Analyzing a Single Hand in Blackjack Using Markov Chains

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Overview

Why Blackjack?

Personal

Blackjack, a.k.a, Twenty-One, has been a game I've played for no money as a kid and very expensive hobby later as an adult

Professional

"Blackjack lends itself to Markov Chain Analysis"- *Wakin and Rozell, A Markov Chain Analysis of Blackjack Strategy*

Mathematical

House (Casino) Edge is 1.5% and lowest compared to other games of chance

Origin, Rules & Warm-up

- Originally called "Vingt-et-un", french for "21"
- E.O. Thorpe's 1963 book *Beat The Dealer* featured player advantage strategies based on basic strategy
- Casino's vary rules, deck sizes, shuffling procedures, etc.
- Cards follow point system regardless of suit
- Dealer follows a fixed strategy to "Hit" on all *hand totals* less than 17

- Player can "Hit" any amount of times, "Stand" on any combination of hand totals, follow basic and/or advanced strategies ("Surrender", "Insurance", "Split" and "Double Down")

- Player(s) commit a unit bet B before the dealer deals at the start of each hand

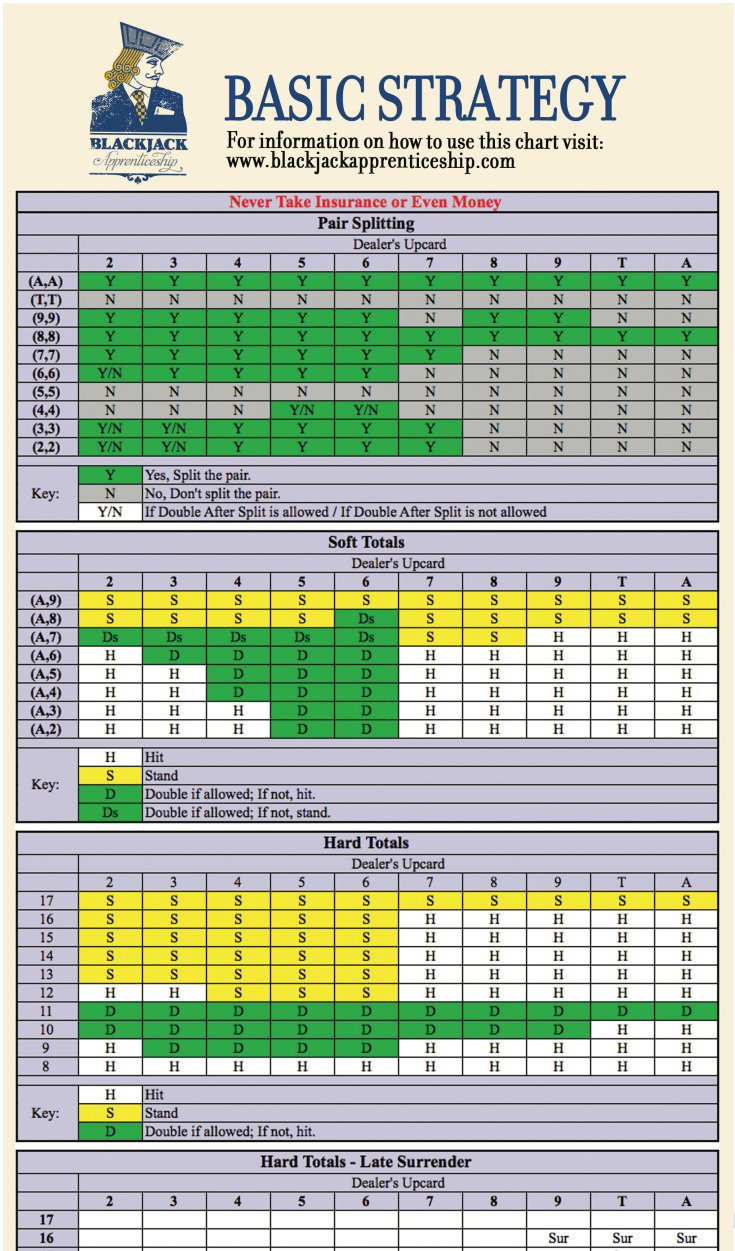
- Dealer Wins (2):

- *Natural*: Acquires a "Blackjack" in first two cards dealt (-B)

- *Player Bust*: Players hand total exceeds 21 (-B)

- Player Wins (2):
- *Natural*: Blackjack in first two cards dealt to the player (1.5*B)
- *Dealer Bust*: Dealers final hand total exceeds 21 (B)
- *Push*: Hand with sum total both equal to the dealers final hand, and still less than or equal to 21 (0)

Objective



BASIC STRATEGY

For information on how to use this chart visit:
www.blackjackapprenticeship.com

Never Take Insurance or Even Money

Pair Splitting

	Dealer's Upcard									
	2	3	4	5	6	7	8	9	T	A
(A,A)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
(T,T)	N	N	N	N	N	N	N	N	N	N
(9,9)	Y	Y	Y	Y	Y	N	Y	Y	N	N
(8,8)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
(7,7)	Y	Y	Y	Y	Y	Y	N	N	N	N
(6,6)	Y/N	Y	Y	Y	Y	N	N	N	N	N
(5,5)	N	N	N	N	N	N	N	N	N	N
(4,4)	N	N	N	Y/N	Y/N	N	N	N	N	N
(3,3)	Y/N	Y/N	Y	Y	Y	Y	N	N	N	N
(2,2)	Y/N	Y/N	Y	Y	Y	Y	N	N	N	N

Key:		
	Y	Yes, Split the pair.
	N	No, Don't split the pair.
	Y/N	If Double After Split is allowed / If Double After Split is not allowed

Soft Totals

	Dealer's Upcard									
	2	3	4	5	6	7	8	9	T	A
(A,9)	S	S	S	S	S	S	S	S	S	S
(A,8)	S	S	S	S	Ds	S	S	S	S	S
(A,7)	Ds	Ds	Ds	Ds	Ds	S	S	H	H	H
(A,6)	H	D	D	D	D	H	H	H	H	H
(A,5)	H	H	D	D	D	H	H	H	H	H
(A,4)	H	H	D	D	D	H	H	H	H	H
(A,3)	H	H	H	D	D	H	H	H	H	H
(A,2)	H	H	H	D	D	H	H	H	H	H

Key:		
	H	Hit
	S	Stand
	D	Double if allowed; If not, hit.
	Ds	Double if allowed; If not, stand.

Hard Totals

	Dealer's Upcard									
	2	3	4	5	6	7	8	9	T	A
17	S	S	S	S	S	S	S	S	S	S
16	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
12	H	H	S	S	S	H	H	H	H	H
11	D	D	D	D	D	D	D	D	D	D
10	D	D	D	D	D	D	D	D	H	H
9	H	D	D	D	D	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H

Key:		
	H	Hit
	S	Stand
	D	Double if allowed; If not, hit.

Hard Totals - Late Surrender

	Dealer's Upcard									
	2	3	4	5	6	7	8	9	T	A
17										
16								Sur	Sur	Sur

Scope

- What is a player's "expected profit" using basic strategy in a single hand?
- Why is it generally -.015*B?
- How do we model Blackjack Strategy as a Markov Chain (MC)?

Key Assumptions, Theorems and Definitions

- Cards dealt from N number of decks are assumed to be independent trials (infinite shuffle) from distribution δ
- Dealer always plays with a fixed strategy, player strategy is always based on dealer's up card
- For simplification, a player cannot "Split" or "Double Down"*

Theorem (A Random Walk Model)

A MC $\{X_n : n > 0\}$ *whose state space is given by the integers* $i = 0, \pm 1, \pm 2...$ *is said to be a random walk for some number* $0 < p < 1$, $P_{i,i+1} = p = 1 - P_{i,i+1}$ *(Ross)*

- Blackjack can be modeled as a Random Walk, Basic Strategy can be determined from Optimal Stopping

- Let ψ_D and D denote the dealer State Space and Transition Matrix
- Let ψ_P and P be the players
- Chains are irreducible (*Wakin and Rozell*)*
- Dealer has absorbing states where integer values correspond with *Standing*, *Blackjack*, or *Bust*
- Player has no absorbing states*

Definition

Optimal Stopping: A stopping rule for a sequence of random variables $X_1, X_2, ...$ maximizes a sequence of reward functions $w_i = w_i(x_1, ..., x_i)$.

Distribution δ , ψ_D and D

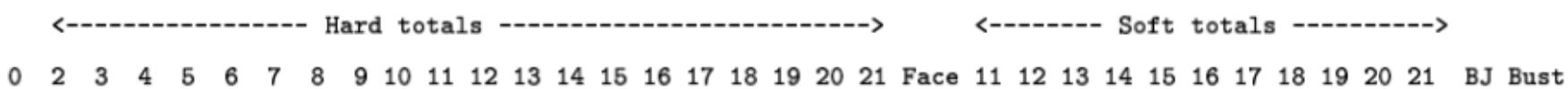
The probability it is an Ace is $\frac{1}{13}$

The probability it is a 2 is $\frac{1}{13}$ ...

The probability it is a 9 is $\frac{1}{13}$

The probability it is a Face card is $\frac{4}{13}$

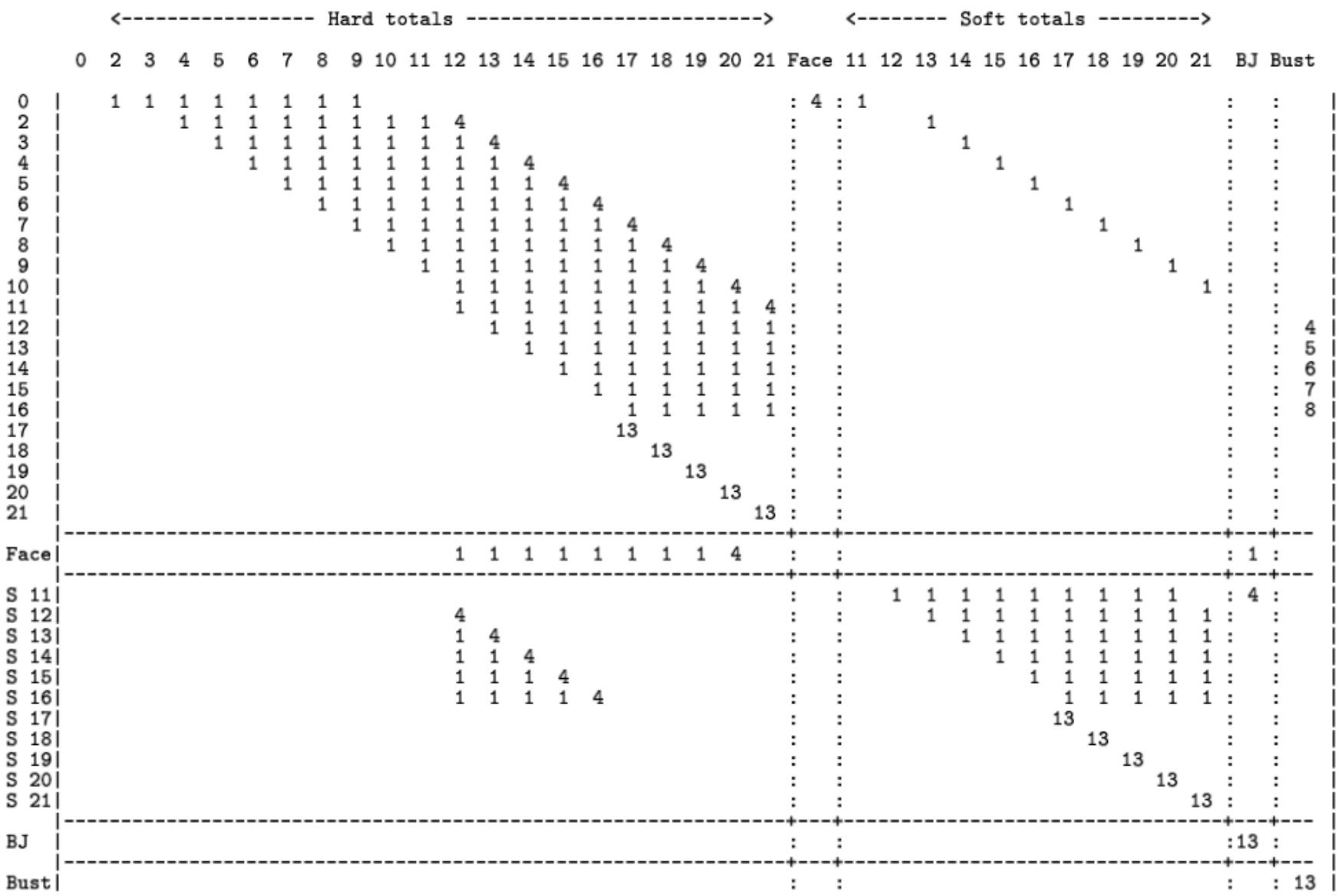
- $\{first_i : i \in \{2, \ldots, 11\}\}$: the dealer holds a single card, valued i. All other states assume the dealer holds more than one card.
- $\{hard_i : i \in \{4, \ldots, 17\}\}$: the dealer holds a hard total of i.
- $\{soft_i : i \in \{12, \ldots, 17\}\}$: the dealer holds a soft total of i.
- $\{stand_i : i \in \{17, \ldots, 21\}\}$: the dealer stands with a total of i.
- bj: the dealer holds a natural.
- $bust$: the dealer busts.



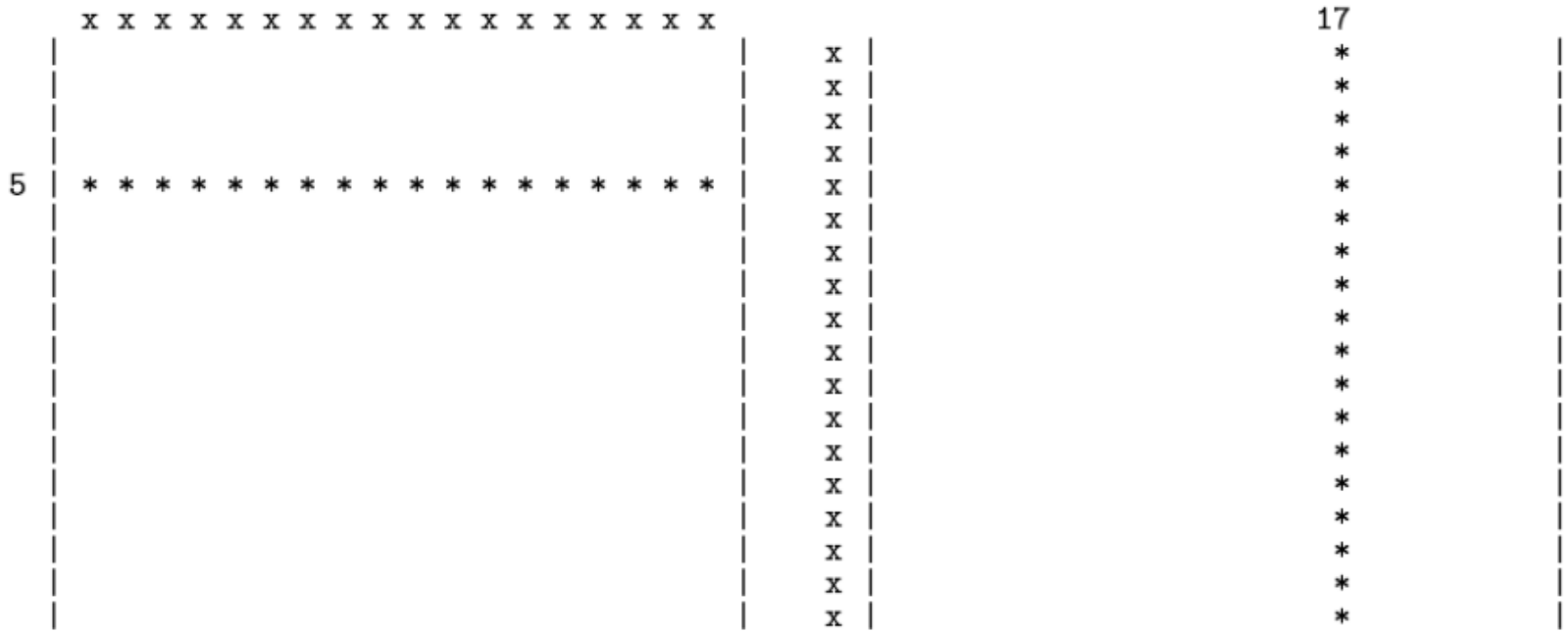
$$d_A = d_2 = d_3 = \cdots = d_9 = 1/13,$$

$$d_{10} = 4/13.$$

<---------- Hard totals ---------->																				<-------- Soft totals ---------->												
0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	Face	11	12	13	14	15	16	17	18	19	20	21	BJ	Bust
0	0	0	0	0	0	1	1	1	1	1	1	1	1	4	0	0	0	0	0	0	: 0	: 0	0	0	0	0	1	0	0	0	0	0	: 0	: 0


<---------- Hard totals ---------->
<-------- Soft totals --------->
0 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 Face 11 12 13 14 15 16 17 18 19 20 21 BJ Bust
0 1 1 1 1 1 1 1 1 : 4 : 1 : :
2 1 1 1 1 1 1 1 1 4 : : 1 : :
3 1 1 1 1 1 1 1 4 : : 1 : :
4 1 1 1 1 1 1 1 1 4 : : 1 : :
5 1 1 1 1 1 1 1 1 4 : : 1 : :
6 1 1 1 1 1 1 1 1 4 : : 1 : :
7 1 1 1 1 1 1 1 1 4 : : 1 : :
8 1 1 1 1 1 1 1 1 4 : : 1 : :
9 1 1 1 1 1 1 1 1 4 : : 1 : :
10 1 1 1 1 1 1 1 1 4 : : 1 : :
11 1 1 1 1 1 1 1 1 4 : : 1 : :
12 1 1 1 1 1 1 1 1 1 : : 1 : : 4
13 1 1 1 1 1 1 1 1 1 : : 1 : : 5
14 1 1 1 1 1 1 1 1 1 : : 1 : : 6
15 1 1 1 1 1 1 1 1 1 : : 1 : : 7
16 1 1 1 1 1 1 1 1 1 : : 1 : : 8
17 13 : : 1 : :
18 13 : : 1 : :
19 13 : : 1 : :
20 13 : : 1 : :
21 13 : : 1 : :
Face 1 1 1 1 1 1 1 1 4 : : 1 : 1 :
S 11 : : 1 1 1 1 1 1 1 1 1 : 4 :
S 12 4 : : 1 1 1 1 1 1 1 1 1 : :
S 13 1 4 : : 1 1 1 1 1 1 1 1 1 : :
S 14 1 1 4 : : 1 1 1 1 1 1 1 1 : :
S 15 1 1 1 4 : : 1 1 1 1 1 1 1 : :
S 16 1 1 1 1 4 : : 1 1 1 1 1 1 : :
S 17 : : 13 : :
S 18 : : 13 : :
S 19 : : 13 : :
S 20 : : 13 : :
S 21 : : 13 : :
BJ : : :13 :
Bust : : : 13

Example



Expected Value Matrix F

		Bust	17	18	19	20	21	Blackjack
Dealer's card	2	0.3536	0.1398	0.1349	0.1297	0.1240	0.1180	0
	3	0.3739	0.1350	0.1305	0.1256	0.1203	0.1147	0
	4	0.3945	0.1305	0.1259	0.1214	0.1165	0.1112	0
	5	0.4164	0.1223	0.1223	0.1177	0.1131	0.1082	0
	6	0.4232	0.1654	0.1063	0.1063	0.1017	0.0972	0
	7	0.2623	0.3686	0.1378	0.0786	0.0786	0.0741	0
	8	0.2447	0.1286	0.3593	0.1286	0.0694	0.0694	0
	9	0.2284	0.1200	0.1200	0.3508	0.1200	0.0608	0
	Face	0.2121	0.1114	0.1114	0.1114	0.3422	0.0345	0.0769
	Ace	0.1153	0.1308	0.1308	0.1308	0.1308	0.0539	0.3077

	Dealer's card									
	2	3	4	5	6	7	8	9	Face	Ace
0	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
2	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
3	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
4	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
5	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
6	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
7	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
8	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
9	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
10	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
11	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
12	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644	-0.6386
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415	-0.3771
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307
Face	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.4989	-0.4617
S 11	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 12	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644	-0.6386
S 18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415	-0.3771
S 19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
S 20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
S 21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307
BJ	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.3846	1.0385
Bust	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000

	Dealer's card									
	2	3	4	5	6	7	8	9	Face	Ace
0	0.0664	0.0938	0.1221	0.1530	0.1827	0.1215	0.0440	-0.0477	-0.1779	-0.3389
2	-0.0759	-0.0498	-0.0221	0.0137	0.0389	-0.0273	-0.1032	-0.1900	-0.3003	-0.4485
3	-0.1005	-0.0689	-0.0363	0.0002	0.0245	-0.0574	-0.1309	-0.2151	-0.3218	-0.4655
4	-0.1149	-0.0826	-0.0494	-0.0124	0.0111	-0.0883	-0.1593	-0.2407	-0.3439	-0.4829
5	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3662	-0.5006
6	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3887	-0.5183
7	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3714	-0.5224
8	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.3071	-0.4441
9	0.0744	0.1013	0.1290	0.1580	0.1960	0.1719	0.0984	-0.0522	-0.2181	-0.3532
10	0.1825	0.2061	0.2305	0.2563	0.2878	0.2569	0.1980	0.1165	-0.0536	-0.2513
11	0.2384	0.2603	0.2830	0.3073	0.3337	0.2921	0.2300	0.1583	0.0334	-0.2087
12	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.4287	-0.5504
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4695	-0.5825
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.5074	-0.6123
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5425	-0.6400
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5752	-0.6657
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644	-0.6386
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415	-0.3771
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307
Face	0.2300	0.2534	0.2775	0.3030	0.3337	0.3011	0.2418	0.1597	-0.0095	-0.1969
S 11	0.5598	0.5768	0.5944	0.6129	0.6396	0.6340	0.5759	0.4940	0.3431	0.1168
S 12	0.0818	0.1035	0.1266	0.1565	0.1860	0.1655	0.0951	0.0001	-0.1415	-0.3219
S 13	0.0466	0.0741	0.1025	0.1334	0.1617	0.1224	0.0541	-0.0377	-0.1737	-0.3474
S 14	0.0224	0.0508	0.0801	0.1119	0.1392	0.0795	0.0133	-0.0752	-0.2057	-0.3727
S 15	-0.0001	0.0292	0.0593	0.0920	0.1182	0.0370	-0.0271	-0.1122	-0.2373	-0.3977
S 16	-0.0210	0.0091	0.0400	0.0734	0.0988	-0.0049	-0.0668	-0.1486	-0.2684	-0.4224
S 17	-0.0005	0.0290	0.0593	0.0912	0.1281	0.0538	-0.0729	-0.1498	-0.2586	-0.4320
S 18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1007	-0.2097	-0.3720
S 19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
S 20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
S 21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307

Player's total	Dealer's card: 2	3	4	5	6	7	8	9	Face	Ace
0	H	H	H	H	H	H	H	H	H	H
2	H	H	H	H	H	H	H	H	H	H
3	H	H	H	H	H	H	H	H	H	H
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	H	H	H	H	H	H	H	H	H
10	H	H	H	H	H	H	H	H	H	H
11	H	H	H	H	H	H	H	H	H	H
12	H	H				H	H	H	H	H
13						H	H	H	H	H
14						H	H	H	H	H
15						H	H	H	H	H
16						H	H	H	H	H
17										
18										
19										
20										
21										
Face	H	H	H	H	H	H	H	H	H	H
S 11	H	H	H	H	H	H	H	H	H	H
S 12	H	H	H	H	H	H	H	H	H	H
S 13	H	H	H	H	H	H	H	H	H	H
S 14	H	H	H	H	H	H	H	H	H	H
S 15	H	H	H	H	H	H	H	H	H	H
S 16	H	H	H	H	H	H	H	H	H	H
S 17	H	H	H	H	H	H	H	H	H	H

Player's total	Dealer's card: 2	3	4	5	6	7	8	9	Face	Ace
0	H	H	H	H	H	H	H	H	H	H
2	H	H	H	H	H	H	H	H	H	H
3	H	H	H	H	H	H	H	H	H	H
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	D	D	D	D	H	H	H	H	H
10	D	D	D	D	D	D	D	D	H	H
11	D	D	D	D	D	D	D	D	H	H
12	H	H				H	H	H	H	H
13						H	H	H	H	H
14						H	H	H	H	H
15						H	H	H	H	H
16						H	H	H	H	H
17										
18										
19										
20										
21										
Face	D	D	D	D	D	D	D	D	H	H
S 11	D	D	D	D	D	D	D	D	D	H
S 12	H	H	H	H	H	H	H	H	H	H
S 13	H	H	H	H	D	H	H	H	H	H
S 14	H	H	H	D	D	H	H	H	H	H
S 15	H	H	H	D	D	H	H	H	H	H
S 16	H	H	D	D	D	H	H	H	H	H
S 17	H	D	D	D	D	H	H	H	H	H

		Dealer's card									
		2	3	4	5	6	7	8	9	Face	Ace
Player's pair	2	S	S	S	S	S	S				
	3	S	S	S	S	S	S				
	4				S	S					
	5										
	6	S	S	S	S	S					
	7	S	S	S	S	S	S				
	8	S	S	S	S	S	S	S	S		
	9	S	S	S	S	S		S	S		
	Face										
	Ace	S	S	S	S	S	S	S	S	S	

```
Following dealer's strategy  :  lose 5.67 cents per $1 bet
Optimal stopping alone       :  lose 2.42 cents per $1 bet
With doubling down           :  lose 1.17 cents per $1 bet
With splitting               :  lose 0.68 cents per $1 bet
```

Conclusion

- A player's expected winnings in a single hand, using Basic Strategy, can be approximated from the Optimal Stopping solution of an MC
- Expected winnings can still vary (lack of simulation)
- The Optimal Stopping of MC's are widely applied in fields such as Gambling, Finance, and Software

Going Forward

- Player MC, Optimal Stopping Method
- Followed through with Simulation
- Used MATLAB to write a subroutine to Blackjack.m

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The End