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Author(s): Robert Wilson

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THE THEORY OF SYNDICATES[1]

BY ROBERT WILSON[2]

A syndicate is defined to be a group of individuals who must make a common decision under uncertainty that will result in a payoff to be shared jointly among them. In this paper, the normative criterion of Pareto optimality is employed to determine the construction, and conditions for existence, of a group utility function and a consensus of the members' probability assessments.

1. INTRODUCTION

WE SHALL DEFINE a *syndicate* to be a group of individual decision makers who must make a common decision under uncertainty, and who, as a result, will receive jointly a payoff to be shared among them. Our concern is to analyze the decision process of a syndicate when the members have diverse risk tolerances and/or diverse probability assessments of the uncertain events affecting the payoff. Of particular interest is the possiblity of constructing a surrogate "group utility function" and a surrogate "group probability assessment." Such constructions potentially have a role in the theory of finance; e.g., for determining the forms of organizational charters and financial instruments, as well as the modes of delegating the group decision process to professional managers. The present treatment, however, is confined to tractable features embodying only a small measure of the complexity of practical situations. Of comparable importance are the ramifications for welfare theory; in particular, we shall be able to specify conditions under which Pareto optimal behavior by the group satisfies the Savage axioms [15] for consistent decision making under uncertainty, and to isolate the inconsistent characteristics in the contrary case.

Arrow's original treatise [1] has been the source of most of the work on group decision theory. Marschak [13], Radner [17], and Bower [6] have considered the case of a team, in which there is a joint utility function for the members. Harsanyi [9] and Theil [16] have considered the criterion that the group decisions satisfy the Von Neumann-Morgenstern axioms, and others. Madansky [12] has imposed the "external Bayes axiom" in the case of a common utility function but differing probability assessments among the members. Christenson [7] has constructed an axiomatic system for the case of an investment banking syndicate that is a special case of the present study, except for certain institutional factors. Borch [3, 4, 5]

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has examined the special case of reinsurance treaties among insurance companies, but his results are quite general and provide the starting point for our work. Eisenberg and Gale's [**8**] results on parimutuel betting systems have some similarities to our results (Theorem 8 below). Luce and Raiffa [**10**] provide a survey of the field through about 1957.

For the most part, the numerous approaches to the group decision problem have reflected a variety of formulations, and of normative criteria for a solution. As yet, there is no evident agreement on these basic considerations. The results in this paper are mainly an extension of Borch's work [**5**] : the distinguishing features are omission of any stronger criterion than Pareto optimality, and detailed analysis of the sharing rule.

2. FORMULATION

The syndicate is faced with a decision under uncertainty. Denote the uncertain event affecting the payoff by the random variable $\tilde{\xi}$ and represent the decision by α. For convenience, the range of the random variable will be taken to be the real line. Let the income accruing to the syndicate from a decision α when the even ξ occurs be given by a payoff function $p(\xi, \alpha)$, which is known to every member.

The individual members of the syndicate will be indexed by $i=1, \ldots, I$. When referring to the syndicate as a whole we shall use the index $i=0$. Presuming that each member adheres to the Savage axioms for consistent decision making under uncertainty [**15**], we can suppose that for each member i there exists a utility function $U_i(x_i)$ for income x_i received from the syndicate and a probability distribution function $F_i(\xi)$ for the uncertain event $\tilde{\xi}$. The expectation operator with respect to the probability measure $F_i(\xi)$ on the variable $\tilde{\xi}$ will be denoted by E_i: sufficient measurability conditions for the expectation operator to be well defined will always be taken for granted. It will be convenient to assume that each of the functions U_i, F_i is twice differentiable. The marginal functions will be denoted by small letters; viz., $u_i(x_i) = U_i'(x_i)$ is the marginal utility function and $f_i(\xi) = F_i'(\xi)$ is the probability density function. Each of the marginal functions is uniformly nonnegative.

Attention will be confined solely to the special case in which each of the members is strictly risk averse; viz., for each member i the utility function U_i is strictly concave, or equivalently, $u_i' = U_i''$ is uniformly negative. We can, therefore, work as well with the *risk tolerance* functions defined by

$$\rho_i(x_i) = -u_i(x_i)/u_i'(x_i) = -U_i'(x_i)/U_i''(x_i)\,.$$

Note that each risk tolerance function is measured in units of income, independently of the irrelevant choices of origin and scale of the utility function from which it is derived, and by assumption it is continuous and uniformly positive. Moreover, each risk tolerance function completely determines the corresponding utility function by the relations $u_i = \exp(-\int 1/\rho_i)$, $U_i = \int \exp(-\int 1/\rho_i)$, since the undetermined constants of integration can generate at most an irrelevant (positive) linear transformation of U_i. Pratt [**14**] has introduced the reciprocal *risk aversion* functions $-U_i''/U_i'$, described as "twice the risk premium per unit of variance for infinitesimal risks"; consequently, we can interpret the risk tolerance $\rho_i(x_i)$ as half the tolerable variance per unit of compensating risk premium for infinitesimal risks when the income of member i is x_i. For example, in the special case of a utility function $U(x)$ with constant risk tolerance ρ, we have $U(x) = -\rho e^{-x/\rho}$; and the certainty equivalent q of a gamble $\tilde{z}$ which is normally distributed with mean $E(\tilde{z})$ and variance $V(\tilde{z})$ is defined by $U(x+q) = EU(x+\tilde{z})$, from which we obtain $q = E(\tilde{z}) - V(\tilde{z})/2\rho$ in accordance with the above interpretation of the risk tolerance ρ. In the general case, Pratt shows that the interpretation holds if the concept of an infinitesimal risk, or a gamble with infinitesimal variance, is employed: in this context, risk tolerance must be considered as a differential, or local, property [**14**].

We shall also have occasion to use the *conjugate utility* functions defined by

$$\omega_i(\mu_i) = \operatorname*{Sup.}_{x_i} [U_i(x_i) - \mu_i x_i] .$$

It is well known [**11**] that each conjugate utility function is convex over the domain for which it is finite (which is a convex set), and that the inverse transform is

$$U_i(x_i) = \operatorname*{Inf.}_{\mu_i} [\omega_i(\mu_i) + \mu_i x_i] .$$

If the supremum is an attained maximum at x_i, then the infimum is an attained minimum at μ_i with $\mu_i = U_i'(x_i)$ and $x_i = -\omega_i'(\mu_i)$. Economists will recognize the conjugate utility $\omega_i(\mu_i)$ as the economic rent or "consumer surplus" to member i, measured in utility units, when a price μ_i in utility units per income unit is charged for income. The concept of rent plays an inherent role in the present analysis due to the use (Theorem 1 below) of an implicit price mechanism generated from Lagrange multipliers to allocate income among the members of the syndicate. Conjugate functions are of central importance in the duality theory of mathematical programming, to which the analysis in Section 4 below is closely related; in an earlier paper, I have shown that conjugate utility functions can be used constructively in the theory of general equilibrium [**20**].

Finally, we shall also have need to use the *dispersion* functions defined by $\varphi_i(\xi) = -f_i'(\xi)/f_i(\xi)$ wherever $f_i(\xi) > 0$. A dispersion function completely determines the density function from which it is derived by the inverse relation $f_i = \exp(-\int \varphi_i)$, where the constant of integration is to be determined to make the total probability equal to unity. Wherever $f_i(\xi) = 0$ we shall take $\varphi_i(\xi) = 0$. The choice of the name "dispersion" function here is arbitrary, and only in the case of a Normal density function does it appear justified (cf. Table II below). Dispersion functions are introduced in the present analysis because they play a symmetric role in relation to the risk tolerance functions (which are of fundamental importance). Also, they are independent of scale factors and enjoy additive properties which considerably simplify our derivations and results. For example, Bayes' Rule is ordinarily expressed in the multiplicative form

$$f(\xi|\theta) = l(\xi|\theta)f(\xi)/\int l(\xi|\theta)f(\xi)\,\mathrm{d}\xi\,,$$

where the likelihood function $l(\xi|\theta)$ is the conditional probability of a sample observation θ given the event ξ, but in terms of dispersion functions it takes the additive form

$$\varphi(\xi|\theta) = \psi(\xi|\theta) + \varphi(\xi)\,,$$

where $\varphi = -f'/f$ and $\psi = -l'/l$. In other words, the notion of a dispersion function is introduced mainly for analytical convenience and no clearcut interpretation is offered or implied.

TABLE I

Type	Domain Restriction	$U_i(x)$	$\omega_i(\mu)$	$u_i(x)$	$\rho_i(x)$
Exponential		$-c_i e^{-x/c_i}$	$-c_i\mu[1-\log\mu]$	e^{-x/c_i}	c_i
Power*	$a_i x > a_i b_i$	$k_i(x-b_i)^{d_i}$	$d_i^{-1}(\mu/a_i)^{1-a_i} - \mu b_i$	$a_i(x-b_i)^{d_i-1}$	$a_i(x-b_i)$
Logarithmic	$x > b_i$	$\log(x-b_i)$	$-[1+\mu b_i + \log\mu]$	$1/(x-b_i)$	$x-b_i$
Quadratic	$x < b_i$	$b_i x - \frac{1}{2}x^2$	$\frac{1}{2}(\mu-b_i)^2$	$b_i - x$	$b_i - x$

* $d_i \neq 0,\ d_i \neq 1,\ k_i = [d_i(1-d_i)]^{-1},\ a_i = (1-d_i)^{-1}$.

The following popular forms of utility functions will be used in examples (Table I). In each case the risk tolerance is linear: exponential utilities yield the special case of constant risk tolerance. In the case of a logarithmic utility, the parameter b_i can be considered a subsistence level, since the risk tolerance vanishes as the income approaches that level. Similarly, in the quadratic case the parameter b_i can be considered a goal level. The logarithmic case corresponds to the singular case $d_i = 0$ of the power utility, and the quadratic is the special case $d_i = 2$.

Two examples of density functions are given in Table II.

TABLE II

Type	Domain Restriction	$f_i(\xi)$	$\varphi_i(\xi)$
Normal	—	$(2\pi v_i)^{-\frac{1}{2}} \exp[(\xi - m_i)^2/2v_i]$	$(\xi - m_i)/v_i$
Gamma	$\xi > 0$	$t_i(t_i\xi)^{r_i} \exp[-t_i\xi]/(r_i!)$	$t_i - r_i/\xi$

3. THE SHARING RULE

In a syndicate, a common decision α is chosen and then, depending upon the outcome ξ of the random variable $\tilde{\xi}$, the resulting payoff $x = p(\xi, \alpha)$ is shared among the members. The rule by which the payoff is shared is an essential feature of the group's decision process, since to every member the merits of a decision depend upon the share he receives from the payoff.

A sharing rule is a set of functions $S = \{s_i(x, \xi) | i = 1, \ldots, I\}$ such that $\Sigma_i s_i(x, \xi) = x$ for all (x, ξ). When we wish to take account of the dependence of the sharing rule on the decision α we shall use the notation $S(\alpha) = \{s_i(x, \xi|\alpha)\}$. When the sharing functions $s_i(x, \xi)$ are independent of the outcome ξ, we shall usually denote them as $s_i^0(x)$.

Our criterion for choosing a sharing rule is that it must be Pareto optimal. A sharing rule is Pareto optimal if there is no alternative sharing rule which would increase the expected utility of some member(s) without decreasing the expected utility of any other member. This criterion presumes, at least, that the syndicate is loosely organized, cooperative, and static, rather than cohesive (a "team") or dominated by longer run considerations, so that there is no overriding goal beyond the immediate welfare of the members. Ordinarily there will exist a variety of Pareto optimal sharing rules, but we will make no further specification of the particular one to be chosen from among these.

Pareto optimality is a necessary, but not a sufficient, condition for determination of the sharing rule. It is well known, nevertheless, that, given Pareto optimality, a necessary and sufficient condition for determination of the sharing rule is an assignment of weights to the members, the reasoning being as follows. The locus of the member's expected utilities obtainable from a decision α by choice of the sharing rule $S(\alpha)$ is a convex set, and therefore to every Pareto optimal sharing rule there corresponds a tangent hyperplane to this set. Thus, corresponding to every Pareto optimal sharing rule $S(\alpha)$ for a given decision α there exists a set of nonnegative weights $\{\lambda_i(\alpha) | i = 1, \ldots, I\}$ such that $S(\alpha) = \{s_i(x, \xi|\alpha)\}$ is obtained as a solution to the following Variational Problem.

VARIATIONAL PROBLEM: By choice of the functions $\{s_i(x, \xi|\alpha)\}$, maximize $\Sigma_i \lambda_i(\alpha) E_i U_i(s_i(p(\xi, \alpha), \xi|\alpha))$ subject to $\Sigma_i s_i(x, \xi|\alpha) = x$ for all $(x, \xi|\alpha)$.

The differential conditions for a solution of the Variational Problem provide the following slight generalization of a result due to Borch [**3**, **5**].

THEOREM 1: *A necessary and sufficient condition for Pareto optimality of the sharing rule* $S(\alpha) = \{s_i(x, \xi|\alpha)\}$ *is that there exist nonnegative weights* $\lambda(\alpha) = \{\lambda_i(\alpha)\}$ *and a function* $\mu_0(x, \xi|\alpha)$ *such that*

$$(1) \qquad \sum_i s_i(x, \xi|\alpha) = x$$

for all $(x, \xi|\alpha)$, *and*

(2) $\lambda_i(\alpha)u_i(s_i(x, \xi|\alpha))f_i(\xi) = \mu_0(x, \xi|\alpha)$

for each i *and almost every* ξ *for which* $\lambda_i(\alpha)f_i(\xi) > 0$.

This result can be derived as an application of a theorem of Zahl [**21**]. One can construe $\mu_0(x, \xi|\alpha)$ as a Lagrange multiplier attached to the constraint (1). Then (2) is obtained by pointwise maximization with respect to s_i and (1) is obtained by pointwise minimization with respect to μ_0 inside the integral of the Lagrangean function,

$$(3) \qquad \int_{-\infty}^{\infty} \{\sum_i [\lambda_i(\alpha)U_i(s_i(p(\xi,\alpha),\ \xi|\alpha))f_i(\xi) - s_i(p(\xi,\alpha),\ \xi|\alpha)\mu_0(p(\xi,\alpha),\ \xi|\alpha)] + p(\xi,\alpha)\mu_0(p(\xi,\alpha),\xi|\alpha)\}d\xi\,,$$

formed from the objective and constraints of the Variational Problem.

It should be clear that the sharing rule does not depend upon the payoff function $p(\xi, \alpha)$, and that it depends upon the decision α and the event ξ only through $\{\lambda_i(\alpha)\}$ and $\{f_i(\xi)\}$, respectively. In particular, it is evident from (2) that if there is an agreed upon probability assessment, $f_i(\xi) = f_0(\xi)$ for every member i, then (2) can be replaced by

(4) $\lambda_i(\alpha)u_i(s_i^0(x|\alpha)) = u_0(x|\alpha)$

for some function u_0 (viz., $u_0 = \mu_0/f_0$ where $f_0 > 0$).

We shall always assume that $\lambda_i(\alpha) > 0$ for all i and α. Using (1) and (2) to define $s_i(x, \xi|\alpha)$ and $\mu_0(x, \xi|\alpha)$ for every ξ (rather than "almost every ξ" as in Theorem 1), it follows from our assumptions in Section 2 that μ_0 and s_i are differentiable in (x, ξ) over every interval of ξ for which $f_i(\xi) > 0$. In what follows we shall treat them as differentiable everywhere, and when there is no possibility of confusion, operate as though $f_i(\xi) > 0$. Indeed, in the sequel we shall not persist in enforcing rigor whenever our amply sufficient assumptions in Section 2 can speak for themselves.

It is instructive to sketch an heuristic proof of Theorem 1. Omitting mention of the decision variable α, let $\mu_{ik} = u_i(s_i(x_k, \xi_k))f_i(\xi_k)$, and suppose that for outcome (x_1, ξ_1), $\lambda_{11}\mu_{11} = \lambda_{21}\mu_{21}$, and that for outcome (x_2, ξ_2), $\lambda_{12}\mu_{12} = \lambda_{22}\mu_{22}$. Then member 1 can give up a small amount ε for $k = 1$ resulting in an expected loss of $\varepsilon\mu_{11}$ while benefiting member 2 by $\varepsilon\mu_{21} = \varepsilon(\lambda_{11}/\lambda_{21})\mu_{11}$. In return, member 2 can give up an amount $\delta = \varepsilon\lambda_{11}\mu_{11}/(\lambda_{21}\mu_{22})$ for $k = 2$, incurring a loss exactly offsetting his gain for $k = 1$, while benefiting member 1 by $\varepsilon(\lambda_{11}\lambda_{22}/\lambda_{12}\lambda_{21})\mu_{11}$. In order that member 1 not be benefited by this exchange, which is a condition for Pareto optimality, we require that $\lambda_{11}/\lambda_{12} = \lambda_{21}/\lambda_{22}$, as, in effect, was concluded in Theorem 1.

Examples of sharing rules in a two member syndicate are given in Table III for the special case of an agreed upon probability assessment. Examples of sharing rules for the general case of divergent probability assessments are given in Table IV in the instance of exponential utilities. The sharing rules in these examples are linear in the payoff x, and linear in the outcome ξ if the distribution "shape" parameters are identical ($v_1 = v_2$ or $r_1 = r_2$). It is worth observing in Table III that the Lagrange multiplier function $u_0(x)$ has the same form as the members' marginal

TABLE III

U_i	$s_1^0(x)$	$u_0(x)$
Exponential*	$\left(\sum_i c_i^{-1}\right)^{-1} \log(\lambda_1/\lambda_2) + xc_1/c_0$	$\left(\prod_i \lambda_i^{c_i/c_0}\right) e^{-x/c_0}$
Power**	$\left(\sum_i \lambda_i^{-a}\right)^{-1} (\lambda_1^{-a} b_1 - \lambda_2^{-a} b_2) + x\lambda_1^a/\lambda_0^a$	$\lambda_0 a(x - b_0)^{d-1}$

* $c_0 = \sum_i c_i$.

** In the power case it is here assumed that $d_i = d = 1 - 1/a$ for all i, and $\lambda_0^a = \Sigma_i \lambda_i^a$, $b_0 = \Sigma_i b_i$.

TABLE IV

f_i	$s_1(x, \xi) - s_1^0(x)$	$\mu_0(x, \xi)/u_0(x)$
Normal	$\left(\sum_i c_i^{-1}\right)^{-1} \frac{1}{2}[-(\xi - m_1)^2/v_1 + (\xi - m_2)^2/v_2 - \log(v_1/v_2)]$	$(2\pi)^{(I-1)/2} \prod_i f_i(\xi)^{c_i/c_0}$
Gamma	$\left(\sum_i c_i^{-1}\right)^{-1} \left[-(t_1 - t_2)\xi + (r_1 - r_2) \log \xi + \log\left(\frac{t_1^{r_1+1}/r_1!}{t_2^{r_2+1}/r_2!}\right)\right]$	$\prod_i f_i(\xi)^{c_i/c_0}$

utility functions. We have also noted in Table IV, without writing out the quantities in full, that $\mu_0(x, \xi)/u_0(x)$ takes the form of a likelihood function composed from the members' probability assessments. We shall see later to what extent these features are general.

It is often appealing to construe the sharing rule as the sum of a *side payment* $s_i^0(0)$, a *dividend* $s_i^0(x) - s_i^0(0)$, and a *side bet* $\beta_i(x, \xi) = s_i(x, \xi) - s_i^0(x)$. For example, in the special case of exponential utilities, Normal probability assessments, and agreement upon the variance ($v_1 = v_2 = v$) we have

$$\beta_1(x, \xi) = \left(\sum_i c_i^{-1}\right)^{-1} \left[\frac{m_1 - m_2}{v^{\frac{1}{2}}}\right] \left[\frac{\xi - (m_1 + m_2)/2}{v^{\frac{1}{2}}}\right],$$

which has an evident interpretation in terms of side bets. We shall have reason later to be particularly interested in the members' *stock* functions, $\sigma_i(x, \xi) = \partial s_i(x, \xi)/\partial x$, which, for the examples presented, are always constants. A member's stock function measures his incremental sharing proportion of the syndicate's income x, in the same fashion that a shareholder's common stock in a corporation measures his sharing proportion; in the more general case considered here, however, the sharing proportion need not be constant with respect either to the income x or the event ξ yielding the payoff. The properties of the stock functions are examined in detail in Sections 5 and 6.

4. THE EVALUATION MEASURE

For every member i of the syndicate there exists an *evaluation measure* $M_i(x_i, \xi) = U_i(x_i)f_i(\xi)$ with the property that, if he must choose an act α_i yielding an uncertain payoff $x_i = p_i(\xi, \alpha_i)$ depending upon the outcome ξ of the random variable $\tilde{\xi}$, then his preferred act maximizes $\int_{-\infty}^{\infty} M_i(p_i(\xi, \alpha_i), \xi)d\xi$. Of course, whenever a person has a complete preference ordering among acts there exists some preference function which his preferred choice maximizes; since each member of the syndicate adheres to the Savage axioms, however, the preference function takes this special form.

Our immediate task is to construct the preference function for the syndicate and determine under what conditions there exists an evaluation measure for the group decision process. (The term "evaluation measure" is employed here to distinguish the objective of the maximization from the ordinal utility functions introduced before Edgeworth's work by Gossen and Jevons [**18**], which were also separable and additive if one were to construe income x_i for distinct events ξ as different goods. More recently, Arrow [**2**] has used such ordinal utilities, though not necessarily separable, in an analysis of risk markets and then demonstrated the stronger results obtainable with the Von Neumann-Morgenstern cardinal utilities [**19**], such as U_i. Here we are particularly concerned with the separable parts $U_i f_i$, and the term evaluation measure is coined for this purpose.)

We assume that, besides the sharing rule, the decision chosen by the syndicate must be Pareto optimal. From this assumption it follows readily that the decision α must maximize the Lagrangean function (3). Hence, we have in total the following characterization of maximizing behavior in the group process (omitting the arguments of functions)

$$\text{(5)} \qquad \operatorname*{Sup.}_{\alpha} \int_{-\infty}^{\infty} \operatorname*{Inf.}_{\mu_0} \left\{ p\mu_0 + \sum_i \operatorname*{Sup.}_{s_i} [\lambda_i U_i f_i - \mu_0 s_i] \right\} d\xi \,.$$

Employing the conjugate utility functions to eliminate the supremum with respect to s_i yields the reduced form

$$\text{(6)} \qquad \operatorname*{Sup.}_{\alpha} \int_{-\infty}^{\infty} \operatorname*{Inf.}_{\mu_0} \left\{ \sum_i \lambda_i f_i \omega_i(\mu_0/\lambda_i f_i) + p\mu_0 \right\} d\xi \,.$$

Defining,

$$\text{(7)} \qquad M_0(x, \xi, \alpha) = \operatorname*{Inf.}_{\mu_0} \left\{ \sum_i \lambda_i(\alpha) f_i(\xi) \omega_i(\mu_0/\lambda_i(\alpha) f_i(\xi)) + x\mu_0 \right\},$$

it follows that the preferred decision α maximizes $\int_{-\infty}^{\infty} M_0(p(\xi, \alpha), \xi, \alpha)d\xi$. Hence M_0

is an evaluation measure if and only if, as defined in (7), it is independent of the decision variable α. This proves the following theorem.

THEOREM 2: *If the weights $\{\lambda_i(\alpha)\}$ are independent of the decision α, then there exists an evaluation measure for the syndicate.*

The hypothesis of Theorem 2 primarily specifies a kind of independence or stability to the effect that the members are motivated only by income and unaffected by the decision itself. The hypothesis also rules out situations, however, in which the set of feasible weights depends crucially upon the decision. Additionally, it excludes some of the proposed arbitration schemes for group decision making [**10**], such as the Nash solution, and perhaps most important, it rules out randomization among decisions.

We shall always assume, henceforth, that an evaluation measure exists for the syndicate (viz., $\lambda_i(\alpha)=\lambda_i$ for each i, and $M_0(x,\xi,\alpha)=M_0(x,\xi)$), independently of the decision variable α. Also, for convenience we shall confine the subsequent analysis to the syndicate's *differential evaluation measure* (DEM), $\partial M_0(x,\xi)/\partial x = \mu_0(x,\xi)$, which is precisely the Lagrange multiplier specified in Theorem 1.

It is well known [**11**] that as defined in (7), the evaluation measure M_0 is a concave function in the payoff variable x. Hence, besides being nonnegative (*cf.* (2)), μ_0 is a nonincreasing function of x. From (2) it follows that if the weights are independent of the decision α then so is the sharing rule S.

5. SURROGATE FUNCTIONS

The differential evaluation measure for an individual member takes the special form $\mu_i(x_i,\xi)=u_i(x_i)f_i(\xi)$; i.e., it is compounded of a marginal utility depending only on the payoff and a probability density for the event. When the syndicate's DEM also takes this special form, $\mu_0(x,\xi)=u_0(x)f_0(\xi)$, we shall say that there exist a *surrogate marginal utility function* $u_0(x)$ and a *surrogate probability assessment* or *consensus* $f_0(\xi)$. The behavioral significance of surrogate functions for a syndicate can be characterized by the observation that Savage's fourth postulate [**15**, p. 31] is equivalent to the existence of surrogate functions.

THEOREM 3: *The choice between two events on which a syndicate will stake a prize does not depend on the prize itself if and only if surrogate functions exist.*

It is routine, and an interesting exercise, to verify that all of Savage's postulates except the fourth are satisfied by a syndicate (in effect, they amount to the existence of an evaluation measure); hence, if the fourth postulate is satisfied then surrogate functions exist. The converse proposition is trivial. Theorem 3 states that, in the absence of surrogate functions, the syndicate's decisions will not evidence consistent probability judgments independently of the consequences.

Our goal now is to identify necessary and sufficient conditions for the existence of surrogate functions. In what follows, whenever a function depends upon (x, ξ) we shall denote differentiation with respect to x by an asterisk (*) and differentiation with respect to ξ by a prime ($'$).

The function $\rho_0(x,\xi) = -\mu_0(x,\xi)/\mu_0^*(x,\xi)$ can be called the *syndicate risk tolerance*, for which we obtain the following simple characterization.

THEOREM 4:

$$\rho_0(x,\xi) = \sum_i \rho_i(s_i(x,\xi)). \tag{8}$$

That is, the syndicate risk tolerance is the sum of the members' risk tolerances. In terms of Pratt's formulation [**14**], this means that a compensating risk premium for an infinitesimal risk is distributed among the members in proportion to the variance each undertakes to absorb. This result is a direct corollary of Theorem 1. Differentiating in (2) yields (omitting the arguments of functions, as we shall do often), $\lambda_i u_i' s_i^* f_i = \mu_0^*$, which with (2) implies $s_i^* = \rho_i/\rho_0$. Since $\Sigma_i s_i^* = 1$, the conclusion (8) follows. In the process we have obtained a characterization of the members' stocks, $\sigma_i(x,\xi) = s_i^*(x,\xi)$:

THEOREM 5:

$$\sigma_i(x,\xi) = \rho_i(s_i(x,\xi))/\rho_0(x,\xi)\,. \tag{9}$$

Thus, a member's incremental sharing proportion is given by his proportion of the syndicate risk tolerance.

The syndicate risk tolerance will be called *determinate* when it is independent of the outcome ξ, and similarly for the members' stocks. It is obvious that, if surrogate functions exist, the syndicate risk tolerance is determinate and $\rho_0(x) = -u_0(x)/u_0'(x)$ defines the surrogate utility function; moreover, the converse is true.

THEOREM 6: *Surrogate functions exist if and only if the syndicate risk tolerance is determinate.*

Let $\mu_0(x,\xi) = u_0(x) f_0(x,\xi)$ where $\rho_0(x) = -u_0(x)/u_0'(x)$. From the relation $\lambda_i u_i' s_i^* f_i = \mu_0^*$ established above, we obtain $\mu_0 s_i^* = -\rho_i \mu_0^*$ and therefore $\mu_0 = -\rho_0 \mu_0^*$. This means, however, that $\mu_0 = [u_0/u_0'][u_0' f_0 + u_0 f_0^*]$, which implies that $f_0^* = 0$ as required. This result shows that the crux of the matter is contained in the attitude toward risk.

We shall say that the sharing rule is *linear* if the stock of every member is independent of the payoff; viz., $\sigma_i(x,\xi) = \sigma_i(\xi)$.

THEOREM 7: *Surrogate functions exist if and only if the sharing rule is linear or there is agreement on the probability assessment.*

If there is agreement on the probability assessment, then the syndicate risk tolerance is determinate, as we can see from (4). In the contrary case, suppose first that surrogate functions exist, so that differentiation in (2) yields $\lambda_i u_i' s_i' f_i + \lambda_i u_i f_i' = u_0 f_0'$. Combining this with (2) to obtain the dispersion functions gives $s_i' = \rho_i \varphi_0 - \rho_i \varphi_i$. But $\Sigma_i s_i' = 0$, so we obtain

$$\varphi_0(\xi) = \sum_i \sigma_i(x, \xi) \varphi_i(\xi) , \tag{10}$$

which, in the absence of agreement on the probability assessment, requires that the sharing rules be linear, $\sigma_i(x, \xi) = \sigma_i(\xi)$, as stated in the theorem. To prove the converse, repeat the derivation of (10) to obtain

$$-\mu_0'(x, \xi)/\mu_0(x, \xi) = \sum_i \sigma_i(x, \xi) \varphi_i(\xi) ,$$

which in the case of a linear sharing rule implies that $\mu_0'(x, \xi)/\mu_0(x, \xi)$ is independent of x and therefore μ_0 has the form $\mu_0(x, \xi) = u_0(x) f_0(\xi)$ as required.

In the process we have obtained a characterization of the surrogate probability assessment,

THEOREM 8: *When a surrogate probability assessment exists, its dispersion function is given by* (10), *independently of the payoff* x.

Also, combining Theorems 6 and 7 shows that surrogate functions exist if and only if each member's risk tolerance is compounded in the form $\rho_i(s_i(x, \xi)) = \rho_0(x)\sigma_i(\xi)$, unless there is agreement on the probability assessment.

A further characterization can be obtained from a concept treated by Pratt [**14**] in terms of "decreasing risk aversion." When the members' utility functions possess third derivatives we shall call $\rho_i'(x_i)$ the member's *cautiousness*.

THEOREM 9: *Surrogate functions exist if and only if the syndicate and the members have identical cautiousness, or there is agreement on the probability assessment.*

This theorem follows from the equivalence of the following relations: $\rho_i(s_i(x, \xi)) = \rho_0(x, \xi)\sigma_i(\xi)$, $\rho_i' s_i^* = \rho_0^* \sigma_i$, and $\rho_i' = \rho_0^*$. Thus, for example, there cannot possibly exist surrogate functions for a syndicate with both a cautious member ($\rho_i' \geqslant 0$ uniformly) and an uncautious member ($\rho_i' < 0$ uniformly), unless they agree on the probability assessment.

In substance, the foregoing results suggest that surrogate functions exist if there is a modicum of homogeneity in the attitudes toward risk among the members. For example, a class of members' utility functions for which surrogate functions always exist is the one with linear risk tolerances, $\rho_i(x_i)=lx_i-k_i$. Theorem 9 shows that agreement on the cautiousness l is mandatory, and Theorem 4 shows that the syndicate risk tolerance is of the same form, $\rho_0(x)=l\Sigma_i s_i(x,\xi)-\Sigma_i k_i= lx-k_0$ where $k_0=\Sigma_i k_i$. Tables III and IV exhibit examples from this class.

6. DETERMINATE SHARING RULES

There are several stronger results which one might hope to hold generally, but which in fact obtain only if the sharing rule is both linear and determinate. It can easily be shown, however, that exponential utilities are the only ones yielding such sharing rules in the absence of agreement on the probability assessment. For completeness, we shall examine two of those results here.

THEOREM 10:

$$\partial\mu_0(x,\xi)/\partial\lambda_j=\lambda_j^{-1}\sigma_j(x,\xi)\mu_0(x,\xi)\,. \tag{11}$$

This relation can be derived by differentiating in (2), dividing the respective sides of the result by (2), and then summing over i. The implication of (11) is that if the sharing rule is both linear and determinate, $\sigma_i(x,\xi)=\sigma_i$, then the weights affect the syndicate evaluation measure only through an extraneous constant of proportionality, and therefore the syndicate's decision will be independent of the weights. The converse is also true. This situation is apparent in the examples in Tables III and IV.

THEOREM 11: *The syndicate's decision is preferred by every member, given his sharing rule, if and only if the sharing rule is both linear and determinate.*

A sharing function $s_i(x,\xi)$ induces an evaluation measure $U_i(s_i(x,\xi))f_i(\xi)$ for member i, and therefore a DEM

$$u_i(s_i(x,\xi))\sigma_i(x,\xi)f_i(\xi)=\lambda_i^{-1}\sigma_i(x,\xi)\mu_0(x,\xi)\,,$$

which differs by an extraneous constant from $\mu_0(x,\xi)$ if and only if his stock $\sigma_i(x,\xi)$ is constant.

Note that when the stocks are constant, (10) implies that $f_0(\xi)=k\Pi_i f_i(\xi)^{\sigma_i}$. It is appealing to interpret this formula as the consequence of sampling the members' opinions in proportion to their stocks, or equivalently, in proportion to their risk tolerances. For example, if each $f_i(\xi)$ is a Normal density function with mean m_i and common variance $v_i=v$ for all i, then $f_0(\xi)$ is a Normal density function with mean $m_0=\Sigma_i\sigma_i m_i$ and variance v.

7. AN EXAMPLE

Suppose that a syndicate has an opportunity to invest its capital of \$1 in a project that will return an uncertain amount $\tilde{\xi}$ per dollar invested. The syndicate can borrow (or lend) capital in any amount α at a certain cost of h, so a payoff $p(\xi, \alpha) = (1+\alpha)\xi - \alpha h$ is available, and the decision problem is to choose an optimal amount of debt. Member i assesses a Normal distribution for ξ with mean m_i and variance v_i, and has an exponential utility with risk tolerance c_i. The sharing rules for this case are given in Tables III and IV. Using (8), the syndicate risk tolerance is $c_0 = \Sigma_i c_i$ and therefore the surrogate utility function is $-c_0 e^{-x/c_0}$. From (10), the surrogate probability assessment is Normal with variance v_0 such that $v_0^{-1} = \Sigma_i \sigma_i v_i^{-1}$, where the stocks are given by $\sigma_i = c_i/c_0$, and mean m_0 such that $m_0/v_0 = \Sigma_i \sigma_i (m_i/v_i)$. It is then readily calculated that the optimal capital to invest is $1+\alpha = c_0(m_0 - h)/v_0$. Theorems 9, 10, and 11 are easily verified for this example.

8. CONCLUSION

In terms of welfare theory, the ramifications of our results are mainly negative.[3] Even though the formulation enforces stringent regularity conditions, we have found that only under very narrow circumstances will the group behavior be consistent with the Savage axioms. In sum, the requirement for consistency is either agreement on the probability assessment or else identical cautiousness among the members. In terms of observable traits, we have found that consistency is uniquely associated with linear Pareto-optimal sharing rules in the absence of agreement on the probability assessment.

On the other hand, inconsistency is uniquely associated with a single observable trait; namely, the group's probability assessments are not independent of the consequences. For purposes of further research, this result casts serious doubt on the appropriateness of Savage's fourth postulate in the study of group decision making under uncertainty; or at least, it indicates that the fourth postulate is not consistent with Pareto optimality. Analytically, inconsistency is associated with an inability to factor the evaluation measure into a utility function and a probability density.

A later paper will apply these results to problems in welfare theory, including the role of institutional factors. In particular, the corporate form of financial organization, with its characteristically linear sharing rule, deserves attention.

Stanford University

[3] The connection with the welfare theory of public goods can be made explicit as follows. Let the group decision be $\alpha = (a, z)$, where a is the amount to be expended for public goods and z is a vector of activity levels denoting the choice of technology; of course, (a, z) may be constrained to lie in a feasible set C. The payoff to member i is then $b_i(a, z; \xi) - t_i(a, z; \xi)$, where b_i is the cash equivalent value to him of the public goods enjoyed by member i and t_i is the tax paid to cover the expenditure. Now, Pareto optimality of the tax assessments, subject to the constraint $\Sigma_i t_i(a, z; \xi) = a$, yields a condition similar to (2). It is easily shown, however, that in this case the Lagrange multiplier μ_0 depends only on the event ξ and the quantity $[b_0(a, z; \xi) - a]$ where $b_0 = \Sigma_i b_i$. Hence, the public goods problem can be formulated in the present context by defining the group payoff as $x = p(\xi, \alpha) = b_0(\alpha; \xi) - a$ with $\alpha = (a, z)$, and by letting the shares be $s_i = b_i - t_i$.

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