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Notes on Bounding Box Construction for Filtering Geographic Data

Nick Halmagyi

Laboratoire de Physique Théorique et Hautes Energies,
Université Pierre et Marie Curie, CNRS UMR 7589,
F-75252 Paris Cedex 05, France

halmagyi@lpthe.jussieu.fr

1 Introduction

A common problem is data-science is the following: given a single geographic location S (the source) and a set of geographic locations $\widehat{T}$ (the targets), determine the closest N locations $t_i \in \widehat{T}$ to S. One can achieve this by computing the distance $d(t_i, S)$ from every target to the source but this is prohibitively slow even for a moderately large set of targets. A better solution is to make a bounding region $\mathcal{B}$ around S and compute the distance $d(t_i, S)$ only for the $t_i \in \mathcal{B}$. For well constructed $\mathcal{B}$ one will find the N closest points within $\mathcal{B}$.

There are numerous ways to store a set of geographic locations in memory or on disk but regardless of which specific method is utilized, the process of filtering $\widehat{T}$ down to the set of $t_i \in \mathcal{B}$ is fast as it only involves a few numerical comparisons per target.

Geographic locations are typically given in the longitude-latitude coordinate system, so $\mathcal{B}$ should be constructed from lines of equal latitude and lines of equal longitude. This results in $\mathcal{B}$ which look roughly square close to the equator and a sort of curved triangle when near the poles.

2 Constructing the Bounding Box

We denote C_d the equidistant surface around S. it is the collection of points on the sphere which are at a distance d from S. To construct the bounding box $\mathcal{B}$, we need to determine the minimum and maximum values of latitide and longitude attained by C_d.

2.1 Haversine Formula

The Haversine formula gives the distance between two points on the sphere in the latitude-longitude coordinate system:

$$\sin^2\left(\frac{d}{2R}\right) = \sin^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) + \cos\varphi_1 \cos\varphi_2 \sin^2\left(\frac{\lambda_1 - \lambda_2}{2}\right) \tag{1}$$

or more conveniently

$$\cos\frac{d}{R} \quad = \quad \sin\varphi_1 \sin\varphi_2 + \cos(\lambda_2 - \lambda_1)\cos\varphi_1 \cos\varphi_2\,. \tag{2}$$

where R is the radius of the sphere.

2.2 Maximum Latitude

The maximum latitude on C_d is attained when the longitude is the same as the longitude of S. When two longitudes are equal

$$\lambda_2 = \lambda_1 \tag{3}$$

the Haversine formula gives

$$\varphi_2 - \varphi_1 = \frac{d}{R} \tag{4}$$

2.3 Maxmimum Longitude

2.3.1 Ellipse

To warm up to the computation of the point of maximum longitude, we first look at an ellipse in $\mathbb{R}^2$:

$$a^2x^2 + b^2y^2 = 1 \tag{5}$$

and look for the maximum value of x.

We solve for x

$$x(y) \quad = \quad \frac{1}{a}\sqrt{(1-b^2y^2)} \tag{6}$$

and compute the variation of x w.r.t. y

$$x'(y) = \frac{1}{a}\frac{2b^2y}{(1-b^2y^2)} \,. \tag{7}$$

The extremum of x are of course at the point where $x'(y) = 0$ and this is at $y = 0$, in particular on the x-axis.

2.3.2 Back to the Sphere

Given a section of equal distance d from the source $S = (\varphi_1, \lambda_1)$ we want to know the maximum longitudinal distance, i.e. the maximum value of $|\lambda_2 - \lambda_1|$. These will be the east and west boundaries of the bounding box.

We first fix d and (φ_1, λ_1) then look for the extremum of λ_2 as a function of φ_2. It is convenient to change coordinates to

$$\Lambda = \cos\big(\lambda_1 - \lambda_2(\varphi_2)\big)\,, \qquad \Psi = \sin\varphi_2 \tag{8}$$

then the equations to be solved are

$$\csc\varphi_1 \cos\frac{d}{R} \quad = \quad \cot\varphi_1 \Lambda\sqrt{1-\Psi^2} + \Psi \tag{9}$$

$$\cot\varphi_1 \Lambda\Psi \quad = \quad \sqrt{1-\Psi^2} \tag{10}$$

The solution is

$$\Lambda \quad = \quad \sec(\varphi_1)\sqrt{\cos^2\big(\frac{d}{R}\big) - \sin^2(\varphi_1)} \tag{11}$$

$$\Psi \quad = \quad \sec\big(\frac{d}{R}\big)\sin\varphi_1 \tag{12}$$

There is a branch point in Λ at one of the following points:

$$\text{b.p.} \quad \varphi_1 \pm \frac{d}{R} = \pm\pi/2 \tag{13}$$

which is where the bounding section touches the north(south) pole. At this point the maximum longitude reached is $\lambda_2 - \lambda_1 = \pi/2$ which is half way around the globe (though not necessarily at the equator).

2.3.3 Equal Latitude

One can also consider a related but inequivalent question, which is for the shortest distance between two points of equal latitude. This is slightly non-trivial since the great circle between two points does not follow lines of equal latitude unless both points are on the equator. Indeed in this interesting blog post the author realises the distinction between great circles and lines of constant latitude but mistakenly assumes that the maximum longitude is attained at the same latitude as the source S.

To compute the distance between two points of equal latitude, we use the Haversine formula with

$$\varphi_1 = \varphi_2 = \varphi\,. \tag{14}$$

Then we find

$$\begin{aligned} d &= 2R\sin^{-1}\left|\cos\varphi\sin\left(\frac{\lambda_1-\lambda_2}{2}\right)\right| && (15)\\ \Rightarrow \quad |\lambda_1-\lambda_2| &= 2\left|\sin^{-1}\left[\frac{\sin\left(\frac{d}{2R}\right)}{\cos(\varphi)}\right]\right|\,. && (16)\end{aligned}$$