A Differential Geometry Basics Cheat Sheet

By Nick Konz

This is designed to be a quick, yet rigorous introduction/reference to the basic principles found within differential geometry, covering only the bare minimum of topics needed. Largely adapted from Spivak's A Comprehensive Introduction to Differential Geometry, 3rd Edition *and Mendelson's* Introduction to Topology, 3rd Edition

1 Prerequisite Knowledge

We only assume basic knowledge of continuity and differentiability in the context of functions and topology. Here, $\forall$ means "for all" or "for each", and $\exists$ means "there exists".

1.1 Some Basic Topology

Definition 1.1. *Neighborhood.* Let $a \in \mathbb{R}^n, \delta > 0$. The $\delta-$**neighborhood** of a is the set

$$U(a,\delta) = \left\{x \in \mathbb{R}^n : \|x-a\| < \delta\right\}. \tag{1}$$

Definition 1.2. *Metric Space.* Let X be some non-empty set and d be the mapping/function $d : X \times X \to \mathbb{R}$. The pair (X,d) is a **metric space** if, $\forall x,y,z \in X$,

1. $d(x,y) \geq 0$
2. $d(x,y) = 0$ if and only if (iff) $x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$ *(triangle inequality).*

Definition 1.3. *Topological Space.* Let X be a non-empty set and $\mathcal{J}$ be a collection of subsets of X. $(X,\mathcal{J})$ is a **topological space** if

1. $X \in \mathcal{J}$
2. $\varnothing \in \mathcal{J}$, where $\varnothing$ is the empty set
3. If $O_1, O_2, \dots, O_n \in \mathcal{J}$, then $\bigcap_{i=1}^n O_i \in \mathcal{J}$
4. If $\forall \alpha \in I$, $O_\alpha \in \mathcal{J}$, then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{J}$ (I is some *indexing set*).

We label X as the **underlying set**, $\mathcal{J}$ as the **topology** on X, and members of $\mathcal{J}$ as **open sets**.

Remark. Topological and metric spaces $(X,\mathcal{J})$ and (X,d), respectively, are sometimes notated simply as X.

Definition 1.4. *Homeomorphism.* Let $(X,\mathcal{J})$ and $(Y,\mathcal{K})$ be topological spaces. $(X,\mathcal{J})$ and $(Y,\mathcal{K})$ are **homeomorphic** if $\exists$ inverse functions, or **homeomorphisms**, $f : X \to Y$ and $g : Y \to X$ such that f, g are continuous.

1.2 Some Linear Algebra

Definition 1.5. *Vector Space.* A **vector space** V is a set, or space, that is closed under addition and scalar multiplication, i.e. the result of performing these operations on some element(s) of V is itself an element of V. We define elements of a vector space as **vectors**.

Definition 1.6. *Basis.* A **basis** for a vector space V is a set of linearly independent vectors (i.e. none of the basis vectors can be written as linear combinations of the others) that span V (i.e. any element of V can be written as a linear combination of the basis vectors). In other words, the basis vectors define a "coordinate system" for V.

Remark. You can convert vectors written in one basis to another via *change-of-basis* matrices (or functions in the case of infinite-dimensional vectors).

2 Manifolds

Definition 2.1. *Manifold.* A **manifold** is a metric space M such that if $x \in M$, $\exists$ some neighborhood U of x and some $n \in \{0,1,2\dots\}$ such that U is homeomorphic to $\mathbb{R}^n$. If $\exists$ such an n that is the same $\forall x \in M$, we say that M is $\boldsymbol{n}$**-dimensional**, which can be notated as M^n.

Remark. Think of a manifold as being a surface that is locally Euclidean.

Definition 2.2. *C^∞-related Homeomorphisms.* Let M be some manifold, and let U, V be open subsets of M. Two homeomorphisms $x : U \to x(U) \subset \mathbb{R}^n$ and $y : V \to y(V) \subset \mathbb{R}^n$ (for some n) are $\boldsymbol{C^\infty}$**-related** if the maps

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V) \tag{2}$$
$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V) \tag{3}$$

are infinitely differentiable, or $\boldsymbol{C^\infty}$.

Definition 2.3. *Atlas.* A family of *mutually* C^∞-related homeomorphisms whose domains cover M (i.e. their union equals M) is an **atlas** of M.

Definition 2.4. *Chart/Coordinate System.* A **chart** or **coordinate system** for some manifold M is a homeomorphism x from some open $U \in M$ to an open subset of $\mathbb{R}^n$, denoted (x,U). A chart of M is a member of some atlas of M.

Remark. Charts/coordinate systems (x,U) create a way of assigning coordinates to points on U, and are sometimes notated simply with x.

Definition 2.5. *Differentiable Manifold.* A **differentiable**, **smooth** or $\boldsymbol{C^\infty}$ manifold is a pair $(M,\mathcal{A})$, where M is some manifold, and $\mathcal{A}$ is some *maximal atlas* for M, i.e. the union of all possible atlases of M.

Remark. To summarize, a manifold is essentially a space that can be covered with coordinate charts, which are invertible, continuous mappings to some subset of $\mathbb{R}^n$. If the mapping between any pair of overlapping charts is differentiable, then the manifold itself is differentiable.

3 The Tangent Bundle

Definition 3.1. *Tangent Space of* $\mathbb{R}^n$. Consider some point $v \in \mathbb{R}^n$, drawn as an arrow with a "reference point" of some $p \in \mathbb{R}^n$; this arrow from p to $p+v$ is denoted (p,v). The set of all such (p,v) is the **tangent space** $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ of $\mathbb{R}^n$. Elements of $T\mathbb{R}^n$ are **tangent vectors** of $\mathbb{R}^n$.

Definition 3.2. *Projection Map on $\mathbb{R}^n$.* The **projection map** $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ recovers the first member of any pair $(p, v) \in T\mathbb{R}^n$, and is defined as

$$\pi(p, v) = p. \tag{4}$$

Remark. The projection map can be thought of as mapping any tangent vector to "where it's at"/"where it comes from".

Definition 3.3. *Fiber of $\mathbb{R}^n$.* For some $p \in \mathbb{R}^n$, the set $\{(p, v) : v \in \mathbb{R}^n\}$ formed by $\pi^{-1}(p)$ is defined as the **fiber** over p.

Remark. The fiber $\pi^{-1}(p)$ can be pictured as all arrows that start at p. The name "fiber" is also intuitive: imagine a cylindrical hairbrush; $\pi(p, v)$ takes any point v on some bristle on the brush and maps it to the root p of that bristle. Such a fiber also forms a vector space, which will come up later.

Definition 3.4. *Directional Derivative* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map (for some n, m). The **directional derivative** $Df(p)(v)$ (the derivative of f along $v \in \mathbb{R}^n$, at $p \in \mathbb{R}^n$) is defined by

$$Df(p)(v) = \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h}. \tag{5}$$

Definition 3.5. *Pushforward Operator for $\mathbb{R}^n$.* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map.. The **pushforward operator** $f_* : T\mathbb{R}^n \to T\mathbb{R}^m$ is defined by

$$f_* = \bigcup_{p \in \mathbb{R}^n} f_{*p}, \tag{6}$$

where f_{*p} is the mapping that takes some $(p, v) \in T\mathbb{R}^n$ to $Df(p)(v)\big|_{f(p)} \in T\mathbb{R}^m$.

Remark. In other words, the pushforward operator f_* takes a directional derivative operator (tangent vector) from within one tangent space $T\mathbb{R}^n$ to a directional derivative operator within another tangent space $T\mathbb{R}^m$, all dependent on the function $f : \mathbb{R}^n \to \mathbb{R}^m$. As such, $\pi \circ f_* = f \circ \pi$.

Definition 3.6. *Tangent Space on a Manifold.*

Definition 3.7. *Vector Bundle.* An n-dimensional **vector bundle** is a five-tuple $\xi = (E, \pi, B, \oplus, \odot)$, where

1. E is the **total space**
2. B is the **base space**
3. $\pi : E \to B$ is a continuous map onto B
4. $\oplus$ and $\odot$ are operators defined to be analogous to addition and scalar multiplication, such that each *fiber* $\pi^{-1}(p)$ $(p \in E)$ is an n-dimensional vector space over $\mathbb{R}$.

Remark. Vector bundles, or bundles for short, are usually just notated $\pi : E \to B$, as the inclusion of $\oplus$ and $\odot$ to form the fiber vector spaces is treated as implicit.

Definition 3.8. *Tangent Bundle.* The **tangent bundle** TM of a manifold M is the (disjoint) union of all tangent spaces on M, i.e.

$$TM = \bigcup_{p \in M} T_p M, \tag{7}$$

and is a type of vector bundle, defined with $\pi : TM \to M$