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NOTES AND COMMENTS

COMMON KNOWLEDGE OF AN AGGREGATE OF EXPECTATIONS

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1. INTRODUCTION

WHEN INDIVIDUALS MAKE ECONOMIC DECISIONS based on diverse sources of information, market aggregates—such as prices—will to a greater or lesser extent reflect the pieces of information received. By observing a market aggregate, each person may learn something about the information possessed by others. In some cases, enough information is reflected to induce everyone to agree on a consensus estimate of the relevant uncertainty. This theme has been extensively explored in the literature on rational expectations.

In this note we provide a theorem giving a set of conditions under which public observation of an aggregate of individual expectations leads to consensus. The key concept underlying the result is that of *common knowledge*, as formalized by Aumann (1976). (An event is said to be common knowledge if everyone knows it, everyone knows that everyone knows it, and so on ad infinitum.) We suppose that n individuals start with a common prior and then form expectations, conditioned on their diverse information sources, of some random variable of interest. The theorem states that if a stochastically monotone aggregate of the n conditional expectations is common knowledge, then the expectations must be identical.

The work presented here builds on a theorem due to McKelvey and Page (1986). First, we extend their result from conditional probabilities of an event to conditional expectations of a random variable. Second, we provide a simple and brief proof of the theorem on conditional expectations. Third, we cover the general case in which information is represented by σ-fields rather than by partitions. The original statement of the theorem on conditional expectations is due to Nielsen (1987a); versions of the present proof were derived independently in three working papers, namely Brandenburger and Geanakoplos (1987), McKelvey and Page (1987), and Nielsen (1987b).

Section 2 reviews the formalization of common knowledge developed by Aumann (1976) and restates Aumann's fundamental theorem on the impossibility of "agreeing to disagree." Section 3 develops the main result of this note.

2. PRELIMINARIES

There is a finite set Ω of states of the world. Subsets of Ω are called events. There are n individuals indexed by $i=1,\dots,n$. Each individual i receives private information about the true state of the world according to a partition Π_i of Ω. So if $\omega \in \Omega$ is the true state, i is informed of the member $\Pi_i(\omega)$ of Π_i that contains ω. Given this information structure, it is natural to say that i knows an event A at ω if $\Pi_i(\omega) \subset A$. The event that i knows A, to be denoted by $K_i(A)$, is then given by $K_i(A) = \{\omega \in \Omega : \Pi_i(\omega) \subset A\}$.

An event A is *common knowledge* at a state ω if everyone knows A at ω, everyone knows that everyone knows A at ω, and so on ad infinitum. If we define an "everyone

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knows" operator by setting $K(A) = \bigcap_{i=1}^{n} K_i(A)$, then the condition that A is common knowledge at ω can be written as $\omega \in K^{\infty}(A)$ where K^{∞} denotes the infinite application of the K operator. Aumann's key insight was to observe that the potentially complex criterion $\omega \in K^{\infty}(A)$ is in fact equivalent to a much simpler condition. Let Π denote the meet (i.e., finest common coarsening) of the partitions $\Pi_1, \ldots, \Pi_n$, and let $\Pi(\omega)$ denote the member of Π that contains ω. Aumann demonstrated that $\omega \in K^{\infty}(A)$ if and only if $\Pi(\omega) \subset A$.

Using this characterization of common knowledge, Aumann went on to prove a fundamental theorem on the impossibility of "agreeing to disagree." Suppose that the n individuals share a common prior probability distribution P on Ω which assigns positive probability to each state and that they are interested in the probability of some event $F \subset \Omega$. If the true state is ω, then individual i's probability assessment of F is given by the conditional probability $P(F|\Pi_i(\omega)) \equiv P(F \cap \Pi_i(\omega))/P(\Pi_i(\omega))$. Aumann proved that if these n conditional probabilities are common knowledge at some ω^*, then they must all in fact be equal. To state this result more precisely, for $i = 1, \ldots, n$ let $A_i(p_i)$ denote the event that i's conditional probability of F is p_i, that is, $A_i(p_i) \equiv \{\omega \in \Omega : P(F|\Pi_i(\omega)) = p_i\}$.

THEOREM 1 (Aumann (1976)): *Suppose the event* $\bigcap_{i=1}^{n} A_i(p_i)$ *is common knowledge at some* $\omega^* \in \Omega$. *Then* $p_1 = p_2 = \cdots p_n = P(F|\Pi(\omega^*))$.

The logic behind Theorem 1 is simply this. If it is common knowledge at ω^* that i's conditional probability of F is p_i, it must be the case that $P(F|\Pi_i(\omega)) = p_i$ for all $\omega \in \Pi(\omega^*)$. But then by the properties of conditional probability, $P(F|\Pi(\omega^*)) = p_i$. Since this is true for all i, the result is established.

As the above argument makes clear, Aumann's theorem extends immediately to the case of conditional expectations: suppose that the n individuals are interested in the expectation of some random variable $X : \Omega \to \mathbb{R}$. If the true state is ω, then individual i calculates the conditional expectation

$$X_i(\omega) \equiv E(X|\Pi_i(\omega)) \equiv \frac{1}{P(\Pi_i(\omega))} \sum_{\omega' \in \Pi_i(\omega)} X(\omega')P(\omega').$$

For $i = 1, \ldots, n$ let $B_i(x_i)$ denote the event that i's conditional expectation of X is x_i, that is, $B_i(x_i) \equiv \{\omega \in \Omega : X_i(\omega) = x_i\}$.

THEOREM 2: *Suppose the event* $\bigcap_{i=1}^{n} B_i(x_i)$ *is common knowledge at some* $\omega^* \in \Omega$. *Then* $x_1 = x_2 = \cdots = x_n = E(X|\Pi(\omega^*))$.

Theorem 2 is stated in Geanakoplos and Sebenius (1983) who prove it in the course of showing that it cannot be common knowledge that two risk neutral individuals both expect to profit from a bet.

3. THE RESULT

In many economic settings, instead of assuming that individual opinions (conditional probabilities or expectations) are common knowledge, it is more natural to suppose that only some *aggregate* of individual opinions (e.g., a price) becomes common knowledge. In this context, McKelvey and Page (1986) proved that if a stochastically monotone (defined below) aggregate of individual conditional probabilities is common knowledge, then it is still true that all the conditional probabilities must be equal. We now show that, as was the case for Aumann's result, McKelvey and Page's result also extends to conditional expectations. At the same time we provide a simple proof of the more general theorem on conditional expectations.

We will call a function $f\colon \mathbb{R}^n \to \mathbb{R}$ *stochastically monotone* if it can be written in the form $f(x_1,\dots,x_n) = \sum_{i=1}^n f_i(x_i)$ where each $f_i : \mathbb{R} \to \mathbb{R}$ is strictly increasing. McKelvey and Page defined stochastic monotonicity differently, but Bergin and Brandenburger (1990) show that the two definitions are equivalent. As before, suppose that the n individuals are interested in the expectation of some random variable $X:\Omega \to \mathbb{R}$. We assume that an aggregate $f(X_1(\omega),\dots,X_n(\omega))$, where f is stochastically monotone, of the n individual conditional expectations of X is common knowledge. To state our result formally, let $A(x)$ denote the event that the stochastically monotone aggregate takes the value x, that is

$$A(x) \equiv \{\omega \in \Omega : f(X_1(\omega),\dots,X_n(\omega)) = x\}.$$

THEOREM 3: *Suppose the event $A(x)$ is common knowledge at some $\omega^* \in \Omega$. Then the n conditional expectations of X at ω^* are equal, that is, $X_1(\omega^*) = X_2(\omega^*) = \cdots = X_n(\omega^*) = E(X|\Pi(\omega^*))$.*

PROOF: For $\omega \in \Omega$ let $X_0(\omega) \equiv E(X|\Pi(\omega))$. Since $f(X_1(\omega),\dots,X_n(\omega))$ (by the common knowledge hypothesis) and $X_0(\omega)$ (by definition) are constant for all $\omega \in \Pi(\omega^*)$,

$$\begin{aligned}
0 &= \sum_{\omega\in\Pi(\omega^*)} f(X_1(\omega),\dots,X_n(\omega))(X(\omega)-X_0(\omega))P(\omega)\\
&= \sum_{i=1}^{n}\sum_{\omega\in\Pi(\omega^*)} f_i(X_i(\omega))(X(\omega)-X_0(\omega))P(\omega)\\
&= \sum_{i=1}^{n}\sum_{\omega\in\Pi(\omega^*)} f_i(X_i(\omega))(X_i(\omega)-X_0(\omega))P(\omega)\\
&= \sum_{i=1}^{n}\sum_{\omega\in\Pi(\omega^*)} [f_i(X_i(\omega))-f_i(X_0(\omega))](X_i(\omega)-X_0(\omega))P(\omega).
\end{aligned}$$

Since f_i is strictly increasing, the terms $f_i(X_i(\omega)) - f_i(X_0(\omega))$ and $X_i(\omega) - X_0(\omega)$ have the same sign. Hence for each i,

$$\sum_{\omega\in\Pi(\omega^*)} [f_i(X_i(\omega))-f_i(X_0(\omega))](X_i(\omega)-X_0(\omega))P(\omega) \geqslant 0,$$

with strict inequality if X_i is nonconstant on $\Pi(\omega^*)$. Since strict inequality leads to a contradiction (the sum over i is equal to 0), each X_i must be constant on $\Pi(\omega^*)$. Equality of the X_i's now follows from precisely the argument underlying Theorem 2. *Q.E.D.*

Even if the aggregate $f(X_1(\omega)),\dots,X_n(\omega))$ is *not* initially common knowledge, the argument in Geanakoplos and Polemarchakis (1982) can be adapted to show that repeated public announcements of the aggregate must eventually lead to common knowledge and hence (by Theorem 3) to consensus. McKelvey and Page (1986) demonstrated this convergence result when conditional probabilities are aggregated. Exactly the same reasoning applies to aggregation of conditional expectations.

A similar proof yields a version of Theorem 3 for general probability spaces, where information is represented by σ-fields rather than by partitions. Such extensions of Theorem 3 are stated and proved in Nielsen (1987a, b) and McKelvey and Page (1987). Bergin (1989) extended McKelvey and Page's original result on conditional probabilities to allow for σ-fields. To consider the general case, let $(\Omega, \mathbb{F}, P)$ be a probability space where Ω is again the set of states of the world, $\mathbb{F}$ is a σ-field of events, and P is the common prior probability measure on $\mathbb{F}$. Each individual i receives private information according to a sub σ-field $\mathbb{F}_i$ of $\mathbb{F}$ and forms the conditional expectation $X_i(\omega) \equiv E(X|\mathbb{F}_i)_\omega$ where $X: \Omega \to \mathbb{R}$ is the (integrable) random variable of interest. Let $\mathbb{F}_0 = \bigcap_{i=1}^n \mathbb{F}_i$ be the meet (i.e., intersection) of the σ-fields $\mathbb{F}_1, \ldots, \mathbb{F}_n$. Write $X_0(\omega) \equiv E(X|\mathbb{F}_0)_\omega$. We say that a random variable $Y: \Omega \to \mathbb{R}$ is common knowledge if it is almost surely (a.s.) equal to an $\mathbb{F}_0$-measurable random variable.[2]

Suppose now that the stochastically monotone aggregate $f(X_1(\omega), \ldots, X_n(\omega))$ is common knowledge. Then, using the properties of conditional expectation,

$$\begin{aligned} 0 &= \int_\Omega f(X_1(\omega), \ldots, X_n(\omega))(X(\omega) - X_0(\omega))\, dP(\omega) \\ &= \sum_{i=1}^n \int_\Omega f_i(X_i(\omega))(X(\omega) - X_0(\omega))\, dP(\omega) \\ &= \sum_{i=1}^n \int_\Omega f_i(X_i(\omega))(X_i(\omega) - X_0(\omega))\, dP(\omega) \\ &= \sum_{i=1}^n \int_\Omega [f_i(X_i(\omega)) - f_i(X_0(\omega))](X_i(\omega) - X_0(\omega))\, dP(\omega).^3 \end{aligned}$$

By the same argument as that in the proof of Theorem 3 we conclude that for each i

$$\int_\Omega [f_i(X_i(\omega)) - f_i(X_0(\omega))](X_i(\omega) - X_0(\omega))\, dP(\omega) \geqslant 0$$

and in fact, since strict inequality is impossible, that $X_i(\omega) = X_0(\omega)$ a.s. Hence the n conditional expectations of X are common knowledge and are equal a.s.

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[2] This is effectively a global definition of common knowledge "at every ω" (or "at almost every ω") as opposed to a local definition of common knowledge "at some particular ω." A local definition is also possible but demands careful treatment of probability-zero events—see Nielsen (1984) and Brandenburger and Dekel (1987).

[3] Further integrability assumptions—such as the integrability of $f_i(X_i(\omega))X(\omega)$—may appear necessary here. However, by introducing an extra conditioning on $\mathbb{F}_0$ and making use of "integrability conditional on $\mathbb{F}_0$" it can be shown that no such further assumptions are required.

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