SYSEN 5200: Systems Analysis Behavior and Optimization

Monte Carlo Simulation

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1. Suppose that you want to generate a random sample from a p.d.f. with the following distribution:

$$f(x) = \begin{cases} \frac{3x^2}{8} & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

the equivalent c.d.f can be derived as:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{0}^{x} \frac{3t^2}{8}\,dt \\ &= \frac{t^3}{8}\Big|_0^x \\ &= \frac{x^3}{8} \end{aligned} \tag{2}$$

Therefore, we have:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^3}{8} & \text{for } 0 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases} \tag{3}$$

when inverted is:

$$F^{-1}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \sqrt[3]{8y} & \text{for } 0 < y \leq 1 \\ 2 & \text{for } y > 1 \end{cases} \tag{4}$$

where $g(y) = F^{-1}(y)$ that recovers the r.v. X with the desired p.d.f. $f(x)$ from $Y \sim U[0,1]$.

2. Suppose that the lifetime of each component i is independent and follows a distribution that is the maximum of zero and a normal random variable with mean μ_i and standard deviation σ_i (both in minutes), whose values are given by: $\mu_1 = 60, \sigma_1 = 15; \mu_2 = 50, \sigma_2 = 10; \mu_3 = 75, \sigma_3 = 15; \mu_4 = 75, \sigma_4 = 20; \mu_5 = 60, \sigma_5 = 15; \mu_6 = 70, \sigma_6 = 15$.

   (a) Considering the structure function of the following system:

$$\phi(x) = \min(\max(\min(x_1, x_2), x_3), x_4, \max(x_5, x_6)) \tag{5}$$

   (b) A simulation was conducted with over 1,000 replications to estimate the probability that the system will be functional for more than 60 minutes with a 95% confidence interval.

```python
## libraries
import numpy as np

## structure func
def phi(x):
    return min(max(min(x[0], x[1]), x[2]), x[3], max(x[4], x[5]))

## simulation
def simulate():
    x = np.maximum(0, np.random.normal(
        [60, 50, 75, 75, 60, 70], ## mu
        [15, 10, 15, 20, 15, 15] ## sigma
        )
    )
    return phi(x) > 60

## compute
n = 1001
results = [simulate() for i in range(n)]
prob = sum(results) / n
lower = prob - 1.96 * np.sqrt(prob * (1 - prob) / n)
upper = prob + 1.96 * np.sqrt(prob * (1 - prob) / n)

## results
print(f"Probability: {prob:.4f}")
print(f"95% CI: ({lower:.4f}, {upper:.4f})")
```

Fig. Python 2(b)

Probability: **0.5834**
95% CI: **(0.5529, 0.6140)**

   (c) Suppose that the error of the estimate is to be bounded by 0.01 with 0.95 probability. Using Chebyshev's inequality, we estimate the number of replications is sufficient to provide this guarantee as:

$$\begin{aligned} P(|\hat{p} - p| \geq 0.01) \leq \frac{Var(\hat{p})}{0.01^2} &= \frac{p(1-p)/n}{0.01^2} \\ n &\geq \frac{p(1-p)}{(0.01^2 \cdot 0.05)} \\ n &\geq \frac{0.5 \cdot 0.5}{(0.01^2 \cdot 0.05)} = 50,000 \end{aligned} \tag{6}$$

   Assuming that the approximation from Central Limit Theorem holds, we have:

$$n \geq \left(\frac{z_\alpha / 2^\sigma}{E}\right)^2 = \left(\frac{1.96}{0.01}\right)^2 \cdot 0.5 \cdot 0.5 \approx 9,604. \tag{7}$$

(d) Another simulation was conducted with over 1,000 replications to estimate the expected lifetime of the system in minutes with a 95% confidence interval.

```python
## libraries
import numpy as np

## structure func
def phi(x):
    return min(max(min(x[0], x[1]), x[2]), x[3], max(x[4], x[5]))

## simulation
def simulate():
    x = np.maximum(0, np.random.normal(
        [60, 50, 75, 75, 60, 70], ## mu
        [15, 10, 15, 20, 15, 15] ## sigma
        )
    )
    return phi(x)

## compute
n = 1001
results = [simulate() for i in range(n)]
mean = np.mean(results)
lower = mean - 1.96 * np.std(results, ddof=1) / np.sqrt(n)
upper = mean + 1.96 * np.std(results, ddof=1) / np.sqrt(n)

## results
print(f"Expected L: {mean:.2f} minutes")
print(f"95% CI: ({lower:.2f}, {upper:.2f}) minutes")
```

Fig. Python 2(d)

Expected L: **61.80** min.
95% CI: **(61.05, 62.56)** min.

(e) Suppose that the error of the estimate is to be bounded by 1 minute with 0.95 probability. Using Chebyshev's inequality, we estimate the number of replications is sufficient to provide this guarantee as:

$$n \geq \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2}{0.05} = \frac{15^2 + 10^2 + 15^2 + 20^2 + 15^2 + 15^2}{0.05} = 2,843 \tag{8}$$

Assuming that the approximation from Central Limit Theorem holds, we have:

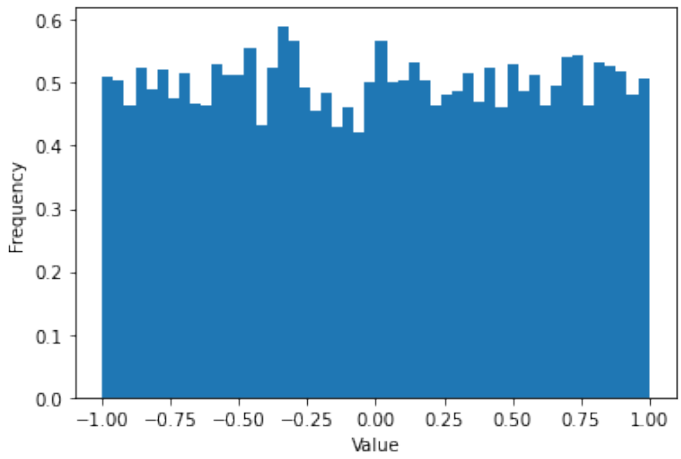
$$n \geq \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 18.8}{1}\right)^2 \approx 546 \tag{9}$$

3. One application of Monte Carlo Simulation is estimating a difficult to evaluate integral. Here we will perform a simple simulation to estimate the area of a unit circle.

   (a) First we will generate 10,000 points uniformly on $U[-1, 1]$ using 50 uniform-sized bins.

```python
## libraries
import numpy as np
import matplotlib.pyplot as plt

## replicate
X = np.random.uniform(-1, 1, 10_000)

## plot
plt.hist(
    x = X,
    bins = 50,
    density = True
)

## plot labels
plt.xlabel('Value')
plt.ylabel('Frequency')

## plot preview
plt.show()
```

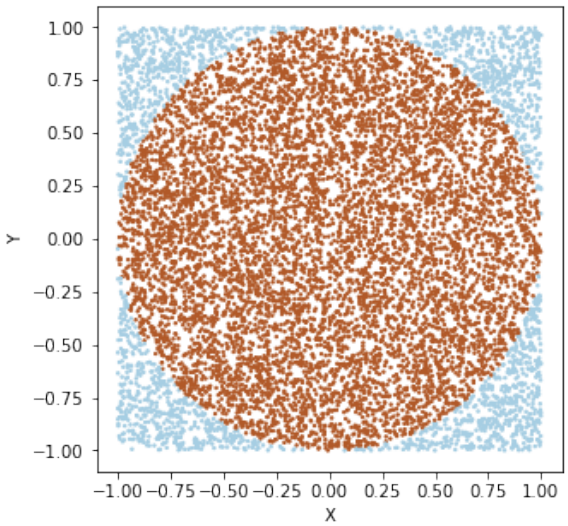
Fig. Python 3(a)



(b) Then we generate 10,000 points uniformly distributed in a square around the origin with corners (-1,-1), (-1,1), (1,-1), and (1,1). We independently generate the X and Y-coordinates of a point to be $U[-1, 1]$. Then we create an additional array Z of length 10,000, which indicates whether each random point lies within a circle of radius 1 centered at the origin.

```python
## libraries
import numpy as np
import matplotlib.pyplot as plt

## replicate
X = np.random.uniform(-1, 1, 10_000)
Y = np.random.uniform(-1, 1, 10_000)

## compute distance
R = np.sqrt(X**2 + Y**2)
Z = np.where(R <= 1, 1, 0)

## plot
fig, ax = plt.subplots(figsize=(5, 5))
ax.scatter(
    x = X,
    y = Y,
    c = Z,
    s = 2.0,
    cmap = plt.cm.Paired
)

## plot labels
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_aspect('equal')

## plot preview
plt.show()
```

Fig. Python 3(b)



(c) Given n random points above, we can estimate π as $\hat{\pi_n} = 4N_{circle}/n$, where N_{circle} is the number of random points which lie within the circle. For n between 1 and 1000, we plot the relative error $|\pi_n - \pi|/\pi$, as well as the function $1/\sqrt{n}$.

```python
## libraries
import numpy as np
import matplotlib.pyplot as plt

## pi
pi_true = np.pi

## number of replications
ns = np.arange(1, 1001)

## initialize estimations of pi and errors
pi_est = np.zeros_like(ns, dtype = float)
rel_err = np.zeros_like(ns, dtype = float)

## replications and estimations
for i, n in enumerate(ns):
    X = np.random.uniform(-1, 1, n)
    Y = np.random.uniform(-1, 1, n)

    ## compute distances
    R = np.sqrt(X**2 + Y**2)
    N_circle = np.count_nonzero(R <= 1)

    ## estimate pi and compute error
    pi_est[i] = 4 * N_circle / n
    rel_err[i] = np.abs(pi_est[i] - pi_true) / pi_true

## plot
fig, ax = plt.subplots()
ax.scatter(ns, rel_err, label = 'Relative Error', alpha = 0.5, s = 5)
ax.plot(ns, 1/np.sqrt(ns), label = '1 / sqrt(n)', color = 'orange')

## plot labels
ax.set_xlabel('n')
ax.set_ylabel('Relative Error')
ax.legend()

## plot preview
plt.show()
```

Fig. Python 3(c)