Math

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Contents

1 Exponents

- Multiply some number many times
- b^n
- b the base
- n the exponent or power of b
- $\exp(x) \equiv e^x$: exponential function base e
- Properties:
  - Multiplying two expressions with like-bases: $b^m b^n \ \equiv \ b^{m+n}$
  - Division can be expressed in the following way: $b^{-1} \ \equiv \ \frac{1}{b}$
  - $\frac{b^m}{b^n} \ \equiv \ b^{m-n}$
  - $(b^m)^n \ \equiv \ b^{mn}$
  - $(ab)^n \ \equiv \ a^a b^n$
  - $\left(\frac{a}{b}\right)^n \ \equiv \ \frac{a^2}{b^n}$
  - $b^{\frac{1}{n}} \ \equiv \ \sqrt[n]{b}$
  - $\sqrt[n]{ab} \ \equiv \ (ab)^{\frac{1}{n}} \ \equiv \ a^{\frac{1}{n}} b^{\frac{1}{n}} \ \equiv \ \sqrt[n]{a}\,\sqrt[n]{b}$
  - $\sqrt[n]{\left(\frac{a}{b}\right)} \ \equiv \ \left(\frac{a}{b}\right)^{\frac{1}{n}} \ \equiv \ \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \ \equiv \ \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

2 Logarithms

- Inverse of eponentiation
- $\log_b(x)$: log of x base b is the inverse of b^x
- $\log_b(x) = m \ \Leftrightarrow \ b^m \ = \ x$
- $\ln(x)$: the natural log; a log with base e; inverse of e^x
- $\log(x) + \log(y) = \log(xy)$
- $\log(x^k)j = k\log(x)$
- $\log(x) - \log(y) = \log\left(\frac{x}{y}\right)$
- $\log_B(x) \ = \ \frac{\log_b(x)}{\log_b(B)}$
- $\log_{10}(S) \ = \ \frac{\log_{10}(S)}{1} \ = \ \frac{\log_{10}(S)}{\log_{10}(10)} \ = \ \frac{\log_2(S)}{\log_2(10)} \ = \ \frac{\ln(S)}{\ln(10)}$

3 Polynomials

$$f(x) = ax^2 + bx + c$$

- Degree of $f(x)$ is the largest power of x.
- Roots of $f(x)$ are the values of x for which $f(x) = 0$.
- A polynomial of nth degree can be written using summation.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... a_2 x^2 + a_0$$
$$f(x) = \sum_{k=0}^{n} a_k x^k$$

3.1 Solving Polynomials

- First degree polynomials

$$P_1(x) = mx + b = 0$$
$$x = \frac{b}{m}$$

- Second degree polynomials

$$P_2(x) = ax^2 + bx + c = 0$$
$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, it involves taking the square root of a negative number, for which no real solution exists.

3.2 Polynomial Example

The revenue of a company: $R(x)$
The cost incurred: $C(x)$
The cost to break even: $R(x) = C(x)$

$$
\begin{aligned}
R(x) &= 2x^2 + 2x \\
C(x) &= x^2 + 5x + 10 \\
2x^2 + 2x &= x^2 + 5x + 10
\end{aligned}
$$

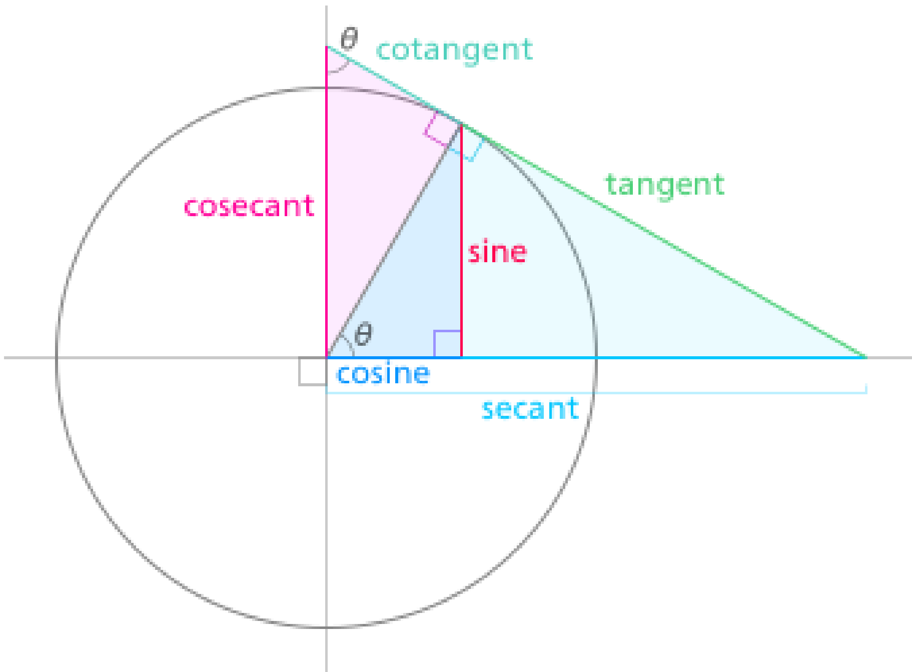
$$
\begin{aligned}
2x^2 + 2x &= x^2 + 5x + 10 \\
x^2 + 2x &= 5x + 10 \\
x^2 - 3x &= 10 \\
x^2 - 3x - 10 &= 0
\end{aligned}
$$

$$a = 1 b = -3 c = -10$$

$$
\begin{aligned}
x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
x_1 &= \frac{3 + \sqrt{9 - 40}}{2} \\
x_1 &= \frac{3 + 7}{2} \\
x_1 &= \frac{10}{2} \\
x_1 &= 5
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
x_2 &= \frac{3 - \sqrt{9 + 40}}{2} \\
x_2 &= \frac{3 - 7}{2} \\
x_2 &= \frac{-4}{2} \\
x_2 &= -2
\end{aligned}
$$

4 Trigonometry



4.1 Basic Concepts

- A, B, C: three vertices
- θ: angle
- $opp \equiv \overline{AB}$: the length of the side opposite θ.
- $adj \equiv \overline{BC}$: the length of the side adjacent θ.
- $hyp \equiv \overline{AC}$: the length of the longest side (hypotenuse)
- h: the height of the triangle.
- $\sin\theta \equiv \frac{opp}{hyp}$: the sine of theta is the ratio of the length of the opposite side and the hypotenuse.
- $\cos\theta \equiv \frac{adj}{hyp}$: the cosine of theta is the ratio of the length of the adjacent side and the hypotenuse.
- $\tan\theta \equiv \frac{\sin\theta}{\sin\theta} \equiv \frac{opp}{adj}$: the tangent is the ratio of the opposite length divided by the adjacent length.

4.1.1 Pythagorean Theorem

The length of the hypotenuse squared is equal to the sum of the squares of the lengths of the opposite and adjacent sides:

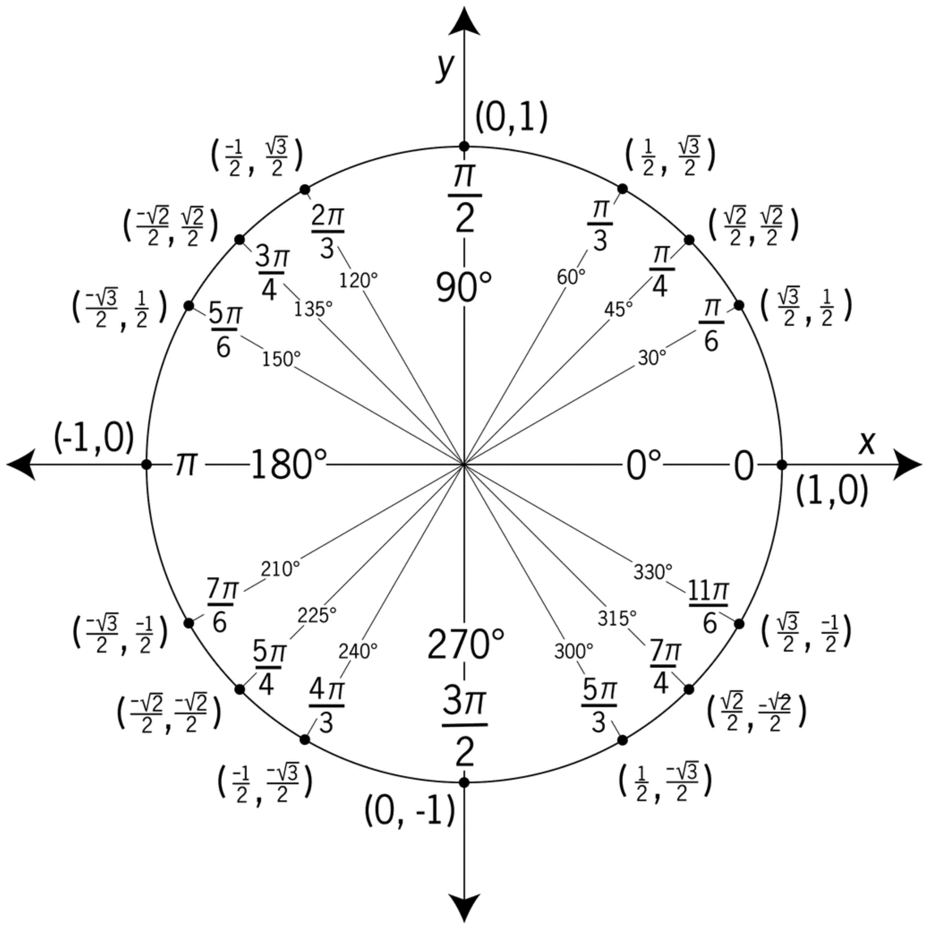
- $|adj|^2 + |opp|^2 = |hyp|^2$
- $\frac{|adj|^2}{|hyp|^2} + \frac{|opp|^2}{|hyp|^2} = 1$
- $\cos^2\theta + \sin^2\theta = 1$

4.2 Radians

Radians are the natureal unit for measurings angles.

- $2\pi = 360°$
- If a circle has a radius $r = 1$, then the arc length is equal to the angle in radians ℓ. $\ell = \theta_{rad}$.
- Measuring radians is equivalent to measuring arc length on a circle of radius 1.

4.3 Unit Circle



- Length of radius is equal to 1.
- P = a point on the unit circle
- $P(\theta) = (P_x(\theta), P_y(\theta)) = (\cos\theta, \sin\theta)$

4.3.1 Polar Coordinates

- Used for circles
- $r\angle\theta$ r = radius, $\angle\theta$ = the angle from the x axis
- Given the form (r, θ)
- Example: $\left(2, \frac{\pi}{6}\right)$

4.4 Sine and Cosine

- Take angles as inputs and output ratios
- Sine: How tall a triangle is
- Cosine: How wide a triangle is
- $\cos^2\theta + \sin^2\theta = 1$ for all angles.
- Knowing that $\sin(30^\circ) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and the previous rule you can determine all other angles.

$$\begin{aligned} \cos(30^\circ) &= \sqrt{1 - \sin^2(30^\circ)} \\ &= \sqrt{1 - \frac{1}{4}} \\ &= \sqrt{\frac{3}{4}} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

- For non unit circles: $Q(\theta) = (Q_x(\theta), Q_y(\theta)) = (r\cos\theta, r\sin\theta)$

4.4.1 Trigonometric Identities

- sico + sico: $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
- coco - sisi: $\cos(a+b)\cos(a)\cos(b) - \sin(a)\sin(b)$

4.4.2 Derived Formulae

Using the above Identities, the following can be derived:

- Double angle formulae:

$$\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(2x) &= 2\cos^2(x) - 1 \\ &= 2(1 - sin^2(x)) - 1 \\ &= 1 - 2\sin^2(x) \end{aligned}$$

- The above could also be rewritten as:

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)) \qquad \sin^2(x) = \frac{1}{2}(1-\cos(2x))$$

- Self similarity

- Sine and Cosine are periodic functions with a period of 2π, adding a multiple of 2π to the input has no change to the function.
- Sine and Cosine are $\frac{\pi}{2}$ shifted versions of each other:

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right) \qquad \sin(x) = \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right)$$

- Sum formulae:

$$\begin{aligned}
\sin(a) + \sin(b) &= 2\sin\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right) \\
\sin(a) - \sin(b) &= 2\sin\left(\frac{1}{2}(a-b)\right)\cos\left(\frac{1}{2}(a+b)\right) \\
\cos(a) + \cos(b) &= 2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right) \\
\cos(a) - \cos(b) &= -2\sin\left(\frac{1}{2}(a+b)\right)\sin\left(\frac{1}{2}(a-b)\right)
\end{aligned}$$

- Product formulae:

$$\begin{aligned}
\sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b) + \sin(a-b)) \\
\sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b) - \cos(a+b)) \\
\cos(a)\cos(b) &= \frac{1}{2}(\cos(a-b) + \cos(a+b))
\end{aligned}$$

5 Geometry

5.1 Triangles

- Area of a triangle with respect to side a: $A = \frac{1}{2}ah_a$
- Perimeter: $P = a + b + c$
- Sine Rule: $\frac{a}{\sin(\alpha)} = \frac{b}{\beta} = \frac{c}{\gamma}$
- Cosine Rules:

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(\alpha) \\
b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\
c^2 &= a^2 + b^2 - 2ab\cos(\gamma)
\end{aligned}$$

- All internal angles add to $180°$

5.2 Spheres

- Described by the equation $x^2 + y^2 + z^2 = r^2$
- Surface area: $A = 4\pi r^2$
- Volume: $V = \frac{4}{3}\pi r^3$

5.3 Cylinders

- Surface area: $A = 2(\pi r^2) + (2\pi r)h$
- Volume: $V = (\pi r^2)h$

5.4 Circle

- Described by the equation: $x^2 + y^2 = r^2$
- Described by a point (p, q) other than the center: $(x-p)^2 + (x-q)^2 = r^2$
- Area: $A = \pi r^2$
- Circumference: $C = 2\pi r$
- Arc Length: $\ell = 2\pi r \frac{\theta}{360}$

5.5 Ellipse

- a: half the length along the x axis
- b: half the length along the y axis
- ϵ: eccentricity (elongation)

$$\epsilon \equiv \sqrt{1 - \frac{b^2}{a^2}}$$

- F_1, F_2: Focal point
- r_1: Distance from a point to F_1
- r_2: Distance from a point to F_2
- The coordinates of the focal points:

$$F_1 = (-a\epsilon, 0)$$
$$F_2 = (a\epsilon, 0)$$

- An ellipse is a set of points that satisfy the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- In polar coordinates an ellipse is described as:

$$r_2(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}$$

5.6 Conic Sections

In polar coordinates all four conic sections can be described by the following equation:

$$r(\theta) = \frac{q(1+\epsilon)}{1+\epsilon\cos(\theta)}$$

Table 1: Conic Sections

Section	**Equation**	**Polar Equation**	**Eccentricity**	**q**
Circle	$x^2+y^2=a^2$	$r(\theta)=a$	$\epsilon=0$	$q=a$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$r(\theta)=\frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}$	$\epsilon=\sqrt{1-\frac{b^2}{a^2}} \in [0,1)$	$q=a(1-\epsilon)$
Parabola	$y^2=4qf^x$	$r(\theta)=\frac{2q}{1+\cos(\theta)}$	$\epsilon=1$	$q=f=$ focal length
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$r(\theta)=\frac{a(\epsilon^2-1)}{1+\epsilon\cos(\theta)}$	$\epsilon=\sqrt{1+\frac{b^2}{a^2}} \in (1,\infty$	$q=a(\epsilon-1)$

6 Systems of Linear Equations

- Equations

$$x+2y=5$$
$$3x+9y=21$$

- Equating

$$\begin{aligned} x &= 5-2y \\ x &= \frac{1}{3}(21-9y) = 7-3y \\ 5-2y &= 7-3y \\ y &= 2 \\ x+2(2) &= 5 \\ x &= 1 \end{aligned}$$

- Substitution

$$\begin{aligned}
x &= 5 - 2y \\
3(5-2y) + 9y &= 21 \\
15 - 6y + 9y &= 21 \\
15 - 3y &= 21 \\
3y &= 6 \\
y &= 2 \\
x + 2(2) &= 5 \\
x &= 1
\end{aligned}$$

- Subtraction

$$\begin{aligned}
3x + 6y &= 15 \\
3x + 9y &= 21 \\
3x - 3x - 6y + 9y &= 21 - 15 \\
3y &= 6 \\
y &= 2 \\
x + 2(2) &= 5 \\
x &= 1
\end{aligned}$$

7 Compound Interest

7.1 Annual Interest

- Loan: \$1000
- Interest: 6% annual
- Interest: $I_1 = \frac{6}{100} \times \$1000 = \$60$
- One year: $L_1 = \left(1 + \frac{6}{100}\right) 1000 = (1 + 0.06)1000 = 1.06 \times 1000 = 1060$
- 6 Years: $L_6 = (1.06)^6 \times 1000 = \$1418.52$

7.2 Monthly Interest

- nAPR: $12 \times r$
- r: monthly interest rate
- $L_1 = \left(1 + \frac{0.5}{100}\right)^{12} \times 1000 = \$1061.68$
- $L_6 = \left(1 + \frac{0.5}{100}\right)^{72} \times 1000 = \$1432.04$

8 <u>Set Notation</u>

- A set is a collection of objects
- $\mathbb{C}$: The set of complex numbers
- $\mathbb{N}$: The set of natural numbers
- $\mathbb{Z}$: The set of integers
- $\mathbb{Q}$: The set of rational numbers
- $\mathbb{R}$: The set of real numbers
- $\{...\}$: A set
- $S \cup T$: Union of sets
- $S \cap T$: Intersection of sets
- $S \setminus T$: Set minus
- $S \subset T$: Is subset of
- $S \subseteq T$: Is subset or equal to
- $S = T$: Is equal to
- $S \equiv T$: Is equivalent to
- $\forall$: For all
- $\exists$: There exists
- $\nexists$: There does not exist
- $|$: Such that
- $\in$: Element of
- $\notin$: Not an element of
- Set of all real positive numbers: $\mathbb{R}_+ \equiv \{$ all x in $\mathbb{R}$ such that x $\geq 0\}$

$$\mathbb{R}_+ \equiv \{x \in \mathbb{R} | x \geq 0\}$$

- Set of all even integers:

$$E \equiv \left\{ n \in \mathbb{Z} | \frac{n}{2} \in \mathbb{Z} \right\}$$

- Set of all odd integers:

$$O \equiv \left\{ n \in \mathbb{Z} | \frac{n+1}{2} \in \mathbb{Z} \right\}$$

9 Physics

9.1 Motion

- UAM (Uniform Acceleration Motion)
  - Acceleration
    $$a(t) = a$$
  - Velocity
    $$\begin{aligned} v(t) &= at + v_i \\ \Delta v &= a\Delta t \\ \Delta v &\equiv v_f - v_i \\ \Delta t &\equiv t_f - t_i \end{aligned}$$
  - Position
    $$x(t) = \frac{1}{2}at^2 + v_i t + x_i$$
  - Final velocity
    $$\begin{aligned} [v(t)]^2 &= v_i^2 + 2a[x(t) - x_i] \\ v_f^2 &= v_i^2 + 2a\Delta x \end{aligned}$$
- UVM (Uniform Velocity Motion)
  - Accelteration
    $$a(= 0)$$
  - Velocity
    $$v(t) = v_i$$
  - Position
    $$x(t) = v_i t + x_i$$
- Free Fall
  - Gravity: $a_y = -9.81m/s^2$
- Examples:

– a ball dropped from height $y_i = 44.145m$

$$
\begin{aligned}
y(t) &= \frac{1}{2}at^2 + v_i t + y_i \\
0 &= y(t_{fall}) \\
0 &= \frac{1}{2}(-9.81)(t_{fall})^2 + 0(t_{fall}) + 44.145 \\
t_{fall} &= \sqrt{\frac{44.145 \times 2}{9.81}} = 3s
\end{aligned}
$$

– a ball thrown (10m/s) from 44.145m high

$$
\begin{aligned}
y(t) &= \frac{1}{2}a_y t^2 + v_i t + y_i \\
y(t) = 0 &= \frac{1}{2}(-9.81)t^2 - 10t + 44.145 \\
t_{fall} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-10 \pm \sqrt{25 + 866.12}}{9.81} = 2.53s
\end{aligned}
$$

10 Vectors

- Describe directions in space
- Ways to denote vectors:
  - Component notation: $\vec{v} = (v_x, v_y)$
  - Unit vector notation (denoted by hats not arrows): $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$
    $\hat{\imath} = (1, 0) \qquad \hat{\jmath} = (0, 1)$
  - Length and direction notation: $\|\vec{v}\|\angle\theta$
- Vector operations: $\vec{u} = (u_x, u_y) \qquad \vec{v} = (v_x, v_y)$
  - Addition: $\vec{u} + \vec{v} = (u_x + v_x, u_y + v_y)$
  - Subtraction: $\vec{u} - \vec{v} = (u_x - v_x, u_y - v_y)$
  - Scaling: $\alpha\vec{u} = (\alpha u_x, \alpha u_y)$
  - Dot Product: $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y$ or geometrically
    $\vec{v} \cdot \vec{w} \equiv \|\vec{v}\|\|\vec{w}\| \cos(\varphi)$
    ϖ is the angle between two vectors; known as the scalar product.
  - Length: $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_x^2 + u_y^2}$
  - Cross product (only for 3-dimension): $\vec{u} \times \vec{v} = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$
    $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin(\varphi)$

- Unit vectors:

$$\begin{aligned}
(\hat{\imath}, \hat{\jmath}, \hat{k}) &\rightarrow (x, y, z) \\
\hat{\imath} &\rightarrow (1,0,0) \qquad 4\hat{\imath} = (4,0,0) \\
\hat{\jmath} &\rightarrow (0,1,0) \qquad 5\hat{\jmath} = (0,5,0) \\
\hat{k} &\rightarrow (0,0,1) \qquad 6\hat{k} = (0,0,6)
\end{aligned}$$

$$v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} = \vec{v} = (v_x, v_y, v_z)$$

- Length and Direction Notation: $r\angle\theta$

  – convert to

$$\begin{aligned}
r_x &= \|\vec{r}\| \cos\theta \\
r_y &= \|\vec{r}\| \sin\theta
\end{aligned}$$

  – convert from

$$\begin{aligned}
r &= \|\vec{r}\| = \sqrt{r_x^2 + r_y^2} \\
\theta &= \tan^{-1}\left(\frac{r_y}{r_x}\right)
\end{aligned}$$

  if $v_x < 0; +\pi(180°)$

- Examples:

  – Compute $\vec{s} = 4\hat{\imath} + 5\angle 30°$ express answer in length and direction notation

$$\begin{aligned}
r_x &= r\cos(30°) \qquad r_y = r\sin(30°) \\
5\angle 30° &= (5\cos 30°)\hat{\imath} + (5\sin 30°)\hat{\jmath} \\
&= 5\frac{\sqrt{3}}{2}\hat{\imath} + \frac{5}{2}\hat{\jmath} \\
\vec{s} &= 4\hat{\imath} + 5\frac{\sqrt{3}}{2}\hat{\jmath} = \left(4 + 5\frac{\sqrt{3}}{2}\right)\hat{\imath} + \left(\frac{5}{2}\right)\hat{\jmath} \\
s_x &= \left(4 + 5\frac{\sqrt{3}}{2}\right) \qquad s_y = \left(\frac{5}{2}\right) \\
\|\vec{s}\| &= \sqrt{s_x^2 + s_y^2} = 8.697 \qquad \theta = \tan^{-1}\left(\frac{s_y}{s_x}\right) = 16.7 \\
\vec{s} &= 8.697\angle 16.7°
\end{aligned}$$

– A block is sliding down an incline, find the net force

$$\vec{W} = 30\angle-90^\circ \qquad \vec{N} = 200\angle-290^\circ \qquad \vec{F_f} = 50\angle 60^\circ$$

$$\sum \vec{F} = \vec{F_{net}} = m\vec{a} \qquad \vec{F_{net}} = \sum \vec{F} = \vec{W} + \vec{N} + \vec{F_f}$$

$$\begin{aligned} F_{net,x} &= W_x + N_x + F_{f,x} \\ &= 30\cos(-90^\circ) + 200\cos(-290^\circ) + 50\cos(60^\circ) \\ &= 93.4 \\ F_{net,y} &= W_y + N_y + F_{f,y} \\ &= 30\sin(-90^\circ) + 200\sin(-290^\circ) + 50\sin(60^\circ) \\ &= 201.2 \\ \vec{F}_{net} &= (F_{net,x}, F_{net,y}) = (93.4, 201.2) = 93.4\hat{\imath} + 201.2\hat{\jmath} \end{aligned}$$

11 Calculus

11.1 Derivative

- A derivative descirbes change over time, or the rate of change, or the slope of a function

$$f\prime(t) \equiv slope_f(t) = \frac{change\ in\ f(t)}{change\ in\ t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

- Denoted by:

$$f\prime(t) = \frac{df}{df} = \frac{d}{dt}f(t) = f$$

11.2 Integral

- The area under a curve

$$A(a,b) \equiv \int_{t=a}^{t=b} f(t)dt$$

- Two important formulae

$$\int_0^\tau a\ dt = a\tau$$

$$\int_0^\tau at\ dt = \frac{1}{2}a\tau^2$$

- compute the area under $h(t) = mt + b$

$$H(\tau) = \int_0^\tau h(t)\ dt = \int_0^\tau (mt+b)dt = \int_0^\tau mt\ dt = \int_0^\tau mt\ dt + \int_0^\tau mt\ dt + \int_0^\tau b\ dt = \frac{1}{2}m\tau^2 + b\tau$$

- Integrating is the opposite of differentiation

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