# SYSEN 6000: Foundations of Complex Systems

## Decision Making Under Uncertainty

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## Two-Stage Stochastic Mixed-Integer Linear Programming

A company is considering to produce a product $C$, which can be manufactured with either Process II or Process III, both of which use raw material $B$. $B$ can be purchased from another company, or manufactured with Process I, which uses raw material $A$.

Process II and Process III are exclusive (at most one of them can be built). There are five possible outcomes of demand and sale price combinations for product $C$. The probability of each scenario corresponding to the demand and sale price, as well as all supplemental data are below:

| | **Demand of $C$ (M tons)** | **Sale Price ($/ton)** | **Probability** |
|---|---|---|---|
| **Scenario 1** | 5 | 2,000 | 0.10 |
| **Scenario 2** | 8 | 1,900 | 0.20 |
| **Scenario 3** | 10 | 1,800 | 0.40 |
| **Scenario 4** | 12 | 1,600 | 0.20 |
| **Scenario 5** | 15 | 1,000 | 0.10 |

| | **Fixed CapEx ($M)** | **Var CapEx ($/ton)** | **Var OpEx ($/ton)** |
|---|---|---|---|
| **Phase I** | 100 | 250 | 100 |
| **Phase II** | 150 | 400 | 150 |
| **Phase III** | 200 | 550 | 200 |

The price for raw material $A$ is 250 ($/ton) and the price for raw material $B$ is 450 ($/ton).

The conversion rate for Process I is 0.90 of $A$ to $B$, Process II is 0.82 of $B$ to $C$, and Process III is 0.95 of $B$ to $C$.

The maximum supply of $A$ is 16 (M tons).

Given the problem, maximizing expected profit can be formulated as:

$$
\begin{aligned}
\max\quad & = -100x_1 - 250z_1 - 150x_2 - 400z_2 - 200x_3 - 550z_3 + \\
& \sum_s p_s((y_{s3} + y_{s4}) \cdot Price_s - 100y_{s1} - 250_{ys1}/0.90 - 450y_{sz} - 150y_{s3} - 200y_{s4}) \\
\text{s.t. } & y_{s1} \leq z_1 \\
& z_1/0.90 \leq 16 \cdot x_{s1} \\
& y_{s1} + y_{s2} = (y_{s3}/0.82) + (y_{s4}/0.95) \\
& y_{s3} + y_{s4} \leq Demand_s \\
& y_{s3} \leq z_2 \\
& z_2 \leq M \cdot x_2 \\
& y_{s4} \leq z_3 \\
& z_3 \leq M \cdot x_3 \\
& x_2 + x_3 = 1
\end{aligned} \tag{1}
$$

where the first stage decision variables are:

$$
\begin{aligned}
x_1 &= \text{Process I } [0,1] \text{ with capacity } z_1 \\
x_2 &= \text{Process II } [0,1] \text{ with capacity } z_2 \\
x_3 &= \text{Process III } [0,1] \text{ with capacity } z_3
\end{aligned} \tag{2}
$$

and the second stage decision variables are:

$$
\begin{aligned}
y_{s1} &= B \text{ manufactured in Process I} \\
y_{s2} &= B \text{ purchased from another company} \\
y_{s3} &= C \text{ manufactured in Process II} \\
y_{s4} &= C \text{ manufactured in Process III}
\end{aligned} \tag{3}
$$

that when solved, the optimal value is **$4,931.098M**, where:

$$
\begin{gathered}
x_1 = 0, x_2 = 1, x_3 = 0 \\
z_2 = 10
\end{gathered} \tag{4}
$$

Considering this solution, it would be prudent to manufacture product $C$ with Process II. The corresponding capacity would be 10M tons.

It would also be prudent to obtain the raw material $B$ by purchasing it from another company, rather than manufacturing it with Process I with raw material $A$, where product $C$ should be sold in the following amounts for each scenario:

| | **Sale of $C$ (M tons)** |
|---|---|
| **Scenario 1** | 5 |
| **Scenario 2** | 8 |
| **Scenario 3** | 10 |
| **Scenario 4** | 10 |
| **Scenario 5** | 10 |

In this case, the *"value of perfect information"* can be reformulated as:

$$\begin{aligned}
\max \quad &= \sum_s p_s(-100x_{s1} - 250z_{s1} - 150x_{s2} - 400z_{s2} - 200x_{s3} - 550z_{s3}) + \\
&\quad \sum_s p_s((y_{s3} + y_{s4}) \cdot Price_s - 100y_{s1} - (250_{ys1}/0.90) - 450y_{sz} - 150y_{s3} - 200y_{s4}) \\
\text{s.t.} \quad & y_{s1} \le z_{s1} \\
& z_{s1}/0.90 \le 16 \cdot x_{s1} \\
& y_{s1} + y_{s2} = (y_{s3}/0.82) + (y_{s4}/0.95) \\
& y_{s3} + y_{s4} \le Demand_s \\
& y_{s3} \le z_{s2} \\
& z_{s2} \le M \cdot x_{s2} \\
& y_{s4} \le z_{s3} \\
& z_{s3} \le M \cdot x_{s3} \\
& x_{s2} + x_{s3} = 1
\end{aligned} \tag{5}$$

where there are no first stage decision variables, and the second stage decision variables are:

$$\begin{aligned}
x_{s1} &= \text{Process I } [0,1] \text{ with capacity } z_{s1} \\
x_{s2} &= \text{Process II } [0,1] \text{ with capacity } z_{s2} \\
x_{s3} &= \text{Process III } [0,1] \text{ with capacity } z_{s3} \\
y_{s1} &= B \text{ manufactured in Process I} \\
y_{s2} &= B \text{ purchased from another company} \\
y_{s3} &= C \text{ manufactured in Process II} \\
y_{s4} &= C \text{ manufactured in Process III}
\end{aligned} \tag{6}$$

that when solved, the optimal value is \$5,590.366M, so that:

$$\$5,590.366\text{M} - \$4,931.098\text{M} = \$659.268\text{M} \tag{7}$$

Hence, the *"value of perfect information"* is **\$659.268M**.

# Robust Two-Stage Stochastic Linear Programming

The following is a derivation of the robust counterpart, given that:

$$\begin{aligned}
\max \quad & 10x_1 + 5x_2 \\
\text{s.t.} \quad & (6 + u_1)x_1 + (2 + u_2)x_2 \le 80, \ \forall (u_1, u_2) \in U
\end{aligned} \tag{8}$$

where $U$ is the box uncertainty set:

$$U = \{(u_1, u_2) : |u_1| \le 1, |u_2| \le 1\} \tag{9}$$

which can begin with the expression:

$$\begin{aligned} \max \ & 10x_1 + 5x_2 \\ \text{s.t.} \ & 6x_1 + 2x_2 + \sqrt{(x_1)^2 + (x_2)^2} \leq 80 \end{aligned} \tag{10}$$

and the box uncertainty set $U$, which can be expressed as a general polyhedral set:

$$U = \left\{ (u_1, u_2) \middle| \begin{aligned} u_1 \leq 1 \\ u_2 \leq 1 \\ -u_1 \leq 1 \\ -u_2 \leq 1 \end{aligned} \right\} = \{(u_1, u_2) | \mathbf{Wu} \leq \mathbf{v}\} \tag{11}$$

where:

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}^T, \mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \tag{12}$$

such that:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, r = 80 \tag{13}$$

This allows for the constraint $(6 + u_1)x_1 + (2 + u_2)x_2 \leq 80$, $\forall (u_1, u_2) \in U$ to be reformulated as:

$$\begin{aligned} & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq -6x_1 - 2x_2 + 80 \\ & \lambda_1 - \lambda_3 = x_1 \\ & \lambda_2 - \lambda_4 = x_2 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned} \tag{14}$$

such that $\lambda_1, \ldots, \lambda_4$ are dual variables corresponding to $\mathbf{Wu} \leq \mathbf{v}$.

Hence, the robust counterpart can be expressed as:

$$\begin{aligned} \max \ & 10x_1 + 5x_2 \\ \text{s.t.} \ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq -6x_1 - 2x_2 + 80 \\ & \lambda_1 - \lambda_3 = x_1 \\ & \lambda_2 - \lambda_4 = x_2 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned} \tag{15}$$