Operads and equivariance

Alexander S. Corner
alex.corner@shu.ac.uk

Nick Gurski
nick.gurski@case.edu

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Abstract

1 Introduction

QQQ Needs an intro. Might be something salvageable from the original papers.

Original paper intro:

Operads were defined by May [**maygeom**] in the early 70's to provide a convenient tool to approach problems in algebraic topology, notably the question of when a space X admits an n-fold delooping Y so that $X \simeq \Omega^n Y$. An operad, like an algebraic theory [**lawvere-thesis**], is something like a presentation for a monad or algebraic structure. The theory of operads has seen great success, and we would like to highlight two reasons. First, operads can be defined in any suitable symmetric monoidal category, so that there are operads of topological spaces, of chain complexes, of simplicial sets, and of categories, to name a few examples. Moreover, symmetric (lax) monoidal functors carry operads to operads, so we can use operads in one category to understand objects in another via transport by such a functor. Second, operads in a fixed category are highly flexible tools. In particular, the categories listed above all have some inherent notion of "homotopy equivalence" which is weaker than that of isomorphism, so we can study operads which are equivalent but not isomorphic. These tend to have algebras which have similar features in an "up-to-homotopy" sense but very different combinatorial or geometric properties arising from the fact that different objects make up these equivalent but not isomorphic operads.

Operads in the category **Cat** of small categories have a unique flavor arising from the fact that **Cat** is not just a category but a 2-category. These 2-categorical aspects have not been widely treated in the literature, although a few examples can be found. Lack [**lack-cod**] mentions braided **Cat**-operads (the reader new to braided operads should refer to the work of Fiedorowicz [**fie-br**]) in his work on coherence for 2-monads, and Batanin [**bat-eh**] uses lax morphisms of operads in **Cat** in order to define the notion of an internal operad. But aside from a few appearances, the basic theory of operads in **Cat** and their 2-categorical properties seems missing. This paper was partly motivated by the need for such a theory to be explained from the ground up.

There were two additional motivations for the work in this paper. In thinking about coherence for monoidal functors, the first author was led to a general study of algebras for multicategories internal to **Cat**. These give rise to 2-monads (or perhaps pseudomonads, depending on how the theory is set up), and checking abstract properties of these 2-monads prompts one to consider the simpler case of operads in **Cat** instead of multicategories. The other motivation was from the second author's attempt to understand the interplay between operads in **Cat**, operads in **Top**, and the passage from (bi)permutative categories to E_∞ (ring) spaces. The first of these motivations raised the issue of when operads in **Cat** are cartesian, while the second led us to consider when an operad in **Cat** possesses a pseudo-commutative structure.

While considering how to best tackle a general discussion of operads in **Cat**, it became clear that restricting attention to the two most commonly used types of operads, symmetric and non-symmetric operads, was both short-sighted and unnecessary. Many theorems apply to both kinds of operads at once, with the difference in proofs being negligible; in fact, most of the arguments which applied to the symmetric case seemed to apply to the case of braided operads as well. This led us to the notion of an action operad **G**, and then to a definition of **G**-operads. In essence, this is merely the general notion of what it means for an operad $P = \{P(n)\}_{n \in \mathbb{N}}$ to have groups of equivariance $\mathbf{G} = \{G(n)\}_{n \in \mathbb{N}}$ such that $G(n)$ acts on $P(n)$. Choosing different natural families of groups **G**, we recover known variants of the definition of operad.

Groups **G**	Type of operad
Terminal groups	Non-symmetric operad
Symmetric groups	Symmetric operad
Braid groups	Braided operad

These definitions have appeared, with minor variations, in two sources of which we are aware. In Wahl's thesis [**wahl-thesis**], the essential definitions appear but not in complete generality as she requires a surjectivity condition. Zhang [**zhang-grp**] also studies these notions[1], once again in the context of homotopy theory, but requires the superfluous condition that $e_1 = \mathrm{id}$ (see Lemma **??**).

This paper consists of the following. In Section 1, we give the definition of an action operad **G** and a **G**-operad. We develop this definition abstractly so as to apply it in any suitable symmetric monoidal category. It is standard to express operads as monoids in a particular functor category using a composition tensor product. In order to show that our **G**-operads fit into this philosophy, we must work abstractly and use the calculus of coends together with the Day convolution product [**day-thesis**]. The reader uninterested in these details can happily skip them, although we find the route taken here to be quite satisfactory in justifying the axioms for an action operad **G** and the accompanying notion of **G**-operad. Many of our calculations are generalizations of those appearing in work of Kelly [**kelly-op**], although there are slight differences in flavor between the two treatments.

Section 2 works through the basic 2-categorical aspects of operads in **Cat**. We explain how every operad gives rise to a 2-monad, and show that all of the various 1-cells between algebras of the associated 2-monad correspond to the obvious sorts of 1-cells one might define between algebras over an operad in **Cat**. Similarly, we show that the algebra 2-cells, using the 2-monadic approach, correspond to the obvious notion of transformation one would define using the operad.

Section 3 studies three basic 2-categorical properties of an operad, namely the property of being finitary, the property of being 2-cartesian, and the coherence property. The first of these always holds, as a simple calculation shows. The second of these turns out to be equivalent to the action of $G(n)$ on $P(n)$ being free for all n, at least up to a certain kernel. In particular, our characterization clearly shows that every non-symmetric operad is 2-cartesian, and that a symmetric operad is 2-cartesian if and only if the symmetric group actions are all free. (It is useful to note that a 2-monad on **Cat** is 2-cartesian if and only if the underlying monad on the category of small categories is cartesian in the usual sense as the (strict) 2-pullback of a diagram is the same as its pullback.) The third property is also easily shown to hold for any **G**-operad on **Cat** using a factorization system argument due to Power [**power-gen**].

Section 4 then goes on to study the question of when a **G**-operad P admits a pseudo-commutative structure. Such a structure provides the 2-category of algebras with a richer structure that includes well-behaved notions of tensor product, internal hom, and multilinear map that fit together much as the analogous notions do in the category of vector spaces. When P is contractible (i.e., each $P(n)$ is equivalent to the terminal category), this structure can be obtained from a collection of elements $t_{m,n} \in G(mn)$ satisfying certain properties. In particular, we show that every contractible symmetric operad is pseudo-commutative, and we prove that there exist such elements $t_{m,n} \in Br_{mn}$ so that every contractible braided operad is pseudo-commutative as well (in fact in two canonical ways). Thus Section 4 can be seen as a continuation, in the operadic context, of the work in [**HP**], and in particular the "geometric" proof of the existence of a pseudo-commutative structure for braided strict monoidal categories

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[1]Zhang calls our action operad a *group operad*. We dislike this terminology as it seems to imply that we are dealing with an operad in the category of groups, which is not the case unless all of the maps $\pi_n : G(n) \to \Sigma_n$ are zero maps.

demonstrates the power of being able to change the groups of equivariance.

Original Borel intro:

Categories of interest are often monoidal: sets, topological spaces, and vector spaces are all symmetric monoidal, while the category of finite ordinals (under ordinal sum) is merely monoidal. But other categories have more exotic monoidal structures. The first such type of structure discovered was that of a braided monoidal category. These arise in categories whose morphisms have a geometric flavor like cobordisms embedded in some ambient space [**js**], in categories produced from double loop spaces [**fie-br**], and categories of representations over objects like quasitriangular (or braided) bialgebras [**street-quantum**] . Another such exotic monoidal structure is that of a coboundary category, arising in examples from the representation theory of quantum groups [**drin-quasihopf**].

Going back to the original work of May on iterated loop spaces [**maygeom**], operads were defined in both symmetric and nonsymmetric varieties. But Fiedorowicz's work on double loop spaces [**fie-br**] showed that there was utility in considering another kind of operad, this time with braid group actions instead of symmetric group actions. There is a clear parallel between these definitions of different types of operads and the definitions of different kinds of monoidal category, with each given by some general schema in which varying an $\mathbb{N}$-indexed collection of groups produced the types of operads or monoidal categories seen in nature. Building on the work in [**cg**], the goal of this paper is to show that this parallel can be upgraded from an intuition to precise mathematics using the notion of action operad.

An action operad $\mathbf{\Lambda}$ is an operad which incorporates all of the essential features of the operad of symmetric groups. Thus $\Lambda(n)$ is no longer just a set, but instead also has a group structure together with a map $\pi_n : \Lambda(n) \to \Sigma_n$. Operadic composition then satisfies an additional equivariance condition using the maps π_n and the group structures. Each action operad $\mathbf{\Lambda}$ produces a notion of $\mathbf{\Lambda}$-operad which encodes equivariance conditions using both the groups $\Lambda(n)$ and the maps π_n. Examples include the symmetric groups, the terminal groups (giving nonsymmetric operads), the braid groups (giving braided operads), and the n-fruit cactus groups [**hk-cobound**] (giving a new notion of operad one might call cactus operads). Using a formula resembling the classical Borel construction for spaces with a group action, we can produce from any action operad $\mathbf{\Lambda}$ a notion of $\mathbf{\Lambda}$-monoidal category, in which the group $\Lambda(n)$ acts naturally on n-fold tensor powers of any object. Thus the categorical Borel construction embeds action operads into a category of monads on **Cat**, and we characterize the image of this embedding as those monads describing monoidal structures of a precise kind.

The paper is organized into the following sections. Section 1 reviews the definition of an action operad, and defines the categorical Borel construction on them. The key result, which reappears in proofs throughout the paper, is Theorem 4.14, characterizing action operads in terms of two new operations mimicking the block sum of permutations and the operation which takes a permutation of n letters and produces a new permutation on $k_1 + k_2 + \cdots + k_n$ letters by permuting the blocks of k_i letters. In Section 2, we use this characterization and Kelly's theory of clubs [**kelly˙club1**, **kelly˙club0**, **kelly˙club2**] to embed action operads into monads on **Cat** and determine the essential image of this embedding. Section 3 gives a construction of the free action operad from a suitable collection of data, and relates this to how clubs can be described using generators and relations. The results of Sections 2 and 3 show that the definitions of symmetric monoidal category or coboundary category, for example,

correspond to the action operad constructed from the corresponding free symmetric monoidal or coboundary category on one object; these and other examples appear in detail in Section 4. Section 5 then extends the definition of $\mathbf{\Lambda}$-operad to that of $\mathbf{\Lambda}$-multicategory and shows that these arise abstractly via a Kleisli construction.

Copied from text: Yau [**yau˙infinity˙2021**] collects together a large number of results on the topic of action operads while also investigating the setting of infinity group operads.

2   Notation and Conventions

Notation 2.1 (Symmetric groups). We denote the symmetric group on the symbols $1, 2, \ldots, n$ by Σ_n. Elements of a symmetric group are usually denoted by lowercase Greek letters or written in cycle notation.

Notation 2.2 (Braid groups). We denote the braid group on n strands by B_n.

Notation 2.3 (Identity elements). The symbol e will generically represent an identity element in a group. If we are considering a set of groups $\{\Lambda(n)\}_{n\in\mathbb{N}}$ indexed by the natural numbers, then e_n is the identity element in $\Lambda(n)$. We will often drop the subscripts and just write e when the index n in $\Lambda(n)$ is either clear from context or unimportant to the argument at hand.

Convention 2.4 (Identity morphisms). We generically write an identity morphism $A \to A$ as either 1 or 1_A.

Notation 2.5 (Group actions). For a group G, a right G-action on a set X will be denoted $(x, g) \mapsto x \cdot g$ or $(x, g) \mapsto xg$. Similar notation will be used for left actions, and for multiplication in a group.

Convention 2.6 (Indexed objects). We generically write $\{\Lambda(n)\}_{n\in\mathbb{N}}$ for a $\mathbb{N}$-indexed family of objects $\Lambda(n)$. We will occasionallywrite Λ_n in place of $\Lambda(n)$, especially in diagrams or when the objects Λ_n have been independently defined, as in Notations 2.1 and 2.2.

Convention 2.7 (Products and quotients). We will often be interested in elements of a product of the form
$$A \times B(1) \times \cdots \times B(n) \times C$$
(or similar, for example without C). We will write elements of this set as $(a; b_1, \ldots, b_n; c)$, where $b_i \in B(i)$. In the case that we need to consider equivalence classes of such elements, these classes will be written as $[a; b_1, \ldots, b_n; c]$. The most common situation in which we consider such equivalence classes is that of a coequalizer of left and right group actions in the following sense. A coequalizer of maps

$$A \times G \times B \mathrel{\mathop{\rightrightarrows}^{1 \times \lambda}_{\rho \times 1}} A \times B \xrightarrow{\;\;\varepsilon\;\;} A \otimes_G B$$

will be written as $A \otimes_G B$, where ρ is a right action of G on A and λ is a left action of G on B. This notation is meant to emphasize the analogy with tensor products of R-modules, even when the monoidal structure involved is cartesian. It also differentiates these coequalizers from pullbacks.

Convention 2.8 (Pullbacks). The pullback of the diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow f \\
Y & \xrightarrow{\;g\;} & A
\end{array}
$$

will be written as $X \times_A Y$.

Definition 2.9.](Underlying permutation)] Suppose that $f\colon G \to \Sigma_n$ is a given group homomorphism, and $x \in G$. The *underlying permutation* of x is the element $f(x) \in \Sigma_n$. If there is likely to be some confusion as to which homomorphism f is being used, we will call $f(x)$ the *underlying permutation with respect to f*.

Notation 2.10 (Applying underlying permutations). Throughout we will be using maps $\pi_n\colon O(n) \to \Sigma_n$, where $O(n)$ is the set of n-ary operations of an operad O and Σ_n is the symmetric group on n elements. For any $\sigma \in O(n)$, we will write $\sigma(i)$ for $\pi_n(\sigma)(i)$, the image of i with respect to the underlying permutation of σ; the notation $\sigma^{-1}(i)$ will be used for the inverse image of i with respect to the underlying permutation of σ.

Definition 2.11 (Block sum). Let $k_1, \ldots, k_n$ be natural numbers and suppose that $\sigma_i \in \Sigma_{k_i}$ are permutations. The *block sum* of $\sigma_1, \ldots, \sigma_n$, written

$$
\beta(\sigma_1, \ldots, \sigma_n),
$$

is the permutation in Σ_K, where $K = \sum_{i=1}^n k_i$, given as described below. For $1 \le j \le K$, define c to be the unique integer such that

$$
k_1 + \cdots + k_c < j \le k_1 + \cdots + k_c + k_{c+1}.
$$

Define

$$
\beta(\sigma_1, \ldots, \sigma_n)(j) = k_1 + \cdots + k_c + \sigma_{c+1}\big(j - (\sum_{i=1}^c k_i)\big).
$$

Remark 2.12. The formula above expresses the idea that $\beta(\sigma_1, \ldots, \sigma_n)$ permutes the first k_1 elements using σ_1, the next k_2 elements using σ_2, and so on.

Definition 2.13 (Duplication). Let $k_1, \ldots, k_n$ be natural numbers, and suppose that $\sigma \in \Sigma_n$ is a permutation. The *duplication* of σ with respect to $k_1, \ldots, k_n$, written

$$
\delta_{n;k_1,\ldots,k_n}(\sigma),
$$

is the permutation in Σ_K, where $K = \sum_{i=1}^n k_i$, given as described below. For $1 \le j \le K$, define c to be the unique integer such that

$$
k_1 + \cdots + k_c < j \le k_1 + \cdots + k_c + k_{c+1}.
$$

Define

$$
\delta_{n;k_1,\ldots,k_n}(\sigma)(j) = \big(\sum_{\sigma(k_i)<\sigma(k_{c+1})} k_i \big) + j - \big(\sum_{i=1}^c k_i \big).
$$

Remark 2.14. The formula above for $\delta_{n;k_1,\ldots,k_n}(\sigma)$ is best explained by drawing the graph of σ as follows. The function σ can be represented by drawing two rows of n dots each, and connecting dot i in the top row to dot $\sigma(i)$ in the bottom row. Then $\delta_{n;k_1,\ldots,k_n}(\sigma)$ is obtained by

- replacing dot i in the top row with k_i dots,
- replacing dot $\sigma(i)$ in the bottom row with k_i dots, and
- connecting these two sets of k_i dots in the unique way that preserves order.

Thus we see that the ith entry for σ is duplicated k_i times in $\delta_{n;k_1,\ldots,k_n}(\sigma)$.

Remark 2.15. Figure out where to put these matrices in **??**.

Convention 2.16 (Superscripts). We generically use superscripts, when needed, to distinguish between operations of the same type associated to different structures. As an example, a monoid homomorphism $f\colon A \to B$ would have axioms written as

$$f(x \cdot^A y) = f(x) \cdot^B f(y),$$
$$f(1^A) = 1^B.$$

Part I: Operads and Action Operads

3   Background: Operads

This section will collect the basic background information on operads that we will later generalize in. We begin with the most common type of operad, a symmetric operad, $\boxed{\text{ref later sections}}$ before defining two more types of operad: plain and braided.

Definition 3.1 (Symmetric operad). A *symmetric operad O* (in the category of sets) consists of

- a set, $O(n)$, for each natural number n,
- for each n, a right Σ_n-action on $O(n)$,
- an element id $\in O(1)$, and
- functions
$$\mu\colon O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying the following three axioms.

1. The element id $\in O(1)$ is a two-sided unit for μ, meaning that

$$\mu(\mathrm{id}; x) = x,$$
$$\mu(x; \mathrm{id}, \ldots, \mathrm{id}) = x$$

   for any $x \in O(n)$.

2. The functions μ are associative, meaning that the diagram below commutes.

$$
\begin{array}{ccc}
O(n) \times \left(\prod_{i=1}^n O(k_i)\right) \times \left(\prod_{i=1}^n \prod_{j=1}^{k_i} O(l_{i,j})\right) & \xrightarrow{\ \cong\ } & O(n) \times \prod_{i=1}^n \left(O(k_i) \times \prod_{j=1}^{k_i} O(l_{i,j})\right) \\
\Big\downarrow{\scriptstyle \mu \times 1} & & \Big\downarrow{\scriptstyle 1 \times \prod \mu} \\
& & O(n) \times \prod_{i=1}^n O\left(\sum_{j=1}^{k_i} l_{i,j}\right) \\
& & \Big\downarrow{\scriptstyle \mu} \\
O\left(\sum_{i=1}^n k_i\right) \times \prod_{i=1}^n \prod_{j=1}^{k_i} O(l_{i,j}) & \xrightarrow{\quad\quad \mu \quad\quad} & O\left(\sum_{i=1}^n \sum_{j=1}^{k_i} l_{i,j}\right)
\end{array}
$$

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3. The functions μ are equivariant with respect to the symmetric group actions, meaning that two equations hold.

   3.1 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\tau_i \in \Sigma_{k_i}$ for $i = 1, \ldots, n$. Then the first equivariance axiom is the requirement that

   $$\mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) = \mu(x; y_1, \ldots, y_n) \cdot \beta(\tau_1, \ldots, \tau_n)$$

   holds, where β is the function from Definition 2.11.

   3.2 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\sigma \in \Sigma_n$. Then the second equivariance axiom is the requirement that

   $$\mu(x \cdot \sigma; y_1, \ldots, y_n) = \mu\left(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \cdot \delta_{n; k_1, \ldots, k_n}(\sigma)$$

   holds, where $\delta_{n; k_1, \ldots, k_n}$ is the function from Definition 2.13.

Terminology 3.2 (Operadic multiplication, composition). The functions μ in Definition 3.1 are called *operadic multiplication* or *operadic composition* maps.

Remark 3.3. One is intended to think that $x \in O(n)$ is a function with n inputs and a single output, as below.
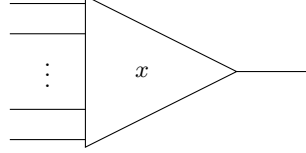
Operadic composition is then a generalization of function composition, with the pictorial representation below being $\mu(x; y_1, y_2)$ for $\mu\colon O(2) \times O(2) \times O(3) \to O(5)$.
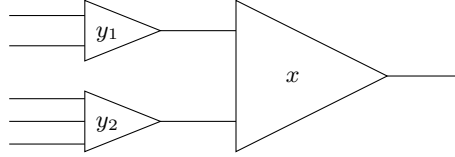
Terminology 3.4 (n-ary operations). The set $O(n)$ in Definition 3.1 is called the set of n-*ary operations* of O.

Remark 3.5. If O is an operad in a category other than **Sets** (see Remark 3.11), then we would call $O(n)$ the *object* of n-ary operations.

Here are two important examples of symmetric operads.

Example 3.6 (Symmetric operad of symmetric groups). The canonical example of *a* symmetric operad is *the* symmetric operad which we write as Σ. The set $\Sigma(n)$ is the set of elements of the symmetric group Σ_n, and the group action is just multiplication on the right. The identity element $\mathrm{id} \in \Sigma(1)$ is just the identity permutation on a one-element set. Operadic composition in Σ will then be given by a function

$$\Sigma(n) \times \Sigma(k_1) \times \cdots \times \Sigma(k_n) \to \Sigma(k_1 + \cdots + k_n)$$

which takes permutations $\sigma \in \Sigma_n, \tau_i \in \Sigma_{k_i}$ and produces the following permutation in $\Sigma_{k_1 + \cdots + k_n}$:
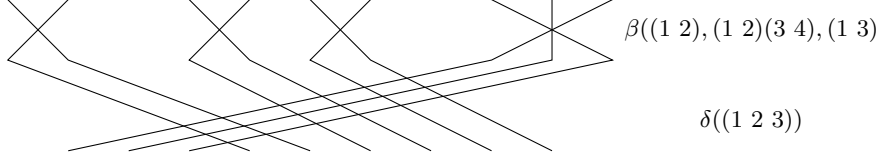
$$\mu(\sigma; \tau_1, \ldots, \tau_n) = \delta(\sigma) \cdot \beta(\tau_1, \ldots, \tau_n),$$

with β and δ as in **??**.

Below we have drawn the permutation for the composition

$$\mu\colon \Sigma(3) \times \Sigma(2) \times \Sigma(4) \times \Sigma(3) \to \Sigma(9)$$

evaluated on the element $((1\ 2\ 3); (1\ 2), (1\ 2)(3\ 4), (1\ 3))$, in terms of β and δ. We expand on this in Theorem 4.14.

$$\beta((1\ 2), (1\ 2)(3\ 4), (1\ 3))$$

$$\delta((1\ 2\ 3))$$

End this example here, with labels, move the rest to after 2.26.

Remark 3.7. Permutations, as elements of Σ_n, can be considered as permutation *matrices*, matrices with exactly one 1 in each row and column. E.g., the permutation $(1\ 3\ 2) \in \Sigma_3$ can be considered as the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which permutes three elements $\begin{bmatrix} a & b & c \end{bmatrix}$ by pre-multiplying by the permutation matrix:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ a \\ b \end{bmatrix}.$$

Then β corresponds to the process of taking the block diagonal matrix of the original permutation matrices. So given elements $(1\ 2) \in \Sigma_2$, $e_1 \in \Sigma_1$, and $(1\ 3\ 2) \in \Sigma_3$, then

$$\beta((1\ 2), e_1, (1\ 2\ 3)) = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 & 0 \\ 0 & [1] & 0 \\ 0 & 0 & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= (1\ 2)(3)(4\ 5\ 6).$$

Similarly, we can describe δ as an operation on permutation matrices. The idea here is that for $\sigma \in \Sigma_n$, $\delta_{n;k_1,\ldots,k_n}(\sigma)$ takes a block diagonal of identity matrices I_{k_1}, ..., I_{k_n} (which corresponds to $\beta(e_{k_1}, \ldots, e_{k_n}) \in \Sigma_{k_1+\ldots+k_n}$), and permutes these according to the effect of the permutation σ. For example, given $\sigma = (1\ 2\ 3)$, then

$$\delta_{3;2,1,3}(\sigma) = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix} * \sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_3 \\ I_2 & 0 & 0 \\ 0 & I_1 & 0 \end{bmatrix}.$$

We make use of a similar interpretation of signed permutations and block diagonal matrices in a counterexample given in **??**.

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Note that $(12)(34) \in \Sigma(4)$ is actually $\mu(e_2; (12), (12))$, where $e_2 \in \Sigma_2$ is the identity permutation. Using this and operad associativity, one can easily check that

$$\mu\left((123); (12), (12)(34), (13)\right) = \mu\left((1234); (12), (12), (12), (13)\right),$$

where now the composition on the right side uses the function

$$\mu \colon \Sigma(4) \times \Sigma(2) \times \Sigma(2) \times \Sigma(2) \times \Sigma(3) \to \Sigma(9).$$

This equality is obvious using the picture above, but verifiable directly using only the algebra of the symmetric operad.

Example 3.8 (Endomorphism operad). Let X be a set. The *endomorphism operad* of X, denoted $\mathcal{E}_X$, consists of

- the sets
$$\mathcal{E}_X(n) = \mathbf{Sets}(X^n, X),$$

- the right group actions $\mathcal{E}_X(n) \times \Sigma_n \to \mathcal{E}_X(n)$ given by
$$(f \cdot \sigma)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$

- the element $\mathrm{id} \in \mathcal{E}_X(1)$ being the identity function $1 \colon X \to X$, and

- operadic multiplication given by
$$\mu(g; f_1, \ldots, f_n) = g \circ (f_1 \times \cdots \times f_n).$$

We leave verification of the axioms to the reader, or recommend. may-GoILS, find specific ref number

Remark 3.9. Intuition in Remark 3.3 connected with example Example 3.8 through the concept of an algebra, see for these. later section

One can also drop the symmetric group actions entirely to obtain the notion of a non-symmetric or plain operad.

Definition 3.10 (Non-symmetric operad). A *non-symmetric operad* O consists of

- a set, $O(n)$, for each natural number n,

- an element $\mathrm{id} \in O(1)$, and

- functions
$$\mu \colon O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n),$$

satisfying axioms 1 and 2 from Definition 3.1.

Remark 3.11. 1. One can change from operads in $\mathbf{Sets}$ to operads in another (symmetric) monoidal category $\mathcal{V}$ by requiring each $O(n)$ to be an object of $\mathcal{V}$ and replacing all instances of cartesian product with the appropriate tensor product in $\mathcal{V}$. One would also the element $\mathrm{id} \in O(1)$ with a map $I \to O(1)$ from the unit object of $\mathcal{V}$ to $O(1)$. In the case of symmetric operads, one would also express the right group actions as homomorphisms

$$\Sigma_n^{op} \to \mathcal{V}\big(O(n), O(n)\big).$$

2. Every symmetric operad has an underlying *symmetric collection* which consists of the natural number-indexed set $\{O(n)\}_{n\in\mathbb{N}}$ together with symmetric group actions, but without a chosen identity element or composition maps. The category of symmetric collections is a presheaf category, and we will equip it with a monoidal structure in which monoids are precisely operads in Theorem 12.5. A similar construction, but without reference to group actions, shows that every non-symmetric operad has an underlying (non-symmetric) collection which is now merely a $\mathbb{N}$-indexed collection of sets.

Example 3.12.

insert example of non-symmetric operad here, how about Trimble's thing in dimension 1?

Example 3.12

In the original topological applications [**maygeom**], symmetric operads were the central figures. A further kind of operad was studied by Fiedorowicz in [**fie-br**], that of a *braided* operad in which the braid groups take the place of the symmetric groups. We sketch that definition below, but postpone details to.

write this out later, ref back

Definition 3.13 (Braided operad, sketch). A *braided operad* consists of

- a non-symmetric operad O and

- for each n, a right action of the nth braid group B_n on $O(n)$,

such that the operadic multiplication functions μ are equivariant with respect to the braid group actions, meaning that two equations hold.

1 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\tau_i \in B_{k_i}$ for $i = 1, \ldots, n$. Then the first equivariance axiom is the requirement that

$$\mu(x; y_1 \cdot \tau_1, \ldots, y_n \cdot \tau_n) = \mu(x; y_1, \ldots, y_n) \cdot \beta(\tau_1, \ldots, \tau_n)$$

holds.

2 Suppose that $x \in O(n)$, $y_i \in O(k_i)$ for $i = 1, \ldots, n$, and $\sigma \in B_n$. Then the second equivariance axiom is the requirement that

$$\mu(x \cdot \sigma; y_1, \ldots, y_n) = \mu\left(x; y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \cdot \delta_{n;k_1,\ldots,k_n}(\sigma)$$

holds.

Remark 3.14. The above sketch omits the definitions of β, δ for braids. Formulas for these can be found in [**yau˙infinity˙2021**], although the geometric interpretations are simple: β takes the disjoint union of braids, and $\delta_{n;k_1,\ldots,k_n}(\tau)$ is obtained by replacing the ith strand of τ by k_i parallel strands. These operations are sometimes referred to as 'cabling' operations for braids, as described in, for example, [**doucot˙local˙2025**].

Example 3.15.

insert example of braided operad here, probably just braid groups??

We conclude this section by defining various categories of operads in **Sets**, although the reader can generalize these to categories of operads in any symmetric monoidal category. We focus on the case of symmetric operads, and explain after how to modify the definitions for non-symmetric or braided operads.

Definition 3.16 (Map of symmetric operads). Let O, O' be symmetric operads in **Sets**. Then a *map of symmetric operads* (or just *operad map* for short, when it is clear that the intent is to respect the symmetric group actions) $f\colon O \to O'$ consists of functions $f_n\colon O(n) \to O'(n)$ for each natural number such that the following axioms hold for all $x \in O(n), y_i \in O(k_i), \sigma \in \Sigma_n$.

$$f\left(\mathrm{id}_O\right) = \mathrm{id}_{O'}$$

$$f\left(\mu^O(x; y_1, \ldots, y_n)\right) = \mu^{O'}\left(f(x); f(y_1), \ldots, f(y_n)\right)$$

$$f(x \cdot \sigma) = f(x) \cdot \sigma$$

The next proposition states that symmetric operads and their maps form a category. We leave the proof to the reader.

> let's use the following prop/notation pair as a template for how to define categories

Proposition 3.17. *There is a category with*

- *objects the symmetric operads O in **Sets**,*
- *morphisms the maps of symmetric operads between them,*
- *identities $1_O\colon O \to O$ given by*

$$(1_O)_n = 1_{O(n)}\colon O(n) \to O(n),$$

and

- *composition given by*

$$(g \circ f)_n = g_n \circ f_n.$$

Notation 3.18 (The category of symmetric operads). The category in Proposition 3.17 is called the *category of symmetric operads (in **Sets**)*, and is denoted $\Sigma\text{-}\mathbf{Op}$.

Remark 3.19 (The category of non-symmetric operads).

Remark 3.20 (The category of braided operads).

4 Action Operads

The axioms for both symmetric and braided operads use the following features.

1. For each n, we have a group Λ_n acting on the set $O(n)$ of n-ary operations of the operad. Each such group is equipped with a homomorphism $\pi_n\colon \Lambda_n \to \Sigma_n$, so that every element of Λ_n has an underlying permutation.

2. The first equivariance axiom requires the additional data of a family of functions

$$\beta\colon \Lambda_{k_1} \times \cdots \Lambda_{k_n} \to \Lambda_{k_1 + \cdots + k_n}.$$

In order for this to be a well-defined function, the right group action axioms force these functions to be group homomorphisms.

3. The second equivariance axiom requires the additional data of a family of functions

$$\delta_{n; k_1, \ldots, k_n}\colon \Lambda_{k_1} \times \cdots \Lambda_{k_n} \to \Lambda_{k_1 + \cdots + k_n}.$$

These functions are not forced to be group homomorphisms, but do satisfy some additional axioms.

In this section, we define *action operads* in Definition 4.1 in order to present a unified treatment of a family of groups satisfying the conditions above. In **??** II, we define for each action operad Λ a notion of Λ-operad; symmetric operads will arise when $\Lambda = \Sigma$, non-symmetric operads will arise when Λ is the action operad of trivial groups, and braided operads will arise when $\Lambda = B$. Our definition of an action operad will not mention β or δ, but will instead use a single axiom relating the group structure, operadic multiplication, and underlying permutations. The main result of this section is Theorem 4.14 in which we prove that action operads can be described entirely in terms of the functions β, δ as above. We will give two examples of action operads (the symmetric groups and the trivial groups) in this section, and postpone the rest to Section 5.

Definition 4.1 (Action operad). An *action operad* (Λ, π) consists of

- an operad $\Lambda = \{\Lambda(n)\}$ in the category of sets such that each $\Lambda(n)$ is equipped with the structure of a group and

- a map $\pi \colon \Lambda \to \Sigma$ which is simultaneously a map of operads and a group homomorphism $\pi_n \colon \Lambda(n) \to \Sigma_n$ for each n

such that one additional axiom holds. Write

$$\mu \colon \Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n) \to \Lambda(k_1 + \cdots + k_n)$$

for the multiplication in the operad Λ. Let $(g; f_1, \ldots, f_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_1) \times \cdots \times \Lambda(k_n)$ and let $(g'; f'_1, \ldots f'_n)$ be an element of the product $\Lambda(n) \times \Lambda(k_{g^{-1}(1)}) \times \cdots \times \Lambda(k_{g^{-1}(1)})$. We require that

$$\mu\left(g'; f'_1, \ldots f'_n\right) \mu\left(g; f_1, \ldots, f_n\right) = \mu\left(g'g; f'_{g(1)} f_1, \ldots, f'_{g(n)} f_n\right) \qquad (1)$$

in the group $\Lambda(k_1 + \cdots + k_n)$.

Notation 4.2. We write an action (Λ, π) as merely Λ. The maps π will be left implicit in the notation, as we will not have reason to study the case of a single operad Λ equipped with two different action operad structures via π and π'.

Remark 4.3. Our definition of an action operad is the same as that appearing in Wahl's thesis [**wahl-thesis**], but without the condition that each π_n is surjective. It is also the same as that appearing in work of Zhang [**zhang-grp**], although we prove later (see **??**) that Zhang's condition of $e_1 \in \Lambda(1)$ being the identity element follows from the rest of the axioms.

We now give the two examples of action operads that have already appeared in this paper: the symmetric groups and the trivial groups.

Example 4.4 (Action operad of symmetric groups). Copied, likely could be improved:

The symmetric operad Σ has a canonical action operad structure. It is given by taking π to be the identity map, and this action operad will be denoted $\boldsymbol{\Sigma}$. This is the terminal object in the category of action operads.

Example 4.5 (Action operad of trivial groups). Copied, likely could be improved:

The terminal operad T in the category of sets has a unique action operad structure, $\mathbf{T}$. Since $T(n)$ is a singleton for each n, the group structure is unique, as is the map π. The single action operad axiom is then automatic as both sides of Equation (1) are the unique element which happens to be the identity. This is the initial object in the category of action operads.

Remark 4.6. • As per Notation 2.10, we write $g(i)$ to mean $\pi(g)(i)$ and $g^{-1}(i)$ to mean $\pi(g)^{-1}(i)$.

• The final axiom is best explained using the operad Σ of symmetric groups. Reading symmetric group elements as permutations from top to bottom, below is a pictorial representation of the final axiom for the map $\mu\colon \Sigma_3 \times \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_6$.

$$\mu((23);(12),(12),e_2)\cdot\mu((132);(12),e_2,(12)) \qquad \mu((23)\cdot(132);e_2\cdot(12),(12)\cdot e_2,(12)\cdot(12))$$

Action operads are themselves the objects of a category, **AOp**. The morphisms of this category are defined below.

Definition 4.7 (Map of action operads). A *map of action operads* $f\colon \Lambda \to \Lambda'$ consists of a map $f\colon \Lambda \to \Lambda'$ of the underlying operads such that

1. $\pi^{\Lambda'} \circ f = \pi^{\Lambda}$ (i.e., f is a map of operads over Σ) and

2. each $f_n\colon \Lambda(n) \to \Lambda'(n)$ is a group homomorphism.

Proposition 4.8. *There is a category with*

- *objects the action operads O in* **Sets***,*
- *morphisms as defined in Definition 4.7,*
- *identities $1_\Lambda\colon \Lambda \to \Lambda$ given by the identity morphism of Λ as an operad, and*
- *composition given by composition of maps of operads.*

Notation 4.9 (The category of action operads). The category in Proposition 4.8 is called the *category of action operads (in* **Sets***)*, and is denoted **AOp**.

Proposition 4.10. *Let (Λ, π) be an action operad. The map $\pi\colon \Lambda \to \Sigma$ is a map of action operads.*

We now study some of the structure on the groups $\Lambda(n)$ for small values of n. Recall from Notation 2.3 that we write e_n for the identity element in the group $\Lambda(n)$. Many of our proofs rely on the following version of the Eckmann-Hilton argument.

Proposition 4.11 ((Eckmann-Hilton argument))**.** *Let G be a group with identity element e, and suppose $\varphi\colon G \times G \to G$ is a function. If φ is a homomorphism, meaning that*

$$\varphi(g',h') \cdot \varphi(g,h) = \varphi(g' \cdot g, h' \cdot h),$$

and $\varphi(g,e) = g = \varphi(e,g)$ for all elements $g \in G$, then

$$\varphi(g,h) = g \cdot h$$

and G is abelian.

Lemma 4.12. *Let Λ be an action operad.*

1. *In $\Lambda(1)$, the identity element for the group structure, e_1, is equal to the identity element for the operad structure, id.*

2. *The equation*
   $$\mu(e_n; e_{i_1}, \ldots, e_{i_n}) = e_I$$
   holds for any natural numbers $n, i_j, I = \sum_{j=1}^{n} i_j$.

15

3. The group $\Lambda(1)$ is abelian.

Proof. For the first claim, we will prove that $\text{id} \cdot e_1 = \text{id} \cdot \text{id}$, so $e_1 = \text{id}$ by cancellation. Note that since the only element of Σ_1 is the identity permutation, the action operad axiom Equation (1) is

$$\mu(g'; f') \cdot \mu(g; f) = \mu(g'g; f'f)$$

when $g, g' \in \Lambda(1)$. Thus we obtain

$$\begin{aligned}
\text{id} \cdot e_1 &= \mu(\text{id}; \text{id}) \cdot \mu(\text{id}; e_1) \\
&= \mu(\text{id} \cdot \text{id}; \text{id} \cdot e_1) \\
&= \mu(\text{id} \cdot \text{id}; \text{id}) \\
&= \text{id} \cdot \text{id}
\end{aligned}$$

using that id is the identity element for operadic multiplication, the $n = 1$ action operad axiom explained above, that e_1 is the identity for group multiplication, and that id is the identity for operadic multiplication again. Therefore $\text{id} \cdot e_1 = \text{id} \cdot \text{id}$ as desired, and $e_1 = \text{id}$.

For the second claim, we write $\mu(e_n; e_{i_1}, \ldots, e_{i_n})$ as $\mu(e; \underline{e})$, and consider the square of this element. We find that

$$\begin{aligned}
\mu(e; \underline{e}) \cdot \mu(e; \underline{e}) &= \mu(e \cdot e; \underline{e} \cdot \underline{e}) \\
&= \mu(e; \underline{e}),
\end{aligned}$$

where the first equality follows from the last action operad axiom together with the fact that e gets mapped to the identity permutation; here $\underline{e} \cdot \underline{e}$ is the sequence $e_{i_1} \cdot e_{i_1}, \ldots, e_{i_n} \cdot e_{i_n}$. Thus $\mu(e; \underline{e})$ is an idempotent element of the group $\Lambda(I)$, so must be the identity element e_I.

For the final claim, note that the specific operadic multiplication map $\mu \colon \Lambda(1) \times \Lambda(1) \to \Lambda(1)$ is a group homomorphism following from the action operad axioms, and $\text{id} = e_1$ is a two-sided unit, so Proposition 4.11 shows that μ is actually group multiplication and that $\Lambda(1)$ is abelian. $\qquad\square$

Lemma 4.13. *Let Λ be an action operad. For any $g, h \in \Lambda(0)$, the equation*

$$g \cdot h = \mu(e_2; g, h)$$

holds. As a consequence, $\Lambda(0)$ is abelian.

Proof. The function $\Lambda(0) \times \Lambda(0) \to \Lambda(0)$ given by

$$g, h \mapsto \mu(e_2; g, h)$$

is a group homomorphism by the action operad axiom Equation (1) as we verify below.

$$\mu(e_2; g', h') \cdot \mu(e_2; g, h) = \mu(e_2 \cdot e_2; g' \cdot g, h' \cdot h) = \mu(e_2; g' \cdot g, h' \cdot h)$$

In order to apply Proposition 4.11 and conclude that $g \cdot h = \mu(e_2; g, h)$, we must verify that

$$\mu(e_2; e_0, g) = g = \mu(e_2; g, e_0)$$

for all $g \in \Lambda(0)$. We will prove the first equality and leave the second to the reader, as it follows in a similar fashion.

By the identity axiom for an operad and the first part of Lemma 4.12, we have $g = \mu(\mathrm{id}; g) = \mu(e_1; g)$ for any $g \in \Lambda(0)$. By the second part of Lemma 4.12, we also have the equality $e_1 = \mu(e_2; e_0, e_1)$. Then

$$
\begin{aligned}
g &= \mu(e_1; g) \\
&= \mu(\mu(e_2; e_0, e_1); g) \\
&= \mu(\mu(e_2; e_0, e_1); \mu(e_1; g)) \\
&= \mu(e_2; e_0, g),
\end{aligned}
$$

where the final equality follows from operadic associativity together with $g = \mu(e_1; g)$. This calculation shows $g = \mu(e_2; e_0, g)$ as desired, and thus proves that the function $\mu(e_2; -, -)$ satisfies the hypotheses in Proposition 4.11. Therefore $g \cdot h = \mu(e_2; g, h)$ and $\Lambda(0)$ is abelian. $\qquad\square$

The symmetric operad structure on the symmetric groups in Example 3.6 was constructed using the functions β, δ from Definition 2.11 and Definition 2.13, respectively. We are now ready to show that any action operad can be described in this way, as promised in the introductory remarks to this section.

Theorem 4.14. *An action operad Λ determines, and is uniquely determined by, the following:*

- *groups $\Lambda(n)$ together with group homomorphisms $\pi_n \colon \Lambda(n) \to \Sigma_n$,*

- *a group homomorphism*

$$
\Lambda(k_1) \times \cdots \times \Lambda(k_n) \xrightarrow{\ \beta\ } \Lambda(k_1 + \cdots + k_n),
$$

  for each $n > 0$ and $k_1, \ldots, k_n$, and

- *a function of sets*

$$
\Lambda(n) \xrightarrow{\ \delta_{n;k_1,\ldots,k_n}\ } \Lambda(k_1 + \cdots + k_n)
$$

  for each $n, k_1, \ldots, k_n$,

subject to the axioms below. In what we write below, we use the following notational conventions.

- *The symbols f, g, h, with or without subscripts, always refer to an element of some group $\Lambda(n)$.*

- *The symbols j, k, m, n, p are all natural numbers, and i is a natural number between 1 and n.*

Axioms:

1. *The homomorphisms β are natural with respect to the maps π_n, where $K = k_1 + \cdots + k_n$.*

$$
\begin{array}{ccc}
\Lambda(k_1) \times \cdots \times \Lambda(k_n) & \xrightarrow{\ \beta\ } & \Lambda(K) \\
{\scriptstyle \pi_1 \times \cdots \times \pi_n} \downarrow & & \downarrow {\scriptstyle \pi} \\
\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} & \xrightarrow[\ \beta\ ]{} & \Sigma_K
\end{array}
$$

2. *The homomorphism $\beta \colon \Lambda(k) \to \Lambda(k)$ is the identity.*

> Used to have: together with the degenerate case of $n = 0$ which then is a group homomorphism $1 \to \Lambda(0)$. Do we need this?

3. The homomorphisms β are associative in the sense that the equation

$$\beta(\underline{h_1}, \ldots, \underline{h_n}) = \beta(\beta(\underline{h_1}), \ldots, \beta(\underline{h_n}))$$

holds, where $\underline{h_i} = h_{i1}, \ldots, h_{ij_i}$.

4. The functions $\delta_{n;k_1,\ldots,k_n}$ are natural with respect to the maps π_n, where $K = k_1 + \cdots + k_n$.

$$
\begin{array}{ccc}
\Lambda(n) & \xrightarrow{\ \ \delta\ \ } & \Lambda(k_1 + \cdots + k_n) \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\Sigma_n & \xrightarrow[\ \ \delta\ \ ]{} & \Sigma_{k_1 + \cdots + k_n}
\end{array}
$$

5. The function $\delta_{n;1,\ldots,1}\colon \Lambda(n) \to \Lambda(n)$ is the identity. The function $\delta_{1;n}\colon \Lambda(1) \to \Lambda(n)$ maps e_1 to e_n.

> I think something more general is true: any δ maps e_n to e_K

6. The equation $\delta_{n;k_1,\ldots,k_n}(g)\delta_{n;j_1,\ldots,j_n}(h) = \delta_{n;j_1,\ldots,j_n}(gh)$ holds when

$$k_i = j_{h^{-1}(i)}.$$

7. The functions δ are associative in the sense that the equation

$$\delta_{m_1+\cdots+m_n;\underline{p_1},\ldots,\underline{p_n}}\left(\delta_{n;m_1,\ldots,m_n}(g)\right) = \delta_{n;P_1,\ldots,P_n}(g)$$

holds, where $P_i = p_{i1} + \cdots + p_{im_i}$ and $\underline{p_i} = p_{i1}, \ldots, p_{im_i}$.

8. The equation

$$\delta_{n;k_1,\ldots,k_n}(g)\beta(h_1,\ldots,h_n) = \beta(h_{g^{-1}(1)},\ldots,h_{g^{-1}(n)})\delta_{n;k_{g^{-1}(1)},\ldots,k_{g^{-1}(n)}}(g)$$

holds, where $h_i \in \Lambda(k_i)$.

9. The equation

$$\beta(\delta_1(g_1),\ldots,\delta_n(g_n)) = \delta_c(\beta(g_1,\ldots,g_n))$$

holds, where $\delta_i(g_i)$ is shorthand for $\delta_{k_i;m_{i1},\ldots,m_{ik_i}}(g_i)$ and δ_c is shorthand for

$$\delta_{k_1+\cdots+k_n;m_{11},m_{12},\ldots,m_{1k_1},m_{21},\ldots,m_{nk_n}}.$$

Proof. Let Λ be an action operad, and define

$$
\begin{aligned}
\beta(g_1,\ldots,g_n) &= \mu(e_n; g_1,\ldots,g_n), \\
\delta_{n;k_1,\ldots,k_n}(g) &= \mu(g; e_{k_1},\ldots,e_{k_n}).
\end{aligned}
$$

Since $\pi\colon \Lambda \to \Sigma$ is an operad map, Axioms 1 and 4 hold by the definition of the operad structure on Σ in Example 3.6. Since Λ is an operad of sets, Axioms 2 and 5 follow from the operad unit axioms and the first part of Lemma 4.12, and Axioms 3, 7, and 9 follow from the operad associativity axiom and the second part of Lemma 4.12. Axioms 6 and 8 are special cases of the additional action operad axiom, as is the fact that β is a group homomorphism.

Conversely, given the data above, we need only define the operad multiplication, verify the operad unit and multiplication axioms, and finally check the action operad axiom. Multiplication is given by

$$\mu(g; h_1,\ldots,h_n) = \delta_{n;k_1,\ldots,k_n}(g)\beta(h_1,\ldots,h_n)$$

where $h_i \in \Lambda(k_i)$. The identity element id for the operad structure is $e_1 \in \Lambda(1)$.

We now verify the operad unit axioms. Let $g, h \in \Lambda(n)$. Then

$$\mu(e_1; g) = \delta(e_1)\beta(g)$$
$$= e_1 \cdot g$$
$$= g,$$
$$\mu(h; e_1, \ldots, e_1) = \delta_{n;1,\ldots,1}(h)\beta(e_1, \ldots, e_1)$$
$$= h \cdot e_n$$
$$= h$$

by Axioms 2 and 5, together with the fact that β is a group homomorphism. Thus e_1 satisfies the identity axioms for operadic multiplication.

For the operad associativity axiom, let

- $f \in \Lambda(m)$,
- $g_i \in \Lambda(n_i)$ for $i = 1, \ldots, m$, and
- $h_{ij} \in \Lambda(p_{i,j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$.

Further, let $P_i = p_{i1} + \cdots + p_{in_i}$ and $\underline{h_i}$ denote the list $h_{i1}, h_{i2}, \ldots, h_{in_i}$. We must then show that

$$\mu\big(f; \mu(g_1; \underline{h_1}), \ldots, \mu(g_m; \underline{h_m})\big) = \mu\big(\mu(f; g_1, \ldots, g_m); \underline{h_1}, \ldots, \underline{h_m}\big).$$

By definition, the left side of this equation is

$$\delta_{m;P_1,\ldots,P_m}(f)\beta\big(\mu(g_1; \underline{h_1}), \ldots, \mu(g_m; \underline{h_m})\big),$$

and

$$\mu\big(g_i; \underline{h_i}\big) = \delta_{n_i;p_{i1},\ldots,p_{in_i}}(g_i)\beta\big(h_{i1}, \ldots, h_{in_i}\big).$$

From this point, we suppress subscripts on the δ's. Since β is a group homomorphism, we can then rewrite the left side as

$$\delta(f)\beta\big(\delta(g_1), \ldots, \delta(g_m)\big)\beta\big(\beta(\underline{h_1}), \ldots, \beta(\underline{h_m})\big)$$

where we have suppressed the subscripts on the δ's. By Axiom 3,

$$\beta\big(\beta(\underline{h_1}), \ldots, \beta(\underline{h_m})\big) = \beta\big(\underline{h_1}, \ldots, \underline{h_m}\big).$$

Furthermore, Axiom 9 above shows that

$$\beta\big(\delta(g_1), \ldots, \delta(g_m)\big) = \delta\big(\beta(g_1, \ldots, g_m)\big).$$

Thus we have shown that the left side of the operad associativity axiom is equal to

$$\delta(f)\delta\big(\beta(g_1, \ldots, g_m)\big)\beta\big(\underline{h_1}, \ldots, \underline{h_m}\big).$$

Now the right side is

$$\mu\big(\mu(f; g_1, \ldots, g_m); \underline{h_1}, \ldots, \underline{h_m}\big),$$

which is by definition

$$\delta\big(\mu(f; g_1, \ldots, g_m)\big)\beta\big(\underline{h_1}, \ldots, \underline{h_m}\big).$$

Cancelling the $\beta\big(\underline{h_1}, \ldots, \underline{h_m}\big)$ terms, verifying the operad associativity axiom reduces to showing

$$\delta(f)\delta\big(\beta(g_1, \ldots, g_m)\big) = \delta\big(\mu(f; g_1, \ldots, g_m)\big). \tag{2}$$

By the definition of μ,

$$\delta\big(\mu(f; g_1, \ldots, g_m)\big) = \delta\big(\delta(f)\beta(g_1, \ldots, g_m)\big)$$

which is itself equal to

$$\delta\big(\delta(f)\big)\delta\big(\beta(g_1, \ldots, g_m)\big) \tag{3}$$

by Axiom 6 above. Now the $\delta(f)$ on the left side of Equation (2) uses $\delta_{n;P_1,\ldots,P_n}$, while the $\delta(\delta(f))$ in Equation (3) is actually

$$\delta_{m_1+\cdots+m_n;q_{ij}}(\delta_{n;m_1,\ldots,m_n}(f))$$

where the q_{ij} are defined, by Axiom 6, to be given by

$$q_{ij} = p_{i,g_i^{-1}(j)}$$

using the compatibility of β and π in Axiom 1. By Axiom 7, this composite of δ's is then $\delta_{n;Q_1,\ldots,Q_n}$ where $Q_i = q_{i1} + \cdots + q_{im_i}$. But by the definition of the q_{ij}, we immediately see that $Q_i = P_i$, so the $\delta(f)$ in Equation (2) is equal to the $\delta(\delta(f))$ appearing in Equation (3), concluding the proof of the operad associativity axiom.

Writing $\mu(g; \underline{h}) = \mu(g; h_1, \ldots, h_n) =$ and $\mu(g'; \underline{h'}) = \mu(g'; h_1', \ldots, h_n')$, the action operad axiom is now the calculation below, and uses Axioms 4 and 8.

$$\begin{aligned}
\mu(g; \underline{h})\mu(g'; \underline{h'}) &= \delta(g)\,\beta(h_1, \ldots, h_n)\,\delta(g')\,\beta(h_1', \ldots, h_n') \\
&= \delta(g)\,\delta(g')\,\beta(h_{g'(1)}, \ldots, h_{g'(n)})\,\beta(h_1', \ldots, h_n') \\
&= \delta(gg')\,\beta(h_{g'(1)}h_1', \ldots, h_{g'(n)}h_n') \\
&= \mu(gg'; h_{g'(1)}h_1', \ldots, h_{g'(n)}h_n')
\end{aligned}$$

$\square$

Proposition 4.15 (Corollary 2.17, [**zhang-grp**]). *Let Λ be an action operad. Then the homomorphisms $\pi_n \colon \Lambda(n) \to \Sigma_n$ are either all surjective or all the zero map.*

Proof. We will prove each case separately. The two cases coincide for $n = 0, 1$ as both Σ_0, Σ_1 are the trivial group and therefore any homomorphism with one of them as its codomain is both surjective and the zero map. Since Σ_2 only has one non-identity element, any homomorphism $G \to \Sigma_2$ must necessarily be surjective or the zero map.

Suppose that $\pi_2 \colon \Lambda(2) \to \Sigma_2$ is surjective, so there exists $g \in \Lambda(2)$ such that $\pi_2(g) = (1\ 2)$. Let $n > 2$. Since Σ_n is generated by the adjacent transpositions $(a\ a+1)$, we will show that each such element is in the image of π_n. Write $\underline{x}^i$ for the i-tuple $x, x, \ldots, x$. Then $(a\ a+1) = \beta(\underline{e_1}^{a-1}, (1\ 2), \underline{e_1}^{n-a-1})$ in Σ, so

$$\begin{aligned}
(a\ a+1) &= \beta(\underline{e_1}^{a-1}, (1\ 2), \underline{e_1}^{n-a-1}) \\
&= \beta(\underline{\pi_1(e_1)}^{a-1}, \pi_2(g), \underline{\pi_1(e_1)}^{n-a-1}) \\
&= \pi_n(\beta(\underline{e_1}^{a-1}, g, \underline{e_1}^{n-a-1}))
\end{aligned}$$

by Axiom 1 of 4.14. Thus π_n is surjective for all $n > 2$ if π_2 is surjective.

Now we will consider the case where π_2 is the zero map. Suppose that there exists $g \in \Lambda(n)$ such that $\pi_n(g) = \sigma \neq e_n$ in Σ_n. Then we can find $1 \leq i < j \leq n$ such that $\sigma(j) < \sigma(i)$. Consider the element

$$h = \delta_{n; \underline{0}^{i-1}, 1, \underline{0}^{j-i-1}, 1, \underline{0}^{n-j}}(g) \in \Lambda(2).$$

By the assumption that π_2 is the zero map, we must have that $\pi_2(h) = e_2$, but by Axiom 4 of 4.14 we also compute

$$\pi_2(h) = \delta_{n; \underline{0}^{i-1}, 1, \underline{0}^{j-i-1}, 1, \underline{0}^{n-j}}\big(\pi_n(g)\big) = \delta_{n; \underline{0}^{i-1}, 1, \underline{0}^{j-i-1}, 1, \underline{0}^{n-j}}(\sigma).$$

The element $\delta_{n;\underline{0}^{i-1},1,\underline{0}^{j-i-1},1,\underline{0}^{n-j}}(\sigma)$ is equal to (1 2) by the choice of i, j and Definition 2.13. These two computations of $\pi_2(h)$ are in contradiction, so there must be no such $g \in \Lambda(n)$. Thus if π_2 is the zero map, so is π_n for all $n > 2$.

$\square$

5 Examples

In this section, we expand our collection of examples and non-examples of action operads. In all but one case, Example 5.4, the examples we provide have appeared elsewhere. The non-examples we provide were largely sourced from questions received after preliminary talks on this research by the authors.

Example 5.1 (Action operad of braid groups). One can form an operad B where $B(n)$ is the underlying set of the nth braid group, B_n. We define the operad structure using the functions β, δ from Remark 3.14. Yau checks that these groups and functions satisfy the axioms of an action operad in [**yau'infinity'2021**], but we note that each of the nine axioms in Theorem 4.14 follows immediately by using the geometric definitions of β, δ.

Definition 5.2. For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, \ldots, t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$b_i \cdot t_i = t_{i+1} \cdot b_i,$$
$$b_i \cdot t_{i+1} = t_i \cdot b_i.$$

The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in **??**, but with some additional rules for dealing with twists. With regards to the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i = e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} = e_n, \qquad t_{(m+m')} = \left(t_{(m)} \otimes t_{(m')}\right) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Much as the symmetric groups can be represented by crossings of a collection of strings, and the braid groups by braidings of strings, the ribbon braid groups deal with the ways that one can braid together several flat ribbons, including the ability to twist a ribbon about its own axis by 360 degrees. The actual definition of the ribbon braid groups is as the fundamental group of a configuration space in which points have labels in the circle, S^1; see [**sal-wahl**].

b $\qquad$ t
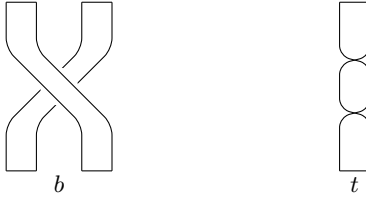
Example 5.3 (Action operad of cactus groups). The operad of n-fruit cactus groups defined by Henriques and Kamnitzer in [**hk-cobound**] has an action operad structure that we will discuss in Section 18.

Example 5.4 (Action operad from an abelian group). Every abelian group A gives rise to an action operad $A^\bullet$ as follows. The group $A^\bullet(n)$ is the direct sum of n copies of A, A^n. The identity element is required to be $e \in A^1$, and the multiplication is defined by

$$\mu((a_1, \ldots, a_n); \underline{b_1}, \ldots, \underline{b_n}) = (a_1 + \underline{b_1}, a_2 + \underline{b_2}, \ldots, a_n + \underline{b_n})$$

where $\underline{b_i}$ is the string $b_{i1}, \ldots, b_{ik_i}$, and $a_i + \underline{b_i}$ is

$$a_i + b_{i1}, a_i + b_{i2}, \ldots, a_i + b_{ik_i}.$$

The map $\pi_n \colon A^\bullet(n) \to \Sigma_n$ is the zero map.

The characterisation of action operads in terms of maps π, β, and δ as in the above Theorem 4.14 allows us to more easily check for counterexamples, as we show in the latter examples below. Some of these, such as the cyclic groups, reflexive groups, and hyperoctahedral groups, do however form crossed simplicial groups as described in Remark 5.7.

Example 5.5 (Non-examples: subgroups of symmetric groups). By Proposition 4.15, the only action operad $\pi \colon \Lambda \to \Sigma$ for which the homomorphisms π_n are injective but not surjective is the action operad of trivial groups. Thus there is no family of proper, nontrivial subgroups of the symmetric groups that admits an action operad structure. In particular, the families of cyclic groups $\{C_n\}$, reflexive groups $\Lambda(n) = C_2$ of [**Kra87**], and alternating groups $\{A_n\}$ do not admit action operad structures.

Example 5.6 (Non-example: hyperoctahedral groups). Copied:

n Example 2.28 of [**zhang-grp**], Zhang describes one way in which the sequence of hyperoctahedral groups $H_n = C_2 \wr \Sigma_n$ do not form an action operad. We clarify that counterexample here, while also describing another. Elements of H_n can be thought of as signed permutations or, equivalently, as $n \times n$ invertible matrices whose entries consist of -1, 0, or 1 and in which each row and column has exactly one non-zero entry. E.g., we can consider an element of H_3 as a permutation matrix and a 3-tuple of elements of $C_2 = \{-1, 1\}$, or simply as a signed permutation matrix:

$$((2\ 3); -1, 1, -1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; -1, 1, -1 \right) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in H_3.$$

In order to describe the hyperoctahedral groups as an action operad, we could use Theorem 4.14 and define maps π, β, and δ. The obvious map $\pi_n \colon H_n \to \Sigma_n$ takes the 'absolute value' of a signed permutation matrix, giving back simply the underlying permutation. It seems obvious then to define β to be the block sum of signed permutation matrices, in much the same way as for the symmetric groups. E.g.,

$$\beta \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

For the maps δ, there seems to be two sensible options to try. The first captures Zhang's counterexample by first taking r_n to be the order-reversing signed permutation matrix where all entries are -1, i.e., the $n \times n$ matrix with -1 in every entry of the anti-diagonal. Then we can define $\delta_{n;k_1,\ldots,k_n}(\sigma)$ to be the block sum $\beta(-r_{k_1},\ldots,-r_{k_n})$ acted on by the product of r_n and σ. For example,

$$\delta_{3;2,1,3}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) = \left(r_3 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) * \begin{bmatrix} -r_2 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & 0 & -r_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} -r_2 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & 0 & -r_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r_1 & 0 \\ 0 & 0 & -r_3 \\ -r_2 & 0 & 0 \end{bmatrix}.$$

In particular, this gives $\delta_{1;n}([-1]) = r_n$ as in [**zhang-grp**]. Taking $\sigma = ((2\ 3); -1, 1, -1)$, as above, we can show that Axiom 8 fails in Theorem 4.14. The left-hand side of the axiom would be

$$\delta_{1;3}([-1])\beta(\sigma) = r_n \cdot \sigma = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

However, the right-hand side of the axiom would be

$$\beta(\sigma)\delta_{1;3}([-1]) = \sigma \cdot r_n = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Clearly defining $\delta_{n;k_1,\ldots,k_n}(\sigma) = (\sigma \cdot r_n) * \beta(-r_{k_1},\ldots,-r_{k_n})$ would run into the same problem.

An alternative way of defining δ is to take $\delta_{n;k_1,\ldots,k_n}(\sigma) = \sigma * \beta(e_{k_1},\ldots,e_{k_n})$, without involving the order-reversing permutation r_n, having the effect of making $\delta_{1;n}([-1]) = -I_n$. This then does satisfy Axiom 8, but fails Axiom 6 instead; working through the following counterexample shows this to be the case:

$$\delta_{3;2,1,3}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right)\delta_{3;3,1,2}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right)$$

$$\neq \delta_{3;3,1,2}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right).$$

Remark 5.7. The crossed simplicial groups of Krasauskas [**Kra87**] and Fiedoriwicz and Loday [**FL91**] are related to action operads in the following way. On objects, we define a functor $C\colon \mathbf{AOp} \to \mathbf{CSGrp}$ from the category of action operads to the category of crossed simplicial groups by defining $C(\Lambda)(n) = \Lambda(n+1)$. The face and degeneracy maps of the underlying simplicial structure are defined using the operadic composition of Λ as in [**Kra96**] or [**ber-simplicial**]. On morphisms, something. This functor is neither faithful nor conservative.

is it just $C(f)(n) = f_{n+1}$? That seems conservative to me

6 Action Operads as Extensions

Proposition 6.1. *There exists a functor $Z\colon \mathbf{Op}(\mathbf{Grp}) \to \mathbf{AOp}$ from the category of operads in groups (using the cartesian monoidal structure) to the category of action operads. This functor is full, faithful, and its image is precisely the collection of action operads Λ with $\pi_n(g) = e_n$ for all $n \in \mathbb{N}$, $g \in \Lambda(n)$.*

Proof. Given an operad in groups P, $Z(P)$ is the action operad with the same underlying operad and each π_n the zero map. It is easy to verify that μ is a group homomorphism if and only if π_n is zero for all n, and further that maps between such action operads are precisely the same thing as maps between the corresponding operads in the category of groups. $\qquad\square$

We have the following corollary.

Corollary 6.2. *For an action operad Λ, the sets*

$$\operatorname{Ker} \pi_n = \{g \in \Lambda(n) \; : \; \pi_n(g) = e_n\}$$

form an action operad for which the inclusion $\operatorname{Ker} \pi \hookrightarrow \Lambda$ is a map of action operads.

Proof. For $\operatorname{Ker} \pi \hookrightarrow \Lambda$ to be a map of action operads, we must define the map $\operatorname{Ker} \pi \to \Sigma$ to be zero, and we must check that the operadic multiplication of elements in the kernel is also in the kernel. This last fact is a trivial consequence of π being an operad map. $\qquad\square$

Corollary 6.3. *For an action operad Λ, the sets*

$$\operatorname{Im} \pi_n = \{\pi_n(g) \; : \; g \in \Lambda(n)\}$$

form an action operad for which the inclusion $\operatorname{Im} \pi \hookrightarrow \Sigma$ is a map of action operads.

Proof. The operad multiplication of elements in the image is also in the image, as is the unit element in $\operatorname{Im} \pi_1$, following again as a consequence of π being an operad map. The map $\operatorname{Im} \pi \hookrightarrow \Sigma$ is an inclusion and so is immediately seen to be a map of action operads. $\qquad\square$

Corollary 6.4. *Every action operad Λ with nontrivial π fits into a short exact sequence*

$$T \to \operatorname{Ker} \pi \hookrightarrow \Lambda \xrightarrow{\ \pi\ } \Sigma \to T,$$

by which we mean this is a sequence of action operad maps which is an exact sequence of groups at each n.

Remark 6.5. Thus we see that an action operad is either an operad in groups, or is an extension of Σ by an operad in groups. This gives a simple proof that the operads of pure braids and pure ribbon braids (those braids or ribbon braids whose underlying permutations are the identity) are both operads in groups.

7 Presentations for Action Operads

A useful method for constructing new examples of some given algebraic structure is through the use of presentations. A presentation consists of generating data together with relations between generators using the operations of the algebra involved. In categorical terms, the generators and relations are both given as free gadgets on some underlying data, and the presentation itself is a coequalizer. This section will establish the categorical structure necessary to give presentations for action operads, and then explain how such a presentation is reflected in the associated club and 2-monad. The most direct route to the desired results uses the theory of locally finitely presentable categories. We recall the main definitions briefly, but refer the reader to [**ar**] for additional details.

Definition 7.1. A *filtered category* is a nonempty category C such that

- if a, b are objects of C, then there exists another object $c \in C$ and morphisms $a \to c, b \to c$; and

- if $f, g \colon a \to b$ are parallel morphisms in C, then there exists a morphism $h \colon b \to c$ such that $hf = hg$.

Definition 7.2. A *filtered colimit* is a colimit over a filtered category.

Definition 7.3. Let C be a category with all filtered colimits. An object $x \in C$ is *finitely presentable* if the representable functor $C(x, -) \colon C \to \mathbf{Sets}$ preserves filtered colimits.

Definition 7.4. A *locally finitely presentable category* is a category C such that

- C is cocomplete and

- there exists a small subcategory $C_{fp} \subseteq C$ of finitely presentable objects such that any object $x \in C$ is the filtered colimit of some diagram in C_{fp}.

The definition of a locally finitely presentable category has many equivalent variants, but we find this one most practicable to work with in this setting.

Theorem 7.5. *The category* $\mathbf{AOp}$ *is locally finitely presentable.*

Proof. First note that we can define a category $\mathbf{Op}^g$, whose objects are operads P in which each $P(n)$ also carries a group structure. This is an equational theory using equations with only finitely many elements, so $\mathbf{Op}^g$ is locally finitely presentable [**ar**]. A slice category of a locally finitely presentable category is itself locally finite presentable [**ar**] and since the symmetric operad is an object of $\mathbf{Op}^g$, the slice category $\mathbf{Op}^g/\Sigma$ is locally finitely presentable.

There is an obvious inclusion functor $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$. Now $\mathbf{AOp}$ is a full subcategory of $\mathbf{Op}^g/\Sigma$ which is closed under products, subobjects. Since any object of $\mathbf{Op}^g/\Sigma$ isomorphic to an action operad is in fact an action operad, the inclusion $\mathbf{AOp} \hookrightarrow \mathbf{Op}^g/\Sigma$ is actually the inclusion of a reflective subcategory. One can easily check that $\mathbf{AOp}$ is in fact closed under all limits and filtered colimits in $\mathbf{Op}^g/\Sigma$, so by the Reflection Theorem (2.48 in [**ar**]), $\mathbf{AOp}$ is locally finitely presentable. $\square$

Definition 7.6. Let $\mathcal{S}$ be the set which is the disjoint union of the underlying sets of all the symmetric groups. Then $\mathbf{Sets}/\mathcal{S}$ is the slice category over $\mathcal{S}$ with objects (X, f) where X is a set and $f \colon X \to \mathcal{S}$ and morphisms $(X_1, f_1) \to (X_2, f_2)$ are those functions $g \colon X_1 \to X_2$ such that $f_1 = f_2 g$. We call an object (X, f) a *collection over* $\mathcal{S}$.

Remark 7.7. In standard presentations of the theory of operads (see, for example, [**mss-op**]), a nonsymmetric operad will have an underlying collection (or $\mathbb{N}$-indexed collection of sets) while a symmetric operad will have an underlying symmetric collection (or $\mathbb{N}$-indexed collection of sets in which the nth set has an action of Σ_n). Our collections over $\mathbb{S}$ more closely resemble the former as there is no group action present.

Example 7.8. One can easily form new action operads from old ones by taking limits. To take a limit of a diagram in **AOp**, one forgets down to the category of operads over Σ and takes the limit there. Concretely, products in **AOp** are computed as products in **Op**/Σ which themselves are (possibly wide) pullbacks in the category of operads. This pullback will then be computed levelwise, showing that at each dimension there is a group structure with a group homomorphism to the appropriate Σ_n and that the final action operad axiom holds since it does in each component. The equalizer of a pair of maps will then just be the levelwise equalizer. This shows that the pointwise product of an action operad P with an action operad of the form $Z(Q)$ (as in Proposition 6.1) is again an action operad, but the pointwise product of two arbitrary action operads might not be.

Theorem 7.9. *There exists a forgetful functor* $U\colon \mathbf{AOp} \to \mathbf{Sets}/\mathbb{S}$ *which preserves all limits and filtered colimits.*

Proof. For a given action operad Λ, we put $U(\Lambda) = \left(\coprod_{\mathbb{N}} \Lambda(n), \coprod_{\mathbb{N}} \pi_n\right)$ and this easily extends to a mapping on morphisms using the universal property of the coproduct. The preservation of filtered colimits follows from the fact that these are computed pointwise, together with the fact that every map between action operads preserves underlying permutations. As equalizers are computed levelwise, and the product $\Lambda \times \Lambda'$ has underlying operad the pullback $\Lambda \times_\Sigma \Lambda'$; this pullback is itself computed levelwise. Together, these imply that U also preserves all limits. $\square$

Corollary 7.10. U *has a left adjoint* $F\colon \mathbf{Sets}/\mathbb{S} \to \mathbf{AOp}$, *the free action operad functor.*

Proof. The category $\mathbf{Sets}/\mathbb{S}$ is locally finitely presentable as it is equivalent to the functor category $[\mathbb{S}, \mathbf{Sets}]$ (here $\mathbb{S}$ is treated as a discrete category) and any presheaf category is locally finitely presentable. The functor U preserves limits and filtered colimits between locally finitely presentable categories, so has a left adjoint (see Theorem 1.66 in [**ar**]). $\square$

Definition 7.11. A *presentation* for an action operad Λ consists of

- a pair of collections over $\mathbb{S}$ denoted $\mathbf{g}, \mathbf{r}$,
- a pair of maps $s_1, s_2\colon F\mathbf{r} \to F\mathbf{g}$ between the associated free action operads, and
- a map $p\colon F\mathbf{g} \to \Lambda$ of action operads exhibiting Λ as the coequalizer of s_1, s_2.

8 Action Operads as Clubs

Kelly's theory of clubs [**kelly˙club1**, **kelly˙club0**, **kelly˙club2**] was designed to simplify and explain certain aspects of coherence results, namely the fact that many coherence results rely on extrapolating information about general free objects for a 2-monad T from information about the specific free object $T1$ where 1 denotes the terminal category. This occurs, for example, in the study of the many different flavors of monoidal category: plain monoidal category, braided monoidal category, symmetric monoidal category, and so on. This section will explain how every action operad gives rise to a club, as well as compute the clubs which arise as the image of this procedure.

We begin by reminding the reader of the notion of a club, or more specifically what Kelly [**kelly˙club1**, **kelly˙club2**] calls a club over **P**. We will only be interested in clubs over **P**, and thusly shorten the terminology to club from this point onward. Defining clubs is accomplished most succinctly using Leinster's terminology of generalized operads [**leinster**].

Definition 8.1. Let C be a category with finite limits.

1. A monad $T\colon C \to C$ is *cartesian* if the functor T preserves pullbacks, and the naturality squares for the unit η and the multiplication μ for T are all pullbacks.

2. The category of T-*collections*, T-**Coll**, is the slice category $C/T1$, where 1 denotes the terminal object.

3. Given a pair of T-collections $X \xrightarrow{x} T1, Y \xrightarrow{y} T1$, their *composition product* $X \circ Y$ is given by the pullback below together with the morphism along the top.

$$
\begin{array}{ccccccc}
X \circ Y & \longrightarrow & TY & \xrightarrow{Ty} & T^2 1 & \xrightarrow{\mu} & T1 \\
\downarrow & \lrcorner & \downarrow{\scriptstyle T!} & & & & \\
X & \xrightarrow{\quad x \quad} & T1 & & & &
\end{array}
$$

4. The composition product, along with the unit of the adjunction $\eta\colon 1 \to T1$, give T-**Coll** a monoidal structure. A T-*operad* is a monoid in T-**Coll**.

Remark 8.2. Everything in the above definition can be **Cat**-enriched without any substantial modifications. Thus we require our ground 2-category to have finite limits in the enriched sense, and the slice and pullbacks are the 2-categorical (and not bicategorical) versions. If we take this 2-category to be **Cat**, then in each case the underlying category of the 2-categorical construction is given by the corresponding 1-categorical version. From this point, we will not distinguish between the 1-dimensional and 2-dimensional theory. Our interest, of course, is in the 2-dimensional version.

Let Σ be the operad of symmetric groups. This is the terminal object of the category of action operads, with each π_n the identity map. Then $\underline{E\Sigma}$ is a 2-monad on **Cat**, and by results in **??** it is cartesian.

Definition 8.3. A *club* is a T-operad in **Cat** for $T = \underline{E\Sigma}$.

Remark 8.4. The category **P** in Kelly's terminology is the result of applying $\underline{E\Sigma}$ to 1, and can be identified with $B\Sigma = \coprod B\Sigma_n$.

It is useful to break down the definition of a club. A club consists of

1. a category K together with a functor $K \to B\Sigma$,

2. a multiplication map $K \circ K \to K$, and

3. a unit map $1 \to K$

satisfying the axioms to be a monoid in the monoidal category of $E\Sigma$-collections. By the definition of $K \circ K$ as a pullback, objects are tuples of objects of K $(x; y_1, \ldots, y_n)$ where $\pi(x) = n$. A morphism

$$(x; y_1, \ldots, y_n) \to (z; w_1, \ldots, w_m)$$

exists only when $n = m$ (since $B\Sigma$ only has endomorphisms) and then consists of a morphism $f\colon x \to z$ in K together with morphisms $g_i\colon y_i \to z_{\pi(x)(i)}$ in K.

Notation 8.5. For a club K and a morphism $(f; g_1, \ldots, g_n)$ in $K \circ K$, we write $f(g_1, \ldots, g_n)$ for the image of the morphism under the functor $K \circ K \to K$.

We will usually just refer to a club by its underlying category K.

Theorem 8.6. *Let Λ be an action operad. Then the map of operads $\pi \colon \Lambda \to \Sigma$ gives the category $B\Lambda = \coprod B\Lambda(n)$ the structure of a club.*

Proof. To give the functor $B\pi \colon B\Lambda \to B\Sigma$ the structure of a club it suffices (see [**leinster**]) to show that

- the induced monad, which we will show to be $\underline{E\Lambda}$, is a cartesian monad on **Cat**,
- the transformation $\tilde{\pi} \colon \underline{E\Lambda} \Rightarrow \underline{E\Sigma}$ induced by the functor $E\pi$ is cartesian, and
- $\tilde{\pi}$ commutes with the monad structures.

The monad $\underline{E\Lambda}$ is always cartesian by results of **??**. The transformation $\tilde{\pi}$ is the coproduct of the maps $\tilde{\pi}_n$ which are induced by the universal property of the coequalizer as shown below.

$$
\begin{array}{ccccc}
E\Lambda(n) \times \Lambda(n) \times X^n & \rightrightarrows & E\Lambda(n) \times X^n & \longrightarrow & E\Lambda(n) \times_{\Lambda(n)} X^n \\
\downarrow{\scriptstyle E\pi \times \pi \times 1} & & \downarrow{\scriptstyle E\pi \times 1} & & \downarrow{\scriptstyle \tilde{\pi}_n} \\
E\Sigma_n \times \Sigma_n \times X^n & \rightrightarrows & E\Sigma_n \times X^n & \longrightarrow & E\Sigma_n \times_{\Sigma_n} X^n
\end{array}
$$

Naturality is immediate, and since π is a map of operads $\tilde{\pi}$ also commutes with the monad structures.

It only remains to show that $\tilde{\pi}$ is cartesian and that the induced monad is actually $\underline{E\Lambda}$. Since the monads $\underline{E\Lambda}$ and $\underline{E\Sigma}$ both decompose into a disjoint union of functors, we only have to show that, for any n, the square below is a pullback.

$$
\begin{array}{ccc}
E\Lambda(n) \otimes_{\Lambda(n)} X^n & \longrightarrow & E\Sigma_n \times_{\Sigma_n} X^n \\
\downarrow & & \downarrow \\
B\Lambda(n) & \longrightarrow & B\Sigma_n
\end{array}
$$

By Lemma 16.2 , this amounts to showing that the square below is a pullback.

$$
\begin{array}{ccc}
(E\Lambda(n) \times X^n)/\Lambda(n) & \longrightarrow & (E\Sigma_n \times X^n)/\Sigma_n \\
\downarrow & & \downarrow \\
B\Lambda(n) & \longrightarrow & B\Sigma_n
\end{array}
$$

Here, $(A \times B)/G$ is the category whose objects are equivalence classes of pairs (a, b) where $(a, b) \sim (ag, g^{-1}b)$, and similarly for morphisms. Now the bottom map is clearly bijective on objects since these categories only have one object. An object in the top right is an equivalence class

$$
[\sigma; x_1, \ldots, x_n] = \left[e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)} \right].
$$

A similar description holds for objects in the top left, with $g \in \Lambda(n)$ replacing σ and $\pi(g)^{-1}$ replacing σ^{-1} in the subscripts. The map along the top sends $[g; x_1, \ldots, x_n]$ to $[\pi(g); x_1, \ldots, x_n]$, and thus sends $[e; x_1, \ldots, x_n]$ to $[e; x_1, \ldots, x_n]$, giving a bijection on objects.

Now a morphism in $(E\Lambda(n) \times X^n)/\Lambda(n)$ can be given as

$$
[e; x_1, \ldots, x_n] \xrightarrow{[!; f_i]} [g; y_1, \ldots, y_n].
$$

Mapping down to $B\Lambda(n)$ gives $ge^{-1} = g$, while mapping over to $(E\Sigma_n \times X^n)/\Sigma_n$ gives $[!; f_i]$ where $!: e \to \pi(g)$ is now a morphism in $E\Sigma_n$. In other words, a morphism in the upper left corner of our putative pullback square is determined completely by its images along the top and lefthand functors. Furthermore, given $g \in \Lambda(n)$, $\tau = \pi(g)$, and morphisms $f_i: x_i \to y_i$ in X, the morphism $[!: e \to g; f_i]$ maps to the pair $(g, [!: e \to \tau; f_i])$, completing the proof that this square is indeed a pullback. $\qquad\square$

The club, which we now denote K_Λ, associated to $E\Lambda$ has the following properties. First, the functor $K_\Lambda \to B\Sigma$ is a functor between groupoids. Second, the functor $K_\Lambda \to B\Sigma$ is bijective-on-objects. We claim that these properties characterize those clubs which arise from action operads. Thus the clubs arising from action operads are very similar to PROPs [**mac˙prop**, **markl˙prop**].

Theorem 8.7. *Let K be a club such that*

- *the map $K \to B\Sigma$ is bijective on objects and*

- *K is a groupoid.*

*Then $K \cong K_\Lambda$ for some action operad Λ. The assignment $\Lambda \mapsto K_\Lambda$ is a full and faithful embedding of the category of action operads **AOp** into the category of clubs.*

Proof. Let K be such a club. Our hypotheses immediately imply that K is a groupoid with objects in bijection with the natural numbers; we will now assume the functor $K \to B\Sigma$ is the identity on objects. Let $\Lambda(n) = K(n, n)$. Now K comes equipped with a functor to $B\Sigma$, in other words group homomorphisms $\pi_n: \Lambda(n) \to \Sigma_n$. We claim that the club structure on K makes the collection of groups $\{\Lambda(n)\}$ an action operad. In order to do so, we will employ Theorem 4.14.

First, we give the group homomorphism β using Notation 8.5. Define

$$\beta(g_1, \ldots, g_n) = e_n(g_1, \ldots, g_n)$$

(see 8.5) where e_n is the identity morphism $n \to n$ in $K(n, n)$. Functoriality of the club multiplication map immediately implies that this is a group homomorphism. Second, we define the function δ in a similar fashion:

$$\delta_{n; k_1, \ldots, k_n}(f) = f(e_1, \ldots, e_n),$$

where here e_i is the identity morphism of k_i in K.

There are now nine axioms to verify in Theorem 4.14. The club multiplication functor is a map of collections, so a map over $B\Sigma$; this fact immediately implies that Axioms (1) (using morphisms in $K \circ K$ with only g_i parts) and (4) (using morphisms in $K \circ K$ with only f parts) hold. The mere fact that multiplication is a functor also implies Axioms (6) (once again using morphisms with only f parts) and (8) (by considering the composite of a morphism with only an f with a morphism with only g_i's). Axiom (2) is the equation $e_1(g) = g$ which is a direct consequence of the unit axiom for the club K; the same is true of Axiom (5). Axioms (3), (7), and (9) all follow from the associativity of the club multiplication.

Finally, we would like to show that this gives a full and faithful embedding

$$K_-: \mathbf{AOp} \to \mathbf{Club}$$

of the category of action operads into the category of clubs. Let $f, f': \Lambda \to \Lambda'$ be maps between action operads. Then if $K_f = K_{f'}$ as maps between clubs, then they must be equal as functors $K_\Lambda \to K_{\Lambda'}$. But these functors are nothing more than the coproducts of the functors

$$B(f_n), B(f'_n): B\Lambda(n) \to B\Lambda'(n),$$

29

and the functor B from groups to categories is faithful, so K_- is also faithful. Now let $f \colon K_\Lambda \to K_{\Lambda'}$ be a maps of clubs. We clearly get group homomorphisms $f_n \colon \Lambda(n) \to \Lambda'(n)$ such that $\pi_n^\Lambda = \pi_n^{\Lambda'} f_n$, so we must only show that the f_n also constitute an operad map. Using the description of the club structure above in terms of the maps β, δ, we are able to see that commuting with the club multiplication implies commuting with both of these, which in turn is equivalent to commuting with operad multiplication. Thus K_- is full as well. $\qquad\square$

Remark 8.8. First, one should note that being a club over $B\Sigma$ means that every K-algebra has an underlying strict monoidal structure. Second, requiring that $K \to B\Sigma$ be bijective on objects ensures that K does not have operations other than $\otimes$, such as duals or internal hom-objects, from which to build new types of objects. Finally, K being a groupoid ensures that all of the "constraint morphisms" that exist in algebras for K are invertible.

These hypotheses could be relaxed somewhat. Instead of having a club over $B\Sigma$, we could have a club over the free symmetric monoidal category on one object (note that the free symmetric monoidal category monad on **Cat** is still cartesian). This would produce K-algebras with underlying monoidal structures which are not necessarily strict. This change should have relatively little impact on how the theory is developed. Changing K to be a category instead of a groupoid would likely have a larger impact, as the resulting action operads would have monoids instead of groups at each level. We have made repeated use of inverses throughout the proofs in the basic theory of action operads, and these would have to be revisited if groups were replaced by monoids in the definition of action operads.

In [**kelly˙club1**], Kelly discusses clubs given by generators and relations. His generators include functorial operations more general than what we are interested in here, and the natural transformations are not required to be invertible. In our case, the only generating operations we require are those of a unit and tensor product, as the algebras for $E\Lambda$ are always strict monoidal categories with additional structure. Tracing through his discussion of generators and relations for a club gives the following theorem.

Theorem 8.9. *Let Λ be an action operad with presentation given by $(\mathbf{g}, \mathbf{r}, s_i, p)$. Then the club $E\Lambda$ is generated by*

- *functors giving the unit object and tensor product, and*
- *natural transformations given by the collection $\mathbf{g}$: each element x of $\mathbf{g}$ with $\pi(x) = \sigma_x \in \Sigma_{|x|}$ gives a natural transformation from the nth tensor power functor to itself,*

subject to relations such that the following axioms hold.

- *The monoidal structure given by the unit and tensor product is strict.*
- *The transformations given by the elements of $\mathbf{g}$ are all natural isomorphisms.*
- *For each element $y \in \mathbf{r}$, the equation $s_1(y) = s_2(y)$ holds.*

Bringing this down to a concrete level we have the following corollary.

Corollary 8.10. *Assume we have a notion $\mathcal{M}$ of strict monoidal category which is given by a set natural isomorphisms*

$$\mathcal{G} = \left\{ (f, \pi_f) \mid x_1 \otimes \cdots \otimes x_n \xrightarrow{\ f\ } x_{\pi_f^{-1}(1)} \otimes \cdots \otimes x_{\pi_f^{-1}(n)} \right\}$$

subject to a set $\mathcal{R}$ of axioms. Each such axiom is given by the data

- *two finite sets $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_m$ of elements of $\mathcal{G}$; and*

- *two formal composites F, F' using only composition and tensor product operations and the f_i, respectively f_i',*

such that the underlying permutation of F equals the underlying permutation of F' (we compute the underlying permutations using the functions β, δ of Theorem 4.14). The element $\left(\underline{f}, \underline{f}', F, F'\right)$ of the set $\mathcal{R}$ of axioms corresponds to the requirement that the composite of the morphisms f_i using F equals the composite of the morphisms f_j' using F' in any strict monoidal category of type $\mathcal{M}$. Then strict monoidal categories of type $\mathcal{M}$ are given as the algebras for the club $E\Lambda$ where Λ is the action operad with

- $\mathbf{g} = \mathcal{G}$,
- $\mathbf{r} = \mathcal{R}$,
- s_1 *given by mapping the generator $\left(\underline{f}, \underline{f}', F, F'\right)$ to the operadic composition of the f_i using F via β, δ, and*
- s_2 *given by mapping the generator $\left(\underline{f}, \underline{f}', F, F'\right)$ to the operadic composition of the f_i' using F' via β, δ.*

Example 8.11. The 2-monad for symmetric strict monoidal categories (or permutative categories, as they are known in the topological literature) is given by $E\Sigma$, so the notion of symmetric strict monoidal categories corresponds to the symmetric operad. While this example is well-known, we go into further detail to set the stage for less common examples.

The 2-monad $\underline{E\Sigma}$ on **Cat** is given by

$$\underline{E\Sigma}(X) = \coprod E\Sigma_n \times_{\Sigma_n} X^n.$$

An object of $E\Sigma_n \times_{\Sigma_n} X^n$ is an equivalence class of the form $[\sigma; x_1, \ldots, x_n]$ where $\sigma \in \Sigma_n$ and $x_i \in X$. The equivalence relation gives

$$[\sigma; x_1, \ldots, x_n] = \left[e; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right],$$

so objects can be identified with finite strings of objects of X. Morphisms are given by equivalence classes of the form

$$[\sigma; x_1, \ldots, x_n] \overset{[!; f_1, \ldots, f_n]}{\longrightarrow} [\tau; y_1, \ldots, y_n].$$

Here $!\colon \sigma \cong \tau$ is the unique isomorphism in $E\Sigma_n$, and $f_i \colon x_i \to y_i$ in X. Using the equivalence relation, we find that morphisms between finite strings

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

are given by a permutation $\rho \in \Sigma_n$ together with maps $f_i \colon x_i \to y_{\rho(i)}$ in X (note that there are no morphisms between strings of different length); this is a special case of the calculation in Lemma 10.13. Thus $E\Sigma(X)$ is easily seen to be the free permutative category generated by X, and therefore Σ-monoidal categories are permutative categories.

Example 8.12. The template above can be used to show that the braid operad B corresponds to the 2-monad for braided strict monoidal categories. The details are almost exactly the same, only we use braids instead of permutations. The equivalence relation on objects gives

$$[\gamma; x_1, \ldots, x_n] = \left[e; x_{\pi(\gamma)^{-1}(1)}, \ldots, x_{\pi(\gamma)^{-1}(n)}\right],$$

where $\gamma \in B_n$ and $\pi(\gamma)$ is its underlying permutation; thus objects of $EB(X)$ are once again finite strings of objects of X. A morphism

$$x_1, \ldots, x_n \to y_1, \ldots, y_n$$

is then given by a braid $\gamma \in B_n$ together with maps $f_i \colon x_i \to y_{\pi(\gamma)(i)}$ in X. Thus one should view a morphism as given by

- a finite ordered set $x_1, \ldots, x_n$ of objects of X as the source,
- another such finite ordered set (of the same cardinality) $y_1, \ldots, y_n$ of objects of X as the target,
- a geometric braid $\gamma \in B_n$ on n strands, and
- for each strand, a morphism in X from the object labeling the source of that strand to the object labeling the target.

This is precisely Joyal and Street's [**js**] construction of the free braided strict monoidal category generated by a category X, and thus B-monoidal categories are braided strict monoidal categories.

This example can be extended to include ribbon braided categories as well. A *ribbon braid* is given, geometrically, in much the same way as a braid except that instead of paths $[0,1] \to \mathbb{R}^3$ making up each individual strand, we use ribbons $[0,1] \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$. This introduces the possibility of performing a full twist on a ribbon, and one can describe ribbon braided categories using generators and relations by introducing a natural twist isomorphism $\tau_A \colon A \to A$ and imposing one relation between the twist and the braid $\gamma_{A,B} \colon A \otimes B \to B \otimes A$. In [**sal-wahl**], the authors show that the ribbon braid groups give an action operad RB, and that (strict) ribbon braided categories are precisely the algebras for ERB.

Part II: Operads with Equivariance

9 Λ-Operads and their Algebras

Just as we had the definitions of operad, symmetric operad, and braided operad, we now come to the general definition of a Λ-operad, where Λ is an action operad. Thereafter we begin to delve into the theory around Λ-operads. Beginning by recasting familiar operads as Λ-operads for particular choices of Λ, we then proceed to shadow standard operadic definitions for algebras and pseudoalgebras in this setting, characterising algebras using endomorphism Λ-operads, and deriving monads from Λ-operads. We show that there is an adjunction between Λ-**Op** and Σ-**Op**, and prove some technical results regarding maps between monads induced by Λ-operads.

Definition 9.1. Let Λ be an action operad. A Λ-*operad* P (in **Sets**) consists of

- a non-symmetric operad P in **Sets** and
- for each n, an action $P(n) \times \Lambda(n) \to P(n)$ of $\Lambda(n)$ on $P(n)$

such that the following two equivariance axioms hold.

- For each $p \in P(n)$, $q_i \in P(k_i)$, and $g_i \in \Lambda(k_i)$:

$$\mu^P(p; q_1 \cdot g_1, \ldots, q_n \cdot g_n) = \mu^P(p; q_1, \ldots, y_n) \cdot \beta^\Lambda(g_1, \ldots, g_n).$$

- For each $p \in P(n)$, $g \in \Lambda(n)$, and $q_i \in \Lambda(k_i)$:

$$\mu^P(p \cdot g; q_1, \ldots, q_n) = \mu^P\left(p; q_{g^{-1}(1)}, \ldots, q_{g^{-1}(n)}\right) \cdot \delta^\Lambda_{n; k_1, \ldots, k_n}(g).$$

Definition 9.2. Let P and Q be Λ-operads. A *morphism* $f \colon P \to Q$ *of* Λ-*operads* consists of an operad map (9.2) which is levelwise equivariant with respect to the $\Lambda(n)$-actions, i.e., for each $n \in \mathbb{N}$ the following diagram commutes.

$$
\begin{array}{ccc}
P(n) \times \Lambda(n) & \xrightarrow{f_n \times 1} & Q(n) \times \Lambda(n) \\
\downarrow & & \downarrow \\
P(n) & \xrightarrow{\quad f_n \quad} & Q(n)
\end{array}
$$

Definition 9.3. The category Λ-**Op** of Λ-operads has objects the Λ-operads and morphisms the Λ-operad maps.

Example 9.4.   1. Let T denote the terminal operad in **Sets** equipped with its unique action operad structure. Then a T-operad is just a non-symmetric operad in **Sets**.

2. Let Σ denote the operad of symmetric groups with $\pi \colon \Sigma \to \Sigma$ the identity map. Then a Σ-operad is a symmetric operad in the category of sets.

3. Let B denote the operad of braid groups with $\pi_n \colon B_n \to \Sigma_n$ the canonical projection of a braid onto its underlying permutation. Then a B-operad is a braided operad in the sense of Fiedorowicz [**fie-br**].

A further example of a Λ-operad is given by the underlying operad, Λ, of Λ itself.

Proposition 9.5. *Let* Λ *be an action operad. Then the operad* Λ *is itself a* Λ-*operad.*

Proof. The underlying operad Λ is of course an operad in **Sets**. The action $\Lambda(n) \times \Lambda(n) \to \Lambda(n)$ is given simply by the group multiplication in $\Lambda(n)$. The two equivariance axioms are then both instances of the action operad axiom of Λ. □

An operad is intended to be an abstract description of a certain type of algebraic structure, and the particular instances of that structure are the algebras for that operad. We begin with the definition of an algebra over a non-symmetric operad and add in the group actions later.

Definition 9.6. Let O be a non-symmetric operad. An *algebra* for O consists of a set X together with maps $\alpha_n \colon O(n) \times X^n \to X$ such that the following axioms hold.

1. The element $\mathrm{id} \in O(1)$ is a unit in the sense that

$$\alpha_1(\mathrm{id}; x) = x$$

   for all $x \in X$.

2. The maps α_n are associative in the sense that the following diagram commutes.

$$
\begin{array}{ccc}
O(n) \times \prod_{i=1}^n \left(O(k_i) \times X^{k_i}\right) & \xrightarrow{1 \times \alpha_{k_1} \times \cdots \alpha_{k_n}} & O(n) \times X^n \\
\cong \downarrow & & \downarrow \alpha_n \\
O(n) \times \left(\prod_{i=1}^n O(k_i)\right) \times \prod_{i=1}^n \left(X^{k_i}\right) & & \\
\mu \times 1 \downarrow & & \\
O(\sum k_i) \times X^{\sum k_i} & \xrightarrow{\alpha_{\sum k_i}} & X
\end{array}
$$

$P(n) \otimes_{\Lambda(n)} X^n$

Moving on to algebras for a Λ-operad, let P be a Λ-operad and let X be any set. Write $P(n) \otimes_{\Lambda(n)} X^n$ for the coequalizer of the pair of maps

$$P(n) \times \Lambda(n) \times X^n \rightrightarrows P(n) \times X^n$$

of which the first map is the action of $\Lambda(n)$ on $P(n)$ and the second map is

$$P(n) \times \Lambda(n) \times X^n \to P(n) \times \Sigma_n \times X^n \to P(n) \times X^n$$

using $\pi_n \colon \Lambda(n) \to \Sigma_n$ together with the canonical action of Σ_n on X^n by permutation of coordinates: $\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$. By the universal property of the coequalizer, a function $f \colon P(n) \otimes_{\Lambda(n)} X^n \to Y$ can be identified with a function $\tilde{f} \colon P(n) \times X^n \to Y$ such that

$$\tilde{f}(p \cdot g; x_1, \dots, x_n) = \tilde{f}\left(p; x_{g^{-1}(1)}, \dots, x_{g^{-1}(n)}\right).$$

Definition 9.7. Let P be a Λ-operad. An *algebra* for P consists of a set X together with maps $\alpha_n \colon P(n) \otimes_{\Lambda(n)} X^n \to X$ such that the maps $\tilde{\alpha}_n$ satisfy the operad algebra axioms given in Definition 9.6.

Remark 9.8. It is worth noting that the equivariance required for a P-algebra is built into the definition above by requiring the existence of the maps α_n to be defined on coequalizers, even though the algebra axioms then only use the maps $\tilde{\alpha}_n$. Since every Λ-operad has an underlying plain operad (see Proposition 9.11, applied to the unique map $T \to \Lambda$), this reflects the fact that the algebras for the Λ-equivariant version are always algebras for the plain version, but not conversely.

Definition 9.9. Let P be a Λ-operad. The category of algebras for P, P-**Alg**, has objects the P-algebras (X, α) and morphisms $f \colon (X, \alpha) \to (Y, \beta)$ those functions $f \colon X \to Y$ such that the following diagram commutes for every n.

$$
\begin{array}{ccc}
P(n) \times X^n & \xrightarrow{\;\;1 \times f^n\;\;} & P(n) \times Y^n \\
{\scriptstyle \tilde{\alpha}_n} \downarrow & & \downarrow {\scriptstyle \tilde{\beta}_n} \\
X & \xrightarrow{\;\;\;f\;\;\;} & Y
\end{array}
$$

Let X be a set. Then the endomorphism operad of X, denoted $\mathcal{E}_X$, is given by the sets $\mathcal{E}_X(n) = \mathbf{Sets}(X^n, X)$ with the identity function in $\mathcal{E}_X(1)$ giving the unit element and composition of functions giving the composition operation. Concretely, composition is given by the formula

$$
\mu(f; g_1, \ldots, g_n) = f \circ (g_1 \times \cdots \times g_n).
$$

Lemma 9.10. *Let Λ be an action operad, and let X be a set. Then $\mathcal{E}_X$ carries a canonical Λ-operad structure.*

Proof. $\mathcal{E}_X$ is a symmetric operad, so we define the actions by

$$
\mathcal{E}_X(n) \times \Lambda(n) \xrightarrow{1 \times \pi_n} \mathcal{E}_X(n) \times \Sigma_n \to \mathcal{E}_X(n).
$$

$\square$

The previous result is really a change-of-structure-groups result. We record the general result as the following proposition.

Proposition 9.11. *Let $f \colon \Lambda \to \Lambda'$ be a map of action operads. Then f induces a functor $f^* \colon \Lambda'$-$\mathbf{Op} \to \Lambda$-$\mathbf{Op}$.*

We can now use endomorphism operads to characterize algebra structures.

Proposition 9.12. *Let X be a set, and P a Λ-operad. Then algebra structures on X are in 1-to-1 correspondence with Λ-operad maps $P \to \mathcal{E}_X$.*

Proof. A map $P(k) \to \mathcal{E}_X(k)$ corresponds, using the closed structure on $\mathbf{Sets}$, to a map $P(k) \times X^k \to X$. The monoid homomorphism axioms give the unit and associativity axioms, and the requirement that $P \to \mathcal{E}_X$ be a map of Λ-operads gives the equivariance condition. $\square$

Remark 9.13. The proposition above holds for P-algebras in any closed symmetric monoidal category. Having a closed structure (in addition to all small colimits) is a stronger condition than the tensor preserving colimits in each variable, but it is a natural one that arises in many examples.

10 Transferring Operads and Λ-Monoidal Categories

Remark 10.1. I moved this here for now.

We have seen (Proposition 11.2) that given any Λ-operad P there is an induced monad $\underline{P} \colon \mathcal{C} \to \mathcal{C}$ and that the category of algebras for the operad P is isomorphic to the category of algebras for the monad $\underline{P}$, following [**maygeom**]. Now we are considering Λ-operads in **Cat**, the induced monad associated to an operad of this sort can be shown to be a 2-monad (see [**KS**] for background on 2-monads) and we will proceed to show that the notions of pseudoalgebra for both the operad and the associated 2-monad correspond precisely, i.e., there is an isomorphism of 2-categories between the 2-category with either strict or pseudo-level cells defined operadically and the 2-category with either strict or pseudo-level cells defined 2-monadically.

The associated monad $\underline{P}$ acquires the structure of a 2-functor as follows. We define $\underline{P}$ on categories much like before as the coproduct

$$\underline{P}(X) = \coprod_n P(n) \otimes_{\Lambda(n)} X^n,$$

whose objects will be written as equivalence classes $[p; x_1, \ldots, x_n]$ where $p \in P(n)$ and each $x_i \in X$, sometimes written as $[p; \underline{x}]$ when there is no confusion. On functors we define $\underline{P}$ in a similar way, exactly as with functions of sets. Given a natural transformation $\alpha \colon f \Rightarrow g$ we define a new natural transformation $\underline{P}(\alpha)$ as follows. The component of $\underline{P}(\alpha)$ at the object

$$[p; x_1, \ldots, x_n]$$

is given by the morphism

$$[1_p; \alpha_{x_1}, \ldots, \alpha_{x_n}]$$

in $\underline{P}(X)$. It is a simple observation that this constitutes a 2-functor, and that the components of the unit and multiplication are functors and are 2-natural.

Remark 10.2. The material in this section can be given a rather more abstract interpretation, in the sense of [**KL97**]. The idea here is that the category of Λ-collections acts on the category **Cat** via a functor $\diamond \colon \Lambda\text{-}\mathbf{Coll} \times \mathbf{Cat} \to \mathbf{Cat}$ which sends (P, X) to $\underline{P}(X)$ as described above. Fixing a Λ-collection P produces an endofunctor $\underline{P} \colon \mathbf{Cat} \to \mathbf{Cat}$ which is then a monad when P is a Λ-operad, just as monoids in Λ-**Coll** are precisely Λ-operads.

First we will set out some conventions and definitions.

Convention 10.3. We will identify maps $\alpha_n \colon P(n) \otimes_{\Lambda(n)} X^n \to X$ with maps $\tilde{\alpha}_n \colon P(n) \times X^n \to X$ which are equivariant with respect to the Λ-actions via the universal property of the coequalizer. The coequalizer in **Cat** also has a 2-dimensional aspect to its universal property, so that a natural transformation $\Gamma \colon \alpha_n \Rightarrow \beta_n$ between functors as above determines and is determined by a transformation $\tilde{\Gamma} \colon \tilde{\alpha}_n \Rightarrow \tilde{\beta}_n$ with the property that the two possible whiskerings of $\tilde{\Gamma}$ with the two functors $P(n) \times \Lambda(n) \times X^n \to P(n) \times X^n$ are equal.

Note also that in the following definitions we will often write the composite

$$P(n) \times \prod_{i=1}^n \left(P(k_i) \times X^{k_i} \right) \to P(n) \times \prod_{i=1}^n P(k_i) \times X^{\Sigma k_i} \xrightarrow{\mu^P \times 1} P(\Sigma_{k_i}) \times X^{\Sigma k_i}$$

simply abbreviated as $\mu^P \times 1$. Furthermore, instead of using an element id $\in P(1)$ as the operadic unit, we will now denote this as $\eta^P \colon 1 \to P(1)$.

Definition 10.4. 1. Let X be a set. We define the *translation category EX* to have objects the elements of X and morphisms consisting of a unique isomorphism between any two objects.

2. Let G be a group. The category BG has a single object $*$, and hom-set $BG(*, *) = G$ with composition and identity given by multiplication and the unit element in the group, respectively.

Definition 10.5. A functor $F\colon X \to Y$ is an *isofibration* if given $x \in X$ and an isomorphism $f\colon y \xrightarrow{\cong} F(x)$ in Y, then there exists an isomorphism $g\colon y \cong x$ in X such that $F(g) = f$.

Proposition 10.6. *There exists a natural transformation $p\colon EU \Rightarrow B$, where U is the underlying set of a group, which is pointwise an isofibration. Applying the classifying space functor to the component p_G gives a universal principal G-bundle.*

Proof. Given a group G, $p_G\colon EUG \to BG$ sends every object of EUG to the unique object of BG. The unique isomorphism $g \to h$ in EUG is mapped to $hg^{-1}\colon * \to *$. It is easy to directly check that this is an isofibration, as well as to see that the classifying spaces of EUG and BG are the spaces classically known as EG, BG, with $|p_G|$ being the standard universal principal G-bundle. $\qquad\square$

We will also need the functors E, B defined for more than just a single set or group, in particular for the sets or groups which make up an operad and are indexed by the natural numbers.

Notation 10.7. Let S be a set which we view as a discrete category.

1. For any functor $F\colon S \to \mathbf{Sets}$, let EF denote the composite $E \circ F\colon S \to \mathbf{Cat}$; we often view F as an indexed set $\{F(s)\}$, in which case EF is the indexed category $\{EF(s)\}$.

2. For any functor $F\colon S \to \mathbf{Grp}$, let BF denote the composite $B \circ F\colon S \to \mathbf{Cat}$; we often view F as an indexed group $\{F(s)\}$, in which case BF is the indexed category $\{BF(s)\}$.

The following lemma is a straightforward verification.

Lemma 10.8. *The functor $E\colon \mathbf{Sets} \to \mathbf{Cat}$ is right adjoint to the set of objects functor. Therefore E preserves all limits, and in particular is a symmetric monoidal functor when both categories are equipped with their cartesian monoidal structures.*

We are additionally interested in Λ-operads in $\mathbf{Cat}$ (or other cocomplete symmetric monoidal categories in which the tensor product preserves colimits in each variable). While the definition above gives the correct notion of a Λ-operad in $\mathbf{Cat}$ if we interpret the two equivariance axioms to hold for both objects and morphisms, it is useful to give a purely diagrammatic expression of these axioms. In the diagrams below, expressions of the form $G \times C$ for a group G and category C mean that the group G is to be treated as a discrete category. This follows the standard method of how one expresses group actions in categories other than $\mathbf{Sets}$ using a copower. Thus the diagrams below are the two equivariance axioms given in Definition 9.1 expressed diagrammatically, where $K = k_1 + \ldots + k_n$.

$$
\begin{array}{ccc}
P(n)\times\left(\prod_{i=1}^{n} P(k_i)\right)\times\left(\prod_{i=1}^{n} \Lambda(k_i)\right) & \xrightarrow{\ \cong\ } & P(n)\times\left(\prod_{i=1}^{n}(P(k_i)\times\Lambda(k_i))\right) \\
{\scriptstyle \mu^P \times \mu^\Lambda(e;-)}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\times\prod_{i=1}^{n}\alpha_i} \\
P(K)\times\Lambda(K) & & P(n)\times\left(\prod_{i=1}^{n} P(k_i)\right) \\
 & \underset{\alpha_K}{\searrow}\quad P(K)\quad\underset{\mu^P}{\swarrow} &
\end{array}
$$

$$
\begin{array}{ccc}
P(n)\times\Lambda(n)\times\prod_{i=1}^{n}P(k_i) & \xrightarrow{\quad\alpha_n\times 1\quad} & P(n)\times\prod_{i=1}^{n}P(k_i)\\[4pt]
{\scriptstyle 1\times\Delta\times 1}\big\downarrow & & \big\downarrow{\scriptstyle \mu^{P}}\\[4pt]
P(n)\times\Lambda(n)\times\Lambda(n)\times\prod_{i=1}^{n}P(k_i) & & \\[4pt]
{\scriptstyle \cong}\big\downarrow & & P(K)\\[4pt]
P(n)\times\Lambda(n)\times\prod_{i=1}^{n}P(k_i)\times\Lambda(n) & & \\[4pt]
{\scriptstyle 1\times\pi_n\times 1}\big\downarrow & & \nearrow{\scriptstyle \alpha_K}\\[4pt]
P(n)\times\Sigma_n\times\prod_{i=1}^{n}P(k_i)\times\Lambda(n) & \xrightarrow[\tilde\mu^{P}\times\mu^{\Lambda}(-;\underline{e})]{} & P(K)\times\Lambda(K)
\end{array}
$$

In the second diagram, the morphism

$$
\tilde\mu^{P}\colon P(n)\times\Sigma_n\times\prod_{i=1}^{n}P(k_i)\to P(K)\times\Lambda(K)
$$

is first the left action of Σ_n on the product followed by the operad multiplication, and $\underline{e}$ is $e_{k_1},\dots,e_{k_n}$.

Definition 10.9. Let Λ be an action operad. Then $B\Lambda$ (see Notation 10.7) is the category with objects the natural numbers and

$$
B\Lambda(m,n)=\left\{
\begin{array}{ll}
\Lambda(n), & m=n\\
\emptyset, & m\neq n,
\end{array}
\right.
$$

where composition is given by group multiplication and the identity morphism is the unit element $e_n\in\Lambda(n)$.

Theorem 10.10. *Let M,N be cocomplete symmetric monoidal categories in which the tensor product preserves colimits in each variable, and let $F\colon M\to N$ be a symmetric lax monoidal functor with unit constraint φ_0 and tensor constraint φ_2. Let Λ be an action operad, and P a Λ-operad in M. Then $FP=\{F(P(n))\}$ has a canonical Λ-operad structure, giving a functor*

$$
F_*\colon \Lambda\text{-}Op(M)\to\Lambda\text{-}Op(N)
$$

from the category of Λ-operads in M to the category of Λ-operads in N.

Proof. The category of Λ-operads in M is the category of monoids for the composition product $\circ_M$ on $[B\Lambda^{\mathrm{op}},M]$ constructed in **??**. Composition with F gives a functor

$$
F_*\colon [B\Lambda^{\mathrm{op}},M]\to[B\Lambda^{\mathrm{op}},N],
$$

and to show that it gives a functor between the categories of monoids we need only prove that F_* is lax monoidal with respect to $\circ_M$ and $\circ_N$. In other words, we must construct natural transformations with components $F_*X\circ_N F_*Y\to F_*(X\circ_M Y)$ and $I_{Op(N)}\to F_*(I_{Op(M)})$ and then verify the lax monoidal functor axioms. We note that in the calculations below, we often write F instead of F_*, but it should be clear from context when we are applying F to objects and morphisms in M and when we are applying F_* to a functor $B\Lambda^{\mathrm{op}}\to M$.

We first remind the reader about copowers in cocomplete categories. For an object X and set S, the copower $X\odot S$ is the coproduct $\coprod_{s\in S}X$. We have natural isomorphisms $(X\odot S)\odot T\cong X\odot(S\times T)$ and $X\odot 1\cong X$, and using these we can define an action of a group G on an object X using a map $X\odot G\to X$. Any functor F between categories with coproducts is lax monoidal with respect to those coproducts: the natural map $FA\coprod FB\to F(A\coprod B)$ is just the map induced by the universal property of

the coproduct using F applied to the coproduct inclusions $A \hookrightarrow A \coprod B, B \hookrightarrow A \coprod B$. In particular, for any functor F there exists an induced map $FX \odot S \to F(X \odot S)$.

The unit object in $[B\Lambda^{\mathrm{op}}, M]$ for $\circ_M$ is the copower $I_M \odot B\Lambda(-, 1)$. Thus the unit constraint for F_* is the composite

$$I_N \odot B\Lambda(-, 1) \xrightarrow{\varphi_0 \odot 1} FI_M \odot B\Lambda(-, 1) \to F(I_M \odot B\Lambda(-, 1)).$$

For the tensor constraint, we will require a map

$$t \colon (FY)^{\star n}(k) \to F(Y^{\star n}(k))$$

where $\star$ is the Day convolution product; having constructed one, the tensor constraint is then the following composite.

$$
\begin{aligned}
(FX \circ FY)(k) &\xrightarrow{\cong} \int^n FX(n) \otimes (FY)^{\star n}(k) \\
&\xrightarrow{\int 1 \otimes t} \int^n FX(n) \otimes F(Y^{\star n}(k)) \\
&\xrightarrow{\int \varphi_2} \int^n F(X(n) \otimes Y^{\star n}(k)) \\
&\longrightarrow F\left(\int^n X(n) \otimes Y^{\star n}(k)\right) \\
&\xrightarrow{\cong} F(X \circ Y)(k)
\end{aligned}
$$

Both isomorphisms in the composite above are induced by universal properties (see **??** for more details) and the unlabeled arrow is induced by the same argument as that for coproducts above but this time using coends. The arrow t is constructed in a similar fashion, and is the composite below.

$$
\begin{aligned}
(FY)^{\star n}(k) = \int^{k_1,\ldots,k_n} & FY(k_1) \otimes \cdots \otimes FY(k_n) \odot B\Lambda(k, \Sigma k_i) \\
\to \int^{k_1,\ldots,k_n} & F(Y(k_1) \otimes \cdots \otimes Y(k_n)) \odot B\Lambda(k, \Sigma k_i) \\
\to \int^{k_1,\ldots,k_n} & F(Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\Lambda(k, \Sigma k_i)) \\
\to F \int^{k_1,\ldots,k_n} & Y(k_1) \otimes \cdots \otimes Y(k_n) \odot B\Lambda(k, \Sigma k_i) \\
= F(Y^{\star n}(k)) &
\end{aligned}
$$

Checking the lax monoidal functor axioms is tedious but entirely routine using the lax monoidal functor axioms for F together with various universal properties of colimits, and we leave the details to the reader. $\qquad\square$

Combining Theorem 10.10 and Proposition 9.5 with Lemma 10.8, we immediately obtain the following.

Corollary 10.11. *Let Λ be an action operad. Then $E\Lambda = \{E(\Lambda(n))\}$ (see Notation 10.7) is a Λ-operad in* **Cat**.

Any Λ-operad P in **Cat** gives rise to a 2-monad on **Cat** which we will often also denote by P or, as in **??**, by $\underline{P}$. In the context of Corollary 10.11, that 2-monad $\underline{E\Lambda}$ is given by

$$X \mapsto \coprod_{n \geq 0} E\Lambda(n) \otimes_{\Lambda(n)} X^n$$

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where the action of $\Lambda(n)$ on $E\Lambda(n)$ is given by the obvious multiplication action on the right, and the action of $\Lambda(n)$ on X^n is given using $\pi_n \colon \Lambda(n) \to \Sigma_n$ together with the standard left action of Σ_n on X^n in any symmetric monoidal category. The 2-monad $\underline{E\Sigma}$ is that for symmetric strict monoidal categories (see **??** for this and further examples).

Definition 10.12. A Λ-*monoidal category* is a strict algebra for the 2-monad $E\Lambda$. A Λ-*monoidal functor* is a strict morphism for this 2-monad $E\Lambda$. The 2-category Λ-**MonCat** is the 2-category $E\Lambda$-**Alg**$_s$ of strict algebras, strict morphisms, and algebra 2-cells for $E\Lambda$.

Strict Λ-monoidal categories, in the sense of being strict algebras for the monad $\underline{E\Lambda}$, can be characterised in more familiar terms by specifying a monoidal structure with appropriate equivariant interaction with the $\Lambda(n)$-actions. Explicit proofs of such can be found in Chapter 19 of [**yau˙infinity˙2021**], along with similar characterisations for strict Λ-monoidal functors and Λ-monoidal categories whose underlying monoidal structure is weak.

It will be useful for our calculations later to give an explicit description of the categories $E\Lambda(n) \otimes_{\Lambda(n)} X^n$. Objects are equivalence classes of tuples $(g; x_1, \ldots, x_n)$ where $g \in \Lambda(n)$ and the x_i are objects of X, with the equivalence relation given by

$$(gh; x_1, \ldots, x_n) \sim \left(g; x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n)} \right);$$

we write these classes as $[g; x_1, \ldots, x_n]$. Morphisms are then equivalence classes of morphisms

$$(!; f_1, \ldots, f_n) \colon (g; x_1, \ldots, x_n) \to \left(h; x'_1, \ldots, x'_n \right).$$

We have two distinguished classes of morphisms, one for which the map $! \colon g \to h$ is the identity and one for which all the f_i's are the identity. Every morphism in $E\Lambda(n) \times X^n$ is uniquely a composite of a morphism of the first type followed by one of the second type. Now $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is a quotient of $E\Lambda(n) \times X^n$ by a free group action, so every morphism of $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is in the image of the quotient map. Using this fact, we can prove the following useful lemma.

Lemma 10.13. *For an action operad Λ and any category X, the set of morphisms from $[e; x_1, \ldots, x_n]$ to $[e; y_1, \ldots, y_n]$ in $E\Lambda(n) \otimes_{\Lambda(n)} X^n$ is*

$$\coprod_{g \in \Lambda(n)} \prod_{i=1}^n X\left(x_i, y_{g(i)} \right).$$

Proof. A morphism with source $(e; x_1, \ldots, x_n)$ in $E\Lambda(n) \times X^n$ is uniquely a composite

$$(e; x_1, \ldots, x_n) \overset{(\mathrm{id}; f_1, \ldots, f_n)}{\longrightarrow} \left(e; x'_1, \ldots, x'_n \right) \overset{(!; \mathrm{id}, \ldots, \mathrm{id})}{\longrightarrow} \left(g; x'_1, \ldots, x'_n \right).$$

Descending to the quotient, this becomes a morphism

$$[e; x_1, \ldots, x_n] \to \left[g; x'_1, \ldots, x'_n \right] = \left[e; x'_{g^{-1}(1)}, \ldots, x'_{g^{-1}(n)} \right],$$

and therefore is a morphism $[e; x_1, \ldots, x_n] \to [e; y_1, \ldots, y_n]$ precisely when $y_i = x'_{g^{-1}(i)}$, and so $f_i \in X(x_i, y_{g(i)})$. $\square$

11   Λ-Operads as Monads

Definition 11.1. Let P be a Λ-operad. Then P induces an endofunctor of **Sets**, denoted $\underline{P}$, by the following formula.

$$\underline{P}(X) = \coprod_n P(n) \otimes_{\Lambda(n)} X^n$$

We now have the following proposition; its proof is standard [**maygeom**], and we leave it to the reader should they be interested in the details.

Proposition 11.2. *Let P be a Λ-operad.*

1. *The Λ-operad structure on P induces a monad structure on $\underline{P}$.*

2. *The category of algebras for the operad P is isomorphic to the category of algebras for the monad $\underline{P}$.*

In the case that we take $P = \Lambda$, we do not get algebras more interesting than monoids.

Proposition 11.3. *Let Λ be an action operad. The category of algebras, Λ-**Alg**, for Λ taken as a Λ-operad, is equivalent to the category of monoids.*

Proof. The key observation here is that, for each $n \in \mathbb{N}$,

$$\Lambda(n) \otimes_{\Lambda(n)} X^n \cong X^n.$$

We will describe this bijection and leave the rest of the proof to the reader, which falls out of the various axioms either for being a monoid or for being a Λ-algebra.

Recall that the elements of $\Lambda(n) \otimes_{\Lambda(n)} X^n$ are equivalence classes of the form $[g; x_1, \ldots, x_n]$ for which

$$(gh; x_1, \ldots, x_n) \sim \left(g; x_{h^{-1}(1)}, \ldots, x_{h^{-1}(n)}\right).$$

There is an obvious map $X^n \to \Lambda(n) \otimes_{\Lambda(n)} X^n$ sending $(x_1, \ldots, x_n)$ to the equivalence class $[e; x_1, \ldots, x_n]$. The inverse to this map is given by the map $\Lambda(n) \otimes_{\Lambda(n)} X^n \to X^n$ sending $[g; x_1, \ldots, x_n]$ to the element $(x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)})$. It is then clear that these are inverses, relying on the equivalence relation to see that

$$[g; x_1, \ldots, x_n] = \left[e; x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}\right].$$

That the second map is well-defined is simple to show. $\qquad\square$

We end this section with a discussion of the relationship between symmetric operads and Λ-operads for an arbitrary action operad Λ.

Theorem 11.4. *Let Λ be an action operad.*

1. *An adjunction exists between the category of Λ-operads and the category of symmetric operads, with right adjoint $\pi^* \colon \Lambda\text{-}\mathbf{Op} \to \Sigma\text{-}\mathbf{Op}$ and left adjoint denoted S.*

2. *The counit of this adjunction is an isomorphism, but the unit is not, thus this adjunction is not an equivalence of categories*

3. *There exists a natural isomorphism of monads between $\underline{P}$ and $\underline{S(P)}$ for a Λ-operad P. In particular, these monads (and hence operads) have isomorphic categories of algebras.*

Proof. Given any map of monoids $f \colon M \to N$ in a monoidal category, there exists an adjunction between right M-modules and right N-modules given by f^* as the right adjoint and $A \mapsto A \otimes_M N$ as the left adjoint. Thus we define

$$S(P)(n) = P(n) \otimes_{\Lambda(n)} \Sigma_n,$$

and this inherits a right Σ_n-action by multiplication. The unit of $S(P)$ is

$$* \xrightarrow{\eta} P(1) \longrightarrow P(1) \otimes_{\Lambda(1)} \Sigma_1 \cong P(1)/\Lambda(1).$$

For the multiplication, let $K = k_1 + \cdots + k_n$, so we must define

$$\mu \colon \big(P(n) \otimes_{\Lambda(n)} \Sigma_n\big) \times \prod_{i=1}^{n} \big(P(k_i) \otimes_{\Lambda(k_i)} \Sigma_{k_i}\big) \to P(K) \otimes_{\Lambda(K)} \Sigma_K.$$

Using the universal property of the coequalizer, this is induced by the following composite.

$$(P(n) \times \Sigma_n) \times \prod_{i=1}^{n} (P(k_i) \times \Sigma_{k_i}) \cong \left(P(n) \times \prod_{i=1}^{n} P(k_i) \right) \times \left(\Sigma_n \times \prod_{i=1}^{n} \Sigma_{k_i} \right)$$
$$\xrightarrow{\mu^P \times \mu^\Sigma} P(K) \times \Sigma_K$$
$$\longrightarrow P(K) \otimes_{\Lambda(K)} \Sigma_K$$

We leave verification of the associativity, unit, and equivariance axioms to the reader; they are simple applications of the same axioms for P and Σ together with some colimit universal properties and the Λ-operad axioms for P. It is then straightforward to check the bijection between Λ-operad maps $P \to \pi^* Q$ and symmetric operad maps $S(P) \to Q$, thus establishing the adjunction.

The second claim is a simple calculation using the coequalizer that defines $S(\pi^* Q)$, using that $Q(n)$ is itself the coequalizer of the obvious pair of maps $Q(n) \times \Sigma_n \times \Sigma_n$.

For the third claim, we find a natural isomorphism

$$P(n) \otimes_{\Lambda(n)} X^n \cong (P(n) \otimes_{\Lambda(n)} \Sigma_n) \times_{\Sigma_n} X^n$$

by the universal property of the colimits involved, so as functors $\underline{P} \cong \underline{S(P)}$. One can then easily verify that this isomorphism commutes with the unit and multiplication of the two monads involved using calculations similar to those used to establish the adjunction. $\square$

Remark 11.5.   1. The adjunction alone is enough to establish that P and $S(P)$ have isomorphic categories of algebras using Proposition 9.12.

2. This theorem shows that semantically, one need never consider any kind of operad aside from symmetric operads: any other kind of operad can be symmetrized without altering the algebras. But as the operad should be considered a finer level of detail than the monad, restricting to symmetric operads misses the structure present in the more nuanced group actions.

3. Furthermore, there is clearly an artifact left from only considering the algebras themselves as objects in a symmetric monoidal category. It is well-known that a braided structure is all that is required for non-symmetric operads, and so one is left to consider that the natural home for algebras over a Λ-operad might be a type of monoidal structure other than symmetric in which case the theorem above gives no insight.

We end this section by presenting some results which allow us to transfer operad or algebra structures to other categories. We will use the following standard definitions of monad maps and transformations, as per [**street-formal**].

Definition 11.6. Let S be a monad on a category C and T be a monad on a category D. A *monad map* of from S to T is a functor $F\colon C \to D$ together with a natural transformation $\alpha\colon TF \Rightarrow FS$ such that the following diagrams commute.

$$
\begin{array}{ccc}
& \xrightarrow{\eta_{FX}^T} & TFX \\
FX & & \Big\downarrow \alpha_X \\
& \xrightarrow{F\eta_X^S} & FSX
\end{array}
\qquad
\begin{array}{ccc}
& \xrightarrow{T\alpha_X} TFSX \xrightarrow{\ \alpha_{SX}\ } FS^2X \\
T^2FX & \qquad\qquad\qquad \Big\downarrow F\mu_X^S \\
& \xrightarrow{\mu_{FX}^T} TFX \xrightarrow{\ \alpha_X\ } FSX
\end{array}
$$

A *transformation* $\Gamma\colon (F,\alpha) \Rightarrow (G,\beta)$ between monad maps is a natural transformation $\Gamma\colon F \Rightarrow G$ such that the following diagram commutes.

$$
\begin{array}{ccc}
TFX & \xrightarrow{\ T\Gamma_X\ } & TGX \\
\alpha_X \Big\downarrow & & \Big\downarrow \beta_X \\
FSX & \xrightarrow{\ \Gamma_{SX}\ } & GSX
\end{array}
$$

Remark 11.7. Every monad map (F,α) induces a functor $S\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}$ on the categories of algebras. An S-algebra (X,σ) is sent to the T-algebra $(FX, F\sigma \cdot \alpha_X)$, as we now describe. For $(FX, F\sigma \cdot \alpha_X)$ to be a T-algebra we require the usual diagrams to commute, shown as the outside of the diagrams below.

$$
\begin{array}{ccccc}
T^2FX & \xrightarrow{T\alpha_X} & TFSX & \xrightarrow{TF\sigma} & TFX \\
\mu_{FX}^T \Big\downarrow & & \alpha_{SX}\Big\downarrow & & \Big\downarrow \alpha_X \\
& & FS^2X & \xrightarrow{FS\sigma} & FSX \\
& & F\mu_X^S \Big\downarrow & & \Big\downarrow F\sigma \\
TFX & \xrightarrow{\ \alpha_X\ } & FSX & \xrightarrow{\ F\sigma\ } & FX
\end{array}
\qquad
\begin{array}{ccc}
FX & \xrightarrow{\ \eta_{FX}^T\ } & TFX \\
& F\eta_X^S \searrow & \Big\downarrow \alpha_X \\
\mathrm{id} & & FSX \\
& \searrow & \Big\downarrow F\sigma \\
& & FX
\end{array}
$$

The first diagram commutes since the left hand side is the second diagram required to commute for (F,α) to be a monad map, the square at the top right is an instance of naturality for α, while the bottom right square commutes since (X,σ) is an S-algebra. The second diagram commutes since the top triangle is again a requirement of α being a transformation, with the lower triangle commuting again as a result of (X,σ) being an S-algebra.

A morphism $f\colon (X,\sigma_X) \to (Y,\sigma_Y)$ of S-algebras is sent to the morphism

$$
Ff\colon (FX, F\sigma_X \cdot \alpha_X) \to (FY, F\sigma_Y \cdot \alpha_Y),
$$

this being a map of T-algebras following from the naturality of α and of f being an S-algebra map. Functoriality follows from that of F.

Throughout the text we make reference to where results can be applied in a more general case where a symmetric monoidal category is cocomplete and for which the tensor product distributes over colimits in each variable. However, we include the following definition to be clear what is meant simply by a cocomplete symmetric monoidal category.

Definition 11.8. A cocomplete symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal category for which the underlying category is cocomplete.

The following three results tie in with material in the coming chapters but are of a general nature which better in the context of this section.

Proposition 11.9. *Let C, D be cocomplete symmetric monoidal categories. Let Λ be an action operad, and P be a Λ-operad in C. Let $F \colon C \to D$ be a symmetric lax monoidal functor. Then FP is a Λ-operad in D, and there exists a monad map $(F, \psi) \colon (C, \underline{P}) \to (D, \underline{FP})$.*

Proof. Theorem 10.10 describes how the functor F can be used to describe a functor

$$\Lambda\text{-}\mathbf{Op}(C) \to \Lambda\text{-}\mathbf{Op}(D),$$

from which we see that FP is a Λ-operad in D.

The functor F constitutes the 1-cell of the monad map, while ψ is required to be a natural transformation as below.

$$
\begin{array}{ccc}
C & \xrightarrow{\quad F \quad} & D \\
\underline{P} \downarrow & \swarrow_{\psi} & \downarrow \underline{FP} \\
C & \xrightarrow{\quad F \quad} & D
\end{array}
$$

We describe the components of this natural transformation at an object X of C below.

$$
\begin{aligned}
\underline{FP}(FX) &= \coprod_{n \in \mathbb{N}} FP(n) \otimes_{\Lambda(n)} (FX)^n \\
&\to \coprod_{n \in \mathbb{N}} F\left(P(n) \otimes_{\Lambda(n)} X^n\right) \\
&\to F\left(\coprod_{n \in \mathbb{N}} P(n) \otimes_{\Lambda(n)} X^n\right) \\
&= F(\underline{P}(X))
\end{aligned}
$$

The first morphism is a composite of the coherence cells of the type

$$FX \otimes FY \to F(X \otimes Y)$$

for the symmetric lax monoidal functor F, while the second morphism is the induced morphism out of the coproduct. Naturality follows from that of the component morphisms. It is then straightforward to see that the monad morphism diagrams commute since the diagrams involved consist of instances of the coherence axioms for F along with naturality of the coherence cells. $\square$

Proposition 11.10. *Let C be a cocomplete symmetric monoidal category. Let Λ be an action operad, and P, Q be Λ-operads in C with a map $\sigma \colon P \to Q$ of Λ-operads between them. Then σ induces a monad map $(id, \sigma^*) \colon (C, \underline{Q}) \to (C, \underline{P})$ and hence a functor on categories of algebras.*

Proof. We will first describe the components of the natural transformation $\sigma^* \colon \underline{P} \Rightarrow \underline{Q}$. The component σ_X^* at an object X of C is a morphism between the coproducts

$$\sigma_X^* \colon \coprod_{n \in \mathbb{N}} P(n) \otimes_{\Lambda(n)} X^n \to \coprod_{n \in \mathbb{N}} Q(n) \otimes_{\Lambda(n)} X^n.$$

This is seen to be induced by the universal properties of the coequalizers and coproducts in the following diagram, where σ_n denotes the n-ary component of the Λ-operad map σ and λ^P, λ^Q, ρ^P, and ρ^Q denote the usual left and right actions.

The upper square which includes λ^P and λ^Q commutes due to σ being a Λ-operad map, while the square with both ρ actions commutes because the σ_n and ρ do not interact. Since c_n^Q coequalizes λ^Q and ρ^Q, then this commutativity shows that $c_n^Q \cdot (\sigma_n \otimes \mathrm{id}) \cdot \lambda^P = c_n^Q \cdot (\sigma_n \otimes \mathrm{id}) \cdot \rho^P$, hence the morphism $\sigma_{n,X}^*$ exists. The morphism σ_X^* is then induced by the universal property of the coproduct $\underline{P}(X)$.

It is then routine to check that these components are natural in X and constitute a monad map. That a functor is then induced on the category of algebras follows from Lemma 6.1.1 of [**leinster**]; the process is described above, following Definition 11.6. $\square$

We can combine these two propositions.

Corollary 11.11. *If C, D, P, F are as in Proposition 11.9, and F is part of a monoidal adjunction (i.e., an adjunction in which both functors are symmetric lax monoidal, and the unit and counit are monoidal transformations) $F \dashv U$, then (F, id) and (U, id) are both monad maps. The unit $\eta \colon 1 \Rightarrow UF$ induces an operad map $\eta \colon P \Rightarrow UFP$, and a transformation between monad maps*

$$(id, id) \Rightarrow (id, \eta^*) \circ (U, \psi^U) \circ \left(F, \psi^F\right).$$

The counit $\varepsilon \colon FU \Rightarrow 1$ induces an operad map $\varepsilon \colon FUFP \Rightarrow FP$, and a transformation between monad maps

$$\left(F, \psi^F\right) \circ (id, \eta^*) \circ \left(U, \psi^U\right) \Rightarrow (id, id).$$

These constitute an adjunction $\left(F, \psi^F\right) \dashv (id, \eta^*) \circ \left(U, \psi^U\right)$ in the 2-category of monads, and hence induce an adjunction between $\underline{P}$-algebras in C and $\underline{FP}$-algebras in D.

12 The Substitution Product

In this section, we will show that Λ-operads are the monoids in the category of Λ-collections equipped with an appropriate substitution product. Such a result is fairly standard [**mss-op**], and in both the symmetric and non-symmetric cases can easily be proven directly. Since we work with an arbitrary action operad, however, it will be more economical to take the abstract approach using coends and Day convolution.

Remark 12.1. It is possible to consider Λ-operads in categories other than the category of sets. In this case we still use the notion of an action operad given above, but then take the operad P to have objects $P(n)$ which are the objects of some closed symmetric monoidal category $\mathcal{V}$. We will rarely use anything that might require the closed structure as such, only the fact that the tensor product distributes over colimits in each variable. This is a consequence of the fact that both $X \otimes -$ and $- \otimes X$ are left adjoints in the case of a closed symmetric monoidal category. Thus while we set up the foundations using only operads in **Sets**, the diligent reader could easily modify this theory for their closed symmetric monoidal category of choice. In fact, we will use the same theory in **Cat** with its cartesian structure, noting only that the same arguments work in **Cat** with essentially no modification.

Definition 12.2. Let Λ be an action operad. The category Λ-**Coll** of Λ-collections has objects $X = \{X(n)\}_{n \in \mathbb{N}}$ which consist of a set $X(n)$ for each natural number n together with an action $X(n) \times \Lambda(n) \to X(n)$ of $\Lambda(n)$ on $X(n)$. A morphism $f \colon X \to Y$ in Λ-**Coll** consists of a $\Lambda(n)$-equivariant map $f_n \colon X(n) \to Y(n)$ for each natural number n.

Remark 12.3. The definition of Λ-**Coll** does not require that Λ be an action operad, only that one has a natural number-indexed set of groups. Given any such collection of groups $\{\Lambda(n)\}_{n \in \mathbb{N}}$, we can form the category $B\Lambda$ whose objects are natural numbers and whose hom-sets are given by $B\Lambda(m, n) = \emptyset$ if $m \neq n$ and $B\Lambda(n, n) = \Lambda(n)$ (where composition and units are given by group multiplication and identity elements, respectively). Then Λ-**Coll** is the presheaf category

$$\widehat{B\Lambda} = [B\Lambda^{\mathrm{op}}, \mathbf{Sets}],$$

with the opposite category arising from our choice of right actions. A key step in explaining how Λ-operads arise as monoids in the category of Λ-collections is to show that being an action operad endows $B\Lambda$ with a monoidal structure.

Definition 12.4. Let Λ be an action operad, and let X, Y be Λ-collections. We define the Λ-collection $X \circ Y$ to be

$$X \circ Y(n) = \left(\left(\coprod_{k_1 + \cdots + k_r = n} X(r) \times Y(k_1) \times \cdots \times Y(k_r) \right) \times \Lambda(n) \right) / \sim$$

where the equivalence relation is generated by

$$(xh; y_1, \ldots, y_r; g) \sim \left(x; y_{h^{-1}(1)}, \ldots, y_{h^{-1}(r)}; \delta(h)g \right),$$
$$(xe; y_1 g_1, \ldots, y_r g_r; g) \sim (x; y_1, \ldots, y_r; \beta(g_1, \ldots, g_r)g).$$

For the first relation above, we must have that the lefthand side is an element of

$$X(r) \times Y(k_1) \times \cdots \times Y(k_r) \times \Lambda(n)$$

while the righthand side is an element of

$$X(r) \times Y\left(k_{h^{-1}(1)}\right) \times \cdots \times Y\left(k_{h^{-1}(r)}\right) \times \Lambda(n);$$

for the second relation, we must have $x \in X(r)$, $y_i \in Y(k_i)$, $f \in \Lambda(r)$, $g_i \in \Lambda(k_i)$, and $g \in \Lambda(n)$. The right $\Lambda(n)$-action on $X \circ Y(n)$ is given by multiplication on the final coordinate.

We will now develop the tools to prove that the category Λ-**Coll** has a monoidal structure given by $\circ$, and that operads are the monoids therein.

Theorem 12.5. *Let Λ be an action operad.*

1. *The category Λ-**Coll** has a monoidal structure with tensor product given by $\circ$ and unit given by the collection I with $I(n) = \emptyset$ when $n \neq 1$, and $I(1) = \Lambda(1)$ with the Λ-action given by multiplication on the right.*

2. *The category $\mathbf{Mon}(\Lambda\text{-}\mathbf{Coll})$ of monoids in Λ-**Coll** is equivalent to the category of Λ-operads.*

While this theorem can be proven by direct calculation using the equivalence relation given above, such a proof is unenlightening. Furthermore, we want to consider Λ-operads in categories other than sets, so an element-wise proof might not apply. Instead we now develop some general machinery that will apply to Λ-operads in any cocomplete symmetric monoidal category in which each of the functors $X \otimes -, - \otimes X$ preserve colimits (as is the case if the monoidal structure is closed). This theory also demonstrates the importance of the final axiom in the definition of an action operad. Our construction of the monoidal structure on the category of Λ-collections will require the Day convolution product [**day-thesis**]. This is a general construction which produces a monoidal structure on the category of presheaves $[\mathcal{V}^{\mathrm{op}}, \mathbf{Sets}]$ from a monoidal structure on the category $\mathcal{V}$. Since the category of Λ-collections is the presheaf category $[B\Lambda^{\mathrm{op}}, \mathbf{Sets}]$, we need to show that $B\Lambda$ has a monoidal structure.

Proposition 12.6. *The action operad structure of Λ gives $B\Lambda$ a strict monoidal structure.*

Proof. The tensor product on $B\Lambda$ is given by addition on objects, with unit object 0. The only thing to do is define the tensor product on morphisms and check naturality for the associativity and unit isomorphisms, which will both be identities. On morphisms, $+$ must be given by a group homomorphism

$$+\colon \Lambda(n) \times \Lambda(m) \to \Lambda(n + m),$$

and this is given by the formula

$$+(g, h) = \mu(e_2; g, h).$$

We need that $+$ is a group homomorphism, and the second part of **??** shows that it preserves identity elements. The final action operad axiom shows that it also preserves group multiplication since $\pi_2(e_2) = e_2$ (each π_n is a group homomorphism) and therefore

$$\begin{aligned}
(+(g, h)) \cdot \left(+(g', h')\right) &= \mu\left(e_2; g, h\right) \cdot \mu\left(e_2; g', h'\right) \\
&= \mu\left(e_2 e_2; gg', hh'\right) \\
&= +\left(gg', hh'\right).
\end{aligned}$$

We now write $+(g, h)$ as $g + h$.

For naturality of the associator, we must have $(f + g) + h = f + (g + h)$. By the operad axioms for both units and associativity, the lefthand side is given by

$$\mu(e_2; \mu(e_2; f, g), h) = \mu(e_2; \mu(e_2; f, g), \mu(\mathrm{id}; h))$$
$$= \mu(\mu(e_2; e_2, \mathrm{id}); f, g, h),$$

while the righthand side is then

$$\mu(e_2; f, \mu(e_2; g, h)) = \mu(\mu(e_2; \mathrm{id}, e_2); f, g, h).$$

By **??**, both of these are equal to $\mu(e_3; f, g, h)$, proving associativity. Naturality of the unit isomorphisms follows similarly, using e_0. □

Now that $B\Lambda$ has a monoidal structure, there is also a monoidal structure on the category of $B\Lambda$-collections

$$[B\Lambda^{\mathrm{op}}, \mathbf{Sets}] = \widehat{B\Lambda}$$

using Day convolution, denoted $\star$. Given collections X, Y, their convolution product $X \star Y$ is given by the coend formula

$$X \star Y(k) = \int^{m, n \in B\Lambda} X(m) \times Y(n) \times B\Lambda(k, m + n)$$

We refer the reader to [**day-thesis**] for further details. We do note, however, that the n-fold Day convolution product of a presheaf Y with itself is given by the following coend formula.

$$Y^{\star n}(k) = \int^{(k_1, \ldots, k_n) \in B\Lambda^n} Y(k_1) \times \cdots \times Y(k_n) \times B\Lambda(k, k_1 + \cdots + k_n)$$

Computations with Day convolution will necessarily involve heavy use of the calculus of coends, and we refer the unfamiliar reader to [**maclane-catwork**] or [**loregian**]. Our goal is to express the substitution tensor product as a coend just as in [**kelly-op**], and to do that we need one final result about the Day convolution product.

Lemma 12.7. *Let Λ be an action operad, let $Y \in \widehat{B\Lambda}$, and let k be a fixed natural number. Then the assignment*

$$n \mapsto Y^{\star n}(k)$$

can be given the structure of a functor $B\Lambda \to \mathbf{Sets}$.

Proof. Since the convolution product is given by a coend, it is the universal object with maps

$$Y(k_1) \times \cdots \times Y(k_n) \times B\Lambda(k, k_1 + \cdots + k_n) \to Y^{\star n}(k)$$

such that the following diagram commutes for every $g_1 \in \Lambda(k_1), \ldots, g_n \in \Lambda(k_n)$, where $K = k_1 + \cdots + k_n$.

$$
\begin{array}{ccc}
\prod_{i=1}^{n} Y(k_i) \times B\Lambda(k, K) & \xrightarrow{(-\cdot g_1, \cdots, -\cdot g_n) \times 1} & \prod_{i=1}^{n} Y(k_i) \times B\Lambda(k, K) \\
{\scriptstyle 1 \times ((g_1 + \cdots + g_n) \cdot -)} \downarrow & & \downarrow \\
\prod_{i=1}^{n} Y(k_i) \times B\Lambda(k, K) & \longrightarrow & Y^{\star n}(k)
\end{array}
$$

The first map along the top acts using the g_i's, while the first map along the bottom is given by

$$h \mapsto \mu(e_n; g_1, \ldots, g_n) \cdot h$$

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in the final coordinate.

Let $f \in \Lambda(n)$, considered as a morphism $n \to n$ in $B\Lambda$. We induce a map $f \bullet -\colon Y^{\star n}(k) \to Y^{\star n}(k)$ using the collection of maps

$$\prod_{i=1}^{n} Y(k_i) \times B\Lambda(k, k_1 + \cdots + k_n) \to \prod_{i=1}^{n} Y(k_{\pi(f)^{-1}(i)}) \times B\Lambda(k, k_1 + \cdots + k_n)$$

by using the symmetry $\pi(f)$ on the first n factors and left multiplication by the element $\mu(f; e_{k_1}, \ldots, e_{k_n})$ on $B\Lambda(k, k_1 + \cdots + k_n)$. To induce a map between the coends, we must show that these maps commute with the two lefthand maps in the diagram above. For the top map, this is merely functoriality of the product together with naturality of the symmetry. For the bottom map, this is the equation

$$\mu(f; \overline{e}) \cdot \mu(e; g_1, \ldots, g_n) = \mu(e; g_{\pi(f)^{-1}1}, \ldots, g_{\pi(f)^{-1}n}) \cdot \mu(f; \overline{e}).$$

Both of these are equal to $\mu(f; g_1, \ldots, g_n)$ by the action operad axiom. Functoriality is then easy to check using that the maps inducing $(f_1 f_2) \bullet -$ are given by the composite of the maps inducing $f_1 \bullet (f_2 \bullet -)$. $\qquad\square$

We are now ready for the abstract description of the substitution tensor product. The following proposition is easily checked directly using the definition of the coend; in fact, the righthand side below should be taken as the definition of $X \circ Y$ as both sides are really the result of some colimiting process.

Proposition 12.8. *Let $X, Y \in \widehat{B\Lambda}$. Then*

$$(X \circ Y)(k) \cong \int^n X(n) \times Y^{\star n}(k).$$

Finally we are in a position to prove Theorem 12.5. We make heavy use of the following consequence of the Yoneda lemma: given any functor $F \colon B\Lambda \to \mathbf{Sets}$ and a fixed object $a \in B\Lambda$, we have a natural isomorphism

$$\int^{n \in B\Lambda} B\Lambda(n, a) \times F(n) \cong F(a);$$

there is a corresponding result for $F \colon B\Lambda^{\mathrm{op}} \to \mathbf{Sets}$ using representables of the form $B\Lambda(a, n)$ instead.

Proof of Theorem 12.5. First we must show that Λ-**Coll** has a monoidal structure using $\circ$. To prove this, we must give the unit and associativity isomorphisms and then check the monoidal category axioms. First, note that the unit object is given as $I = B\Lambda(-, 1)$. Then for the left unit isomorphism, we find that

$$\begin{aligned} I \circ Y(k) &= \int^n B\Lambda(n, 1) \times Y^{\star n}(k) \\ &\cong Y^{\star 1}(k) \\ &\cong Y(k) \end{aligned}$$

using only the properties of the coend. For the right unit isomorphism, we have that

$$X \circ I(k) = \int^n X(n) \times I^{\star n}(k)$$

$$\cong \int^n X(n) \times \int^{k_1,\ldots,k_n} B\Lambda(k_1, 1) \times \cdots \times B\Lambda(k_n, 1) \times B\Lambda(K)$$

$$\cong \int^n X(n) \times B\Lambda(k, 1 + \cdots + 1)$$

$$= \int^n X(n) \times B\Lambda(k, n)$$

$$\cong X(k)$$

using the same methods and where $K = k_1 + \cdots + k_n$.

For associativity, we compute $(X \circ Y) \circ Z$ and $X \circ (Y \circ Z)$.

$$((X \circ Y) \circ Z)(k) = \int^m X \circ Y(m) \times Z^{\star m}(k)$$

$$= \int^m \left(\int^l X(l) \times Y^{\star l}(m) \right) \times Z^{\star m}(k)$$

$$\cong \int^{m,l} X(l) \times Y^{\star l}(m) \times Z^{\star m}(k)$$

$$\cong \int^l X(l) \times \int^m Y^{\star l}(m) \times Z^{\star m}(k)$$

The first isomorphism is from products distributing over colimits and hence coends, and the second is that fact plus the Fubini Theorem for coends [**maclane-catwork**]. A similar calculation shows

$$(X \circ (Y \circ Z))(k) \cong \int^l X(l) \times (Y \circ Z)^{\star l}(k).$$

Thus the associativity isomorphism will be induced once we construct an isomorphism $\int^m Y^{\star l}(m) \times Z^{\star m} \cong (Y \circ Z)^{\star l}$. We do this by induction, with the $l = 1$ case being the isomorphism $Y^{\star 1} \cong Y$ together with the definition of $\circ$. Assuming true for l, we prove the case for $l + 1$ by the calculations below.

$$(Y \circ Z)^{\star(l+1)} \cong (Y \circ Z) \star (Y \circ Z)^{\star l}$$

$$\cong (Y \circ Z) \star \left(\int^m Y^{\star l}(m) \times Z^{\star m} \right)$$

$$= \left(\int^n Y(n) \times Z^{\star n} \right) \star \left(\int^m Y^{\star l}(m) \times Z^{\star m} \right)$$

$$= \int^{a,b} \left(\int^n Y(n) \times Z^{\star n}(a) \right) \times \left(\int^m Y^{\star l}(m) \times Z^{\star m}(b) \right) \times B\Lambda(-, a + b)$$

$$\cong \int^{a,b,n,m} Y(n) \times Y^{\star l}(m) \times Z^{\star n}(a) \times Z^{\star m}(b) \times B\Lambda(-, a + b)$$

$$\cong \int^{n,m} Y(n) \times Y^{\star l}(m) \times Z^{\star(n+m)}$$

$$\cong \int^j \int^{n,m} Y(n) \times Y^{\star l}(m) \times B\Lambda(j, n + m) \times Z^{\star j}$$

$$\cong \int^j Y^{\star(l+1)}(j) \times Z^{\star j}$$

Each isomorphism above arises from the symmetric monoidal structure on **Sets** using products, the monoidal structure on presheaves using $\star$, the properties of the coend, or the fact that products distribute over colimits.

For the monoidal category axioms on $\widehat{B\Lambda}$, we only need to note that the unit and associativity isomorphisms arise, using the universal properties of the coend, from the unit and associativity isomorphisms on the category of sets together with the interaction between products and colimits. Hence the monoidal category axioms follow by those same axioms in **Sets** together with the universal property of the coend.

Now we must show that monoids in $(\widehat{B\Lambda}, \circ)$ are operads. By the Yoneda lemma, a map of Λ-collections $\eta \colon I \to X$ corresponds to an element id $\in X(1)$ since $I = B\Lambda(-, 1)$. A map $\mu \colon (X \circ X) \to X$ is given by a collection of $\Lambda(k)$-equivariant maps $X \circ X(k) \to X(k)$. By the universal property of the coend, this is equivalent to giving maps

$$\mu_{n,K} \colon X(n) \times X(k_1) \times \cdots \times X(k_n) \times B\Lambda(k, k_1 + \cdots + k_n) \to X(k)$$

which are compatible with the actions of $\Lambda(k)$ (using the hom-set in the source, and the standard right action in the target) as well as each of $\Lambda(n), \Lambda(k_1), \ldots, \Lambda(k_n)$. The hom-set in $B\Lambda$ is nonempty precisely when $k = k_1 + \cdots + k_n$, so we define the operad multiplication μ for X to be

$$\mu(x; y_1, \ldots, y_n) = \mu_{n,K}(x; y_1, \ldots, y_n; e_k).$$

Compatibility with the actions of the $\Lambda(n), \Lambda(k_1), \ldots, \Lambda(k_n)$ give the equivariance axioms, and the unit and associativity for the monoid structure give the unit and associativity axioms for the operad structure. Finally, it is easy to check that a map of monoids is nothing more than an operad map which is appropriately equivariant for each n. $\qquad\square$

Remark 12.9. The above result can be interpreted for Λ-operads in an arbitrary cocomplete symmetric monoidal category $\mathcal{V}$ in which the tensor product distributes over colimits in each variable. In order to do so, the following changes must be made. First, cartesian products of objects $X(k)$ must be replaced by the tensor product in $\mathcal{V}$ of the same objects. Second, any product with a hom-set from $B\Lambda$ must be replaced by a copower with the same set (recall that the copower of a set S with an object X is given by the formula $S \odot X = \coprod_S X$). The same changes also allow one to interpret the results in the following chapter about algebras in such a category, unless noted otherwise.

Part III: Operads in Categories

13 Background: 2-Monads and their Algebras

Definition 13.1. Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A *T-pseudoalgebra* consists of an object X, a 1-cell $\alpha\colon TX \to X$, and invertible 2-cells

$$
\begin{array}{ccc}
T^2X & \xrightarrow{T\alpha} & TX \\
{\scriptstyle \mu_X}\downarrow & \Downarrow\Phi & \downarrow{\scriptstyle \alpha} \\
TX & \xrightarrow{\alpha} & X
\end{array}
\qquad\qquad
\begin{array}{c}
X \xrightarrow{\ 1_X\ } \\
{\scriptstyle \eta_X}\searrow \quad \Downarrow\Phi_\eta \\
TX \xrightarrow{\alpha} X
\end{array}
$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

- The following pasting diagram is an identity.

Definition 13.2. Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. A *strict T-algebra* consists of a pseudoalgebra in which all of the isomorphisms Φ are identities.

Definition 13.3. Let T be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$, $(Y, \beta, \Psi, \Psi_\eta)$ be T-pseudoalgebras. A *pseudomorphism* $(f, \bar{f})$ between these pseudoalgebras consists of a 1-cell $f\colon X \to Y$ along with an invertible 2-cell

$$
\begin{array}{ccc}
TX & \xrightarrow{Tf} & TY \\
{\scriptstyle \alpha}\downarrow & \Downarrow\bar{f} & \downarrow{\scriptstyle \beta} \\
X & \xrightarrow{f} & Y
\end{array}
$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

- The following equality of pasting diagrams holds.

Definition 13.4. Let T be a 2-monad and let $(X, \alpha, \Phi, \Phi_\eta)$ and $(Y, \beta, \Psi, \Psi_\eta)$ be T-pseudoalgebras. A *strict morphism* $(f, \bar{f})$ consists of a pseudomorphism in which $\bar{f}$ is an identity.

Remark 13.5. Once again, the strict algebras and strict morphisms are exactly the same as algebras and morphisms for the underlying monad on the underlying category of $\mathcal{K}$.

Definition 13.6. Let $(f, \overline{f}), (g, \overline{g}) \colon X \to Y$ be pseudomorphisms of T-algebras. A T-*transformation* consists of a 2-cell $\gamma \colon f \Rightarrow g$ such that the following equality of pasting diagrams holds.

Once again, we have 2-categories defined using the different kinds of cells.

Definition 13.7. Let T be a 2-monad.

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- The 2-category $T\text{-}\mathbf{Alg}_s$ consists of strict T-algebras, strict morphisms, and T-transformations.

- The 2-category $\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$ consists of T-pseudoalgebras, pseudomorphisms, and T-transformations.

14 Λ-Operads in Cat as 2-Monads

Definition 14.1. Let P be a Λ-operad. A *pseudoalgebra* for P consists of:

- a category X,

- a family of functors
$$\left(\alpha_n \colon P(n) \otimes_{\Lambda(n)} X^n \to X\right)_{n \in \mathbb{N}},$$

- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism $\phi_{k_1,\ldots,k_n}$ (corresponding, via Conventions 10.3) to a natural isomorphism

$$
\begin{array}{ccc}
P_n \times \prod_{i=1}^{n}\left(P_{k_i} \times X^{k_i}\right) & \xrightarrow{\ 1 \times \prod \tilde{\alpha}_{k_i}\ } & P_n \times X^n \\
\downarrow & & \\
P_n \times \prod_{i=1}^{n} P_{k_i} \times X^{\Sigma k_i} & \Downarrow \tilde{\phi}_{k_1,\ldots,k_n} & \tilde{\alpha}_n \\
\mu^P \times 1 \downarrow & & \downarrow \\
P_{\Sigma k_i} \times X^{\Sigma k_i} & \xrightarrow[\ \tilde{\alpha}_{\Sigma k_i}\ ]{} & X
\end{array}
$$

- and a natural isomorphism ϕ_η corresponding to a natural isomorphism

$$
\begin{array}{c}
X \\
\cong \downarrow \quad \searrow^{1} \\
1 \times X \qquad \Downarrow \tilde{\phi}_\eta \\
\eta^P \times 1 \downarrow \\
P(1) \times X \xrightarrow[\ \tilde{\alpha}_1\ ]{} X
\end{array}
$$

satisfying the following axioms.

- For all $n, k_i, m_{ij} \in \mathbb{N}$, the following equality of pasting diagrams holds.

$$P_n \times \prod_i \left(P_{k_i} \times \prod_j \left(P_{m_{ij}} \times X^{m_{ij}}\right)\right) \xrightarrow{1 \times \prod \left(1 \times \prod \tilde{\alpha}_{m_{ij}}\right)} P_n \times \prod_i \left(P_{k_i} \times X^{k_i}\right)$$

$$\mu^P \times 1 \qquad \Downarrow 1 \times \prod_i \tilde{\phi}_{m_{i1},\ldots,m_{ik_i}} \qquad 1 \times \prod \tilde{\alpha}_{k_i}$$

$$P_n \times \prod_i \left(P_{\Sigma m_{ij}} \times X^{\Sigma m_{ij}}\right) \xrightarrow{1 \times \prod \tilde{\alpha}_{\Sigma m_{ij}}} P_n \times X^n$$

$$P_{\Sigma k_i} \times \prod_i \prod_j \left(P_{m_{ij}} \times X^{m_{ij}}\right)$$

$$\mu^P \times 1 \qquad \mu^P \times 1 \qquad \Downarrow \tilde{\phi}_{\Sigma m_{1j},\ldots,\Sigma m_{nj}} \qquad \tilde{\alpha}_n$$

$$P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \xrightarrow{\tilde{\alpha}_{\Sigma\Sigma m_{ij}}} X$$

$$\parallel$$

$$P_n \times \prod_i \left(P_{k_i} \times \prod_j \left(P_{m_{ij}} \times X^{m_{ij}}\right)\right) \xrightarrow{1 \times \prod \left(1 \times \prod \tilde{\alpha}_{m_{ij}}\right)} P_n \times \prod_i \left(P_{k_i} \times X^{k_i}\right)$$

$$\mu^P \times 1 \qquad \mu^P \times 1 \qquad 1 \times \prod \tilde{\alpha}_{k_i}$$

$$P_{\Sigma k_i} \times \prod_i \prod_j \left(P_{m_{ij}} \times X^{m_{ij}}\right) \xrightarrow{1 \times \prod \prod \tilde{\alpha}_{m_{ij}}} P_{\Sigma k_i} \times X^{\Sigma k_i} \qquad P_n \times X^n$$

$$\Downarrow \tilde{\phi}_{k_1,\ldots,k_n}$$

$$\mu^P \times 1 \qquad \Downarrow \tilde{\phi}_{m_{11},\ldots,m_{nk_n}} \qquad \tilde{\alpha}_{\Sigma k_i} \qquad \tilde{\alpha}_n$$

$$P_{\Sigma\Sigma m_{ij}} \times X^{\Sigma\Sigma m_{ij}} \xrightarrow{\tilde{\alpha}_{\Sigma\Sigma m_{ij}}} X$$

- Each pasting diagram of the following form is an identity.

$$P_n \times X^n \xrightarrow{1}$$

$$\cong \qquad \Downarrow 1 \times \tilde{\phi}^n_\eta$$

$$P_n \times (1 \times X)^n$$

$$1 \times \left(\eta^P \times 1\right)^n$$

$$P_n \times (P_1 \times X)^n \xrightarrow{1 \times \tilde{\alpha}^n_1} P_n \times X^n$$

$$\cong \Big\downarrow$$

$$1 \qquad P_n \times P_1^n \times X^n \qquad \Downarrow \tilde{\phi}_{1,\ldots,1} \qquad \tilde{\alpha}_n$$

$$\mu^P \times 1 \Big\downarrow$$

$$P_n \times X^n \xrightarrow{\tilde{\alpha}_n} X$$

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Remark 14.2. The requirement in Definition 14.1 of a natural isomorphism φ_η is to induce a natural isomorphism $\tilde{\varphi}_\eta$. This requirement is really of a natural isomorphism

$$
\begin{array}{ccc}
1 \times_{\Lambda(1)} X & & \\
& \searrow & \\
\eta^P \times_{\Lambda(1)} 1 \downarrow & \Swarrow_{\varphi_\eta} & \\
& & \searrow \\
P(1) \times_{\Lambda(1)} X & \xrightarrow{\alpha_1} & X
\end{array}
$$

where $1 \times_{\Lambda(1)} X$ is the coequalizer of the trivial right action of $\Lambda(1)$ on 1 and the usual left action of $\Lambda(1)$ on X. This induces a natural isomorphism

$$
\begin{array}{ccc}
1 \times X & & \\
& \searrow & \\
\eta^P \times 1 \downarrow & \Swarrow_{\tilde{\varphi}_\eta} & \\
& & \searrow \\
P(1) \times X & \xrightarrow{\tilde{\alpha}_1} & X
\end{array}
$$

which can be whiskered with the isomorphism $X \to 1 \times X$. We make the convention of referring to this whiskered natural isomorphism as $\tilde{\varphi}_\eta$, since no confusion will arise in practice.

Definition 14.3. Let P be a Λ-operad. A *strict algebra* for P consists of a pseudoalgebra in which all of the isomorphisms ϕ are identities.

Definition 14.4. Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a Λ-operad P. A *pseudomorphism* of P-pseudoalgebras consists of:

- a functor $f \colon X \to Y$
- for each $n \in \mathbb{N}$, a natural isomorphism f_n (corresponding, via Conventions 10.3) to a natural isomorphism

$$
\begin{array}{ccc}
P_n \times X^n & \xrightarrow{\tilde{\alpha}_n} & X \\
{\scriptstyle 1 \times f^n} \downarrow & \Downarrow \bar{f}_n & \downarrow {\scriptstyle f} \\
P_n \times Y^n & \xrightarrow{\tilde{\beta}_n} & Y
\end{array}
$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$
\begin{array}{ccc}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) & \xrightarrow{1 \times \prod(1 \times f^{k_i})} & P_n \times \prod_i (P_{k_i} \times Y^{k_i})
\end{array}
$$

with maps $\mu^P \times 1$, $1 \times \prod \tilde\beta_{k_i}$, $\tilde\psi_{k_1,\dots,k_n}$, $1 \times f^{\Sigma k_i}$, $P_{\Sigma k_i} \times X^{\Sigma k_i}$, $P_{\Sigma k_i} \times Y^{\Sigma k_i}$, $P_n \times Y^n$, $\tilde\alpha_{\Sigma k_i}$, $\overline{f}_n$, $\tilde\beta_{\Sigma k_i}$, $\tilde\beta_n$, $X \xrightarrow{f} Y$

$$\|$$

$$
\begin{array}{ccc}
P_n \times \prod_i (P_{k_i} \times X^{k_i}) & \xrightarrow{1 \times \prod(1 \times f^{k_i})} & P_n \times \prod_i (P_{k_i} \times Y^{k_i})
\end{array}
$$

with maps $\mu^P \times 1$, $1 \times \prod \tilde\alpha_{k_i}$, $1 \times \prod \overline{f}_{k_i}$, $1 \times \prod \tilde\beta_{k_i}$, $P_n \times X^n \xrightarrow{1 \times f^n} P_n \times Y^n$, $\tilde\phi_{k_1,\dots,k_n}$, $P_{\Sigma k_i} \times X^{\Sigma k_i}$, $\tilde\alpha_n$, $\overline{f}_n$, $\tilde\beta_n$, $\tilde\alpha_{\Sigma k_i}$, $X \xrightarrow{f} Y$

- The following equality of pasting diagrams holds.

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\cong \downarrow & & \cong \downarrow \\
1 \times X & \xrightarrow{1 \times f} & 1 \times Y \\
\eta^P \times 1 \downarrow & \eta^P \times 1 \downarrow & \Downarrow \tilde\psi_\eta \quad 1 \\
P_1 \times X & \xrightarrow{1 \times f} & P_1 \times Y \\
\tilde\alpha_1 & \Downarrow \overline{f}_1 & \tilde\beta_1 \\
& X \xrightarrow{f} Y &
\end{array}
\quad = \quad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\cong \downarrow & & \\
1 \times X & & 1 \quad 1 \\
\eta^P \times 1 \downarrow & \Downarrow \tilde\phi_\eta & \\
P_1 \times X & & \\
\tilde\alpha_1 & & \\
& X \xrightarrow{f} Y &
\end{array}
$$

Definition 14.5. Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a Λ-operad P. A *strict morphism* of P-pseudoalgebras consists of a pseudomorphism in which all of the isomorphisms $\overline{f}_n$ are identities.

Remark 14.6. A strict algebra for a Λ-operad P in **Cat** is precisely the same thing as an algebra for P considered as an operad in the *category* of small categories and functors. A strict morphism between strict algebras is then just a map of P-algebras

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in the standard sense. We could also consider the notion of a lax algebra for an operad, or a lax morphism of algebras, simply by considering natural transformations in place of isomorphisms in the definitions.

In Definition 14.4 of a pseudomorphism we did not originally make it clear that the isomorphisms $\overline{f}_n$ should satisfy an equivariance condition. This was highlighted in Remark 2.22 of Rubin's thesis [**rubin-thesis**]. Similarly, this is also explicity stated as Definition 2.23 of [**guillou˙symmetric**], as mentioned in [**guillou˙multiplicative**]. That we don't include an explicit equivariance axiom is due to Conventions 10.3. In Definition 14.4 we require the existence of natural isomorphisms f_n in order to induce corresponding natural isomorphisms $\overline{f}_n$. That the $\overline{f}_n$ are induced by the f_n corresponds to the fact that the $\overline{f}_n$ satisfy an equivariance condition, namely that for $(\sigma, g, x_1, \ldots, x_n) \in P(n) \times \Lambda(n) \times X^n$, we have

$$\left(\overline{f}_n\right)_{(\sigma \cdot g, x_1, \ldots, x_n)} = \left(\overline{f}_n\right)_{\left(\sigma, x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}\right)}.$$

Definition 14.7. Let P be a Λ-operad and let $f, g : (X, \alpha, \phi, \phi_\eta) \to (Y, \beta, \psi, \psi_\eta)$ be pseudomorphisms of P-pseudoalgebras. A P-transformation is then a natural transformation $\gamma : f \Rightarrow g$ such that the following equality of pasting diagrams holds, for all n.

We can form various 2-categories using these cells.

Definition 14.8. Let P be a Λ-operad.

- The 2-category $P\text{-}\mathbf{Alg}_s$ consists of strict P-algebras, strict morphisms, and P-transformations.

- The 2-category $\mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}$ consists of P-pseudoalgebras, pseudomorphisms, and P-transformations.

Our main result in this section is the following, showing that one can consider algebras and higher cells, in either strict or pseudo strength, using either the operadic or 2-monadic incarnation of a Λ-operad P. This extends Proposition 11.2.

Theorem 14.9. Let P be a Λ-operad in $\mathbf{Cat}$.

- There is an isomorphism of 2-categories

$$P\text{-}\mathbf{Alg}_s \cong \underline{P}\text{-}\mathbf{Alg}_s.$$

- There is an isomorphism of 2-categories

$$\mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$$

extending the one above.

Proof. We begin by noting that we suppress the difference between 2-cells Γ and those $\tilde{\Gamma}$ as in Conventions 10.3, implicitly always using 2-cells defined on a coequalizer which are appropriately equivariant with respect to the group actions involved.

A proof of the first statement follows from our proof of the second by inserting identities where appropriate. Thus we begin by constructing a 2-functor $R\colon \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}$. We map a $\underline{P}$-pseudoalgebra $(X, \alpha, \Phi, \Phi_\eta)$ to the following P-pseudoalgebra on the same category X. First we define the functor α_n to be the composite

$$\alpha_n\colon P(n) \otimes_{\Lambda(n)} X^n \lhook\joinrel\longrightarrow \underline{P}(X) \xrightarrow{\ \alpha\ } X.$$

The isomorphisms $\phi_{k_1,\ldots,k_n}$ are defined using Φ as in the following diagram

whilst Φ_η is simply sent to itself, since the composition of α with the composite of the coequalizer and inclusion map from $P(1) \times X$ into $\underline{P}(X)$ is just $\tilde{\alpha}_1$. Checking the axioms here is most easily done on components and it can easily seen that the axioms required of this data to be a P-pseudoalgebra are precisely those that they satisfy by virtue of X being a $\underline{P}$-pseudoalgebra.

For a 1-cell $(f, \overline{f})\colon (X, \alpha) \to (Y, \beta)$, we send f to itself whilst sending $\overline{f}$ to the obvious family of isomorphisms, as follows.

It is easy to check that the above data satisfy the axioms for being a pseudomorphism of P-pseudoalgebras, following from the axioms for $(f, \overline{f})$ being a pseudomorphism of $\underline{P}$-pseudoalgebras. A $\underline{P}$-transformation $\gamma\colon (f, \bar{f}) \Rightarrow (g, \bar{g})$ immediately gives a P-transformation $\bar{\gamma}$ between the families of isomorphisms we previously defined, with the components of $\bar{\gamma}$ being precisely those of γ. It is then easily shown that R is a 2-functor.

For there to be an isomorphism of 2-categories, we require an inverse to R, namely a 2-functor $S\colon \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$. Now assume that $(X, \alpha_n, \phi_{\underline{k}_i}, \phi_\eta)$ is a P-pseudoalgebra. We will give the same object X a $\underline{P}$-pseudoalgebra structure. We can

induce a functor $\alpha \colon \underline{P}(X) \to X$ by using the universal property of the coproduct.

$$P(n) \times X^n \longrightarrow P(n) \otimes_{\Lambda(n)} X^n \lhook\joinrel\longrightarrow \underline{P}(X)$$

with α_n, $\tilde{\alpha}_n$, $\exists!\alpha$ arrows to X.

Of course, this can be induced using either α_n or $\tilde{\alpha}_n$, each giving the same functor α by uniqueness. The components of the isomorphism $\Phi \colon \alpha \circ \underline{P}(\alpha) \Rightarrow \alpha \circ \mu_X$ can be given as follows. Let $|\underline{x}_i|$ denote the number of objects in the list $\underline{x}_i$. Then define the component of Φ at the object

$$[p; [q_1; \underline{x}_1], \ldots, [q_n; \underline{x}_n]]$$

to be the component of $\phi_{|\underline{x}_1|, \ldots, |\underline{x}_n|}$ at the same object. To make this clearer, consider the object $[p; [q_1; x_{11}], [q_2; x_{21}, x_{22}], [q_3; x_{31}]]$. The component of Φ at this object is given by the component of $\phi_{1,2,1}$ at the same object. The isomorphism ϕ_η is again sent to itself.

Now given a 1-cell f with structure 2-cells $\overline{f}_n$ we define a 1-cell $(F, \overline{F})$ with underlying 1-cell f and structure 2-cell $\overline{F}$ with components

$$\overline{F}_{[p; x_1, \ldots, x_n]} := \left(\overline{f}_n\right)_{(p; x_1, \ldots, x_n)}.$$

For example, the component of $\overline{F}$ at the object $[p; x_1, x_2, x_3]$ would be the component of f_3 at the object $(p; x_1, x_2, x_3)$.

The mapping for 2-cells is just the identity as before. These mappings again constitute a 2-functor in the obvious way and from how they are defined it is also clear that this is an inverse to R. $\qquad\square$

Remark 14.10. Another interpretation of pseudoalgebras can be given in terms of pseudomorphisms of operads. Algebras for an operad P can be identified with a morphism of operads $F \colon P \to \mathcal{E}_X$, where $\mathcal{E}_X$ is the endomorphism operad (Proposition 9.12). We can similarly define pseudomorphisms for a **Cat**-enriched Λ-operad and identify pseudoalgebras with pseudomorphisms into the endomorphism operad.

If P, Q are Λ-operads then a *pseudomorphism* of Λ-operads $F \colon P \to Q$ consists of a family of Λ-equivariant functors

$$(F_n \colon P(n) \to Q(n))_{n \in \mathbb{N}}$$

together with isomorphisms instead of the standard algebra axioms. For example, the associativity isomorphism has the following form.

$$
\begin{array}{ccc}
P(n) \times \prod_i P(k_i) & \xrightarrow{F_n \times \prod_i F_{k_i}} & Q(n) \times \prod_i Q(k_i) \\
{\scriptstyle \mu^P} \downarrow & \Downarrow \psi_{k_1, \ldots, k_n} & \downarrow {\scriptstyle \mu^Q} \\
P(\Sigma k_i) & \xrightarrow[F_{\Sigma k_i}]{} & Q(\Sigma k_i)
\end{array}
$$

These isomorphisms are then required to satisfy their own axioms, and these ensure that we have a weak map of 2-monads $\underline{P} \Rightarrow \underline{Q}$. In particular, one can show that a pseudomorphism from P into the endomorphism operad $\mathcal{E}_X$ produces pseudoalgebras for the operad P using the closed structure on **Cat**. While abstractly pleasing, we do not pursue this argument any further here.

15 Coherence

This section will be concerned with characterizing various properties of those 2-monads induced by Λ-operads in **Cat**. We first show that these 2-monads are finitary. Second, we show that the coherence theorem in [**lack-cod**] applies to all such 2-monads and allows us to show that each pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra (and so similarly, by our previous results, to the pseudoalgebras for a Λ-operad P). Both of these results are simple extensions of well-known results about operads. Finally, we give conditions for these 2-monads to be 2-cartesian, describing how they interact with certain limits, namely 2-pullbacks. Operads do not always yield 2-cartesian 2-monads, and giving a complete characterization of when they do is more involved than our results on accessibility or coherence.

For a 2-monad T, the 2-categories **Ps**-T-**Alg** (of pseudoalgebras and weak morphisms) and T-**Alg**$_s$ (of strict algebras and strict morphisms) are of particular interest. The behavior of colimits in both of these 2-categories can often be deduced from properties of T, the most common being that T is finitary. In practice, one thinks of a finitary monad as one in which all operations take finitely many inputs as variables. If T is finitary, then T-**Alg**$_s$ will be cocomplete by standard results given in [**BKP**]. There are additional results of a purely 2-dimensional nature concerning finitary 2-monads, detailed in [**lack-cod**] and extending those in [**BKP**], namely the existence of a left adjoint

$$\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}_s$$

to the forgetful 2-functor which regards a strict algebra as a pseudoalgebra with identity structure isomorphisms.

We begin by showing each associated 2-monad is finitary.

Proposition 15.1. *Let P be a Λ-operad. Then $\underline{P}$ is finitary.*

Proof. To show that $\underline{P}$ is finitary we must show that it preserves filtered colimits or, equivalently, that it preserves directed colimits (see [**ar**]). Consider some directed colimit, $\mathrm{colim}X_i$ say, in **Cat**. Then consider the following sequence of isomorphisms:

$$\underline{P}(\mathrm{colim}X_i) = \coprod_n P(n) \otimes_{\Lambda(n)} (\mathrm{colim}X_i)^n$$

$$\cong \coprod_n P(n) \times_{\Lambda(n)} \mathrm{colim}(X_i^n)$$

$$\cong \coprod_n \mathrm{colim}(P(n) \otimes_{\Lambda(n)} X_i^n)$$

$$\cong \mathrm{colim} \coprod_n P(n) \otimes_{\Lambda(n)} X_i^n$$

$$= \mathrm{colim}\underline{P}(X_i).$$

Since **Cat** is locally finitely presentable then directed colimits commute with finite limits, giving the first isomorphism. The second isomorphism follows from this fact as well as that colimits commute with coequalizers. The third isomorphism is simply coproducts commuting with other colimits. $\qquad\square$

The next part of this section is motivated by the issue of coherence. At its most basic, a coherence theorem is a way of describing when a notion of weaker structure is in some way equivalent to a stricter structure. The prototypical case here is the coherence theorem for monoidal categories. In a monoidal category we require associator isomorphisms

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

for all objects in the category. The coherence theorem tells us that, for any monoidal category M, there exists a strict monoidal category which is equivalent to M. In other words, we can treat the associators in M as identities, and similarly for the unit isomorphisms.

The abstract approach to coherence considers when the pseudoalgebras for a 2-monad T are equivalent to strict T-algebras, with the most comprehensive account appearing in [**lack-cod**]. Lack gives a general theorem which provides sufficient conditions for the existence of a left adjoint to the forgetful 2-functor

$$U \colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps\text{-}}T\text{-}\mathbf{Alg}$$

for which the components of the unit of the adjunction are equivalences. We focus on one version of this general result which has hypotheses that are quite easy to check in practice. First we require that the base 2-category $\mathcal{K}$ has an enhanced factorization system. This is much like an orthogonal factorization system on a 2-category, consisting of two classes of maps $(\mathcal{L}, \mathcal{R})$, satisfying the lifting properties on 1-cells and 2-cells as follows. Given a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & C \\
{\scriptstyle l}\downarrow & & \downarrow{\scriptstyle r} \\
B & \xrightarrow{\ g\ } & D
\end{array}
$$

where $l \in \mathcal{L}$ and $r \in R$, there exists a unique morphism $m \colon B \to C$ such that $rm = g$ and $ml = f$. Similarly, given two commuting squares for which $rf = gl$ and $rf' = f'l$, along with 2-cells $\delta \colon f \Rightarrow f'$ and $\gamma \colon g \Rightarrow g'$ for which $\gamma * 1_l = 1_r * \delta$, there exists a unique 2-cell $\mu \colon m \Rightarrow m'$, where m and m' are induced by the 1-cell lifting property, satisfying $\mu * 1_l = \delta$ and $1_r * \mu = \gamma$. However, there is an additional 2-dimensional property of the factorization system which says that given maps $l \in \mathcal{L}$, $r \in \mathcal{R}$ and an invertible 2-cell $\alpha \colon rf \Rightarrow gl$

$$
\begin{array}{ccccccc}
A & \xrightarrow{\ f\ } & C & & A & \xrightarrow{\ f\ } & C \\
{\scriptstyle l}\downarrow & \Swarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle r} & = & {\scriptstyle l}\downarrow & {\scriptstyle m}\nearrow\ \Swarrow{\scriptstyle \beta} & \downarrow{\scriptstyle r} \\
B & \xrightarrow{\ g\ } & D & & B & \xrightarrow{\ g\ } & D
\end{array}
$$

there exists a unique pair (m, β) where $m \colon B \to C$ is a 1-cell and $\beta \colon rm \Rightarrow g$ is an invertible 2-cell such that $ml = f$ and $\beta * 1_l = \alpha$.

Further conditions require that T preserve $\mathcal{L}$ maps and that whenever $r \in \mathcal{R}$ and $rk \cong 1$, then $kr \cong 1$. In our case we are considering 2-monads on the 2-category $\mathbf{Cat}$, which has the enhanced factorization system where $\mathcal{L}$ consists of bijective-on-objects functors and $\mathcal{R}$ is given by the full and faithful functors. This, along with the 2-dimensional property making it an enhanced factorization system, is described in [**power-gen**]. The last stated condition, involving isomorphisms and maps in $\mathcal{R}$, is then clearly satisfied and so the only thing we need to check in order to satisfy the conditions of the coherence result are that the induced 2-monads $\underline{P}$ preserve bijective-on-objects functors, which follows simply from the fact that the set of objects functor, $\mathrm{Ob} \colon \mathbf{Cat} \to \mathbf{Set}$, preserves colimits, being left adjoint to the indiscrete category functor, $E \colon \mathbf{Set} \to \mathbf{Cat}$, as described in Lemma 10.8.

Proposition 15.2. *For any Λ-operad P, the 2-monad $\underline{P}$ preserves bijective-on-objects functors.*

Corollary 15.3. *Every pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra.*

16 Group Actions and Cartesian 2-Monads

Why do people care??

We finally turn to a discussion of the interaction between operads and pullbacks. The monads arising from a non-symmetric operad are always cartesian, as described in [**leinster**]. The monads that arise from symmetric operads, however, are not always cartesian and so it is useful to be able to characterize exactly when they are. An example of where this fails is the symmetric operad for which the algebras are commutative monoids. In the case of 2-monads we can consider the strict 2-limit analogous to the pullback, the 2-pullback, and characterize when the induced 2-monad from a Λ-operad is 2-cartesian, as we now describe.

Definition 16.1. A 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ is said to be 2-*cartesian* if

- the 2-category $\mathcal{K}$ has 2-pullbacks,
- the functor T preserves 2-pullbacks, and
- the naturality squares for the unit and multiplication of the 2-monad are 2-pullbacks.

It is important to note that the 2-pullback of a diagram is actually the same as the ordinary pullback in **Cat**, see [**kelly-elem**]. Since we will be computing with coequalizers of the form $A \otimes_\Lambda B$ repeatedly, we give the following useful lemma.

Lemma 16.2. *Let G be a group and let A, B be categories for which A has a right action by G and B has a left action by G. An action of G on the product $A \times B$ can then be defined by*

$$(a,b) \cdot g \colon = \left(a \cdot g, g^{-1} \cdot b\right).$$

If this action of G on $A \times B$ is free, then the category $(A \times B)/G$, consisting of the equivalence classes of this action, is isomorphic to the coequalizer $A \otimes_G B$.

Proof. The category $A \times_G B$ is defined as the coequalizer

$$A \times G \times B \mathrel{\substack{\xrightarrow{\lambda} \\[-0.3em] \xrightarrow[\rho]{}}} A \times B \xrightarrow{\;\;\varepsilon\;\;} A \otimes_G B$$

where $\lambda(a,g,b) = (a \cdot g, b)$ and $\rho(a,g,b) = (a, g \cdot b)$. However, the map $A \times B \to (A \times B)/G$, sending (a,b) to the equivalence class $[a,b] = [a \cdot g, g^{-1} \cdot b]$, also coequalizes λ and ρ since

$$[a \cdot g, b] = \left[(a \cdot g) \cdot g^{-1}, g \cdot b\right] = [a, g \cdot b].$$

Given any other category X and a functor $\chi\colon A \times B \to X$ which coequalizes λ and ρ, we define a functor $\phi\colon (A \times B)/G \to X$ by $\phi[a,b] = \chi(a,b)$. That this is well-defined is clear, since

$$\phi \left[a \cdot g, g^{-1} \cdot b\right] = \chi \left(a \cdot g, g^{-1} \cdot b\right) = \chi \left(a \cdot \left(gg^{-1}\right), b\right) = \chi(a,b) = \phi[a,b].$$

This is also unique and so we find that $(A \times B)/G$ satisfies the universal property of the coequalizer. $\qquad\square$

We begin our study of the cartesian property in the context of symmetric operads.

Proposition 16.3. *Let P be a symmetric operad. Then the unit $\eta\colon \mathrm{id} \Rightarrow \underline{P}$ for the associated monad is a cartesian transformation.*

Proof. In order to show that η is cartesian, we must prove that for a functor $f\colon X \to Y$, the pullback of the following diagram is the category X.

$$
\begin{array}{ccc}
& & Y \\
& & \downarrow{\scriptstyle \eta_Y} \\
\coprod P(n) \otimes_{\Sigma_n} X^n & \xrightarrow[\underline{P}(f)]{} & \coprod P(n) \otimes_{\Sigma_n} Y^n
\end{array}
$$

The pullback of this diagram is isomorphic to the coproduct of the pullbacks of diagrams of the following form.

$$
\begin{array}{ccc}
& Y & \\
& \downarrow & \\
P(1) \times X \xrightarrow[1 \times f]{} & P(1) \times Y &
\end{array}
\qquad\qquad
\begin{array}{ccc}
& \emptyset & \\
& \downarrow & \\
P(n) \otimes_{\Sigma_n} X^n \xrightarrow[1 \times f^n]{} & P(n) \otimes_{\Sigma_n} Y^n & \\
& n \neq 1 &
\end{array}
$$

It is easy then to see that X is the pullback of the $n = 1$ cospan, and that the empty category is the pullback of each of the other cospans, making X the pullback of the original diagram and verifying that η is cartesian. $\qquad\square$

Proposition 16.4. *Let P be a symmetric operad. Then the 2-monad $\underline{P}$ preserves pullbacks if and only if Σ_n acts freely on $P(n)$ for all n.*

Proof. Consider the following pullback of discrete categories.

$$
\begin{array}{ccc}
\{(x,y),(x,y'),(x',y),(x',y')\} & \longrightarrow & \{y,y'\} \\
\downarrow & & \downarrow \\
\{x,x'\} & \longrightarrow & \{z\}
\end{array}
$$

Letting $\mathbf{4}$ denote the pullback and similarly writing $\mathbf{2}_X = \{x, x'\}$ and $\mathbf{2}_Y = \{y, y'\}$, the following diagram results as the image of this pullback square under $\underline{P}$.

$$
\begin{array}{ccc}
\coprod P(n) \otimes_{\Sigma_n} \mathbf{4}^n & \longrightarrow & \coprod P(n) \otimes_{\Sigma_n} \mathbf{2}_Y^n \\
\downarrow & & \downarrow \\
\coprod P(n) \otimes_{\Sigma_n} \mathbf{2}_X^n & \longrightarrow & \coprod P(n)/\Sigma_n
\end{array}
$$

The projection map $\underline{P}(\mathbf{4}) \to \underline{P}(\mathbf{2}_Y)$ maps an element

$$[p; (x_1, y_1), \ldots, (x_n, y_n)]$$

to the element

$$[p; y_1, \ldots, y_n]$$

and likewise for the projection to $\underline{P}(\mathbf{2}_X)$.

Now assume that, for some n, the action of Σ_n on $P(n)$ is not free. Then find some $p \in P(n)$ along with a nonidentity element $g \in \Sigma_n$ such that $p \cdot g = p$. We will show that the existence of g proves that $\underline{P}$ is not cartesian.

Now $g \neq e$, so there exists an i such that $g(i) \neq i$; without loss of generality, we may take $i = 1$. Using this g we can find two distinct elements

$$\big[p; (x', y), (x, y), \ldots, (x, y), (x, y'), (x, y), \ldots, (x, y)\big]$$

and

$$\big[p; (x, y), \ldots, (x, y), (x', y'), (x, y), \ldots, (x, y)\big]$$

in $\underline{P}(\mathbf{4})$. In the first element we put (x', y) in the first position and (x, y') in position $g(1)$, whilst in the second element we put (x', y') in position $g(1)$. Both of these elements, however, are mapped to the same elements in $\underline{P}(\mathbf{2}_X)$, since

$$\begin{aligned}
\big[p; x', x, \ldots, x\big] &= \big[p \cdot g; (x', x, \ldots, x)\big] \\
&= \big[p; g \cdot (x', x, \ldots, x)\big] \\
&= \big[p; x, x, \ldots, x', \ldots, x\big].
\end{aligned}$$

Similarly, both of the elements are mapped to the same element in $\underline{P}(\mathbf{2}_Y)$, simply

$$\big[p; y, \ldots, y', \ldots, y\big].$$

The pullback of this diagram, however, has a unique element which is projected to the ones we have considered, so $\underline{P}(\mathbf{4})$ is not a pullback. Hence $\underline{P}$ does not preserve pullbacks if for some n the action of Σ_n on $P(n)$ is not free.

Now assume that each Σ_n acts freely on $P(n)$. Given a pullback

$$\begin{array}{ccc}
A & \xrightarrow{\ F\ } & B \\
{\scriptstyle R}\downarrow & & \downarrow{\scriptstyle S} \\
C & \xrightarrow[\ H\ ]{} & D
\end{array}$$

we must show that the image of the diagram under $\underline{P}$ is also a pullback. Now this will be true if and only if each individual diagram

$$\begin{array}{ccc}
P(n) \otimes_{\Sigma_n} A^n & \xrightarrow{\ 1 \times F^n\ } & P(n) \otimes_{\Sigma_n} B^n \\
{\scriptstyle 1 \times R^n}\downarrow & & \downarrow{\scriptstyle 1 \times S^n} \\
P(n) \otimes_{\Sigma_n} C^n & \xrightarrow[\ 1 \times H^n\ ]{} & P(n) \otimes_{\Sigma_n} D^n
\end{array}$$

is also a pullback. The pullback of the functors $1 \times H^n$ and $1 \times S^n$ is a category consisting of pairs of objects $[p; \underline{c}]$ and $[q; \underline{b}]$, where $\underline{b}$ and $\underline{c}$ represent lists of elements in B and C, respectively. These pairs are then required to satisfy the property that

$$\Big[p; \underline{H(c)}\Big] = \Big[q; \underline{S(b)}\Big].$$

Using Lemma 16.2, we know that a pair

$$\big([p; \underline{c}]\, , [q; \underline{b}]\big)$$

is in the pullback if and only if there exists an element $g \in \Sigma_n$ such that $p \cdot g = q$ and $Hc_i = (Sb_{g^{-1}(i)})$. Using this we can define mutual inverses between $P(n) \otimes_{\Sigma_n} A^n$ and the pullback Q'. Considering the category A as the pullback of the diagram we started with, we can consider objects of $P(n) \otimes_{\Sigma_n} A^n$ as being equivalence classes

$$[p; (b_1, c_1), \ldots, (b_n, c_n)]$$

where $p \in P(n)$ and $Hc_i = Sb_i$ for all i.

Taking such an object, we send it to the pair

$$([p; c_1, \ldots, c_n], [p; b_1, \ldots, b_n])$$

which lies in the pullback since the identity in Σ_n satisfies the condition given earlier. An inverse to this sends a pair of equivalence classes in Q' to the single equivalence class

$$[p; (c_1, b_{g^{-1}(1)}), \ldots, (c_n, b_{g^{-1}(n)})]$$

in $P(n) \otimes_{\Sigma_n} A^n$. If we apply the map into Q' we get the pair

$$([p; c_1, \ldots, c_n], [p; b_{g^{-1}(1)}, \ldots, b_{g^{-1}(n)}])$$

which is equal to the original pair since $p \cdot g = q$; the other composite is trivially an identity. A similar calculation on morphisms finishes the proof that $P(n) \otimes_{\Sigma_n} A^n$ is the pullback as required. $\square$

Proposition 16.5. *Let P be a symmetric operad. If the Σ_n-actions are all free, then the multiplication $\mu \colon \underline{P}^2 \Rightarrow \underline{P}$ of the associated monad is a cartesian transformation.*

Proof. Note that if all of the diagrams

$$
\begin{array}{ccc}
\underline{P}^2(X) & \xrightarrow{\underline{P}^2(!)} & \underline{P}^2(1) \\
{\scriptstyle \mu_X} \downarrow & & \downarrow {\scriptstyle \mu_1} \\
\underline{P}(X) & \xrightarrow[\underline{P}(!)]{} & \underline{P}(1)
\end{array}
$$

are pullbacks then the outside of the diagram

$$
\begin{array}{ccccc}
\underline{P}^2(X) & \xrightarrow{\underline{P}^2(f)} & \underline{P}^2(Y) & \xrightarrow{\underline{P}^2(!)} & \underline{P}^2(1) \\
{\scriptstyle \mu_X} \downarrow & & {\scriptstyle \mu_Y} \downarrow & & \downarrow {\scriptstyle \mu_1} \\
\underline{P}(X) & \xrightarrow[\underline{P}(f)]{} & \underline{P}(Y) & \xrightarrow[\underline{P}(!)]{} & \underline{P}(1)
\end{array}
$$

is also a pullback and so each of the naturality squares for μ must therefore be a pullback. Now we can split up the square above, much like we did for η, and prove that each of the squares below is a pullback.

$$
\begin{array}{ccc}
\coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i) \otimes_{\Sigma_{k_i}} X^{k_i} \right) & \longrightarrow & \coprod P(m) \times_{\Sigma_m} \prod_i \left(P(k_i)/\Sigma_{k_i} \right) \\
\downarrow & & \downarrow \\
P(n) \otimes_{\Sigma_n} X^n & \longrightarrow & P(n)/\Sigma_n
\end{array}
$$

The map along the bottom is the obvious one, sending $[p; x_1, \ldots, x_n]$ simply to the equivalence class $[p]$. Along the right hand side the map is the one corresponding to operadic composition, sending $[q; [p_1], \ldots, [p_m]]$ to $[\mu^P(q; p_1, \ldots, p_n)]$. The pullback of these maps would be the category consisting of pairs

$$([p; x_1, \ldots, x_{\Sigma k_i}], [q; [p_1], \ldots, [p_n]]),$$

where $q \in P(n)$, $p_i \in P(k_i)$, $p \in P(\Sigma k_i)$, and for which $[p] = [\mu^P(q; p_1, \ldots, p_n)]$. The upper left category in the diagram, which we will refer to here as Q, has objects

$$[q; [p_1; \underline{x}_1], \ldots, [p_n; \underline{x}_n]] \,.$$

There are obvious maps out of Q making the diagram commute and as such inducing a functor from Q into the pullback via the universal property. This functor sends an object such as the one just described to the pair

$$\left(\left[\mu^P(q; p_1, \ldots, p_n); \underline{x} \right], [q; [p_1], \ldots, [p_n]] \right) .$$

Given an object in the pullback, we then have a pair, as described above, which has $[p] = [\mu^P(q; p_1, \ldots, p_n)]$ meaning that we can find an element $g \in \Sigma_{\Sigma k_i}$ such that $p = \mu^P(q; p_1, \ldots, p_n) \cdot g$. Thus we can describe an inverse to the induced functor by sending a pair in the pullback to the object

$$[q; [p_1; \pi(g)(\underline{x})_1], \ldots, [p_n; \pi(g)(\underline{x})_n]],$$

where $\pi(g)(\underline{x})_i$ denotes the ith block of $\underline{x}$ after applying the permutation $\pi(g)$. For example, if $\underline{x} = (x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31})$ and $\pi(g) = (1 \ 3 \ 5)$, then

$$\pi(g)(\underline{x}) = (x_{23}, x_{12}, x_{11}, x_{22}, x_{21}, x_{31}).$$

Thus $\pi(g)(\underline{x})_1 = (x_{23}, x_{12})$, $\pi(g)(\underline{x})_2 = (x_{11}, x_{22}, x_{21})$ and $\pi(g)(\underline{x})_3 = (x_{31})$.

Now applying the induced functor we find that we get back an object in the pullback for which the first entry is $[q; [p_1], \ldots, [p_n]]$ and whose second entry is

$$\left[\mu^P(q; p_1, \ldots, p_n); \pi(g)(\underline{x}) \right] = \left[\mu^P(q; p_1, \ldots, p_n) \cdot g; \underline{x} \right] = [p; \underline{x}],$$

which is what we started with. Showing the other composite is an identity is similar, here using the fact that the identity acts trivially on $\mu^P(q; p_1, \ldots, p_n)$. Taking the coproduct of these squares then gives us the original diagram that we wanted to show was a pullback and, since each individual square is a pullback, so is the original. $\qquad\square$

Collecting these results together gives the following corollary.

Corollary 16.6. *The 2-monad associated to a symmetric operad P is 2-cartesian if and only if the action of Σ_n is free on each $P(n)$.*

We require one simple technical lemma before giving a complete characterization of Λ-operads which induce cartesian 2-monads.

Lemma 16.7. *Let C be a category with a right action of some group Λ, and let $\pi \colon \Lambda \to \Sigma$ be a group homomorphism to any other group Σ. Then the right Σ-action on $C \otimes_\Lambda \Sigma$ is free if and only if the only elements of Λ which fix an object of C lie in the kernel of π.*

Proof. First, note that a group action on a category is free if and only if it is free on objects as fixing a morphism requires fixing its source and target. Thus our arguments need only concern the objects involved.

Since the set of objects functor preserves colimits, the objects of $C \otimes_\Lambda \Sigma$ are equivalence classes $[c; g]$ where $c \in C$ and $g \in \Sigma$, with $[c \cdot r; g] = [c; \pi(r)g]$. First assume the Σ-action is free. Then noting that $[c; e] \cdot g = [c; g]$, we have if $[c; g] = [c; e]$ then $g = e$. Let $r \in \Lambda$ be an element such that $c \cdot r = c$. Then

$$[c; e] = [c \cdot r; e] = [c; \pi(r)],$$

so $\pi(r) = e$.

Now assume that every element of Λ fixing an object lies in the kernel of π. Let $\tau \in \Sigma$, and assume it fixes $[p; \sigma]$. Without loss of generality, we can take $\sigma = e$, so that

$$[p; \tau] = [p; e] \cdot \tau = [p; e].$$

Since the objects of $C \otimes_\Lambda \Sigma$ are equivalence classes as above, there exists an element $r \in \Lambda$ such that $p \cdot r^{-1} = p$ and $\tau = \pi(r)$. But by assumption, we must have r^{-1}, and hence r, in the kernel, so $\tau = e$ and the Σ-action is free. $\qquad\square$

Theorem 16.8. *The 2-monad associated to a Λ-operad P is 2-cartesian if and only if whenever $p \cdot g = p$ for an object $p \in P(n)$, $g \in Ker\,\pi(n)$.*

Proof. Since the monad $\underline{P}$ is isomorphic to $\underline{S(P)}$, we need only verify when $\underline{S(P)}$ is 2-cartesian. Thus the theorem is a direct consequence of Lemma 16.7 and Corollary 16.6.
$\qquad\square$

17 Pseudocommutativity

This section gives conditions sufficient to equip the 2-monad $\underline{P}$ induced by a Λ-operad P in **Cat** with a pseudo-commutative structure. Such a pseudo-commutativity will then give the 2-category **Ps-$\underline{P}$-Alg** some additional structure that we briefly explain here. For a field k, the category **Vect** of vector spaces over k has many nice features. Of particular interest to us are the following three structures. First, the category **Vect** is monoidal using the tensor product $\otimes_k$. Second, the set of linear maps $V \to W$ is itself a vector space which we denote $[V, W]$. Third, there is a notion of multilinear map $V_1 \times \cdots \times V_n \to W$, with linear maps being the 1-ary version. While these three structures are each useful in isolation, they are tied together by natural isomorphisms

$$\mathbf{Vect}(V_1 \otimes V_2, W) \cong \mathbf{Vect}(V_1, [V_2, W]) \cong \mathbf{Bilin}(V_1 \times V_2, W)$$

expressing that $\otimes$ gives a closed monoidal structure which represents the multicategory of multilinear maps. Moreover, the adjunction between **Vect** and **Sets** respects all of this structure in the appropriate way. This incredibly rich interplay between the tensor product, the internal mapping space, and the multicategory of multilinear maps all arises from the free vector space monad on **Sets** being a *commutative* monad [**kock-monads**, **kock-closed**, **kock-strong**]. The notion of a pseudo-commutative 2-monad [**HP**] is then a generalization of this machinery to a 2-categorical context, and can be viewed as a starting point for importing tools from linear algebra into category theory.

The aim of this section is to give conditions that ensure that the 2-monad $\underline{P}$ associated to a Λ-operad P has a pseudo-commutative structure. We give the definition of pseudo-commutativity as in [**HP**] but before doing so we require the definition of a strength for a 2-monad.

Definition 17.1. A *strength* for an endo-2-functor $T \colon \mathcal{K} \to \mathcal{K}$ on a 2-category with products and terminal object 1 consists of a 2-natural transformation d with components

$$d_{A,B} \colon A \times TB \to T(A \times B)$$

satisfying the following unit and associativity axioms [**kock-monads**].

$$
\begin{array}{ccc}
(A \times B) \times TC & \xrightarrow{\quad d_{AB,C} \quad} & T((A \times B) \times C) \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle Ta} \\
A \times (B \times TC) \xrightarrow{1 \times d_{B,C}} A \times T(B \times C) \xrightarrow{d_{A,BC}} & & T(A \times (B \times C))
\end{array}
$$

$$
\begin{array}{ccc}
A \times T^2 B \xrightarrow{\ d_{A,TB}\ } T(A \times TB) \xrightarrow{\ Td_{A,B}\ } & & T^2(A \times B) \\
{\scriptstyle 1 \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
A \times TB & \xrightarrow{\qquad d_{A,B} \qquad} & T(A \times B)
\end{array}
$$

Similarly, a *costrength* for T consists of a 2-natural transformation d^* with components

$$
d^*_{A,B} \colon TA \times B \to T(A \times B)
$$

again satisfying unit and associativity axioms.

The strength and costrength for the associated 2-monad $\underline{P}$ are quite simple to define. We define the strength d for $\underline{P}$ as follows. The component $d_{A,B}$ is a functor

$$
d_{A,B} \colon A \times \left(\amalg P(n) \times_{\Lambda(n)} B^n \right) \to \amalg P(n) \times_{\Lambda(n)} (A \times B)^n
$$

which sends an object $(a, [p; b_1, \ldots, b_n])$ to the object $[p; (a, b_1), \ldots, (a, b_n)]$. We also define the costrength similarly, sending an object $([p; a_1, \ldots, a_n], b)$ to the object which is an equivalence class $[p; (a_1, b), \ldots, (a_n, b)]$. Both the strength and the costrength are defined in the obvious way on morphisms.

Remark 17.2. It is crucial to note that the strength d and the costrength d^* do not depend on the Λ-actions in the following sense. The Λ-operad P has an underlying non-symmetric operad that we also denote P, and it has a strength

$$
d_{A,B} \colon A \times (\amalg P(n) \times B^n) \to \amalg P(n) \times (A \times B)^n
$$

given by essentially the same formula:

$$
(a; (p; b_1, \ldots, b_n)) \mapsto (p; (a, b_1), \ldots, (a, b_n)) .
$$

The strength for the Λ-equivariant P is just the induced functor between coequalizers.

Definition 17.3. Given a 2-monad $T \colon \mathcal{K} \to \mathcal{K}$ with strength d and costrength d^*, a *pseudo-commutativity* consists of an invertible modification γ with components

$$
\begin{array}{ccc}
TA \times TB \xrightarrow{\ d^*_{A,TB}\ } T(A \times TB) \xrightarrow{\ Td_{A,B}\ } & & T^2(A \times B) \\
{\scriptstyle d_{TA,B}}\downarrow & \Downarrow{\scriptstyle \gamma_{A,B}} & \downarrow{\scriptstyle \mu_{A\times B}} \\
T(TA \times B) \xrightarrow{Td^*_{A,B}} T^2(A \times B) \xrightarrow{\mu_{A\times B}} & & T(A \times B)
\end{array}
$$

satisfying the following three strength axioms, two unit (or η) axioms, and two multiplication (or μ) axioms for all A, B, and C.

1. $\gamma_{A \times B, C} * (d_{A,B} \times 1_{TC}) = d_{A, B \times C} * (1_A \times \gamma_{B,C})$.
2. $\gamma_{A, B \times C} * (1_{TA} \times d_{B,C}) = \gamma_{A \times B, C} * (d^*_{A,B} \times 1_{TC})$.

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3. $\gamma_{A,B\times C} * (1_{TA} \times d^*_{B,C}) = d^*_{A\times B,C} * (\gamma_{A,B} \times 1_C)$.

4. $\gamma_{A,B} * (\eta_A \times 1_{TB})$ is the identity on d.

5. $\gamma_{A,B} * (1_{TA} \times \eta_B)$ is the identity on d^*.

6. $\gamma_{A,B} * (\mu_A \times 1_{TB})$ is equal to the pasting below.

$$
\begin{array}{ccccccc}
T^2A\times TB & \xrightarrow{d^*_{TA,TB}} & T(TA\times TB) & \xrightarrow{Td^*_{A,TB}} & T^2(A\times TB) & \xrightarrow{T^2 d_{A,B}} & T^3(A\times B) \\
\downarrow{\scriptstyle d_{T^2A,B}} & & \downarrow{\scriptstyle Td_{TA,B}} & & \Downarrow{\scriptstyle T\gamma_{A,B}} & & \downarrow{\scriptstyle T\mu_{A\times B}} \\
T(T^2A\times B) & \Downarrow{\scriptstyle \gamma_{TA,B}} & T^2(TA\times B) & \xrightarrow{T^2 d^*_{A,B}} & T^3(A\times B) & \xrightarrow{T\mu_{A\times B}} & T^2(A\times B) \\
\downarrow{\scriptstyle Td^*_{TA,B}} & & \downarrow{\scriptstyle \mu_{TA\times B}} & & \downarrow{\scriptstyle \mu_{T(A\times B)}} & & \downarrow{\scriptstyle \mu_{A\times B}} \\
T^2(TA\times B) & \xrightarrow{\mu_{TA\times B}} & T(TA\times B) & \xrightarrow{Td^*_{A,B}} & T^2(A\times B) & \xrightarrow{\mu_{A\times B}} & T(A\times B)
\end{array}
$$

7. $\gamma_{A,B} * (1_{TA} \times \mu_B)$ is equal to the pasting below.

$$
\begin{array}{ccccc}
TA\times T^2B & \xrightarrow{d^*_{A,T^2B}} & T(A\times T^2B) & \xrightarrow{Td_{A,TB}} & T^2(A\times TB) \\
\downarrow{\scriptstyle d_{TA,TB}} & & \Downarrow{\scriptstyle \gamma_{A,TB}} & & \downarrow{\scriptstyle \mu_{A\times TB}} \\
T(TA\times TB) & \xrightarrow{Td^*_{A,TB}} & T^2(A\times TB) & \xrightarrow{\mu_{A\times TB}} & T(A\times TB) \\
\downarrow{\scriptstyle Td_{TA,B}} & & \downarrow{\scriptstyle T^2 d_{A,B}} & & \downarrow{\scriptstyle Td_{A,B}} \\
T^2(TA\times B) & \Downarrow{\scriptstyle T\gamma_{A,B}} & T^3(A\times B) & \xrightarrow{\mu_{T(A\times B)}} & T^2(A\times B) \\
\downarrow{\scriptstyle T^2 d^*_{A,B}} & & \downarrow{\scriptstyle T\mu_{A\times B}} & & \downarrow{\scriptstyle \mu_{A\times B}} \\
T^3(A\times B) & \xrightarrow{T\mu_{A\times B}} & T^2(A\times B) & \xrightarrow{\mu_{A\times B}} & T(A\times B)
\end{array}
$$

Remark 17.4. It is noted in [**HP**] that this definition has some redundancy and therein it is shown that any two of the strength axioms immediately implies the third. Furthermore, the three strength axioms are equivalent when the η and μ axioms hold, as well as the following associativity axiom:

$$\gamma_{A,B\times C} \circ (1_{TA} \times \gamma_{B,C}) = \gamma_{A\times B,C} \times (\gamma_{A,B} \times 1_{TC}).$$

We need some further notation before stating our main theorem. Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two lists. Then the set $\{(a_i, b_j)\}$ has mn elements, and two natural lexicographic orderings. One of these we write as $(\underline{a}, \underline{b})$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} p < r, \text{ or} \\ p = r \text{ and } q < s. \end{cases}$$

The other we write as $\underline{(a, b)}$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} q < s, \text{ or} \\ q = s \text{ and } p < r. \end{cases}$$

The notation $(a, \underline{b})$ is meant to indicate that there is a single a but a list of b's, so then $(\underline{a}, \underline{b})$ would represent a list which itself consists of lists of that form. There exists a

unique permutation $\tau_{m,n} \in \Sigma_{mn}$ which has the property that $\tau_{m,n}(i) = j$ if the ith element of the ordered set $(a, \underline{b})$ is equal to the jth element of the ordered set $(\underline{a}, b)$. By construction, we have $\tau_{n,m} = \tau_{m,n}^{-1}$. We illustrate these permutations with a couple of examples.

$$\tau_{2,3} \qquad\qquad\qquad \tau_{4,2}$$

Note then that $\tau_{m,n}$ is the permutation given by taking the transpose of the $m \times n$ matrix with entries (a_i, b_j). This has the effect of rearranging m groups of n things into n groups of m things. E.g., the permutation represented below is $\tau_{5,3}$ and corresponds to taking the transpose of the matrix shown.

$$\begin{bmatrix} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) \\ (a_4, b_1) & (a_4, b_2) & (a_4, b_3) \\ (a_5, b_1) & (a_5, b_2) & (a_5, b_3) \end{bmatrix}^T = \begin{bmatrix} (a_1, b_1) & (a_2, b_1) & (a_3, b_1) & (a_4, b_1) & (a_5, b_1) \\ (a_1, b_2) & (a_2, b_2) & (a_3, b_2) & (a_4, b_2) & (a_5, b_2) \\ (a_1, b_3) & (a_2, b_3) & (a_3, b_3) & (a_4, b_3) & (a_5, b_3) \end{bmatrix}$$
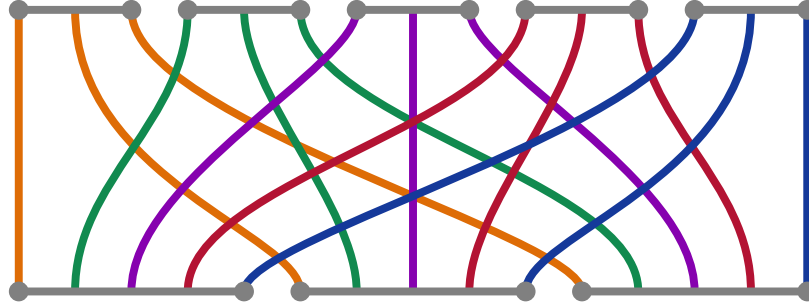
We now define what it means for a Λ-operad to be pseudo-commuative, before then showing that such an operad yields a pseudo-commutative structure on the corresponding 2-monad $\underline{P}$. Let $\mathbb{N}_+$ denote the set of positive integers.

QQQ Rewrite all this using the beta and delta operations, so it's consistent with before.

Definition 17.5. Let P be a Λ-operad. A pseudo-commutative structure on P consists of:

- For each pair $(m, n) \in \mathbb{N}_+^2$, an element $t_{m,n} \in \Lambda(mn)$ such that $\pi(t_{m,n}) = \tau_{m,n}$.
- For each $p \in P(n)$, $q \in P(m)$, a natural isomorphism

$$\lambda_{p,q} \colon \mu(p; q, \ldots, q) \cdot t_{m,n} \cong \mu(q; p, \ldots, p).$$

We write this as $\lambda_{p,q} \colon \mu(p; \underline{q}) \cdot t_{m,n} \cong \mu(q; \underline{p})$.

These are required to satisfy the following axioms:

1. For all $n \in \mathbb{N}_+$,
$$t_{1,n} = e_n = t_{n,1}$$
and for all $p \in P(n)$, the isomorphism $\lambda_{p,\mathrm{id}} \colon p \cdot e_n \cong p$ is the identity map.

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2. For all $l, m_1, \ldots, m_l, n \in \mathbb{N}_+$, with $M = \Sigma m_i$,

$$\mu^\Lambda \left(e_l; t_{m_1,n}, \ldots, t_{m_l,n}\right) \cdot \mu^\Lambda \left(t_{l,n}; \underline{e_{m_1}, \ldots, e_{m_l}}\right) = t_{n,M}.$$

QQQ New version:

$$\beta(t_{n,m_1}, \ldots, t_{n,m_l}) \cdot \delta_{\underline{m_i}, \ldots, \underline{m_i}}(t_{n,l}) = t_{n,M}.$$

Here $\underline{e_{m_1}, \ldots, e_{m_l}}$ is the list $e_{m_1}, \ldots, e_{m_l}$ repeated n times.

3. For all $l, m, n_1, \ldots, n_m \in \mathbb{N}_+$, with $N = \Sigma n_i$,

$$\mu^\Lambda \left(t_{m,l}; \underline{e_{n_1}}, \ldots, \underline{e_{n_m}}\right) \cdot \mu^\Lambda \left(e_m; t_{n_1,l}, \ldots, t_{n_m,l}\right) = t_{N,l}.$$

QQQ New version

$$\delta_{\underline{n_1}, \ldots, \underline{n_m}}(t_{m,l}) \cdot \beta(t_{n_1,l}, \ldots, t_{n_m,l}) = t_{N,l}.$$

Here $\underline{e_{n_i}}$ indicates that each e_{n_i} is repeated l times.

4. For any $l, m_i, n \in \mathbb{N}_+$, with $1 \le i \le n$, and $p \in P(l)$, $q_i \in P(m_i)$ and $r \in P(n)$, the following diagram commutes. (Note that we maintain the convention that anything underlined indicates a list, and double underlining indicates a list of lists. Each instance should have an obvious meaning from context and the equations appearing above.)

5. For any $l, m, n_i \in \mathbb{N}_+$, with $1 \le i \le m$, and $p \in P(l)$, $q \in P(m)$ and $r_i \in P(n_i)$,

the following diagram commutes.

$$\mu\left(\mu(p;\underline{q})\cdot t_{m,l};\underline{r_i}\right)\cdot\mu(e_m;\underline{t_{n_i,l}}) =\!=\!=\!= \mu\left(\mu(p;\underline{q});\underline{r_i}\right)\cdot\mu(t_{m,l};\underline{e_{n_i}})\mu(e_m;\underline{t_{n_i,l}})$$

$$\mu(\lambda_{p,q};1)\cdot 1 \Big\downarrow$$

$$\mu\left(\mu(q;\underline{p});\underline{r_1},\ldots,\underline{r_m}\right)\cdot\mu(e_m;\underline{t_{n_i,l}})$$

$$\mu\left(p;\underline{\mu(q;\underline{r_i})}\right)\cdot\mu(t_{m,l};\underline{e_{n_i}})\mu(e_m;\underline{t_{n_i,l}})$$

$$\mu\left(p;\underline{\mu(q;r_1,\ldots,r_m)}\right)\cdot t_{N,l}$$

$$\Big\downarrow \lambda_{p,\mu(q;r_1,\ldots,r_m)}$$

$$\mu\left(q;\underline{\mu(p;r_i)}\right)\cdot\mu(e_m;\underline{t_{n_i,l}})$$

$$\mu\left(\mu(q;r_1,\ldots,r_m);\underline{p}\right)$$

$$\mu\left(q;\underline{\mu(p;\underline{r_i})\cdot t_{n_i,l}}\right)\xrightarrow[\mu(1;\lambda_{p,r_i})]{}\mu\left(q;\underline{\mu(r_i;\underline{p})}\right)$$

Remark 17.6. Remark 11.2 of [**guillou˙multiplicative**] describes the need for an extra equivariance axiom in the pseudo-commutative structure definition, some detail of which is also described in [**guillou˙symmetric**]. We also believed this to be true until a realisation that the equivariance axiom of [**guillou˙multiplicative**][11.1, Axiom (iii)] becomes the following requirement:

$$\lambda_{p\cdot g,q\cdot h}\circ\left(\mu^P\left(\mathrm{id}_p\cdot g;\underline{\mathrm{id}_q\cdot h}\right)\cdot t_{m,n}\right)=\mu^P\left(\mathrm{id}_q\cdot h;\underline{\mathrm{id}_p\cdot g}\right)\circ\lambda_{p,q}.$$

On closer inspection, this appears to be an instance of naturality for the natural isomorphisms λ, as shown in the naturality square below.

$$
\begin{array}{ccc}
\mu^P\left(p;\underline{q}\right)\cdot t_{m,n} & \xrightarrow{\lambda_{p,q}} & \mu^P\left(q;\underline{p}\right)\\
{\scriptstyle\mu^P\left(\mathrm{id}_p\cdot g;\underline{\mathrm{id}_q\cdot h}\right)\cdot t_{m,n}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mu^P\left(\mathrm{id}_q\cdot h;\underline{\mathrm{id}_p\cdot g}\right)}\\
\mu^P\left(p\cdot g;\underline{q\cdot h}\right)\cdot t_{m,n} & \xrightarrow[\lambda_{p\cdot g,q\cdot h}]{} & \mu^P\left(q\cdot h;\underline{p\cdot g}\right)
\end{array}
$$

Theorem 17.7. *Let P be a Λ-operad equipped with a pseudo-commutative structure. Then $\underline{P}$ has a pseudo-commutativity.*

Proof. We refer to the Axioms in Definition 17.5 throughout. We begin the proof by defining an invertible modification γ for the pseudo-commutativity for which the components are natural transformations $\gamma_{A,B}$. Such a transformation $\gamma_{A,B}$ has components with source

$$\left[\mu\left(p;\underline{q}\right);\underline{(a,\underline{b})}\right]$$

and target

$$\left[\mu\left(q;\underline{p}\right);\underline{(\underline{a},b)}\right].$$

Now $\lambda_{p,q}\colon \mu(p;q,\ldots,q)\cdot t_{m,n}\cong\mu(q;p,\ldots,p)$ gives rise to another map by multiplication on the right by $t_{m,n}^{-1}$,

$$\lambda_{p,q}\cdot t_{m,n}^{-1}\colon \mu(p;q,\ldots,q)\cong\mu(q;p,\ldots,p)\cdot t_{m,n}^{-1},$$

so we define $(\gamma_{A,B})_{[p;a_1,\ldots,a_n],[q;b_1,\ldots,b_m]}$ to be the morphism which is the image of $(\lambda_{p,q} \cdot t_{m,n}^{-1}, 1)$ under the map

$$\coprod P(n) \times (A \times B)^n \to \coprod P(n) \otimes_{\Lambda(n)} A \times B^n.$$

Naturality of $\gamma_{A,B}$ follows from that of each $\lambda_{p,q}$. We will write this morphism as $[\lambda_{p,q}t_{m,n}^{-1}, 1]$. In the case that either p or q is an identity then we choose the component of γ to be the isomorphism involving the appropriate identity element using Axiom 1 above.

There are two things to note about the definition above before we continue. First, it is easy to check that

$$t_{m,n}^{-1} \cdot \underline{(a, \underline{b})} = \underline{(a, b)}$$

since $\pi(t_{m,n}) = \tau_{m,n}$; this ensures that γ has the correct target. Second, the morphism above has second component the identity. This is actually forced upon us by the requirement that γ be a modification: in the case that A, B are discrete categories, the only possible morphism is an identity, and the modification axiom then forces that statement to be true for general A, B by considering the inclusion $A_0 \times B_0 \hookrightarrow A \times B$ where A_0, B_0 are the discrete categories with the same objects as A, B.

We show that this is a modification by noting that it does not rely on objects in the lists $a_1, \ldots, a_n$ or $b_1, \ldots, b_m$, only on their lengths and the operations p and q. As a result, if there are functors $f \colon A \to A'$ and $g \colon B \to B'$, then it is clear that

$$(\underline{P}(f \times g) \circ \gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]} = [\lambda_{p,q}, \underline{1}] = (\gamma_{A',B'} \circ (\underline{P}f \times \underline{P}g))_{[p;\underline{a}],[q;\underline{b}]}.$$

As such we can simply write $(\gamma_{A,B})_{[p;\underline{a}],[q;\underline{b}]}$ in shorthand as $\gamma_{p,q}$.

There are now seven axioms to check for a pseudo-commutativity: three strength axioms, two unit axioms, and two multiplication axioms. For the first strength axiom, we must verify that two different 2-cells of shape

$$A \times TB \times TC \overset{\Downarrow}{\longrightarrow} T(A \times B \times C)$$

are equal. The first of these is γ precomposed with $d \times 1$, and so is the component of γ at an object

$$([p; (a, b_1), \ldots, (a, b_n)], [q; c_1, \ldots, c_m]).$$

The second of these is d applied to the component of $1 \times \gamma$ at

$$(a, ([p; b_1, \ldots, b_n], [q; c_1, \ldots, c_m])).$$

It is straightforward to compute that each of these maps is the image of $\left(\lambda_{p,q} \cdot t_{m,n}^{-1}, 1\right)$ under the functor

$$\coprod P(n) \times (A \times B)^n \to \coprod P(n) \otimes_{\Lambda(n)} A \times B^n.$$

The other two strength axioms follow by analogous calculations for other whiskerings of γ with d or d^*.

For the unit axioms, we must compute the components of γ precomposed with $\eta \times 1$ for the first axiom and $1 \times \eta$ for the second. Thus for the first unit axiom, we must compute the component of γ at $([e; a], [p; b_1, \ldots, b_m])$. By definition, this is the image of $(\lambda_{e,p} \cdot t_{m,1}^{-1}, 1)$ under the map to the coequalizer, and by Axiom 1 of 17.5 know that $t_{m,1}^{-1}$ is the identity element and this isomorphism is the identity as well, so this component of γ is also the identity. The second unit axiom follows similarly, using that $t_{1,n}^{-1}$ is the identity.

For the multiplication axioms, first note that Axiom 2 is necessary in order to ensure the existence of the top horizontal equality in the diagram of Axiom 4 for the pseudo-commutative structure; the same goes for Axioms 3 and 5. We now explain how Axioms 2 and 4 for the pseudo-commutative structure ensure that the first multiplication axiom holds, with the same reasoning showing that Axioms 3 and 5 imply the second multiplication axiom.

We begin by studying the pasting diagram in the first multiplication axiom, but computing its values using the strength and costrength for the non-symmetric operad underlying P; this means that we evaluate on objects of the form $(p; a_1, \ldots, a_n)$ rather than on their equivalence classes. Let $p \in P(l), q_i \in P(m_i)$ for $1 \leq i \leq l$, and $r \in P(n)$. Computing the top and right leg around the pasting diagram gives the function on objects which sends

$$\big((p; (q_1; \underline{a_1}), \ldots, (q_l; \underline{a_l})), (r; \underline{b})\big)$$

to

$$\Big(\mu(p; \mu(q_1; \underline{r}), \ldots, \mu(q_l; \underline{r})); ((\underline{a_{1\bullet}}, \underline{b})), \ldots, ((\underline{a_{l\bullet}}, \underline{b}))\Big),$$

where $((\underline{a_{i\bullet}}, \underline{b}))$ is the list of pairs

$$(a_{i1}, b_1), \ldots, (a_{i1}, b_m), (a_{i2}, b_1), \ldots, (a_{in_i}, b_m).$$

Then $\underline{P}\gamma$ is the image of the morphism which is the identity on the (a_{ij}, b_k)'s, and is the morphism

$$\mu\left(1; \lambda_{q_1, r} t_{n, m_1}^{-1}, \ldots, \lambda_{q_l, r} t_{n, m_l}^{-1}\right)$$

on the first component with domain and codomain shown below.

$$\mu\left(p; \mu\left(q_1; \underline{r}\right), \ldots, \mu\left(q_n; \underline{r}\right)\right) \longrightarrow \mu\left(p; \mu\left(r; \underline{q_1}\right) t_{n, m_1}^{-1}, \ldots, \mu\left(r; \underline{q_l}\right) t_{n, m_l}^{-1}\right)$$

By the Λ-operad axioms, the target of this morphism is equal to

$$\mu\left(p; \mu\left(r; \underline{q_1}\right), \ldots, \mu\left(r; \underline{q_l}\right)\right) \mu\left(e_l; t_{n, m_1}^{-1}, \ldots, t_{n, m_l}^{-1}\right).$$

Note that this is not the same object as one obtains by computing $T\mu \circ T^2 d^* \circ Td \circ d^*$ using the underlying non-symmetric operad of P as we are required to use the Λ-equivariance to ensure that the target of γ is the correct one.

Next we compute the source of $(\mu \circ Td^*) * \gamma$, the other 2-cell in the pasting appearing in the first multiplication axiom. We compute this once again using the strength and costrength for the underlying non-symmetric operad, and note once again that this will not match our previous calculations precisely, but only up to an application of Λ-equivariance. This functor has its map on objects given by

$$\big((p; (q_1; \underline{a_1}), \ldots, (q_l; \underline{a_l})), (r; \underline{b})\big) \mapsto \Big(\mu(\mu(p; \underline{r}); \underline{q_1}, \ldots, \underline{q_l}); (\underline{a_1}, b_\bullet), \ldots, (\underline{a_l}, b_\bullet)\Big).$$

Note that if we apply Λ-equivariance, this matches the target computed above. Once again the component of γ is the image of a morphism which is the identity on the (a_{ij}, b_k)'s, and its first component is

$$\mu\left(\mu(p; \underline{r}); \underline{q_1}, \ldots, \underline{q_l}\right) \xrightarrow{\mu\left(\lambda_{p, r} \cdot t_{n, l}^{-1}; 1, \ldots, 1\right)} \mu\left(\mu(r; \underline{p}) \cdot t_{n, l}^{-1}; \underline{q_1}, \ldots, \underline{q_l}\right).$$

We cannot compose these morphisms in $\coprod P(n) \times (A \times B)^n$ as they do not have matching source and target, but we can in $\coprod P(n) \times_\Lambda (A \times B)^n$. The resulting morphism has first component given by the image of

$$\mu(p; \mu(q_1; \underline{r}), \ldots, \mu(q_n; \underline{r})) \xrightarrow{\mu\left(1; \lambda_{q_1, r} t_{n, m_1}^{-1}, \ldots, \lambda_{q_1, r} t_{n, m_l}^{-1}\right)} \mu\left(p; \mu\left(r; \underline{q_1}\right) t_{n, m_1}^{-1}, \ldots, \mu\left(r; \underline{q_l}\right) t_{n, m_l}^{-1}\right)$$

$$\xrightarrow{\mu\left(\lambda_{p, r} \cdot t_{n, l}^{-1}; 1, \ldots, 1\right) \cdot \mu\left(e_l; t_{n, m_1}^{-1}, \ldots, t_{n, m_l}^{-1}\right)} \mu\left(\mu\left(r; \underline{p}\right) \cdot t_{n, l}^{-1}; \underline{q_1}, \ldots, \underline{q_l}\right) \cdot \mu\left(e_l; t_{n, m_1}^{-1}, \ldots, t_{n, m_l}^{-1}\right),$$

where we have made use of the operad axioms in identifying the target of the first map with the source of the second. Using the Λ-operad axioms again on the target, we find that

$$\mu\left(\mu(r;\underline{p})\cdot t_{n,l}^{-1};\underline{q_1},\ldots,\underline{q_l}\right)\cdot\mu(e_l;t_{n,m_1}^{-1},\ldots,t_{n,m_l}^{-1})$$

is equal to

$$\mu\left(\mu(r;\underline{p});\underline{q_1},\ldots,\underline{q_l}\right)\cdot\mu(t_{n,l}^{-1};\underline{e})\cdot\mu(e_l;t_{n,m_1}^{-1},\ldots,t_{n,m_l}^{-1}).$$

This composite of two morphisms, together with the necessary identities coming from operad axioms, is precisely the left and bottom leg of the diagram in Axiom 4. Using the same method, one then verifies that $\gamma * (\mu \times 1)$ has its first component the image of the morphism appearing along the top and right leg of the diagram in 4. The second component of these morphisms are all identities arising from Λ-equivariance, so the first multiplication axiom is a consequence of Axioms 2 and 4 for the pseudo-commutative structure. We leave the calculations for the second multiplication axiom to the reader as they are of the same nature, using Axioms 3 and 5. $\qquad\square$

Corollary 17.8. *Let P be a non-symmetric operad. Then the induced monad $\underline{P}$ is never pseudo-commutative.*

Proof. In the non-symmetric case, the 2-monad is just given using coproducts and products, i.e., there is no coequalizer. In order to define γ, we then need an isomorphism

$$\left(\mu(p;\underline{q});(\underline{a},\underline{b})\right) \cong \left(\mu(q;\underline{p});(\underline{a},b)\right).$$

When A, B are discrete, there is no isomorphism $(\underline{a},\underline{b}) \cong (\underline{a}, b)$, and therefore no such γ can exist. $\qquad\square$

A further property that a pseudo-commutativity can possess is that of symmetry. This symmetry is then reflected in the monoidal structure on the 2-category of algebras, which will then also have a symmetric tensor product (in a suitable, 2-categorical sense).

Definition 17.9. Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad on a symmetric monoidal 2-category $\mathcal{K}$ with symmetry c. We then say that a pseudo-commutativity γ for T is *symmetric* when the following is satisfied for all $A, B \in \mathcal{K}$:

$$Tc_{A,B} \circ \gamma_{A,B} \circ c_{TB,TA} = \gamma_{B,A}.$$

With the notion of symmetry at hand we are able to extend the above theorem, stating when $\underline{P}$ is symmetric.

Theorem 17.10. *The pseudo-commutativity of $\underline{P}$ given by Theorem 17.7 is symmetric if for all $m,n \in \mathbb{N}_+$ the two conditions below hold.*

1. $t_{m,n} = t_{n,m}^{-1}$.

2. *The following diagram commutes:*

$$
\begin{array}{ccc}
\mu\left(p;\underline{q}\right)\cdot t_{m,n}t_{n,m} & = & \mu\left(p;\underline{q}\right)\cdot e_{mn} \\
\lambda_{p,q}\cdot 1 \downarrow & & \| \\
\mu\left(q;\underline{p}\right)\cdot t_{n,m} & \xrightarrow{\lambda_{q,p}} & \mu\left(p;\underline{q}\right)
\end{array}
$$

Proof. The commutativity of the diagram above ensures that the first component of the symmetry axiom commutes in $P(n)$ before taking equivalence classes in the coequalizer, just as in the proof of Theorem 17.7. $\qquad\square$

Definition 17.11. Let P be a Λ-operad in **Cat**. We say that P is *contractible* if each category $P(n)$ is equivalent to the terminal category.

Corollary 17.12. *If P is contractible and there exist $t_{m,n}$ as in Theorem 17.7, then $\underline{P}$ acquires a pseudo-commutativity. Furthermore, it is symmetric if $t_{n,m} = t_{m,n}^{-1}$.*

Proof. The only thing left to define is the collection of natural isomorphisms $\lambda_{p,q}$. But since each $P(n)$ is contractible, $\lambda_{p,q}$ must be the unique isomorphism between its source and target, and furthermore the last two axioms hold since any pair of parallel arrows are equal in a contractible category. $\square$

Corollary 17.13. *If P is a contractible symmetric operad then $\underline{P}$ has a symmetric pseudo-commutativity.*

Proof. We choose $t_{m,n} = \tau_{m,n}$. $\square$

Remark 17.14. If a Λ-operad P is contractible, it is not the case that its symmetrization $S(P)$ (see Theorem 11.4) will also be contractible. The category $P(n) \otimes_{\Lambda(n)} \Sigma_n$ will necessarily be a groupoid as it is a colimit of groupoids: contractible categories are always groupoids, and both $\Lambda(n)$ and Σ_n are discrete. Let $g \in \ker \pi_n$ be any non-identity element, and let $p \in P(n)$ be any object. Then

$$[p \cdot g, e] = [p, \pi(g)e] = [p, e],$$

but unless $p \cdot g = p$ in $P(n)$, there will be a unique isomorphism between them that will not be the identity, and hence will define a nontrivial automorphism of $[p, e]$ in $P(n) \otimes_{\Lambda(n)} \Sigma_n$. The existence of such ensures that $P(n) \otimes_{\Lambda(n)} \Sigma_n$ is not contractible.

18 Extended Example: Coboundary Categories

We now turn to an example which is not as widely known in the categorical literature, that of coboundary categories [**drin-quasihopf**]. These arise in the representation theory of quantum groups and in the theory of crystals [**hk-cobound**, **hk-quantum**]. Our goal here is to refine the relationship between coboundary categories and the operad of n-fruit cactus groups in [**hk-cobound**] by using the theory of action operads and our Borel construction. We begin by recalling the definition of a coboundary category.

Definition 18.1. A *coboundary category* is a monoidal category C equipped with a natural isomorphism $\sigma_{x,y} \colon x \otimes y \to y \otimes x$ (called the *commutor*) such that

- $\sigma_{y,x} \circ \sigma_{x,y} = 1_{x \otimes y}$ and
- the diagram

$$
\begin{array}{ccccc}
(x \otimes y) \otimes z & \longrightarrow & x \otimes (y \otimes z) & \xrightarrow{\;1\sigma_{y,z}\;} & x \otimes (z \otimes y) \\
{\scriptstyle \sigma_{x,y}1}\downarrow & & & & \downarrow{\scriptstyle \sigma_{x,zy}} \\
(y \otimes x) \otimes z & \xrightarrow[\;\sigma_{yx,z}\;]{} & z \otimes (y \otimes x) & \longrightarrow & (z \otimes y) \otimes x
\end{array}
$$

commutes (in which the unlabeled morphisms are an associator and an inverse associator).

Example 18.2.   1. As noted by Savage [**savage-braidcob**], any braiding automatically satisfies the cactus relation (the diagram in Definition 18.1). However, since braidings need not be involutions this does not mean that any braided monoidal category is a coboundary category. However, it should then be clear that any symmetric monoidal category is also a coboundary category.

2. The name coboundary category comes from the original work of Drinfeld [**drin-quasihopf**] in which he shows that the category of representations of a coboundary Hopf algebra has the structure of coboundary category.

3. Henriques and Kamnitzer [**hk-cobound**] show that the category of crystals for a finite dimensional complex reductive Lie algebra has the structure of a coboundary category.

Our interest is in strict coboundary categories by which we mean coboundary categories with strict underlying monoidal category. Under the assumption of strictness, the second axiom above does not include associations for the tensor product and reduces to a square. To show that every coboundary category is equivalent to a strict coboundary category, we must introduce the 2-category **CobCat** of coboundary categories.

Definition 18.3. Let $(C, \sigma), (C', \sigma')$ be coboundary categories. A *coboundary functor* $F \colon C \to C'$ is a strong monoidal functor (with invertible constraints φ_0 for the unit and $\varphi_{x,y}$ for the tensor product) such that the following diagram commutes for all objects $x, y \in \mathcal{C}$.

$$
\begin{array}{ccc}
Fx \otimes Fy & \xrightarrow{\;\varphi_{x,y}\;} & F(x \otimes y) \\
\downarrow{\scriptstyle \sigma_{Fx,Fy}} & & \downarrow{\scriptstyle F\sigma_{x,y}} \\
Fy \otimes Fx & \xrightarrow[\;\varphi_{y,x}\;]{} & F(y \otimes x)
\end{array}
$$

Coboundary functors are composed just as strong monoidal functors are, giving the following.

Lemma 18.4. *Coboundary categories, coboundary functors, and monoidal transformations form a 2-category, which we denote* **CobCat**.

Proposition 18.5. *Let (C, σ) be a coboundary category. Then there exists a strict coboundary category (C', σ') which is equivalent to (C, σ) in* **CobCat**.

Proof. Consider the underlying monoidal category of (C, σ), which we will just write as C. We can find a strict monoidal category C' by coherence for monoidal categories together with an equivalence, as monoidal categories, between C and C'. By standard methods [**maclane-catwork**], this can be improved to an adjoint equivalence between C and C' in the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. Let $F \colon C \to C', G \colon C' \to C$ be the functors in this adjoint equivalence, and $\eta \colon 1 \Rightarrow FG$ the unit (which we note for emphasis is invertible since it the unit of an adjoint equivalence). For objects $x, y \in C'$, we define a commutor σ' for C' as the following composite.

$$
xy \xrightarrow{\eta \otimes \eta} FGxFGy \xrightarrow{\cong} F(GxGy) \xrightarrow{F\sigma} F(GyGx) \xrightarrow{\cong} FGyFGx \xrightarrow{\eta^{-1} \otimes \eta^{-1}} yx
$$

We then leave to the reader, if they wish, the computations to show that σ' is a commutor for C' and that F, G become coboundary functors using σ'. $\square$

We now turn to the operadic description of strict coboundary categories; we note from this point onwards, all our coboundary categories are assumed to be strict.

Definition 18.6. Fix $n > 1$, and let $1 \le p < q \le n$, $1 \le k < l \le n$.

1. $p < q$ is *disjoint* from $k < l$ if $q < k$ or $l < p$.
2. $p < q$ *contains* $k < l$ if $p \le k < l \le q$.

Definition 18.7. Let $1 \le p < q \le n$, and define $\widehat{s}_{p,q} \in \Sigma_n$ to be the permutation defined below.

$$
\begin{array}{c|ccccccccccccc}
i & 1 & 2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & q-1 & q & q+1 & \cdots & n \\
\hline
\widehat{s}_{p,q}(i) & 1 & 2 & \cdots & p-1 & q & q-1 & q-2 & \cdots & p+1 & p & q+1 & \cdots & n
\end{array}
$$

The n-fruit cactus group is then defined as follows.

Definition 18.8. Let J_n be the group generated by symbols $s_{p,q}$ for $1 \le p < q \le n$ subject to the following relations.

1. For all $p < q$, $s_{p,q}^2 = e$.
2. If $p < q$ is disjoint from $k < l$, then $s_{p,q} s_{k,l} = s_{k,l} s_{p,q}$.
3. If $p < q$ contains $k < l$, then $s_{p,q} s_{k,l} = s_{m,n} s_{p,q}$ where
   - $m = \widehat{s}_{p,q}(l)$ and
   - $n = \widehat{s}_{p,q}(k)$.

It is easy to check that the elements $\widehat{s}_{p,q} \in \Sigma_n$ satisfy the three relations in Definition 18.8, so $s_{p,q} \mapsto \widehat{s}_{p,q}$ extends to a group homomorphism $\pi_n \colon J_n \to \Sigma_n$. This is the first step in proving the following.

Theorem 18.9. *The collection of groups $J = \{J_n\}$ form an action operad.*

Proof. **There is an issue with the corrected version of Axiom 5 and the t's that needs some fixing. AC: It needs to be that $\delta_{1;n}(e_1) = e_n$, but δ is only defined on the symbols $s_{p,q}$. Could just define $\delta_{1;n}(e_1) = e_n$ and then check that this doesn't cause any problems with the other characterisation?**

We will use Theorem 4.14 to determine the rest of the action operad structure. Thus we must give, for any collection of natural numbers $n, k_1, \ldots, k_n$ and $K = \sum k_i$, group homomorphisms $\beta \colon J_{k_1} \times \cdots \times J_{k_n} \to J_K$ and functions $\delta \colon J_n \to J_K$ satisfying nine axioms. We define both of these on generators, starting with β.

Let $s_{p_i, q_i} \in J_{k_i}$. Let $r_i = k_1 + k_2 + \cdots + k_{i-1}$ for $i > 1$. Define β by

$$
\beta(s_{p_1, q_1}, \ldots, s_{p_n, q_n}) = s_{p_1, q_1} s_{p_2 + r_2, q_2 + r_2} \cdots s_{p_n + r_n, q_n + r_n}.
$$

Note that $s_{p_i + r_i, q_i + r_i}$ and $s_{p_j + r_j, q_j + r_j}$ are disjoint when $i \ne j$.

It is easy to check that this disjointness property ensures that β gives a well-defined group homomorphism

$$
J_{k_1} \times \cdots \times J_{k_n} \to J_K.
$$

To define $\delta \colon J_n \to J_K$ for natural numbers $n, k_1, \ldots, k_n$ and $K = \sum k_i$, let $t_k = s_{1,k} \in J_k$. Then we start by defining

$$
\delta(t_n) = t_K \cdot \beta(t_{k_1}, t_{k_2}, \ldots, t_{k_n}).
$$

Note that, by Axiom 8 of Theorem 4.14, this is equal to

$$
\beta(t_{k_n}, t_{k_{n-1}}, \ldots, t_{k_1}) \cdot t_K.
$$

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Now $s_{p,q} \in J_n$ is equal to $\beta(e_{p-1}, t_{q-p+1}, e_{n-q})$ (here e_i is the identity element in J_i) by definition of the t_i and β, so we can define δ on any generator $s_{p,q}$ by

$$\delta(s_{p,q}) = \beta(e_A, M, e_B)$$

with

- $A = k_1 + k_2 + \cdots + k_{p-1}$,
- $M = t_{k_p + \cdots + k_q} \cdot \beta(t_{k_p}, t_{k_{p+1}}, \ldots, t_{k_q})$, and
- $B = k_{q+1} + k_{q+2} + \cdots + k_n$.

Unpacking this yields the following formula:

$$\delta(s_{p,q}) = s_{k_1 + \cdots + k_{p-1} + 1, k_1 + \cdots + k_q} \cdot \beta(e_{k_1 + \cdots + k_{p-1}}, t_{k_p}, \ldots, t_{k_q}, e_{k_{q+1} + \cdots + k_n}).$$

We extend δ to products of generators using Axiom 6 of Theorem 4.14. As before, we must check that this gives a well-defined function on products of two generators in each of the relations of the cactus groups, and we must also check that this is well-defined on products of three or more generators. Thus we define

$$\delta_{n; j_1, \ldots, j_n}(gh) = \delta_{n; k_1, \ldots, k_n}(g) \delta_{n; j_1, \ldots, j_n}(h)$$

where $k_i = j_{\pi(h)^{-1}(i)}$. There are three relations we must verify for compatibility.

- We must show that $\delta_{n; j_1, \ldots, j_n}\left(s_{p,q}^2\right) = e$. By definition, we have

$$\delta_{n; j_1, \ldots, j_n}\left(s_{p,q}^2\right) = \delta_{n; k_1, \ldots, k_n}\left(s_{p,q}\right) \delta_{n; j_1, \ldots, j_n}\left(s_{p,q}\right)$$

  which is

$$t_{\underline{j}} \beta(t_{j_n}, \ldots, t_{j_1}) t_{\underline{j}} \beta(t_{j_1}, \ldots, t_{j_n}).$$

  (QQQ What is $t_{\underline{j}}$?) By the remarks above in the definition of δ and the fact that $s_{p,q}^2 = e$, the element above is easily seen to be the identity.

- We must show that $\delta(s_{p,q} s_{k,l}) = \delta(s_{k,l} s_{p,q})$ when (p,q) is disjoint from (k,l). This is another simple calculation using the definition of δ and the disjointness of the terms involved.

- We must show that $\delta(s_{p,q} s_{k,l}) = \delta(s_{a,b} s_{p,q})$, where $a = \widehat{s}_{p,q}(l), b = \widehat{s}_{p,q}(k)$, if $p < k < l < q$. In this case, we use all of the relations in the cactus groups to show that each side is equal to

$$\beta\left(\underline{e}, t_{j_p + \cdots + j_q} \cdot \beta\left(t_{j_p}, \ldots t_{j_{k-1}}, t_{j_k + \cdots + j_l}, t_{j_{l+1}}, \cdots, t_{j_q}\right), t_{j_{q+1}}, \ldots, t_{j_n}\right)$$

  where $\underline{e} = e_{j_1}, \ldots, e_{j_{p-1}}$.

In order to show that this gives a well-defined function on products of three or more generators, one proceeds inductively to show that $\delta\left((fg)h\right) = \delta\left(f(gh)\right)$ using the formula above. This is simply a matter of keeping track of the permutations used to define the subscripts for the different δ's and we leave it to the reader, should they desire to see the details. This concludes the definition of the family of functions $\delta_{n; j_i}$.

There are now nine axioms to check in Theorem 4.14. Axioms 1 - 3 all concern β, and are immediate from the defining formula. Axiom 4 is obvious for the elements t_k, from which it follows in general by the formulas defining δ. For Axiom 5, one can check easily that

$$\delta_{n; 1, \ldots, 1}(t_n) = t_n, \quad \delta_{1; n}(e_1) = t_n$$

and once again the general case follows from these. Axiom 6 holds by the construction of δ. Axiom 8 can be verified with only one h_i nontrivial at a time, and then it is a simple consequence of the second and third relations for J_n.

Axiom 9 is straightforward to check when only a single g_i is a generator and the rest are identities using the defining formulas, and the general case then follows using Axiom 6. Using Axiom 9, we can then prove Axiom 7 as follows; we suppress the subscripts on different δ's for clarity. We must show

$$\delta_{m_1+\cdots+m_n;p_{11},\ldots,p_{1m_1},p_{21},\ldots,p_{nm_m}}\left(\delta_{n;m_1,\ldots,m_n}(f)\right) = \delta_{n;P_1,\ldots,P_n}(f),$$

and we do so on t_n. By definition, we have

$$\delta\left(\delta(t_n)\right) = \delta\left(t_K\beta(t_{k_1},\ldots,t_{k_n})\right),$$

which by Axiom 6 is equal to

$$t_{P_1+\cdots+P_n}\cdot\beta(t_{p_{11}},\ldots,t_{p_n,m_n})\cdot\delta\left(\beta(t_{k_1},\ldots,t_{k_n})\right).$$

Now this last term is equal to $\beta\left(\delta(t_{k_1}),\ldots,\delta(t_{k_n})\right)$ by Axiom 9, which is then equal to

$$\beta\left(t_{P_1}\cdot\beta(t_{p_{11}},\ldots,t_{p_{1,m_1}}),\ldots,t_{P_n}\cdot\beta(t_{p_{n1}},\ldots,t_{p_{1,m_n}})\right).$$

Taken all together, the left hand side of Axiom 9 is then

$$t_{P_1+\cdots+P_n}\cdot\beta(t_{p_{11}},\ldots,t_{p_n,m_n})\cdot\beta\left(t_{P_1}\cdot\beta(\underline{t_{p_1}}),\ldots,t_{P_n}\cdot\beta(\underline{t_{p_n}})\right).$$

where $\underline{t_{p_i}} = t_{p_{i,1}},\cdots,t_{i,m_i}$ All of the terms coming from an $t_{p_{ij}}$ can be collected together, and since $s_{p,q}^2 = e$ for all p,q, these cancel. This leaves

$$t_{P_1+\cdots+P_n}\cdot\beta\left(t_{P_1},\ldots,t_{P_n}\right)$$

which is the right hand side of Axiom 9 as desired. $\qquad\square$

Lemma 18.10. *The 2-monad C for strict coboundary categories is a club.*

Proof. This is obvious by Corollary 8.10. $\qquad\square$

Theorem 18.11. *The free coboundary category on one element, $C1$, is isomorphic to $BJ = \coprod BJ_n$.*

Proof. The universal property we desire is with respect to strict coboundary functors (i.e., coboundary functors whose underlying monoidal functor is strict), so we must give BJ the structure of a strict coboundary category and then check that to give a strict coboundary functor $BJ \to X$ to any other strict coboundary category is the same as giving an object of X.

The category BJ has natural numbers as objects, and addition as its tensor product. The tensor product of two morphisms is given by β as in Theorem 18.9, and it is simple to check that this is a strict monoidal structure. The commutor $\sigma_{m,n}$ is $s_{1,m+n}s_{1,m}s_{m+1,m+n}$. Using the relations in J_n, it is clear that $\sigma_{m,n}\sigma_{n,m}$ is the identity, so we only have one more axiom to verify in order to give a coboundary structure. By definition, this axiom is equivalent to the equation

$$\sigma_{m,p+n}\cdot\beta(e_m,\sigma_{n,p}) = \sigma_{n+m,p}\cdot\beta(\sigma_{m,n},e_p)$$

holding for all m,n,p. Each side has six terms when written out using the definitions of σ and β, two terms on each side cancel using $s_{p,q}^2 = e$ and the disjointness relation, and the other four terms match after using the disjointness relation. This establishes the coboundary structure on BJ; note that $\sigma_{1,1} = s_{1,2}$, the nontrivial element of $J(2)$.

Every strict coboundary functor $F\colon BJ \to X$ determines an object of X by evaluation at 1. Conversely, given an object x of a strict coboundary category X, there

is an action of J_n on $X(x^n, x^n)$ by Theorem 7 of [**hk-cobound**] and therefore a strict monoidal functor $\bar{x}: BJ \to X$ with $\bar{x}(1) = x$. By construction, this strict monoidal functor is in fact a strict coboundary functor since it sends the commutor $\sigma_{1,1}$ in BJ to $\sigma_{x,x}$ in X. In fact, the calculations in [**hk-cobound**] leading up to Theorem 7 show that every element of J_n is given as an operadic composition of σ's, so requiring $\bar{x}$ to be a strict coboundary functor with $\bar{x}(1) = x$ determines the rest of the functor uniquely. This establishes the bijection between strict coboundary functors $F: BJ \to X$ and objects of X which proves that BJ is the free strict coboundary category on one object. $\square$

Corollary 18.12. *The 2-monad C for coboundary categories corresponds, using Theorem 8.7, to the action operad J.*

19 Extended Example: Braided Monoidal Categories

We conclude with a computation using Theorem 17.7. This result (19.1 below) was only conjectured in [**HP**], but we are able to prove it quite easily with the machinery developed thus far. Our strategy is to construct a Λ-operad which is contractible together with the group elements required in Theorem 17.7. Note that the symmetrized version of this operad will not be contractible, and we do not know of a proof using the structure of the symmetrized operad.

Theorem 19.1. *The 2-monad $\underline{B}$ for braided strict monoidal categories on **Cat** has two pseudo-commutative structures on it, neither of which are symmetric.*

In order to apply our theory, the 2-monad $\underline{B}$ must arise from a Λ-operad. The following proposition describes it as such, and can largely be found as Example 3.2 in the work of Fiedorowicz [**fie-br**].

Proposition 19.2. *The 2-monad $\underline{B}$ is the 2-monad associated to the B-operad B with the category $B(n)$ having objects the elements of the nth braid group B_n and a unique isomorphism between any pair of objects; the action of B_n on $B(n)$ is given by right multiplication on objects and is then uniquely determined on morphisms.*

The interested reader could easily verify that algebras for the B-operad B are braided strict monoidal categories. The objects of $\underline{B}(X)$ can be identified with finite lists of objects of X, and morphisms are generated by the morphisms of X together with new isomorphisms

$$x_1, \ldots, x_n \xrightarrow{\gamma} x_{\gamma^{-1}(1)}, \ldots, x_{\gamma^{-1}(n)}$$

where $\gamma \in B_n$ and the notation $\gamma^{-1}(i)$ means, as before, that we take the preimage of i under the permutation $\pi(\gamma)$ associated to γ. This shows that $\underline{B}(X)$ is the free braided strict monoidal category generated by X according to [**js**], and it is easy to verify that the 2-monad structure on $\underline{B}$ arising from the B-operad structure on B is the correct one to produce braided strict monoidal categories as algebras.

Definition 19.3. A braid $\gamma \in B_n$ is *positive* if it is an element of the submonoid of B_n generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

Definition 19.4. A braid $\gamma \in B_n$ is *minimal* if no pair of strands in γ cross twice.

For our purposes, we are interested in braids which are both positive and minimal. A proof of the following lemma can be found in [**EM2**].

Lemma 19.5. *Let PM_n be the subset of B_n consisting of positive, minimal braids. Then the function sending a braid to its underlying permutation is a bijection of sets $PM_n \to \Sigma_n$.*

Remark 19.6. It is worth noting that this bijection is not an isomorphism of groups, since PM_n is not a group or even a monoid. The element $\sigma_1 \in B_n$ is certainly in PM_n, but σ_1^2 is not as the first two strands cross twice. Thus we see that the product of two minimal braids is generally not minimal, but by definition the product of positive braids is positive.

Proof of Theorem 19.1. We refer to the Axioms of Definition 17.5 throughout the proof. In order to use Theorem 17.7 with the action operad being the braid operad B, we must first construct elements $t_{m,n} \in B_{mn}$ satisfying certain properties. Using Lemma 19.5, we define $t_{m,n}$ to be the unique positive minimal braid such that $\pi(t_{m,n}) = \tau_{m,n}$. Since $\tau_{1,n} = e_n = \tau_{n,1}$ in Σ_n and the identity element $e_n \in B_n$ is positive and minimal, we find that $t_{1,n} = e_n = t_{n,1}$ in B_n, satisfying Axiom 1. Thus in order to verify the remaining hypotheses, we must check two equations, each of which states that some element $t_{m,n}$ can be expressed as a product of operadic compositions of other elements.

Let $l, m_1, \ldots, m_l, n$ be natural numbers, and let $N = \sum m_i$. We must check Axiom 3 is satisfied, i.e., that

$$\beta(t_{n,m_1}, \ldots, t_{n,m_l}) \cdot \delta(t_{n,l}) = t_{N,l}$$

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu\left(t_{n,l}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}\right) = t_{N,l}$$

in B_{lN}. These braids have the same underlying permutations by construction, and both are positive since each operadic composition on the left is positive. The braid on the right is minimal by definition, so if we prove that the braid on the left is also minimal, they are necessarily equal. Now $\mu\left(t_{n,l}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}\right)$ is given by the braid for $t_{n,l}$ but with the first strand replaced by m_1 strands, the second strand replaced by m_2 strands, and so on for the first l strands of $t_{n,l}$, and then repeating for each group of l strands. In particular, since strands $i, i+l, i+2l, \ldots, i+(n-1)l$ never cross in $t_{n,l}$, none of the m_i strands that each of these is replaced with cross. The braid $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ consists of the disjoint union of the braids for each t_{n,m_i}, so if two strands cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ then they must both cross in some t_{n,m_i}. The strands in t_{n,m_i} are those numbered from $n(m_1 + \cdots + m_{i-1}) + 1$ to $n(m_1 + \cdots + m_{i-1} + m_i)$. This is a consecutive collection of nm_i strands, and it is simple to check that these strands are precisely those connected (via the group operation in B_{Nl}, concatenation) to the duplicated copies of strands $i, i+l, i+2l, \ldots, i+(n-1)l$ in $t_{n,l}$. Thus if a pair of strands were to cross in $\beta(t_{n,m_1}, \ldots, t_{n,m_l})$, that pair cannot also have crossed in $\delta(t_{n,l})$, showing that the resulting product braid

$$\beta(t_{n,m_1}, \ldots, t_{n,m_l}) \cdot \delta(t_{n,l})$$

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu\left(t_{n,l}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}\right)$$

is minimal. The calculation for Axiom 2, showing that

$$\delta(t_{m,l}) \cdot \beta(t_{n_1,l}, \ldots, t_{n_m,l})$$

$$\mu\left(t_{m,l}; \underline{e_1}, \ldots, \underline{e_{n_m}}\right) \mu\left(e_m; t_{n_1,l}, \ldots, t_{n_m,l}\right)$$

is also minimal, follows from the same argument, showing that it is equal to $t_{N,l}$ (here N is the sum of the n_i, where once again i ranges from 1 to l).

QQQ Where are Axioms 4 and 5 checked?

These calculations show, using Theorem 17.7, that the B-operad B induces a 2-monad which has a pseudo-commutative structure. As noted before, B-algebras are precisely braided strict monoidal categories. The second pseudo-commutative structure arises by using negative, minimal braids instead of positive ones, and proceeds using the same arguments. This finishes the first part of the proof of Theorem 19.1.

We will now show that neither of these pseudo-commutative structures is symmetric. The symmetry axiom in this case reduces to the fact that, in some category which is given as a coequalizer, the morphism with first component

$$f \colon \mu\left(p; \underline{q}\right) \cdot t_{n,m} t_{m,n} \to \mu\left(q; \underline{p}\right) \cdot t_{m,n} \to \mu\left(p; \underline{q}\right)$$

is the identity. Now it is clear that $t_{n,m}$ is not equal to $t_{m,n}^{-1}$ in general: taking $m = n = 2$, we note that $t_{2,2} = \sigma_2$, and this element is certainly not of order two in B_4. $B(4)$ is the category whose objects are the elements of B_4 with a unique isomorphism between any two pair of objects, and B_4 acts by multiplication on the right; this action is easily shown to be free and transitive. We recall (see Lemma 16.2) that in a coequalizer of the form $A \otimes_G B$, a morphism $[f_1, f_2]$ equals $[g_1, g_2]$ if and only if there exists an $x \in G$ such that

$$f_1 \cdot x = g_1,$$
$$x^{-1} \cdot f_2 = g_2.$$

For the coequalizer in question, for f to be the first component of an identity morphism would imply that $f \cdot x$ would be a genuine identity in $B(4)$ for some x. But $f \cdot x$ would have source $\mu\left(p; \underline{q}\right) t_{n,m} t_{m,n} x$ and target $\mu\left(p; \underline{q}\right) x$, which requires $t_{n,m} t_{m,n}$ to be the identity group element for all n, m. In particular, this would force $t_{2,2}$ to have order two, which as noted above does not hold in B_4, thus giving a contradiction. $\qquad\square$

Remark 19.7. The pseudo-commutativities given above are not necessarily the only ones that exist for the B-operad B, but we do not know a general strategy for producing others.

Example 19.8. Non-Example: Cactus operad.

Begin by defining $t_{2,2} = s_{2,3}$, which has underlying permutation $\pi_4(t_{2,2}) = (2\ 3) = \tau_{2,3}$, as required. This seems to be a sensible choice to then demonstrate that we can describe all other $t_{m,n}$ required for a pseudo-commutativity on J. In particular, we should be able to describe $t_{2,4}$ which would have underlying permutation $\tau_{2,4} = (2\ 3\ 5)(4\ 7\ 6)$. If the required elements $t_{m,m}$ existed for the cactus operad J, then we would be able to apply the axioms from Definition 17.5 to the element $t_{2,4}$ in order to see how it is constructed from $t_{2,2} = s_{2,3}$.

By Axiom 2 of Definition 17.5 we should be able to split the element $t_{2,4}$ up as follows.

$$
\begin{aligned}
t_{2,4} &= t_{2,2+2} \\
&= \beta(t_{2,2}, t_{2,2}) \cdot \delta_{2,2,2,2}(t_{2,2}) \\
&= s_{2,3} \cdot s_{5,7} \cdot \delta_{2,2,2,2}(s_{2,3}) \\
&= s_{2,3} \cdot s_{5,7} \cdot s_{2,6} \cdot \beta(e_2, s_{1,2}, s_{1,2}, e_2) \\
&= s_{2,3} \cdot s_{5,7} \cdot s_{2,6} \cdot s_{3,4} \cdot s_{5,6}.
\end{aligned}
$$

Here we have used δ as defined for generators $s_{p,q}$ in Theorem 18.9. This element has underlying permutation

$$(2\ 3)(5\ 7)(2\ 6)(3\ 5)(3\ 4)(5\ 6) = (2\ 6)(3\ 4\ 7\ 5)$$

which is not equal to $\tau_{2,4} = (2\ 3\ 5)(4\ 7\ 6)$. Hence if J were to have a pseudo-commutative structure, then it cannot arise in this way.

Remark 19.9. I commented out the profunctors stuff, but it is still in the file.