Maths Problems

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Chapter 1

Introductory Problems

1.1 A Singular Square

1.1.1 Problem

$$\overline{AABB} = m^2$$

Find the four digit perfect square where both the first two digits are the same and the last two digits are the same.

1.1.2 Solution

This can be solved in a long winded way by checking whether each of the possible numbers of the form $\overline{AABB}$ are square. However, I think we would prefer to be a little more clever about it. Firstly, a good start with problems of this type is to write our number algebraically:

$$\begin{aligned}
n &= 1000A + 100A + 10B + 1B \\
n &= 1100A + 11B \\
n &= 11(100A + B)
\end{aligned}$$

So we can see that our number, n, must be divisible by 11. Also, as n is a perfect square, we can also conclude that it must be divisible by $11^2 = 121$. Let:

$$n = m^2$$

As we know that n is divisible by 121 we can see that m must be divisible by 11. At this point we can try all the multiples of 11 from 33 to 99, and we will discover our answer. This has reduced our search size by more than a factor of 10. Continuing mathematically, we know,

$$n = 11(100A + B)$$

As n is divisible by 121, we can see that $100A+B$ is divisible by 11. However,

$$100A + B = 99A + (A + B)$$

So, $A + B$ must be divisible by 11. As A and B are both less than 10, $A + B = 11$. From here we have,

$$\begin{aligned} n &= 11(99A + 11) \\ \frac{n}{121} &= 9A + 1 \end{aligned}$$

We know n is a perfect square, so $9A+1$ must also be a perfect square. The only possibility is $A = 7 \Rightarrow 9A + 1 = 64$. This gives us,

$$n = 121 \times 64 = 7744 = 88^2$$

1.2 Selective Shaking

1.2.1 Problem

At a recent party, everyone shook hands exactly once with everyone else. Halfway through the party Magda arrived and shook hands only with the people she likes.

In all there were 201 handshakes.

How many people does Magda like?

1.2.2 Solution

If everyone in a group of n people shakes hands with everyone else, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ handshakes.

People at party	Handshakes
19	$\binom{19}{2} = 171$
20	$\binom{20}{2} = 190$
21	$\binom{21}{2} = 210$

We can see that 19 is too few, as Magda would have to shake hands with more people than are present at the party. Similarly, 21 is too many, as there would have already been 210 handshakes before she arrived.

Therefore, there are 20 people at the party, and Magda likes $201 - 190 = \boxed{11}$ of them.

1.3 Water into Wine

1.3.1 Problem

One glass contains 100ml of water, and a second glass contains 100ml of wine. 10ml of water is taken from the first glass and put in the second. This mixture is stirred thoroughly, and 10ml is taken and placed back in the first glass.

At the end of this procedure, will the amount of wine in the first glass be greater or smaller than the amount of water in the second glass?

1.3.2 Solution

It is possible to show through a calculation that the amount of water in the second glass is the same as the amount of wine in the first glass. However, there is another way to look at this situation:

At the end of the procedure, the glasses both contain 100ml of liquid. Therefore, any water that is in the second glass must have come from the first glass, and must have been replaced in the first glass by an equal amount of wine.

In fact, the mixing thoroughly step is unnecessary: the result will hold even with substandard mixing practices.

1.4 Trigonometric Intersections

1.4.1 Problem

What is the angle between the graphs of $\tan x$ and $\cos x$ at their points of intersection?

1.4.2 Solution

Solution 1

First, find the points of intersection:

$$\begin{aligned} \cos(x) &= \tan(x) \\ \cos(x) &= \frac{\sin(x)}{\cos(x)} \\ \cos^2(x) &= \sin(x) \\ 1 - \sin^2(x) &= \sin(x) \end{aligned}$$

Solve the quadratic to find:

$$\sin(x) = \frac{1 - \sqrt{5}}{2}$$

Happily, we don't need to find the co-ordinates. Differentiating:

$$\begin{aligned} d/dx(\cos(x)) &= -\sin(x) \\ d/dx(\tan(x)) &= -\sec^2(x) \end{aligned}$$

So, at the intersection, the gradient of $\cos(x)$ is:

$$-\sin(x) = -\frac{1 - \sqrt{5}}{2}$$

Using the standard labels for a right-angled triangle (O,A,H), we know:

$$\begin{aligned} \sin(x) = \frac{1 - \sqrt{5}}{2} &= \frac{O}{H} \\ H(\frac{1 - \sqrt{5}}{2}) &= O \end{aligned} \tag{1.1}$$

We know from the problem:

$$\frac{A}{H} = \frac{O}{A}$$
$$A^2 = OH$$

Substitute equation (1):

$$H^2(\frac{1-\sqrt{5}}{2}) = A^2$$
$$\frac{H^2}{A^2} = \sec^2(x) = \frac{2}{1-\sqrt{5}}$$

So, we see that $\tan(x)$ and $\cos(x)$ are perpendicular at the intersection points, giving us an answer of $\pi/2$ for the angle between them.

Solution 2

If we know that the tangents are perpendicular, we can prove it in a very succinct way:

$$f(x) = \cos(x)$$
$$g(x) = \tan(x)$$
$$f'(x) = -\sin(x)$$
$$g'(x) = \sec^2(x)$$

At the points of intersection, $f(x) = g(x)$

$$\cos(x) = \frac{\sin(x)}{\cos(x)}$$
$$1 = \frac{\sin(x)}{\cos^2(x)}$$

So we have

$$f'(x)g'(x) = -1$$

Which implies the tangents at this point are perpendicular.

1.5 Factorial Factors

1.5.1 Problem

How many numbers are factors of 21! but not of 20!?

1.5.2 Solution

Solution 1

A naive approach would suggest that we find the prime decomposition of 20!

$$20! = 2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19$$

We can see the prime decomposition of 21! will be the same, but with an additional 3 and an additional 7. We are therefore looking for numbers which have either 3^9, 7^3 or both as a factor. Our aim is to count the number of factors of 21! that meet these criteria.

For any number we can count its factors by considering its prime composition. Take, for example:

$$n = 2^x \times 3^y \times 5^z$$

n has $(x+1)(y+1)(z+1)$ factors. This is because there are $x+1$ posibilities for the number of 2s in each factor (as there can be any number from 0 to x of them), and a similar thing is true for 3s and 5s.

Therefore, if we limit the factors of 21! to those which have either 3^9, 7^3 or both as a factor, we find:

Fixed	2	3	5	7	11	13	17	19	Total
3^9	19	1	5	3	2	2	2	2	4560
7^3	19	9	5	1	2	2	2	2	13680
$3^9 \times 7^3$	19	1	5	1	2	2	2	2	1520

Totalling these gives us 19760 numbers which are factors of 21! but not 20!

Solution 2

We could also count the number of factors of 21! and subtract the number of factors of 20!. Using the method from solution 1 we find:

	2	3	5	7	11	13	17	19	Total
21!	19	10	5	4	2	2	2	2	60800
20!	19	9	5	3	2	2	2	2	41040

Which gives us the same answer of 19760.

1.6 More or Less

1.6.1 Problem

Do the majority of positive integers below 10000000 contain the digit 1?

1.6.2 Solution

As is often the case, it's easier to count the integers below 10000000 which do **not** contain the digit 1.

This leaves 9 possibilities for each digit: 0 and 2 to 9.

So, for the integers up to 9999999, we have $9^7 - 1 = 4782968$ integers which do not contain the digit 1. We had to subtract 1 because we had counted 0 among our possibilities.

Therefore, $9999999 - 4782968 = 5217031$ integers contain the digit 1, which is the majority.

1.6.3 Extension

What is the first integer below which the majority of positive integers contain the digit 1?