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Analyzing a Single Hand in Blackjack Using Markov Chains

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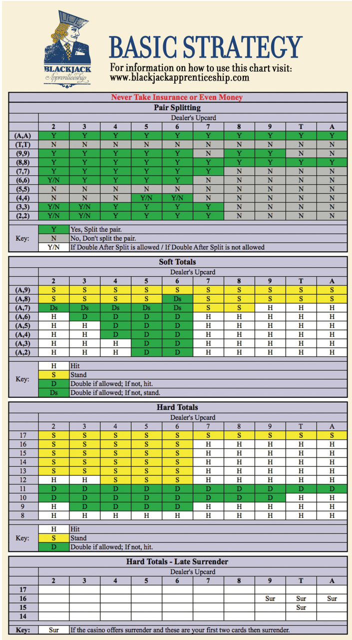
1 INTRODUCTION

Blackjack, a.k.a, Twenty-One, has been a game I've played for no money as a kid and very expensive hobby later as an adult. Originally called "Vingt-et-un", french for "21", Blackjack was brought to America by french colonists. It was not, however, till the 1963 book *Beat The Dealer* by mathematician and professional blackjack player E.O. Thorpe that Blackjack rose in popularity[3]. Thorpe's book featured complex strategies based off of the popular "basic strategy" which further decreased the Casino Edge over a player. More on edge later in this paper.



BASIC STRATEGY

For information on how to use this chart visit: www.blackjackapprenticeship.com

Never Take Insurance or Even Money

Pair Splitting

	Dealer's Upcard									
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**T**	**A**
(A,A)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
(T,T)	N	N	N	N	N	N	N	N	N	N
(9,9)	Y	Y	Y	Y	Y	N	Y	Y	N	N
(8,8)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
(7,7)	Y	Y	Y	Y	Y	Y	N	N	N	N
(6,6)	Y/N	Y	Y	Y	Y	N	N	N	N	N
(5,5)	N	N	N	N	N	N	N	N	N	N
(4,4)	N	N	N	Y/N	Y/N	N	N	N	N	N
(3,3)	Y/N	Y/N	Y	Y	Y	Y	N	N	N	N
(2,2)	Y/N	Y/N	Y	Y	Y	Y	N	N	N	N

Key:	Y	Yes, Split the pair.
	N	No, Don't split the pair.
	Y/N	If Double After Split is allowed / If Double After Split is not allowed

Soft Totals

	Dealer's Upcard									
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**T**	**A**
(A,9)	S	S	S	S	S	S	S	S	S	S
(A,8)	S	S	S	S	Ds	S	S	S	S	S
(A,7)	Ds	Ds	Ds	Ds	Ds	S	S	H	H	H
(A,6)	H	D	D	D	D	H	H	H	H	H
(A,5)	H	H	D	D	D	H	H	H	H	H
(A,4)	H	H	D	D	D	H	H	H	H	H
(A,3)	H	H	H	D	D	H	H	H	H	H
(A,2)	H	H	H	D	D	H	H	H	H	H

Key:	H	Hit
	S	Stand
	D	Double if allowed; If not, hit.
	Ds	Double if allowed; If not, stand.

Hard Totals

	Dealer's Upcard									
	2	3	4	5	6	7	8	9	T	A
17	S	S	S	S	S	S	S	S	S	S
16	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
12	H	H	S	S	S	H	H	H	H	H
11	D	D	D	D	D	D	D	D	D	D
10	D	D	D	D	D	D	D	D	H	H
9	H	D	D	D	D	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H

Key:	H	Hit
	S	Stand
	D	Double if allowed; If not, hit.

Hard Totals - Late Surrender

	Dealer's Upcard									
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**T**	**A**
17										
16								Sur	Sur	Sur
15									Sur	
14										

Key:	Sur	If the casino offers surrender and these are your first two cards then surrender.

Figure 1.1: Basic Strategy Chart

Since his book people have engineered various strategies to decrease the Casino edge during a hand of play. The most used strategy, regarded as the optimal player strategy, is basic strategy which limits the casino edge over a player to 1.5%. What does the statement about edge mean quantitatively? How is basic strategy determined? What is the mathematics behind basic strategy? This paper does its best to answer the above inquiries.

In order to fully comprehend the strategies behind blackjack, one must know the rules of the game and how they affect play. The overall premise of the game is for the player to beat the dealer by obtaining a sum of cards, a hand, less than or equal to 21 and at the same time have a higher hand than the dealer. The game consists of one dealer and up to seven other players, and the number of card decks, denoted by N, which ranges from N equals 1 to 8 held in a shoe. All number cards equal their respective values, and each face card (jack, queen, king) is valued at ten points.

An ace can be worth one point, denoted a hard ace, and eleven, a soft ace, whichever value makes the player's hand better. Blackjack, sometimes referred to as a "Natural" occurs when the first two cards dealt to a player are an ace and a card valued at ten. Blackjacks are paid at 3/2 the original bet B, 1.5 B. The game starts with the dealer dealing two cards to each player from left to right, the dealer last, and in turn, each player can stay (keep the hand they have), or hit (draw another card) as many times as desired. A player busts if the total is over 21, and automatically loses the hand to the dealer.

The rules for this game will be similar to those used in Las Vegas, having the following criteria. A player initially bets a number B for certain hand, and may double down (place double the original bet) upon receiving the first two cards. If the player receives a pair, they may split the hand into two separate hands and place an additional bet,B. If the dealer's up card (every hand the dealer's first card is placed face up) is an ace, the player can make an insurance side bet of that the dealer gets a blackjack, and if this happens, the player gets B returned to him. The rules for the dealer are very specific; they must continue to hit until they have a total of 17 or greater, whether or not the dealer busts.

2 MOTIVATION

My motivation for interest and application of MC's in the context of Blackjack is threefold: 1) to enhance what i know about the game, 2) to apply a probabilistic method to a game I am passionate about, and 3) to learn more about the mathematics behind the determined strategy. Due to subject matter and time constraints, we will limit our analysis to a single hand.

Being an experienced player, I feel the the .015 advantage the house or casino has over the player varies with the combination of hand totals both a player and dealer and strategy used. Again, a .015 edge means that in the longterm we are expected to lose .015 off your unit bet per hand. The value of interest to this analysis is the players advantage per hand which can be computed by structuring one hand of play as an MC.

The scope of the analysis aims to answer three specific questions:

1 What is a player's "expected profit" using basic strategy in a single hand?

2 Why is it generally -.015*B?

3 How do we model Blackjack Strategy as a Markov Chain (MC)?

3 Assumptions

We will make both a few guidelines (and some simplifying assumptions) before we define the Blackjack problem. We will not be considering advanced strategies such as "Surrender", "Insurance", "Split" and "Double Down". Here, both the player and dealer can only "Hit" and "Stand".

Before modeling Blackjack strategy as a MC, we have to make the following assumptions:

a Cards dealt from N number of decks are assumed to be independent trials (infinite shuffle) from distribution δ

b Dealer always plays with a fixed strategy, player strategy is always based on dealer's up card

c For simplification, a player cannot "Split" or "Double Down"*

d Blackjack can be modeled as a Random Walk, Basic Strategy can be determined from Optimal Stopping

4 Problem Formulation

Now we formally define the problem from by applying basic properties of MC's. For any stochastic process, a MC is defined to be:

$$\{X_t : t > 0\}$$

MC's generalize that the value of a state belonging to a stochastic process at step n is independent of all past states when subjected to random walk[4]. I.e., a future state is conditioned on the present state and not the past states. Discrete-time MC's (DTMC's) are excellent for modeling Blackjack[2]. Because we assumed each draw from distribution δ, the result of a player or dealer "Hit", we can now model a single hand of Blackjack.

4.1 Dealer

The game of blackjack can be modeled as a MC. Specifically, we can model the game as a Discrete-time MC (DTMC) which is defined as a family of random variables $X_1, i \in \mathbb{N}$, Explicitly,

$$\{X_n : n > 0\}.$$

Starting with the Dealer, Let ψ_D and D denote the dealer State Space and Transition Matrix. All chains are irreducible and thus enable us to determine long-term distributions[2]. Here, the probability of transitioning from state i to state $i+1$ is defined by the matrix $\underline{\mathbf{D}}$ which is called the one-step transition matrix[1]. $\underline{\mathbf{D}}$ can also be expressed as the Markov property which is the probability distribution of the value of each state that depends on the value of the state preceding it, and is memoryless with respect to how the system arrived in the preceding state[4]:

$$\underline{\mathbf{D}} = (P^{(1)}_{i,j}),$$

where the Markov property is easily recognized when expressed equivalently as the one-step transition matrix[1]. Explicitly,

$$(P^{(1)}_{i,j}) \equiv \left\{X_{n+1} = j \mid X_{n+1} = i\right\}$$

is the for $n > 0$.

In Blackjack, state space can be modeled as all possible hands that a player may have: hard 4 through 21, soft 12 through 21, and bust (22 or more)[3]. Each time indexed by i represents the player drawing another card: X_1 is the two-card hand the player is originally dealt, X_2 is his hand after one hit, and so on. The chain stops at X_N for some finite N when the player either chooses to stand or goes bust. For $m > N$, define X_m to equal X_N. It is clear from the rules of the game that it is irrelevant how a player arrived at his current hand[2].

- $\{first_i : i \in \{2, \ldots, 11\}\}$: the dealer holds a single card, valued i. All other states assume the dealer holds more than one card.
- $\{hard_i : i \in \{4, \ldots, 17\}\}$: the dealer holds a hard total of i.
- $\{soft_i : i \in \{12, \ldots, 17\}\}$: the dealer holds a soft total of i.
- $\{stand_i : i \in \{17, \ldots, 21\}\}$: the dealer stands with a total of i.
- bj: the dealer holds a natural.
- $bust$: the dealer busts.

Figure 4.1: Dealer State Space

Notice that in figure 4.1 that the integer totals corresponding to the state space indicate that we will be doing random walk on all possible combinations of hand totals. The transition matrix for the dealer is a 35 by 35 matrix with absorbing states where integer values correspond with *Standing*, *Blackjack*, or *Bust*. Again one transition corresponds to a draw of a card. The probabilities of drawing specific cards are shown below in figure 4.2.

$$d_A = d_2 = d_3 = \cdots = d_9 = 1/13,$$
$$d_{10} = 4/13.$$

Figure 4.2: Shoe Distribution

Thus the one-step transition matrix is illustrated by figure 4.3. The matrix governing the dealer transition should only transition a maximum of 20 times. For example, playing with an infinite deck, a dealer can draw 20 aces in a row given that she drew 1 ace to begin with. Figure 4.4 shows the relevant probabilities after transitioning a maximum amount of times.

<---------------- Hard totals -------------------------> <-------- Soft totals --------->

	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	Face	11	12	13	14	15	16	17	18	19	20	21	BJ	Bust
0		1	1	1	1	1	1	1	1													4	1												
2				1	1	1	1	1	1	1	1	4													1										
3					1	1	1	1	1	1	1	1	4													1									
4						1	1	1	1	1	1	1	1	4													1								
5							1	1	1	1	1	1	1	1	4													1							
6								1	1	1	1	1	1	1	1	4													1						
7									1	1	1	1	1	1	1	1	4													1					
8										1	1	1	1	1	1	1	1	4													1				
9											1	1	1	1	1	1	1	1	4													1			
10												1	1	1	1	1	1	1	1	4													1		
11												1	1	1	1	1	1	1	1	1	4														
12													1	1	1	1	1	1	1	1	1														4
13														1	1	1	1	1	1	1	1														5
14															1	1	1	1	1	1	1														6
15																1	1	1	1	1	1														7
16																	1	1	1	1	1														8
17																	13																		
18																		13																	
19																			13																
20																				13															
21																					13														
Face												1	1	1	1	1	1	1	1	4														1	
S 11																								1	1	1	1	1	1	1	1	1		4	
S 12												4													1	1	1	1	1	1	1	1	1		
S 13												1	4													1	1	1	1	1	1	1	1		
S 14												1	1	4													1	1	1	1	1	1	1		
S 15												1	1	1	4													1	1	1	1	1	1		
S 16												1	1	1	1	4													1	1	1	1	1		
S 17																													13						
S 18																														13					
S 19																															13				
S 20																																13			
S 21																																	13		
BJ																																		13	
Bust																																			13

Figure 4.3: Dealer Transition Matrix

Dealer's card	Bust	17	18	19	20	21	Blackjack
2	0.3536	0.1398	0.1349	0.1297	0.1240	0.1180	0
3	0.3739	0.1350	0.1305	0.1256	0.1203	0.1147	0
4	0.3945	0.1305	0.1259	0.1214	0.1165	0.1112	0
5	0.4164	0.1223	0.1223	0.1177	0.1131	0.1082	0
6	0.4232	0.1654	0.1063	0.1063	0.1017	0.0972	0
7	0.2623	0.3686	0.1378	0.0786	0.0786	0.0741	0
8	0.2447	0.1286	0.3593	0.1286	0.0694	0.0694	0
9	0.2284	0.1200	0.1200	0.3508	0.1200	0.0608	0
Face	0.2121	0.1114	0.1114	0.1114	0.3422	0.0345	0.0769
Ace	0.1153	0.1308	0.1308	0.1308	0.1308	0.0539	0.3077

Figure 4.4: Relevant probabilities after maximum transitions

5 Analyzing a Single Hand

Now that we have defined ψ_D and D, the players transition matrix and state space, ψ_P and P, can be modeled in a similar way. Note that the player does not have absorbing states and a max amount of hits so we need to use optimal stopping to determine the max amount of times a player should hit. We start by computing the expected value matrix F which defines the expected profit given a dealers up card. This is the final part of the analysis because I was unable to progress past calculating F. However, a paper by Zirbel explains how F and the players MC are used to compute the Basic Strategy chart seen in Figure 1.1. He goes into detail on how Optimal Stopping is used to determine the basic strategy.

	Dealer's card									
	2	3	4	5	6	7	8	9	Face	Ace
0	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
2	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
3	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
4	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
5	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
6	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
7	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
8	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
9	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
10	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
11	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
12	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644	-0.6386
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415	-0.3771
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307
Face	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.4989	-0.4617
S 11	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 12	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5758	-0.7694
S 17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644	-0.6386
S 18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415	-0.3771
S 19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187	-0.1155
S 20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.4350	0.1461
S 21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.8117	0.3307
BJ	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.3846	1.0385
Bust	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000

Figure 5.1: Matrix of Expected values from player standing

The elements in the expected value matrix in figure 5.1 are the expected profits for the player when standing on a total given the corresponding dealer up card.

6 CONCLUSION

This analysis showed how the i) the internal structure of blackjack like other real world processes can be modeled as a MC, and ii) basic strategy can determined from the optimal stopping of dealer and player MC's, iii) the casino edge does change with the combination of a dealer's up card and player's hand total.

By modeling the game of casino blackjack as a Markov chain, we derive the optimal strategy for the game without using simulations. Going further I would like to study optimal stopping and finish the proposed analysis.

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