SYSEN 5200: Systems Analysis Behavior and Optimization

Central Limit Theorem & Reliability Analysis

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Feburary 12, 2023

1. Suppose that X_1 is a r.v. $N(3, 15)$, and X_2 is a r.v. $N(10, 6)$ independent from X_1. The probabilities of $P(X_1 + X_2 \geq z)$, as a standard normal can be expressed such that:

   Z is a normal r.v. where:

$$\begin{aligned} \mu &= \mathbb{E}[X_1 + X_2] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] \\ &= 3 + 10 \\ &= 13 \end{aligned} \tag{1}$$

$$\begin{aligned} \sigma^2 &= Var(X_1 + X_2) \\ &= Var(X_1) + Var(X_2) \\ &= 15 + 6 \\ &= 21 \end{aligned} \tag{2}$$

   Therefore $Z \sim N(13, 21)$ such that:

$$\begin{aligned} P(X_1 + X_2 \geq z) &= P\Big(Z \geq \frac{z - \mu}{\sigma}\Big) \\ &= P\Big(Z \geq \frac{z - 13}{\sqrt{21}}\Big) \end{aligned} \tag{3}$$

2. Suppose that X is a r.v. with mean 6, variance 10, and $X_1, \ldots, X_{100}$ are i.i.d. copies of X.

   (a) Applying the Central Limit Theorem to approximate $P\Big(\sum_{i=1}^{100} X_i \geq 1000\Big)$ we have:

   The same mean:

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ &= \frac{1}{100}\sum_{i=1}^{100} X_i \end{aligned} \tag{4}$$

The probability:

$$P\Big(\sum_{i=1}^{100} X_i \geq 1000\Big) = P\Big(100\,\bar{X} \geq 1000\Big) \tag{5}$$

The Central Limit Theorem:

$$\lim_{n\to\infty} \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1) \tag{6}$$

when applied is:

$$\begin{aligned} P\Big(\sum_{i=1}^{100} X_i \geq 1000\Big) &= P\Big(Z \geq \frac{1000-600}{\sqrt{1000}}\Big) \\ &= P\Big(4\sqrt{10}\Big) \end{aligned} \tag{7}$$

(b) Suppose that $\sum_{i=1}^{n} X_i = 28$ and $n = 100$. Applying the Central Limit Theorem to compute $(1-\alpha)$ and approximate confidence intervals for $\alpha = 0.05$, we have:

The same mean:

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ &= \frac{1}{100}\sum_{i=1}^{100} X_i \\ &= \frac{28}{100} = 0.28 \end{aligned} \tag{8}$$

The approximate Confidence Interval:

$$\begin{aligned} P\Big(-Z\alpha/2 \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq Z\alpha/2\Big) &\approx 1-\alpha \\ P\Big(\bar{X} - Z\alpha/2\cdot\sigma/\sqrt{n} \leq \mu \leq \bar{X} + Z\alpha/2\cdot\sigma/\sqrt{n}\Big) &\approx 1-\alpha \\ &\approx 0.95 \end{aligned} \tag{9}$$

and that:

$$Z\alpha/2 = 1.96 \tag{10}$$

when solved is:

$$\begin{aligned} &\Big(\bar{X} \pm 1.96\cdot\sigma/\sqrt{n}\Big) \\ &\Big(0.28 \pm 1.96\cdot\sigma/10\Big) \\ &\Big(0.28 \pm 0.14\Big) \end{aligned} \tag{11}$$

3. Considering a system where the lifetimes of the first and second components are geometrically distributed with mean 2 hours ($Geo(1/2)$), the lifetime of the third component is geometrically distributed with mean 4 hours ($Geo(1/4)$) and the lifetime of the fourth component is geometrically distributed with mean 3 hour ($Geo(1/3)$).

   (a) The structure function of this system is such that:

$$\begin{aligned}\phi(x) &= max(x_4, min(x_3, max(x_2, x_1)))) \\ &= 1 - (1 - x_4)(1 - x_3(1 - (1 - x_2)(1 - x_1)))\end{aligned} \tag{12}$$

   (b) The probability that the system survives for more than 2 hours is such that:

$$\begin{aligned}\mathbb{P}(L > 2) &= (1 - \lambda_1)^t(1 - \lambda_3)^t + (1 - \lambda_2)^t(1 - \lambda_3)^t + (1 - \lambda_4)^t \\ &- (1 - \lambda_1)^t(1 - \lambda_2)^t(1 - \lambda_3)^t(1 - \lambda_4)^t \\ &= L_1L_3 + L_2L_3 + L_4 - L_1L_2L_3L_4 \\ &= 0.71\end{aligned} \tag{13}$$

   where:

$$\begin{aligned}L_1 = L_2 &\sim \left(1 - \frac{1}{2}\right)^2 \\ &= \frac{1}{4} \\ L_3 &\sim \left(1 - \frac{1}{4}\right)^2 \\ &= \frac{9}{16} \\ L_4 &\sim \left(1 - \frac{1}{3}\right)^2 \\ &= \frac{4}{9}\end{aligned} \tag{14}$$

   (c) The expected lifetime of the system in hours using the geometric series is such that:

$$\begin{aligned}\mathbb{E}[L] &= \sum_{t=0}^{\infty} \mathbb{P}(L > t) \\ &= \frac{1}{1 - (1 - \lambda_1)(1 - \lambda_3)} + \frac{1}{1 - (1 - \lambda_2)(1 - \lambda_3)} + \frac{1}{1 - (1 - \lambda_4)} \\ &- \frac{1}{1 - (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)(1 - \lambda_4)} \\ &= 3.45\end{aligned} \tag{15}$$