A Differential Geometry Basics Cheat Sheet

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This is designed to be a quick, yet rigorous introduction/reference to the basic principles found within differential geometry, covering only the bare minimum of topics needed.
Largely adapted from Spivak's A Comprehensive Introduction to Differential Geometry, 3rd Edition and *Mendelson's* Introduction to Topology, 3rd Edition

1 Prerequisite Knowledge

We only assume basic knowledge of continuity and differentiability in the context of functions.

1.1 Some Basic Topology

Definition 1.1. *Neighborhood.* Let $a \in \mathbb{R}^n, \delta > 0$. The $\delta-$**neighborhood** of a is the set

$$U(a, \delta) = \left\{ x \in \mathbb{R}^n : \|x - a\| < \delta \right\}. \qquad (1)$$

Definition 1.2. *Metric Space.* Let X be some non-empty set and d be the mapping/function $d : X \times X \to \mathbb{R}$. The pair (X, d) is a **metric space** if, $\forall x, y, z \in X$,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if (iff) $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

Definition 1.3. *Topological Space.* Let X be a non-empty set and $\mathcal{J}$ be a collection of subsets of X. $(X, \mathcal{J})$ is a **topological space** if

1. $X \in \mathcal{J}$
2. $\varnothing \in \mathcal{J}$, where $\varnothing$ is the empty set
3. If $O_1, O_2, \dots, O_n \in \mathcal{J}$, then $\bigcap_{i=1}^{n} O_i \in \mathcal{J}$
4. If $\forall \alpha \in I$, $O_\alpha \in \mathcal{J}$, then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{J}$ (I is some *indexing set*).

We label X as the **underlying set**, $\mathcal{J}$ as the **topology** on X, and members of $\mathcal{J}$ as **open sets**.

Definition 1.4. *Homeomorphism.* Let $(X, \mathcal{J})$ and $(Y, \mathcal{K})$ be topological spaces. $(X, \mathcal{J})$ and $(Y, \mathcal{K})$ are **homeomorphic** if $\exists$ inverse functions, or **homeomorphisms**, $f : X \to Y$ and $g : Y \to X$ such that f, g are continuous.

Definition 1.5. *Manifold.*