A Differential Geometry Basics Cheat Sheet

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This is designed to be a quick, yet rigorous introduction/reference to the basic principles found within differential geometry, covering only the bare minimum of topics needed. Largely adapted from Spivak's A Comprehensive Introduction to Differential Geometry, 3rd Edition, *Mendelson's* Introduction to Topology, 3rd Edition and *Michor's* Topics in Differential Geometry.

1 Prerequisite Knowledge

We only assume basic knowledge of continuity and differentiability in the context of functions and topology. Here, ∀ means "for all" or "for each", and ∃ means "there exists".

1.1 Some Basic Topology

Definition 1.1. *Neighborhood.* Let $a \in \mathbb{R}^n, \delta > 0$. The $\delta-$**neighborhood** of a is the set

$$U(a, \delta) = \left\{ x \in \mathbb{R}^n : \|x - a\| < \delta \right\}. \tag{1}$$

Definition 1.2. *Metric Space.* Let X be some non-empty set and d be the mapping/function $d : X \times X \to \mathbb{R}$. The pair (X, d) is a **metric space** if, $\forall x, y, z \in X$,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if (iff) $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

Definition 1.3. *Topological Space.* Let X be a non-empty set and $\mathcal{J}$ be a collection of subsets of X. $(X, \mathcal{J})$ is a **topological space** if

1. $X \in \mathcal{J}$
2. $\varnothing \in \mathcal{J}$, where $\varnothing$ is the empty set
3. If $O_1, O_2, \ldots, O_n \in \mathcal{J}$, then $\bigcap_{i=1}^n O_i \in \mathcal{J}$
4. If $\forall \alpha \in I$, $O_\alpha \in \mathcal{J}$, then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{J}$
   (I is some *indexing set*).

We label X as the **underlying set**, $\mathcal{J}$ as the **topology** on X, and members of $\mathcal{J}$ as **open sets**.

Remark. Topological and metric spaces $(X, \mathcal{J})$ and (X, d), respectively, are sometimes notated simply as X.

Definition 1.4. *Homeomorphism.* Let $(X, \mathcal{J})$ and $(Y, \mathcal{K})$ be topological spaces. $(X, \mathcal{J})$ and $(Y, \mathcal{K})$ are **homeomorphic** if ∃ inverse functions, or **homeomorphisms**, $f : X \to Y$ and $g : Y \to X$ such that f, g are continuous.

1.2 Some Linear Algebra

Definition 1.5. *Vector Space.* A **vector space** V is a set, or space, that is closed under addition and scalar multiplication, i.e. the result of performing these operations on some element(s) of V is itself an element of V. We define elements of a vector space as **vectors**.

Definition 1.6. *Basis.* A **basis** for a vector space V is a set of linearly independent vectors (i.e. none of the basis vectors can be written as linear combinations of the others) that span V (i.e. any element of V can be written as a linear combination of the basis vectors). In other words, the basis vectors define a "coordinate system" for V.

Remark. You can convert vectors written in one basis to another via *change-of-basis* matrices (or functions in the case of infinite-dimensional vectors).

1.3 Some Discrete Math

Definition 1.7. *Equivalence Relation.* A **relation** on some set X is a subset R of $X \times X$. We write xRy to mean $(x, y) \in R$, i.e. *x is related to y.* The relation is an **equivalence relation** if it is

1. **reflexive**: $aRa \quad \forall a \in X$
2. **symmetric**: $aRb \Rightarrow bRa \quad \forall a, b \in X$
3. **transitive**: $aRb,\ bRc \Rightarrow aRc \quad \forall a, b, c \in X$.

Definition 1.8. *Equivalence Class.* Given an equivalence relation R on some set X, the **equivalence class** of some $y \in X$ is the set $\{x \in X : xRy\}$.

2 Manifolds

Definition 2.1. *Manifold.* A **manifold** is a metric space M such that if $x \in M$, ∃ some neighborhood U of x and some $n \in \{0, 1, 2 \ldots\}$ such that U is homeomorphic to $\mathbb{R}^n$. If ∃ such an n that is the same $\forall x \in M$, we say that M is ***n*-dimensional**, which can be notated as M^n.

Remark. Think of a manifold as being a surface that is locally Euclidean.

Definition 2.2. *C^∞-related Homeomorphisms.* Let M be some manifold, and let U, V be open subsets of M. Two homeomorphisms $x : U \to x(U) \subset \mathbb{R}^n$ and $y : V \to y(V) \subset \mathbb{R}^n$ (for some n) are $\boldsymbol{C^\infty}$**-related** if the maps

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V) \tag{2}$$
$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V) \tag{3}$$

are infinitely differentiable, or $\boldsymbol{C^\infty}$.

Definition 2.3. *Atlas.* A family of *mutually* C^∞-related homeomorphisms whose domains cover M (i.e. their union equals M) is an **atlas** of M.

Definition 2.4. *Chart/Coordinate System.* A **chart** or **coordinate system** for some manifold M is a homeomorphism x from some open $U \in M$ to an open subset of $\mathbb{R}^n$, denoted (x, U). A chart of M is a member of some atlas of M.

Remark. Charts/coordinate systems (x, U) create a way of assigning coordinates to points on U, and are sometimes notated simply with x.

Definition 2.5. *Differentiable Manifold.* A **differentiable**, **smooth** or $\boldsymbol{C^\infty}$ manifold is a pair $(M, \mathcal{A})$, where M is some manifold, and $\mathcal{A}$ is some *maximal atlas* for M, i.e. the union of all possible atlases of M.

Definition 2.6. *Differentiable Map Between Manifolds.* Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be differentiable/smooth manifolds (paired explicitly here with their atlases). A function $f : M \to N$ is **differentiable/smooth** if for all coordinate systems (x, U) for M and (y, V) for N, the map

$$y \circ f \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^m, \tag{4}$$

where n and m are the dimensionalities of Euclidean space that U and V are homeomorphic to, respectively. We use $'$ to notate differentiation, i.e. the derivative of some f is f'.

Remark. To summarize, a manifold is essentially a space that can be covered with coordinate charts, which are invertible, continuous mappings to some subset of $\mathbb{R}^n$. If the mapping between any pair of overlapping charts is differentiable, then the manifold itself is differentiable.

We usually don't explicitly notate the atlas of a (differentiable) manifold, i.e. $(M, \mathcal{A})$ vs. simply M.

3 The Tangent Bundle

Definition 3.1. *Tangent Space on* $\mathbb{R}^n$*.* Consider some point $v \in \mathbb{R}^n$, drawn as an arrow with a "reference point" of some $p \in \mathbb{R}^n$; this arrow from p to $p + v$ is denoted (p, v). The set of all such (p, v) is the **tangent space** $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ of $\mathbb{R}^n$. Elements of $T\mathbb{R}^n$ are **tangent vectors** of $\mathbb{R}^n$.

Definition 3.2. *Projection Map on* $\mathbb{R}^n$*.* The **projection map** $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ recovers the first member of any pair $(p, v) \in T\mathbb{R}^n$, and is defined as

$$\pi(p, v) = p. \tag{5}$$

Remark. The projection map can be thought of as mapping any tangent vector to "where it's at"/"where it comes from".

Definition 3.3. *Fiber of* $\mathbb{R}^n$*.* For some $p \in \mathbb{R}^n$, the set $\{(p, v) : v \in \mathbb{R}^n\}$ formed by $\pi^{-1}(p)$ is defined as the **fiber** over p.

Remark. The fiber $\pi^{-1}(p)$ can be pictured as all arrows that start at p. The name "fiber" is also intuitive: imagine a cylindrical hairbrush; $\pi(p, v)$ takes any point v on some bristle on the brush and maps it to the root p of that bristle. Such a fiber also forms a vector space, which will come up later.

Definition 3.4. *Directional Derivative.* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map (for some n, m). The **directional derivative** $Df(p)(v)$ (the derivative of f along $v \in \mathbb{R}^n$, at $p \in \mathbb{R}^n$) is defined by

$$Df(p)(v) = \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h}. \tag{6}$$

$$= \big(D_1 f(p)(v), \ldots, D_i f(p)(v), \ldots, D_n f(p)(v)\big) \tag{7}$$

Definition 3.5. *Pushforward Operator on* $\mathbb{R}^n$*.* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. The **pushforward operator** $f_* : T\mathbb{R}^n \to T\mathbb{R}^m$ is defined by

$$f_* = \bigcup_{p \in \mathbb{R}^n} f_{*p}, \tag{8}$$

where f_{*p} is the mapping that takes some $(p, v) \in T\mathbb{R}^n$ to $Df(p)(v)\big|_{f(p)} \in T\mathbb{R}^m$.

Remark. In other words, the pushforward operator f_* takes a directional derivative operator (tangent vector) from within one tangent space $T\mathbb{R}^n$ to a directional derivative operator within another tangent space $T\mathbb{R}^m$, all dependent on the function $f : \mathbb{R}^n \to \mathbb{R}^m$. As such, $\pi \circ f_* = f \circ \pi$.

Definition 3.6. *Tangent Space on a Manifold.* Let M be some differentiable manifold with some chart $x : U \to \mathbb{R}^n$, and choose some $p \in M$. Suppose that we have two curves $c_1, c_2 : (-\varepsilon, \varepsilon) \to M$ with $c_1(0) = c_2(0) = p$ for some $\varepsilon \in \mathbb{R} > 0$, such that both $x \circ c_1, x \circ c_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ are differentiable.

We say that c_1 and c_2 are **equivalent** at 0 iff. these two derivatives coincide at 0, defining an equivalence relation on the set of all differentiable curves initialized at p. This forms an equivalence class of such curves, known as the **tangent vectors** of M at p. The class of any such curve c is written $c'(0)$. The **tangent space** of M at p, denoted T_pM, is the set of all tangent vectors at p.

Remark. Think of T_p as the (vector) space of all possible directions through which p can be passed through "tangential" to M. Note that T_pM does *not* depend on the choice of coordinate system x.

Remark. Vector bundles, or bundles for short, are usually just notated $\pi : E \to B$, as the inclusion of $\oplus$ and $\odot$ to form the fiber vector spaces is treated as implicit.

Definition 3.7. *Pushforward Operator between Manifolds.* Let $f : M \to N$ be a differentiable/smooth map between smooth manifolds M and N, and let $p \in M$ (so then $f(p) \in N$). Then, the pushforward operator (at p) is given by

$$f_{*p}(c'(0)) = (f \circ c)'(0), \tag{9}$$

where c is a curve along M with $c(0) = p$ (see the previous definition of tangent spaces).

Remark. If some $f : M \to N$ is a mapping between some manifolds M and N, then the pushforward $f_* : M \to N$ is a mapping between their tangent spaces, at $p \in M$ and $f(p) \in N$, respectively.

Definition 3.8. *Vector Bundle.* An n-dimensional **vector bundle** is a five-tuple $\xi = (E, \pi, B, \oplus, \odot)$, where

1. E is the **total space**
2. B is the **base space**
3. $\pi : E \to B$ is a continuous map onto B
4. $\oplus$ and $\odot$ are operators defined to be analogous to addition and scalar multiplication, such that each **fiber** $\pi^{-1}(p)$ $(p \in E)$ is an n-dimensional vector space over $\mathbb{R}$.

Remark. Vector bundles, or bundles for short, are usually just notated $\pi : E \to B$, as the inclusion of $\oplus$ and $\odot$ to form the fiber vector spaces is treated as implicit.

Definition 3.9. *Tangent Bundle.* The **tangent bundle** TM of a manifold M is the (disjoint) union of all tangent spaces on M, i.e.

$$TM = \bigcup_{p \in M} T_pM, \tag{10}$$

and is a type of vector bundle, defined with $\pi : TM \to M$.

Definition 3.10. *Derivation.* For some manifold M, we can define a tangent vector at $p \in M$ to be a linear operator ℓ, called a **derivation** at p, that operates on all C^∞ functions $f, g, \ldots$ such that

$$\ell(fg)(p) = f(p)\ell(g)(p) + g(p)\ell(f)(p). \tag{11}$$

Definition 3.11. *Partial Derivative.* Let $f : M \to \mathbb{R}$ for some smooth manifold M with coordinate chart (x, U). The **partial derivative** operator $\ell = \left.\frac{\partial}{\partial x^i}\right|_p$ is a derivation, defined as

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial f}{\partial x^i}(p) = D_i(f \circ x^{-1})(x(p)). \tag{12}$$

Theorem 3.1. *Partial Derivatives as a Basis for the Tangent Space. Let M^n be some $n-$dimensional smooth manifold, and let (x, U) be some coordinate system around some $p \in M^n$. The set of all linear derivations at p is an $n-$dimensional vector space, that is spanned by the partial derivatives*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p. \tag{13}$$

This set is equivalent to T_pM, and the set of all derivations at all points of M is equivalent to the tangent bundle TM. As such, ***the partial derivatives form a basis for the tangent space****.*

Definition 3.12. *Section.* Let ξ be some vector bundle defined by $\pi : E \to B$. Then a **section** of ξ is a continuous map $s : B \to E$ such that

$$\pi(s(x)) = x \quad \forall x \in B. \tag{14}$$

Remark. In the context of manifolds, as sections pick out points in the total space that lie above the point on the manifold they're mapping from, we can project back down with π to recover the original point. Going back to the hairbrush analogy, if the projection map maps any point along a brush fiber to its base, then the section maps the point on the base to some point on the fiber.