping coends to ends, the copowering of collections over sets, and the identification of the set of natural transformations as an end. Each of these isomorphisms is visibly natural in all three variables, so $[Y, -]$ is right adjoint to $- \circ Y$, completing the proof of the first claim.

The second claim follows immediately from the first, as $X \mapsto X^{\star n}$ is a colimit, hence preserved by $- \circ Y$. ∎

Lemma 9.13. *Let I be the Λ-collection defined by*

$$I(k) = \begin{cases} \emptyset & k \neq 1, \\ \Lambda(1) & k = 1. \end{cases}$$

Then $I^{\star n}(k)$ is empty unless $k = n$, and then is isomorphic to $\Lambda(n)$.

Proof. By definition, we have

$$I^{\star n}(k) = \int^{k_1 + \cdots + k_n = k} I(k_1) \times \cdots \times I(k_n) \times \Lambda(k).$$

The only non-empty terms appear when $k_1 = k_2 = \cdots = k_n = 1$, from which we derive $k = n$. The coend is therefore the coequalizer $\Lambda(1)^n \otimes_{\Lambda(1)^n} \Lambda(n)$, where $\Lambda(1)^n$ acts on itself by right multiplication and on $\Lambda(n)$ by

$$(g_1, \ldots, g_n) \cdot h = \beta(g_1, \ldots, g_n) h.$$

This coequalizer is again $\Lambda(n)$, via the map above, completing the proof. ∎

Finally we are in a position to prove Theorem 9.4.

Proof of Theorem 9.4. First we must show that Λ-**Coll** has a monoidal structure using $\circ$. To prove this, we must give the unit and associativity isomorphisms and then check the monoidal category axioms. Define the unit object to be $I = B\Lambda(-, 1)$. Then for the left unit isomorphism, we find that

$$\begin{aligned}
I \circ Y(k) &= \int^{n} B\Lambda(n, 1) \times Y^{\star n}(k) \\
&\cong Y^{\star 1}(k) \\
&\cong Y(k),
\end{aligned}$$

where both isomorphisms are induced by the universal property of the coend. For the right unit isomorphism, we have that

$$\begin{aligned} X \circ I(k) &= \int^n X(n) \times I^{\star n}(k) \\ &\cong X(k) \end{aligned}$$

by Lemma 9.13.

Next we turn to constructing the associativity isomorphisms. We first compute that

$$\begin{aligned} \big[Y, [Z, W]\big](k) &= \Lambda\text{-}\mathbf{Coll}\big(Y^{\star k}, [Z, W]\big) \\ &\cong \Lambda\text{-}\mathbf{Coll}(Y^{\star k} \circ Z, W) \\ &\cong \Lambda\text{-}\mathbf{Coll}\big((Y \circ Z)^{\star k}, W\big) \\ &= \Lambda\text{-}\mathbf{Coll}\big[Y \circ Z, W\big](k) \end{aligned}$$

by the definition of the internal hom from the first part of Corollary 9.12 and the preservation of colimits from the second part of Corollary 9.12. These isomorphisms are compatible with the right $\Lambda(k)$-actions, so constitute an isomorphism that we denote

$$\overline{a}\colon \big[Y, [Z, W]\big] \cong \big[Y \circ Z, W\big].$$

The associativity isomorphism is defined to be the one induced, by Yoneda, from the composite below, in which each unmarked isomorphism is obtained from an adjunction of the form $- \circ A \dashv [A, -]$.

$$\begin{aligned} \Lambda\text{-}\mathbf{Coll}\big((X \circ Y) \circ Z, W\big) &\cong \Lambda\text{-}\mathbf{Coll}\big(X \circ Y, [Z, W]\big) \\ &\cong \Lambda\text{-}\mathbf{Coll}\big(X, \big[Y, [Z, W]\big]\big) \\ &\overset{\overline{a}}{\cong} \Lambda\text{-}\mathbf{Coll}\big(X, \big[Y \circ Z, W\big]\big) \\ &\cong \Lambda\text{-}\mathbf{Coll}\big(X \circ (Y \circ Z), W\big) \end{aligned}$$

In order to finish the proof that $(\Lambda\text{-}\mathbf{Coll}, \circ, I)$, with the unit and associativity isomorphisms above, is a monoidal category, we must check two axioms. These axioms follow immediately from the fact that the unit and associativity isomorphisms were all induced by the universal property of the colimit constructing their domains.

Now we must show that monoids in $(\Lambda\text{-}\mathbf{Coll}, \circ, I)$ are operads. By the Yoneda lemma, a map of Λ-collections $\eta\colon I \to X$ corresponds to an element $\mathrm{id} \in X(1)$ since $I = B\Lambda(-, 1)$. A map $\mu\colon X \circ X \to X$ is given by, for each k, a $\Lambda(k)$-equivariant map $(X \circ X)(k) \to X(k)$. By the universal property of the coend, this is equivalent to giving maps

$$\mu_{n;k_1,\ldots,k_n;k}\colon X(n) \times X(k_1) \times \cdots \times X(k_n) \times \Lambda(k) \to X(k)$$

that are compatible with the following group actions as specified.

- $\Lambda(n)$ acts on $X(n)$ on the right by the given action, and on $X(k_1) \times \cdots \times X(k_n) \times \Lambda(k)$ on the left by permutations and δ. The map μ must coequalize these.
- The group $\Lambda(k_i)$ acts on the factor $X(k_i)$ on the right by the given action, and on the left of $\Lambda(k)$ by group multiplication and β. The map μ must coequalize these.
- $\Lambda(k)$ acts on the right of $\Lambda(k)$ by group multiplication, and on $X(k)$ on the right by the given action. The map μ must preserve this action.

Given such a monoid structure, we define the operadic multiplication on the Λ-collection X by

$$\mu(x; y_1, \ldots, y_n) = \mu_{n;k_1,\ldots,k_n;k}(x; y_1, \ldots, y_n; e_k).$$

Conversely, given an operad P, we make the underlying Λ-collection into a monoid under $\circ$ by defining

$$\mu_{n;k_1,\ldots,k_n;k}(x; y_1, \ldots, y_n; g) = \mu(x; y_1, \ldots, y_n) \cdot g.$$

We leave checking the remaining details to the reader. $\square$

Part II: Operads in Categories

10 Background: 2-monads and their Algebras

insert something like: we assume basic familiarity with 2-categories, but still review 2-monads stuff

To investigate operads in **Cat** we will make use of 2-monads and their algebras, specifically the notion of a pseudoalgebra for a 2-monad. We assume familiarity with basic 2-category theory, but cover the required definitions and theory related to 2-monads here. For further reference, we refer the reader to [4] and [40].

cite johnson-yau book, kelly-street paper

Definition 10.1 (2-monad). Let $\mathcal{K}$ be a 2-category. A 2-*monad* on $\mathcal{K}$ consists of

- a strict 2-functor $T \colon \mathcal{K} \to \mathcal{K}$,
- a 2-natural transformation $\mu \colon T^2 \Rightarrow T$,
- a 2-natural transformation $\eta \colon \mathrm{id}_{\mathcal{K}} \Rightarrow T$,

satisfying the following axioms.

- The following diagram commutes.

$$\begin{tikzcd}
T^3X \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_x"] \\
T^2X \arrow[r, "\mu_X"'] & TX
\end{tikzcd}$$

- The following diagram commutes.

$$\begin{tikzcd}
TX \arrow[r, "\eta_{TX}"] \arrow[dr, "\mathrm{id}_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "T\eta_X"'] \arrow[dl, "\mathrm{id}TX"] \\
& TX &
\end{tikzcd}$$

Definition 10.2 (Pseudoalgebra, 2-monad version). Let $T \colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A T-*pseudoalgebra* consists of an object X, a 1-cell $\alpha \colon TX \to X$ in $\mathcal{K}$, and invertible 2-cells of $\mathcal{K}$

$$\begin{tikzcd}
T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "\eta_X"'] \arrow[drr, bend left, "1_X"] & \arrow[d, phantom, "\Downarrow\Phi_\eta"] & \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
T^3X \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TX}"'] \arrow[dr, "T\mu_X"] & T^2X \arrow[dr, "T\alpha"] \arrow[d, phantom, "T\Phi\Downarrow"] & \\
T^2X \arrow[dr, "\mu_X"'] & T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
=
\begin{tikzcd}
T^3X \arrow[r, "T^2\alpha"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_X"'] \arrow[dr, "T\alpha"] & \\
T^2X \arrow[r, "T\alpha"'] \arrow[dr, "\mu_X"'] \arrow[r, phantom, shift right=6, "\Downarrow\Phi"] & TX \arrow[dr, "\alpha"'] \arrow[r, phantom, "\Phi\Downarrow"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
$$

- The following pasting diagram is an identity.

$$
\begin{tikzcd}
TX \arrow[dr, "T\eta_X"'] \arrow[drr, bend left, "1_{TX}"] \arrow[ddr, bend right, "1_{TX}"'] & \arrow[d, phantom, "\Downarrow T\Phi_\eta"] & \\
& T^2X \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & TX \arrow[d, "\alpha"] \\
& TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
$$

Remark 10.3 (Omitted third axiom). Power's definition of a pseudoalgebra includes a third axiom relating to the unit of the 2-monad [40, Definition 2.4, Axiom 2.1]. However, following an argument of Marmolejo [38, Lemma 9.1] this extra axiom is redundant and is omitted here.

Definition 10.4 (Strict algebra, 2-monad version). Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A *strict T-algebra* is a pseudoalgebra in which all of the isomorphisms Φ are identities.

Definition 10.5 (Pseudomorphism, 2-monad version). Let T be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$, $(Y, \beta, \Psi, \Psi_\eta)$ be T-pseudoalgebras. A *pseudomorphism* $(f, \bar{f})$ between these pseudoalgebras consists of a 1-cell $f\colon X \to Y$ along with an invertible 2-cell

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
T^2X \arrow[r, "T^2f"] \arrow[d, "\mu_X"'] \arrow[dr, "T\alpha"] \arrow[r, phantom, shift right=4ex, "\Downarrow T\bar{f}"] & T^2Y \arrow[dr, "T\beta"] & \\
TX \arrow[dr, "\alpha"'] \arrow[r, phantom, "\Downarrow\Phi"] & TX \arrow[r, "Tf"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TY \arrow[d, "\beta"] \\
& TX \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
T^2X \arrow[r, "T^2f"] \arrow[d, "\mu_X"'] & T^2Y \arrow[d, "\mu_Y"'] \arrow[dr, "T\beta"] & \\
TX \arrow[r, "Tf"'] \arrow[dr, "\alpha"'] \arrow[dr, phantom, shift left=4ex, "\Downarrow\bar{f}"] & TX \arrow[dr, "\beta"] \arrow[r, phantom, "\Psi\Downarrow"] & TY \arrow[d, "\beta"] \\
& TX \arrow[r, "f"'] & Y
\end{tikzcd}
$$

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\eta_X"'] & & Y \arrow[d, "\eta_Y"'] \arrow[ddr, bend left, "1_Y"] & \\
TX \arrow[rr, "Tf"'] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\Downarrow\bar{f}"] & TY \arrow[dr, "\beta"] \arrow[r, phantom, "\Downarrow\Psi_\eta"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\eta_X"'] \arrow[ddr, bend left, "1_X"] & & Y \arrow[ddr, bend left, "1_Y"] & \\
TX \arrow[dr, "\alpha"'] \arrow[r, phantom, "\Downarrow\Phi_\eta"] & & & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

Definition 10.6 (Strict morphism, 2-monad verison). Let T be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$ and $(Y, \beta, \Psi, \Psi_\eta)$ be T-pseudoalgebras. A *strict morphism* $(f, \bar{f})$ consists of a pseudomorphism in which $\bar{f}$ is an identity.

Remark 10.7. The strict algebras and strict morphisms are exactly the same as algebras and morphisms for the underlying monad on the underlying category of $\mathcal{K}$.

Definition 10.8 (T-transformation, 2-monad version). Let $(f, \overline{f}), (g, \overline{g})\colon X \to Y$ be pseudomorphisms of T-algebras. A *T-transformation* consists of a 2-cell $\gamma\colon f \Rightarrow g$ such that the following equality of pasting diagrams holds.

$$
\begin{tikzcd}
TX \arrow[r, bend left=40, "Tf"] \arrow[r, "Tg"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow T\gamma"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & TY \arrow[d, "\beta"] \\
X \arrow[r, "f"] \arrow[r, bend right=40, "g"'] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\gamma"] & Y
\end{tikzcd}
$$

There are many different possible choices of 2-categories in which the objects are some kind of algebra over a 2-monad T. Here are the two that will be the most important for us.

Definition 10.9 (2-categories of algebras, 2-monad version). Let T be a 2-monad.

- The 2-category T-$\mathbf{Alg}_s$ consists of strict T-algebras, strict morphisms, and T-transformations.
- The 2-category $\mathbf{Ps}$-T-$\mathbf{Alg}$ consists of T-pseudoalgebras, pseudomorphisms, and T-transformations.

11 Λ-Operads in Cat as 2-monads

This section begins our study of algebras over a Λ-operad P in **Cat**. This theory blends together standard results in both 2-monad theory and operad theory.

lack codescent, bkp

may geometry of iterated loops

Convention 11.1 (Sets and discrete categories). Throughout the rest of this text, Λ will be an action operad in the category of sets. By abuse of notation, any set S will be identified with the discrete, small category dS with object set S. In this way, we also view any action operad Λ as an operad in **Cat**, and we view any group G as a discrete, strict monoidal category.

Convention 11.2 (Group actions on categories). A group action on a category is meant in the strict sense, not in the up-to-isomorphism sense. Thus if G acts on C, the equations

$$\begin{aligned} g \cdot (h \cdot x) &= (gh) \cdot x, \\ 1 \cdot x &= x \end{aligned}$$

hold for all x, where x is allowed to be either an object or morphism of C.

Definition 11.3 (Free actions). Suppose that a group G acts on a category C. We say that the action is *free* if...

Lemma 11.4. *Let G be a group and let A, B be categories for which A has a right action by G and B has a left action by G. An action of G on the product $A \times B$ can then be defined by*

$$(a, b) \cdot g \colon = \left(a \cdot g, g^{-1} \cdot b\right).$$

If this action of G on $A \times B$ is free, then the category $(A \times B)/G$, consisting of the equivalence classes of this action, is isomorphic to the coequalizer $A \otimes_G B$.

Proof. The category $A \times_G B$ is defined as the coequalizer

$$\begin{tikzcd} A \times G \times B \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & A \times B \arrow[rr, "\varepsilon"] & & A \otimes_G B \end{tikzcd}$$

where $\lambda(a, g, b) = (a \cdot g, b)$ and $\rho(a, g, b) = (a, g \cdot b)$. However, the map $A \times B \to (A \times B)/G$, sending (a, b) to the equivalence class $[a, b] = [a \cdot g, g^{-1} \cdot b]$, also coequalizes λ and ρ since

$$[a \cdot g, b] = \left[(a \cdot g) \cdot g^{-1}, g \cdot b\right] = [a, g \cdot b].$$

Given any other category X and a functor $\chi \colon A \times B \to X$ which coequalizes λ and ρ, we define a functor $\phi \colon (A \times B)/G \to X$ by $\phi[a, b] = \chi(a, b)$. That this is well-defined is clear, since

$$\phi\left[a \cdot g, g^{-1} \cdot b\right] = \chi\left(a \cdot g, g^{-1} \cdot b\right) = \chi\left(a \cdot \left(gg^{-1}\right), b\right) = \chi(a, b) = \phi[a, b].$$

This is also unique and so we find that $(A \times B)/G$ satisfies the universal property of the coequalizer. $\square$

Remark 11.5 (Λ-operads in Cat). Here we explicitly describe the structure of a Λ-operad P in **Cat**, following **??**. A Λ-operad P in **Cat** consists of

Hmm, this remark seems to have been removed, probably need a replacement

- a category, $P(n)$, for each natural number n,
- for each n, a right $\Lambda(n)$-action on $P(n)$ as per Convention 11.2,
- an object $\mathrm{id} \in P(1)$, and
- functors
$$\mu\colon P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n),$$

satisfying the first two axioms from Definition 3.1 and the two equivariance axioms from Definition 8.1.

Definition 11.6 (Pseudoalgebra, Λ-operad version). Let P be a Λ-operad. A *pseudoalgebra* for P consists of:

- a category X,
- a family of functors
$$\big(\alpha_n\colon P(n) \otimes_{\Lambda(n)} X^n \to X\big)_{n\in\mathbb{N}},$$
- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism $\phi_{k_1,\ldots,k_n}$ (corresponding, via Conventions **??**) to a natural isomorphism

$$\begin{tikzcd}
P_n \times \prod_{i=1}^n \big(P_{k_i} \times X^{k_i}\big) \arrow[r, "1\times\prod\tilde{\alpha}_{k_i}"] \arrow[d] & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_n \times \prod_{i=1}^n P_{k_i} \times X^{\Sigma k_i} \arrow[d, "\mu^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\phi}_{k_1,\ldots,k_n}"] & \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[r, "\tilde{\alpha}_{\Sigma k_i}"'] & X
\end{tikzcd}$$

- and a natural isomorphism ϕ_η corresponding to a natural isomorphism

$$\begin{tikzcd}
X \arrow[d, "\cong"'] \arrow[ddr, "1"] & \\
1 \times X \arrow[d, "\eta^P\times 1"'] \arrow[r, phantom, "\Downarrow\tilde{\phi}_\eta"] & \\
P(1) \times X \arrow[r, "\tilde{\alpha}_1"'] & X
\end{tikzcd}$$

satisfying the following axioms.

- For all $n, k_i, m_{ij} \in \mathbb{N}$, the following equality of pasting diagrams holds.

$$
\begin{tikzcd}
P_n \times \prod_i \big(P_{k_i} \times \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big)\big) \arrow[rr, "1\times\prod\left(1\times\prod \tilde{\alpha}_{m_{i}j}\right)"] \arrow[dd, "\mu^P\times 1"'] \arrow[dr, "\mu^P\times 1"] & & P_n \times \prod_i \big(P_{k_i} \times X^{k_i}\big) \arrow[dr, "1\times\prod \tilde{\alpha}_{k_i}"] \arrow[d, phantom, "\Downarrow 1\times\prod_i \tilde{\phi}_{m_{i1},\dots,m_{ik_i}}"] & \\
& P_n \times \prod_i \big(P_{\Sigma m_{ij}} \times X^{\Sigma m_{ij}}\big) \arrow[rr, "1\times\prod \tilde{\alpha}_{\Sigma m_{ij}}"] \arrow[dd, "\mu^P\times 1"] \arrow[ddrr, phantom, "\Downarrow \tilde{\phi}_{\Sigma m_{1j},\dots,\Sigma m_{nj}}"] & & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_{\Sigma k_i} \times \prod_i \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big) \arrow[dr, "\mu^P\times 1"'] & & & \\
& P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \arrow[rr, "\tilde{\alpha}_{\Sigma\Sigma m_{ij}}"'] & & X
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
P_n \times \prod_i \big(P_{k_i} \times \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big)\big) \arrow[rr, "1\times\prod\left(1\times\prod \tilde{\alpha}_{m_{i}j}\right)"] \arrow[dd, "\mu^P\times 1"'] & & P_n \times \prod_i \big(P_{k_i} \times X^{k_i}\big) \arrow[dr, "1\times\prod \tilde{\alpha}_{k_i}"] \arrow[dd, "\mu^P\times 1"'] & \\
& & & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \arrow[dl, phantom, "\Downarrow \tilde{\phi}_{k_1,\dots,k_n}"] \\
P_{\Sigma k_i} \times \prod_i \prod_j \big(P_{m_{ij}} \times X^{m_{ij}}\big) \arrow[rr, "1\times\prod\prod \tilde{\alpha}_{m_{ij}}"] \arrow[dr, "\mu^P\times 1"'] \arrow[drr, phantom, "\Downarrow \tilde{\phi}_{m_{11},\dots,m_{nk_n}}"] & & P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"] & \\
& P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \arrow[rr, "\tilde{\alpha}_{\Sigma\Sigma m_{ij}}"'] & & X
\end{tikzcd}
$$

- Each pasting diagram of the following form is an identity.

$$
\begin{tikzcd}
P_n \times X^n \arrow[dr, "\cong"] \arrow[ddrrr, bend left, "1"] \arrow[ddddrr, bend right, "1"'] & & & \\
& P_n \times (1 \times X)^n \arrow[dr, "1\times\left(\eta^P\times 1\right)^n"] \arrow[r, phantom, "\Downarrow 1\times\tilde{\phi}^n_\eta"] & {} & \\
& & P_n \times (P_1 \times X)^n \arrow[r, "1\times\tilde{\alpha}^n_1"] \arrow[d, "\cong"'] & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
& & P_n \times P_1^n \times X^n \arrow[d, "\mu^P\times 1"'] \arrow[r, phantom, "\Downarrow \tilde{\phi}_{1,\dots,1}"] & {} \\
& & P_n \times X^n \arrow[r, "\tilde{\alpha}_n"'] & X
\end{tikzcd}
$$

Remark 11.7.

reread this, and resolve whatever issue it is addressing

The requirement in **??** of a natural isomorphism φ_η is to induce a natural isomorphism $\tilde{\varphi}_\eta$. This requirement is really of a natural isomorphism

$$\begin{tikzcd}
1 \times_{\Lambda(1)} X \arrow[d, "\eta^P \times_{\Lambda(1)} 1"'] \arrow[dr] \arrow[d, phantom, shift left=6ex, "\Swarrow \varphi_\eta"] & \\
P(1) \times_{\Lambda(1)} X \arrow[r, "\alpha_1"'] & X
\end{tikzcd}$$

where $1 \times_{\Lambda(1)} X$ is the coequalizer of the trivial right action of $\Lambda(1)$ on 1 and the usual left action of $\Lambda(1)$ on X. This induces a natural isomorphism

$$\begin{tikzcd}
1 \times X \arrow[d, "\eta^P \times 1"'] \arrow[dr] \arrow[d, phantom, shift left=6ex, "\Swarrow \tilde{\varphi}_\eta"] & \\
P(1) \times X \arrow[r, "\tilde{\alpha}_1"'] & X
\end{tikzcd}$$

which can be whiskered with the isomorphism $X \to 1 \times X$. We make the convention of referring to this whiskered natural isomorphism as $\tilde{\varphi}_\eta$, since no confusion will arise in practice.

Definition 11.8 (Strict algebra, Λ-operad version). Let P be a Λ-operad. A *strict algebra* for P consists of a pseudoalgebra in which all of the isomorphisms ϕ are identities.

Definition 11.9 (Pseudomorphism, Λ-operad version). Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a Λ-operad P. A *pseudomorphism* of P-pseudoalgebras consists of:

- a functor $f\colon X \to Y$
- for each $n \in \mathbb{N}$, a natural isomorphism f_n (corresponding, via Conventions **??**) to a natural isomorphism

$$\begin{tikzcd}
P_n \times X^n \arrow[r, "\tilde{\alpha}_n"] \arrow[d, "1 \times f^n"'] \arrow[dr, phantom, "\Downarrow \overline{f}_n"] & X \arrow[d, "f"] \\
P_n \times Y^n \arrow[r, "\tilde{\beta}_n"'] & Y
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) \arrow[rr, "1 \times \prod (1 \times f^{k_i})"] \arrow[dd, "\mu^P \times 1"'] & & P_n \times \prod_i (P_{k_i} \times Y^{k_i}) \arrow[dr, "1 \times \prod \tilde{\beta}_{k_i}"] \arrow[dd, "\mu^P \times 1"'] & \\
& & \arrow[r, phantom, "\Downarrow \tilde{\psi}_{k_1,\dots,k_n}"] & P_n \times Y^n \arrow[dd, "\tilde{\beta}_n"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[rr, "1 \times f^{\Sigma k_i}"] \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"'] & \arrow[r, phantom, "\Downarrow \overline{f}_n"] & P_{\Sigma k_i} \times Y^{\Sigma k_i} \arrow[dr, "\tilde{\beta}_{\Sigma k_i}"'] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) \arrow[rr, "1 \times \prod (1 \times f^{k_i})"] \arrow[dd, "\mu^P \times 1"'] \arrow[dr, "1 \times \prod \tilde{\alpha}_{k_i}"] & \arrow[d, phantom, "\Downarrow 1 \times \prod \overline{f}_{k_i}"] & P_n \times \prod_i (P_{k_i} \times Y^{k_i}) \arrow[dr, "1 \times \prod \tilde{\beta}_{k_i}"] & \\
& P_n \times X^n \arrow[rr, "1 \times f^n"] \arrow[dd, "\tilde{\alpha}_n"'] \arrow[dl, phantom, "\Downarrow \tilde{\phi}_{k_1,\dots,k_n}"] & & P_n \times Y^n \arrow[dd, "\tilde{\beta}_n"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[dr, "\tilde{\alpha}_{\Sigma k_i}"'] & & \arrow[phantom, "\Downarrow \overline{f}_n"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"'] \arrow[ddd, bend left, "1"] & \\
1 \times X \arrow[r, "1 \times f"] \arrow[d, "\eta^P \times 1"'] & 1 \times Y \arrow[d, "\eta^P \times 1"'] \arrow[r, phantom, "\Downarrow \tilde{\psi}_\eta"] & \\
P_1 \times X \arrow[r, "1 \times f"] \arrow[dr, "\tilde{\alpha}_1"'] & P_1 \times Y \arrow[dr, "\tilde{\beta}_1"] \arrow[d, phantom, "\Downarrow \overline{f}_1"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] \arrow[dddr, bend left, "1"] & Y \arrow[dddr, bend left, "1"] & \\
1 \times X \arrow[d, "\eta^P \times 1"'] \arrow[r, phantom, "\Downarrow \tilde{\phi}_\eta"] & & \\
P_1 \times X \arrow[dr, "\tilde{\alpha}_1"'] & & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Definition 11.10 (Strict morphism, Λ-operad version). Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a Λ-operad P. A *strict morphism* of P-pseudoalgebras consists of a pseudomorphism in which all of the isomorphisms $\overline{f}_n$ are identities.

Remark 11.11 (Insert description here). A strict algebra for a Λ-operad P in **Cat** is precisely the same thing as an algebra for P considered as an operad in the *category* of small categories and functors. A strict morphism between strict algebras is then just a map of P-algebras in the standard sense. We could also consider the notion of a lax algebra for an operad, or a lax morphism of algebras, simply by considering natural transformations in place of isomorphisms in the definitions.

In **??** of a pseudomorphism we did not originally make it clear that the isomorphisms $\overline{f}_n$ should satisfy an equivariance condition. This was highlighted in Remark 2.22 of Rubin's thesis [42]. Similarly, this is also explicity stated as Definition 2.23 of [12], as mentioned in [13]. That we don't include an explicit equivariance axiom is due to Conventions **??**. In **??** we require the existence of natural isomorphisms f_n in order to induce corresponding natural isomorphisms $\overline{f}_n$. That the $\overline{f}_n$ are induced by the f_n corresponds to the fact that the $\overline{f}_n$ satisfy an equivariance condition, namely that for $(\sigma, g, x_1, \ldots, x_n) \in P(n) \times \Lambda(n) \times X^n$, we have

$$\left(\overline{f}_n\right)_{(\sigma \cdot g, x_1, \ldots, x_n)} = \left(\overline{f}_n\right)_{\left(\sigma, x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}\right)}.$$

Definition 11.12 (P-transformation, Λ-operad version). Let P be a Λ-operad and let $f, g\colon (X, \alpha, \phi, \phi_\eta) \to (Y, \beta, \psi, \psi_\eta)$ be pseudomorphisms of P-pseudoalgebras. A *P-transformation* is then a natural transformation $\gamma\colon f \Rightarrow g$ such that the following equality of pasting diagrams holds, for all n.

$$\begin{tikzcd}
P_n \times X^n \arrow[rr, bend left, "1\times f^n"] \arrow[rr, phantom, bend left=20, "\Downarrow 1\times\gamma^n"] \arrow[rr, "1\times g^n"'] \arrow[d, "\tilde{\alpha}_n"'] \arrow[drr, phantom, "\Downarrow\overline{g}_n"] & & P_n \times Y^n \arrow[d, "\tilde{\beta}_n"] \\
X \arrow[rr, "g"'] & & Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
P_n \times X^n \arrow[rr, "1\times f^n"] \arrow[d, "\tilde{\alpha}_n"'] \arrow[drr, phantom, "\Downarrow\overline{f}_n"] & & P_n \times Y^n \arrow[d, "\tilde{\beta}_n"] \\
X \arrow[rr, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, bend right=20, "\Downarrow\gamma"] & & Y
\end{tikzcd}$$

We can form various 2-categories using these cells.

Definition 11.13 (2-categories of algebras, Λ-operad version). Let P be a Λ-operad.

- The 2-category P-$\mathbf{Alg}_s$ consists of strict P-algebras, strict morphisms, and P-transformations.
- The 2-category $\mathbf{Ps}$-P-$\mathbf{Alg}$ consists of P-pseudoalgebras, pseudomorphisms, and P-transformations.

Our main result in this section is the following, showing that one can consider algebras and higher cells, in either strict or pseudo strength, using either the operadic or 2-monadic incarnation of a Λ-operad P. This extends **??**.

Theorem 11.14. *Let P be a Λ-operad in* $\mathbf{Cat}$.

- *There is an isomorphism of* 2*-categories*

$$P\text{-}\mathbf{Alg}_s \cong \underline{P}\text{-}\mathbf{Alg}_s.$$

- *There is an isomorphism of* 2*-categories*

$$\mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$$

extending the one above.

Proof.

hopefully shorten

We begin by noting that we suppress the difference between 2-cells Γ and those $\tilde{\Gamma}$ as in Conventions **??**, implicitly always using 2-cells defined on a coequalizer which are appropriately equivariant with respect to the group actions involved.

A proof of the first statement follows from our proof of the second by inserting identities where appropriate. Thus we begin by constructing a 2-functor $R\colon \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}$. We map a $\underline{P}$-pseudoalgebra $(X, \alpha, \Phi, \Phi_\eta)$ to the following P-pseudoalgebra on the same category X. First we define the functor α_n to be the composite

$$\alpha_n\colon P(n) \otimes_{\Lambda(n)} X^n \hookrightarrow \underline{P}(X) \xrightarrow{\alpha} X.$$

The isomorphisms $\phi_{k_1,\ldots,k_n}$ are defined using Φ as in the following diagram

$$\begin{tikzcd}
P_n \times \prod_{i=1}^n \left(P_{k_i} \times X^{k_i}\right) \arrow[r] \arrow[dd] & P_n \times \prod_i \left(P_{k_i} \times_{\Lambda_{k_i}} X^{k_i}\right) \arrow[r, hook] & P_n \times \underline{P}(X)^n \arrow[r, "1 \times \alpha^n"] \arrow[d] & P_n \times X^n \arrow[d] \\
 & & P_n \times_{\Lambda_n} \underline{P}(X)^n \arrow[d, hook] & P_n \times_{\Lambda_n} X^n \arrow[d, hook] \\
P_n \times \prod_i P_{k_i} \times X^{\Sigma k_I} \arrow[d, "\mu^P \times 1"'] & & \underline{P}^2(X) \arrow[r, "\underline{P}\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & \underline{P}(X) \arrow[d, "\alpha"] \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[r] & P_{\Sigma k_i} \times_{\Lambda_{\Sigma k_i}} X^{\Sigma k_i} \arrow[r, hook] & \underline{P}(X) \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

whilst Φ_η is simply sent to itself, since the composition of α with the composite of the coequalizer and inclusion map from $P(1) \times X$ into $\underline{P}(X)$ is just $\tilde{\alpha_1}$. Checking the axioms here is most easily done on components and it can easily seen that the axioms required of this data to be a P-pseudoalgebra are precisely those that they satisfy by virtue of X being a $\underline{P}$-pseudoalgebra.

For a 1-cell $(f, \overline{f})\colon (X, \alpha) \to (Y, \beta)$, we send f to itself whilst sending $\overline{f}$ to the obvious family of isomorphisms, as follows.

$$\begin{tikzcd}
P(n) \times X^n \arrow[r] \arrow[d, "1 \times f^n"'] & P(n) \otimes_{\Lambda(n)} X^n \arrow[r, hook] \arrow[d, "1 \times f^n"'] & \underline{P}(X) \arrow[r, "\alpha"] \arrow[d, "\underline{P}f"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & X \arrow[d, "f"] \\
P(n) \times Y^n \arrow[r] & P(n) \otimes_{\Lambda(n)} Y^n \arrow[r, hook] & \underline{P}(Y) \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

It is easy to check that the above data satisfy the axioms for being a pseudomorphism of P-pseudoalgebras, following from the axioms for $(f, \overline{f})$ being a pseudomorphism of $\underline{P}$-pseudoalgebras. A $\underline{P}$-transformation $\gamma\colon (f, \bar{f}) \Rightarrow (g, \bar{g})$ immediately gives a P-transformation $\bar{\gamma}$ between the families of isomorphisms we previously defined, with the components of $\bar{\gamma}$ being precisely those of γ. It is then easily shown that R is a 2-functor.

For there to be an isomorphism of 2-categories, we require an inverse to R, namely a 2-functor $S\colon \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$. Now assume that $(X, \alpha_n, \phi_{\underline{k}_i}, \phi_\eta)$ is a P-pseudoalgebra. We will give the same object X a $\underline{P}$-pseudoalgebra structure. We can induce a functor $\alpha\colon \underline{P}(X) \to X$ by using the universal property of the coproduct.

$$\begin{tikzcd} P(n) \times X^n \arrow[r] \arrow[drr, "\tilde{\alpha}_n"'] & P(n) \otimes_{\Lambda(n)} X^n \arrow[r, hook] \arrow[dr, "\alpha_n"] & \underline{P}(X) \arrow[d, "\exists!\alpha"] \\ & & X \end{tikzcd}$$

Of course, this can be induced using either α_n or $\tilde{\alpha}_n$, each giving the same functor α by uniqueness. The components of the isomorphism $\Phi\colon \alpha \circ \underline{P}(\alpha) \Rightarrow \alpha \circ \mu_X$ can be given as follows. Let $|\underline{x}_i|$ denote the number of objects in the list $\underline{x}_i$. Then define the component of Φ at the object

$$[p; [q_1; \underline{x}_1], \ldots, [q_n; \underline{x}_n]]$$

to be the component of $\phi_{|\underline{x}_1|,\ldots,|\underline{x}_n|}$ at the same object. To make this clearer, consider the object $[p; [q_1; x_{11}], [q_2; x_{21}, x_{22}], [q_3; x_{31}]]$. The component of Φ at this object is given by the component of $\phi_{1,2,1}$ at the same object. The isomorphism ϕ_η is again sent to itself.

Now given a 1-cell f with structure 2-cells $\overline{f}_n$ we define a 1-cell $(F, \overline{F})$ with underlying 1-cell f and structure 2-cell $\overline{F}$ with components

$$\overline{F}_{[p;x_1,\ldots,x_n]} := \left(\overline{f}_n\right)_{(p;x_1,\ldots,x_n)}.$$

For example, the component of $\overline{F}$ at the object $[p; x_1, x_2, x_3]$ would be the component of f_3 at the object $(p; x_1, x_2, x_3)$.

The mapping for 2-cells is just the identity as before. These mappings again constitute a 2-functor in the obvious way and from how they are defined it is also clear that this is an inverse to R. □

Remark 11.15. Every category C determines an endomorphism operad $\mathcal{E}_C$ in **Cat** by defining

$$\mathcal{E}_C(n) = [C^n, C],$$

where the square brackets indicate the functor category. While $\mathcal{E}_C$ is naturally a symmetric operad, it can be given the structure of a Λ-operad for any action operad (Λ, π) using π^* from **??**. The reader can verify that strict P-algebra structures are in bijection with strict maps of Λ-operads $P \to \mathcal{E}_C$, and pseudo-P-algebra structures are in bijection with pseudomorphisms of Λ-operads $P \to \mathcal{E}_C$. It is possible to develop analogues of Lemma 8.18 and Corollary 8.19, but we do not pursue this line of research here.

We finish this section by studying a special case of algebras over a Λ-operad in **Cat** that we call Λ-monoidal categories. These generalize the various kinds of monoidal categories (plain, symmetric, and braided) to any action operad Λ. In order to define Λ-monoidal categories, we must first construct the operads for which they will be algebras.

Definition 11.16.

1. Let X be a set. We define the *translation category* EX to have objects the elements of X and morphisms consisting of a unique isomorphism between any two objects.
2. Let G be a group. The category BG has a single object $*$, and hom-set $BG(*, *) = G$ with composition and identity given by multiplication and the unit element in the group, respectively.

The following lemma is straightforward to verify.

Lemma 11.17. *The functor $E\colon \mathbf{Sets} \to \mathbf{Cat}$ is right adjoint to the set of objects functor. Therefore E preserves all limits, and in particular is a symmetric monoidal functor when both categories are equipped with their cartesian monoidal structures.*

Corollary 11.18. *Let Λ be an action operad. Then $E\Lambda = \{E(\Lambda(n))\}$ (see **??**) is a Λ-operad in* **Cat**.

Proof.

this needs to follow the same structure as Remark 11.5

□

Definition 11.19 (Λ-monoidal categories, functors, and transformations). Let Λ be an action operad.

- A Λ-*monoidal category* is a strict algebra for the Λ-operad $E\Lambda$.
- A Λ-*monoidal functor* is a strict morphism for the Λ-operad $E\Lambda$.
- A Λ-*transformation* is an $E\Lambda$-transformation.

Remark 11.20 ($\underline{E\Lambda}$-algebras are $E\Lambda$-algebras). In each of the items above, we could have expressed the same concept using the 2-monad $\underline{E\Lambda}$ instead of the Λ-operad $E\Lambda$ by Theorem 11.14. The same substitution can be made throughout without changing any of the results. We have just chosen to state definitions and results in terms of operads rather than 2-monads.

Remark 11.21 (Strictness of Λ-monoidal categories).

we only do strict, so like permutative rather than symmetric monoidal cats. can always strictify

Definition 11.22 (The 2-category of Λ-monoidal categories). The 2-category Λ-**MonCat** is the 2-category $E\Lambda$-$\mathbf{Alg}_s$ of strict algebras, strict morphisms, and algebra 2-cells for $E\Lambda$.

start here, had just copied stuff up and it needs to be improved/shortened

The 2-monad $\underline{E\Lambda}$ has underlying 2-functor given by

$$X \mapsto \underline{E\Lambda}(X) = \coprod_{n\geq 0} E\Lambda(n) \otimes_{\Lambda(n)} X^n.$$

We will eventually show that Λ-monoidal categories can be given in more familiar terms, as in Chapter 19 of [48]. Our final result of this section, Proposition 11.23, will give a complete description of the free algebras $\underline{E\Lambda}(X)$.

Proposition 11.23. *For an action operad Λ and any category X, the set of morphisms from $[e; x_1, \ldots, x_n]$ to $[e; y_1, \ldots, y_n]$ in $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is*

Rework into a full description of the free algebra $\underline{E\Lambda}(X)$

$$\coprod_{g\in\Lambda(n)} \prod_{i=1}^{n} X\left(x_i, y_{g(i)}\right).$$

Proof.

copied from exposition

It will be useful for our calculations later to give an explicit description of the categories $E\Lambda(n) \otimes_{\Lambda(n)} X^n$. Objects are equivalence classes of tuples $(g; x_1, \ldots, x_n)$ where $g \in \Lambda(n)$ and the x_i are objects of X, with the equivalence relation given by

$$(gh; x_1, \ldots, x_n) \sim \left(g; x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n)}\right);$$

we write these classes as $[g; x_1, \ldots, x_n]$. Morphisms are then equivalence classes of morphisms

$$(!; f_1, \ldots, f_n) \colon (g; x_1, \ldots, x_n) \to \left(h; x_1', \ldots, x_n'\right).$$

We have two distinguished classes of morphisms, one for which the map $! \colon g \to h$ is the identity and one for which all the f_i's are the identity. Every morphism in $E\Lambda(\underline{n}) \times X^n$ is uniquely a composite of a morphism of the first type followed by one of the second type. Now $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is a quotient of $E\Lambda(\underline{n}) \times X^n$ by a free group action, so every morphism of $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is in the image of the quotient map. Using this fact, we can prove the following useful lemma.

old proof

A morphism with source $(e; x_1, \ldots, x_n)$ in $E\Lambda(\underline{n}) \times X^n$ is uniquely a composite

$$(e; x_1, \ldots, x_n) \xrightarrow{(\mathrm{id}; f_1, \ldots, f_n)} \left(e; x_1', \ldots, x_n'\right) \xrightarrow{(!; \mathrm{id}, \ldots, \mathrm{id})} \left(g; x_1', \ldots, x_n'\right).$$

Descending to the quotient, this becomes a morphism

$$[e; x_1, \ldots, x_n] \to \left[g; x_1', \ldots, x_n'\right] = \left[e; x'_{g^{-1}(1)}, \ldots, x'_{g^{-1}(n)}\right],$$

and therefore is a morphism $[e; x_1, \ldots, x_n] \to [e; y_1, \ldots, y_n]$ precisely when $y_i = x'_{g^{-1}(i)}$, and so $f_i \in X(x_i, y_{g(i)})$. □

Remark 11.24.

moved to the end, shorten/rewrite

The material in this section can be given a rather more abstract interpretation, in the sense of [23]. The idea here is that the category of Λ-collections acts on the category **Cat** via a functor $\diamond \colon \Lambda$-**Coll** $\times$ **Cat** $\to$ **Cat** which sends (P, X) to $\underline{P}(X)$ as described above. Fixing a Λ-collection P produces an endofunctor $\underline{P} \colon$ **Cat** $\to$ **Cat** which is then a monad when P is a Λ-operad, just as monoids in Λ-**Coll** are precisely Λ-operads.

12 Coherence

This section addresses questions of coherence for 2-monads induced by Λ-operads in **Cat**. Coherence theorems take various forms, and we will primarily be concerned with strictification-style coherence theorems. The prototypical example here is the coherence theorem for monoidal categories. In a monoidal category we require associator isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

for all objects in the category. The coherence theorem tells us that, for any monoidal category M, there exists a strict monoidal category which is equivalent to M. In other words, we can treat the associators in M as identities, and similarly for the unit isomorphisms.

By Theorem 11.14, we can study the algebras for a Λ-operad P directly, or do so by studying the algebras for the corresponding 2-monad $\underline{P}$. We first note that the 2-monads induced by Λ-operads are finitary, using standard arguments. Second, we show that the coherence theorem in [29] applies to all such 2-monads and allows us to show that each pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra. cite john power general coherence

Proposition 12.1. *Let P be a Λ-operad. Then $\underline{P}$ is finitary.*

Proof. The argument is identical to that for braided operads in Section 4.1 of [29]. $\square$

We now give an abstract coherence theorem for algebras over a Λ-operad P in **Cat** following the method of John Power [40]. In order to do so, we recall the notion of an enhanced factorization system and Power's coherence result.

Definition 12.2 (Enhanced factorization system).

fix this definition

First we require that the base 2-category $\mathcal{K}$ has an enhanced factorization system. This is much like an orthogonal factorization system on a 2-category, consisting of two classes of maps $(\mathcal{L}, \mathcal{R})$, satisfying the lifting properties on 1-cells and 2-cells as follows. Given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

where $l \in \mathcal{L}$ and $r \in R$, there exists a unique morphism $m\colon B \to C$ such that $rm = g$ and $ml = f$. Similarly, given two commuting squares for which $rf = gl$ and $rf' = f'l$, along with 2-cells $\delta\colon f \Rightarrow f'$ and $\gamma\colon g \Rightarrow g'$ for which $\gamma * 1_l = 1_r * \delta$, there exists a unique 2-cell $\mu\colon m \Rightarrow m'$, where m and m' are induced by the 1-cell lifting property, satisfying $\mu * 1_l = \delta$ and $1_r * \mu = \gamma$. However, there is an additional 2-dimensional property of the factorization system which says that given maps $l \in \mathcal{L}$, $r \in \mathcal{R}$ and an invertible 2-cell $\alpha\colon rf \Rightarrow gl$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\Swarrow\alpha"] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \arrow[d, phantom, shift right=3, "\Swarrow\beta"] \\ B \arrow[r, "g"'] \arrow[ur, "m"] & D \end{tikzcd}$$

there exists a unique pair (m, β) where $m\colon B \to C$ is a 1-cell and $\beta\colon rm \Rightarrow g$ is an invertible 2-cell such that $ml = f$ and $\beta * 1_l = \alpha$.

Theorem 12.3 (cite power)**.** *Let $\mathcal{K}$ be a 2-category, and T be a 2-monad on $\mathcal{K}$. If*

below is copied, fix

Further conditions require that T preserve $\mathcal{L}$ maps and that whenever $r \in \mathcal{R}$ and $rk \cong 1$, then $kr \cong 1$.

resuming

Then the inclusion 2-functor

$$U\colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$$

has a left 2-adjoint, and the components of the unit of the adjunction are equivalences in **Ps**-T-**Alg**. *In particular, every pseudo-T-algebra is equivalent to a strict one.*

Lemma 12.4 (cite power again)**.** *The 2-category* **Cat** *has an enhanced factorization system in which the class $\mathcal{L}$ consists of the functors that are bijective on objects and the class $\mathcal{R}$ consists of the functors that are full and faithful.*

Proposition 12.5. *For any Λ-operad P, the* 2*-monad $\underline{P}$ preserves bijective-on-objects functors.*

Proof.

copied, fix

The last stated condition, involving isomorphisms and maps in $\mathcal{R}$, is then clearly satisfied and so the only thing we need to check in order to satisfy the conditions of the coherence result are that the induced 2-monads $\underline{P}$ preserve bijective-on-objects functors, which follows simply from the fact that the set of objects functor, $\mathrm{Ob}\colon$ **Cat** $\to$ **Set**, preserves colimits, being left adjoint to the indiscrete category functor, $E\colon$ **Set** $\to$ **Cat**, as described in **??**. ☐

Corollary 12.6. *Every pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra.*

Remark 12.7 (Unpacking pseudo-$\underline{P}$-algebras).

explain how these are unbiased and weak

Notation 12.8 (Standard association). Let $(M, \otimes, I, a, l, r)$ be a monoidal category. The *standard association* of a tuple $x_1, \ldots, x_n$ of objects is defined inductively as follows.

1. The standard association of the empty tuple, written $\underline{\emptyset}$, is the unit object I.
2. The standard association of a single object x, written $\underline{x}$, is x itself.
3. Assume that the standard association of n objects $x_1, \ldots, x_n$ has been given as $\underline{x_1 \cdots x_n}$. The standard association of $n+1$ objects $x_1, \ldots, x_{n+1}$ is defined by the formula
$$\underline{x_1 \cdots x_{n+1}} = x_1 \otimes \underline{x_2 \cdots x_{n+1}}.$$

Definition 12.9 (Weak Λ-monoidal categories). A *weak Λ-monoidal category* consists of

- a monoidal category $(M, \otimes, I, a, l, r)$ and
- a natural isomorphism
$$[g]\colon \underline{x_1 \ldots x_n} \cong \underline{x_{g^{-1}(1)} \cdots x_{g^{-1}(n)}}$$
for each $g \in \Lambda(n)$

satisfying the following two axioms.

1. For every pair $g, h \in \Lambda(n)$, the composite $[h] \circ [g]$ shown below
$$\underline{x_1 \ldots x_n} \xrightarrow{[g]} \underline{x_{g^{-1}(1)} \cdots x_{g^{-1}(n)}} \xrightarrow{[h]} \underline{x_{g^{-1}(h^{-1}(1))} \cdots x_{g^{-1}(h^{-1}(n))}}$$
equals $[hg]$, where $hg \in \Lambda(n)$ is given by multiplication using the group structure.
2. then one about operadic composition, or maybe do two more with deltas and betas

now need weak Λ-monoidal functors, then 2-category Wk-Λ-**MonCat** of weak lmc's, weak lmf's, and monoidal transformations

Theorem 12.10. *Let Λ be an action operad.*

1. *There is an inclusion 2-functor*

$$i\colon \Lambda\text{-}\mathbf{MonCat} \to \mathit{Wk}\text{-}\Lambda\text{-}\mathbf{MonCat},$$

the image of which consists of those weak Λ-monoidal categories for which the underlying monoidal category is strict.

2. *Every weak Λ-monoidal category is equivalent, in Wk-Λ-**MonCat**, to one in the image of i.*

Remark 12.11 (Pseudo-$E\Lambda$-algebras versus weak Λ-monoidal categories). compare the pseudoalgebras and the stuff above

13 Group Actions and Cartesian 2-monads

Why do people care??

rewrite this section intro, maybe the whole point is the relationship between cartesian-ness and free group actions

We finally turn to a discussion of the interaction between operads and pullbacks. The monads arising from a non-symmetric operad are always cartesian, as described in [32]. The monads that arise from symmetric operads, however, are not always cartesian and so it is useful to be able to characterize exactly when they are. An example of where this fails is the symmetric operad for which the algebras are commutative monoids. In the case of 2-monads we can consider the strict 2-limit analogous to the pullback, the 2-pullback, and characterize when the induced 2-monad from a Λ-operad is 2-cartesian, as we now describe.

Definition 13.1 (2-pullback). Let $\mathcal{K}$ be a 2-category,

fill in

Definition 13.2 (Preserving 2-pullbacks). Let $\mathcal{K}, \mathcal{L}$ be a 2-categories with all 2-pullbacks, and $F\colon \mathcal{K} \to \mathcal{L}$ a 2-functor between them. Then F *preserves 2-pullbacks* if the morphism

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$

induced by the universal property is an isomorphism.

Definition 13.3 (2-cartesian 2-monad). A 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ is said to be 2-*cartesian* if

- the 2-category $\mathcal{K}$ has 2-pullbacks,
- the functor T preserves 2-pullbacks, and
- the naturality squares for the unit and multiplication of the 2-monad are 2-pullbacks.

It is important to note that the 2-pullback of a diagram is actually the same as the ordinary pullback in **Cat**, see [19].

We begin our study of the cartesian property in the context of symmetric operads.

Proposition 13.4. *Let P be a symmetric operad. Then the unit $\eta\colon id \Rightarrow \underline{P}$ for the associated monad is a cartesian transformation.*

Proof. In order to show that η is cartesian, we must prove that for a functor $f\colon X \to Y$, the pullback of the diagram below is the category X.

$$
\begin{tikzcd}
 & Y \arrow[d, "\eta_Y"] \\
\coprod P(n) \otimes_{\Sigma_n} X^n \arrow[r, "\underline{P}(f)"'] & \coprod P(n) \otimes_{\Sigma_n} Y^n
\end{tikzcd}
$$

The pullback of this diagram is isomorphic to the coproduct of the pullbacks of diagrams of the following form.

$$
\begin{tikzcd}
 & Y \arrow[d] \\
P(1) \times X \arrow[r, "1\times f"'] & P(1) \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \emptyset \arrow[d] \\
P(n) \otimes_{\Sigma_n} X^n \arrow[r, "1\times f^n"'] & P(n) \otimes_{\Sigma_n} Y^n
\end{tikzcd}
$$

$$n \neq 1$$

It is easy then to see that X is the pullback of the $n = 1$ cospan, and that the empty category is the pullback of each of the other cospans, making X the pullback of the original diagram and verifying that η is cartesian. □

Proposition 13.5. *Let P be a symmetric operad. Then the 2-monad $\underline{P}$ preserves pullbacks if and only if Σ_n acts freely on $P(n)$ for all n.*

Proof. Consider the following pullback of discrete categories.

$$
\begin{tikzcd}
\{(x,y),(x,y'),(x',y),(x',y')\} \arrow[r] \arrow[d] & \{y,y'\} \arrow[d] \\
\{x,x'\} \arrow[r] & \{z\}
\end{tikzcd}
$$

Letting $\mathbf{4}$ denote the pullback and similarly writing $\mathbf{2}_X = \{x, x'\}$ and $\mathbf{2}_Y = \{y, y'\}$, the following diagram results as the image of this pullback square under $\underline{P}$.

$$
\begin{tikzcd}
\coprod P(n) \otimes_{\Sigma_n} \mathbf{4}^n \arrow[r] \arrow[d] & \coprod P(n) \otimes_{\Sigma_n} \mathbf{2}_Y^n \arrow[d] \\
\coprod P(n) \otimes_{\Sigma_n} \mathbf{2}_X^n \arrow[r] & \coprod P(n)/\Sigma_n
\end{tikzcd}
$$

The projection map $\underline{P}(\mathbf{4}) \to \underline{P}(\mathbf{2}_Y)$ maps an element

$$[p; (x_1, y_1), \ldots, (x_n, y_n)]$$

to the element

$$[p; y_1, \ldots, y_n]$$

and likewise for the projection to $\underline{P}(\mathbf{2}_X)$.

Now assume that, for some n, the action of Σ_n on $P(n)$ is not free. Then find some $p \in P(n)$ along with a nonidentity element $g \in \Sigma_n$ such that $p \cdot g = p$. We will show that the existence of g proves that $\underline{P}$ is not cartesian.

Now $g \neq e$, so there exists an $i \in \{1, \ldots, n\}$ such that $g(i) \neq i$; without loss of generality, we may take $i = 1$. Using this g we can find two distinct elements

$$\big[p; (x', y), (x, y), \ldots, (x, y), (x, y'), (x, y), \ldots, (x, y)\big]$$

and

$$\big[p; (x, y), \ldots, (x, y), (x', y'), (x, y), \ldots, (x, y)\big]$$

in $\underline{P}(\mathbf{4})$. In the first element we put (x', y) in the first position and (x, y') in position $g(1)$, whilst in the second element we put (x', y') in position $g(1)$. Both of these elements, however, are mapped to the same elements in $\underline{P}(\mathbf{2}_X)$, since

$$\begin{aligned}
\big[p; x', x, \ldots, x\big] &= \big[p \cdot g; (x', x, \ldots, x)\big] \\
&= \big[p; g \cdot (x', x, \ldots, x)\big] \\
&= \big[p; x, x, \ldots, x', \ldots, x\big] .
\end{aligned}$$

Similarly, both of the elements are mapped to the same element in $\underline{P}(\mathbf{2}_Y)$, simply

$$\big[p; y, \ldots, y', \ldots, y\big] .$$

The pullback of this diagram, however, has a unique element which is projected to the ones we have considered, so $\underline{P}(\mathbf{4})$ is not a pullback. Hence $\underline{P}$ does not preserve pullbacks if for some n the action of Σ_n on $P(n)$ is not free.

Now assume that each Σ_n acts freely on $P(n)$. Given a pullback

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "R"'] & B \arrow[d, "S"] \\
C \arrow[r, "H"'] & D
\end{tikzcd}$$

we must show that the image of the diagram under $\underline{P}$ is also a pullback. Now this will be true if and only if each individual diagram

$$\begin{tikzcd}
P(n) \otimes_{\Sigma_n} A^n \arrow[r, "1 \otimes_{\Sigma_n} F^n"] \arrow[d, "1 \otimes_{\Sigma_n} R^n"'] & P(n) \otimes_{\Sigma_n} B^n \arrow[d, "1 \otimes_{\Sigma_n} S^n"] \\
P(n) \otimes_{\Sigma_n} C^n \arrow[r, "1 \otimes_{\Sigma_n} H^n"'] & P(n) \otimes_{\Sigma_n} D^n
\end{tikzcd}$$

is also a pullback.

rework the rest of this slightly to just check the pb universal property directly

The pullback of the functors $1 \otimes_{\Sigma_n} H^n$ and $1 \otimes_{\Sigma_n} S^n$ is a category consisting of pairs of objects $[p; \underline{c}]$ and $[q; \underline{b}]$, where $\underline{b}$ and $\underline{c}$ represent lists of elements in B and C, respectively, that satisfy the property that

$$\Big[p; \underline{H(c)}\Big] = \Big[q; \underline{S(b)}\Big]$$

in $P(n) \otimes_{\Sigma_n} D^n$. Since the action is free by hypothesis, Lemma 11.4 implies that a pair

$$([p; \underline{c}], [q; \underline{b}])$$

is in the pullback if and only if there exists a necessarily unique element $g \in \Sigma_n$ such that $p \cdot g = q$ and $Hc_i = (Sb_{g^{-1}(i)})$. The morphisms in the pullback admit a similar description.

Using this we can define mutual inverses between $P(n) \otimes_{\Sigma_n} A^n$ and the pullback Q'. Considering the category A as the pullback of the diagram we started with, we can consider objects of $P(n) \otimes_{\Sigma_n} A^n$ as being equivalence classes

$$[p; (b_1, c_1), \dots, (b_n, c_n)]$$

where $p \in P(n)$ and $Hc_i = Sb_i$ for all i.

Taking such an object, we send it to the pair

$$([p; c_1, \dots, c_n], [p; b_1, \dots, b_n]),$$

and note that it lies in the pullback since the identity in Σ_n satisfies the condition given earlier. An inverse to this sends a pair of equivalence classes in Q' to the single equivalence class

$$\left[p; \left(c_1, b_{g^{-1}(1)}\right), \dots, \left(c_n, b_{g^{-1}(n)}\right)\right]$$

in $P(n) \otimes_{\Sigma_n} A^n$. If we apply the map into Q' we get the pair

$$\left([p; c_1, \dots, c_n], \left[p; b_{g^{-1}(1)}, \dots, b_{g^{-1}(n)}\right]\right)$$

which is equal to the original pair since $p \cdot g = q$; the other composite is trivially an identity. A similar calculation on morphisms finishes the proof that $P(n) \otimes_{\Sigma_n} A^n$ is the pullback as required. $\square$

Proposition 13.6. *Let P be a symmetric operad. If the Σ_n-actions are all free, then the multiplication $\mu \colon \underline{P}^2 \Rightarrow \underline{P}$ of the associated monad is a cartesian transformation.*

Proof. Note that if all of the diagrams

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_X"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

are pullbacks then the outside of the diagram

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(f)"] \arrow[d, "\mu_X"'] & \underline{P}^2(Y) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_Y"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(f)"'] & \underline{P}(Y) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

is also a pullback and so each of the naturality squares for μ must therefore be a pullback. Now we can split up the square above, much like we did for η, and prove that each of the squares below is a pullback.

$$
\begin{tikzcd}
\coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i) \otimes_{\Sigma_{k_i}} X^{k_i} \right) \arrow[r] \arrow[d] & \coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i)/\Sigma_{k_i} \right) \arrow[d] \\
P(n) \otimes_{\Sigma_n} X^n \arrow[r] & P(n)/\Sigma_n
\end{tikzcd}
$$

The map along the bottom is the obvious one, sending $[p; x_1, \dots, x_n]$ simply to the equivalence class $[p]$. Along the right hand side the map is the one corresponding to operadic composition, sending $[q; [p_1], \dots, [p_m]]$ to $[\mu^P(q; p_1, \dots, p_n)]$. The pullback of these maps would be the category consisting of pairs

$$([p; x_1, \dots, x_{\Sigma k_i}], [q; [p_1], \dots, [p_n]]) \, ,$$

where $q \in P(n)$, $p_i \in P(k_i)$, $p \in P(\Sigma k_i)$, and for which $[p] = [\mu^P(q; p_1, \dots, p_n)]$. The upper left category in the diagram, which we will refer to here as Q, has objects

$$[q; [p_1; \underline{x}_1], \dots, [p_n; \underline{x}_n]] \, .$$

There are obvious maps out of Q making the diagram commute and as such inducing a functor from Q into the pullback via the universal property. This functor sends an object such as the one just described to the pair

$$\left(\left[\mu^P(q; p_1, \dots, p_n); \underline{x} \right], [q; [p_1], \dots, [p_n]] \right) .$$

Given an object in the pullback, we then have a pair, as described above, which has $[p] = [\mu^P(q; p_1, \dots, p_n)]$ meaning that we can find an element $g \in \Sigma_{\Sigma k_i}$ such that $p = \mu^P(q; p_1, \dots, p_n) \cdot g$. Thus we can describe an inverse to the induced functor by sending a pair in the pullback to the object

$$[q; [p_1; \pi(g)(\underline{x})_1], \dots, [p_n; \pi(g)(\underline{x})_n]],$$

where $\pi(g)(\underline{x})_i$ denotes the ith block of $\underline{x}$ after applying the permutation $\pi(g)$. For example, if $\underline{x} = (x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31})$ and $\pi(g) = (1\ 3\ 5)$, then

$$\pi(g)(\underline{x}) = (x_{23}, x_{12}, x_{11}, x_{22}, x_{21}, x_{31}).$$

Thus $\pi(g)(\underline{x})_1 = (x_{23}, x_{12})$, $\pi(g)(\underline{x})_2 = (x_{11}, x_{22}, x_{21})$ and $\pi(g)(\underline{x})_3 = (x_{31})$.

Now applying the induced functor we find that we get back an object in the pullback for which the first entry is $[q; [p_1], \dots, [p_n]]$ and whose second entry is

$$\left[\mu^P(q; p_1, \dots, p_n); \pi(g)(\underline{x}) \right] = \left[\mu^P(q; p_1, \dots, p_n) \cdot g; \underline{x} \right] = [p; \underline{x}],$$

which is what we started with. Showing the other composite is an identity is similar, here using the fact that the identity acts trivially on $\mu^P(q; p_1, \dots, p_n)$. Taking the coproduct of these squares then gives us the original diagram that we wanted to show was a pullback and, since each individual square is a pullback, so is the original. $\square$

Collecting these results together gives the following corollary.

Corollary 13.7. *The 2-monad associated to a symmetric operad P is 2-cartesian if and only if the action of Σ_n is free on each $P(n)$.*

We require one simple technical lemma before giving a complete characterization of Λ-operads that induce cartesian 2-monads.

Lemma 13.8. *Let C be a category with a right action of some group Λ, and let $\pi\colon \Lambda \to \Sigma$ be a group homomorphism to any other group Σ. Then the right Σ-action on $C \otimes_\Lambda \Sigma$ is free if and only if the only elements of Λ that fix an object of C lie in the kernel of π.*

Proof. First, note that a group action on a category is free if and only if it is free on objects as fixing a morphism requires fixing its source and target. Thus our arguments need only concern the objects involved.

Since the set of objects functor preserves colimits, the objects of $C \otimes_\Lambda \Sigma$ are equivalence classes $[c; g]$ where $c \in C$ and $g \in \Sigma$, with $[c \cdot r; g] = [c; \pi(r)g]$. First assume the Σ-action is free. Then noting that $[c; e] \cdot g = [c; g]$, we have if $[c; g] = [c; e]$ then $g = e$. Let $r \in \Lambda$ be an element such that $c \cdot r = c$. Then

$$[c; e] = [c \cdot r; e] = [c; \pi(r)],$$

so $\pi(r) = e$.

Now assume that every element of Λ fixing an object lies in the kernel of π. Let $\tau \in \Sigma$, and assume it fixes $[p; \sigma]$. Without loss of generality, we can take $\sigma = e$, so that

$$[p; \tau] = [p; e] \cdot \tau = [p; e].$$

Since the objects of $C \otimes_\Lambda \Sigma$ are equivalence classes as above, there exists an element $r \in \Lambda$ such that $p \cdot r^{-1} = p$ and $\tau = \pi(r)$. But by assumption, we must have r^{-1}, and hence r, in the kernel, so $\tau = e$ and the Σ-action is free. $\square$

Theorem 13.9. *The 2-monad $\underline{P}$ associated to a Λ-operad P is 2-cartesian if and only if whenever $p \cdot g = p$ for an object $p \in P(n)$, $g \in Ker\,\pi(n)$.*

Proof. Since the monad $\underline{P}$ is isomorphic to $\underline{S(P)}$, we need only verify when $\underline{S(P)}$ is 2-cartesian. Thus the theorem is a direct consequence of **??** and **??**. $\square$

Corollary 13.10. *Let Λ be an action operad in* **Sets**. *Then the 2-monad $\underline{E\Lambda}$ is 2-cartesian.*

Proof. The action of $\Lambda(n)$ on $E\Lambda(n)$ is free for all n, so in particular satisfies the conditions in Theorem 13.9. $\square$

14 Action Operads as Clubs

Kelly's theory of clubs [18, 20, 21] was designed to simplify and explain how coherence results for a 2-monad T can often be extracted from information about the specific free object $T1$ where 1 denotes the terminal category. This occurs, for example, in the study of the many different types of monoidal category: plain monoidal category, braided monoidal category, symmetric monoidal category, and so on. This section will explain how every action operad gives rise to a club, as well as compute the clubs that arise as the image of this procedure.

We begin by reminding the reader of the notion of a club, or more specifically what Kelly [18, 21] calls a club over **P**. We will only be interested in clubs over **P**, and thusly shorten the terminology to club from this point onward. We define clubs succinctly using Leinster's terminology of generalized operads [32].

Definition 14.1. Let C be a category with finite limits.

1. A monad $T\colon C \to C$ is *cartesian* if the functor T preserves pullbacks, and the naturality squares for the unit η and the multiplication μ for T are all pullbacks.

2. The category of T-*collections*, T-**Coll**, is the slice category $C/T1$, where 1 denotes the terminal object.
3. Given a pair of T-collections $X \xrightarrow{x} T1, Y \xrightarrow{y} T1$, their *composition product* $X \circ Y$ is given by the pullback below together with the morphism along the top.

$$\begin{tikzcd}
X \circ Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TY \arrow[r, "Ty"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu"] & T1 \\
X \arrow[r, "x"'] & T1
\end{tikzcd}$$

4. The composition product, along with the unit of the adjunction $\eta\colon 1 \to T1$, give T-**Coll** a monoidal structure. A T-*operad* is a monoid in T-**Coll**.

Remark 14.2. Everything in the above definition can be **Cat**-enriched without any substantial modifications. Thus we require our ground 2-category to have finite limits in the enriched sense, and the slice and pullbacks are the 2-categorical (and not bicategorical) versions. If we take this 2-category to be **Cat**, then in each case the underlying category of the 2-categorical construction is given by the corresponding 1-categorical version. From this point, we will not distinguish between the 1-dimensional and 2-dimensional theory. Our interest, of course, is in the 2-dimensional version.

Let Σ be the operad of symmetric groups. This is the terminal object of the category of action operads, with each π_n the identity map. Then $\underline{E\Sigma}$ is a 2-monad on **Cat**, and by Corollary 13.10 it is cartesian.

Definition 14.3. A *club* is a T-operad in **Cat** for $T = \underline{E\Sigma}$.

Remark 14.4. The category **P** in Kelly's terminology is the result of applying $\underline{E\Sigma}$ to 1, and can be identified with the category $B\Sigma$ of Definition 9.1.

It is useful to break down the definition of a club. A club consists of

1. a category K together with a functor $\pi\colon K \to B\Sigma$,
2. a multiplication map $K \circ K \to K$, and
3. a unit map $1 \to K$,

satisfying the axioms to be a monoid in the monoidal category of $E\Sigma$-collections. By the definition of $K \circ K$ as a pullback, objects are tuples of objects of K $(x; y_1, \ldots, y_n)$ where $\pi(x) = n$. A morphism

$$(x; y_1, \ldots, y_n) \to (z; w_1, \ldots, w_m)$$

exists only when $n = m$ (since $B\Sigma$ only has endomorphisms) and then consists of a morphism $f\colon x \to z$ in K together with morphisms $g_i\colon y_i \to z_{x(i)}$ in K.

Notation 14.5. For a club K and a morphism $(f; g_1, \ldots, g_n)$ in $K \circ K$, we write $f(g_1, \ldots, g_n)$ for the image of the morphism under the functor $K \circ K \to K$.

We will usually just refer to a club by its underlying category K.

Theorem 14.6. *Let Λ be an action operad. Then the map of operads $\pi\colon \Lambda \to \Sigma$ gives the category $B\Lambda = \coprod B\Lambda(n)$ the structure of a club.*

Proof.

This probably needs a bit of a rewrite, using the to-be-updated 12.23

To give the functor $B\pi\colon B\Lambda \to B\Sigma$ the structure of a club it suffices (see [32]) to show that

find specific thm number

- the induced monad, which we will show to be $\underline{E\Lambda}$, is a cartesian monad on **Cat**,
- the transformation $\tilde{\pi} \colon \underline{E\Lambda} \Rightarrow \underline{E\Sigma}$ induced by the functor $E\pi$ is cartesian, and
- $\tilde{\pi}$ commutes with the monad structures.

The monad $\underline{E\Lambda}$ is always cartesian by Corollary 13.10. The transformation $\tilde{\pi}$ is the coproduct of the maps $\tilde{\pi}_n$ that are induced by the universal property of the coequalizer as shown below.

$$\begin{tikzcd}
E\Lambda(\underline{n}) \times \Lambda(n) \times X^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "E\pi \times \pi \times 1"'] & E\Lambda(\underline{n}) \times X^n \arrow[r] \arrow[d, "E\pi \times 1"'] & E\Lambda(\underline{n}) \otimes_{\Lambda(n)} X^n \arrow[d, dotted, "\tilde{\pi}_n"] \\
E\Sigma_n \times \Sigma_n \times X^n \arrow[r, shift left] \arrow[r, shift right] & E\Sigma_n \times X^n \arrow[r] & E\Sigma_n \otimes_{\Sigma_n} X^n
\end{tikzcd}$$

Naturality is immediate, and since π is a map of operads $\tilde{\pi}$ also commutes with the monad structures.

It only remains to show that $\tilde{\pi}$ is cartesian and that the induced monad is actually $\underline{E\Lambda}$. Since the monads $\underline{E\Lambda}$ and $\underline{E\Sigma}$ both decompose into a disjoint union of functors, we only have to show that, for any n, the square below is a pullback.

$$\begin{tikzcd}
E\Lambda(n) \otimes_{\Lambda(n)} X^n \arrow[r] \arrow[d] & E\Sigma_n \otimes_{\Sigma_n} X^n \arrow[d] \\
B\Lambda(n) \arrow[r] & B\Sigma_n
\end{tikzcd}$$

By **??** , this amounts to showing that the square below is a pullback.

$$\begin{tikzcd}
\left(E\Lambda(\underline{n}) \times X^n\right)/\Lambda(n) \arrow[r] \arrow[d] & \left(E\Sigma_n \times X^n\right)/\Sigma_n \arrow[d] \\
B\Lambda(n) \arrow[r] & B\Sigma_n
\end{tikzcd}$$

Here, $(A \times B)/G$ is the category whose objects are equivalence classes of pairs (a, b) where $(a, b) \sim (ag, g^{-1}b)$, and similarly for morphisms. Now the bottom map is clearly bijective on objects since these categories only have one object. An object in the top right is an equivalence class

$$[\sigma; x_1, \ldots, x_n] = \left[e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right].$$

A similar description holds for objects in the top left, with $g \in \Lambda(n)$ replacing σ and $\pi(g)^{-1}$ replacing σ^{-1} in the subscripts. The map along the top sends $[g; x_1, \ldots, x_n]$ to $[\pi(g); x_1, \ldots, x_n]$, and thus sends $[e; x_1, \ldots, x_n]$ to $[e; x_1, \ldots, x_n]$, giving a bijection on objects.

Now a morphism in $(E\Lambda(\underline{n}) \times X^n)/\Lambda(n)$ can be given as

$$[e; x_1, \ldots, x_n] \xrightarrow{[!; f_i]} [g; y_1, \ldots, y_n].$$

Mapping down to $B\Lambda(n)$ gives $ge^{-1} = g$, while mapping over to $(E\Sigma_n \times X^n)/\Sigma_n$ gives $[!; f_i]$ where $! \colon e \to \pi(g)$ is now a morphism in $E\Sigma_n$. In other words, a morphism in the upper left corner of our putative pullback square is determined completely by its images along the top and lefthand functors. Furthermore, given $g \in \Lambda(n)$, $\tau = \pi(g)$, and morphisms $f_i \colon x_i \to y_i$ in X, the morphism $[! \colon e \to g; f_i]$ maps to the pair $(g, [! \colon e \to \tau; f_i])$, completing the proof that this square is indeed a pullback. $\square$

The club, which we now denote K_Λ, associated to $E\Lambda$ has the following properties. First, the functor $K_\Lambda \to B\Sigma$ is a functor between groupoids. Second, the functor $K_\Lambda \to B\Sigma$ is bijective-on-objects. We claim that these properties characterize those clubs which arise from action operads. Thus the clubs arising from action operads are a special class of PROPs [34, 36].

Theorem 14.7. *Let K be a club such that*

- *the map $K \to B\Sigma$ is bijective on objects and*
- *K is a groupoid.*

Then $K \cong K_\Lambda$ for some action operad Λ. The assignment $\Lambda \mapsto K_\Lambda$ is a full and faithful embedding of the category of action operads **AOp** *into the category of clubs.*

Proof.

shorten?

Let K be such a club. Our hypotheses immediately imply that K is a groupoid with objects in bijection with the natural numbers; we will now assume the functor $K \to B\Sigma$ is the identity on objects. Let $\Lambda(n) = K(n,n)$. Now K comes equipped with a functor to $B\Sigma$, in other words group homomorphisms $\pi_n \colon \Lambda(n) \to \Sigma_n$. We claim that the club structure on K makes the collection of groups $\{\Lambda(n)\}$ an action operad. In order to do so, we will employ Theorem 4.15.

First, we give the group homomorphism β using Notation 14.5. Define

$$\beta(g_1, \ldots, g_n) = e_n(g_1, \ldots, g_n)$$

(see 14.5) where e_n is the identity morphism $n \to n$ in $K(n,n)$. Functoriality of the club multiplication map immediately implies that this is a group homomorphism. Second, we define the function δ in a similar fashion:

$$\delta_{n;k_1,\ldots,k_n}(f) = f(e_1, \ldots, e_n),$$

where here e_i is the identity morphism of k_i in K.

There are now nine axioms to verify in Theorem 4.15. The club multiplication functor is a map of collections, so a map over $B\Sigma$; this fact immediately implies that Axioms (1) (using morphisms in $K \circ K$ with only g_i parts) and (4) (using morphisms in $K \circ K$ with only f parts) hold. The mere fact that multiplication is a functor also implies Axioms (6) (once again using morphisms with only f parts) and (8) (by considering the composite of a morphism with only an f with a morphism with only g_i's). Axiom (2) is the equation $e_1(g) = g$ which is a direct consequence of the unit axiom for the club K; the same is true of Axiom (5). Axioms (3), (7), and (9) all follow from the associativity of the club multiplication.

Finally, we would like to show that this gives a full and faithful embedding

$$K_- \colon \mathbf{AOp} \to \mathbf{Club}$$

of the category of action operads into the category of clubs. Let $f, f' \colon \Lambda \to \Lambda'$ be maps between action operads. Then if $K_f = K_{f'}$ as maps between clubs, then they must be equal as functors $K_\Lambda \to K_{\Lambda'}$. But these functors are nothing more than the coproducts of the functors

$$B(f_n), B(f'_n) \colon B\Lambda(n) \to B\Lambda'(n),$$

and the functor B from groups to categories is faithful, so K_- is also faithful. Now let $f \colon K_\Lambda \to K_{\Lambda'}$ be a maps of clubs. We clearly get group homomorphisms $f_n \colon \Lambda(n) \to \Lambda'(n)$ such that $\pi_n^\Lambda = \pi_n^{\Lambda'} f_n$, so we must only show that the f_n also constitute an operad map. Using the description of the club structure above in terms of the maps β, δ, we are able to see that commuting with the club multiplication implies commuting with both of these, which in turn is equivalent to commuting with operad multiplication. Thus K_- is full as well. $\square$

Remark 14.8.

shorten?

First, one should note that being a club over $B\Sigma$ means that every K-algebra has an underlying strict monoidal structure. Second, requiring that $K \to B\Sigma$ be bijective on objects ensures that K does not have operations other than $\otimes$, such as duals or internal hom-objects, from which to build new types of objects. Finally, K being a groupoid ensures that all of the "constraint morphisms" that exist in algebras for K are invertible.

These hypotheses could be relaxed somewhat. Instead of having a club over $B\Sigma$, we could have a club over the free symmetric monoidal category on one object (note that the free symmetric monoidal category monad on **Cat** is still cartesian). This would produce K-algebras with underlying monoidal structures which are not necessarily strict. This change should have relatively little impact on how the theory is developed. Changing K to be a category instead of a groupoid would likely have a larger impact, as the resulting action operads would have monoids instead of groups at each level. We have made repeated use of inverses throughout the proofs in the basic theory of action operads, and these would have to be revisited if groups were replaced by monoids in the definition of action operads.

need specific references in Kelly

In [18], Kelly discusses clubs given by generators and relations. His generators include functorial operations more general than what we are interested in here, and the natural transformations are not required to be invertible. In our case, the only generating operations we require are those of a unit and tensor product, as the algebras for $E\Lambda$ are always strict monoidal categories with additional structure. Tracing through his discussion of generators and relations for a club gives the following theorem.

I worry about the rigor of the next two results

Theorem 14.9. *Let Λ be an action operad with presentation given by $(\mathbf{g}, \mathbf{r}, s_i, p)$. Then the club $E\Lambda$ is generated by*

- *functors giving the unit object and tensor product, and*
- *natural transformations given by the collection $\mathbf{g}$: each element x of $\mathbf{g}$ with $\pi(x) = \sigma_x \in \Sigma_{|x|}$ gives a natural transformation from the nth tensor power functor to itself,*

subject to relations such that the following axioms hold.

- *The monoidal structure given by the unit and tensor product is strict.*
- *The transformations given by the elements of $\mathbf{g}$ are all natural isomorphisms.*
- *For each element $y \in \mathbf{r}$, the equation $s_1(y) = s_2(y)$ holds.*

Bringing this down to a concrete level we have the following corollary.

Corollary 14.10. *Assume we have a notion $\mathcal{M}$ of strict monoidal category which is given by a set natural isomorphisms*

$$\mathcal{G} = \Big\{ (f, \pi_f) \,|\, x_1 \otimes \cdots \otimes x_n \xrightarrow{f} x_{\pi_f^{-1}(1)} \otimes \cdots \otimes x_{\pi_f^{-1}(n)} \Big\}$$

subject to a set $\mathcal{R}$ of axioms. Each such axiom is given by the data

- *two finite sets* $f_1, \ldots, f_n$ *and* $f_1', \ldots, f_m'$ *of elements of* $\mathcal{G}$*; and*
- *two formal composites* F, F' *using only composition and tensor product operations and the* f_i*, respectively* f_i'*,*

such that the underlying permutation of F *equals the underlying permutation of* F' *(we compute the underlying permutations using the functions* β, δ *of Theorem 4.15). The element* $\left(\underline{f}, \underline{f}', F, F'\right)$ *of the set* $\mathcal{R}$ *of axioms corresponds to the requirement that the composite of the morphisms* f_i *using* F *equals the composite of the morphisms* f_j' *using* F' *in any strict monoidal category of type* $\mathcal{M}$*. Then strict monoidal categories of type* $\mathcal{M}$ *are given as the algebras for the club* $E\Lambda$ *where* Λ *is the action operad with*

- $\mathbf{g} = \mathcal{G}$*,*
- $\mathbf{r} = \mathcal{R}$*,*
- s_1 *given by mapping the generator* $\left(\underline{f}, \underline{f}', F, F'\right)$ *to the operadic composition of the* f_i *using* F *via* β, δ*, and*
- s_2 *given by mapping the generator* $\left(\underline{f}, \underline{f}', F, F'\right)$ *to the operadic composition of the* f_i' *using* F' *via* β, δ*.*

Example 14.11.

shorten?

The 2-monad for symmetric strict monoidal categories (or permutative categories, as they are known in the topological literature) is given by $E\Sigma$, so the notion of symmetric strict monoidal categories corresponds to the symmetric operad. While this example is well-known, we go into further detail to set the stage for less common examples.

The 2-monad $\underline{E\Sigma}$ on **Cat** is given by

$$\underline{E\Sigma}(X) = \coprod E\Sigma_n \times_{\Sigma_n} X^n.$$

An object of $E\Sigma_n \times_{\Sigma_n} X^n$ is an equivalence class of the form $[\sigma; x_1, \ldots, x_n]$ where $\sigma \in \Sigma_n$ and $x_i \in X$. The equivalence relation gives

$$[\sigma; x_1, \ldots, x_n] = \left[e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right],$$

so objects can be identified with finite strings of objects of X. Morphisms are given by equivalence classes of the form

$$[\sigma; x_1, \ldots, x_n] \xrightarrow{[!; f_1, \ldots, f_n]} [\tau; y_1, \ldots, y_n].$$

Here $! \colon \sigma \cong \tau$ is the unique isomorphism in $E\Sigma_n$, and $f_i \colon x_i \to y_i$ in X. Using the equivalence relation, we find that morphisms between finite strings

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

are given by a permutation $\rho \in \Sigma_n$ together with maps $f_i \colon x_i \to y_{\rho(i)}$ in X (note that there are no morphisms between strings of different length); this is a special case of the calculation in **??**. Thus $E\Sigma(X)$ is easily seen to be the free permutative category generated by X, and therefore Σ-monoidal categories are permutative categories.

Example 14.12.

shorten?

The template above can be used to show that the braid operad B corresponds to the 2-monad for braided strict monoidal categories. The details are almost exactly the same, only we use braids instead of permutations. The equivalence relation on objects gives

$$[\gamma; x_1, \ldots, x_n] = \left[e; x_{\pi(\gamma)^{-1}(1)}, \ldots, x_{\pi(\gamma)^{-1}(n)}\right],$$

where $\gamma \in B_n$ and $\pi(\gamma)$ is its underlying permutation; thus objects of $EB(X)$ are once again finite strings of objects of X. A morphism

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

is then given by a braid $\gamma \in B_n$ together with maps $f_i \colon x_i \to y_{\pi(\gamma)(i)}$ in X. Thus one should view a morphism as given by

- a finite ordered set $x_1, \ldots, x_n$ of objects of X as the source,
- another such finite ordered set (of the same cardinality) $y_1, \ldots, y_n$ of objects of X as the target,
- a geometric braid $\gamma \in B_n$ on n strands, and
- for each strand, a morphism in X from the object labeling the source of that strand to the object labeling the target.

This is precisely Joyal and Street's [17] construction of the free braided strict monoidal category generated by a category X, and thus B-monoidal categories are braided strict monoidal categories.

This example can be extended to include ribbon braided categories as well. A *ribbon braid* is given, geometrically, in much the same way as a braid except that instead of paths $[0,1] \to \mathbb{R}^3$ making up each individual strand, we use ribbons $[0,1] \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$. This introduces the possibility of performing a full twist on a ribbon, and one can describe ribbon braided categories using generators and relations by introducing a natural twist isomorphism $\tau_A \colon A \to A$ and imposing one relation between the twist and the braid $\gamma_{A,B} \colon A \otimes B \to B \otimes A$. In [43], the authors show that the ribbon braid groups give an action operad RB, and that (strict) ribbon braided categories are precisely the algebras for ERB.

15 Pseudocommutativity

shorter intro, get to the point here, maybe move some of this to paper intro

This section gives conditions sufficient to equip the 2-monad $\underline{P}$ induced by a Λ-operad P in **Cat** with a pseudo-commutative structure. Such a pseudo-commutativity will then give the 2-category **Ps**-$\underline{P}$-**Alg** some additional structure that we briefly explain here. For a field k, the category **Vect** of vector spaces over k has many nice features. Of particular interest to us are the following three structures. First, the category **Vect** is monoidal using the tensor product $\otimes_k$. Second, the set of linear maps $V \to W$ is itself a vector space which we denote $[V, W]$. Third, there is a notion of multilinear map $V_1 \times \cdots \times V_n \to W$, with linear maps being the 1-ary version. While these three structures are each useful in isolation, they are tied together by natural isomorphisms

$$\mathbf{Vect}(V_1 \otimes V_2, W) \cong \mathbf{Vect}(V_1, [V_2, W]) \cong \mathbf{Bilin}(V_1 \times V_2, W)$$

expressing that $\otimes$ gives a closed monoidal structure which represents the multicategory of multilinear maps. Moreover, the adjunction between **Vect** and **Sets** respects all of this structure in the appropriate way. This incredibly rich interplay between the tensor product, the internal mapping space, and the multicategory of multilinear maps all arises from the free vector space monad on **Sets** being a *commutative* monad [25, 24, 26]. The notion of a pseudo-commutative 2-monad [16] is then a generalization of this machinery to a 2-categorical context, and can be viewed as a starting point for importing tools from linear algebra into category theory.

The aim of this section is to give conditions that ensure that the 2-monad $\underline{P}$ associated to a Λ-operad P has a pseudo-commutative structure. We give the definition of pseudo-commutativity as in [16] but before doing so we require the definition of a strength for a 2-monad.

Definition 15.1.

left/right strength terminology?

A *strength* for an endo-2-functor $T\colon \mathcal{K} \to \mathcal{K}$ on a 2-category with products and terminal object 1 consists of a 2-natural transformation d with components

$$d_{A,B}\colon A \times TB \to T(A \times B)$$

satisfying the following unit and associativity axioms [25].

$$\begin{tikzcd}
1 \times TA \arrow[r, "d_{1,A}"] \arrow[dr, "\cong"'] & T(1 \times A) \arrow[d, "\cong"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "1\times\eta"] \arrow[dr, "\eta"'] & A \times TB \arrow[d, "d_{A,B}"] \\
& T(A \times B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times TC \arrow[rr, "d_{AB,C}"] \arrow[d, "a"'] & & T\left((A \times B) \times C\right) \arrow[d, "Ta"] \\
A \times (B \times TC) \arrow[r, "1\times d_{B,C}"'] & A \times T(B \times C) \arrow[r, "d_{A,BC}"'] & T\left(A \times (B \times C)\right)
\end{tikzcd}$$

$$\begin{tikzcd}
A \times T^2 B \arrow[r, "d_{A,TB}"] \arrow[d, "1\times\mu"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] & T^2(A \times B) \arrow[d, "\mu"] \\
A \times TB \arrow[rr, "d_{A,B}"'] & & T(A \times B)
\end{tikzcd}$$

Similarly, a *costrength* for T consists of a 2-natural transformation d^* with components

$$d^*_{A,B}\colon TA \times B \to T(A \times B)$$

again satisfying unit and associativity axioms.

The strength and costrength for the associated 2-monad $\underline{P}$ are quite simple to define. We define the strength d for $\underline{P}$ as follows. The component $d_{A,B}$ is a functor

$$d_{A,B}\colon A \times \left(\amalg P(n) \times_{\Lambda(n)} B^n\right) \to \amalg P(n) \times_{\Lambda(n)} (A \times B)^n$$

which sends an object $(a, [p; b_1, \ldots, b_n])$ to the object $[p; (a, b_1), \ldots, (a, b_n)]$. We also define the costrength similarly, sending an object $([p; a_1, \ldots, a_n], b)$ to the object which is an equivalence class $[p; (a_1, b), \ldots, (a_n, b)]$. Both the strength and the costrength are defined in the obvious way on morphisms.

Remark 15.2. It is crucial to note that the strength d and the costrength d^* do not depend on the Λ-actions in the following sense. The Λ-operad P has an underlying non-symmetric operad that we also denote P, and it has a strength

$$d_{A,B} \colon A \times (\amalg P(n) \times B^n) \to \amalg P(n) \times (A \times B)^n$$

given by essentially the same formula:

$$(a; (p; b_1, \ldots, b_n)) \mapsto (p; (a, b_1), \ldots, (a, b_n))\,.$$

The strength for the Λ-equivariant P is just the induced functor between coequalizers.

Definition 15.3. Given a 2-monad $T \colon \mathcal{K} \to \mathcal{K}$ with strength d and costrength d^*, a *pseudo-commutativity* consists of an invertible modification γ with components

$$\begin{tikzcd}
TA \times TB \arrow[r, "d^*_{A,TB}"] \arrow[d, "d_{TA,B}"'] & T(A \times TB) \arrow[r, "Td_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma_{A,B}"] & T^2(A \times B) \arrow[d, "\mu_{A\times B}"] \\
T(TA \times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A \times B) \arrow[r, "\mu_{A\times B}"'] & T(A \times B)
\end{tikzcd}$$

satisfying the following three strength axioms, two unit (or η) axioms, and two multiplication (or μ) axioms for all A, B, and C.

1. $\gamma_{A\times B,C} * (d_{A,B} \times 1_{TC}) = d_{A,B\times C} * (1_A \times \gamma_{B,C})$.
2. $\gamma_{A,B\times C} * (1_{TA} \times d_{B,C}) = \gamma_{A\times B,C} * (d^*_{A,B} \times 1_{TC})$.
3. $\gamma_{A,B\times C} * (1_{TA} \times d^*_{B,C}) = d^*_{A\times B,C} * (\gamma_{A,B} \times 1_C)$.
4. $\gamma_{A,B} * (\eta_A \times 1_{TB})$ is the identity on d.
5. $\gamma_{A,B} * (1_{TA} \times \eta_B)$ is the identity on d^*.
6. $\gamma_{A,B} * (\mu_A \times 1_{TB})$ is equal to the pasting below.

$$\begin{tikzcd}
T^2A\times TB \arrow[r, "d^*_{TA,TB}"] \arrow[d, "d_{T^2A,B}"'] & T(TA\times TB) \arrow[r, "Td^*_{A,TB}"] \arrow[d, "Td_{TA,B}"'] & T^2(A\times TB) \arrow[r, "T^2d_{A,B}"] \arrow[d, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A\times B) \arrow[d, "T\mu_{A\times B}"] \\
T(T^2A\times B) \arrow[r, phantom, "\Downarrow\gamma_{TA,B}"] \arrow[d, "Td^*_{TA,B}"'] & T^2(TA\times B) \arrow[r, "T^2d^*_{A,B}"'] \arrow[d, "\mu_{TA\times B}"'] & T^3(A\times B) \arrow[r, "T\mu_{A\times B}"'] \arrow[d, "\mu_{T(A\times B)}"'] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\
T^2(TA\times B) \arrow[r, "\mu_{TA\times B}"'] & T(TA\times B) \arrow[r, "Td^*_{A,B}"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B)
\end{tikzcd}$$

7. $\gamma_{A,B} * (1_{TA} \times \mu_B)$ is equal to the pasting below.

$$\begin{tikzcd}
TA \times T^2B \arrow[r, "d^*_{A,T^2B}"] \arrow[d, "d_{TA,TB}"'] & T(A \times T^2B) \arrow[r, "Td_{A,TB}"] \arrow[d, phantom, "\Downarrow\gamma_{A,TB}"] & T^2(A \times TB) \arrow[d, "\mu_{A\times TB}"] \\
T(TA \times TB) \arrow[r, "Td^*_{A,TB}"'] \arrow[d, "Td_{TA,B}"'] & T^2(A \times TB) \arrow[r, "\mu_{A\times TB}"'] \arrow[d, "T^2d_{A,B}"] & T(A \times TB) \arrow[d, "Td_{A,B}"] \\
T^2(TA \times B) \arrow[d, "T^2d^*_{A,B}"'] \arrow[r, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A \times B) \arrow[r, "\mu_{T(A\times B)}"] \arrow[d, "T\mu_{A\times B}"] & T^2(A \times B) \arrow[d, "\mu_{A\times B}"] \\
T^3(A \times B) \arrow[r, "T\mu_{A\times B}"'] & T^2(A \times B) \arrow[r, "\mu_{A\times B}"'] & T(A \times B)
\end{tikzcd}$$

Remark 15.4. It is noted in [16, Proposition 1] that this definition has some redundancy and therein it is claimed that any two of the strength axioms (Axioms 1-3) immediately implies the third. Furthermore, the three strength axioms are equivalent when the η and μ axioms hold (Axioms 4-6), as well as the following associativity axiom:

$$\gamma_{A,B\times C} \circ (1_{TA} \times \gamma_{B,C}) = \gamma_{A\times B,C} \times (\gamma_{A,B} \times 1_{TC}).$$

We need some further notation before stating our main theorem.

Notation 15.5 (Lexicographic and colexicographic orderings). Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two ordered finite lists. We use the following notation for the lexicographic and colexicographic orderings on the set $\underline{a} \times \underline{b} = \{(a_i, b_j)\}$.

1. The *lexicographic ordering* is denoted $\underline{(a, \underline{b})}$, and has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} p < r, \text{ or} \\ p = r \text{ and } q < s. \end{cases}$$

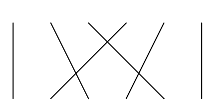
2. The *colexicographic ordering* is denoted $\underline{(\underline{a}, b)}$, and has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} q < s, \text{ or} \\ q = s \text{ and } p < r. \end{cases}$$

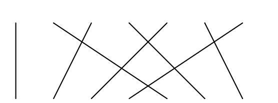
Remark 15.6. The notation $(a, \underline{b})$ is meant to indicate that there is a single a but a list of b's, so then $\underline{(a, \underline{b})}$ would represent a list which itself consists of lists of that form.

Definition 15.7 (The transposition permutation, τ). Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two ordered finite lists. The permutation $\tau_{m,n} \in \Sigma_{mn}$ is defined uniquely by the property that $\tau_{m,n}(i) = j$ if the ith element of the ordered set $\underline{(a, \underline{b})}$ is equal to the jth element of the ordered set $\underline{(\underline{a}, b)}$.

By construction, we have $\tau_{n,m} = \tau^{-1}_{m,n}$. We illustrate these permutations with a couple of examples.



$\tau_{2,3}$



$\tau_{4,2}$

Note then that $\tau_{m,n}$ is the permutation given by taking the transpose of the $m \times n$ matrix with entries (a_i, b_j). This has the effect of rearranging m groups of n things into n groups of m things.

We now define what it means for a Λ-operad to be pseudo-commuative, before then showing that such an operad yields a pseudo-commutative structure on the corresponding 2-monad $\underline{P}$.

Notation 15.8. Let $\mathbb{N}_+$ denote the set of positive integers.

Need to look back at how we structured other definitions, make it consistent

Definition 15.9. Let P be a Λ-operad in **Cat**. A pseudo-commutative structure on P consists of the following data.

From before: QQQ Rewrite all this using the beta and delta operations, so it's consistent with before. This seems to be partially done, but need to recheck details

- For each pair $(m, n) \in \mathbb{N}_+^2$, an element $t_{m,n} \in \Lambda(mn)$ such that $\pi(t_{m,n}) = \tau_{m,n}$.
- For each $p \in P(n)$, $q \in P(m)$, a natural isomorphism natural where/how?

$$\lambda_{p,q} \colon \mu(p; q, \dots, q) \cdot t_{m,n} \cong \mu(q; p, \dots, p).$$

We write this as $\lambda_{p,q} \colon \mu(p; \underline{q}) \cdot t_{m,n} \cong \mu(q; \underline{p})$.

These are required to satisfy the following axioms:

1. For all $n \in \mathbb{N}_+$,
$$t_{1,n} = e_n = t_{n,1}$$
and for all $p \in P(n)$, the isomorphism $\lambda_{p,\mathrm{id}} \colon p \cdot e_n \cong p$ is the identity map.
2. For all $l, m_1, \dots, m_l, n \in \mathbb{N}_+$, with $M = \Sigma m_i$,
$$\mu^\Lambda(e_l; t_{m_1,n}, \dots, t_{m_l,n}) \cdot \mu^\Lambda\left(t_{l,n}; \underline{e_{m_1}, \dots, e_{m_l}}\right) = t_{n,M}.$$

check below is same as above, then delete above

$$\beta(t_{n,m_1}, \dots, t_{n,m_l}) \cdot \delta_{\underline{m_i},\dots,\underline{m_i}}(t_{n,l}) = t_{n,M}.$$
Here $\underline{e_{m_1}, \dots, e_{m_l}}$ is the list $e_{m_1}, \dots, e_{m_l}$ repeated n times.
3. For all $l, m, n_1, \dots, n_m \in \mathbb{N}_+$, with $N = \Sigma n_i$,
$$\mu^\Lambda\left(t_{m,l}; \underline{e_{n_1}}, \dots, \underline{e_{n_m}}\right) \cdot \mu^\Lambda(e_m; t_{n_1,l}, \dots, t_{n_m,l}) = t_{N,l}.$$

check below is same as above, then delete above

$$\delta_{\underline{n_1},\dots,\underline{n_m}}(t_{m,l}) \cdot \beta(t_{n_1,l}, \dots, t_{n_m,l}) = t_{N,l}.$$
Here $\underline{e_{n_i}}$ indicates that each e_{n_i} is repeated l times.

4. For any $l, m_i, n \in \mathbb{N}_+$, with $1 \le i \le n$, and $p \in P(l)$, $q_i \in P(m_i)$ and $r \in P(n)$, the following diagram commutes. (Note that we maintain the convention that anything underlined indicates a list, and double underlining indicates a list of lists. Each instance should have an obvious meaning from context and the equations appearing above.)

$$\begin{tikzcd}
\mu\left(p; \underline{\mu(q_i; \underline{r})}\right) \cdot \mu(e_l; \underline{t_{n,m_i}})\mu(t_{n,l}; \underline{\underline{e_{m_i}}}) \arrow[r, equal] \arrow[d, equal] & \mu\left(p; \underline{\mu(q_i; \underline{r})}\right) \cdot t_{n,M} \arrow[d, equal] \\
\mu\left(p; \underline{\mu(q_i; \underline{r}) \cdot t_{n,m_i}}\right) \cdot \mu(t_{n,l}; \underline{e_{m_1}, \dots, e_{m_l}}) \arrow[d, "\mu(1; \underline{\lambda_{q_i,r}}) \cdot 1"'] & \mu\left(\mu(p; q_1, \dots, q_n); \underline{\underline{r}}\right) \cdot t_{n,M} \arrow[d, "\lambda_{\mu(p;q_1,\dots,q_n),r}"] \\
\mu\left(p; \underline{\mu(r; \underline{q_i})}\right) \cdot \mu(t_{n,l}; \underline{e_{m_1}, \dots, e_{m_l}}) \arrow[d, equal] & \mu\left(\mu(p; q_1, \dots, q_n); \underline{\underline{r}}\right) \arrow[dd, equal] \\
\mu\left(\mu(p; \underline{r}) \cdot t_{n,l}; \underline{q_1, \dots, q_n}\right) \arrow[d, "\mu(\lambda_{p,r}; 1)"'] & \\
\mu\left(\mu(r; \underline{p}); \underline{q_1, \dots, q_n}\right) \arrow[r, equal] & \mu\left(r; \underline{\mu(p; q_1, \dots, q_n)}\right)
\end{tikzcd}$$

5. For any $l, m, n_i \in \mathbb{N}_+$, with $1 \le i \le m$, and $p \in P(l)$, $q \in P(m)$ and $r_i \in P(n_i)$, the following diagram commutes.

$$\begin{tikzcd}
\mu\left(\mu(p; \underline{q}) \cdot t_{m,l}; \underline{\underline{r_i}}\right) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[r, equal] \arrow[d, "\mu(\lambda_{p,q}; 1) \cdot 1"'] & \mu\left(\mu(p; \underline{q}); \underline{\underline{r_i}}\right) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\left(\mu(q; \underline{p}); \underline{r_1}, \dots, \underline{r_m}\right) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] & \mu\left(p; \underline{\mu(q; \underline{r_i})}\right) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\left(q; \underline{\underline{\mu(p; r_i)}}\right) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[dd, equal] & \mu\left(p; \underline{\mu(q; r_1, \dots, r_m)}\right) \cdot t_{N,l} \arrow[d, "\lambda_{p,\mu(q;r_1,\dots,r_m)}"] \\
 & \mu\left(\mu(q; r_1, \dots, r_m); \underline{\underline{p}}\right) \arrow[d, equal] \\
\mu\left(q; \underline{\mu(p; \underline{r_i}) \cdot t_{n_i,l}}\right) \arrow[r, "\mu(1; \underline{\lambda_{p,r_i}})"'] & \mu\left(q; \underline{\mu(r_i; \underline{p})}\right)
\end{tikzcd}$$

Remark 15.10. Remark 11.2 of [13] describes the need for an extra equivariance axiom in the pseudo-commutative structure definition, some detail of which is also described in [12]. We also believed this to be true until a realisation that the equivariance axiom of [13][11.1, Axiom (iii)] becomes the following requirement:

$$\lambda_{p\cdot g, q\cdot h} \circ \left(\mu^P\left(\mathrm{id}_p \cdot g; \underline{\mathrm{id}_q \cdot h}\right) \cdot t_{m,n}\right) = \mu^P\left(\mathrm{id}_q \cdot h; \underline{\mathrm{id}_p \cdot g}\right) \circ \lambda_{p,q}.$$

On closer inspection, this equation is an instance of naturality for λ, as shown in the naturality square below.

What are the morphisms $p \to p \cdot g$ that this is naturality wrt? Probably the unique $e \to g$ acting on p, but we should explain.

$$\begin{tikzcd}
\mu^P\left(p; \underline{q}\right) \cdot t_{m,n} \arrow[r, "\lambda_{p,q}"] \arrow[d, "\mu^P\left(\mathrm{id}_p \cdot g; \underline{\mathrm{id}_q \cdot h}\right) \cdot t_{m,n}"'] & \mu^P\left(q; \underline{p}\right) \arrow[d, "\mu^P\left(\mathrm{id}_q \cdot h; \underline{\mathrm{id}_p \cdot g}\right)"] \\
\mu^P\left(p \cdot g; \underline{q \cdot h}\right) \cdot t_{m,n} \arrow[r, "\lambda_{p \cdot g, q \cdot h}"'] & \mu^P\left(q \cdot h; \underline{p \cdot g}\right)
\end{tikzcd}$$

Theorem 15.11. *Let P be a Λ-operad in* **Cat** *equipped with a pseudo-commutative structure. Then $\underline{P}$ has a pseudo-commutativity.*

Proof.

go through details again

We refer to the Axioms in Definition 15.9 throughout. We begin the proof by defining an invertible modification γ for the pseudo-commutativity for which the components are natural transformations $\gamma_{A,B}$. Such a transformation $\gamma_{A,B}$ has components with source

$$\left[\mu\left(p; \underline{q}\right); \underline{(a, \underline{b})}\right]$$

and target

$$\left[\mu\left(q; \underline{p}\right); \underline{(\underline{a}, b)}\right].$$

Now $\lambda_{p,q}\colon \mu(p; q, \ldots, q) \cdot t_{m,n} \cong \mu(q; p, \ldots, p)$ gives rise to another map by multiplication on the right by $t_{m,n}^{-1}$,

$$\lambda_{p,q} \cdot t_{m,n}^{-1}\colon \mu(p; q, \ldots, q) \cong \mu(q; p, \ldots, p) \cdot t_{m,n}^{-1},$$

so we define $(\gamma_{A,B})_{[p;a_1,\ldots,a_n],[q;b_1,\ldots,b_m]}$ to be the morphism which is the image of $(\lambda_{p,q} \cdot t_{m,n}^{-1}, 1)$ under the map

$$\coprod P(n) \times (A \times B)^n \to \coprod P(n) \otimes_{\Lambda(n)} A \times B^n.$$

Naturality of $\gamma_{A,B}$ follows from that of each $\lambda_{p,q}$. We will write this morphism as $[\lambda_{p,q} t_{m,n}^{-1}, 1]$. In the case that either p or q is an identity then we choose the component of γ to be the isomorphism involving the appropriate identity element using Axiom 1 above.

There are two things to note about the definition above before we continue. First, it is easy to check that

$$t_{m,n}^{-1} \cdot \underline{(a, \underline{b})} = \underline{(\underline{a}, b)}$$

since $\pi(t_{m,n}) = \tau_{m,n}$; this ensures that γ has the correct target. Second, the morphism above has second component the identity. This is actually forced upon us by the requirement that γ be a modification: in the case that A, B are discrete categories, the only possible morphism is an identity, and the modification axiom then forces that statement to be true for general A, B by considering the inclusion $A_0 \times B_0 \hookrightarrow A \times B$ where A_0, B_0 are the discrete categories with the same objects as A, B.

We show that this is a modification by noting that it does not rely on objects in the lists $a_1, \ldots, a_n$ or $b_1, \ldots, b_m$, only on their lengths and the operations p and q. As a result, if there are functors $f\colon A \to A'$ and $g\colon B \to B'$, then it is clear that

$$(\underline{P}(f \times g) \circ \gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]} = [\lambda_{p,q}, \underline{1}] = (\gamma_{A',B'} \circ (\underline{P} f \times \underline{P} g))_{[p;\underline{a}],[q;\underline{b}]}.$$

As such we can simply write $(\gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]}$ in shorthand as $\gamma_{p,q}$.

There are now seven axioms to check for a pseudo-commutativity: three strength axioms, two unit axioms, and two multiplication axioms. For the first strength axiom, we must verify that two different 2-cells of shape

$$\begin{tikzcd} A \times TB \times TC \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] & & T(A \times B \times C) \end{tikzcd}$$

are equal. The first of these is γ precomposed with $d \times 1$, and so is the component of γ at an object

$$([p;(a,b_1),\ldots,(a,b_n)],[q;c_1,\ldots,c_m])\,.$$

The second of these is d applied to the component of $1 \times \gamma$ at

$$(a,([p;b_1,\ldots,b_n],[q;c_1,\ldots,c_m]))\,.$$

It is straightforward to compute that each of these maps is the image of $\left(\lambda_{p,q} \cdot t^{-1}_{m,n}, 1\right)$ under the functor

$$\coprod P(n) \times (A \times B)^n \to \coprod P(n) \otimes_{\Lambda(n)} A \times B^n.$$

The other two strength axioms follow by analogous calculations for other whiskerings of γ with d or d^*.

For the unit axioms, we must compute the components of γ precomposed with $\eta \times 1$ for the first axiom and $1 \times \eta$ for the second. Thus for the first unit axiom, we must compute the component of γ at $([e;a],[p;b_1,\ldots,b_m])$. By definition, this is the image of $(\lambda_{e,p} \cdot t^{-1}_{m,1}, 1)$ under the map to the coequalizer, and by Axiom 1 of 15.9 know that $t^{-1}_{m,1}$ is the identity element and this isomorphism is the identity as well, so this component of γ is also the identity. The second unit axiom follows similarly, using that $t^{-1}_{1,n}$ is the identity.

For the multiplication axioms, first note that Axiom 2 is necessary in order to ensure the existence of the top horizontal equality in the diagram of Axiom 4 for the pseudo-commutative structure; the same goes for Axioms 3 and 5. We now explain how Axioms 2 and 4 for the pseudo-commutative structure ensure that the first multiplication axiom holds, with the same reasoning showing that Axioms 3 and 5 imply the second multiplication axiom.

We begin by studying the pasting diagram in the first multiplication axiom, but computing its values using the strength and costrength for the non-symmetric operad underlying P; this means that we evaluate on objects of the form $(p;a_1,\ldots,a_n)$ rather than on their equivalence classes. Let $p \in P(l), q_i \in P(m_i)$ for $1 \le i \le l$, and $r \in P(n)$. Computing the top and right leg around the pasting diagram gives the function on objects which sends

$$\big((p;(q_1;\underline{a_1}),\ldots,(q_l;\underline{a_l})),(r;\underline{b})\big)$$

to

$$\Big(\mu(p;\mu(q_1;\underline{r}),\ldots,\mu(q_l;\underline{r}));(\underline{(a_{1\bullet},\underline{b})}),\ldots,(\underline{(a_{l\bullet},\underline{b})})\Big)\,,$$

where $(\underline{(a_{i\bullet},\underline{b})})$ is the list of pairs

$$(a_{i1},b_1),\ldots,(a_{i1},b_m),(a_{i2},b_1),\ldots,(a_{in_i},b_m).$$

Then $\underline{P}\gamma$ is the image of the morphism which is the identity on the (a_{ij},b_k)'s, and is the morphism

$$\mu\left(1;\lambda_{q_1,r}t^{-1}_{n,m_1},\ldots,\lambda_{q_l,r}t^{-1}_{n,m_l}\right)$$

on the first component with domain and codomain shown below.

$$\mu\left(p;\mu\left(q_1;\underline{r}\right),\ldots,\mu\left(q_n;\underline{r}\right)\right) \longrightarrow \mu\left(p;\mu\left(r;\underline{q_1}\right)t^{-1}_{n,m_1},\ldots,\mu\left(r;\underline{q_l}\right)t^{-1}_{n,m_l}\right)$$

By the Λ-operad axioms, the target of this morphism is equal to

$$\mu\left(p; \mu\left(r; \underline{q_1}\right), \dots, \mu\left(r; \underline{q_l}\right)\right) \mu\left(e_l; t_{n,m_1}^{-1}, \dots, t_{n,m_l}^{-1}\right).$$

Note that this is not the same object as one obtains by computing $T\mu \circ T^2 d^* \circ Td \circ d^*$ using the underlying non-symmetric operad of P as we are required to use the Λ-equivariance to ensure that the target of γ is the correct one.

Next we compute the source of $(\mu \circ Td^*) * \gamma$, the other 2-cell in the pasting appearing in the first multiplication axiom. We compute this once again using the strength and costrength for the underlying non-symmetric operad, and note once again that this will not match our previous calculations precisely, but only up to an application of Λ-equivariance. This functor has its map on objects given by

$$\big((p; (q_1; \underline{a_1}), \dots, (q_l; \underline{a_l})), (r; \underline{b})\big) \mapsto \Big(\mu(\mu(p; \underline{r}); \underline{q_1}, \dots, \underline{q_l}); \underline{(\underline{a_1}, b_\bullet)}, \dots, \underline{(\underline{a_l}, b_\bullet)}\Big).$$

Note that if we apply Λ-equivariance, this matches the target computed above. Once again the component of γ is the image of a morphism which is the identity on the (a_{ij}, b_k)'s, and its first component is

$$\mu\left(\mu(p; \underline{r}); \underline{q_1}, \dots, \underline{q_l}\right) \xrightarrow{\mu\left(\lambda_{p,r} \cdot t_{n,l}^{-1}; 1, \dots, 1\right)} \mu\left(\mu(r; \underline{p}) \cdot t_{n,l}^{-1}; \underline{q_1}, \dots, \underline{q_l}\right).$$

We cannot compose these morphisms in $\coprod P(n) \times (A \times B)^n$ as they do not have matching source and target, but we can in $\coprod P(n) \times_\Lambda (A \times B)^n$. The resulting morphism has first component given by the image of

$$\mu(p; \mu(q_1; \underline{r}), \dots, \mu(q_n; \underline{r})) \xrightarrow{\mu\left(1; \lambda_{q_1,r} t_{n,m_1}^{-1}, \dots, \lambda_{q_1,r} t_{n,m_l}^{-1}\right)} \mu\left(p; \mu\left(r; \underline{q_1}\right) t_{n,m_1}^{-1}, \dots, \mu\left(r; \underline{q_l}\right) t_{n,m_l}^{-1}\right)$$

$$\xrightarrow{\mu\left(\lambda_{p,r} \cdot t_{n,l}^{-1}; 1, \dots, 1\right) \cdot \mu\left(e_l; t_{n,m_1}^{-1}, \dots, t_{n,m_l}^{-1}\right)} \mu\left(\mu\left(r; \underline{p}\right) \cdot t_{n,l}^{-1}; \underline{q_1}, \dots, \underline{q_l}\right) \cdot \mu\left(e_l; t_{n,m_1}^{-1}, \dots, t_{n,m_l}^{-1}\right),$$

where we have made use of the operad axioms in identifying the target of the first map with the source of the second. Using the Λ-operad axioms again on the target, we find that

$$\mu\left(\mu(r; \underline{p}) \cdot t_{n,l}^{-1}; \underline{q_1}, \dots, \underline{q_l}\right) \cdot \mu(e_l; t_{n,m_1}^{-1}, \dots, t_{n,m_l}^{-1})$$

is equal to

$$\mu\left(\mu(r; \underline{p}); \underline{q_1, \dots, q_l}\right) \cdot \mu(t_{n,l}^{-1}; \underline{e}) \cdot \mu(e_l; t_{n,m_1}^{-1}, \dots, t_{n,m_l}^{-1}).$$

This composite of two morphisms, together with the necessary identities coming from operad axioms, is precisely the left and bottom leg of the diagram in Axiom 4. Using the same method, one then verifies that $\gamma * (\mu \times 1)$ has its first component the image of the morphism appearing along the top and right leg of the diagram in 4. The second component of these morphisms are all identities arising from Λ-equivariance, so the first multiplication axiom is a consequence of Axioms 2 and 4 for the pseudo-commutative structure. We leave the calculations for the second multiplication axiom to the reader as they are of the same nature, using Axioms 3 and 5. $\square$

Corollary 15.12. *Let P be a non-symmetric operad. Then the induced monad $\underline{P}$ is never pseudo-commutative.*

Proof.

go through details again

In the non-symmetric case, the 2-monad is just given using coproducts and products, i.e., there is no coequalizer. In order to define γ, we then need an isomorphism

$$\left(\mu(p;\underline{q});\underline{(a,\underline{b})}\right) \cong \left(\mu(q;\underline{p});\underline{(\underline{a},b)}\right).$$

When A, B are discrete, there is no isomorphism $\underline{(a,\underline{b})} \cong \underline{(\underline{a},b)}$, and therefore no such γ can exist. □

Hyland and Power also define a symmetry for a pseudo-commutative structure on a 2-monad T. This symmetry is then reflected in the monoidal structure on the 2-category of algebras, which will then also have a symmetric tensor product (in a suitable, 2-categorical sense).

include specific reference

This is prop 18 in HP. But they are talking about the *multicategory* T-Alg.

Definition 15.13. Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad on a symmetric monoidal 2-category $\mathcal{K}$ with symmetry c. We then say that a pseudo-commutativity γ for T is *symmetric* when the following is satisfied for all A, $B \in \mathcal{K}$:

$$Tc_{A,B} \circ \gamma_{A,B} \circ c_{TB,TA} = \gamma_{B,A}.$$

With the notion of symmetry at hand we are able to extend the above theorem, stating when $\underline{P}$ is symmetric.

Theorem 15.14. *The pseudo-commutativity of $\underline{P}$ given by* **??** *is symmetric if for all $m, n \in \mathbb{N}_+$ the two conditions below hold.*

1. $t_{m,n} = t_{n,m}^{-1}$.
2. *The following diagram commutes:*

$$\begin{tikzcd}
\mu\left(p;\underline{q}\right)\cdot t_{m,n}t_{n,m} \arrow[r, equal] \arrow[d, "\lambda_{p,q}\cdot 1"'] & \mu\left(p;\underline{q}\right)\cdot e_{mn} \arrow[d, equal] \\
\mu\left(q;\underline{p}\right)\cdot t_{n,m} \arrow[r, "\lambda_{q,p}"'] & \mu\left(p;\underline{q}\right)
\end{tikzcd}$$

Proof. The commutativity of the diagram above ensures that the first component of the symmetry axiom commutes in $P(n)$ before taking equivalence classes in the coequalizer, just as in the proof of **??**. □

Definition 15.15. Let P be a Λ-operad in **Cat**. We say that P is *contractible* if each category $P(n)$ is equivalent to the terminal category.

Corollary 15.16. *If P is contractible and there exist $t_{m,n}$ as in* **??**, *then $\underline{P}$ acquires a pseudo-commutativity. Furthermore, it is symmetric if $t_{n,m} = t_{m,n}^{-1}$.*

Proof. The only thing left to define is the collection of natural isomorphisms $\lambda_{p,q}$. But since each $P(n)$ is contractible, $\lambda_{p,q}$ must be the unique isomorphism between its source and target, and furthermore the last two axioms hold since any pair of parallel arrows are equal in a contractible category. □

Corollary 15.17. *If P is a contractible symmetric operad then $\underline{P}$ has a symmetric pseudo-commutativity.*

Proof. We choose $t_{m,n} = \tau_{m,n}$. □

Remark 15.18. If a Λ-operad P is contractible, it is not the case that its symmetrization $S(P)$ (see **??**) will also be contractible.

now just give an example, do braids work?

Thus we see that a given Λ-operad P might satisfy the hypotheses of Corollary 15.16 without its symmetrization $S(P)$ satisfying the hypotheses of Corollary 15.17.

This is exactly the kind of thing that "keeping more complicated action operads around lets you do more stuff" needs to reference!

16 Extended Example: Coboundary Categories

This section could still move somewhere else. I have a presentation for the symmetric groups as an action operad in section 7, and this doesn't use much more than that.

We now turn to an example that is not as widely known in the categorical literature, that of coboundary categories [9]. These arise in the representation theory of quantum groups and in the theory of crystals [14, 15]. Our goal here is to refine the relationship between coboundary categories and the operad of n-fruit cactus groups in [14] by using the theory of action operads and our Borel construction. We begin by recalling the definition of a coboundary category. [I think in particular we want to do this via presentations]

Definition 16.1. A *coboundary category* is a monoidal category C equipped with a natural isomorphism $\sigma_{x,y} \colon x \otimes y \to y \otimes x$ (called the *commutor*) such that

- $\sigma_{y,x} \circ \sigma_{x,y} = 1_{x \otimes y}$ and
- the diagram

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r] \arrow[d, "\sigma_{x,y} 1"'] & x \otimes (y \otimes z) \arrow[r, "1\sigma_{y,z}"] & x \otimes (z \otimes y) \arrow[d, "\sigma_{x,zy}"] \\
(y \otimes x) \otimes z \arrow[r, "\sigma_{yx,z}"'] & z \otimes (y \otimes x) \arrow[r] & (z \otimes y) \otimes x
\end{tikzcd}$$

  commutes (in which the unlabeled morphisms are an associator and an inverse associator).

Example 16.2.
1. As noted by Savage [44], any braiding automatically satisfies the cactus relation (the diagram in Definition 16.1). However, since braidings need not be involutions this does not mean that any braided monoidal category is a coboundary category. However, it should then be clear that any symmetric monoidal category is also a coboundary category.
2. The name coboundary category comes from the original work of Drinfeld [9] in which he shows that the category of representations of a coboundary Hopf algebra has the structure of coboundary category.
3. Henriques and Kamnitzer [14] show that the category of crystals for a finite dimensional complex reductive Lie algebra has the structure of a coboundary category.

replace what follows with a discussion/reference to Theorem 12.10

Our interest is in strict coboundary categories by which we mean coboundary categories with strict underlying monoidal category. Under the assumption of strictness, the second axiom above does not include associations for the tensor product and reduces to a square. To show that every coboundary category is equivalent to a strict coboundary category, we must introduce the 2-category **CobCat** of coboundary categories.

Definition 16.3. Let $(C, \sigma), (C', \sigma')$ be coboundary categories. A *coboundary functor* $F\colon C \to C'$ is a strong monoidal functor (with invertible constraints φ_0 for the unit and $\varphi_{x,y}$ for the tensor product) such that the following diagram commutes for all objects x, $y \in \mathcal{C}$.

$$\begin{tikzcd}
Fx \otimes Fy \arrow[r, "\varphi_{x,y}"] \arrow[d, "\sigma_{Fx,Fy}"'] & F(x \otimes y) \arrow[d, "F\sigma_{x,y}"] \\
Fy \otimes Fx \arrow[r, "\varphi_{y,x}"'] & F(y \otimes x)
\end{tikzcd}$$

Coboundary functors are composed just as strong monoidal functors are, giving the following.

Lemma 16.4. *Coboundary categories, coboundary functors, and monoidal transformations form a 2-category, which we denote* **CobCat***.*

Proposition 16.5. *Let (C, σ) be a coboundary category. Then there exists a strict coboundary category (C', σ') which is equivalent to (C, σ) in* **CobCat***.*

Proof. Consider the underlying monoidal category of (C, σ), which we will just write as C. We can find a strict monoidal category C' by coherence for monoidal categories together with an equivalence, as monoidal categories, between C and C'. By standard methods [35], this can be improved to an adjoint equivalence between C and C' in the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. Let $F\colon C \to C', G\colon C' \to C$ be the functors in this adjoint equivalence, and $\eta\colon 1 \Rightarrow FG$ the unit (which we note for emphasis is invertible since it the unit of an adjoint equivalence). For objects $x, y \in C'$, we define a commutor σ' for C' as the following composite.

$$xy \xrightarrow{\eta\otimes\eta} FGxFGy \xrightarrow{\cong} F(GxGy) \xrightarrow{F\sigma} F(GyGx) \xrightarrow{\cong} FGyFGx \xrightarrow{\eta^{-1}\otimes\eta^{-1}} yx$$

We then leave to the reader, if they wish, the computations to show that σ' is a commutor for C' and that F, G become coboundary functors using σ'. $\square$

We now turn to the operadic description of strict coboundary categories; we note from this point onwards, all our coboundary categories are assumed to be strict.

Definition 16.6. Fix $n > 1$, and let $1 \leq p < q \leq n$, $1 \leq k < l \leq n$.

1. $p < q$ is *disjoint* from $k < l$ if $q < k$ or $l < p$.
2. $p < q$ *contains* $k < l$ if $p \leq k < l \leq q$.

Definition 16.7. Let $1 \leq p < q \leq n$, and define $\widehat{s}_{p,q} \in \Sigma_n$ to be the permutation defined below.

$$\begin{array}{r|cccccccccccc}
i & 1 & 2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & q-1 & q & q+1 & \cdots & n \\
\widehat{s}_{p,q}(i) & 1 & 2 & \cdots & p-1 & q & q-1 & q-2 & \cdots & p+1 & p & q+1 & \cdots & n
\end{array}$$

The n-fruit cactus group is then defined as follows.

Definition 16.8. Let J_n be the group generated by symbols $s_{p,q}$ for $1 \leq p < q \leq n$ subject to the following relations.

1. For all $p < q$, $s_{p,q}^2 = e$.
2. If $p < q$ is disjoint from $k < l$, then $s_{p,q}s_{k,l} = s_{k,l}s_{p,q}$.
3. If $p < q$ contains $k < l$, then $s_{p,q}s_{k,l} = s_{m,n}s_{p,q}$ where
   - $m = \widehat{s}_{p,q}(l)$ and
   - $n = \widehat{s}_{p,q}(k)$.

It is easy to check that the elements $\widehat{s}_{p,q} \in \Sigma_n$ satisfy the three relations in Definition 16.8, so $s_{p,q} \mapsto \widehat{s}_{p,q}$ extends to a group homomorphism $\pi_n \colon J_n \to \Sigma_n$. This is the first step in proving the following.

Theorem 16.9. *The collection of groups $J = \{J_n\}$ form an action operad.*

Proof.

This is the part where instead we could check that the group presentations match the action operad presentation that we extract from the definition/the end of Sec 15

There is an issue with the corrected version of Axiom 5 and the t's that needs some fixing. AC: It needs to be that $\delta_{1;n}(e_1) = e_n$, but δ is only defined on the symbols $s_{p,q}$. Could just define $\delta_{1;n}(e_1) = e_n$ and then check that this doesn't cause any problems with the other characterisation?

We will use Theorem 4.15 to determine the rest of the action operad structure. Thus we must give, for any collection of natural numbers $n, k_1, \ldots, k_n$ and $K = \sum k_i$, group homomorphisms $\beta \colon J_{k_1} \times \cdots \times J_{k_n} \to J_K$ and functions $\delta \colon J_n \to J_K$ satisfying nine axioms. We define both of these on generators, starting with β.

Let $s_{p_i,q_i} \in J_{k_i}$. Let $r_i = k_1 + k_2 + \cdots + k_{i-1}$ for $i > 1$. Define β by

$$\beta(s_{p_1,q_1}, \ldots, s_{p_n,q_n}) = s_{p_1,q_1} s_{p_2+r_2,q_2+r_2} \cdots s_{p_n+r_n,q_n+r_n}.$$

Note that $s_{p_i+r_i,q_i+r_i}$ and $s_{p_j+r_j,q_j+r_j}$ are disjoint when $i \neq j$.

It is easy to check that this disjointness property ensures that β gives a well-defined group homomorphism

$$J_{k_1} \times \cdots \times J_{k_n} \to J_K.$$

To define $\delta \colon J_n \to J_K$ for natural numbers $n, k_1, \ldots, k_n$ and $K = \sum k_i$, let $t_k = s_{1,k} \in J_k$. Then we start by defining

$$\delta(t_n) = t_K \cdot \beta(t_{k_1}, t_{k_2}, \ldots, t_{k_n}).$$

Note that, by Axiom 8 of Theorem 4.15, this is equal to

$$\beta(t_{k_n}, t_{k_{n-1}}, \ldots, t_{k_1}) \cdot t_K.$$

Now $s_{p,q} \in J_n$ is equal to $\beta(e_{p-1}, t_{q-p+1}, e_{n-q})$ (here e_i is the identity element in J_i) by definition of the t_i and β, so we can define δ on any generator $s_{p,q}$ by

$$\delta(s_{p,q}) = \beta(e_A, M, e_B)$$

with

- $A = k_1 + k_2 + \cdots + k_{p-1}$,
- $M = t_{k_p+\cdots+k_q} \cdot \beta(t_{k_p}, t_{k_{p+1}}, \ldots, t_{k_q})$, and

- $B = k_{q+1} + k_{q+2} + \cdots + k_n$.

Unpacking this yields the following formula:

$$\delta(s_{p,q}) = s_{k_1+\cdots+k_{p-1}+1,k_1+\cdots+k_q} \cdot \beta(e_{k_1+\cdots+k_{p-1}}, t_{k_p}, \ldots, t_{k_q}, e_{k_{q+1}+\cdots+k_n}).$$

We extend δ to products of generators using Axiom 6 of Theorem 4.15. As before, we must check that this gives a well-defined function on products of two generators in each of the relations of the cactus groups, and we must also check that this is well-defined on products of three or more generators. Thus we define

$$\delta_{n;j_1,\ldots,j_n}(gh) = \delta_{n;k_1,\ldots,k_n}(g)\delta_{n;j_1,\ldots,j_n}(h)$$

where $k_i = j_{\pi(h)^{-1}(i)}$. There are three relations we must verify for compatibility.

- We must show that $\delta_{n;j_1,\ldots,j_n}\left(s_{p,q}^2\right) = e$. By definition, we have

  $$\delta_{n;j_1,\ldots,j_n}\left(s_{p,q}^2\right) = \delta_{n;k_1,\ldots,k_n}\left(s_{p,q}\right)\delta_{n;j_1,\ldots,j_n}\left(s_{p,q}\right)$$

  which is

  $$t_{\underline{j}}\beta(t_{j_n}, \ldots, t_{j_1})t_{\underline{j}}\beta(t_{j_1}, \ldots, t_{j_n}).$$

  By the remarks above in the definition of δ and the fact that $s_{p,q}^2 = e$, the element above is easily seen to be the identity.

  What is $t_{\underline{j}}$?

- We must show that $\delta(s_{p,q}s_{k,l}) = \delta(s_{k,l}s_{p,q})$ when (p,q) is disjoint from (k,l). This is another simple calculation using the definition of δ and the disjointness of the terms involved.

- We must show that $\delta(s_{p,q}s_{k,l}) = \delta(s_{a,b}s_{p,q})$, where $a = \widehat{s}_{p,q}(l), b = \widehat{s}_{p,q}(k)$, if $p < k < l < q$. In this case, we use all of the relations in the cactus groups to show that each side is equal to

  $$\beta\left(\underline{e}, t_{j_p+\cdots+j_q} \cdot \beta\left(t_{j_p}, \ldots t_{j_{k-1}}, t_{j_k+\cdots+j_l}, t_{j_{l+1}}, \cdots, t_{j_q}\right), t_{j_{q+1}}, \ldots, t_{j_n}\right)$$

  where $\underline{e} = e_{j_1}, \ldots, e_{j_{p-1}}$.

In order to show that this gives a well-defined function on products of three or more generators, one proceeds inductively to show that $\delta\left((fg)h\right) = \delta\left(f(gh)\right)$ using the formula above. This is simply a matter of keeping track of the permutations used to define the subscripts for the different δ's and we leave it to the reader, should they desire to see the details. This concludes the definition of the family of functions $\delta_{n;j_i}$.

There are now nine axioms to check in Theorem 4.15. Axioms 1 - 3 all concern β, and are immediate from the defining formula. Axiom 4 is obvious for the elements t_k, from which it follows in general by the formulas defining δ. For Axiom 5, one can check easily that

$$\delta_{n;1,\ldots,1}(t_n) = t_n, \quad \delta_{1;n}(e_1) = t_n$$

and once again the general case follows from these. Axiom 6 holds by the construction of δ. Axiom 8 can be verified with only one h_i nontrivial at a time, and then it is a simple consequence of the second and third relations for J_n.

Axiom 9 is straightforward to check when only a single g_i is a generator and the rest are identities using the defining formulas, and the general case then follows using Axiom 6. Using Axiom 9, we can then prove Axiom 7 as follows; we suppress the subscripts on different δ's for clarity. We must show

$$\delta_{m_1+\cdots+m_n;p_{11},\ldots,p_{1m_1},p_{21},\ldots,p_{nm_m}}\left(\delta_{n;m_1,\ldots,m_n}(f)\right) = \delta_{n;P_1,\ldots,P_n}(f),$$

and we do so on t_n. By definition, we have

$$\delta\left(\delta(t_n)\right) = \delta\left(t_K \beta(t_{k_1}, \ldots, t_{k_n})\right),$$

which by Axiom 6 is equal to

$$t_{P_1+\cdots+P_n} \cdot \beta(t_{p_{11}}, \ldots, t_{p_{n,m_n}}) \cdot \delta\left(\beta(t_{k_1}, \ldots, t_{k_n})\right).$$

Now this last term is equal to $\beta\left(\delta(t_{k_1}), \ldots, \delta(t_{k_n})\right)$ by Axiom 9, which is then equal to

$$\beta\left(t_{P_1} \cdot \beta(t_{p_{11}}, \ldots, t_{p_{1,m_1}}), \ldots, t_{P_n} \cdot \beta(t_{p_{n1}}, \ldots, t_{p_{1,m_n}})\right).$$

Taken all together, the left hand side of Axiom 9 is then

$$t_{P_1+\cdots+P_n} \cdot \beta(t_{p_{11}}, \ldots, t_{p_{n,m_n}}) \cdot \beta\left(t_{P_1} \cdot \beta(\underline{t_{p_1}}), \ldots, t_{P_n} \cdot \beta(\underline{t_{p_n}})\right).$$

where $\underline{t_{p_i}} = t_{p_{i,1}}, \cdots, t_{i,m_i}$ All of the terms coming from an $t_{p_{ij}}$ can be collected together, and since $s_{p,q}^2 = e$ for all p, q, these cancel. This leaves

$$t_{P_1+\cdots+P_n} \cdot \beta\left(t_{P_1}, \ldots, t_{P_n}\right)$$

which is the right hand side of Axiom 9 as desired. □

Lemma 16.10. *The 2-monad C for strict coboundary categories is a club.*

Proof. This is obvious by **??**. □

Theorem 16.11. *The free coboundary category on one element, $C1$, is isomorphic to $BJ = \coprod BJ_n$.*

Proof. The universal property we desire is with respect to strict coboundary functors (i.e., coboundary functors whose underlying monoidal functor is strict), so we must give BJ the structure of a strict coboundary category and then check that to give a strict coboundary functor $BJ \to X$ to any other strict coboundary category is the same as giving an object of X.

The category BJ has natural numbers as objects, and addition as its tensor product. The tensor product of two morphisms is given by β as in Theorem 16.9, and it is simple to check that this is a strict monoidal structure. The commutor $\sigma_{m,n}$ is $s_{1,m+n}s_{1,m}s_{m+1,m+n}$. Using the relations in J_n, it is clear that $\sigma_{m,n}\sigma_{n,m}$ is the identity, so we only have one more axiom to verify in order to give a coboundary structure. By definition, this axiom is equivalent to the equation

$$\sigma_{m,p+n} \cdot \beta(e_m, \sigma_{n,p}) = \sigma_{n+m,p} \cdot \beta(\sigma_{m,n}, e_p)$$

holding for all m, n, p. Each side has six terms when written out using the definitions of σ and β, two terms on each side cancel using $s_{p,q}^2 = e$ and the disjointness relation, and the other four terms match after using the disjointness relation. This establishes the coboundary structure on BJ; note that $\sigma_{1,1} = s_{1,2}$, the nontrivial element of $J(2)$.

Every strict coboundary functor $F\colon BJ \to X$ determines an object of X by evaluation at 1. Conversely, given an object x of a strict coboundary category X, there is an action of J_n on $X(x^n, x^n)$ by Theorem 7 of [14] and therefore a strict monoidal functor $\overline{x}\colon BJ \to X$ with $\overline{x}(1) = x$. By construction, this strict monoidal functor is in fact a strict coboundary functor since it sends the commutor $\sigma_{1,1}$ in BJ to $\sigma_{x,x}$ in X. In fact, the calculations in [14] leading up to Theorem 7 show that every element of J_n is given as an operadic composition of σ's, so requiring $\overline{x}$ to be a strict coboundary functor with $\overline{x}(1) = x$ determines the rest of the functor uniquely. This establishes the bijection between strict coboundary functors $F\colon BJ \to X$ and objects of X which proves that BJ is the free strict coboundary category on one object. □

Corollary 16.12. *The 2-monad C for coboundary categories corresponds, using Theorem 14.7, to the action operad J.*

17 Extended Example: Braided Monoidal Categories

We conclude with a computation using **??**. This result (17.1 below) was only conjectured in [16], but we are able to prove it quite easily with the machinery developed thus far. Our strategy is to construct a Λ-operad which is contractible together with the group elements required in **??**. Note that the symmetrized version of this operad will not be contractible, and we do not know of a proof using the structure of the symmetrized operad.

Theorem 17.1. *The* 2*-monad* $\underline{B}$ *for braided strict monoidal categories on* **Cat** *has two pseudo-commutative structures on it, neither of which are symmetric.*

In order to apply our theory, the 2-monad $\underline{B}$ must arise from a Λ-operad. The following proposition describes it as such, and can largely be found as Example 3.2 in the work of Fiedorowicz [10].

I think we should also rework this using our presentations stuff

Proposition 17.2. *The* 2*-monad* $\underline{B}$ *is the* 2*-monad associated to the* B*-operad* B *with the category* $B(n)$ *having objects the elements of the nth braid group* B_n *and a unique isomorphism between any pair of objects; the action of* B_n *on* $B(n)$ *is given by right multiplication on objects and is then uniquely determined on morphisms.*

Yeah this paragraph should go, it doesn't really help that much. Make sure to keep the references though

The interested reader could easily verify that algebras for the B-operad B are braided strict monoidal categories. The objects of $\underline{B}(X)$ can be identified with finite lists of objects of X, and morphisms are generated by the morphisms of X together with new isomorphisms

$$x_1, \ldots, x_n \xrightarrow{\gamma} x_{\gamma^{-1}(1)}, \ldots, x_{\gamma^{-1}(n)}$$

where $\gamma \in B_n$ and the notation $\gamma^{-1}(i)$ means, as before, that we take the preimage of i under the permutation $\pi(\gamma)$ associated to γ. This shows that $\underline{B}(X)$ is the free braided strict monoidal category generated by X according to [17], and it is easy to verify that the 2-monad structure on $\underline{B}$ arising from the B-operad structure on B is the correct one to produce braided strict monoidal categories as algebras.

Definition 17.3. A braid $\gamma \in B_n$ is *positive* if it is an element of the submonoid of B_n generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

Definition 17.4. A braid $\gamma \in B_n$ is *minimal* if no pair of strands in γ cross twice.

For our purposes, we are interested in braids which are both positive and minimal. A proof of the following lemma can be found in [41].

specific ref

Lemma 17.5. *Let* PM_n *be the subset of* B_n *consisting of positive, minimal braids. Then the function sending a braid to its underlying permutation is a bijection of sets* $PM_n \to \Sigma_n$*.*

Remark 17.6. It is worth noting that this bijection is not an isomorphism of groups, since PM_n is not a group or even a monoid. The element $\sigma_1 \in B_n$ is certainly in PM_n, but σ_1^2 is not as the first two strands cross twice. Thus we see that the product of two minimal braids is generally not minimal, but by definition the product of positive braids is positive.

Proof of Theorem 17.1. We refer to the Axioms of Definition 15.9 throughout the proof. In order to use **??** with the action operad being the braid operad B, we must first construct elements $t_{m,n} \in B_{mn}$ satisfying certain properties. Using Lemma 17.5, we define $t_{m,n}$ to be the unique positive minimal braid such that $\pi(t_{m,n}) = \tau_{m,n}$. Since $\tau_{1,n} = e_n = \tau_{n,1}$ in Σ_n and the identity element $e_n \in B_n$ is positive and minimal, we find that $t_{1,n} = e_n = t_{n,1}$ in B_n, satisfying Axiom 1. Thus in order to verify the remaining hypotheses, we must check two equations, each of which states that some element $t_{m,n}$ can be expressed as a product of operadic compositions of other elements.

Let $l, m_1, \ldots, m_l, n$ be natural numbers, and let $N = \sum m_i$. We must check Axiom 3 is satisfied, i.e., that

$$\beta(t_{n,m_1}, \ldots, t_{n,m_l}) \cdot \delta(t_{n,l}) = t_{N,l}$$

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu\left(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}\right) = t_{N,l}$$

in B_{lN}. These braids have the same underlying permutations by construction, and both are positive since each operadic composition on the left is positive. The braid on the right is minimal by definition, so if we prove that the braid on the left is also minimal, they are necessarily equal. Now $\mu\left(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}\right)$ is given by the braid for $t_{n,l}$ but with the first strand replaced by m_1 strands, the second strand replaced by m_2 strands, and so on for the first l strands of $t_{n,l}$, and then repeating for each group of l strands. In particular, since strands $i, i+l, i+2l, \ldots, i+(n-1)l$ never cross in $t_{n,l}$, none of the m_i strands that each of these is replaced with cross. The braid $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ consists of the disjoint union of the braids for each t_{n,m_i}, so if two strands cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ then they must both cross in some t_{n,m_i}. The strands in t_{n,m_i} are those numbered from $n(m_1 + \cdots + m_{i-1}) + 1$ to $n(m_1 + \cdots + m_{i-1} + m_i)$. This is a consecutive collection of nm_i strands, and it is simple to check that these strands are precisely those connected (via the group operation in B_{Nl}, concatenation) to the duplicated copies of strands $i, i+l, i+2l, \ldots, i+(n-1)l$ in $t_{n,l}$. Thus if a pair of strands were to cross in $\beta(t_{n,m_1}, \ldots, t_{n,m_l})$, that pair cannot also have crossed in $\delta(t_{n,l})$, showing that the resulting product braid

$$\beta(t_{n,m_1}, \ldots, t_{n,m_l}) \cdot \delta(t_{n,l})$$

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu\left(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}\right)$$

is minimal. The calculation for Axiom 2, showing that

$$\delta(t_{m,l}) \cdot \beta(t_{n_1,l}, \ldots, t_{n_m,l})$$

$$\mu\left(t_{m,l}; \underline{e_1}, \ldots, \underline{e_{n_m}}\right)\mu\left(e_m; t_{n_1,l}, \ldots, t_{n_m,l}\right)$$

is also minimal, follows from the same argument, showing that it is equal to $t_{N,l}$ (here N is the sum of the n_i, where once again i ranges from 1 to l).

Where are Axioms 4 and 5 checked?

These calculations show, using **??**, that the B-operad B induces a 2-monad which has a pseudo-commutative structure. As noted before, B-algebras are precisely braided strict monoidal categories. The second pseudo-commutative structure arises by using negative, minimal braids instead of positive ones, and proceeds using the same arguments. This finishes the first part of the proof of Theorem 17.1.

We will now show that neither of these pseudo-commutative structures is symmetric. The symmetry axiom in this case reduces to the fact that, in some category which is given as a coequalizer, the morphism with first component

$$f \colon \mu\left(p; \underline{q}\right) \cdot t_{n,m}t_{m,n} \to \mu\left(q; \underline{p}\right) \cdot t_{m,n} \to \mu\left(p; \underline{q}\right)$$

is the identity. Now it is clear that $t_{n,m}$ is not equal to $t_{m,n}^{-1}$ in general: taking $m = n = 2$, we note that $t_{2,2} = \sigma_2$, and this element is certainly not of order two in B_4. $B(4)$ is the category whose objects are the elements of B_4 with a unique isomorphism between any two pair of objects, and B_4 acts by multiplication on the right; this action is easily shown to be free and transitive. We recall (see **??**) that in a coequalizer of the form $A \otimes_G B$, a morphism $[f_1, f_2]$ equals $[g_1, g_2]$ if and only if there exists an $x \in G$ such that

$$\begin{aligned} f_1 \cdot x &= g_1, \\ x^{-1} \cdot f_2 &= g_2. \end{aligned}$$

For the coequalizer in question, for f to be the first component of an identity morphism would imply that $f \cdot x$ would be a genuine identity in $B(4)$ for some x. But $f \cdot x$ would have source $\mu\left(p; \underline{q}\right) t_{n,m} t_{m,n} x$ and target $\mu\left(p; \underline{q}\right) x$, which requires $t_{n,m} t_{m,n}$ to be the identity group element for all n, m. In particular, this would force $t_{2,2}$ to have order two, which as noted above does not hold in B_4, thus giving a contradiction. □

Remark 17.7. The pseudo-commutativities given above are not necessarily the only ones that exist for the B-operad B, but we do not know a general strategy for producing others.

Example 17.8.

Move this into coboundary section, reword this example somewhat

Non-Example: Cactus operad.

Begin by defining $t_{2,2} = s_{2,3}$, which has underlying permutation $\pi_4(t_{2,2}) = (2\ 3) = \tau_{2,3}$, as required. This seems to be a sensible choice to then demonstrate that we can describe all other $t_{m,n}$ required for a pseudo-commutativity on J. In particular, we should be able to describe $t_{2,4}$ which would have underlying permutation $\tau_{2,4} = (2\ 3\ 5)(4\ 7\ 6)$. If the required elements $t_{m,m}$ existed for the cactus operad J, then we would be able to apply the axioms from Definition 15.9 to the element $t_{2,4}$ in order to see how it is constructed from $t_{2,2} = s_{2,3}$.

By Axiom 2 of Definition 15.9 we should be able to split the element $t_{2,4}$ up as follows.

$$\begin{aligned} t_{2,4} &= t_{2,2+2} \\ &= \beta(t_{2,2}, t_{2,2}) \cdot \delta_{2,2,2,2}(t_{2,2}) \\ &= s_{2,3} \cdot s_{5,7} \cdot \delta_{2,2,2,2}(s_{2,3}) \\ &= s_{2,3} \cdot s_{5,7} \cdot s_{2,6} \cdot \beta(e_2, s_{1,2}, s_{1,2}, e_2) \\ &= s_{2,3} \cdot s_{5,7} \cdot s_{2,6} \cdot s_{3,4} \cdot s_{5,6}. \end{aligned}$$

Here we have used δ as defined for generators $s_{p,q}$ in Theorem 16.9. This element has underlying permutation

$$(2\ 3)(5\ 7)(2\ 6)(3\ 5)(3\ 4)(5\ 6) = (2\ 6)(3\ 4\ 7\ 5)$$

which is not equal to $\tau_{2,4} = (2\ 3\ 5)(4\ 7\ 6)$. Hence if J were to have a pseudo-commutative structure, then it cannot arise in this way.

Remark 17.9. I commented out the profunctors stuff, but it is still in the file.

References

[1] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories.* Vol. 189. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1994, pp. xiv+316.

[2] M. A. Batanin. "The Eckmann-Hilton argument and higher operads". In: *Adv. Math.* 217.1 (2008), pp. 334–385.

[3] A. J. Berrick and E. Hanbury. "Simplicial structures and normal forms for mapping class groups and braid groups". In: *Algebraic & Geometric Topology* 14.6 (Jan. 2014). Publisher: MSP, pp. 3259–3280.

[4] R. Blackwell, G. M. Kelly, and A. J. Power. "Two-dimensional monad theory". In: *J. Pure Appl. Algebra* 59.1 (1989), pp. 1–41.

[5] Eugenia Cheng and Nick Gurski. "Towards an n-category of cobordisms". In: *Theory and Applications of Categories* 18.10 (2007), pp. 274–302.

[6] Eugenia Cheng and Aaron Lauda. "Higher-dimensional categories: an illustrated guide book". In: *Preprint* (2004).

[7] Brian Day. *Construction of biclosed categories.* Thesis (Ph.D.)–University of New South Wales. 1970.

[8] Jean Douçot, Gabriele Rembado, and Matteo Tamiozzo. *Local wild mapping class groups and cabled braids.* preprint, arXiv:2204.08188. 2025.

[9] V. G. Drinfeld. "Quasi-Hopf algebras". In: *Algebra i Analiz* 1.6 (1989), pp. 114–148.

[10] Zbigniew Fiedorowicz. *The symmetric bar construction.* preprint.

[11] Zbigniew Fiedorowicz and Jean-Louis Loday. "Crossed simplicial groups and their associated homology". In: *Transactions of the American Mathematical Society* 326.1 (1991), pp. 57–87.

[12] Bertrand J Guillou, J Peter May, Mona Merling, and Angélica M Osorno. "Symmetric monoidal G-categories and their strictification". In: *The Quarterly Journal of Mathematics* 71.1 (Mar. 2020), pp. 207–246.

[13] Bertrand J. Guillou, J. Peter May, Mona Merling, and Angélica M. Osorno. "Multiplicative equivariant K-theory and the Barratt-Priddy-Quillen theorem". en. In: *Advances in Mathematics* 414 (Feb. 2023), p. 108865.

[14] A. Henriques and J. Kamnitzer. "Crystals and coboundary categories". In: *Duke Math. J.* 132.2 (2006), pp. 191–216.

[15] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases.* Vol. 42. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. xviii+307.

[16] Martin Hyland and John Power. "Pseudo-commutative monads and pseudo-closed 2-categories". In: *J. Pure Appl. Algebra* 175.1-3 (2002). Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 141–185.

[17] André Joyal and Ross Street. "Braided tensor categories". In: *Adv. Math.* 102.1 (1993), pp. 20–78.

[18] G. M. Kelly. “An abstract approach to coherence”. In: *Coherence in categories*. Springer, Berlin, 1972, 106–147. Lecture Notes in Math., Vol. 281.

[19] G. M. Kelly. “Elementary observations on 2-categorical limits”. In: *Bull. Austral. Math. Soc.* 39.2 (1989), pp. 301–317.

[20] G. M. Kelly. “Many-variable functorial calculus. I”. In: *Coherence in categories*. Springer, Berlin, 1972, 66–105. Lecture Notes in Math., Vol. 281.

[21] G. M. Kelly. “On clubs and doctrines”. In: *Category Seminar (Proc. Sem., Sydney, 1972/1973)*. Springer, Berlin, 1974, 181–256. Lecture Notes in Math., Vol. 420.

[22] G. M. Kelly. “On the operads of J. P. May”. In: *Repr. Theory Appl. Categ.* 13.1 (2005), pp. 1–13.

[23] G. M. Kelly and Stephen Lack. “On property-like structures”. In: *Theory Appl. Categ* 3.9 (1997), pp. 213–250.

[24] Anders Kock. “Closed categories generated by commutative monads”. In: *J. Austral. Math. Soc.* 12 (1971), pp. 405–424.

[25] Anders Kock. “Monads on symmetric monoidal closed categories”. In: *Arch. Math. (Basel)* 21 (1970), pp. 1–10.

[26] Anders Kock. “Strong functors and monoidal monads”. In: *Arch. Math. (Basel)* 23 (1972), pp. 113–120.

[27] R Krasauskas. “Crossed simplicial groups of framed braids and mapping class groups of surfaces”. In: *Lithuanian Mathematical Journal* 36.3 (1996), pp. 263–281.

[28] R. Krasauskas. “Skew-simplicial groups”. en. In: *Lithuanian Mathematical Journal* 27.1 (Jan. 1987), pp. 47–54.

[29] Stephen Lack. “Codescent objects and coherence”. In: *J. Pure Appl. Algebra* 175.1-3 (2002). Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 223–241.

[30] F. William Lawvere. *Functional semantics of algebraic theories*. Thesis (Ph.D.)–Columbia University. ProQuest LLC, Ann Arbor, MI, 1963.

[31] Tom Leinster. “A survey of definitions of n-category”. In: *arXiv preprint math/0107188* (2001).

[32] Tom Leinster. *Higher operads, higher categories*. Vol. 298. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2004, pp. xiv+433.

[33] Fosco Loregian. *(Co)end Calculus*. Vol. 468. Cambridge University Press, 2021.

[34] Saunders Mac Lane. “Categorical algebra”. In: *Bull. Amer. Math. Soc.* 71 (1965), pp. 40–106.

[35] Saunders MacLane. *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262.

[36] M. Markl. “Operads and PROPs”. In: *Handbook of algebra. Vol. 5*. Vol. 5. Handb. Algebr. Elsevier/North-Holland, Amsterdam, 2008, pp. 87–140.

[37] M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*. Vol. 96. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2002, pp. x+349.

[38] Francisco Marmolejo. "Doctrines whose structure forms a fully faithful adjoint string". In: *Theory and applications of categories* 3.2 (1997), pp. 24–44.

[39] J. P. May. *The geometry of iterated loop spaces*. Lectures Notes in Mathematics, Vol. 271. Berlin: Springer-Verlag, 1972, pp. viii+175.

[40] A. J. Power. "A general coherence result". In: *J. Pure Appl. Algebra* 57.2 (1989), pp. 165–173.

[41] Elsayed A. El-Rifai and H. R. Morton. "Algorithms for positive braids". In: *Quart. J. Math. Oxford Ser. (2)* 45.180 (1994), pp. 479–497.

[42] Jonathan Rubin. *Equivariant Categorical Coherence Theory*. Thesis (Ph.D.)–University of Chicago. 2018.

[43] Paolo Salvatore and Nathalie Wahl. "Framed discs operads and Batalin-Vilkovisky algebras". In: *Q. J. Math.* 54.2 (2003), pp. 213–231.

[44] Alistair Savage. "Braided and coboundary monoidal categories". In: *Algebras, representations and applications* 483 (2009), pp. 229–251.

[45] Ross Street. *Quantum groups*. Vol. 19. Australian Mathematical Society Lecture Series. A path to current algebra. Cambridge University Press, Cambridge, 2007, pp. xviii+141.

[46] Ross Street. "The formal theory of monads". In: *Journal of Pure and Applied Algebra* 2.2 (1972), pp. 149–168.

[47] Nathalie Wahl. *Ribbon braids and related operads*. Thesis (Ph.D.)–University of Oxford. 2001.

[48] Donald Yau. *Infinity Operads and Monoidal Categories with Group Equivariance*. World Scientific, 2021.

[49] Wenbin Zhang. *Group operads and homotopy theory*. preprint, arXiv:1111.7090v2. 2012.