RTB | RWANDA TVET BOARD

RQF LEVEL 4

TRADE:

MODULE CODE: GENBA402

TEACHER'S GUIDE

Module name: BASIC MATHEMATICAL ANALYSIS

MODULE NAME:GENAM402 BASIC MATHEMATICAL ANLYSIS

2022

Table of Contents

Introduction

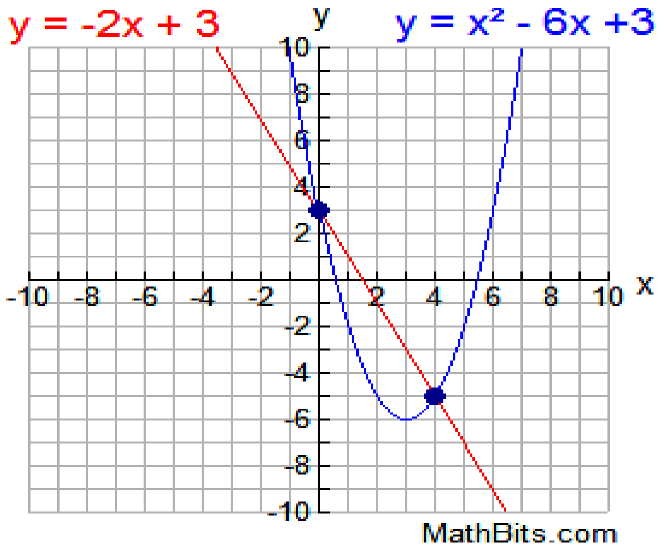
This general module describes the knowledge, skills and attitude required to apply basic mathematical analysis. The ability to do basic mathematical analysis is absolutely vital to successfully passing any field. At the end of this module, the trainee of Level Four will be able to solve algebraically and graphically linear or quadratic equations and inequalities. He/she will also be able to determine analyse algebraic functions, and to apply fundamentals of differentiation. As Algebra and fundamentals of differentiation are tools of different field. Therefore, this module will be useful to trainees as a means of analysis and improving their understanding of Mathematics and he/she will be prepared to perform well in any fields that require some knowledge of mathematics especially algebra and fundamentals of differentiation as well as working in daily mathematical logic and problem solving, financial and economics in hospitality sector for an effective performance in critical thinking, and so on.

Module Code and Title: GENBA 402:APPLY BASIC MATHEMATICAL ANALYSIS

Learning Units:

1. Solve algebraically or graphically linear and quadratic equations or inequalities.
2. Determine and analyse algebraic functions.
3. Apply fundamentals of differentiation.

Learning Unit 1: Solve algebraically or graphically linear and quadratic equations or inequalities.



STRUCTURE OF LEARNING UNIT

Learning outcomes:

1.1. Solve algebraically or graphically linear equation or inequality.
1.2. Solve parametric equations
1.3. Solve algebraically or graphically two simultaneous linear equations
1.4. Solve algebraically or graphically a quadratic equation
1.5. Solve algebraically or graphically a quadratic inequality

Learning outcome 1.1. Solve algebraically or graphically linear equation or inequality.

Duration: 10 hrs

Learning outcome 1.1 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to solve a linear equation by algebraic method as applied in basic mathematical analysis.

2. Describe appropriately how to solve a linear equation by graphical method as applied in basic mathematical analysis.

3. Describe clearly how to solve a quadratic equation by algebraic method as applied in basic mathematical analysis.

4. Describe correctly how to solve a quadratic equation by graphical method as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books	**Didactic materials such as manila paper** **Chalk board** **White board**	**Geometric instruments (Ruler, T-square)** **Handouts on worked examples** **Chalk** **Internet** **Marker pen**

Advance preparation:

. Refer to linear equation manual

. calculator

Content1: Solving a linear equation

Definition: A linear equation is equation of a straight line. The general form of the linear equation with one variable is $ax + b = 0$

Where a, b $\in \mathbb{R}$: and $a \neq 0$ the value of x in which the equality is verified is called the **root. (solution** of the equation)

Examples:

1. $y = 2x + 1$
2. $5x = 6 + 3y$
3. $\frac{y}{2} = 3 - x$

.

How to solve a linear equation?

- A linear equation is a polynomial of degree 1.
- In order to solve for the unknown variable, you must isolate the variable.
- In the order of operation, multiplication and division are completed before addition and subtraction.

The linear equation can be solved **algebraically** or **graphically**.

✓ **Algebraic method**

Example

Solve the equation: $4x – 7 = 9$

Solution

(i) Isolate x to one side of the equation

$4x-7+7 = 9+7$

$4x = 16$

(ii) Divide both side by 4

$\frac{4x}{4} = \frac{16}{4}$

$X = 4$

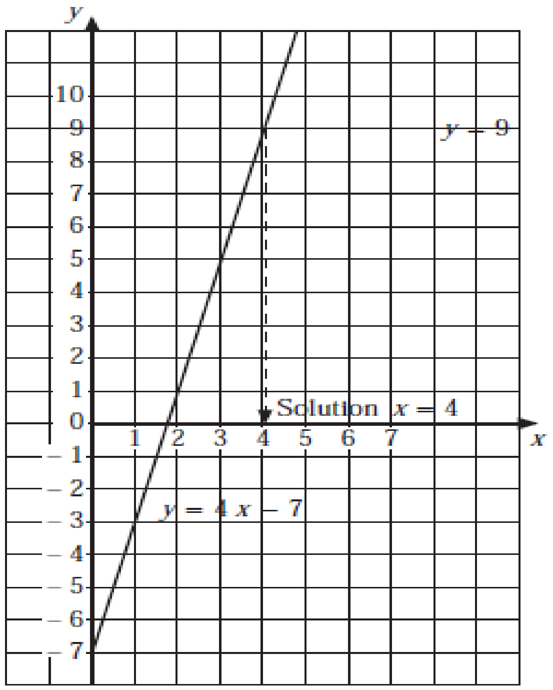
The root square is 4

This method used is known as **algebraic method**

✓ Graphical method

$4x - 7 = 9$

Draw the lines $y = 4x - 7$ and $y = 9$.



The solution is given by the value on the x-axis immediately below the point where $y = 4x - 7$ and $y = 9$ cross.

The solution is $x = 4$.

Theoretical learning Activity

- ✓ Group discuss on solving linear equation by different methods

Practical learning Activity

- ✓ Solve the following by algebraic and graphical method.
  a) y= 2y+2
  b) 2x+4= 4

Points to Remember (Take home message)

To solve a word problem in which a number is to be found:

- ✓ Introduce a letter to stand for the number to be found (the unknown).
- ✓ Form an equation involving this letter by expressing the given information in symbols instead of words.
- ✓ Solve the equation to get the required number.

Content2 : Solving a linear inequality

✓ **Algebraic method**

They are solved as linear equations except that:

(a) When we multiply an inequality by a negative real number the sign will be reversed
(b) When we interchange the right side and the left side, the sign will be reversed.

Example:

Solve: $-2(x+3) < 10$

Solution

$$-2x - 6 < 10$$

$$-2x - 6 + 6 < 10 + 6$$

$$2x < 16$$

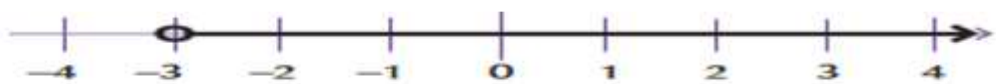
$$\frac{-2x}{-2} > \frac{16}{-2}$$

$$x > -8$$

✓ **Graphical method**

The graph of a linear inequality in one variable is a number line. We use an unshaded circle for < and > and a shaded circle for ≤ and ≥.

The graph for x > -3:



Solved example

Solve the following inequality:

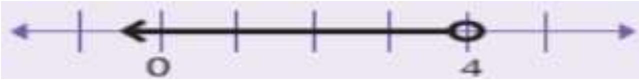
2x – 6 < 2.

Solution: Add 6 to both sides: 2x – 6+7 <2+6

Divide both sides by 2: $\frac{2}{2}x < \frac{8}{2}$

$$x < 4$$

Open circle at 4 (since x cannot equal 4) and an arrow to the left (because we want values less than 4).



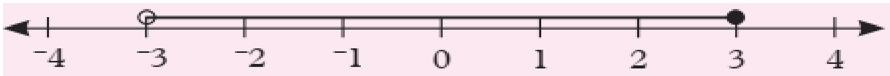
Write the following pairs of simple inequality statements as compound statements and illustrate them on number lines.

$x \leq 3, x > -3$

Solution

$x \leq 3, x > -3$ becomes $-3 < x \leq 3$

∴ x lies between –3 and 3, and 3 is included.



Theoretical learning Activity

- In groups of three students, discuss and answer the following questions

1.Describe the procedure of constructing the number line?

2.Discuss on methods of solving inequalities?

Practical learning Activity

- ✓ **Solve the following inequalities and represent your solution on a number line.**
  - i) $3x - 4 \geq 5$
  - ii) $\frac{1}{4}x + 5 \leq 14$

Points to Remember (Take home message)

- ✓ **Inequalities / inequations** – these are mathematical statements that involves phrases such as “less than”, “greater than” , “smaller to” , “greater or equal to”.

Learning out come1 : Formative Assessment

Practical assessment

- ✓ Task to be performed:

  Solve and graph the following:

1. y ≤ 3	4. y ≥ -2
2. y = $\frac{1}{2}x - 3$	5. 2y – x ≤ 6
3. y = -3x + 2	6. $\frac{y}{2} + 2 > x$

Learning outcome 1.2. Solve parametric equations

Duration: 10 hrs

Learning outcome 1.2 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Define correctly the terms "**Parameter** and **Parametric equations**" as applied in basic mathematical analysis.

2. Describe clearly the steps of solving parametric equations as applied in basic mathematical analysis.

3. Discuss appropriately on all the steps required to solve a parametric equation as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books	**Didactic materials such as manila paper**	**Handouts on worked examples**

Advance preparation:

. Refer to manual to introduce parametric equations

Content1 : Definition of terminologies

- **Definitions:**
  - ✓ Parameter
  - ✓ Parametric equation

In case certain coefficients of equations contain one or several letter variables, the equation is called **parametric** and the letters are called **real parameters**. In this case, we solve and discuss the equation (for parameters only).

There are also a great many curves that we cannot even write down as a single equation in terms of only **x** and **y**. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining **y** in terms of **x** i.e. $y = f(x)$) or **x** in terms of **y** i.e. $x = h(y)$ we define both x and y in terms of a third variable called a parameter as follows:

$x = f(t)$ $y = g(t)$

This third variable is usually denoted by **t** (but does not have to be). Sometimes we will restrict the values of **t** that we shall use and at other times we will not.
If the coefficients of an equation contain one or several letters (variables) the equation is called parametric and the letters are called **real parameters**. In this case, we solve and discuss the equation (for parameters only).
Each value of t defines a point (x, y) = (f(t),g(t)) that we can plot. The collection of points that we get by letting **t** be all possible values is the graph of the parametric equations and is called the **parametric curve**.

Theoretical learning Activity

In groups of three members

- Find the meaning of parametric equations and inequalities.

Practical learning Activity

In groups of three members

- Discuss your findings using suitable examples

Points to Remember (Take home message)

1. The general form of a linear equation is **ax + b = 0.**
2. The inequalities of one unknown are of the form $ax + b \geq 0$ or $ax + b \leq 0$ with a $\in$, b $\in$.
3. In the case where certain coefficients of an equation contain one or several variables, the equation is called **parametric** and the letters are called **real parameters**.

Content2: Steps of Solving parametric equations

- <u>**Solving steps**</u>

Solve and discuss the equation $(2-3m)x+1=m^2(1-x)$

Solution

$(2-3m)x + 1 = m^2(1-x)$

$2x - 3mx + 1 = m^2 - m^2x$

$2x - 3mx + m^2x - m^2 + 1 = 0$

$x(2-3m+m^2) - m^2 + 1 = 0$

$x(2-3m+m^2) = m^2 - 1$

$x = \frac{m^2-1}{2-3m+m^2}$

$= \frac{(m-1)(m+1)}{(m-1)(m-2)} = \frac{m+1}{m-2}$

If m = 2, then there is no solution.

If $m \neq 2$, then the solution is $x = \frac{m+1}{m-2}$

Notes

In the example above we can see that after finding the value of x, it follows a discussion so that we can validate the solution

Parametric equations in one unknown

If at least one of the coefficients a, b and c depend on the real parameter which is not determined, the root of the parametric quadratic equation depends on the values attributed to that parameter.

Example

Find the values of k for which the equation $x^2 + (k+1)x + 1 = 0$ has:

(a) two distinct real roots
(b) no real roots.

Solution

Δ = $(k+1)^2 - 4(1)(1) = (k+1)^2 - 4$

$= k^2+2k+1-4 = k^2+2k-3 = (k+3)(k-1) = 0$ then $k = -3$ or $k = 1$.

Table of sign of $\Delta = k^2 + 2k - 3 = (k+3)(k-1)$

k / Factors	$-\infty$		-3		1		$+\infty$
$k+3$		$-$	0	$+$		$+$	
$k-1$		$-$		$-$	0	$+$	
$(k+3)(k-1)$		$+$	0	$-$	0	$+$	

a) For two distinct real roots; $\Delta > 0$ and so $k < -3$ or $k > 1$.

b) For no real roots $\Delta < 0$ and so $-3 < k < 1$.

Exercises

1. Find the range of values of m for which the equation $(m-3)x^2 - 8x + 4 = 0$ has:

(a) two real roots (b) no real root

(c) one double root.

2. Find the range of values of k for which the equation $x^2 - 2(k+1)x + k^2 = 0$ has:

(a) two real roots (b) no real root

(c) one double root.

3. Find the range of values of m for which the equation $2x^2 - 5x + 3m - 1 = 0$ has:

(a) two real roots (b) no real root

(c) one double root.

4. Find the set of values of m for which $x^2 + 3mx + m$ is a positive for all real values of x.

Theoretical learning Activity

- ✓ Discuss on methods of solving parametric equation.

Practical learning Activity

- ✓ Solve $(m + 3)x \geq 2$.

Points to Remember (Take home message)

- ✓ . **A product equation** is one of the form $(ax + b)(cx + d) = 0$
  2. The general form of a **fractional equation** of the first degree is: $\frac{ax+b}{cx+d} = 0$

Learning outcome 1.2: Formative Assessment

Practical assessment

- ✓ Task to be performed:

Sketch the parametric curve for the following set of parametric equations.

a) $x = t^2 + t$

b) $y = 2t - 1$; $-1 \leq t \leq 1$

Learning outcome 1.3. Solve algebraically or graphically two simultaneous linear equations

Duration: 10 hrs

Learning outcome 1.3 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to solve algebraically two simultaneous linear equations as applied in basic mathematical analysis.

2. Describe correctly how to solve graphically two simultaneous linear equations as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet, Laptop, Projector**	**Didactic materials such as manila paper**	**Handouts on worked examples, Flipchart, marker pen, chalk board, chalk**

Advance preparation:

. Reference manual discussing on solving of two simultaneous equations.

CONTENT

Content1: Solving algebraically two simultaneous linear equations

- **<u>Solving algebraically two simultaneous linear equations</u>**

What are simultaneous linear equations? How do we solve them?

A linear equation in two variables x and is an equation of the form

Ax + by =c where a ≠ 0, b≠ 0 and a, b, c are real numbers.

Let us consider such equation $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1,b_1,c_1,a_2,b_2 and c_2 are

Constants. We say that we have two simultaneous linear equations in two unknowns or a system of two linear equations in two unknowns.

The pair (x,y) satisfying both equations is the solution of the given equation.

Consider the following example:

Claire and Laura are Sisters : we know that

(i) Claire is the elder sister ;
(ii) Their ages added together give 20 years ;

Let x = Claire's age ,in years and y = Laura's age ,in years.

$$x + y = 20$$

$$x - y = 2$$

This is an example of a pair of simultaneous equations.

We can solve such systems of linear equations by using one of the following methods:

1. substitution method
2. Elimination method
3. Comparison method
4. Cramer's rule

<u>1. Substitution method</u>

This method is used when one of the variables is given in terms of the other.

Example : Find the simultaneous solution of the following pair of equations : y=2x – 1, y=x + 3

Solution

Note that the system can also be written as $\begin{cases} y = 2x - 1 \\ y = x + 3 \end{cases}$, then

2x – 1 = x + 3

x = 4

And so y = 4 + 3

y = 7

So, the simultaneous solution is x = 4 and y= 7.

2. **<u>Elimination method</u>**

Elimination method is used to solve simultaneous equations where neither variable is given as the subject of another.

Solve simultaneously, by elimination: $\begin{cases} 5x + 3y = 12 \\ 7x + 2y = 19 \end{cases}$

Solution

$$\begin{cases} \mathbf{5x + 3y = 12} \ \ldots\ldots(\mathbf{1}) \\ \mathbf{7x + 2y = 19} \ \ldots\ldots(\mathbf{2}) \end{cases}$$

We multiply (1) by 2 and (2) by -3 :

$$\begin{cases} 10x + 6y = 24 \\ -21x - 6y = -57 \end{cases}$$

Adding the two equations term by term gives :

-11x = -33

X = 3

Substituting x = 3 into (1) gives :

5(3) + 3y = 12

15 + 3y = 12

3y = -3

y = -1

Hence x = 3, y = -1 is the solution to the system of equations.

3. Comparison method

Let's consider the following simultaneous equations

3x – 2y = 2

7x + 3y = 43

Steps to solve the system of linear equations by using the comparison method to find the value of **x** and **y**.

3x – 2y = 2 ---------- (i)

7x + 3y = 43 --------- (ii)

Now for solving the above simultaneous linear equations by using the method of comparison follow the instructions and the method of solution.

Step I: From equation 3x – 2y = 2 --------- (i), express **x** in terms of **y**.

Likewise, from equation 7x + 3y = 43 -------- (ii), express **x** in terms of **y**.

From equation (i) 3x – 2y = 2 we get;

3x – 2y + 2y = 2 + 2y (adding both sides by 2y)

or, 3x = 2 + 2y

or, 3x/3 = (2 + 2y)/3 (dividing both sides by 3)

or, x = (2 + 2y)/3

Therefore, x = (2y + 2)/3 ---------- (iii)

From equation (ii) 7x + 3y = 43 we get;

7x + 3y – 3y = 43 – 3y (subtracting both sides by 3y)

or, 7x = 43 – 3y

or, 7x/7 = (43 – 3y)/7 (dividing both sides by 7)

or, x = (43 – 3y)/7
Therefore, x = (–3y + 43)/7 ---------- (iv)

Step II: Equate the values of **x** in equation (iii) and equation (iv) forming the equation in **y**
From equation (iii) and (iv), we get;
(2y + 2)/3 = (–3y + 43)/7 ---------- (v)

Step III: Solve the linear equation (v) in **y**

(2y + 2)/3 = (–3y + 43)/7 ---------- (v) Simplifying we get;
or, 7(2y + 2) = 3(–3y + 43)
or, 14y + 14 = –9y + 129
or, 14y + 14 – 14 = –9y + 129 – 14
or, 14y = -9y + 115
or, 14y + 9y = –9y + 9y + 115
or, 23y = 115
or, 23y/23 = 115/23
Therefore, y = 5

Step IV: Putting the value of **y** in equation (iii) or equation (iv), find the value of **x**
Putting the value of **y = 5** in equation (iii) we get;

x = $\frac{(2\times5+2)}{3}$

or, x = $\frac{(10+2)}{3}$

or, x = $\frac{12}{3}$
Therefore, x = 4

Step V: Required solution of the two equations

Therefore, x = 4 and y = 5

Therefore, we have compared the values of **x** obtained from equation (i) and (ii) and formed an equation in **y**, so this method of solving simultaneous equations is known as the comparison method. Similarly, comparing the two values of **y**, we can form an equation in **x**.

04. Cramer's method

Step1 : Fund the determinant, D,by using the x and y values from the problem.

Step2 : Find the determinant ,$D_{X,}$ by replacing the x- values in the first column with the values after the equal sign leaving the y column unchanged.

Step3 : Find the determinant, D_Y, by replacing the y- values in the second column with the values after the equal sign leaving the x column unchanged.

Step 4 : Use Cramer's Rule to find the values of x and y.

Example 1 : Use Cramer's Rule to solve : $\begin{aligned} 3x - 2y &= 17 \\ 4x + 5y &= -8 \end{aligned}$

Step 1 : Find the determinant, D, by using the x and y values from the problem.

$$D = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 15 - (-8) = 23$$

Step 2 : Find the determinant, Dx, by replacing the x- values in the first column with the values after the equal sign leaving the y column unchanged.

$$D_x = \begin{vmatrix} 17 & -2 \\ -8 & 5 \end{vmatrix} = 85 - 16 = 69$$

Step 3 : Find the determinant, Dy, by replacing the y- values in the second column with the values after the equal sign leaving the x column unchanged.

$$D_y = \begin{vmatrix} 3 & 17 \\ 4 & -8 \end{vmatrix} = -24 - 68 = -92$$

Step 4 : Use Cramer's Rule to find the values of x and y.

$$x = \frac{D_x}{D} = \frac{69}{23} = 3$$

$$y = \frac{D_y}{D} = \frac{-92}{23} = -4$$

The answer written as an ordered pair is (3, -4)

Theoretical learning Activity

- ✓ Discuss on how to solve simultaneous equations**?**

Practical learning Activity

- ✓ Solve the inequalities 2x < x + 5 and x + 4 > 3.
  - Represent the two solutions on the same number line.
  - What can you say about the two solutions?
  - Express these solutions as a single inequality?

Points to Remember (Take home message)

- A system of two linear equations in two unknowns is of the form:
- $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, b_1, c_1, a_2, b_2 and c_2 are constants.

Content 2 : Solving graphically two simultaneous linear equations

- **Solving graphically two simultaneous linear equations**

Let's consider the following example:

Use a graph to solve the simultaneous equations:

$x+y = 20$

$x-y = 2$

Solution

We can rewrite the first equation to make y the subject:

$x+y = 20$

$y= 20-x$

For the second equation.

$x-y=2$

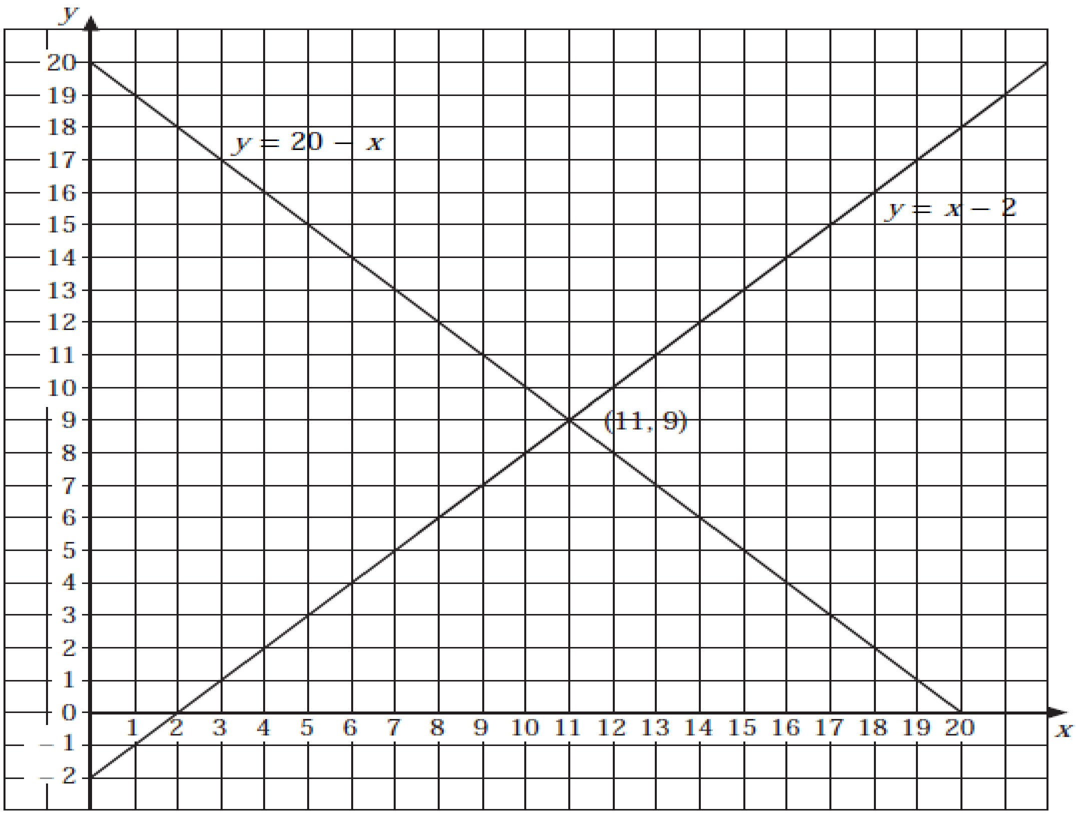
$x=y+2$

$x-2 = y$

or

$y= x-2$

Now draw the graphs $y= 20-x$ and $y= x-2$.



The lines cross at the point with coordinates (11,9), so the solution of the pair of simultaneous equation is x = 11, y = 9.

Example 2

Use a graph to solve the simultaneous equations:

$$x + 2y = 18$$
$$3x - y = 5$$

Solution

First rearrange the equations in the form y= ….

x + 2y = 18

2y = 18 – x

y = $\frac{18 - x}{2}$

y = 9 - $\frac{x}{2}$

3x – y = 5

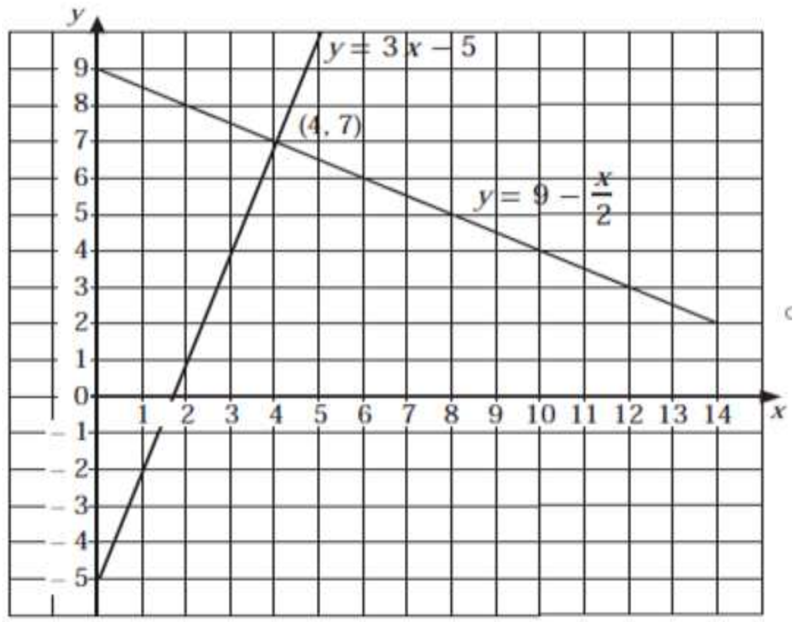
3x = y + 5

3x – 5 = y

or

y = 3x – 5

Now draw these two graphs:



The lines cross at the point with coordinates (4,7), so the solution is x = 4, y = 7.

Theoretical learning Activity

- ✓ Discuss on how to solve simultaneous equation?

Practical learning Activity

- ✓ Solve the following simultaneous inequalities and represent each solution on a number line.

(a) $2x < 10$, $5x \geq 15$

(b) $3x \leq 9$, $2x > 0$

Points to Remember (Take home message)

Inequalities that must be satisfied at the same time are called simultaneous inequalities.

- ✓ Definition of simultaneous equation.
- ✓ Methods of solving

Learning outcome 1.3: Formative Assessment

Practical assessment

- ➢ Solve the following simultaneous inequalities and represent each solution on a number line.

(a) $x + 7 < 0$, $x - 2 > -10$
(b) $x \geq 3$, $2x - 1 \leq 13$
(c) $4x - 33 < -1$, $-2 < 3x + 1$

Learning outcome 1.4. Solve algebraically or graphically a quadratic equation.

Duration: 10 hrs

Learning outcome 1.4. Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly what is "a quadratic equation" as applied in basic mathematical analysis.

2. Describe appropriately methods of solving algebraically a quadratic equation as applied in basic mathematical analysis.

3. Describe correctly how to solve graphically a quadratic equation as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books	**Didactic materials such as manila paper** **Geometric instruments** **(Ruler, T-square)**	**Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on quadratic equations

Content 1: Methods of solving algebraically a quadratic equation

What is a quadratic equation?

The term **quadratic** comes from the word *quad* meaning **square,** because the variable gets squared (like x^2). It is also called an "**equation of degree 2**" because of the "**2"** on the x.

The standard form of a quadratic looks like this:

$$ax^2 + bx + c = 0$$

where **a, b and c** are known values and **a** cannot be 0.

"x" is the **variable** or the unknown.

Here are some more examples of quadratic equations:

$\mathbf{2x^2 + 5x + 3 = 0}$ In this one, **a = 2, b = 5 and c = 3**

$\mathbf{x^2 - 3x = 0}$ For this, **a = 1, b = -3, and c = 0,** so 1 is not shown.

$\mathbf{5x - 3 = 0}$ This one is **not** a quadratic equation. It is missing a value in x^2 i.e. **a = 0,**

Which means it cannot be quadratic.

- **Different method of solving algebraically a quadratic equation**

✓Factorizing method

Let us use an example, $x^2 - 5x + 6$. To solve $x^2 - 5x + 6 = 0$ we must first factorise

$X^2 - 5x + 6$. To do this we have to find two numbers with a sum of -5 and a product of 6.

The numbers required are -2 and -3, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

In solving this, we use the following low:

When the product of two or more numbers is zero, then at least one of them must be zero. So if ab=0 or b=0.

Example

Solve for x:

$x^2 - 3x + 2 = 0$

Solution

$x^2 - 3x + 2 = 0$

We need two numbers with sum -3 and product 2. These are -1 and -2.

$x^2 - 3x + 2 = (x - 1)(x - 2) = 0$

$x - 1 = 0$ or $x - 2 = 0$

x = 1 or 2

✓ Square root property

This property states: if A and B are algebraic expressions such that $A^2 = B$, then $A = \pm\sqrt{B}$. This method is used if the form of the equation is: $x^2 = k$ or $(ax + b)^2 = k$ where k represents a constant

Steps to solve quadratic equations by the square root property:

1. Transform the equation so that a perfect square is on one side and a constant is on the other side of the equation.

2. Use the square root property to find the square root of each side. **REMEMBER** that finding the square root of a constant yields positive and negative values.
3. Solve each resulting equation. (If you are finding the square root of a negative number, there is no real solution and imaginary numbers are necessary.)

Example

Solve the quadratic equation $(\mathbf{x}+1)^2 = 49$

$(\mathbf{x}+1)^2 = 49$

Ansuer

x + 1 = 7 or x + 1 = -7

x = 6 or x = -8

✔ **Completing the square method**

Let's consider the equation $\mathbf{x}^2 + 2\mathbf{x} - 8 = 0$

Step 1- find the completed square form of $x^2 + 2x - 8$

$x^2 + 2x - 8$

Halve the coefficient of x (which here is 2) and add to x in a bracket squared

$(x + 1)^2$

Expand out the bracket

$(x + 1)^2 = x^2 + 2x + 1$

Subtract the 1 from both sides

$(x + 1)^2 - 1 = x^2 + 2x$

Now substitute this back into $x^2 + 2x - 8$ for the first two terms

$x^2 + 2x - 8 = (x + 1)^2 - 1 - 8 = 0$

$(x + 1)^2 - 9 = 0$

Step 2- solve this quadratic equation for x

$(x + 1)^2 - 9 = 0$

Add 9 to both sides

$(x + 1)^2 = 9$

Square root both sides

$x + 1 = \pm 3$

Subtract 1 from both sides

$x = -1 \pm 3$

x = - 1 – 3 or x = - 1 + 3

Solutions x = - 4 or 2

✓ **Quadratic formula**

Example

Use the quadratic formula to solve the equation $x^2 + 2x - 8 = 0$

Step 1- get the values of a, b and c to use in the formula

$ax^2 + bx + c = 0$

$x^2 + 2x - 8 = 0$

Therefore

a = 1, b = 2, c = -8

Step 2- substitute these values for a, b, and c into the quadratic formula and go on to simplify and solve for x

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{-2 \pm \sqrt{((2)^2 - ((4)(1)(-8))}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{(4-(-32))}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2}$$

$$x = \frac{-2 \pm 6}{2}$$

$$x = \frac{-2-6}{2} \quad \text{or} \quad x = \frac{-2+6}{2}$$

Solutions: x = -4 or 2

Exercises

Solve by factoring and then solve by completing the square.

1. $x^2 + 2x - 8 = 0$

2. $x^2 - 8x + 15 = 0$

3. $y^2 + 2y - 24 = 0$

4. $y^2 - 12y + 11 = 0$

5. $t^2 + 3t - 28 = 0$

6. $t^2 - 7t + 10 = 0$

7. $2x^2 + 3x - 2 = 0$

8. $3x^2 - x - 2 = 0$

9. $2y^2 - y - 1 = 0$

Theoretical learning Activity

- ✓ Discuss on how to solve quadratic equation.

Practical learning Activity

- ✓ **Solve the following quadratic equations:**
  1. $16x^2 – 8x + 1 = 0$
  2. $x^2 – 3x + 72 = 0$

Points to Remember (Take home message)

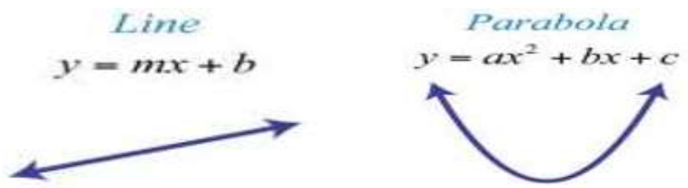
- ✓ If at least one of the coefficients a, b and c depend on the real parameter which is not determined, the root of the parametric quadratic equation depends on the values attributed to that parameter.
- ✓ Definition of quadratic equation
- ✓ Methods of solving quadratic equation

Content2: Solving graphically a quadratic equation

- **<u>Graphical resolution of a quadratic equation</u>**

✓ **Construction of a parabola**

We know that any linear equation with two variables can be written in the form $y = mx + b$ and that its graph is a line. In this section, we will see that any quadratic equation of the form $y = ax^2 + bx + c$ has a curved graph called a **parabola.**



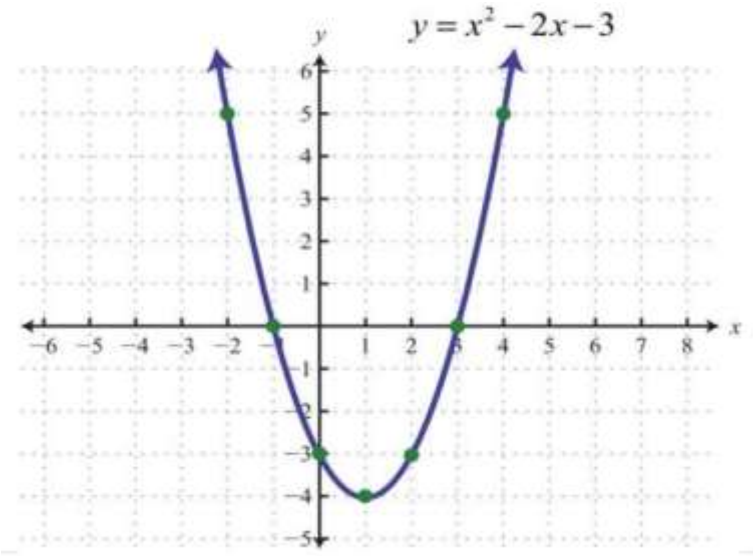
Example 1: Graph by plotting points: $y = x^2 – 2x -3$

Solution: In this example, choose the x-values { -2, -2, 0, 1, 2, 3, 4} and calculate the corresponding y-values.

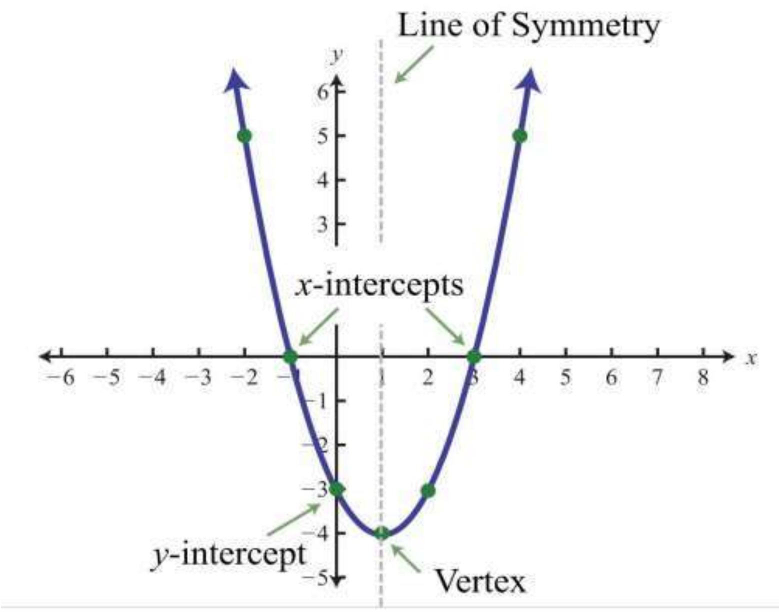
x	y		Points
-2	5	$y=(-2)^2-2(-2)-3=4+4-3=5$	$(-2, 5)$
-1	0	$y=(-1)^2-2(-1)-3=1+2-3=0$	$(-1, 0)$
0	-3	$y=(0)^2-2(0)-3=0-0-3=-3$	$(0, -3)$
1	-4	$y=(1)^2-2(1)-3=1-2-3=-4$	$(1, -4)$
2	-3	$y=(2)^2-2(2)-3=4-4-3=-3$	$(2, -3)$
3	0	$y=(3)^2-2(3)-3=9-6-3=0$	$(3, 0)$
4	5	$y=(4)^2-2(4)-3=16-8-3=5$	$(4, 5)$

Plot these points and determine the shape of the graph.

Answer:



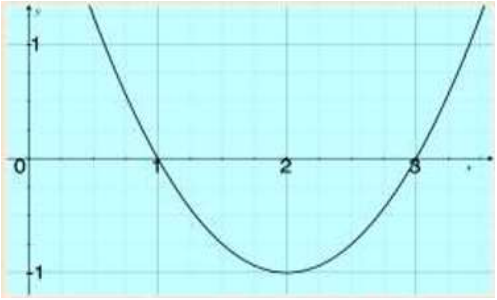
When graphing, we want to include certain special points in the graph.
The **y – intercept** is the point where the graph intersects the y-axis. **The x –intercept** is the point where the graph intersects the x-axis. **The vertex** is the point that defines the **minimum** or **maximum** of the graph. Lastly, **the line of symmetry** (also called the axis of symmetry) is the vertical line through the vertex about which the parabola is symmetric.



✓ **Determination of solution**

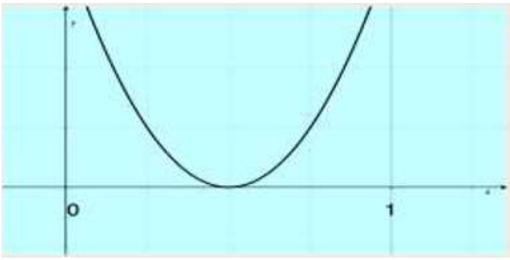
In using this method, we draw a graph of a quadratic function by creating a table of values. The solutions or roots are obtained by reading-off the x-coordinates of the point of intersection of the curve and the horizontal axis (when the equation = 0) Recall that the quadratic can have a maximum of two roots – this occurs when the graph cuts the x-axis at two distinct points.
If the x-axis is a tangent to the curve, then the two roots are equal to each other and so there is just one solution. If the curve does not cut or touch the x-axis, there are no solutions. These cases are illustrated below.

Graph of $x^2 - 4x + 3 = 0$



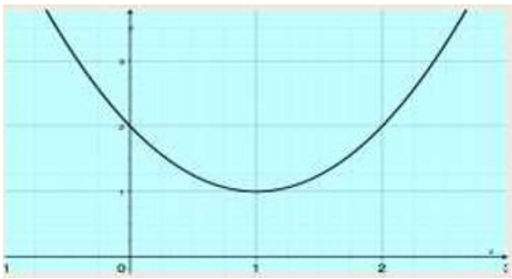
The roots are x = 1 and x = 3

Graph of $4x^2 - 4x + 1 = 0$



There is only one root, $x = \frac{1}{2}$

Graph of $x^2 + 2x + 8 = 0$



There are no solutions – the graph does not cut or touch the x-axis

Theoretical learning Activity

- ✓ Discuss on graphical method for solving a quadratic equation?

Practical learning Activity

- ✓ Draw the graph of y = x2 – 4x + 4 for values of x from –1 to +5. Solve from your graph the equations:

  (a) $x^2 - 4x + 4 = 0$

(b) $x^2 - 4x + 1 = 0$

(c) $x^2 - 4x - 1 = 0$

Points to Remember (Take home message)

In a quadratic function graph, the x-coordinate of the point where the graph cuts x-axis gives the solution to the quadratic equation represented by the function.

(a) When the graph cuts the x-axis at one point, then the equation has one repeated solution.
(b) When the graph cuts x-axis at two points, then the equation has two different solutions.
(c) When the graph does not cut x-axis at any point, then the equation has no solution in the field of real numbers

Learning outcome1.4 : formative assessment

Practical assessment

- ✓ Task to be performed:

  Sketch the graph of $y = -2x^2 - 6x - 9$.

Learning outcome 1.5. Solve algebraically or graphically a quadratic inequality

Duration: 10hrs

Learning outcome 1.5. Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe appropriately all the steps followed to solve algebraically a quadratic inequality as applied in basic mathematical analysis.

2. Describe correctly how to solve graphically a quadratic inequality as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books	**Didactic materials such as manila paper** **Geometric instruments (Ruler, T-square)**	**Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on solving graphically quadratic inequality.

Content1: Steps of solving algebraically a quadratic inequality

• Solving algebraically a quadratic inequality

✓ Factorization of the given inequality
✓ Determination of roots
✓ Study of sign
✓ Determination of interval of solutions

Form of a quadratic inequality:

After rearrangement, quadratic inequality has the following standard form

$ax^2 + bx + c > 0$ $\boxed{\geq, \quad <, \quad \leq}$

Example 1

Solve the inequality (x + 3) (x – 2) > 0.

Solution

The critical values are x = -3, x = 2.

The required sign diagram is:

Factors \ x	$-\infty$		-3		2		$+\infty$
$x + 3$		$-$	0	$+$		$+$	
$x - 2$		$-$		$-$	0	$+$	
$(x + 3)(x - 2)$		$+$	0	$-$	0	$+$	

The answer is **x < -3** or **x > 2** ⇔ **x** ∈ **]**-∞, −3**[** ∪ **]2, +∞[**

Theoretical learning Activity

- ✓ Discuss on how to solve quadratic inequality

Practical learning Activity

- ✓ **Solve the following quadratic inequalities:**
  a) $2x^2 + x - 2 > 0$

b) $x^2 - 3x + 2 \geq 0$

Points to Remember (Take home message)

1. A quadratic equation in the unknown x is an equation of the form $ax^2 + bx + c > 0$, where a, b and c are given real numbers, with $a \neq 0$.
2. The product of two factors is positive if and only if $a > 0$ and $b > 0$ or $a \leq 0$ and $b < 0$.

Content2: Graphical resolution of quadratic inequality

- <u>**Graphical resolution of quadratic inequality**</u>

✓ Plotting a parabola
✓ Shading the region satisfying the given inequality
✓ Determination of interval of solutions

Example

Solve graphically the following quadratic inequality

$x^2 - 4x + 3 > 0$

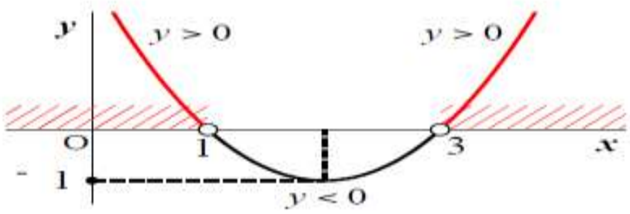
Solution

The standard form $y = (x-2)^2 - 1$

By factoring, we have $y = (x-1)(x-3)$
therefore, the roots are $x = 1,\ x = 3$

The inequality is satisfied in the shaded domain.

The solution is
$x < 1, \quad x > 3$



Exercises

Solve the following quadratic inequalities

1. $y^2 - 17y + 70 < 0$

2. $x^2 + 9x + 13 > -7$

3. $x(x+1) > 112 - 5x$

4. $a^2 + 3a + 2 < -3(a+2)$

5. $2x^2 \leq 5x - 2$

6. $10 - 9y \geq -2y^2$

7. $b(b+3) \geq -2$

8. $a^2 \leq 4(2a-3)$

9. $y^2 - 17y + 70 < 0$

10. $x^2 + 9x + 13 > -7$

11. $x(x+1) > 112 - 5x$

12. $a^2 + 25 < 10a$

Theoretical learning Activity

- ✓ Discuss on definition of quadratic inequality

Practical learning Activity

- ✓ Sketch the graph of y ≥ −2x2 − 6x − 9.

Points to Remember (Take home message)

- ✓ Vertex of a quadratic function: Every quadratic function has vertex. The graph turns at its vertex. The vertex is the coordinate ([h, f(h)) where x = h is the axis of symmetry.
- ✓ For the expression $y \geq ax^2 + bx + c$, if the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the "∪"-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the "∩"-shape

Learning outcome 1.5: formative assessment

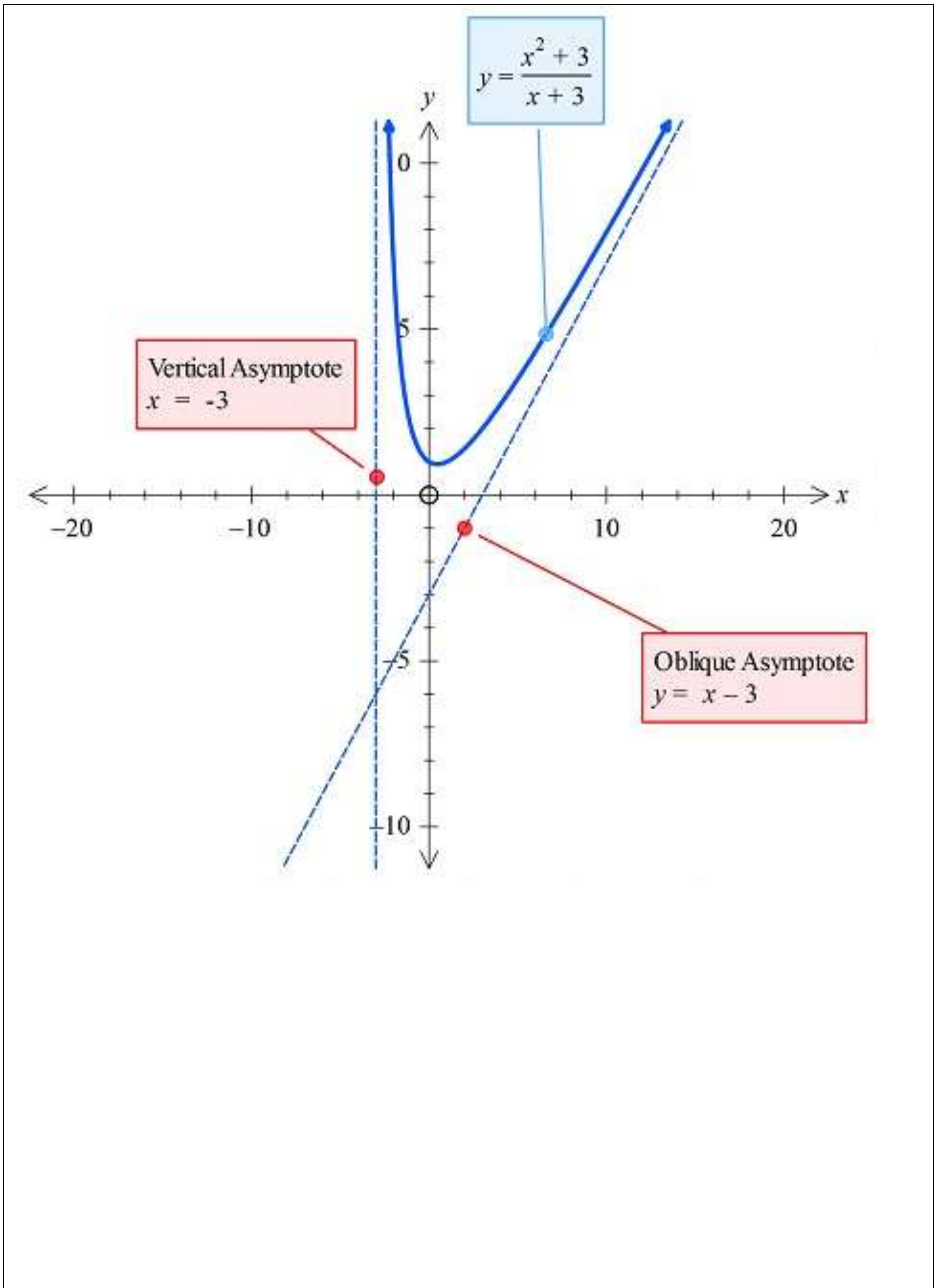
Practical assessment

- ✓ Task to be performed:

  Solve the following inequalities: **a)** $\dfrac{(x-2)(x-1)}{(x+1)(x-3)} > 0$

  b) $x^2 + (k+1)x + 1 > 0$

Learning Unit 2: Determine and analyze algebraic functions



STRUCTURE OF LEARNING UNIT

Learning outcomes:
2.1. Determine the domain and range of algebraic function. **2.2. Identify the symmetry of algebraic function.** **2.3. Determine limits of a function.** **2.4. Determine the asymptotes to the rational and polynomial functions.**

2.1. Determine the domain and range of algebraic function.

Duration: 5 hrs

Learning outcome 2.1 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Define correctly the terms: “Domain and Range” of a function as applied in basic mathematical analysis.

2. Describe clearly how to determine the domain of a function as applied in basic mathematical analysis.

3. Describe clearly how to determine the range of a function as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper**	**Geometric instruments (Ruler, T-square)** **Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on algebraic function

CONTENT

Content1 : Definitions of terminologies

- **Definitions:**
  - ✓ Existence condition
  - ✓ Domain of definition of a function
  - ✓ Range of a function

Algebraic function: **For any two subsets A and B of the real line, algebraic function is a rule that assigns exactly one element y in set B to each element x in set A.**

x is the independent variable, $f(x)$ is the image of x under function f, $y = f(x)$ is the dependent variable.

We write $f: A \longrightarrow B: x \longmapsto f(x)$

Examples of numerical functions include:

$f: \mathbb{N} \rightarrow \mathbb{Z}: x \longmapsto y = 2x - 1$; $\quad h: \mathbb{R} \rightarrow \mathbb{R}: x \longmapsto y = \sqrt{x}$

$g: \mathbb{R} \rightarrow \mathbb{R}: x \longmapsto y = \frac{x}{x^2+1}$; $\quad t: \mathbb{R} \rightarrow \mathbb{R}: x \longmapsto y = \frac{x}{\sqrt{x^3 - 27}}$

The domain of a numerical function f

Is the largest set of real numbers for which the function is defined? We write Df

Thus $D_f = \{x \in \mathbb{R} ; f(x) \in \mathbb{R}\}$.

The set of all values $f(x)$ is called the range of the numerical function f, it is denoted by Imf.

Range of a function

We have already discussed the domain of a function $f(x)$ i.e. the values of x for which $f(x)$ is defined. Next we consider the values $f(x)$ we get as x varies over the domain. This is, not surprisingly, called the range of $f(x)$.

Thus $\text{Imf} = \{f(x); x \in D_f\}$.

The **graph** consists of all points (x, y), where x is in the domain of f and where $y = f(x)$.

Theoretical learning Activity

- ✓ Discuss on how to find the domain and range of algebraic function?

Practical learning Activity

- ✓ By using symbols, describe how to apply the domain and range of algebraic function?

Points to Remember (Take home message)

1.The rational and irrational numbers together make up the set of **real numbers** denoted by $\mathbb{R}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all subsets of .

2.A **rational number** is one which can be expressed in the form $\frac{q}{p}$ where p and q are integers and $q \neq 0$.

3. If a and b are any two real numbers, then either $a < b$ or $b < a$ or $a = b$.

Content 2. Determination of Range and Domain of functions

- **Calculations**

✓ Domain of definition of a function
✓ Range of a function

1. A polynomial

A polynomial function is any function that can be written in the form.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and n is a positive integer (the **degree** of the polynomial).

Note that the domain of every polynomial function is the entire real line.

Further, recognise that the graph of $f(x) = ax + b$ is a **straight line**, and the graph of $f(x) = ax^2 + bx + c, a \neq 0$, is a **parabola**.

The following are examples of polynomials:

$f: \mathbb{R} \to \mathbb{R}: x \mapsto y = 2x^3 + \frac{1}{2}x + 3;\ D_f = \mathbb{R},$

$g: \mathbb{R} \to \mathbb{R}: x \mapsto y = 5x^4 - 3x^2 + x - 5;\ D_f = \mathbb{R}.$

2. A rational function

Any function that can be written in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with no common factors other than 1, is called a **rational function**.

Notice that since $P(x)$ and $Q(x)$ are polynomials, they are both defined for all x, and so, the rational function $y = \frac{P(x)}{Q(x)}$ is defined for all x, for which $Q(x) \neq 0$.

Thus, $D_f = \{x \in \mathbb{R}; Q(x) \neq 0\}$ or $D_f = \mathbb{R} - \{x \in \mathbb{R}; Q(x) = 0\}$.

Examples:

Determine the domain of each of the following numerical functions:

a) $f(x) = \frac{x+1}{2x-4}$

b) $f(x) = \frac{2x+3}{x^3 + 2x^2 - x - 2}$

c) $f(x) = \frac{x}{x^2+1}$

j

Solutions

a) $D_f = \mathbb{R} - \{x \in \mathbb{R};\ 2x - 4 = 0\}$

$= \mathbb{R} - \{2\}$

$=]-\infty, 2[\cup]2, +\infty[$

b) $Df = \mathbb{R} - \{x \in \mathbb{R};\ x^3 + 2x^2 - x - 2 = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ x^2(x+2) - (x+2) = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ (x^2 - 1)(x+2) = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ (x-1)(x+1)(x+2) = 0\}$

$= \mathbb{R} - \{1; -1; -2\}$

$=]-\infty, -2[\cup]-2, -1[\cup]-1, 1[\cup]1, +\infty[$

c) $D_f = \mathbb{R} - \{x \in \mathbb{R};\ x^2 + 1 = 0\}$

$= \mathbb{R} - \varnothing$: $x^2 + 1 \neq 0$ for all $x \in \mathbb{R}$

$= \mathbb{R}$

$=]-\infty, +\infty[$

3. An irrational function

Any function that can be written as $f(x) = \sqrt[n]{P(x)}$, where $n \in \mathbb{N} - \{0; 1\}$ and $P(x)$ is a polynomial is called an irrational function.

We have the following:

	n:odd	n:even
$P(x)$ is a polynomial	$D_f = R$	$D_f = \{x \in \mathbb{R};\ P(x) \geq 0\}$
$P(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials	$D_f = \{x \in \mathbb{R};\ D(x) \neq 0\}$	$D_f = \{x \in \mathbb{R};\ \frac{N(x)}{D(x)} \geq 0\}$

Examples

Find the domain of each of the following numerical functions:

a) $f(x) = \sqrt[3]{\dfrac{x+1}{x^2 - 3x + 2}}$

b) $f(x) = \sqrt[3]{x^3 - 3x + 2}$

c) $f(x) = \sqrt{x^2 - 3x - 10}$

d) $f(x) = \sqrt[4]{x^2 - 2x + 3}$

e) $f(x) = \sqrt{-4x^2 - 3x - 5}$

f) $f(x) = \sqrt{\dfrac{-x^2 - 2x + 3}{2x + 8}}$

g) $f(x) = \sqrt{\dfrac{-x^2 - x - 2}{9 - x^2}}$

Solution

a) $D_f = \{x \in \mathbb{R};\ x^2 - 3x + 2 \neq 0\}$
$= \mathbb{R} - \{1, 2\}$
$=]-\infty, 1[\text{u}]1, 2[\text{u}]2, +\infty[$

b) $D_f = \mathbb{R}$

c) $D_f = \{x \in \mathbb{R};\ x^2 - 3x - 10 \geq 0\}$
$=]-\infty, -2]\ \text{u}\ [5, +\infty[$
(from the sign of $x^2 - 3x - 10$:

x	$-\infty \quad -2 \quad 5 \quad +\infty$
$x^2 - 3x - 10$	$+ \quad 0 \quad - \quad 0 \quad +$

d) $D_f = \{x \in \mathbb{R};\ x^2 - 2x + 3 \geq 0\}$
$=]-\infty, +\infty[$ ($x^2 - 2x + 3 \geq 0$ for all $x \in \mathbb{R}$ since $\Delta = (-2)^2 - 4(1)(3) < 0$)

e) $D_f = \{x \in \mathbb{R};\ -4x^2 + 3x - 5 \geq 0\}$
$= \varnothing$; $-4x^2 + 3x - 5 < 0$ for all $x \in \mathbb{R}$

f)

x	$-\infty \quad -4 \quad -3 \quad 1 \quad +\infty$
$-x^2 + 2x + 3$	$- \quad - \quad 0 \quad + \quad 0 - - -$
$2x + 8$	$- \quad 0 \quad + \quad + \quad +$
$\dfrac{-x^2 - 2x + 3}{2x + 8}$	$+ \quad \vert\vert \quad - \quad 0 \quad + \quad 0 \quad -$

$D_f = \{x \in \mathbb{R};\ \dfrac{-x^2 - 2x + 3}{2x + 8} \geq 0\}$

$=]-\infty, -4[\cup[-3, 1]$

g)

x	$-\infty \quad -3 \quad -1 \quad 2 \quad 3 \quad +\infty$
$-x^2 + x + 2$	$- \quad - \quad -0 \quad + \quad 0 - - - \quad -$
$9 - x^2$	$- - - 0 \quad + \quad + + + 0 - - -$
$\dfrac{-x^2 + x + 2}{9 - x^2}$	$+ \quad \vert\vert \quad - \quad 0 \quad + \quad 0 \quad - \quad \vert\vert \quad - \quad +$

$D_f = \{x \in \mathbb{R};\ \dfrac{-x^2 + x + 2}{9 - x^2} \geq 0\}$

$D_f =]-\infty, -3[\text{u}[-1, 2]\text{u}]\ 3, +\infty[$

Examples for finding the range of functions:

(a) f(x) = x. The domain is ℝ i.e. all numbers and the range is also ℝ.

(b) $f(x) = x^2$. The domain is once again ℝ, but the range is all positive numbers as $x^2 \geq 0$ i.e.[0,∞] .

(c) $g(x) = \sin(x)$. The domain is ℝ, but the range is given by [-1, 1] as $-1 \leq \sin(x) \leq 1$.

(d) $h(t) = \sqrt{t}$. Remember that this is the positive square root. The domain is [0,∞] as is the range.

Exercises

Find the ranges of the following functions:

1. $f(x) = 3 - 2x$
2. $f(x) = 3x^2 - 2$

Solutions:

(a) $f(x) = 3 - 2x$. the domain is ℝ.

To find the range, let f be a value in the range then

$$3 - 2x = f \Rightarrow$$

$$x = (3 - f) / 2.$$

This shows than no matter what value f we choose we can find x such that f(x) = f, hence the range is also ℝ.

(b) $f(x) = 3x^2 - 2$. The domain is once again ℝ, but the range is all $f \geq -2$ as given $f \geq -2$ then $f = 3x^2 - 2 \Rightarrow x = \sqrt{f + 2/3}$ gives f(x) = f i.e. the range is [-2 , ∞) .

Theoretical learning Activity

- ✓ Distinguish range and domain accordingly?

Practical learning Activity

- ✓ Determine the domain and range of the given function: $y = -\sqrt{2x + 3}$

Points to Remember (Take home message)

- Domain of a function is the set of all real numbers for which the expression of the function is defined as a real number.
- Let f: A → B be a function. The range of f, denoted by Im(f) is the image of A under f, that is, Im (f) = f [A]. The range consists of all possible values the function f can have.

Learning outcome 2.1: formative assessment

Practical assessment

- ✓ Task to be performed:

Determine the domain and range of the given function: $y = \frac{x^2+x-2}{x^2-x-2}$

Learning outcome 2.2. Identify the symmetry of algebraic function.

<table>
<tr><td colspan="3">Duration: 5 hrs</td></tr>
<tr><td colspan="3">Learning outcome 2.2 Objectives:
By the end of the learning outcome, the trainees will be able to:
1. Define correctly the terms: “Even function and Odd function” as applied in basic mathematical analysis.
2. Describe clearly how to identify a function as Even as applied in basic mathematical analysis.
3. Describe clearly how to identify a function as Odd as applied in basic mathematical analysis.</td></tr>
<tr><td colspan="3" align="center">Resources</td></tr>
<tr><td align="center">Equipment</td><td align="center">Tools</td><td align="center">Materials</td></tr>
<tr><td>Reference books
Internet</td><td>Didactic materials such as manila paper</td><td>Handouts on worked examples</td></tr>
<tr><td colspan="3">Advance preparation:
. Refer to manual discussing on how to identify symmetry of algebraic function.</td></tr>
</table>

CONTENT

Content 1: Definition of terminologies

- **Definitions:**

- ✓ Even function
- ✓ Odd function

Let f be an algebraic function whose domain is Df.

f is even if and only if:

$$:(\forall x \in D_f); -x \in D_f \text{ and } f(-x) = f(x);$$

f is odd if and only if:

$$:(\forall x \in D_f); -x \in D_f \text{ and } f(-x) = -f(x);$$

Theoretical learning Activity

- ✓ Differentiate the even from odd function?

Practical learning Activity

- ✓ State whether the following function is even or odd: $f(x) = x^2 - 1$

Points to Remember (Take home message)

- ✓ Let f be a function of in we say that f is even if ∀ x ∈ Dom (f), (–x) ∈ Dom(f); f(–x) = f(x)
- ✓ We say that a function f is odd if ∀x ∈ Dom (f), (–x) ∈Dom(f); f(–x) = –f(x)

2: Identification of functions

- **Even function**
- **Odd function**

Examples

1. Determine whether f is odd or even in each of the following cases:

a) $f(x) = 2x^2 + 1$

b) $f(x) = \cos x$

c) $f(x)\ x^3 + 2x$

d) $f(x) = \sin x$

Solution

a) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = 2(-x)^2 + 1$

$= 2x^2 + 1$

$= f(x)$

Therefore, f is even.

b) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = \cos(-x)$

$= \cos x$

$= f(-x)$

Therefore, f is even.

c) $D_f = \mathbb{R};$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(x) = (-x)^3 + 2(-x)$

$= -x^3 - 2x$

$= -(x^3 + 2x)$

$= -f(x)$

Therefore, f is odd.

d) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$

Therefore, f is odd.

Theoretical learning Activity

- ✓ What do you understand by even and odd functions?

Practical learning Activity

- ✓ State whether the following function is even or odd: $g(x) = \frac{x^3 + x}{5}$

Points to Remember (Take home message)

- ✓ f: $\mathbb{R} \to \mathbb{R}$: $x \to f(x) = x^2$ is an even function because $\mathbf{f}(-x) = (-x)^2 = x^2 = f(x)$
- ✓ f: $\mathbb{R} \to \mathbb{R}$: $x \to f(x) = x^3$ is odd function because $\mathbf{f}(-x) = (-x)^3 = -x^3 = -f(x)$

Learning outcome 2.2: Formative Assessment

Practical assessment

- ✓ Task to be performed:

State whether the following function is even or odd: $h(x) = \frac{x^3 + x + 2}{x^2}$

Learning outcome 2.3. Determine limits of a function.

<table>
<tr><td colspan="3">Duration: 10 hrs</td></tr>
<tr><td colspan="3">Learning outcome 2.3 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to determine the limits of functions as applied in basic mathematical analysis.

2. Describe correctly how to remove indeterminate cases as applied in basic mathematical analysis.</td></tr>
<tr><td colspan="3" align="center">Resources</td></tr>
<tr><td align="center">Equipment</td><td align="center">Tools</td><td align="center">Materials</td></tr>
<tr><td>Reference Books

Internet</td><td>Didactic materials such as manila paper</td><td>Hand-out notes</td></tr>
<tr><td colspan="3">Advance preparation:

. Refer to manual discussing on how to determine the limits of a function.</td></tr>
</table>

CONTENT

1: Determination of function limits

Introduction

Consider the functions f(x) = $\frac{x^2-4}{x-1}$ and g(x) = $\frac{x^2-5}{x-2}$. Notice that both functions are undefined at x = 2. But the two functions have quite different beahaviours in the vicinity of x = 2.

From numerical approach, for f(x) = $\frac{x^2-4}{x-1}$, we have :

1.

x	f(x)

1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

2.

x	f(x)
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x – values get closer to 2, but are all less than 2. We use the notation $x \to 2^-$ to indicate that x approaches 2 from the left side. f(x) is getting closer and closer to 4. In view of this, we say that the limit of f(x) as x approaches 2 from the left is 4, written $\lim_{x\to 2^-} f(x) = 4$.

The second table suggests that as x gets closer and closer to 2 (with x > 2), f(x) is getting closer and closer to 4.

In view of this, we say that the limit of f(x) at x approaches 2 from the right is 4, written $\lim_{x\to 2^+} f(x) = 4^-$.

We call $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ one sided limits of f(x), if they are the same, then we summarize the results by saying that limit of f(x) as x approaches 2 is 4, *written* $\lim_{x\to 2} f(x) = 4$.

In general if

$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = l$ we conclude that $\lim_{x\to a} f(x) = l$

If

$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ we conclude that $\lim_{x\to a} f(x)$ doesn't exist

Operations on limits

(1) For any constant c and any real number a, $\lim_{x\to a} c = c$.

(2) For any real number a, $\lim_{x\to a} x = a$.

(3) Suppose that $\lim_{x\to a} (x)$ and $\lim_{x\to a} g(x)$ both exist and let c be any constant.

The following then apply:

(i) $\lim_{x\to a} [c.f(x)] = c.\ \lim_{x\to a} f(x)$;

(ii) $\lim\limits_{x\to a} [f(x)\pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$,

(iii) $\lim\limits_{x\to a} [(x).g(x)] = [\lim\limits_{x\to a} f(x)].[\lim\limits_{x\to a} g(x)]$,

(iv) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, if $\lim\limits_{x\to a} g(x)\neq 0$.

Observe that by applying part (iii) of rule (3), within $g(x) = f(x)$, we get that , whenever $\lim\limits_{x\to a} f(x)$ exists,

$$\begin{aligned}\lim_{x\to a} [f(x)]^2 &= \lim_{x\to a} [f(x).f(x)] \\ &= [\lim_{x\to a} f(x)].[\lim_{x\to a} f(x)] \\ &= [\lim_{x\to a} f(x)]^2.\end{aligned}$$

Likewise, for any positive integer n, $\lim\limits_{x\to a} [f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$.

(4) For any polynomial $P(x)$, and any real number a, $\lim\limits_{x\to a} P(x) = P(a)$

Proof:

Suppose $P(x)$ is the polynomial of degree $n \geq 0$, $P(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$.

$$\begin{aligned}\text{Then, } \lim_{x\to a} P(x) &= \lim_{x\to a} (c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0) \\ &= c_n \lim_{x\to a} x^n + c_{n-1} \lim_{x\to a} x^{n-1} + \ldots + c_1 \lim_{x\to a} x + \lim_{x\to a} c_0 \\ &= c_n a^n + c_{n-1}a^{n-1} + \ldots + c_1 a + c_0 \\ &= P(a).\end{aligned}$$

(5) Suppose that $\lim\limits_{x\to a} f(x) = L$, and n is any positive integer, $n \geq 2$.

Then, $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$, where for n even , we assume that L>0.

(6) For any real number a, we have:

(i) $\lim\limits_{x\to a} \sin x = \sin a$,

(ii) $\lim\limits_{x\to a} \cos x = \cos a$,

(iii) $\lim\limits_{x\to a} \sin^{-1} x = \sin^{-1} a$, for $-1<a<1$,

(iv) $\lim\limits_{x\to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(v) $\lim\limits_{x\to a} \tan^{-1} x = \tan^{-1} a, a \in \mathbb{R}$,

(vi) If $P(x)$ is a polynomial and $\lim\limits_{x\to P(a)} f(x) = L$, then $\lim\limits_{x\to a} f[P(x)] = L$.

- **Finite limits**

Examples

Find the following limits:

(a) $\lim\limits_{x \to 2} (3x^2 - 5x + 4)$ (b) $\lim\limits_{x \to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$

(c) $\lim\limits_{x \to 2} \sqrt[5]{3x^2 - 2x}$ (d) $\lim\limits_{x \to 0} \sin^{-1}\left(\dfrac{x+1}{2}\right)$

Solution

(a) $\lim\limits_{x \to 2} (3x^2 - 5x + 4) = 3(2)^2 - 5(2) + 4$

$= 6$

(b) $\lim\limits_{x \to 2} \dfrac{x^3 - 5x + 4}{x^2 - 2} = \dfrac{3^3 - 5(3) + 4}{3^2 - 2}$

$= \dfrac{16}{7}$

(c) $\lim\limits_{x \to 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{3(2)^2 - 2(2)}$

$= \sqrt[5]{8}$

(d) $\lim\limits_{x \to 0} \sin^{-1}\left(\dfrac{x+1}{2}\right) = \text{Sin}^{-1}\dfrac{1}{2}$

$= \dfrac{\pi}{6}.$

Exercises

Evaluate:

(1) $\lim\limits_{x \to 1} \dfrac{x+1}{\sqrt{x^2}}$ (2) $\lim\limits_{x \to \frac{\pi}{6}} (\sin x + \cos x)$

(3) $\lim\limits_{x \to \frac{\pi}{2}} 2(\sin x - \cos x + \cos^2 x)$ (4) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin 2x + \tan x}{\cos 3x + 4}$

- **Infinite limits**

In general, any limit of the six types

$$\lim_{x \to a^-} f(x) = -\infty, \qquad \lim_{x \to a^-} f(x) = \infty,$$
$$\lim_{x \to a^+} f(x) = -\infty, \qquad \lim_{x \to a^+} f(x) = \infty,$$
$$\lim_{x \to a} f(x) = -\infty, \qquad \lim_{x \to a} f(x) = \infty,$$

Is called an **infinite limit.**

Examples

Evaluate:

(a) $\lim\limits_{x \to 5} \frac{1}{(x-5)^2}$

(b) $\lim\limits_{x \to -2^-} \frac{x+1}{(x-3)\,(x+2)}$ and $\lim\limits_{x \to -2^+} \frac{x+1}{(x-3)\,(x+2)}$

Solution

(a) $\lim\limits_{x \to 5} \frac{1}{(x-5)^2} = \frac{1^+}{0^+} = +\infty$.

(b) $\lim\limits_{x \to -2^-} \frac{x+1}{(x-3)\,(x+2)} = \frac{-2+1}{(-2-3)\,.\,0^-} = \frac{-1}{0^+} = -\infty$

$\lim\limits_{x \to -2^+} \frac{x+1}{(x-3)\,(x+2)} = \frac{-1}{0^-} = +\infty$

$\lim\limits_{x \to -2} \frac{x+1}{(x-3)\,(x+2)}$ does not exist.

• **Limits at infinity**

In this section we are intended in examining the limiting behaviour of functions as x increases without bound (written $x \to +\infty$) or as x decreases without bound (written $x \to -\infty$).

Example1

Calulate, from numerical approach $\lim\limits_{x \to +\infty} \frac{1}{x}$

X	$\frac{1}{x}$
1	1
10	0.1
100	0.01
1000	0.001
10 000	0.0001

Conclusion : $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$

Example2

Evaluate $\lim_{x\to\infty}\sqrt{\dfrac{2x^3-5x^2+4x-6}{6x^3+2x}}$.

Solution Because the limit of the rational function inside the radical exists and is positive, we can write

$$\lim_{x\to\infty}\sqrt{\frac{2x^3-5x^2+4x-6}{6x^3+2x}}=\sqrt{\lim_{x\to\infty}\frac{2x^3-5x^2+4x-6}{6x^3+2x}}=\sqrt{\lim_{x\to\infty}\frac{2x^3}{6x^3}}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}.$$

Theoretical learning Activity

- ✓ **Discuss on how to find the limit of a function?**

Practical learning Activity

- ✓ **Evaluate the following limits:** $\lim_{x\to 2} 2x+1$

Points to Remember (Take home message)

1. A neighbourhood of a real number is any interval that contains a real number a and some point below and above it.
2. If x is taking values sufficiently close to and greater than a, then we say that x tends to a from above and the limiting value is then what we call the **right-sided** limit. It is written as $\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a\\ >}} f(x)$
3. If x is taking values sufficiently close to and less than a, then we say that x tends to a from below and the limiting value is then what we call the **left-sided** limit. It is written as $\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a\\ <}} f(x)$
4. If the f(x) tends closer to a value L as x approaches the value a from either side, then L is the limit of f(x) as x approaches . We use the following notation: $\lim_{x\to a} f(x) = L$

CONTENT

2 : Remove of indeterminate cases

- **Indeterminate cases**

✓ **Indeterminate form** $\frac{0}{0}$

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation by:

- factoring
- rationalizing
- using trigonometric transformation

Examples

1. Calculate:

(a) $\lim\limits_{x\to 1}\frac{x^2-1}{1-x}$ (b) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$

(c) $\lim\limits_{x\to 3}\frac{3x+9}{x^2-9}$

Solution

(a) $\lim\limits_{x\to 1}(x^2-1) = 1^2-1 = 0$ and $\lim\limits_{x\to 1}(1-x) = 0$

We have: $\lim\limits_{x\to 1}\frac{x^2-1}{1-x} = \lim\limits_{x\to 1}\frac{(x-1)\,(x+1)}{-(x-1)}$

$= \lim\limits_{x\to 1}[-(x+1)]$

$= -(1+1) = -2.$

(b) (b) $\lim\limits_{x\to 1}(x^2-4) = 0$ and $\lim\limits_{x\to 1}(x-2) = 0$

Since the expression in the numerator factors,

$\lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \lim\limits_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$

$\lim\limits_{x\to 3}\frac{3x+9}{x^2-9} = \frac{3(-3)+9}{(-3)^2-9} = \frac{0}{0}$ indeterminate form.

$\lim\limits_{x\to 3}\frac{3x+9}{x^2-9} = \lim\limits_{x\to 3}\frac{3(x+3)}{(x-3)(x+3)}$

$\lim\limits_{x\to 3}\frac{3}{x-3}$

$= \frac{3}{-6}$

$= -\frac{1}{2}$.

2. Evaluate:

(a) $\lim\limits_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$ (b) $\lim\limits_{x\to 0}\frac{2x}{3\sqrt{x+9}}$

Solution

(a) $\lim\limits_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \frac{\sqrt{2}-\sqrt{2}}{0} = \frac{0}{0}$ indeterminate form.

$\lim\limits_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim\limits_{x\to 0}\frac{\left(\sqrt{x+2}-\sqrt{2}\,\right)\left(\sqrt{x+2}+\sqrt{2}\,\right)}{x\left(\sqrt{x+2}\right)}$

$= \lim\limits_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\,\right)}$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2}\,)}$$

$$= \lim_{x\to 0} \frac{1}{(\sqrt{x+2}+\sqrt{2}\,)}$$

$$= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}\,.$$

(b) $\lim_{x\to 0} \frac{2x}{3\sqrt{x+9}} = \frac{0}{3-3} = \frac{0}{0}$ indeterminate form

$$= \lim_{x\to 0} \frac{2x\left(3+\sqrt{x+9}\right)}{\left(3-\sqrt{x+9}\right)\left(3+\sqrt{x+9}\right)}$$

$$= \lim_{x\to 0} \frac{2x\left(3+\sqrt{x+9}\right)}{9-(x+9)}$$

$$= \lim_{x\to 0} \frac{2x\left(3+\sqrt{x+9}\right)}{-x}$$

$$= -2 \lim_{x\to 0}\left(3+\sqrt{x+9}\right)$$

$$= -2\ (6) = -12\ .$$

✓ **Indeterminate form** $\infty - \infty$

If the evaluation of a limit leads to:

$(+\infty) - (+\infty)$; $\quad (-\infty) - (-\infty)$;

$(+\infty) + (-\infty)$; $\quad (-\infty) + (+\infty)$.

then the limit is an indeterminate form.

To remove the indetermination $\infty - \infty$, multiply and divide the expression by the conjugate.

Example

Calculate: $\lim_{x\to +\infty}\left(x - \sqrt{x^2+4x-1}\right)$

Solution

$$\lim_{x\to +\infty}\left(x - \sqrt{x^2+4x-1}\right) =$$

$$= \lim_{x\to +\infty}\left(x - \sqrt{x^2}\right) = \lim_{x\to +\infty}\left(x - |x|\right)$$

$$= \lim_{x\to +\infty}(x - x)$$

$$= (+\infty) - (+\infty) \qquad = \infty - \infty : \text{indeterminate form.}$$

$$\lim_{x\to +\infty}\left(x - \sqrt{x^2+4x-1}\right) = \lim_{x\to +\infty} \frac{\left(x-\sqrt{x^2+4x-1}\right)\left(x+\sqrt{x^2+4x-1}\right)}{\left(x+\sqrt{x^2+4x-1}\right)}$$

$$= \lim_{x\to +\infty} \frac{x^2-\left(x^2+4x-1\right)}{x+\sqrt{x^2+4x-1}} = \lim_{x\to +\infty} \frac{-4x}{x+x} = -2\ .$$

✓ Indeterminate form $\frac{\infty}{\infty}$

Example1 Evaluate $\lim\limits_{x\to\infty} \frac{x+1}{x^2+3x+1}$

Solution: $\lim\limits_{x\to\infty} \frac{x+1}{x^2+3x+1} = \frac{x\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} \Rightarrow \lim\limits_{x\to\infty} \frac{x+1}{x^2+3x+1} = \lim\limits_{x\to\infty} \frac{1}{x} \frac{\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} = \frac{1}{\infty} = 0$

✓ Indeterminate form $0\times\infty$

Example1 Evaluate $\lim\limits_{x\to 0^+} x\ln x$

Solution $\lim\limits_{x\to 0^+} x\ln x = 0\times\infty$ which is indeterminate form.

Theoretical learning Activity

- ✓ Discuss on what do you understand by **"indefinite forms"**

Practical learning Activity

- ✓ Evaluate the limit: $\lim\limits_{x\to\pm\infty} \sqrt{x^2+x+2} - \sqrt{x^2-x+3}$

Points to Remember (Take home message)

Suppose that f(x) → 0 and g(x) as x → a. Then the limit of the quotient $\frac{f(x)}{g(x)}$ as x → a is said to give an indeterminate form, sometimes denoted by 00 . It may be that the limit of $\frac{f(x)}{g(x)}$ can be found by some methods such as factor method, rationalisation method, l'Hôspital's rule, etc...

Similarly, if f(x)→ ∞ and g(x) → ∞ as x → a , then the limit of $\frac{f(x)}{g(x)}$ gives an indeterminate form, denoted by $\frac{\infty}{\infty}$. Also, if f(x) → 0 and g(x) → ∞ as x → a, then the limit of the product f(x)g(x) gives an indeterminate form 0 **x** ∞.

Learning outcome 2.3: Formative Assessment

Practical assessment

- ✓ Task to be performed:

Find the following limits, if they exist.

1. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 3x + 2}$ 2. $\lim_{x\to 3} \frac{\sqrt{25 - x^2} - 4}{x-3}$ 3. $\lim_{z\to 0} \frac{\sqrt{2z + 4} - 2}{z}$

Learning outcome 2.4. Determine the asymptotes to the rational and polynomial functions.

Duration: 10 hrs

Learning outcome 2.4. Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly on the definitions of different types of asymptotes as applied in basic mathematical analysis.

2. Describe appropriately how to determine different types of asymptotes of a function as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper** **Geometric instruments** **(Ruler, T-square)**	**Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on how to determine the asymptotes to the rational and polynomial functions.

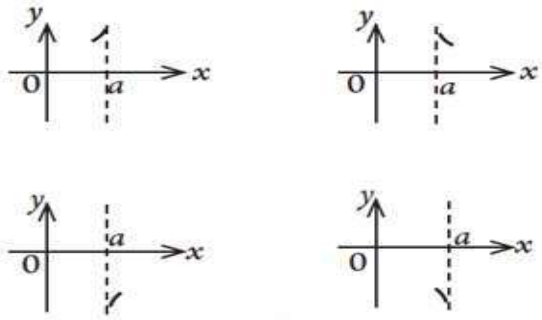
Content 1: Definitions of asymptotes

- **<u>Definitions</u>**

A line (l) is an asymptote to a curve if the distance from a point P to the line (l) tends to zero as P tends to infinity along some unbounded part of the curve.

✓ **Boundaries of domain of definition**

✓ **Vertical asymptotes**

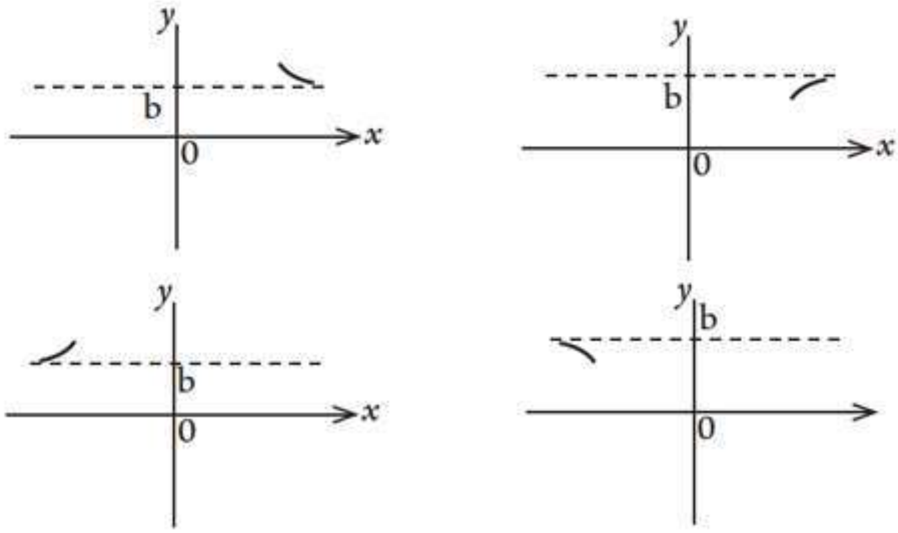


line $x = a$, where $a \in \mathbb{R}$, is said to be vertical asymptote to curve $y = f(x)$ if and only if $\lim\limits_{x \to a} f(x) = \infty$.

More exactly, if at least one of the following is verified:

$$\lim_{x \to a+} f(x) = +\infty;\ \lim_{x \to a-} f(x) = +\infty;\ \lim_{x \to a-} f(x) = -\infty;\ \lim_{x \to a} f(x) = -\infty.$$

✓ **Horizontal asymptotes**

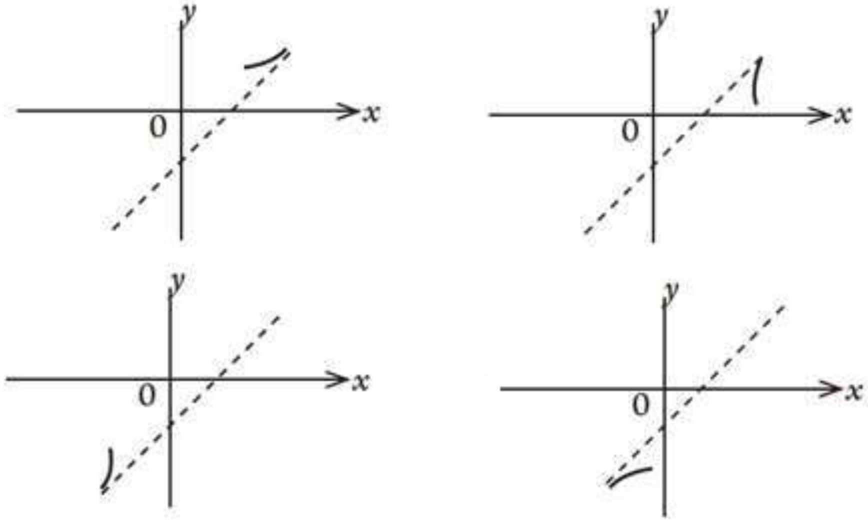


Line $y = \mathrm{b}$, where $\mathrm{b} \in \mathbb{R}$, is said to be horizontal asymptote to curve $y = f(x)$ if and only if $\lim\limits_{x \to \infty} f(x) = \mathrm{b}$.

More exactly,

$$\lim_{x \to -\infty} f(x) = \mathrm{b} \text{ or } \lim_{x \to +\infty} f(x) = \mathrm{b},\ \mathrm{b} \in \mathbb{R}.$$

✓ **Oblic asymptotes**



Line $y = mx + p$, where m and p are real numbers and $m \neq 0$ is said to be oblique asymptote of the graph of function $y = f(x)$ if and only if:

$$m = \lim_{x \to \infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \to \infty} [f(x) - mx].$$

Theoretical learning Activity

- ✓ Describe the types of asymptotes?

Practical learning Activity

- ✓ What is an asymptote? Carry out research and find out the meaning. Also, find out the types of asymptotes.

Points to Remember (Take home message)

- ✓ A line **L** is an asymptote to a curve if the distance from a point P of the curve to the line L tends to zero as P tends to infinity along some unbounded part of the curve. We have three kinds of asymptotes: vertical asymptote, horizontal asymptote and oblique asymptote.

Content 2 : Determination of asymptotes

- **Calculations**

✓ Vertical asymptote
✓ Horizontal asymptote
✓ Oblique asymptote

Example 1:

Find the vertical asymptotes of function f(x) = $\dfrac{x^3}{1-x^2}$ and the position of the graph with respect to the vertical asymptotes.

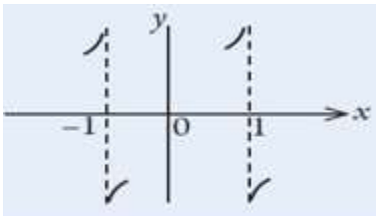
Solution

Vertical asymptotes: x = -1 and x = 1.

To determine the position of the curve with respect to the vertical asymptotes, we consider the sign of f(x) in the vicinity of x = -1 and x = 1.

x	$-\infty$		-1		0		1		$+\infty$
x^3		$-$	$-$	$-$	0	$+$	$+$	$+$	
$1-x^2$		$-$	0	$+$	$+$	$+$	0	$-$	
$f(x)$		$+$	$\|\|$	$-$	0	$+$	$\|\|$	$-$	

$$\lim_{x \to -1^-} f(x) = +\infty \ ; \ \lim_{x \to -1^+} f(x) = -\infty \ ; \ \lim_{x \to 1^-} f(x) = +\infty \ ; \ \lim_{x \to 1^+} f(x) = -\infty \ .$$



Example 2:

Find the equations of vertical and horizontal asymptotes of function f(x) = $\dfrac{x^2+3x}{4-x^2}$

Solution

Vertical asymptotes: x = -2 and x = 2

x	$-\infty \quad -3 \quad -2 \quad 0 \quad 2 \quad -\infty$
x^2+3x	$+ \; + \; 0 \; - \; - \; - \; 0 \; + \; + \; + \; +$
$4-x^2$	$- \; - \; - \; - \; 0 \; + \; + \; + \; 0 \; - \; - \; - \; -$
$f(x)$	$- \; 0 \; + \; \| \; - \; 0 \; + \; \| \; -$

$\lim\limits_{x \to -2^-} f(x) = +\infty$;

$\lim\limits_{x \to -2^+} f(x) = -\infty$;

$\lim\limits_{x \to 2^-} f(x) = +\infty$;

$\lim\limits_{x \to 2^+} f(x) = \infty$.

Horizontal asymptote:

y = -1 ($\lim\limits_{x \to \infty} f(x) = -1$)

Example 3:

Find the asymptotes of the function f(x) = $\frac{x^2-x-2}{x-2}$

Solution

Let y = mx + p be the equation of oblique asymptote:

Then m = $\lim\limits_{x\to\infty} \frac{f(x)}{x}$

$= \lim\limits_{x\to\infty} \frac{x^2-x-2}{x^2+2x} = 1$

P = $\lim\limits_{x\to\infty} [f(x) - mx]$

$= \lim\limits_{x\to\infty} \left(\frac{x^2-x-2}{x^2+2x} - x\right)$

$= \lim\limits_{x\to\infty} \frac{x^2-x-2-x^2-2x}{x+2}$

$= \lim\limits_{x\to\infty} \frac{-3x-2}{x+2} = -3$

Therefore, the oblique asymptote: y = x - 3

Practice:

Lines $x = 2$ and $y = 3$ are asymptotes of function $f(x) = \frac{ax + 5}{bx + 4}$. Find a and b.

Line $x + 3 = 0$ is asymptote of function $f(x) = \frac{3x + 5}{2x + a}$. Find the value of a.

Find the vertical and horizontal asymptotes of the following functions:

(a) $f(x) = \frac{x^2-x-6}{x^2-x-20}$

(b) $g(x) = \frac{x+1}{(x+3)(x+5)}$ (c) $h(x) = \frac{(x+1)^2}{x^2+4x+3}$

Theoretical learning Activity

- ✓ Describe the types of asymptotes?

Practical learning Activity

- ✓ Graph the following and find its asymptotes $y = \frac{x^2-x-2}{x-2}$

Points to Remember (Take home message)

- ✓ An asymptote can be in a negative direction, the curve can approach from any side (such as from above or below for a horizontal asymptote), or may actually cross over (possibly many times), and even move away and back again.
  The important point is that:
  The distance between the curve and the asymptote tends to zero as they head to infinity (or - infinity)

Learning outcome 2.4: Formative Assessment

Practical assessment

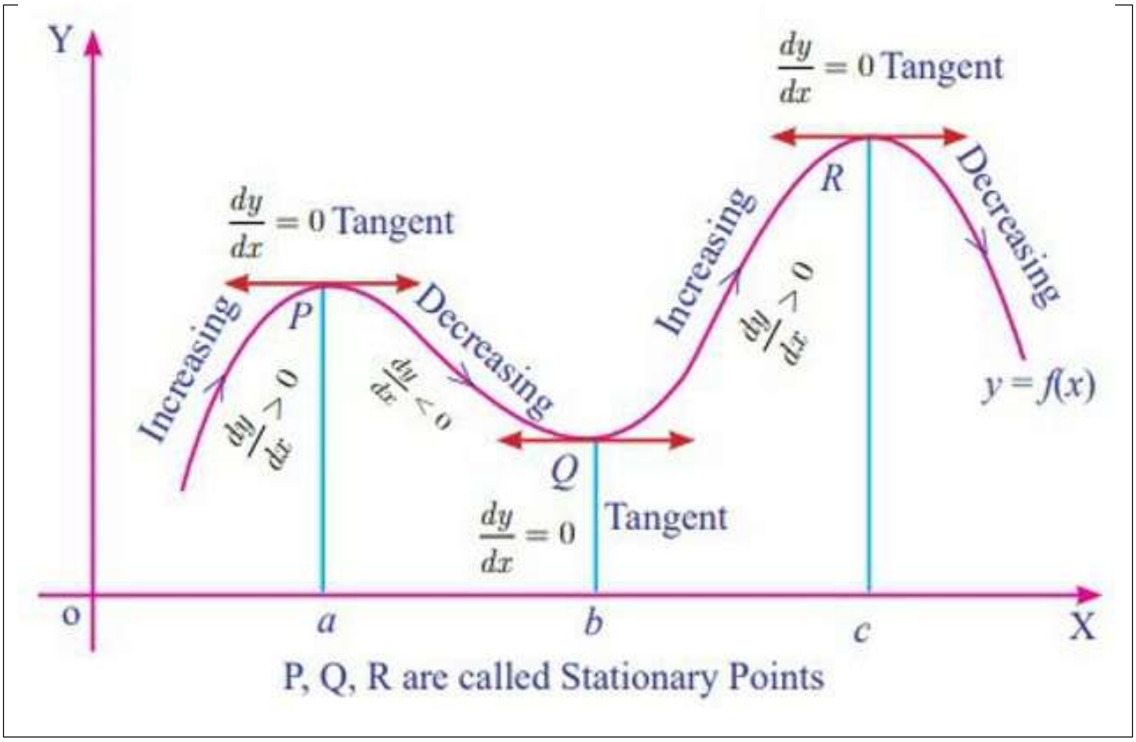
- ✓ Task to be performed:

1. **Find the asymptotes of** $\frac{(x^2-3x)}{(2x-2)}$ **and sketch the graph.**

2. Graph the following and find their asymptotes.

1. $y = \frac{x+2}{x^2+1}$ 2. $y = \frac{x^3-8}{x^2+5x+6}$

Learning Unit 3: Apply fundamentals of differentiation



STRUCTURE OF LEARNING UNIT

Learning outcomes:
3.1. Determine derivative of a function. **3.2. Interpret derivative of a function.** **3.3. Apply derivative.** **3.4. Sketch graph of a function.**

Learning outcome 3.1. Determine derivative of a function.

Duration: 5 hrs
Learning outcome 3.1. Objectives: By the end of the learning outcome, the trainees will be able to: 1. Define correctly the term: "Derivative" of a function as applied in basic mathematical analysis. 2. Describe clearly how to determine the derivatives of different types of functions as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper**	**Geometric instruments (Ruler, T-square)** **Handouts on worked examples**

Advance preparation: . Refer to manual describing fundamentals of differentiation

Content 1: Definition of derivative

- **Definition of derivative**

The **derivative of f(x)** is the function f '(x) given by f '(x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists.

The process of computing a derivative is called **differentiation.**

Further, f is differentiable on interval I if f is differentiable at every point in I.

Theoretical learning Activity

✓ Find out the meaning of the term derivative in mathematics?

Practical learning Activity

✓ What is the derivative of f(x) ?

Points to Remember (Take home message)

✓ The derivative of a function, also known as slope of a function, or derived function or simply the derivative, is defined as f '(x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Content 2. Determination of derivatives

• **Calculation of derivatives**

✓ **Derivative of function at a given point**

The derivative of function $y = f(x)$ at $x = a$ is defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, provided the limit exists.

We say that f is differentiable at $x = a$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}.$$

By calculating the derivative in this way, we say that f is differentiated using definition or **first principle.**

1. Calculate the derivative of f(x) = 3x^2 + 2x -1 at x = 1.

Solution

We have: f ' (1) = $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h \to 0} \frac{[3\,(1+h)^2+2\,(1+h)-1]-4}{h}$$

$$= \lim_{h \to 0} \frac{3h^2+8h}{h}$$

$$= \lim_{h \to 0}(3h+8) = 11 .$$

Find the derivative of f(x) = x^2 – 5x + 3 at an unspecified value of x. Then evaluate f ' (0) , f ' (1)

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) + 3 - x^2 + 5x - 3}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 5)}{h}$$

$$= \lim_{h \to 0} (2x + h - 5) = (2x - 5)$$

f ' (0) = 2 (0) – 5 = -5

f ' (1) = 2 (1) – 5 = -3

Properties:

If $y = f(x)$, then

$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$

The expression $\frac{d}{dx}$ is called a differential operator and tells us that the derivative is with respect to x (that is by considering x as a variable).

(1) For any constant c, $\frac{d}{dx}\ c = 0$.

(2) $\frac{d}{dx}\ x = 1$.

(3) For any integer n > 0, $\frac{d}{dx} x^n = n\, x^{n-1}$. The formula is generalised for any real number r, $\frac{d}{dx}\ (x^r) = r\, x^{r-1}$.

(4) If f(x) and g(x) are differentiable at x and c is any constant, then.

(i) $\frac{d}{dx} [f(x) \pm g(x)] = f\ '(x) \pm g'\ (x)$;

(ii) $\frac{d}{dx}\ [c\, f(x)] = c\, f\ '\ (x)$.

✓ **Derivative of a polynomial function**

If f(x) and g(x) are differentiable at x and c is any constant, then.

(i) $\frac{d}{dx}\ [f(x) \pm g(x)]$ = f ' (x) $\pm$ g ' (x) ;

(ii) $\frac{d}{dx}\ [c\, f(x)] = c\, f\ '\ (x)$.

Example

Find the derivative of f **(**x**)** = **2** + **3**x + **6**x^2 + **21**x^3

Solution

f **'(**x**)** = **0** + **3**+**12**x + **63**x^2

Product rule

Suppose that f and g are differentiable functions at x. Then

$$\frac{d}{dx}[f(x).g(x)] = f'(x).g(x) + f(x).g'(x).$$

Example

Find f ' (x) if f(x) = $(2x^4 - 3x + 5)\left(x^2 - \sqrt{x} + \frac{2}{x}\right)$:

Solution

f ' (x) = $(8x^3 - 3)\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + (2x^4 - 3x + 5)\left(2x - \frac{1}{2\sqrt{x}} - \frac{2}{x^2}\right)$.

✓ **Derivative of a rational function**

Quotient rule

Suppose that f and g are differentiable at x and $g(x) \neq 0$.

Then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$.

Example

Find f ' (x) if f(x) $= \frac{x^2-2}{x^2+1}$.

Solution

$$f'(x) = \frac{(x^2-2)'\,(x^2+1) - (x^2+1)'\,(x^2-2)}{(x^2+1)^2}$$

$$= \frac{2x\,(x^2+1) - 2x\,(x^2-2)}{(x^2+1)^2}$$

$= \frac{6x}{(x^2+1)^2}$.

Exercises

Find the derivative of each function:

1. f (x) = $(x^2 + 3)(x^2 - 3x + 1)$. **2.** f(x) $= \frac{3x-2}{5x+1}$.

3. f(x) $= \frac{(x+1)(x-2)}{x^2 - 5x + 1}$. **4.** f(x) $= \frac{6x - \frac{2}{x}}{x^2 + \sqrt{x}}$.

Trigonometric formulas

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x, \qquad \frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

✓ **Derivative of an irrational function**

Formula

f(x) = $\sqrt[n]{u} \rightarrow f\,'(x) = \frac{u'}{n\sqrt[n]{u^{n-1}}}$

Example

f(x) = $\sqrt{x^2 - 3x} \rightarrow$ f ' = $\frac{2x-3}{2\sqrt{x^2-x3}}$ f(x) = $\sqrt[3]{x^2+1} \rightarrow f'$(x) = $\frac{2x}{3\sqrt[3]{(x^2+1)^2}}$

f(x) = $\sqrt[3]{(x^2-3x)^2} \rightarrow \ f'(x) = \frac{2 \cdot (2x-3) \cdot (x^2-3x)}{3 \cdot \sqrt[3]{(x^2-3x)^4}}$

✓ **Successive derivatives**

Let $y = f(x)$

$$y' = \frac{dy}{dx} : \text{the first derivative of } y$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2} : \text{the second derivative of } y$$

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^3y}{dx^3} : \text{the third order derivative of } y$$

$$y^{(n)} = \frac{d^ny}{dx^n} : \text{the derivative of order n.}$$

Example

Let y = $3x^4 - 2x^2 + 1$.

Compute as many derivatives as possible

Solution

y' = $12x^3 - 4x$; y'' = $36x^2 - 4$; y''' = $72x$; $y^{(4)} = 72$; $y^{(5)} = 0$; $y^{(n)} = 0 \;\; for\; n \geq 5$.

Theoretical learning Activity

- ✓ Discuss on the first principle of derivative?

Practical learning Activity

- ✓ Find the derivative of the following function using the definition of the derivative: g(t) = $\sqrt{t}$

Points to Remember (Take home message)

- ✓ **The derivative of a function f(x), also known as slope of a function, or derived function or simply the derivative, is defined as**

  f '(x) = $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$
- ✓ From the first principles, the derivative functions of: **f(x) = x^2**

Solution

f(x) = x^2

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0}(2x+h) = 2x + 0 = 2x$$

Learning outcome 3.1: Formative Assessment

Practical assessment

- ✓ Task to be performed:

Determine from first principles the derivative of the following functions:

(a) 2 **(b)** $x + x^3$ **(c)** $x^3 + 2x + 3$

Learning outcome 3.2. Interpret derivative of a function.

Duration: 5 hrs

Learning outcome 3.2 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly on Geometric interpretation of a derivative of a function at a point as applied in basic mathematical analysis.
2. Describe properly the kinematical meaning of a derivative as applied in basic mathematical analysis.

Resources

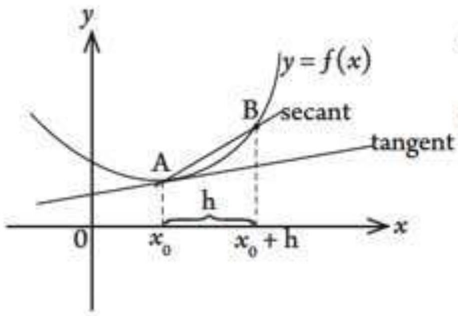
Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper**	**Handouts on worked examples** **Geometric instruments**

Advance preparation:

. Refer to manual describing interpretation of derivative

Content 1 : Geometric interpretation of derivative of a function at a point

• **Geometric interpretation of derivative at a point**



$A(x_0, f(x_0))$

$B(x_0 + h, f(x_0 + h))$

The gradient of the secant line AB is $\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{f(x_0+h)-f(x_0)}{h}$.

As $h \to 0$, B moves on the graph $y = f(x)$ towards A, the secant line approaches the tangent line to the graph $y = f(x)$ at x_0. Thus, the gradient (the slope) of the tangent is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$. Therefore, the equation of the tangent to curve $y = f(x)$ at point $(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

Since, the normal is the perpendicular to the tangent, the equations of the normal to curve $y = f(x)$ at point $(x_0, f(x_0))$ is $y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$.

Examples

Find the equation of the tangent to graph:

(a) $y = x^2 + 1$ at $x = 1$

(b) $y = \frac{2}{x}$ at $x = 2$

Solution

(a) $x = 1$; $y = 1^2 + 1 = 2$

$y' = 2x$; at $x = 1$, $y' = 2$

The equation of the tangent is

$y - 2 = 2(x - 1)$; $y = 2x$.

(b) $x = 2$; $y = 1$

$y' = -\frac{2}{x^2}$; $at\ x = 2,\ y' = -\frac{1}{2}$.

The equation of the tangent is

$y - 1 = -\frac{1}{2}(x-2);\ y = -\frac{1}{2}x + 2$.

Exercises

1. Find the equation of the tangent line to the graph of function $f(x) = 4 - 4x + \frac{2}{x}$

2. Find the equation of the normal to the graph of function

$f(x) = (x^4 - 3x^2 + 2x)(x^3 - 2x + 3)$ at the origin.

Theoretical learning Activity

- ✓ **Discuss on the interpretation of a derivative of a certain function at any given point?**

Practical learning Activity

- ✓ Find from first principles, the slope of the tangent to the following functions at a given value of x: $f(x) = 2x^2 + 3$ at $x = 2$

Points to Remember (Take home message)

- ✓ The slope (gradient) of the tangent to a curve of f(x) is defined as the slope of the curve f(x), and is the instantaneous rate of change in y with respect to x.
  - Finding the slope using the limit method is said to be using first principles.
  - A chord (secant) of curve is a straight line segment which joins any two points on the curve.
  - A tangent is straight line which touches curve at point.
- ✓ The slope of the tangent at the point x = a is defined as the slope of the curve at the point where x= a, and is the instantaneous rate of change in y with respect to x at that point.

Content2: Kinematical meaning of a derivative

• **<u>Kinematical meaning of a derivative</u>**

Velocity

Suppose that the function $f(t)$ gives the position at time t of an object moving along a straight line. That is, $f(t)$ gives the displacement from a fixed reference point, so that $f(t) < 0$ means that the object is located $|f(t)|$ away from the reference point, but in the negative direction. Then, for two times a and b (where $a < b$), $f(b) - f(a)$ gives the signed distance between the positions $f(a)$ and $f(b)$.

The average velocity is given by

$$v = \frac{f(b) - f(a)}{b - a}.$$

If $f(t)$ represents the position of an object with respect to some fixed point at time t as it moves on a straight line, then the instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$ provided the limit exists.

Example

The height of a falling object, t seconds after being dropped from a height of 64 meters is given by $f(t) = 64 - 16t^2$. Find the average velocity between times t = 1 and t = 2, and find the instantaneous velocity at time t = 2.

Solution

Average velocity = $\frac{f(2)-f(1)}{2-1} = \frac{0-48}{1}$ = - 48m/s .

Instantaneous velocity:

$f'(t) = -32t$

$t = 2$, $f'(2) = -32(2) = -64$ m/s .

If s(t), v(t) and a(t) are position, velocity and acceleration at time t respectively, we have $v = \frac{ds}{dt}$ and $a = \frac{d^2s}{dt^2}$ or $a = \frac{dv}{dt}$.

✓ A body moves along the x-axis so that at time t seconds $x(t) = t^3 + 3t^2 - 9t$. Find:
(a) the position and velocity of the body at t = 0. 1, 2
(b) where and when the body comes to rest
(c) the maximum speed of the body in the first 1 second of motion
(d) the maximum velocity of the body in the first 1 second of motion
(e) the total distance travelled by the body in the first 2 seconds of motion.

✓ **Solution**

(a) $x(t) = t^3 + 3t^2 - 9t$; $v(t) = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t + 3)(t - 1)$

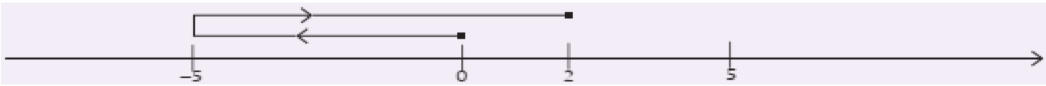
When t = 0, x = 0 and v = –9; when t = 1, x = –5 and v = 0; when t = 2, x = 2 and v = 5 .

At t = 0, the body is at the origin with velocity of $-9ms^{-1}$.

At t = 1, the body is 5m to the left of 0 with velocity of $0ms^{-1}$.

At t = 2, the body is 2m to the right of 0 with velocity of 15 ms^{-1}.

(b) The body is at rest when v = 0. This occurs when t = 1 (t ≥ 0). At this time the body is 5m to the left of the origin.
(c) The velocity is increasing in the interval [0, 1] since v′(t) = 6t + 6 > 0. v(0) = –9 and v(1) = 0. Therefore the maximum speed in the first 1 second is $9ms^{-1}$.
(d) From part (c), the maximum velocity is $0ms^{-1}$.
(e) The following diagram illustrates the position of the body from t = 0 to t = 2.



From the diagram the total distance travelled is 12 m.

Exercises

1. The height of a falling object t seconds after being released is given by, $s(t) = 640 - 20t - 16t^2$. Find the acceleration at time t.

2. Use the position function to find the velocity at time t:

(a) $s(t) = t^2 - \sin 2t$; $t = 0$. (b) $s(t) = \frac{\cos t}{t}$; $t = \pi$.

(c) $s(t) = 4 + 3 \sin t$; $t = \pi$.

<u>Indeterminate forms and Hospital's rule</u>

Suppose that f and g are differentiable on the interval $]a, b[$, except possibly at some fixed point $c \in]a, b[$ and that $g'(x) \neq 0$ on $]a, b[$, except possibly at c.

Suppose further that $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ has the indetermination $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim\limits_{x \to c} \frac{f'x)}{g'(x)} = L$ (or ∞).

Then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'x)}{g'(x)}$.

The conclusion holds if $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ is replaced with any of the limits

$$\lim_{x \to c^+} \frac{f(x)}{g(x)}, \quad \lim_{x \to c^-} \frac{f(x)}{g(x)} \quad \text{or} \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)} .$$

Example

Evaluate : $\lim\limits_{x \to 0} \frac{1-\cos x}{\sin x}$.

Solution

$\lim\limits_{x \to 0} \frac{1-\cos x}{\sin x} = \frac{1-\cos 0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$; indeterminate form.

$\lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\frac{(1-\cos x)'}{(\sin x)'} = \lim_{x\to 0}\frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$.

Remark: Before applying Hospital's rule, make sure you have indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theoretical learning Activity

- ✓ Discuss on what is Kinematical meaning of a derivative?

Practical learning Activity

- ✓ A particle is moving along the x-axis such that its position, x(t) metres to the right of the origin at time t seconds, is given by $x(t) = t^3 - 9t^2 + 24t - 18$. Describe the particle motion during the first five seconds and calculate the distance travelled in that time.

Points to Remember (Take home message)

If the function y = f(x) is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of y with respect to x. Since $f''(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called convexity (bending) of a curve.

If x = t is time and if y = s(t) is displacement function of moving object, then $s'(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity i.e. the second derivative of the displacement function is $s''(t)$ or $s''(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity function, which is, the acceleration function.

Learning outcome 3.2: Formative Assessment

Practical assessment

- ✓ Task to be performed:

- ✓ A body moves along the x-axis so that its position is x(t) metres to the right of the origin at time t seconds.
  a) If $x(t) = t^3 - 3t^2$ explain why the total distance travelled in the first three seconds of motion is not equal to the displacement in that time.
  b) If $x(t) = t^3 - 3t^2 + 3t$ explain why the distance travelled in that first three seconds of motion is now equal to the displacement in that time.

Learning outcome 3.3. Apply derivative.

Duration: 5 hrs

Learning outcome 3.3 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to determine the equation of tangent line at a given point as applied in basic mathematical analysis.

2. Describe correctly how to determine the equation of normal line at a given point as applied in basic mathematical analysis.

3. Demonstrate appropriately the increasing and decreasing intervals for a function as applied in basic mathematical analysis.

4. Describe properly the maximum and minimum points of a function as applied in basic mathematical analysis.

5. Discuss clearly the concavity, inflection point on a graph as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference Books **Internet**	**Didactic materials such as manila paper**	**Hand-out notes** **Geometric instruments**

Advance preparation:

. Refer to manual discussing on application of derivative.

Content 1: Determination of equation of tangent line at a given point

- <u>**Tangent and normal at a point a of a function**</u>

•Determination of equation of normal line at a given point

The equation of the tangent line at x = a can be expressed as:

$$\boxed{f(x) - f(a) = \mathbf{f'(a)}(x - a)}$$

The equation of the normal line at x = a can be expressed as:

$$f(x) - f(a) = \frac{-1}{f'(a)}(x - a)$$

Example 1: Find the equation of the tangent and normal lines of the function $f(x) = \sqrt{2x-1}$ at the point (5, 3).

Solution:

a) Equation of the Tangent Line.

Step 1: Find the slope of the function by solving for its first derivative.	$f(x) = \sqrt{2x-1}$ $f(x) = (2x-1)^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}(2x-1)^{\frac{-1}{2}}(2)$ $f'(x) = \frac{1}{\sqrt{2x-1}}$

Step 2: Knowing $f'(a)$, solve for the slope of the tangent at $a = 5$.	$f'(5) = \frac{1}{\sqrt{2(5)-1}}$ $f'(5) = \frac{1}{3}$
Step 3: Solve for $f(a)$.	$f(5) = \sqrt{2(5)-1}$ $f(5) = 3$
Step 4: Substitute found values into the equation of a tangent line.	$f(x) - f(a) = f'(a)(x - a)$ $f(x) - 3 = \frac{1}{3}(x - 5)$

b) Equation of the Normal Line.

Step 1: Find the slope of the normal line $\frac{-1}{f'(a)}$.	Since $f'(5) = \frac{1}{3}$, then $\frac{-1}{f'(a)} = -3$
Step 2: Given the equation of a tangent line, swap slopes.	$f(x) - f(a) = \frac{-1}{f'(a)}(x - a)$ $f(x) - 3 = -3(x - 5)$

Theoretical learning Activity

- ✓ Discuss on what is a tangent line at normal point?

Practical learning Activity

- ✓ Find the equation of the tangent to f(x) = x^2 + 2 at the point where x = 1.

Points to Remember (Take home message)

- ✓ **Tangent line to a curve of function**
  Consider a curve y = f(x) . If P is the point with x-coordinate a, then the slope of the tangent at this point is f′(a) . The equation of the tangent is by equating slopes $\frac{y - f(a)}{x - a} = f'(a)$ or $y - f(a) = f'(a)(x - a)$
- ✓ **Normal line to a curve of function**
  A normal to a curve is a line which is perpendicular to the tangent at the point of contact. Therefore, if the slope of the tangent at x = a is f′(a) , then the slope of a normal at x = a is $-\frac{1}{f'(a)}$ This comes from the fact that the product of gradients of two perpendicular lines is –1.
- ✓ **Note**: If a tangent touches y = f(x) at (a, b) then it has equation
  $\frac{y - b}{x - a} = f'(a)$ or $y - b = f'(a)(x - a)$
- ✓ Vertical and horizontal lines have equations of the form x = k and y = c respectively, where c and k are constants

CONTENT

Content 2: Increasing and decreasing intervals for a function

A function f is **strictly increasing** on an interval I if for every x_1, x_2 in I, with $x_1<$
$f(x_1)<f(x_2)$, that is $f(x)$ gets larger as x gets larger. For convenience, we place ascenc
arrows; ↗.

A function f is **strictly decreasing** on an interval I if for every x_1, x_2, in I , with $x_1<$
$f(x_1)>f(x_2)$, that is $f(x)$ gets smaller as x gets larger. For convenience, we place descenc
arrows; ↘.

Suppose that f is differentiable on an interval I:

(i) If $f'(x) > 0$ for all $x \in$ I, then f is increasing,

(ii) If $f'(x) < 0$, 0 for all $x \in$ I, then f is decreasing.

Example

Find the interval where *f (x)* = $2x^3 + 9x^2 - 24x - 10$ is:

(a) increasing,

(b) decreasing,

Solution

f′(x) = $6x^2 + 18x - 24$

= $6(x - 1)(x + 4)$

x	$-\infty$		-4		1		$+\infty$
$f'(x)$	+ +	+	0	− − −	0	+ +	+
	↗			↘		↗	

(a) f is increasing for $x \in]-\infty, -4[\cup]1, +\infty[$.

(b) f is decreasing for $x \in]-4,1[$.

Note that the critical numbers (1 and -4) are the only possible locations for local extrema.

Theoretical learning Activity

- ✓ Describe what is an increasing and decreasing function?

Practical learning Activity

- ✓ Given the function $f(x) = 2x^3 + 9x^2 + 12x + 20$. Determine the interval where the graph of the function is increasing and where it is decreasing.

Points to Remember (Take home message)

- ✓ A real function f is **increasing** in or on an interval I if $f(x_1) \leq f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly increasing** if $f(x1) < f(x_2)$ whenever $x_1 < x_2$.
  A real function f is **decreasing** in or on an interval I if $f(x_1) \geq f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Content3 : Maximum and minimum points of a function

- **Maximum and minimum points of a function**

First derivative test

Suppose that f is continuous on the interval $[a, b]$ and $c \in]a, b[$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in]a, c[$ and $f'(x) < 0$ for all $x \in]c, b[$ (that is f' changes the sign from positive to negative) then $f(c)$ is a local maximum.

(ii) If $f'(x) < 0$ for all $x \in]a, c[$ and $f'(x) > 0$ for all $x \in]c, b[$, then $f(c)$ is a local minimum.

(iii) If $f'(x)$ has the same sign on $]a, c[$ and $]c, b[$, then $f(c)$ is not a local extremum.

Example

Find the local extrema of function *f(x)* = $2x^3 - 3x^2 - 12x + 13$, and state the nature of each of them. Find intervals when *f* is increasing, decreasing.

Solution

f'(x) = $6x^2 - 6x - 12$

= $6(x^2 - x - 2)$

= $6(x - 2)(x + 1)$

f'(x) = 0 if and only if x = -1 or x = 2

x	$-\infty$		-1		2		$+\infty$
$f'(x)$	+	+	0 −	−	0 +	+	
		↗		↘		↗	

f is increasing if x ∈ $]-\infty, -1\,[\ \cup\ ]2, +\infty\,[$, *f* is decreasing if x ∈ $]-1, 2[$
maximum (-1, 20) , minimum (2, -7).

Theoretical learning Activity

- ✓ Describe on what is maximum and minimum points of inflection?

Practical learning Activity

- ✓ Find the maximum profit that a company can make, if the profit function is given by P(x) = 4 + 24x – $18x^2$

Points to Remember (Take home message)

- ✓ **Stationary point**

  This is a point on the graph y = f(x) at which f is differentiable and f′(x) = 0. The term is also used for the number c such that f′(c). The corresponding value f(c) is a stationary value. A stationary point c can be classified as one of the following, depending on the behaviour of f in the neighbourhood of c:

  (i) A local maximum, if f′(x) > 0 to the left of c and f′(x) < 0 to the right of c,

  (ii) A local minimum, if f′(x) < 0 to the left of c and f′(c) > 0 to the right of c,

  (iii) Neither local maximum nor minimum, if (i) and (ii) are not satisfied.

  Note: Maximum and minimum values are termed as extreme values.

Content4 : Concavity, inflection point on a graph

- **Concavity, inflection point on a curve**

For a function f that is differentiable on an interval I, the graph of f is:

(i) **Concave up** on I if $f''(x) > 0$ (that is f' is increasing on I): indicated by $\smile$

(ii) **Concave down** on I if $f''(x) < 0$ (that is $f'$is decreasing on I): indicated by $\frown$

Suppose that f is continuous on the interval $]a, b[$ and that the graph changes the concavity at a point $c \in]a, b[$.

Then the point $(c, f(c))$ is called an **inflection point** of f.

Second derivative test

Suppose that f is continuous on the interval $]a, b[$ and $f'(c) = 0$, for some number $c \in]a, b[$,

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum,

(ii) If $f''(c) > 0$, then $f(c)$ is a local maximum.

Example

1. Use the second derivative test to find the nature of the local extrema of function $f(x) = x^4 - 8x^2 + 10$.

Solution

$f'(x) = 4x^3 - 16x$

$= 4x(x-2)(x+2)$

Thus the critical numbers are x = 0; x = 2; x = -2.

We also have : $f''(x) = 12x^2 - 16$

$f''(0) = -16 < 0$

$f''(-2) = 32 > 0$

$f''(2) = 32 > 0$

So by the second derivative test, $f(0)$ is a local maximum and $f(-2)$ and $f(2)$ are local minima.

2. Determine where the graph of $f(x) = x^4 - 6x^2 + 1$ is concave up and concave down, and find the inflection point.

Solution

$f'(x) = 4x^3 - 12x$

$= 4x(x - \sqrt{3})(x + \sqrt{3})$

$f''(x) = 12x^2 - 12$

$= 12(x - 1)(x + 1)$

x	$-\infty$		-1		1		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
		$\smile$	I_1	$\frown$	I_2	$\smile$	

The graph is concave up if $x \in]-\infty, 1[\cup]1, +\infty[$, concave down if $x \in]-1, 1[$. The graph has two inflection points: I_1 $(-1, -4)$ and I_2 $(1, -4)$.

Exercises

1. Determine the intervals where the graph of the given function is concave up and concave down:

(a) $f(x) = x^3 - 3x^2 + 4x - 1$ (b) $f(x) = x^4 - 6x^2 + 2x + 3$

2. Find the local extrema of function:

(a) $y = 3x^4 - 4x^2$. (b) $y = \frac{1}{3}x^3 - 2x^2 + 3x + 1$.

3. Given the function $f(x) = -x^3 + 3x + 4$, find:

(a) The first and second derivatives, study their signs.

(b) The intervals where f is increasing, decreasing.

(c) The local extrema and precise the nature of each of them.

(d) The intervals where the graph of f is concave up, concave down.

(e) The inflection point.

Theoretical learning Activity

- ✓ Discuss on concavity and inflection point of a function**?**

Practical learning Activity

- ✓ Given the function : $y = f(X) = \frac{x^3}{3} + \frac{x^2}{2}$

  (a) State the values of x for which f is increasing

  (b) Find the x-coordinate of each extreme point of f.

(c) State the values of x for which the curve of f is concave upwards.

(d) Find the x-coordinate of each point of inflection.

(e) Sketch the general shape of the graph of f indicating the extreme points and points of inflection.

Points to Remember (Take home message)

- ✓ A curve is said to be concave downwards (or concave) in an interval]a, b[If $f''(x) < 0$ for all $x \in]a,b[$.
- ✓ A curve is said to be concave upwards (or convex)
- ✓ A point of inflection is a point on a graph $y = f(x)$ at which the concavity changes. If f' is continuous at a , then for $y = f(x)$ to have a point of inflection at a it is necessary that $f''(a) = 0$, and so this is the usual method of finding possible points of inflection. in an interval]a, b[if $f''(x) > 0$ for all $x \in]a,b[$.

Learning outcome 3.3: Formative Assessment

Practical assessment

- ✓ Task to be performed:
  1. Find the inflection point of the function defined by: $f(X) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$
  2. State the values of x for which the curve of f is concave upwards

Learning outcome 3.4. Sketch graph of a given function.

Duration: 5 hrs		
Learning outcome 3.4. Objectives: By the end of the learning outcome, the trainees will be able to: 1. Discuss clearly how to establish required parameters as applied in basic mathematical analysis. 2. Describe appropriately how to sketch a graph as applied in basic mathematical analysis.		
Resources		
Equipment	**Tools**	**Materials**

Reference books **Internet** **Scientific calculator**	**Didactic materials such as manila paper** **Geometric instruments** **(Ruler, T-square)**	**Handouts on worked examples**
Advance preparation: . Refer to a manual describing how to sketch a graph of a function.		

Content 1 : Establishing required parameters

Introduction

A rule that defines a function can be given by: an equation connecting the independent variable x and the dependent variable y = ƒ(x) or a graph: the set of all points (x, y) in the xy– plane that satisfy the equation y = ƒ(x).

- **Parameters required**

✓ Variation table

By definition, a variation table is the table which contains the results from the first derivative and second derivative.

✓ Additional points

Additional points are other points which shows where the curve of a function must pass

Theoretical learning Activity

✓ Discuss on the parameters required for graphing a given function

Practical learning Activity

✓ Describe on a variation table and addition points for graphing a function f(x)

Points to Remember (Take home message)

- ✓ **Variation table:** is the table which contains the results from the first and second derivative.
- ✓ **Additional** other points which shows where the curve of a function must pass points

Content 2 : Sketching graph

- **Curve sketching**

x

When trying to draw the graph of **y** = f(**x**), start by gathering information about the graph through the following tests:

1. Domain
2. Limits at the end points of the domain and asymptotes
3. First derivative information
4. Second derivative.
5. Variation table
6. Intercepts
7. Supplementary points
8. Sketch the curve.

✓ Curve sketching of a polynomial function

Example 1

Investigate fully the graph of function if: $\mathbb{R} \rightarrow \mathbb{R}: x \rightarrow f(x) = -x^4 + 5x^2 - 4$:

Solution

1. Domain:

$D_f =]-\infty, +\infty[$

$\forall\, x \in D_{f'} - x \in D_f$ and $f(-x) = -(-x)^4 + 5(-x)^2 - 4$

$= -x^4 + 5x^2 - 4$

$= f(x)$

Therefore, f is an even function.

The graph of y = *f(x)* is symmetrical about the y-axis: It is sufficient to graph the function on $\mathbb{R}^+ = [0, +\infty[$ and complete it on $]-\infty, 0[$ from symmetry.

2. Limits at the end points of the domain

$= \lim\limits_{x\to+\infty}(-x^4 + 5x^2 - 4)$

$= \lim\limits_{x\to+\infty}(-x^4)$

$= -(+\infty)^4$

$= -\infty$

From symmetry, $\lim\limits_{x\to-\infty}(-x^4 + 5x^2 - 4) = -\infty$.

The graph has no asymptote.

3. First derivative information.

f'(x) = $-4x^3 + 10x = -2x(2x^2 - 5)$

f''(x) = 0 , if and only if x = 0 or x = $\frac{\sqrt{10}}{2}$ (on $[0, +\infty[$)

x	0		$\frac{\sqrt{10}}{2}$		$+\infty$
y'	0	$+$	0	$-$	
y	-4	$\nearrow$	$\frac{9}{4}$	$\searrow$	$-\infty$

Function f is increasing for x ∈ $\left]0, \frac{\sqrt{10}}{2}\right[$ and *f* is decreasing for x ∈ $\left]\frac{\sqrt{10}}{2}, +\infty\right[$.

The graph has a maximum at M ($\frac{\sqrt{10}}{2}$, $\frac{9}{4}$) .

4. Second derivative information:

f'' (x) = $-12x^2 + 10$

f'' (x) = 0 if and only if x = $\frac{\sqrt{30}}{6}$

x	0		$\frac{\sqrt{30}}{6}$		$+\infty$
y''		$+$	0	$-$	
y		$\smile$	$-\frac{19}{36}$	$\frown$	

The graph is concave up if x ∈ $\left]0, \frac{\sqrt{30}}{6}\right[$ and concave down if x ∈ $\left]\frac{\sqrt{30}}{6}, +\infty\right[$.

The graph has an inflection point at I_1 ($\frac{\sqrt{30}}{6}$, $-\frac{19}{36}$).

The first and second derivatives information can be summarized in the table below:

x	0	$\frac{\sqrt{30}}{6}$		$\frac{\sqrt{10}}{2}$	$+\infty$
y'	0	+	+	0	−
y''	+	0	−	−	−
y	-4	$-\frac{19}{36}$		$\frac{9}{4}$	$-\infty$

Intercepts of the functions:

The x-intercepts: $\begin{cases} y = -x^4 + 5x^2 - 4 \\ y = 0 \end{cases}$

$\Leftrightarrow \quad \begin{cases} x = 1 \\ y = 0 \end{cases} \quad \begin{cases} x = 2 \\ y = 0 \end{cases}$

The x-intercepts are (1, 0) and (2, 0) on $[0, +\infty[$

The y-intercept

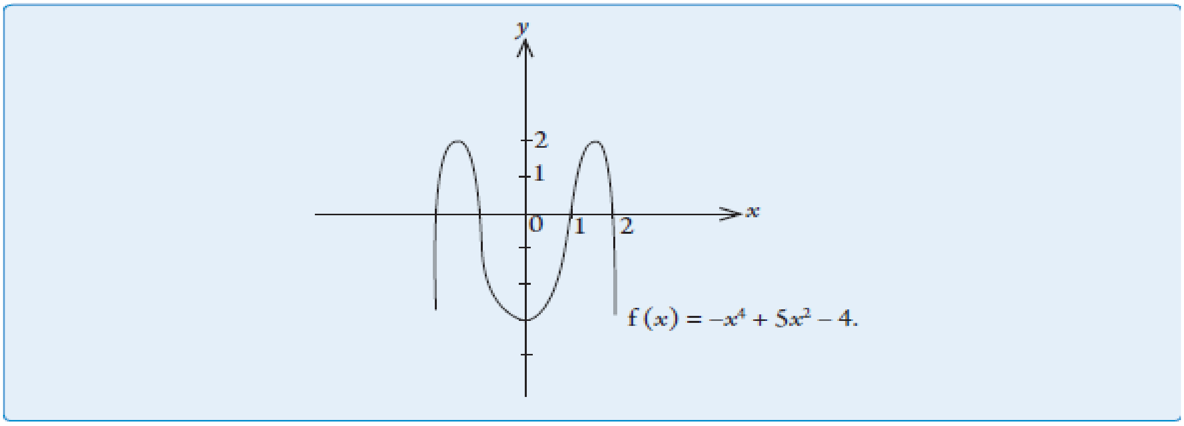
$\begin{cases} x = 0 \\ y = -x^4 + 5x^2 - 4 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = -4 \end{cases}$

The y-intercept is (0,-4) .

Some other points on the graph are;

x	Y	point
0.5	-2.8	(0.5; -2.8)
1.5	2.2	(1.5; 2.2)
2.5	-11.8	(2.5; -11.8)



✓ Curve sketching of a rational function

Example 2

Investigate fully the graph of function if $\mathbb{R} \to \mathbb{R}$: $x \to f(x) = \frac{2x-3}{x+1}$.

Solution

(1) $D_f =]-\infty,-1] \cup [-1,+\infty[$

f is neither even, nor odd, nor periodic.

(2) Limits at the end points of the domain.

$\lim\limits_{x\to+\infty} \frac{2x-3}{x+1} = 2$

$\lim\limits_{x\to-1^-} \frac{2x-3}{x+1} = +\infty$

$\lim\limits_{x\to-1^+} \frac{2x-3}{x+1} = -\infty$

$\lim\limits_{x\to+\infty} \frac{2x-3}{x+1} = 2$

Asymptotes : x = -1 and y = 2 .

(3) First derivative information:

$f' = \frac{5}{(x+1)^2}$; $f'(x) \neq 0$, for all x .

x	$-\infty$		-1		$+\infty$
y'		+	||	+	
y	2	↗	$+\infty$ || $-\infty$	↗	2

f is increasing for for $x \in]-\infty,-1[\cup]-1,+\infty[$.

No maximum, no minimum.

4. Second derivative information:

$f''(x) = -\frac{10}{(x+1)^3}$

x	$-\infty$		-1		$+\infty$
y''		+	||	−	
y		⌣	||	⌢	

The graph is concave up if $x \in]-\infty,-1[$ and concave down if $x \in]-1,+\infty[$.

No inflection point.

The first and second derivatives information can be summarised in the table below:

x	$-\infty$		-1		$+\infty$
y'		+	||	+	
y''		+	||	−	
y	2	↗ ⌣	$+\infty$ || $-\infty$	↗ ⌢	2

5. Intercepts

x-intercept: $\begin{cases} y = \frac{2x-3}{x+1} \\ y = 0 \end{cases}$ $\qquad \begin{cases} y = \frac{3}{2} \\ y = 0 \end{cases}$

The x-intercept is $\left(\frac{3}{2}, 0\right)$

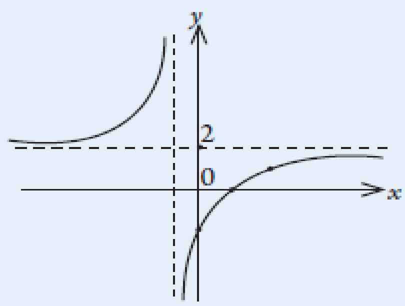
x-intercept: $\begin{cases} y = \frac{2x-3}{x+1} \\ x = 0 \end{cases}$ $\qquad \begin{cases} y = -3 \\ x = 0 \end{cases}$

The y-intercept is (0, -3) .

Some other points on the graph are;

X	Y	Point
-3	$\frac{9}{2}$	$(-3, \frac{9}{2})$
-2	7	(-2, 7)
1	$-\frac{1}{2}$	$(1, -\frac{1}{2})$
2	$\frac{1}{3}$	$(2, \frac{1}{3})$
3	$\frac{3}{4}$	$(3, \frac{3}{4})$
4	1	(4, 1)

The needed graph is;



✓ Curve sketching of an irrational function

Example

Investigate fully the graph of the function:

$f:\mathbb{R} \to \mathbb{R} : x \to f(x) = \sqrt{4x^2 - 2x - 6}$

Solution

1. Domain: $D_f =]-\infty, -1] \cup \left[\frac{3}{2}, +\infty\right[$
2. Limits at the boundaries of the domain and asymptotes:

$$\lim_{x \to -\infty} \sqrt{4x^2 - 2x - 6} = +\infty$$

$$\lim_{x \to -1^-} \sqrt{4x^2 - 2x - 6} = 0$$

$$\lim_{x \to \frac{3}{2}^+} \sqrt{4x^2 - 2x - 6} = 0$$

$$\lim_{x \to +\infty} \sqrt{4x^2 - 2x - 6} = +\infty$$

There are two oblique asymptotes:

y= 2x - $\frac{1}{2}$ $(at\ +\infty)$.

and y = -2x + $\frac{1}{2}$ $(at\ -\infty)$.

3. First derivative information:

$f'(x) = \frac{4x-1}{\sqrt{4x^2-2x-6}}$

$f'(x) = 0 \Leftrightarrow x = \frac{1}{4}$

x	$-\infty$	-1		$\frac{3}{2}$	$+\infty$
$f'(x)$	$-$	$\|\|$		$\|\|$ $+$	
$f(x)$	$+\infty \searrow$	0		0	$\nearrow +\infty$

Function f is increasing if $x > \frac{3}{2}$ and decreasing if $x < -1$.

The graph of f is:

Concave down for $x < -1$ or $x > \frac{3}{2}$.

4. Intercepts

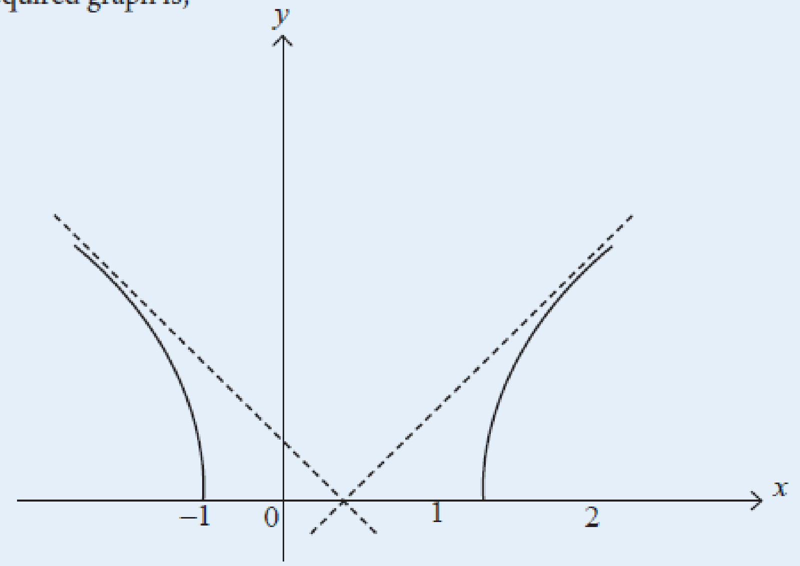
The x – intercepts are (-1,0) and $(\frac{3}{2}, 0)$

No y – intercepts exist.

Some other points are;

x	y	**Point**
-3	6	(-3,6)
-2	$\sqrt{14}$	(2, $\sqrt{14}$)
3	$\sqrt{24}$	(3, $\sqrt{24}$)
4	$\sqrt{50}$	(4, $\sqrt{50}$)

The required graph is;



Exercises

Investigate fully each of the following

1. $f(x) = -x^2$ - 3x - 2
2. $f(x) = \frac{x+2}{x-4}$
3. $f(x) = \frac{x^2-2x-8}{x-1}$

Theoretical learning Activity

- ✓ Discuss on solving and graphing a function?

Practical learning Activity

- ✓ Given the function $f(x) = \frac{x^2}{3} + \frac{x^2}{2}$
  (a) State the values of x for which f is increasing

(b) Find the x-coordinate of each extreme point of f.

(c) State the values of x for which the curve of f is concave upwards.

(d) Find the x-coordinate of each point of inflection.

(e) Sketch the general shape of the graph of f indicating the extreme points and points of inflection.

Points to Remember (Take home message)

- ✓ Steps for solving a function
- ✓ Sketching a graph of a function

Learning outcome 3.4: Formative Assessment

Practical assessment

- ✓ Task to be performed:
- ✓ Given the function. Determine: $f(X) = \frac{X+1}{X-3}$

  a) the domain of f

  b) the x-intercept(s) and y-intercept(s)

  c) all asymptotes to the curve of the function f.

  d) the first derivative f′(x) and the second derivative f′′(x)

  e) the extrema point(s) (local minimum or local maximum)

  f) interval(s) on which f is increasing or decreasing.

  g) inflection point(s)

  h) interval(s) of upward or downward concavity.

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