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$\coprod_{n \in \mathbb{N}} P(n) \times_{G(n)} X^n$

OPERADS WITH GENERAL GROUPS OF EQUIVARIANCE, AND SOME 2-CATEGORICAL ASPECTS OF OPERADS IN Cat

ALEXANDER S. CORNER AND NICK GURSKI

ABSTRACT. We give a definition of an operad with general groups of equivariance suitable for use in any symmetric monoidal category with appropriate colimits. We then apply this notion to study the 2-category of algebras over an operad in **Cat**. We show that any operad is finitary, that an operad is cartesian if and only if the group actions are nearly free (in a precise fashion), and that the existence of a pseudo-commutative structure largely depends on the groups of equivariance. We conclude by showing that the operad for strict braided monoidal categories has two canonical pseudo-commutative structures.

CONTENTS

INTRODUCTION

when a space X admits...

Operads were defined by May [15] in the early 70's to provide a convenient tool to approach problems in algebraic topology, notably the question of when does a space X admit an n-fold delooping Y so that $X \simeq \Omega^n Y$? An operad, like an algebraic theory [13], is something like a presentation of a monad or algebraic structure. The theory of operads has seen great success, and we would like to highlight two reasons. First, operads can be defined in any suitable symmetric monoidal category, so that there are operads of topological spaces, of chain complexes, of simplicial sets, and of categories, to name a few examples. Moreover, symmetric (lax) monoidal functors carry operads to operads, so we can use operads in one category to understand objects in another via transport by such a functor. Second, operads in a fixed category are highly flexible tools. In particular, the categories listed above all have some inherent notion of "homotopy equivalence" which is weaker than that of isomorphism, so we can study operads which are equivalent but not isomorphic. These tend to have algebras which have similar features in an "up-to-homotopy" sense but very different combinatorial or geometric properties arising from the different operads.

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Operads in the category **Cat** of small categories have a unique flavor arising from the fact that **Cat** is not just a category but a 2-category. These 2-categorical aspects have not been widely treated in the literature, although a few examples can be found. Lack [12] mentions braided **Cat**-operads (the reader new to braided operads should refer to the work of Fiedorowicz [5]) in his work on coherence for 2-monads, and Batanin [1] uses lax morphisms of operads in **Cat** in order to define the notion of an internal operad. But aside from a few appearances, the basic theory of operads in **Cat** and their 2-categorical properties seems missing. This paper was partly motivated by the need for such a theory to be explained from the ground up.

There were two additional motivations for the work in this paper. In thinking about coherence for monoidal functors, the first author was led to a general study of algebras for multicategories internal to **Cat**. These give rise to 2-monads (or perhaps pseudomonads, depending on how the theory is set up), and checking abstract properties of these 2-monads prompts one to consider the simpler case of operads in **Cat** instead of multicategories. The other motivation was from the second author's attempt to understand the interplay between operads in **Cat**, operads in **Top**, and the passage from (bi)permutative categories to E_∞ (ring) spaces. The first of these motivations raised the issue of when operads in **Cat** are cartesian, while the second led us to the question of when does an operad in **Cat** possess a pseudo-commutative structure.

While considering how to best tackle a general discussion of operads in **Cat**, it became clear that restricting attention to the two most commonly used types of operads, symmetric and non-symmetric operads, was both short-sighted and unnecessary. Many theorems apply to both kinds of operads at once, with the difference in proofs being negligible; in fact, most of the arguments which applied to the symmetric case seemed to apply to the case of braided operads as well. This led us to the notion of an action operad **G**, and then to a definition of **G**-operads. In essence, this is merely the general notion of what it entails for an operad $P = \{P(n)\}$ to have groups of equivariance $\mathbf{G} = \{G(n)\}$ such that $G(n)$ acts on $P(n)$. Choosing different natural families of groups **G**, we recover known variants of the definition of operad.

$n \in \mathbb{N}$?

$n \in \mathbb{N}$?

Groups **G**	Type of operad
Terminal groups	Non-symmetric operad
Symmetric groups	Symmetric operad
Braid groups	Braided operad

These definitions have appeared, with minor variations, in two sources of which we are aware. In Wahl's thesis [17], the essential definitions appear but not in complete generality as she requires a surjectivity condition. Zhang [18] also studies these notions[1], once again in the context of homotopy theory, but requires the superfluous condition that $e_1 = \mathrm{id}$ (see Lemma 1.11).

This paper consists of the following. In Section 1, we give the definition of an action operad **G** and a **G**-operad. We develop this definition abstractly so as to

[1]Zhang calls our action operad a *group operad*. We dislike this terminology as it seems to imply that we are dealing with an operad in the category of groups, which is not the case unless all of the maps $\pi_n : G(n) \to \Sigma_n$ are zero maps.

apply it in any suitable symmetric monoidal category. In order to show that **G**-operads are monoids for a particular monoidal structure on a category of collections, we must use the calculus of coends and the Day convolution product [3]. The reader uninterested in these details can happily skip them, although we find the route taken here to be quite satisfactory in justifying the axioms for an action operad **G** and the accompanying notion of **G**-operad.

Section 2 works through the basic 2-categorical aspects of operads in **Cat**. We explain how every operad gives rise to a 2-monad, and show that all of the various 1-cells between algebras of the associated 2-monad correspond to the obvious sorts of 1-cells one might define between algebras over an operad in **Cat**. Similarly, we show that the algebra 2-cells, using the 2-monadic approach, correspond to the obvious notion of transformation one would define using the operad.

Section 3 studies three basic 2-categorical properties of an operad, namely the property of being finitary (which every operad in **Cat** possesses), the property of being 2-cartesian, and the coherence property. The second of these turns out to be equivalent to the action of $G(n)$ on $P(n)$ being free for all n, at least up to a certain kernel. In particular, our characterization clearly shows that every non-symmetric operad is 2-cartesian, and that a symmetric operad is 2-cartesian if and only if the action of Σ_n on $P(n)$ is free for all n. (It is useful to note that a 2-monad on **Cat** is 2-cartesian if and only if the underlying monad on the category of small categories is cartesian in the usual sense as the (strict) 2-pullback of a diagram is the same as its pullback.)

Section 4 then goes on to study the question of when a **G**-operad P admits a pseudo-commutative structure. Such a structure provides the 2-category of algebras with a richer structure that includes well-behaved notions of tensor product, internal hom, and multilinear map that fit together much as the analogous notions do in the category of vector spaces. When P is contractible (i.e., each $P(n)$ is equivalent to the terminal category), this structure can be obtained from a collection of elements $t_{m,n} \in G(mn)$ satisfying certain properties. In particular, we show that every contractible symmetric operad is pseudo-commutative, and we prove that there exist such elements $t_{m,n} \in B_{mn}$ so that every contractible braided operad is pseudo-commutative as well (in fact in two canonical ways).

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1. Operads in symmetric monoidal categories

Wording.

In this section, we will explore the general definition of an operad P in a symmetric monoidal category which is equipped with groups of equivariance $G(n)$. The group $G(n)$ will act on the right on the object $P(n)$, and the operad structure of P will be required to respect this action. For certain choices of the groups $G(n)$, we will recover standard notions of operads such as symmetric operads, non-symmetric operads, and braided operads. We note that unless otherwise stated, the term operad will mean a non-symmetric operad.

Define Σ_n's (sym. groups).

We begin with the definition of an action operad which underlies the notion of an operad with given groups of equivariance.

Definition 1.1. Am *action operad* **G** consists of

An

$\{G(n)\}_{n\in\mathbb{N}}$?

define $P(n) \xrightarrow{f_n} Q(n)$ s.t. ...

- an operad $G = \{G(n)\}$ in the category of sets such that each $G(n)$ is equipped with the structure of a group and
- a map $\pi : G \to \Sigma$ which is simultaneously a map of operads and a group homomorphism $\pi_n : G(n) \to \Sigma_n$ for each n $\in\mathbb{N}$

such that one additional axiom holds. Write

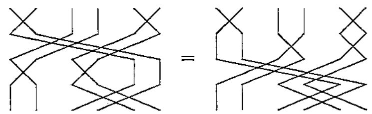
$$\mu : G(n) \times G(k_1) \times \cdots \times G(k_n) \to G(k_1 + \cdots + k_n)$$

for the multiplication in the operad G. Let $(g; f_1, \ldots f_n)$ be an element of the product $G(n) \times G(k_{\pi(g')^{-1}(1)}) \times \cdots \times G(k_{\pi(g')^{-1}(1)})$ and $(g'; f_1', \ldots, f_n')$ be an element of the product $G(n) \times G(k_1) \times \cdots \times G(k_n)$. We require that

$$\mu(g; f_1, \ldots f_n)\mu(g'; f_1', \ldots, f_n') = \mu(gg'; f_{\pi(g')(1)}f_1', \ldots, f_{\pi(g')(n)}f_n')$$

in the group $G(k_1 + \cdots + k_n)$.

Remark 1.2.
- The final axiom is best explained using the operad Σ of symmetric groups. Reading symmetric group elements as permutations from top to bottom, below is a pictorial representation of the final axiom for the map $\mu : \Sigma_3 \times \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \Sigma_6$.



$\mu\big((23);(12),(12),\mathrm{id}\big)\cdot\mu\big((132);(12),\mathrm{id},(12)\big)$ $\qquad$ $\mu\big((23)\cdot(132);\mathrm{id}\cdot(12),(12)\cdot\mathrm{id},(12)\cdot(12)\big)$

- Our definition of an action operad is the same as that appearing in Wahl's thesis [17], but without the condition that each π_n is surjective. It is also the same as that appearing in work of Zhang [18], although we prove later (see Lemma 1.11) that the condition $e_1 = \mathrm{id}$ in Zhang's definition follows from the rest of the axioms.

Example 1.3.
(1) There are two trivial examples of action operads. The first is the symmetric operad $\mathbf{G} = \Sigma$ with the identity map; this is the terminal object in the category of action operads (with the obvious morphisms). The second is $\mathbf{G} = *$ with the π_n's being given by the inclusion of identity elements; this is the initial object in the category of action operads.
(2) Two less trivial examples are given by the braid groups, $\mathbf{G} = \mathbf{Br}$, and the ribbon braid groups, $\mathbf{G} = \mathbf{RBr}$.

Definition 1.4. Let **G** be an action operad. A **G**-*operad* P (in **Sets**) consists of

- an operad P in **Sets** and
- for each n, an action $P(n) \times G(n) \to P(n)$ of $G(n)$ on $P(n)$

such that the following two equivariance axioms hold for P.

$$\mu^P(x; y_1, \ldots, y_n) \cdot \mu^G(e; g_1, \ldots, g_n) = \mu^P(x; y_1 \cdot g_1, \ldots, y_n \cdot g_n)$$
$$\mu^P(x \cdot g; y_1, \ldots, y_n) = \mu^P(x; y_{\pi(g)^{-1}(1)}, \ldots, y_{\pi(g)^{-1}(n)}) \cdot \mu^G(g; e_1, \ldots, e_n)$$

Remark 1.5. It is possible to consider **G**-operads in categories other than the category of sets. In this case we still use the notion of an action operad given above, but then take the operad P to have objects $P(n)$ which are the objects of some closed symmetric monoidal category $\mathcal{V}$. We will rarely use anything that might require the closed structure as such, only the fact that the tensor product distributes over colimits in each variable. This is a consequence of the fact that both $X \otimes -$ and $- \otimes X$ are left adjoints in the case of a closed symmetric monoidal category. Thus while we set up the foundations using only operads in **Sets**, the diligent reader can easily modify this theory for their closed symmetric monoidal category of choice. In fact, we later use the same theory in **Cat** with its cartesian structure, noting only that the same arguments work in **Cat** with essentially no modification.

Cite Kelly?

Example 1.6. (1) Let **T** denote the terminal operad in **Sets** equipped with its unique structure of an action operad. Then a **T**-operad is just a non-symmetric operad in **Sets**.

(2) Let $\mathbf{\Sigma}$ denote the operad of symmetric groups with $\pi : \Sigma \to \Sigma$ the identity map. Then a $\mathbf{\Sigma}$-operad is a symmetric operad in the category of sets.

each $\pi_n : \Sigma_n \to \Sigma_n$ being...

(3) Let **Br** denote the operad of braid groups with each $\pi_n : Br_n \to \Sigma_n$ the canonical projection of a braid onto its underlying permutation. Then a **Br**-operad is a braided operad in the sense of Fiedorowicz [5].

Definition 1.7. Let **G** be an action operad. The category **G-Coll** of **G**-collections has objects $X = \{X(n)\}$ which consist of a set $X(n)$ for each natural number n together with an ~~action~~ $X(n) \times G(n) \to X(n)$ of $G(n)$ on $X(n)$. A morphism $f : X \to Y$ in **G-Coll** consists of a $G(n)$-equivariant map $X(n) \to Y(n)$ for each natural number n.

Remark 1.8. The definition of **G-Coll** does not require that **G** be an action operad, only that one has a natural number-indexed set of groups. Given any such collection of groups $\{G(n)\}$, we can form the category $\mathbb{G}$ whose objects are natural numbers and whose hom-sets are given by $\mathbb{G}(m,n) = \emptyset$ if $m \neq n$ and $\mathbb{G}(n,n) = G(n)$. Then **G-Coll** is the presheaf category

$$\hat{\mathbb{G}} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Sets}],$$

with the opposite category arising from our choice of right actions. A key step in explaining how **G**-operads arise as monoids in the category of **G**-collections is to show ~~that being~~ an action operad endows $\mathbb{G}$ with a monoidal structure.

Definition 1.9. Let **G** be an action operad, and let X, Y be **G**-collections. Then ~~there is~~ a **G**-collection $X \circ Y$ given by

there is...?

$$X \circ Y(n) = \Big(\coprod_{k_1+\cdots+k_r=n} X(r) \times Y(k_1) \times \cdots \times Y(k_r) \Big) \times G(n) / \sim$$

where the equivalence relation is generated by

Maybe put in display math $\Big[\ldots \Big]$

(1) $(xh; y_1, \ldots, y_r; g) \sim (x; y_{\pi(h)^{-1}(1)}, \ldots, y_{\pi(h)^{-1}(r)}; \mu(h; e, \ldots, e)g)$ where the lefthand side is an element of $X(r) \times Y(k_1) \times \cdots \times Y(k_r) \times G(n)$ and the righthand side is an element of $X(r) \times Y(k_{\pi(h)^{-1}(1)}) \times \cdots \times Y(k_{\pi(h)^{-1}(r)}) \times G(n)$; and

(2) $(x; y_1, \ldots, y_r; \mu(e; g_1, \ldots, g_r)g) \sim (xe; y_1g_1, \ldots, y_rg_r; g)$ where $x \in X(r)$, $y_i \in Y(k_i)$, $f \in G(r)$, $g_i \in G(k_i)$, and $g \in G(n)$.

We will now develop the tools to prove that the category **G-Coll** has a monoidal structure given by $\circ$, and that operads are the monoids therein.

Theorem 1.10. *Let* **G** *be an action operad.*

(1) *The category* **G-Coll** *has a monoidal structure with tensor product given by $\circ$ and unit given by the collection I with $I(n) = \emptyset$ when $n \neq 1$, and $I(1) = G(1)$ with the* **G**-*action given by multiplication on the right.*

(2) *The category* Mon(**G-Coll**) *of monoids in* **G-Coll** *is equivalent to the category of* **G**-*operads with morphisms being those operad maps $P \to Q$ for which each $P(n) \to Q(n)$ is $G(n)$-equivariant.*

While this theorem can be proven by direct calculation using the equivalence relation given above, such a proof is unenlightening. Furthermore, we want to consider **G**-operads in categories other than sets, so an element-wise proof might not apply. Instead we now develop some general machinery that will apply to **G**-operads in any cocomplete symmetric monoidal category in which each of the functors $X \otimes -, - \otimes X$ preserve colimits (as is the case if the monoidal structure is closed). This theory also demonstrates the importance of the final axiom in the definition of an action operad. We begin with a calculational lemma.

Lemma 1.11. *Let* **G** *be an action operad, and write e_n for the unit element in the group $G(n)$.*

(1) *In $G(1)$, the unit element e_1 is equal to the identity element for the operad structure,* id.

(2) *The equation*
$$\mu(e_n; e_{i_1}, \ldots, e_{i_n}) = e_I$$
holds for any natural numbers $n, i_j, I = \sum i_j$.

(3) *The group $G(1)$ is abelian.*

Proof. For the first claim, let $g \in G(1)$. Then

$$\begin{aligned} g &= g \cdot e_1 \\ &= \mu(g; \mathrm{id}) \cdot \mu(\mathrm{id}; e_1) \\ &= \mu(g \cdot \mathrm{id}; \mathrm{id} \cdot e_1) \\ &= \mu(g \cdot \mathrm{id}; \mathrm{id}) \\ &= g \cdot \mathrm{id} \end{aligned}$$

using that e_1 is the unit element for the group structure, that id is a two-sided unit for operad multiplication, and the final axiom for an action operad together with the fact that the only element of the symmetric group Σ_1 is the identity permutation. Thus $g = g \cdot \mathrm{id}$, so $\mathrm{id} = e_1$.

For the second claim, write the operadic product as $\mu(e; \overline{e})$, and consider the square of this element. We have

$$\begin{aligned} \mu(e; \overline{e}) \cdot \mu(e; \overline{e}) &= \mu(e \cdot e; \overline{e} \cdot \overline{e}) \\ &= \mu(e; \overline{e}) \end{aligned}$$

where the first equality follows from the last action operad axiom together with the fact that e gets mapped to the identity permutation; $\overline{e} \cdot \overline{e}$ is the sequence $e_{i_1} \cdot e_{i_1}, \ldots, e_{i_n} \cdot e_{i_n}$. Thus $\mu(e; \overline{e})$ is an idempotent element of the group G_I, so must be the identity element e_I.

For the final claim, note that operadic multiplication $\mu : G(1) \times G(1) \to G(1)$ is a group homomorphism by the action operad axioms, and $\mathrm{id} = e_1$ is a two-sided unit, so the Eckmann-Hilton argument shows that μ is actually group multiplication and that G_1 is abelian. $\square$

Our construction of the monoidal structure on the category of **G**-collections will require the Day convolution product [3]. This is a general construction which produces a monoidal structure on the category of presheaves $[\mathcal{V}^{op}, \mathbf{Sets}]$ from a monoidal structure on the category $\mathcal{V}$. Since the category of **G**-collections is the presheaf category $[\mathbb{G}^{op}, \mathbf{Sets}]$, we need to show that $\mathbb{G}$ has a monoidal structure.

Proposition 1.12. *The operad structure on G gives $\mathbb{G}$ a strict monoidal structure.*

Proof. The tensor product on $\mathbb{G}$ is given by addition on objects, with unit object 0. The only thing to do is define the tensor product on morphisms and check naturality for the associativity and unit isomorphisms, which will both be identities. On morphisms, $+$ must be given by a group homomorphism $+ : G(n) \times G(m) \to G(n+m)$, and this is given by the formula

$$+(g,h) = \mu(e_2; g, h);$$

that this is a group homomorphism follows from the final axiom of an action operad and the second part of Lemma 1.11. We now write $+(g,h)$ as $g+h$.

For naturality of the associator, we must have $(f+g)+h = f+(g+h)$. By operad associativity, the lefthand side is given by

$$\mu(\mu(e_2; e_2, \mathrm{id}); f, g, h),$$

while the righthand side is then

$$\mu(\mu(e_2; \mathrm{id}, e_2); f, g, h).$$

By Lemma 1.11, both of these are equal to $\mu(e_3; f, g, h)$, proving associativity. Naturality of the unit follows similarly, using e_0. $\square$

Now that $\mathbb{G}$ has a monoidal structure, we get a monoidal structure on $[\mathbb{G}^{op}, \mathbf{Sets}] =$ $\hat{\mathbb{G}}$ using Day convolution, denoted $\star$. We refer the reader to [3] for further details. We do note, however, that the n-fold Day convolution product of a presheaf Y with itself is given by the following coend formula.

$$Y^{\star n}(k) = \int^{(k_1,\ldots,k_n)\in \mathbb{G}^n} Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n)$$ s.t. $k = \sum_i k_i$

In order to express the substitution tensor product as a coend, we need one final result about the Day convolution product.

Lemma 1.13. *Let **G** be an action operad, let $Y \in \hat{\mathbb{G}}$, and let k be a fixed natural number. Then the assignment*

$$n \mapsto Y^{\star n}(k)$$

*can be given the structure of a functor $\mathbb{G} \to$ **Sets**.*

Proof. Since the convolution product is given by a coend, it is the universal object with maps

$$Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n) \to Y^{\star n}(k)$$

such that the following diagram commutes for every $g_1 \in G(k_1), \ldots, g_n \in G(k_n)$.

$$\begin{tikzcd}[column sep=small]
 & Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n) \arrow[dr] & \\
Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n) \arrow[ur, "(-\cdot g_1, \ldots, -\cdot g_n)\times 1"] \arrow[dr, "1\times((g_1+\cdots+g_n)\cdot -)"'] & & Y^{\star n}(k) \\
 & Y(k_1) \times \cdots \times Y(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n) \arrow[ur] &
\end{tikzcd}$$

The first map along the top acts using the g_i's, while the first map along the bottom is given by

$$h \mapsto \mu(e_n; g_1, \ldots, g_n) \cdot h$$

in the final coordinate.

Let $f \in G(n)$, in other words a morphism $n \to n$ in $\mathbb{G}$. We induce a map $f \bullet - : Y^{\star n}(k) \to Y^{\star n}(k)$ using the collection of maps

$$\prod_{i=1}^{n} Y(k_i) \times \mathbb{G}(k, k_1 + \cdots + k_n) \to \prod_{i=1}^{n} Y(k_{\pi(f)^{-1}(i)}) \times \mathbb{G}(k, k_1 + \cdots + k_n)$$

by using the symmetry $\pi(f)$ on the first n factors and left multiplication by the element $\mu(f; e_{k_1}, \ldots, e_{k_n})$ on $\mathbb{G}(k, k_1 + \cdots + k_n)$. To induce a map between the coends, we must show that these maps commute with the two lefthand maps in the diagram above. For the top map, this is merely functoriality of the product together with naturality of the symmetry. For the bottom map, this is the equation

$$\mu(f; \overline{e}) \cdot \mu(e; g_1, \ldots, g_n) = \mu(e; g_{\pi(f)^{-1}1}, \ldots, g_{\pi(f)^{-1}n}) \cdot \mu(f; \overline{e}).$$

Both of these are equal to $\mu(f; g_1, \ldots, g_n)$ by the action operad axiom. Functoriality is then easy to check using that the maps inducing $(f_1 f_2) \bullet -$ are given by the composite of the maps inducing $f_1 \bullet (f_2 \bullet -)$. $\square$

We are now ready for the abstract description of the substitution tensor product. The following proposition is easily checked directly using the definition of the coend.

Proposition 1.14. *Let $X, Y \in \hat{\mathbb{G}}$. Then*

$$(X \circ Y)(k) \cong \int^n X(n) \times Y^{\star n}(k).$$

Finally we are in a position to prove Theorem 1.10.

Proof of 1.10. For the first claim, we must give the unit and associativity isomorphisms and then check the monoidal category axioms. First, note that $I = \mathbb{G}(-, 1)$. Then for the left unit isomorphism, we have

$$\begin{aligned} I \circ Y(k) &\cong \textstyle\int^n \mathbb{G}(n,1) \times Y^{\star n}(k) \\ &\cong Y^{\star 1}(k) \\ &\cong Y(k) \end{aligned}$$

using only the properties of the coend. For the right unit isomorphism, we have

$$\begin{aligned} X \circ I(k) &\cong \int^n X(n) \times I^{*n}(k) \\ &\cong \int^n X(n) \times \int^{k_1,\ldots,k_n} \mathbb{G}(k_1,1) \times \cdots \times \mathbb{G}(k_n,1) \times \mathbb{G}(k, k_1 + \cdots + k_n) \\ &\cong \int^n X(n) \times \mathbb{G}(k,n) \\ &\cong X(k) \end{aligned}$$

using the same methods.

For associativity, we compute $(X \circ Y) \circ Z$ and $X \circ (Y \circ Z)$.

$$\begin{aligned} (X \circ Y) \circ Z(k) &= \int^m X \circ Y(m) \times Z^{*m}(k) \\ &= \int^m \left(\int^l X(l) \times Y^{*l}(m) \right) \times Z^{*m}(k) \\ &\cong \int^{m,l} X(l) \times Y^{*l}(m) \times Z^{*m}(k) \\ &\cong \int^l X(l) \times \int^m Y^{*l}(m) \times Z^{*m}(k) \end{aligned}$$

The first isomorphism is from products distributing over colimits and hence coends, and the second is that fact plus the Fubini Theorem for coends [14]. A similar calculation shows

$$X \circ (Y \circ Z)(k) \cong \int^l X(l) \times (Y \circ Z)^{*l}(k).$$

Thus the associativity isomorphism will be induced once we construct an isomorphism $\int^m Y^{*l}(m) \times Z^{*m} \cong (Y \circ Z)^{*l}$. We do this by induction, with the $l = 1$ case being the isomorphism $Y^{*1} \cong Y$ together with the definition of $\circ$. Assuming true for l, we prove the case for $l+1$ by the calculations below.

$$\begin{aligned} (Y \circ Z)^{*l+1} &\cong (Y \circ Z) * (Y \circ Z)^{*l} \\ &\cong (Y \circ Z) * \left(\int^m Y^{*l}(m) \times Z^{*m} \right) \\ &= \left(\int^n Y(n) \times Z^{*n} \right) * \left(\int^m Y^{*l}(m) \times Z^{*m} \right) \\ &= \int^{a,b} \left(\int^n Y(n) \times Z^{*n}(a) \right) \times \left(\int^m Y^{*l}(m) \times Z^{*m}(b) \right) \times \mathbb{G}(-, a+b) \\ &\cong \int^{a,b,n,m} Y(n) \times Y^{*l}(m) \times Z^{*n}(a) \times Z^{*m}(b) \times \mathbb{G}(-, a+b) \\ &\cong \int^{n,m} Y(n) \times Y^{*l}(m) \times Z^{*(n+m)} \\ &\cong \int^j \int^{n,m} Y(n) \times Y^{*l}(m) \times \mathbb{G}(j, n+m) \times Z^{*j} \\ &\cong \int^j Y^{*(l+1)}(j) \times Z^{*j} \end{aligned}$$

Each isomorphism above arises from the symmetric monoidal structure, the properties of the coend, or the fact that products distribute over colimits.

For the monoidal category axioms on $\hat{\mathbb{G}}$, we only need to note that the unit and associativity isomorphisms arise, using the universal properties of the coend, from the same isomorphisms on the category of sets together with the interaction between products and colimits. Hence the monoidal category axioms follow by those same axioms in **Sets** together with the universal property of the coend.

Now we must show that monoids in $(\hat{\mathbb{G}}, \circ)$ are operads. By the Yoneda lemma, a map of $\mathbb{G}$-collections $\eta : I \to X$ corresponds to an element id $\in X(1)$ since $I = \mathbb{G}(-,1)$. A map $\mu : X \circ X \to X$ is given by a collection of $G(k)$-equivariant maps $X \circ X(k) \to X(k)$. By the universal property of the coend, this is equivalent to giving maps

$$\mu_{n,\underline{k}} : X(n) \times X(k_1) \times \cdots \times X(k_n) \times \mathbb{G}(k, k_1 + \cdots + k_n) \to X(k)$$

which are compatible with the actions of $G(k)$ (using the hom-set in the source, and the standard right action in the target) as well as each of $G(n), G(k_1), \ldots, G(k_n)$.

The hom-set in $\mathbb{G}$ is nonempty precisely when $k = k_1 + \cdots + k_n$, so we define the operad multiplication μ for X to be

$$\mu(x; y_1, \ldots, y_n) = \mu_{n,\underline{k}}(x; y_1, \ldots, y_n; e_k).$$

Compatibility with the actions of the $G(n), G(k_1), \ldots, G(k_n)$ give the equivariance axioms, and the unit and associativity for the monoid structure give the unit and associativity axioms for the operad structure. Finally, it is easy to check that a map of monoids is nothing more than an operad map which is appropriately equivariant for each n. □

Remark 1.15. The above result can be interpreted for $\mathbf{G}$-operads in an arbitrary cocomplete symmetric monoidal category $\mathcal{V}$ in which tensor distributes over colimits in each variable. In order to do so, the following changes must be made. First, cartesian products of objects $X(k)$ must be replaced by the tensor product in $\mathcal{V}$ of the same objects. Second, any product with a hom-set from $\mathbb{G}$ must be replaced by a copower with the same set (recall that the copower of a set S with an object X is given by the formula $S \odot X = \coprod_S X$). The same changes also allow one to interpret the results below about algebras in such a category, unless noted otherwise.

Let P be a G-operad, and let X be any set. Write $P(n) \times_{G(n)} X^n$ for the coequalizer of the pair of maps

$$P(n) \times G(n) \times X^n \rightrightarrows P(n) \times X^n$$

of which the first map is the action of $G(n)$ on $P(n)$ and the second map is

$$P(n) \times G(n) \times X^n \to P(n) \times \Sigma_n \times X^n \to P(n) \times X^n$$

using $\pi_n : G(n) \to \Sigma_n$ together with the canonical action of Σ_n on X^n by permutation of coordinates: $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. By the universal property of the coequalizer, a function $f : P(n) \times_{G(n)} X^n \to Y$ can be identified with a function $\tilde{f} : P(n) \times X^n \to Y$ such that

$$\tilde{f}(p \cdot g, x_1, \ldots, x_n) = \tilde{f}(p, x_{\pi(g)^{-1}(1)}, \ldots, x_{\pi(g)^{-1}(n)}).$$

Definition 1.16. Let P be a G-operad. An *algebra* for P consists of a set X together with maps $\alpha_n : P(n) \times_{G(n)} X^n \to X$ such that the maps $\tilde{\alpha}_n$ satisfy the usual operad algebra axioms.

Definition 1.17. The category of algebras for P, P-**Alg**, has objects the P-algebras (X, α) and morphisms $f : (X, \alpha) \to (Y, \beta)$ those functions $f : X \to Y$ such that the following diagram commutes for every n.

$$\begin{tikzcd} P(n) \times X^n \arrow[r, "1 \times f^n"] \arrow[d, "\tilde{\alpha}_n"'] & P(n) \times Y^n \arrow[d, "\tilde{\beta}_n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Let X be a set. Then the endomorphism operad of X, denoted $\mathcal{E}_X$, is given by the sets $\mathcal{E}_X(n) = \mathbf{Sets}(X^n, X)$ with the identity function in $\mathcal{E}_X(1)$ giving the unit element and composition of functions giving the composition operation.

Lemma 1.18. *Let G be an action operad, and let X be a set. Then $\mathcal{E}_X$ carries a canonical $\mathbf{G}$-operad structure.*

Proof. $\mathcal{E}_X$ is a symmetric operad, so we define the actions by
$$\mathcal{E}_X(n) \times G(n) \to \mathcal{E}_X \times \Sigma_n \to \mathcal{E}_X.$$
□

The previous result is really a change-of-structure groups result. We record the general result as the following proposition and note that the proof is essentially the same as that for the lemma.

Slice category

op/Σ

$P \xrightarrow{\pi} \Sigma$

Proposition 1.19. *Let $f : \mathbf{G} \to \mathbf{G}'$ be a map of action operads (i.e., a map of operads over Σ which is also a group homomorphism in each dimension). Then f induces a functor f^* from the category of $\mathbf{G}'$-operads to the category of $\mathbf{G}$-operads.*

We can now use endomorphism operads to characterize algebra structures.

Proposition 1.20. *Let X be a set, and P a $\mathbf{G}$-operad. Then algebra structures on X are in 1-to-1 correspondence with $\mathbf{G}$-operad maps $P \to \mathcal{E}_X$.*

Proof. A map $P(k) \to \mathcal{E}_X(k)$ corresponds, using the closed structure on **Sets**, to a map $P(k) \times X^k \to X$. The monoid homomorphism axioms give the unit and associativity axioms, and the requirement that $P \to \mathcal{E}_X$ be a map of $\mathbf{G}$-operads gives the equivariance axioms. □

Remark 1.21. The proposition above holds for P-algebras in any closed symmetric monoidal category. Having a closed structure (in addition to all small colimits) is a stronger condition than the tensor preserving colimits in each variable, but it is a natural one that arises in many examples.

Definition 1.22. Let P be a $\mathbf{G}$-operad. Then P induces an endofunctor of **Sets**, denoted $\underline{P}$, by the following formula.
$$\underline{P}(X) = \coprod_n P(n) \times_{G(n)} X^n$$

We now have the following proposition; its proof is standard, and we leave it to the reader.

Proposition 1.23. *Let P be a $\mathbf{G}$-operad.*

(1) *The operad structure on P induces a monad structure on $\underline{P}$.*
(2) *The category of algebras for the operad P is isomorphic to the category of algebras for the monad $\underline{P}$.*

2. Operads in Cat

In this section we will be considering $\mathbf{G}$-operads P for which each $P(n)$ is a category. It then follows that $\underline{P}$ is not only a monad but in fact a 2-monad (see [8] for background on 2-monads). We can see how $\underline{P}$ acquires the structure of a 2-functor as follows. We define $\underline{P}$ on objects (which are now categories rather than sets) much like before as the coproduct
$$\underline{P}(X) = \coprod_n P(n) \times_{G(n)} X^n,$$
whose objects will be written as equivalence classes $[p; x_1, \ldots, x_n]$ where $p \in P(n)$ and each $x_i \in X$, sometimes written as $[p; \underline{x}]$ where there is no confusion. On

and

Extend this.

Comment about operads in 2-categories?

maps: I or id?

objects: 1, * or I?

functors we define $\underline{P}$ in a similar way, exactly as with sets. Given a natural transformation $\alpha\colon f \Rightarrow g$ we define a new natural transformation $\underline{P}(\alpha)$ with components
$$\boxed{\underline{P}(\alpha)_{[p;\underline{x}]} = [p; \underline{\alpha_x}].}$$
– Write explicitly

It is a simple observation that this constitutes a 2-functor and that the components of the unit and multiplication are functors and are 2-natural.

The algebras for this induced 2-monad then correspond to algebras for the operad. Since we are working with a 2-monad we can also consider the pseudoalgebras and, as the operad is Cat-enriched, a notion of pseudoalgebra for an operad makes sense ~~and corresponds~~ precisely with those for the monad. This happens similarly for the morphisms of algebras and the appropriate 2-cells. For much of this section, for space considerations, we will denote $P(n)$ by P_n and similarly for the $G(n)$. Throughout we will denote the unit map for an operad P as $\eta^P : 1 \to P(1)$. First we will set out some definitions.

in diagrams

Just as before, we will identify maps $\alpha_n\colon P(n) \times_{G(n)} X^n \to X$ with maps $\tilde{\alpha}_n\colon P(n) \times X^n \to X$ via the universal property of the coequalizer. Note also that in the following definitions we will often write the composite
$$P(n) \times \prod (P(k_i) \times X^{k_i}) \to P(n) \times \prod P(k_i) \times X^{\Sigma k_i} \xrightarrow{\mu^P \times 1} P(\Sigma_{k_i}) \times X^{\Sigma k_i}$$
simply abbreviated as $\mu^P \times 1$.

Definition 2.1. Let P be a $\mathbf{G}$-operad. A pseudoalgebra for P consists of:

- a category X
- a family of functors
$$(\alpha_n : P(n) \times_{G(n)} X^n \to X)_{n \in \mathbb{N}}$$
- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism

$$\begin{tikzcd}
P_n \times \prod_{i=1}^n (P_{k_i} \times X^{k_i}) \arrow[rr, "1 \times \prod \tilde{\alpha}_{k_i}"] \arrow[d] & & P_n \times X^n \arrow[dd, "\tilde{\alpha}_n"] \\
P_n \times \prod_{i=1}^n P_{k_i} \times X^{\Sigma k_i} \arrow[d, "\mu^P \times 1"'] \arrow[rr, phantom, "\Downarrow \phi_{k_1,\ldots,k_n}"] & & \\
P_{\Sigma k_i} \times X^{\Sigma k_i} \arrow[rr, "\tilde{\alpha}_{\Sigma k_i}"'] & & X
\end{tikzcd}$$

Σ has no indices

$\boxed{\prod_{i=1}^n \text{ vs. } \Sigma}$

- and a natural isomorphism

$$\begin{tikzcd}
X \arrow[d, "\cong"'] \arrow[ddr, "1"] & \\
1 \times X \arrow[d, "\eta^P \times 1"'] \arrow[r, phantom, "\Downarrow \phi_\eta"] & \\
P(1) \times X \arrow[r, "\tilde{\alpha}_1"'] & X
\end{tikzcd}$$

satisfying the following axioms.

$\prod_{i=1}^{n}$ or $\prod_i$?

- For all $n, k_i, m_{ij} \in \mathbb{N}$, the following equality of pasting diagrams holds.

$$\begin{tikzcd}
P_n\times\prod_i\left(P_{k_i}\times\prod_j\left(P_{m_{ij}}\times X^{m_{ij}}\right)\right) \arrow[rr, "1\times\prod(1\times\prod\bar\alpha_{m_{ij}})"] \arrow[dr, "\mu^P\times 1"] \arrow[dd, "\mu^P\times 1"'] & \arrow[d, phantom, "\Downarrow 1\times\prod_i\phi_{m_{i1},\dots,m_{ik_i}}"] & P_n\times\prod_i\left(P_{k_i}\times X^{k_i}\right) \arrow[dr, "1\times\prod\bar\alpha_{k_i}"] & \\
& P_n\times\prod_i\left(P_{\Sigma m_{ij}}\times X^{\Sigma m_{ij}}\right) \arrow[rr, "1\times\prod\bar\alpha_{\Sigma m_{ij}}"] \arrow[dd, "\mu^P\times 1"] & \arrow[d, phantom, "\Downarrow\phi_{\Sigma m_{1j},\dots,\Sigma m_{nj}}"] & P_n\times X^n \arrow[dd, "\bar\alpha_n"] \\
P_{\Sigma k_i}\times\prod_i\prod_j\left(P_{m_{ij}}\times X^{m_{ij}}\right) \arrow[dr, "\mu^P\times 1"'] & & & \\
& P_{\Sigma\Sigma m_{ij}}\times X^{\Sigma\Sigma m_{ij}} \arrow[rr, "\bar\alpha_{\Sigma\Sigma m_{ij}}"'] & & X
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
P_n\times\prod_i\left(P_{k_i}\times\prod_j\left(P_{m_{ij}}\times X^{m_{ij}}\right)\right) \arrow[r, "1\times\prod(1\times\prod\bar\alpha_{m_{ij}})"] \arrow[dd, "\mu^P\times 1"'] & P_n\times\prod_i\left(P_{k_i}\times X^{k_i}\right) \arrow[dr, "1\times\prod\bar\alpha_{k_i}"] \arrow[dd, "\mu^P\times 1"'] & \\
& \arrow[r, phantom, "\Downarrow\phi_{k_1,\dots,k_n}"] & P_n\times X^n \arrow[dd, "\bar\alpha_n"] \\
P_{\Sigma k_i}\times\prod_i\prod_j\left(P_{m_{ij}}\times X^{m_{ij}}\right) \arrow[r, "1\times\prod\prod\bar\alpha_{m_{ij}}"] \arrow[d, "\mu^P\times 1"'] \arrow[dr, phantom, "\Downarrow\phi_{m_{11},\dots,m_{nk_n}}"] & P_{\Sigma k_i}\times X^{\Sigma k_i} \arrow[dr, "\bar\alpha_{\Sigma k_i}"] & \\
P_{\Sigma\Sigma m_{ij}}\times X^{\Sigma\Sigma m_{ij}} \arrow[rr, "\bar\alpha_{\Sigma\Sigma m_{ij}}"'] & & X
\end{tikzcd}$$

- Each pasting diagram of the following form is an identity.

$$\begin{tikzcd}
P_n\times X^n \arrow[dr, "\cong"] \arrow[ddrr, bend left, "1"] \arrow[ddddr, bend right, "1"'] & & \\
& P_n\times(1\times X)^n \arrow[d, "1\times(\eta^P\times 1)^n"] \arrow[r, phantom, "\Downarrow 1\times\phi_\eta^n"] & {} \\
& P_n\times(P_1\times X)^n \arrow[r, "1\times\bar\alpha_1^n"] \arrow[d, "\cong"] & P_n\times X^n \arrow[dd, "\bar\alpha_n"] \\
& P_n\times P_1^n\times X^n \arrow[d, "\mu^P\times 1"] \arrow[r, phantom, "\Downarrow\phi_{1,\dots,1}"] & {} \\
& P_n\times X^n \arrow[r, "\bar\alpha_n"'] & X
\end{tikzcd}$$

Definition 2.2. Let $(X, \alpha_n, \phi, \phi_\eta)$ and $(Y, \beta_n, \psi, \psi_\eta)$ be pseudoalgebras for a G-operad P. A weak morphism of P-algebras consists of:

- a functor $f\colon X \to Y$
- a family of natural isomorphisms

$$\begin{tikzcd}
P_n\times X^n \arrow[r, "\bar\alpha_n"] \arrow[d, "1\times f^n"'] \arrow[dr, phantom, "\Downarrow\bar f_n"] & X \arrow[d, "f"] \\
P_n\times Y^n \arrow[r, "\bar\beta_n"'] & Y
\end{tikzcd}$$

satisfying the following axioms.

- The following equality of pasting diagrams holds.

$$\begin{tikzcd}
P_n\times\prod_i\left(P_{k_i}\times X^{k_i}\right) \arrow[rr, "1\times\prod(1\times f^{k_i})"] \arrow[dd, "\mu^P\times 1"'] & & P_n\times\prod_i\left(P_{k_i}\times Y^{k_i}\right) \arrow[dr, "1\times\prod\bar\beta_{k_i}"] \arrow[dd, "\mu^P\times 1"'] & \\
& & \arrow[r, phantom, "\Downarrow\psi_{k_1,\dots,k_n}"] & P_n\times Y^n \arrow[dd, "\bar\beta_n"] \\
P_{\Sigma k_i}\times X^{\Sigma k_i} \arrow[rr, "1\times f^{\Sigma k_i}"] \arrow[dr, "\bar\alpha_{\Sigma k_i}"'] & \arrow[d, phantom, "\Downarrow\bar f_n"] & P_{\Sigma k_i}\times Y^{\Sigma k_i} \arrow[dr, "\bar\beta_{\Sigma k_i}"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
P_n\times\prod_i\left(P_{k_i}\times X^{k_i}\right) \arrow[rr, "1\times\prod(1\times f^{k_i})"] \arrow[dr, "1\times\prod\bar\alpha_{k_i}"] \arrow[dd, "\mu^P\times 1"'] & \arrow[d, phantom, "\Downarrow 1\times\prod\bar f_{k_i}"] & P_n\times\prod_i\left(P_{k_i}\times Y^{k_i}\right) \arrow[dr, "1\times\prod\bar\beta_{k_i}"] & \\
\arrow[r, phantom, "\Downarrow\phi_{k_1,\dots,k_n}"] & P_n\times X^n \arrow[rr, "1\times f^n"] \arrow[dd, "\bar\alpha_n"] & \arrow[d, phantom, "\Downarrow\bar f_n"] & P_n\times Y^n \arrow[dd, "\bar\beta_n"] \\
P_{\Sigma k_i}\times X^{\Sigma k_i} \arrow[dr, "\bar\alpha_{\Sigma k_i}"'] & & {} & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

- The following equality of pasting diagrams holds.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \arrow[dddr, bend left, "1"] & \\
1 \times X \arrow[r, "1\times f"] \arrow[d, "\eta^P\times 1"'] & 1 \times Y \arrow[d, "\eta^P\times 1"] \arrow[r, phantom, "\Downarrow\psi_\eta"] & {} \\
P_1 \times X \arrow[r, "1\times f"] \arrow[dr, "\bar\alpha_1"'] & P_1 \times Y \arrow[dr, "\bar\beta_1"] \arrow[d, phantom, "\Downarrow\bar f_1"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\cong"'] \arrow[ddr, bend left, "1"] \arrow[dr, phantom, "\Downarrow\phi_\eta"] & Y \arrow[dd, bend left=50, "1"] \\
1 \times X \arrow[d, "\eta^P\times 1"'] & \\
P_1 \times X \arrow[r, "\bar\alpha_1"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Definition 2.3. Let $(f,\bar f),(g,\bar g)\colon (X,\alpha,\phi,\phi_\eta)\to(Y,\beta,\psi,\psi_0)$ be weak morphisms of pseudoalgebras. A P-transformation is then a natural transformation $\gamma\colon f\Rightarrow g$ such that the following following equality of pasting diagrams holds, for all n.

$$
\begin{tikzcd}
P_n \times X^n \arrow[r, bend left=50, "1\times f^n"] \arrow[r, "1\times g^n"'] \arrow[r, phantom, bend left=25, "\Downarrow 1\times\gamma^n"] \arrow[d, "\bar\alpha_n"'] \arrow[dr, phantom, "\Downarrow\bar g_n"] & P_n \times Y^n \arrow[d, "\bar\beta_n"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
P_n \times X^n \arrow[r, "1\times f^n"] \arrow[d, "\bar\alpha_n"'] \arrow[dr, phantom, "\Downarrow\bar f_n"] & P_n \times Y^n \arrow[d, "\bar\beta_n"] \\
X \arrow[r, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, bend right=25, "\Downarrow\gamma"] & Y
\end{tikzcd}
$$

Remark 2.4. We could also consider the notion of a lax algebra for an operad, or a lax morphism of algebras, simply by considering natural transformations in place of isomorphisms in the definitions. Of course, the usual definitions of algebra and algebra morphisms are retrieved if we were to consider identities in place of the isomorphisms.

In the previous section we noted that there is an isomorphism of categories between the category of algebras for the operad P and the category of algebras for the associated monad $\underline{P}$. This can again be done in the case of the induced 2-monads that we have been talking about.

Proposition 2.5. *Let P be a* **G**-*operad. Then there is an isomorphism of 2-categories*

$$P\text{-}\mathbf{Alg}_s \cong \underline{P}\text{-}\mathbf{Alg}_s.$$

Proof. The isomorphism on the underlying categories is obvious, as in the previous section. All that remains to be seen is that there is a bijection of 2-cells between each of the hom-categories. A P-transformation between strict morphisms is exactly as in the previous definition except that the squares in the pasting diagrams now commute. It is clear that a 2-cell in either of the 2-categories of algebras will satisfy the properties required of it in the other 2-category. □

We now turn from the 2-category of algebras and strict morphisms to the 2-category of pseudoalgebras and pseudomorphisms (see [2] for background on the different 2-categories of algebras for a 2-monad).

Proposition 2.6. *Let P be a* **G**-*operad. Then there is an isomorphism of 2-categories*

$$\mathbf{Ps}\text{-}P\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}.$$

Proof. We begin by constructing a 2-functor $R\colon \mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg} \to \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}$. We take a pseudo-$\underline{P}$-algebra $(X,\alpha,\Phi,\Phi_\eta)$, with Φ being the structure isomorphism for the pseudoalgebra as in the diagram below, to the following. First we define functor α_n to be the composite

$$\alpha_n\colon P(n)\times_{G(n)} X^n \hookrightarrow \underline{P}(X) \xrightarrow{\alpha} X.$$

The isomorphisms $\phi_{k_1,\dots,k_n}$ are defined using Φ as in the following diagram

$$
\begin{tikzcd}[column sep=small]
P_n\times\prod_{i=1}^n\left(P_{k_i}\times X^{k_i}\right) \arrow[r] \arrow[dd] & P_n\times\prod_i\left(P_{k_i}\times_{G_{k_i}} X^{k_i}\right) \arrow[r, hook] & P_n\times\underline{P}(X)^n \arrow[r, "1\times\alpha^n"] \arrow[d] & P_n\times X^n \arrow[d] \\
& & P_n\times_{G_n}\underline{P}(X)^n \arrow[d, hook] & P_n\times_{G_n} X^n \arrow[d, hook] \\
P_n\times\prod_i P_{k_i}\times X^{\Sigma k_I} \arrow[d, "\mu^P\times 1"'] & & \underline{P}^2(X) \arrow[r, "\underline{P}\alpha"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Downarrow\Phi"] & \underline{P}(X) \arrow[d, "\alpha"] \\
P_{\Sigma k_i}\times X^{\Sigma k_i} \arrow[r] & P_{\Sigma k_i}\times_{G_{\Sigma k_i}} X^{\Sigma k_i} \arrow[r, hook] & \underline{P}(X) \arrow[r, "\alpha"'] & X
\end{tikzcd}
$$

whilst Φ_η is simply sent to itself, since the composition of α with the composite of the coequalizer and inclusion map from $P(1)\times X$ into $\underline{P}(X)$ is just $\bar{\alpha}_1$. Checking the axioms here is most easily done on components and it is easily seen that the axioms required of this data to be a pseudo-P-algebra are precisely those that they satisfy by virtue of X being a pseudo-$\underline{P}$-algebra.

For a 1-cell $(f,\bar f)\colon (X,\alpha)\to(Y,\beta)$, we send f to itself whilst sending $\bar f$ to the obvious family of isomorphisms, as follows.

$$
\begin{tikzcd}
P(n)\times X^n \arrow[r] \arrow[d, "1\times f^n"'] & P(n)\times_{G(n)} X^n \arrow[r, hook] \arrow[d, "1\times f^n"'] & \underline{P}(X) \arrow[r, "\alpha"] \arrow[d, "\underline{P}f"'] \arrow[dr, phantom, "\Downarrow\bar f"] & X \arrow[d, "f"] \\
P(n)\times Y^n \arrow[r] & P(n)\times_{G(n)} Y^n \arrow[r, hook] & \underline{P}(Y) \arrow[r, "\beta"'] & Y
\end{tikzcd}
$$

This collection then satisfies the axioms for being a weak morphism of pseudo-P-algebras, following from the axioms for it being a weak morphism of pseudo-$\underline{P}$-algebras. The 2-cells are sent in the same way as for strict algebras and can immediately be seen to satisfy the appropriate axioms. That R is a 2-functor is now obvious.

For there to be an isomorphism of 2-categories, we require an inverse to R, namely a 2-functor $S\colon \mathbf{Ps}\text{-}P\text{-}\mathbf{Alg}\to\mathbf{Ps}\text{-}\underline{P}\text{-}\mathbf{Alg}$. If $(X,\alpha_n,\phi_{\underline{k}_i},\phi_\eta)$ is a pseudo-P-algebra then we need to give a pseudo-$\underline{P}$-algebra. We can induce a functor

R; S

$\alpha\colon \underline{P}(X) \to X$ by using the universal property of the coproduct.

$$\begin{tikzcd} P(n) \times X^n \arrow[r] \arrow[drr, "\tilde{\alpha}_n"'] & P(n) \times_{G(n)} X^n \arrow[r, hook] \arrow[dr, "\alpha_n"] & \underline{P}(X) \arrow[d, "\exists!\alpha"] \\ & & X \end{tikzcd}$$

$\Phi : \alpha \circ \underline{P}(\alpha) \Rightarrow \alpha \cdot \mu_X$

Of course, this can be induced using either α_n or $\tilde{\alpha}_n$, each giving the same functor α by uniqueness. The components of the isomorphism $\Phi\colon \alpha\underline{P}\alpha \Rightarrow \alpha\mu_X$ can be given as follows. Let $|\underline{x}_i|$ denote the number of objects in the list $\underline{x}_i$. Then define the component of Φ at the object

$$[p; [q_1; \underline{x}_1], \ldots, [q_n; \underline{x}_n]]$$

to be component of $\phi_{|\underline{x}_1|,\ldots,|\underline{x}_n|}$ at the same object. The isomorphism ϕ_η is again sent to itself.

Now given a 1-cell f with structure 2-cells $\overline{f}_n$: we can define a 1-cell $(F, \overline{F}) = (f, \overline{F})$ as

$$\overline{F}_{[p;\underline{x}]} := \left(\overline{f}_{|\underline{x}|}\right)_{p;\underline{x}}.$$

Clunky wording

Define a weak map w/ und. 1-cell f & structure 2-cells $\overline{f}$ as follows.

A mapping for 2-cells is just the identity as before. These mappings again constitute a 2-functor in the obvious way and from how they are defined it is also clear that this is an inverse to R. □

Another interpretation of pseudoalgebras can be given in terms of pseudomorphisms of operads. Just as algebras for an operad P can be identified with a morphism of operads $F\colon P \to \mathcal{E}_X$, where $\mathcal{E}_X$ is the endomorphism operad, we can define pseudomorphisms for a **Cat**-enriched **G**-operad and identify pseudoalgebras with pseudomorphisms into the endomorphism operad as defined in the previous section.

Definition 2.7. Let P, Q be **G**-operads. A pseudomorphism of **G**-operads $F\colon P \to Q$ consists of

- a family of **G**-equivariant functors
$$(F_n\colon P(n) \to Q(n))_{n\in\mathbb{N}},$$
- for each $n, k_1, \ldots, k_n \in \mathbb{N}$, a natural isomorphism
$$\begin{tikzcd} P(n)\times\prod_i P(k_i) \arrow[r, "F_n\times\prod_i F_{k_i}"] \arrow[d, "\mu^P"'] \arrow[dr, phantom, "\Downarrow\psi_{k_1,\ldots,k_n}"] & Q(n)\times\prod_i Q(k_i) \arrow[d, "\mu^Q"] \\ P(\Sigma k_i) \arrow[r, "F_{\Sigma k_i}"'] & Q(\Sigma k_i) \end{tikzcd}$$

and

- a natural isomorphism
$$\begin{tikzcd} 1 \arrow[d, "\eta^P"'] \arrow[dr, "\eta^Q"] \arrow[d, phantom, shift left=4ex, "\Downarrow\psi_\eta"] & \\ P(1) \arrow[r, "F_1"'] & Q(1) \end{tikzcd}$$

satisfying axioms analogous to those of a pseudoalgebra for an operad.

If we now consider a pseudomorphism from P into the endomorphism operad $\mathcal{E}_X$ then we have functors

$$F_n\colon P(n) \to [X^n, X],$$

a family of isomorphisms

$$\begin{tikzcd} P(n)\times\prod_i P(k_i) \arrow[r, "F_n\times\prod_i F_{k_i}"] \arrow[d, "\mu^P"'] \arrow[dr, phantom, "\Downarrow\psi_{k_1,\ldots,k_n}"] & {[X^n,X]\times\prod_i[X^{k_i},X]} \arrow[d, "\mu^{\mathcal{E}_X}"] \\ P(\Sigma k_i) \arrow[r, "F_{\Sigma k_i}"'] & {[X^{\Sigma k_i},X]} \end{tikzcd}$$

and an isomorphism

$$\begin{tikzcd} 1 \arrow[d, "\eta^P"'] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\Downarrow\psi_\eta"] & \\ P(1) \arrow[r, "F_1"'] & {[X,X]} \end{tikzcd}$$

which then satisfy precisely the axioms for pseudoalgebras for the operad P. We characterise this with the following proposition.

Proposition 2.8. *There is a correspondence between pseudomorphisms $P \to \mathcal{E}_X$ and pseudoalgebra structures on X.*

3. Basic properties

This section will look at characterizing various properties of the 2-monads $\underline{P}$ induced by a **G**-operad P. First of all, we consider when the 2-monads are finitary as this describes how they interact with colimits. We will then give conditions for these 2-monads to be cartesian, describing how they interact with some limits. Finally in this section we will discuss the extra structure that algebras for 2-monads can have. The theory of 2-monads can be partly seen as the theory of **Cat**-enriched monads but this doesn't capture the full picture. In particular, 2-monads can not only have strict algebras but also weaker notions of pseudoalgebras and lax algebras, along with appropriate notions of morphism between them. These give rise to a number of 2-categories whose 0-cells are algebras of a set level of strictness and for which we can take morphisms of the same, or differing, level of strictness. We will use the coherence theorem given in [12] to show that each pseudo-$\underline{P}$-algebra is equivalent to strict $\underline{P}$-algebra.

a

The 2-categories **Ps**-T-**Alg** (of pseudoalgebras and weak morphisms) and T-$\mathbf{Alg}_s$ (of strict algebras and strict morphisms) are of particular interest. The behavior of colimits in both of these 2-categories can often be deduced from properties of the 2-monad T, the most common being that T is finitary. In practice, one thinks of a finitary monad as one in which all operations take finitely many inputs as variables. If T is finitary, then T-$\mathbf{Alg}_s$ will be cocomplete by standard results. There are additional results of a purely 2-dimensional nature concerning finitary 2-monads, namely the existence of a left adjoint

$$\mathbf{Ps}\text{-}T\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}_s$$

to the forgetful 2-functor which regards a strict algebra as a pseudoalgebra with identity structure isomorphisms (see [2]).

Proposition 3.1. *Let P be a* **G**-*operad. Then $\underline{P}$ is finitary.*

Proof. To show that $\underline{P}$ is finitary we must show that it preserves filtered colimits or, equivalently, that it preserves directed colimits. Consider some directed colimit, $\mathrm{colim}X_i$ say, in **Cat**. Then consider the following sequence of isomorphisms:

$$\begin{aligned}\underline{P}(\mathrm{colim}X_i) &= \coprod_n P(n) \times_{G(n)} (\mathrm{colim}X_i)^n \\ &\cong \coprod_n P(n) \times_{G(n)} \mathrm{colim}(X_i^n) \\ &\cong \coprod_n \mathrm{colim}(P(n) \times_{G(n)} X_i^n) \\ &\cong \mathrm{colim} \coprod_n P(n) \times_{G(n)} X_i^n = \mathrm{colim}\underline{P}(X_i).\end{aligned}$$

Since **Cat** is locally finitely presentable then directed colimits commute with finite limits, giving the first two isomorphisms. The third isomorphism is simply coproducts commuting with other colimits. □

The monads arising from a non-symmetric operad are always cartesian. The monads that arise from symmetric operads, however, are not always cartesian and so it is useful to be able to characterise exactly when they are. An example of where this fails is the symmetric operad for which the algebras are commutative monoids. In the case of 2-monads we can consider the analogous strict 2-limit to the pullback, the 2-pullback, and characterise when the induced 2-monad from a **G**-operad is 2-cartesian, as we now describe.

Definition 3.2. A 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ is said to be 2-cartesian if:

- the 2-category $\mathcal{K}$ has 2-pullbacks;
- the functor T preserves 2-pullbacks;
- the naturality squares for the unit and multiplication of the 2-monad are 2-pullbacks.

It is important to note that the 2-pullback of a diagram is actually the same as the ordinary pullback in **Cat**, see [7]. Since we will be computing with coequalizers of the form $A \times_G B$ repeatedly, we give the following useful lemma.

Lemma 3.3. *Given categories A and B and a group G, for which A has a right action of G and B has a left action by G, then $A \times B/G$ satisfies the universal property of the coequalizer $A \times_G B$. The equivalence classes of $A \times B/G$ correspond to the action of G on $A \times B$ for which $(a,b) \sim (a\cdot g, g^{-1}\cdot b)$.*

Proof. The category $A \times_G B$ is defined as the coequalizer

$$\begin{tikzcd} A \times G \times B \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & A \times B \arrow[r, "\varepsilon"] & A \times_G B \end{tikzcd}$$

where $\lambda(a,g,b) = (a\cdot g, b)$ and $\rho(a,g,b) = (a, g\cdot b)$. However, the category $A \times B/G$ also coequalizes λ and ρ by sending (a,b) to the equivalence class $[a,b] = [a\cdot g, g^{-1}\cdot b]$. The equivalence relation then gives us that $[a\cdot g, b] = [a, g^{-1}\cdot b]$.

Now given any other category X and a functor $\chi\colon A \times B \to X$ which coequalizes λ and ρ, we get a functor $\phi\colon A \times B/G \to X$ defined by $\phi[a,b] = \chi(a,b)$. That this is well defined is clear, since

$$\phi[a\cdot g, g^{-1}\cdot b] = \chi(a\cdot g, g^{-1}\cdot b) = \chi(a\cdot(gg^{-1}), b) = \chi(a,b) = \phi[a,b].$$

This is also unique and so we find that $A \times B/G$ satisfies the universal property of the coequalizer and $A \times_G B \cong A \times B/G$. □

Proposition 3.4. *The 2-monad $\underline{P}$ induced by an operad P is 2-cartesian if and only if the action of each $G(n)$ on $P(n)$ has the following property:*

- *if $p \in P(n)$ and $g \in G(n)$ such that $p \cdot g = p$, then $g \in \ker\pi_n$, where $\pi_n\colon G(n) \to \Sigma_n$.*

Proof. Assume, in order to derive a contradiction, that, for some n, the action of $G(n)$ on $P(n)$ does not have the prescribed property. Find some $p \in P(n)$ along with $g \notin \ker\pi_n$ such that $p\cdot g = p$. We will show that the existence of g proves that $\underline{P}$ is not cartesian.

Consider the following pullback of discrete categories.

$$\begin{tikzcd} \{(x,y),(x,y'),(x',y),(x',y')\} \arrow[r] \arrow[d] & \{y,y'\} \arrow[d] \\ \{x,x'\} \arrow[r] & \{z\} \end{tikzcd}$$

Letting **4** denote the pullback and similarly writing $\mathbf{2}_X = \{x,x'\}$ and $\mathbf{2}_Y = \{y,y'\}$, we get the following diagram as the image of this pullback square under $\underline{P}$.

$$\begin{tikzcd} \coprod P(n) \times_{G(n)} \mathbf{4}^n \arrow[r] \arrow[d] & \coprod P(n) \times_{G(n)} \mathbf{2}_Y^n \arrow[d] \\ \coprod P(n) \times_{G(n)} \mathbf{2}_X^n \arrow[r] & \coprod P(n)/G(n) \end{tikzcd}$$

Using g we can find two distinct elements

$$[p; (x',y), (x,y), \dots, (x,y), (x,y'), (x,y), \dots, (x,y)]$$

and

$$[p; (x,y), \dots, (x,y), (x',y'), (x,y), \dots, (x,y)]$$

in $\underline{P}(\mathbf{4})$, where the first has (x,y') in the $\pi(g)(1)^{th}$ position and the second similarly has (x',y') in the $\pi(g)(1)^{th}$ position. Both of these elements, however, are mapped to the same elements in $\underline{P}(\mathbf{2}_X)$, since

$$\begin{aligned}[p; x', x, \dots, x] &= [p\cdot g, x', x, \dots, x] \\ &= [p; \pi(g)(x', x, \dots, x)] \\ &= [p; x, \dots, x, x', x, \dots, x].\end{aligned}$$

Similarly, both of the elements are mapped to the same element in $\underline{P}(\mathbf{2}_Y)$, simply

$$[p; y, \dots, y].$$

The pullback of this diagram, however, has a unique element which is projected to the ones we have considered, so $\underline{P}(4)$ is not a pullback. Hence $\underline{P}$ does not preserve pullbacks if for some n the action of $G(n)$ on $P(n)$ does not have the given property.

For the rest of the proof we will assume that each $G(n)$ acts on $P(n)$ in the prescribed way. We require that the naturality squares for η and μ are 2-pullbacks. In the case of η this is to require that for a functor $f\colon X \to Y$, the pullback of the following diagram is the category X.

$$\begin{tikzcd}
 & Y \arrow[d, "\eta_Y"] \\
\coprod P(n) \times_{G(n)} X^n \arrow[r, "\underline{P}(f)"'] & \coprod P(n) \times_{G(n)} Y^n
\end{tikzcd}$$

The pullback of this diagram is then isomorphic to the coproduct of the pullbacks of diagrams of the following form.

$$\begin{tikzcd}
 & Y \arrow[d] \\
P(n) \times_{G(n)} X^n \arrow[r, "1\times f^n"'] & P(n) \times_{G(n)} Y^n
\end{tikzcd}$$

Since η_Y lands in $P(1) \times_{G(1)} Y$, which is isomorphic to $(P(1)/G(1)) \times Y$ since this satisfies the universal property of the coequalizer, then this amounts to checking that X is the pullback of the above cospan in the case that $n = 1$. That we can write the given objects in such a way follows from $G(1)$ acting trivially on the categories X and Y (each element of G gives a trivial permutation under π_1) meaning that for any $([p;x])$ with $p \in P(1)$, $x \in X$ we can write this as $([p], x)$. The pullback of that diagram would then be a category consisting of pairs $(([p], x), y)$ such that $([p], f(x)) = ([1], y)$. Such pairs exist only when $y = f(x)$ and $[p] = [1]$, giving the category X as the pullback. So naturality squares for η are pullbacks.

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For μ we will use the fact that if all of the diagrams

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_X"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

are pullbacks then the outside of the diagram

where $x \longmapsto ! \ldots$

$$\begin{tikzcd}
\underline{P}^2(X) \arrow[r, "\underline{P}^2(f)"] \arrow[d, "\mu_X"'] & \underline{P}^2(Y) \arrow[r, "\underline{P}^2(!)"] \arrow[d, "\mu_Y"'] & \underline{P}^2(1) \arrow[d, "\mu_1"] \\
\underline{P}(X) \arrow[r, "\underline{P}(f)"'] & \underline{P}(Y) \arrow[r, "\underline{P}(!)"'] & \underline{P}(1)
\end{tikzcd}$$

is also a pullback and so each of the naturality squares for μ must therefore be a pullback. Now we can split up the square above, much like we did for η, and prove

that each of the squares

$$\begin{tikzcd}
\coprod P(m) \times_{G(m)} \prod_i \left(P(k_i) \times_{G(k_i)} X^{k_i}\right) \arrow[r] \arrow[d] & \coprod P(m) \times_{G(m)} \prod_i \left(P(k_i)/G(k_i)\right) \arrow[d] \\
P(n) \times_{G(n)} X^n \arrow[r] & P(n)/G(n)
\end{tikzcd}$$

?

is a pullback. The map along the bottom is the obvious one, sending $[p; x_1, \ldots, x_n]$ simply to the equivalence class $[p]$. Along the right hand side the map is the one corresponding to operadic composition, sending $[q; [p_1], \ldots, [p_m]]$ to $[\mu^P(q; p_1, \ldots, p_n)]$. The pullback of these maps would be the category consisting of pairs

$$([p; x_1, \ldots, x_n], [q; [p_1], \ldots, [p_n]])$$

for which $[p] = [\mu^P(q; p_1, \ldots, p_n)]$. The upper left category in the diagram, which we will refer to here as Q, which has objects

$$[q; [p_1; \underline{x}_1], \ldots, [p_n; \underline{x}_n]].$$

There are obvious maps out of Q making the diagram commute and as such inducing a functor from Q into the pullback via the universal property. This functor sends an object such as the one just described to the pair

$$([\mu^P(q; p_1, \ldots, p_n); \underline{x}], [q; [p_1], \ldots, [p_n]]).$$

Given an object in the pullback, we then have a pair, as described above, which has $[p] = [\mu^P(q; p_1, \ldots, p_n)]$ meaning that we can find an element $g \in G(\Sigma k_i)$ such that $p = \mu^P(q; p_1, \ldots, p_n) \cdot g$. Thus we can describe an inverse to the induced functor by sending a pair in the pullback to the object

$$[q; [p_1; \pi(g)(\underline{x})_1], \ldots, [p_n; \pi(g)(\underline{x})_n]],$$

where $\pi(g)(\underline{x})_i$ denotes the k_ith block of $\underline{x}$ after applying the permutation $\pi(g)$. Applying the induced functor we find that we get back an object in the pullback for which the first entry is $[q; [p_1], \ldots, [p_n]]$ and whose second entry is

$$[\mu^P(q; p_1, \ldots, p_n); \pi(g)(\underline{x})] = [\mu^P(q; p_1, \ldots, p_n) \cdot g; \underline{x}] = [p; \underline{x}],$$

which is what we started with. Showing the other composite is an identity is similar, here using the fact that the identity acts trivially on $\mu^P(q; p_1, \ldots, p_n)$. Taking the coproduct of these squares then gives us the original diagram that we wanted to show was a pullback and, since each individual square is a pullback, so is the original.

To finish we must also show that $\underline{P}$ preserves pullbacks. Given a pullback

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "R"'] & B \arrow[d, "S"] \\
C \arrow[r, "H"'] & D
\end{tikzcd}$$

we must show that the image of the diagram under $\underline{P}$ is also a pullback. Now this will be true if and only if each individual diagram

$$\begin{tikzcd} P(n) \times_{G(n)} A^n \arrow[r, "1\times F^n"] \arrow[d, "1\times R^n"'] & P(n) \times_{G(n)} B^n \arrow[d, "1\times S^n"] \\ P(n) \times_{G(n)} C^n \arrow[r, "1\times H^n"'] & P(n) \times_{G(n)} D^n \end{tikzcd}$$

is also a pullback. The pullback of the functors $1 \times H^n$ and $1 \times S^n$ is a category consisting of pairs of objects $[p; \underline{c}]$ and $[q; \underline{b}]$, where $\underline{b}$ and $\underline{c}$ represent lists of elements in B and C, respectively. These pairs are then required to satisfy the property that $[p; H(\underline{c})] = [q; S(\underline{b})]$. Using the lemma, we know that a pair $([p; \underline{c}], [q; \underline{b}])$ is in the pullback if and only if there exists an element $g \in G(n)$ such that $p \cdot g = q$ and $Hc_i = (Sb_{\pi(g)^{-1}(i)})$. Using this we can define mutual inverses between $P(n) \times_{G(n)} A^n$ and the pullback Q'. Considering the category A as the pullback of the diagram we started with, we can consider objects of $P(n) \times_{G(n)} A^n$ as being equivalence classes

$$[p; (b_1, c_1), \ldots, (b_n, c_n)]$$

where $p \in P(n)$ and $Hc_i = Sb_i$ for all i.

Taking such an object, we send it to the pair

$$([p; c_1, \ldots, c_n], [p; b_1, \ldots, b_n])$$

which lies in the pullback since the identity in $G(n)$ satisfies the condition given earlier. An inverse to this sends a pair of equivalence classes in Q' to the single equivalence class

$$[p; (c_1, b_{\pi(g)^{-1}(1)}), \ldots, (c_n, b_{\pi(g)^{-1}(n)})]$$

in $P(n) \times_{G(n)} A^n$. If we apply the map into Q' we get the pair

$$([p; c_1, \ldots, c_n], [p; b_{\pi(g)^{-1}(1)}, \ldots, b_{\pi(g)^{-1}(n)}])$$

which is equal to the original pair since $p \cdot g = q$. The other composite is trivially an identity since the identity in $G(n)$ has trivial permutation. $\square$

Corollary 3.5. *The monad for a symmetric operad P is cartesian if and only if the action of Σ_n is free on $P(n)$.*

The final part of this section is motivated by the issue of coherence. At its most basic, a coherence theorem is a way of describing when a notion of weaker structure is in some way equivalent to a stricter structure. The prototypical case here is the coherence theorem for monoidal categories. In a monoidal category we require associator isomorphisms

$$(A \otimes B)\, C \cong A \otimes (B \otimes C)$$

for all objects in the category. The coherence theorem tells us that, for any monoidal category M, there is a strict monoidal category which is equivalent to M. In other words, we can treat the associators in M as identities, and similarly for the unit isomorphisms. This result can also be regarded as a specific case of a more general theory, considering when pseudoalgebras for 2-monads are equivalent to strict algebras for the same 2-monad. Monoidal categories are essentially the pseudoalgebras for the free monoid 2-monad on **Cat** and one is able to apply

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\gamma"] & D \end{tikzcd}$$

$1_r * \delta = \gamma * 1_l$

the following theory to very nearly retrieve the coherence theorem for monoidal categories.

In [12], sufficient conditions are described for when the pseudoalgebras of a 2-monad are equivalent to strict ones. More precisely, the theorem states conditions for the existence of a left adjoint to the forgetful 2-functor

$$U\colon T\text{-}\mathbf{Alg}_s \to \mathbf{Ps}\text{-}T\text{-}\mathbf{Alg}$$

for which the components of the unit of the adjunction are equivalences. We focus on one version of this general result which has hypotheses that are quite easy to check in practice. First we require that the base 2-category $\mathcal{K}$ has an enhanced factorization system. This is much like an orthogonal factorization system on a 2-category, consisting of two classes of maps $(\mathcal{L}, \mathcal{R})$, satisfying the lifting properties on 1-cells and 2-cells as follows. Given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

g'

where $l \in \mathcal{L}$ and $r \in R$, there exists a unique morphism $m\colon B \to C$ such that $rm = g$ and $ml = f$. Similarly, given two commuting squares for which $rf = gl$ and $rf' = f'l$, along with 2-cells $\delta\colon f \Rightarrow f'$ and $\gamma\colon g \Rightarrow g'$ for which $\gamma * 1_l = 1_r * \gamma$, there exists a unique 2-cell $\mu\colon m \Rightarrow m'$, where m and m' are induced by the 1-cell lifting property, satisfying $\mu * 1_l = \delta$ and $1_r * \mu = \gamma$. However, there is an additional 2-dimensional property of the factorization system which says that given maps $l \in \mathcal{L}$, $r \in \mathcal{R}$ and an invertible 2-cell $\alpha\colon rf \Rightarrow gl$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\swarrow\alpha"] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r, "g"'] \arrow[ur, "m"] & D \arrow[ul, phantom, "\swarrow\beta"] \end{tikzcd}$$

there is a unique pair (m, β) where $m\colon C \to B$ is a 1-cell and $\beta\colon rm \Rightarrow g$ is an invertible 2-cell such that $ml = f$ and $\beta * 1_l = \alpha$.

Further conditions require that T preserve $\mathcal{L}$ maps and that whenever $r \in \mathcal{R}$ and $rk \cong 1$, then $kr \cong 1$. In our case we are considering 2-monads on the 2-category **Cat**, which has the enhanced factorization system where $\mathcal{L}$ consists of bijective-on-objects functors and $\mathcal{R}$ is given by the full and faithful functors. This, along with the 2-dimensional property making it an enhanced factorization system, is described in [16]. The last stated condition, involving isomorphisms and maps in $\mathcal{R}$, is then clearly satisfied and so the only thing we need to check in order to satisfy the conditions of the coherence result are that the induced 2-monads $\underline{P}$ preserve bijective-on-objects functors, which is a simple exercise.

Proposition 3.6. *For any* **G**-*operad P, the 2-monad $\underline{P}$ preserves bijective-on-objects functors.*

Corollary 3.7. *Every pseudo-$\underline{P}$-algebra is equivalent to a strict $\underline{P}$-algebra.*

4. Pseudo-commutativity

This final section gives sufficient conditions to equip the 2-monad $\underline{P}$ induced by a **G**-operad P in **Cat** with a pseudo-commutative structure. Such a pseudo-commutativity will then give the 2-category Ps-$\underline{P}$-Alg some additional structure that we briefly explain here. For a field k, the category **Vect** of vector spaces over k has many nice features. Of particular importance to us are the following three structures. First, the category **Vect** is monoidal using the tensor product $\otimes_k$. Second, the set of linear maps $V \to W$ is itself a vector space which we denote $[V, W]$. Third, there is a notion of multilinear map $V_1 \times \cdots \times V_n \to W$, with linear maps being the 1-ary version. While these three structures are each useful in isolation, they are tied together by natural isomorphisms

$$\mathbf{Vect}(V_1 \otimes V_2, W) \cong \mathbf{Vect}(V_1, [V_2, W]) \cong \mathbf{Bilin}(V_1 \times V_2, W)$$

expressing that $\otimes$ gives a closed monoidal structure which represents the multicategory of multilinear maps. Moreover, the adjunction between **Vect** and **Sets** respects all of this structure in the appropriate way. This incredibly rich interplay between the tensor product, the internal mapping space, and the multicategory of multilinear maps all arises from the free vector space monad on **Sets** being a *commutative* monad [9, 10, 11]. The notion of a pseudo-commutative 2-monad [6] is then a generalization of this machinery to a 2-categorical context, and can be viewed as a starting point for importing tools from linear algebra into category theory.

The aim of this section is to give conditions that ensure that the 2-monad $\underline{P}$ associated to a **G**-operad P has a pseudo-commutative structure. We give the definition of pseudo-commutativity as in [6] but before doing so we note what we mean by a strength for a 2-monad.

Definition 4.1. A *strength* for an endo-2-functor $T\colon \mathcal{K} \to \mathcal{K}$ consists of a 2-natural transformation t with components

$$t_{A,B}\colon A \times TB \to T(A \times B)$$

satisfying unit and associativity axioms. Similarly, a costrength for T consists of a 2-natural transformation t^* with components

$$t^*_{A,B}\colon TA \times B \to T(A \times B)$$

again satisfying unit and associativity axioms.

The strength and costrength for the associated 2-monad $\underline{P}$ are quite simple to define. We define a strength t for $\underline{P}$ as follows. The component $t_{A,B}$ is a functor

$$t_{A,B}\colon A \times \left(\amalg P(n) \times_{G(n)} B^n\right) \to \amalg P(n) \times_{G(n)} (A \times B)^n$$

which sends an object $(a, [p; b_1, \ldots, b_n])$ to the object $[p; (a, b_1), \ldots, (a, b_n)]$. We can also define a costrength which works in a similar way, sending an object $([p; a_1, \ldots, a_n], b)$ to the object $[p; (a_1, b), \ldots, (a_n, b)]$. Both the strength and the costrength work similarly on morphisms.

Definition 4.2. Given a 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ with strength t and costrength t^*, a *pseudo-commutativity* consists of an invertible modification γ with components

$$\begin{tikzcd}
TA \times TB \arrow[r, "t^*_{A,TB}"] \arrow[d, "t_{TA,B}"'] & T(A \times TB) \arrow[r, "Tt_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma_{A,B}"] & T^2(A \times B) \arrow[d, "\mu_{A\times B}"] \\
T(TA \times B) \arrow[r, "Tt^*_{A,B}"'] & T^2(A \times B) \arrow[r, "\mu_{A\times B}"'] & T(A \times B)
\end{tikzcd}$$

satisfying the following three strength axioms, two unit (or η) and two multiplication (or μ) axioms for all A, B, and C.

(1) $\gamma_{A\times B,C} \circ (t_{A,B} \times 1_{TC}) = t_{A,B\times C} \circ (1_A \times \gamma_{B,C})$
(2) $\gamma_{A,B\times C} \circ (1_{TA} \times t_{B,C}) = \gamma_{A\times B,C} \circ (t^*_{A,B} \times 1_{TC})$
(3) $\gamma_{A,B\times C} \circ (1_{TA} \times T^*_{B,C}) = t^*_{A\times B,C} \circ (\gamma_{A,B} \times 1_C)$
(4) $\gamma_{A,B} \circ (\eta_A \times 1_{TB})$ is an identity.
(5) $\gamma_{A,B} \circ (1_{TA} \times \eta_B)$ is an identity.
(6) $\gamma_{A,B} \circ (\mu_A \times 1_{TB})$ is equal to the pasting below.

$$\begin{tikzcd}
T^2A{\times}TB \arrow[r, "t^*_{TA,TB}"] \arrow[d, "t_{T^2A,B}"'] & T(TA{\times}TB) \arrow[r, "Tt^*_{A,TB}"] \arrow[d, "Tt_{TA,B}"] & T^2(A{\times}TB) \arrow[r, "T^2t_{A,B}"] \arrow[d, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A{\times}B) \arrow[d, "T\mu_{A\times B}"] \\
T(T^2A{\times}B) \arrow[r, phantom, "\Downarrow\gamma_{TA,B}"] \arrow[d, "Tt^*_{TA,B}"'] & T^2(TA{\times}B) \arrow[r, "T^2t^*_{A,B}"'] \arrow[d, "\mu_{TA\times B}"] & T^3(A{\times}B) \arrow[r, "T\mu_{A\times B}"'] \arrow[d, "\mu_{T(A\times B)}"] & T^2(A{\times}B) \arrow[d, "\mu_{A\times B}"] \\
T^2(TA{\times}B) \arrow[r, "\mu_{TA\times B}"'] & T(TA{\times}B) \arrow[r, "Tt^*_{A,B}"'] & T^2(A{\times}B) \arrow[r, "\mu_{A\times B}"'] & T(A{\times}B)
\end{tikzcd}$$

(7) $\gamma_{A,B} \circ (1_{TA} \times \mu_B)$ is equal to the pasting below.

$$\begin{tikzcd}
TA{\times}T^2B \arrow[r, "t^*_{A,T^2B}"] \arrow[d, "t_{TA,TB}"'] & T(A{\times}T^2B) \arrow[r, "Tt_{A,TB}"] \arrow[d, phantom, "\Downarrow\gamma_{A,TB}"] & T^2(A{\times}TB) \arrow[d, "\mu_{A\times TB}"] \\
T(TA{\times}TB) \arrow[r, "Tt^*_{A,TB}"'] \arrow[d, "Tt_{TA,B}"'] & T^2(A{\times}TB) \arrow[r, "\mu_{A\times TB}"'] \arrow[d, "T^2t_{A,B}"] & T(A{\times}TB) \arrow[d, "Tt_{A,B}"] \\
T^2(TA{\times}B) \arrow[r, phantom, "\Downarrow T\gamma_{A,B}"] \arrow[d, "T^2t^*_{A,B}"'] & T^3(A{\times}B) \arrow[r, "\mu_{T(A\times B)}"] \arrow[d, "T\mu_{A\times B}"] & T^2(A{\times}B) \arrow[d, "\mu_{A\times B}"] \\
T^3(A{\times}B) \arrow[r, "T\mu_{A\times B}"'] & T^2(A{\times}B) \arrow[r, "\mu_{A\times B}"'] & T(A{\times}B)
\end{tikzcd}$$

Remark 4.3. It is noted in [6] that this definition has some redundancy and therein it is shown that any two of the strength axioms immediately implies the third. Furthermore, the three strength axioms are equivalent, when the η and μ axioms hold, when the following associativity equation also holds:

$$\gamma_{A,B\times C} \circ (1_{TA} \times \gamma_{B,C}) = \gamma_{A\times B,C} \times (\gamma_{A,B} \times 1_{TC}).$$

We need some notation before stating our main theorem. Let $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_n$ be two lists. Then the set $\{(a_i, b_j)\}$ has mn elements, and two natural lexicographic orderings. One of these we write as $\underline{(a, b)}$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} p < r, \text{ or} \\ p = r \text{ and } q < s. \end{cases}$$

The other we write as $\underline{(a, b)}$, and it has the order given by

$$(a_p, b_q) < (a_r, b_s) \text{ if } \begin{cases} q < s, \text{ or} \\ q = s \text{ and } p < r. \end{cases}$$

The notation $(a, \underline{b})$ is meant to indicate that we have a single a but a list of b's, so then $\underline{(a, \underline{b})}$ would represent a list which itself consists of lists of that form. There is a unique permutation $\tau_{m,n} \in \Sigma_{mn}$ which has the property that $\tau_{m,n}(i) = j$ if the ith element of the ordered set $\underline{(a, b)}$ is equal to the jth element of the ordered set $\underline{(a, b)}$. By construction, we have $\tau_{nm} = \tau_{mn}^{-1}$. We illustrate these permutations with a couple of examples.

$\tau_{2,3}$ $\tau_{4,2}$

Note then that $\tau_{m,n}$ is given by taking the transpose of the $m \times n$ matrix with entries (a_i, b_j).

We now state when a monad $\underline{P}$ for a **G**-operad P acquires a pseudo-commutative structure. Let $\mathbb{N}_+$ denote the set of positive natural numbers.

Theorem 4.4. *Let P be a **G**-operad. Then the following equip $\underline{P}$ with a pseudo-commutative structure.*

1. *For each pair $(m, n) \in \mathbb{N}_+^2$, we are given an element $t_{m,n} \in G(mn)$ such that $\pi(t_{m,n}) = \tau_{m,n}$.*
2. *For each $p \in P(n)$, $q \in P(m)$, we are given a natural isomorphism*

$$\lambda_{p,q} \colon \mu(p; q, \ldots, q) \cdot t_{m,n} \cong \mu(q; p, \ldots, p).$$

We write this as $\lambda_{p,q} : \mu(p; \underline{q}) \cdot t_{m,n} \cong \mu(q; \underline{p})$.

These must satisfy the following:

- *For all $n \in \mathbb{N}_+$,*

$$t_{1,n} = e_n = t_{n,1}$$

and for all $p \in P(n)$, $p \cdot e_n \cong p$ is the identity map.

- *For all $l, m_1, \ldots, m_l, n \in \mathbb{N}_+$, with $M = \Sigma m_i$,*

$$\mu(e_l; t_{m_1,n}, \ldots, t_{m_l,n})\mu(t_{l,n}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}) = t_{n,M}.$$

Here $\underline{e_{m_1}}, \ldots, \underline{e_{m_l}}$ is the list $e_{m_1}, \ldots, e_{m_l}$ repeated n times.

- *For all $l, m, n_1, \ldots, n_m \in \mathbb{N}_+$, with $N = \Sigma n_i$,*

$$\mu(t_{m,l}; \underline{e_{n_1}}, \ldots, \underline{e_{n_m}})\mu(e_m; t_{n_1,l}, \ldots, t_{n_m,l}) = T_{N,l}.$$

Here $\underline{e_{n_i}}$ indicates that each e_{n_i} is repeated l times.

- *For any $l, m_i, n \in \mathbb{N}_+$, with $1 \leq i \leq n$, and $p \in P(l)$, $q_i \in P(m_i)$ and $r \in P(n)$, the following diagram commutes. (Note that we maintain the convention that anything underlined indicates a list, and double underlining indicates a list of lists. Each instance should have an obvious meaning from context and the equations appearing above.)*

$$\begin{tikzcd}
\mu\Big(p; \underline{\mu(q_i; \underline{r})}\Big) \cdot \mu(e_l; \underline{t_{n,m_i}})\mu(t_{n,l}; \underline{\underline{e_{m_i}}}) \arrow[r, equal] \arrow[d, equal] & \mu\Big(p; \underline{\mu(q_i; \underline{r})}\Big) \cdot t_{n,M} \arrow[d, equal] \\
\mu\Big(p; \underline{\mu(q_i; \underline{r}) \cdot t_{n,m_i}}\Big) \cdot \mu(t_{n,l}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}) \arrow[d, "\mu(1;\underline{\lambda_{q_i,r}})^{-1}"'] & \mu\Big(\mu(p; q_1, \ldots, q_n); \underline{r}\Big) \cdot t_{n,M} \arrow[d, "\lambda_{\mu(p;q_1,\ldots,q_n),r}"] \\
\mu\Big(p; \underline{\mu(r; \underline{q_i})}\Big) \cdot \mu(t_{n,l}; \underline{e_{m_1}}, \ldots, \underline{e_{m_l}}) \arrow[d, equal] & \mu\Big(\mu(p; q_1, \ldots, q_l); \underline{r}\Big) \arrow[dd, equal] \\
\mu\Big(\mu(p; \underline{r}) \cdot t_{n,l}; \underline{q_1, \ldots, q_l}\Big) \arrow[d, equal] & \\
\mu\Big(\mu(r, \underline{p}); \underline{q_1, \ldots, q_n}\Big) \arrow[r, "\mu(\lambda_{p,r};1)"'] & \mu\Big(r; \underline{\mu(p; q_1, \ldots, q_l)}\Big)
\end{tikzcd}$$

- *For any $l, m, n_i \in \mathbb{N}_+$, with $1 \leq i \leq m$, and $p \in P(l)$, $q \in P(m)$ and $r_i \in P(n_i)$, the following diagram commutes:*

$$\begin{tikzcd}
\mu\Big(\mu(p; \underline{q}) \cdot t_{m,l}; \underline{\underline{r_i}}\Big) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[r, equal] \arrow[d, "\mu(\lambda_{p,q};1)^{-1}"'] & \mu\Big(\mu(p; \underline{q}); \underline{\underline{r_i}}\Big) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\Big(\mu(q; \underline{p}); \underline{r_1}, \ldots, \underline{r_m}\Big) \cdot \mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] & \mu\Big(p; \underline{\mu(q; \underline{r_i})}\Big) \cdot \mu(t_{m,l}; \underline{\underline{e_{n_i}}})\mu(e_m; \underline{t_{n_i,l}}) \arrow[d, equal] \\
\mu\Big(q; \underline{\underline{\mu(p; r_i)}}\Big) \cdot mu(e_m; \underline{t_{n_i,l}}) \arrow[dd, equal] & \mu\Big(p; \underline{\mu(q; r_1, \ldots, r_m)}\Big) \cdot t_{N,l} \arrow[d, "\lambda_{p,\mu(q;r_1,\ldots,r_m)}"] \\
& \mu\Big(\mu(q; r_1, \ldots, r_m); \underline{p}\Big) \arrow[d, equal] \\
\mu\Big(q; \underline{\mu(p; \underline{r_i}) \cdot t_{n_i,l}}\Big) \arrow[r, "\mu(1;\underline{\lambda_{p,r_i}})"'] & \mu\Big(q; \underline{\mu(r_i; \underline{p})}\Big)
\end{tikzcd}$$

Proof. We begin the proof by defining an invertible modifcation γ for the pseudo-commutativity for which the components are natural transformations $\gamma_{A,B}$. Such a transformation $\gamma_{A,B}$ has components with source

$$[\mu(p; \underline{q}); \underline{(x, \underline{y})}]$$

and target

$$[\mu(q; \underline{p}); \underline{(\underline{x}, y)}].$$

Proof of Theorem 4.10. In order to use Theorem 4.4 with the action operad being the braid operad $\mathbf{B} = \{B_n\}$, we must first construct elements $t_{m,n} \in B_{mn}$ satisfying certain properties. Using Lemma 4.13, we define $t_{m,n}$ to be the unique positive braid such that $\pi(t_{m,n}) = \tau_{m,n}$. Since $\tau_{1,n} = e_n = \tau_{n,1}$ in Σ_n and the identity element $e_n \in B_n$ is positive and minimal, we have that $t_{1,n} = e_n = t_{n,1}$ in B_n. Thus in order to verify the remaining hypotheses, we must check two equations, each of which states that some element $t_{m,n}$ can be expressed as a product of operadic compositions of other elements.

Let $l, m_1, \ldots, m_l, n$ be natural numbers, and let $M = \sum m_i$. We must check that

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}}) = t_{N,l}$$

in B_{lN}. These braids have the same underlying permutations by construction, and both are positive since each operadic composition on the left is positive. The braid on the right is minimal by definition, so if we prove that the braid on the left is also minimal, they are necessarily equal. Now $\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$ is given by the braid for $t_{n,l}$ but with the first strand replaced by m_1 strands, the second strand replaced by m_2 strands, and so on for the first l strands of $t_{n,l}$, and then repeating for each group of l strands. In particular, since strands $i, i+l, i+2l, \ldots, i+(n-1)l$ never cross in $t_{n,l}$, none of the m_i strands that each of these is replaced with cross. The braid $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ consists of the disjoint union of the braids for each t_{n,m_i}, so if two strands cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$ then they must both cross in some t_{n,m_i}. The strands in t_{n,m_i} are those numbered from $n(m_1 + \cdots + m_{i-1}) + 1$ to $n(m_1 + \cdots + m_{i-1} + m_i)$. This is a consecutive collection of nm_i strands, and it is simple to check that these strands are precisely those connected (via the group operation in B_{Nl}, concatenation) to the duplicated copies of strands $i, i+l, i+2l, \ldots, i+(n-1)l$ in $t_{n,l}$. Thus if a pair of strands were to cross in $\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})$, that pair cannot also have crossed in $\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$, showing that the resulting product braid

$$\mu(e_l; t_{n,m_1}, \ldots, t_{n,m_l})\mu(t_{n,l}; \underline{e_{m_1}, \ldots, e_{m_l}})$$

is minimal. The calculation showing that

$$\mu(t_{m,l}; \underline{e_1}, \ldots, \underline{e_{n_m}})\mu(e_m; t_{n_1,l}, \ldots, t_{n_m,l})$$

is also minimal follows from the same argument, showing that it is equal to $t_{N,l}$ (here N is the sum of the n_i, where once again i ranges from 1 to l).

Now strict braided monoidal categories are the algebras for a contractible $\mathbf{B}$-operad Br following Fiedorowicz [5], therefore the construction above gives a pseudo-commutative structure. The second pseudo-commutative structure arises by using negative, minimal braids instead of positive ones, and proceeds using the same arguments.

We will now show that neither of these pseudo-commutative structures is symmetric. The symmetry axiom in this case reduces to the fact that, in some category which is given as a coequalizer, the morphism with first component

$$f : \mu(p; \underline{q}) \cdot t_{n,m}t_{m,n} \to \mu(q; \underline{p}) \cdot t_{m,n} \to \mu(p; \underline{q})$$

is the identity. Now it is clear that that $t_{n,m}$ is not equal to $t_{m,n}^{-1}$ in general: taking $m = n = 2$, we note that $t_{2,2} = \sigma_2$, and this element is certainly not of order two in B_4. $Br(4)$ is the category EB_4 (the category whose elements are the elements of B_4 with a unique isomorphism between any two pair of objects), and B_4 acts by

multiplication on the right; this action is clearly free and transitive. We recall (see Lemma 3.3) that in a coequalizer of the form $A \times_G B$, we have that a morphism $[f_1, f_2]$ equals $[g_1, g_2]$ if and only if there exists an $x \in G$ such that

$$\begin{aligned} f_1 \cdot x &= g_1, \\ x^{-1} \cdot f_2 &= g_2. \end{aligned}$$

For the coequalizer in question, for f to be the first component of an identity morphism would imply that $f \cdot x$ would be a genuine identity in EB_4 for some x. But $f \cdot x$ would have source $\mu(p; \underline{q})t_{n,m}t_{m,n}x$ and target $\mu(p; \underline{q})x$, which requires $t_{n,m}t_{m,n}$ to be the identity group element for all n, m, a contradiction. □

Remark 4.15. The pseudo-commutativities given above are not necessarily the only ones that exist for the B-operad Br, but we do not know a general strategy for producing others.

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These are given by the isomorphisms

$$(\gamma_{A,B})_{[p;a_1,\dots,a_n],[q;b_1,\dots,b_m]} = [\lambda_{p,q},\underline{1}],$$

by which we mean the composite

$$[\mu(p;\underline{q});\underline{(x,\underline{y})}] = [\mu(p;\underline{q})\cdot t_{m,n}; t_{m,n}^{-1}\cdot\underline{(x,\underline{y})}] \xrightarrow{[\lambda,1]} [\mu(q;\underline{p});\underline{(\underline{x},y)}].$$

In the case that either p or q is an identity then we choose the component of γ to be the isomorphism involving the appropriate identity element, as assumed to exist in the statement of the theorem. Two things to note about the definition above before we continue. First, it is easy to check that

$$t_{m,n}^{-1}\cdot\underline{(x,\underline{y})} = \underline{(\underline{x},y)}$$

since $\pi(t_{m,n}) = \tau_{m,n}$. Second, the morphism above has second component the identity. This is actually forced upon us by the requirement that γ be a modification: in the case that A, B are discrete categories, the only possible morphism is an identity, and the modification axiom then forces that statement to be true for general A, B by considering the inclusion $A_0 \times B_0 \hookrightarrow A \times B$ where A_0, B_0 are the discrete categories with the same objects as A, B.

We show that this is a modification by noting that it does not rely on objects in the lists $a_1,\dots,a_n$ or $b_1,\dots,b_m$, only on their lengths and the operations p and q. As a result, if we have functors $f\colon X \to X'$ and $g\colon Y \to Y'$, then it is clear that

$$(\underline{P}(f\times g)\circ\gamma_{X,Y})_{[p;\underline{x}],[q;\underline{y}]} = [\lambda_{p,q},\underline{1}] = (\gamma_{X',Y'}\circ(\underline{P}f\times\underline{P}g))_{[p;\underline{x}],[q;\underline{y}]}.$$

As such we can simply write $(\gamma_{X,Y})_{[p;\underline{x}],[q;\underline{y}]}$ in shorthand as $\gamma_{p,q}$.

The three strength axioms are immediately satisfied, again since $\gamma_{p,q}$ has no dependence on the objects in the lists and as such the isomorphisms are the same. The unit axioms follow from the assumption that $t_{1,n} = e_n = t_{n,1}$ and that components of γ involving an identity operation are also identity maps. The multiplication axioms follow from the two diagrams assumed to commute in the statement of the theorem. If we consider each axiom to consist of two equations, one in $P(n)$ and one in some power $(A\times B)^n$, then the two diagrams at the end of the statement of the theorem actually force the first components of the two multiplication axioms to hold in $P(n)$ before taking equivalence classes in the coequalizer. □

A further property that a pseudo-commutativity can possess is that of symmetry. This symmetry is then reflected in the monoidal structure on the 2-category of algebras, which will then also have a symmetric tensor product (in a suitable, 2-categorical sense).

Definition 4.5. Let $T\colon \mathcal{K}\to\mathcal{K}$ be a 2-monad on a symmetric monoidal 2-category $\mathcal{K}$ with symmetry c. We then say that a pseudo-commutativity γ for T is *symmetric* when the following is satisfied for all $A, B \in \mathcal{K}$:

$$Tc_{A,B}\circ\gamma_{A,B}\circ c_{TB,TA} = \gamma_{B,A}.$$

With the notion of symmetry at hand we are able to extend the above theorem, stating when $\underline{P}$ is symmetric.

Theorem 4.6. *The pseudo-commutative structure for $\underline{P}$ given by Theorem 4.4 is symmetric if for all $m, n \in \mathbb{N}_+$ the two conditions below hold.*

(1) $t_{m,n} = t_{n,m}^{-1}$.

(2) *The following diagram commutes:*

$$\begin{tikzcd}
\mu(p;\underline{q})\cdot t_{m,n}t_{n,m} \arrow[r, equal] \arrow[d, dashed, "\lambda_{p,q}\cdot 1"'] & \mu(p;\underline{q})\cdot e_{mn} \arrow[d, equal] \\
\mu(q;\underline{p})\cdot t_{n,m} \arrow[r, "\lambda_{q,p}"'] & \mu(p;\underline{q})
\end{tikzcd}$$

Proof. The commutativity of the diagram above ensures that the first component of the symmetry axiom commutes in $P(n)$ before taking equivalence classes in the coequalizer, just as in Theorem 4.4. □

Corollary 4.7. *If P is contractible ($P(n) \simeq *$ for all n) and there exist $t_{m,n}$ as in Theorem 4.4, then $\underline{P}$ acquires a pseudo-commutativity. Furthermore, it is symmetric if $t_{n,m} = t_{m,n}^{-1}$.*

Proof. The only thing left to define is the collection of natural isomorphisms $\lambda_{p,q}$. But since each $P(n)$ is contractible, $\lambda_{p,q}$ must be the unique isomorphism between its source and target, and furthermore the last two axioms hold since any pair of parallel arrows are equal in a contractible category. □

Corollary 4.8. *If P is a contractible symmetric operad then $\underline{P}$ has a symmetric pseudo-commutativity.*

Proof. We choose $t_{m,n} = \tau_{m,n}$. □

Corollary 4.9. *Let P be a non-symmetric operad. Then $\underline{P}$ is never pseudo-commutative.*

Proof. There is no isomorphism $\underline{(x,\underline{y})} \cong \underline{(\underline{x},y)}$ when A, B are discrete. □

We conclude with a computation using Theorem 4.4. This result was only conjectured in [6], but we are able to prove it quite easily with the machinery developed thusfar.

Theorem 4.10. *The 2-monad $\underline{Br}$ for strict braided monoidal categories on* **Cat** *has two pseudo-commutative structures on it, neither of which are symmetric.*

Definition 4.11. A braid $\gamma \in B_n$ is *positive* if it is an element of the submonoid of B_n generated by the elements $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$.

Definition 4.12. A braid $\gamma \in B_n$ is *minimal* if no pair of strands in γ cross twice.

For our purposes, we are interested in braids which are both positive and minimal. A proof of the following lemma can be found in [4].

Lemma 4.13. *Let PM_n be the subset of B_n consisting of positive, minimal braids. Then the function sending a braid to its underlying permutation is a bijection of sets $PM_n \to \Sigma_n$.*

Remark 4.14. It is worth noting that this bijection is not an isomorphism of groups, since PM_n is not a group or even a monoid. The element $\sigma_1 \in B_n$ is certainly in PM_n, but σ_1^2 is not as the first two strands cross twice. Thus we see that the product of two minimal braids is generally not minimal, but by definition the product of positive braids is positive.

School of Mathematics and Statistics, University of Sheffield, Sheffield, UK, S3 7RH
E-mail address: a.s.corner@sheffield.ac.uk

School of Mathematics and Statistics, University of Sheffield, Sheffield, UK, S3 7RH
E-mail address: nick.gurski@sheffield.ac.uk