# SYSEN 6000: Foundations of Complex Systems

## Non-Linear Dynamics & Chaos

Nick Kunz [NetID: nhk37] nhk37@cornell.edu

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## SIR Model

The Susceptible, Infected, and Removed (SIR) Model is commonly applied in epidemiology to model disease transmission. It can be expressed as:

$$\begin{aligned} S' &= -\alpha \cdot SI \\ I' &= \alpha \cdot SI - \beta I \\ R' &= \beta I \end{aligned} \tag{1}$$

When $n = 800$, $\alpha = 2.2 \cdot 10^{-3}$, and $\beta = 0.4$, the phase plane solution can be illustrated as:

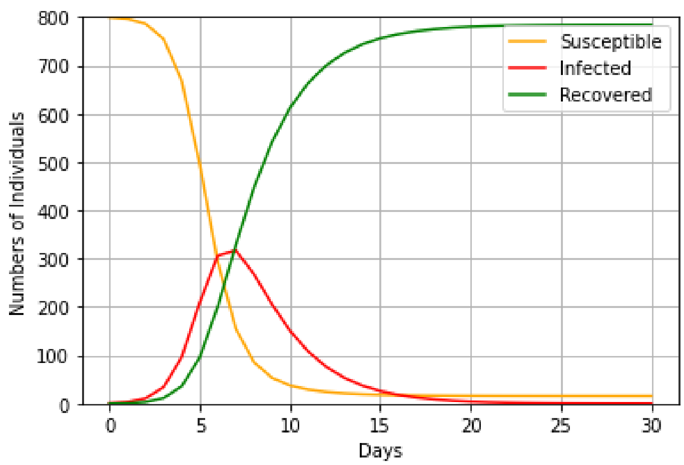



Figure 1: SIR Model General Application

Computing $\rho$, when $\rho = \beta/\alpha$, we see that:

$$\begin{aligned} \rho &= 0.4/2.2 \cdot 10^{-3} \\ &= 181.82 \end{aligned} \tag{2}$$

In the English Boarding School case of 1978, when $n = 1,000$, the phase plane solution can be illustrated as:

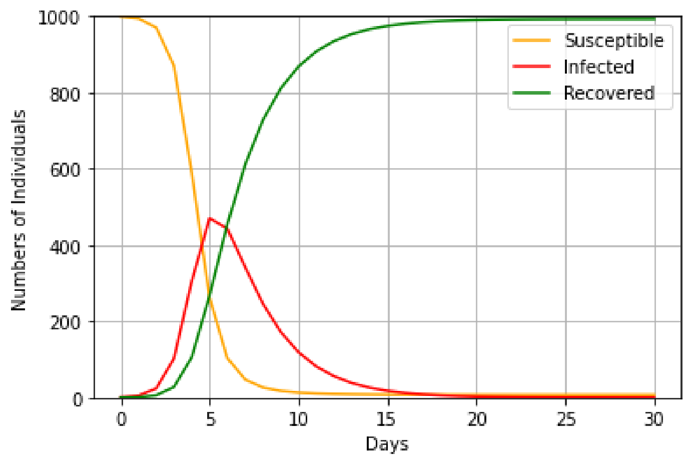



Figure 2: SIR Model English Boarding School House of 1978

The SIR Model could exhibit chaos like below by changing variables in higher dimensions (1) (2).

## Chaos Theory

Considering the Rossler model for chemical reactions, the dynamics can be represented by dimensions $X, Y, Z$, which have quadratic non-linearity similar to the SIR model and can be expressed as:

$$\begin{aligned} X' &= -(Y+Z) \\ Y' &= X + \alpha Y \\ Z' &= \beta + (X-c)Z \end{aligned} \tag{3}$$

Integrating through the first two period doubling of the Rossler model can be illustrated as:

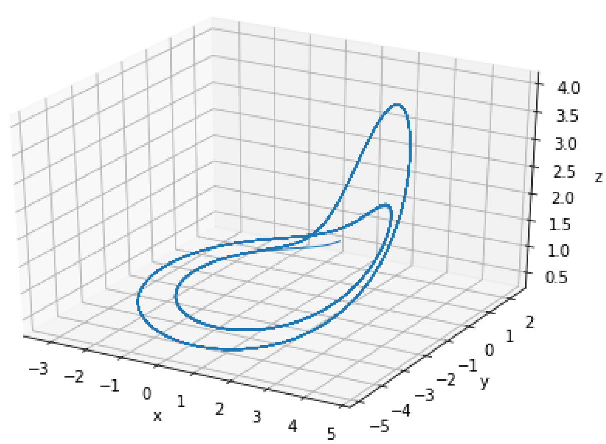



Figure 3: Rossler's Model Phase Space (3)

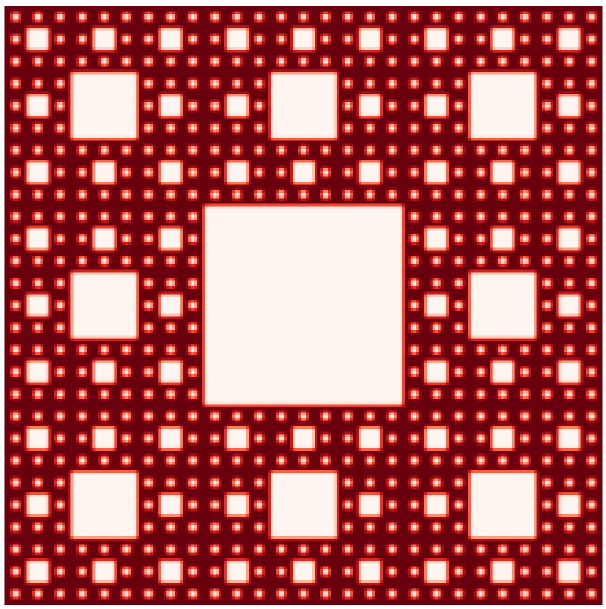
## Fractal Geometry

The *"Sierpinksi Carpet"* begins with a square and is divided into nine sub-squares. The central square is removed and repeated for each of the remaining sub-squares.

It's dimension can be expressed simply as:

$$-\lim_{m\to\infty}\frac{\log N(\epsilon_m)}{\log \epsilon_m} = -\lim_{m\to\infty} = \frac{m\log 8}{m\log 3} = \frac{\log 8}{\log 3} \tag{4}$$

and can be illustrated as such:



# References

[1] P. G. Barrientos, J. Á. Rodríguez, and A. Ruiz-Herrera, "Chaotic Dynamics in the Seasonally Forced SIR Epidemic Model," *Journal of Mathematical Biology*, vol. 75, no. 6, pp. 1655–1668, 2017.

[2] H. L. Smith, "Subharmonic Bifurcation in an S-I-R Epidemic Model," *Journal of Mathematical Biology*, vol. 17, no. 2, pp. 163–177, 1983.

[3] F. C. Moon, *Chaotic and Fractal Dynamics.* Nashville, Tennessee: John Wiley & Sons, 2 ed., Oct. 1992.

# Appendix

1. SIR Model

```python
## libraries
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.integrate import odeint

## params
n = 1000
alpha = 0.002
beta = 0.4

## init
i = 1
s = n - i
r = 0
t_max = 30
t = np.linspace(0, t_max, t_max + 1)

## sir model
def sir_der(y, t, alpha, beta):

    s, i, r  = y
    ds_dt = -alpha * s * i
    di_dt = alpha * s * i - beta * i
    dr_dt = beta * i

    return [ds_dt, di_dt, dr_dt]

## derive
y = s, i, r
res = odeint(sir_der, y, t, args = (alpha, beta))
s, i, r = res.T

## plot
plt.figure()
plt.grid()
plt.plot(t, s, 'orange', label = 'Susceptible')
plt.plot(t, i, 'r', label = 'Infected')
plt.plot(t, r, 'green', label = 'Recovered')
plt.xlabel('Days')
plt.ylabel('Numbers of Individuals')
plt.ylim([0, n])
plt.legend()
plt.show()
```

Python 3: SIR Model for English Boarding School House of 1978

2. Rossler Model

```python
## libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

## dynamics
def rossler(x, y, z, a, b, c):

    x_dot = -(y + z)
    y_dot = x + a * y
    z_dot = b + (x - c) * z

    return x_dot, y_dot, z_dot

## differ rate
def rossler_differ(step, dt):

    xs = np.empty((step + 1,))
    ys = np.empty((step + 1,))
    zs = np.empty((step + 1,))

    xs[0], ys[0], zs[0] = (1.0, 1.0, 1.0)

    for i in range(0, step):
        x_dot, y_dot, z_dot = rossler(
            x = xs[i],
            y = ys[i],
            z = zs[i],
            a = 0.35,
            b = 2.00,
            c = 4.00
        )

        xs[i+1] = xs[i] + (x_dot * dt)
        ys[i+1] = ys[i] + (y_dot * dt)
        zs[i+1] = zs[i] + (z_dot * dt)

    return xs, ys, zs

## comp traj
xs, ys, zs = rossler_differ(
    step = 100000,
    dt = 0.001
)

## plot
fig = plt.figure()
ax = Axes3D(fig)
ax.plot(xs, ys, zs, lw = 1)
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')
plt.show()
```

Python 3: Rossler Model for Chemical Reactions

3. Fractal Geometry

```python
## libraries
import numpy as np
import matplotlib.pyplot as plt

## make square
def square(x, y, len, img):
    for i in range(x, x + len):
        for j in range(y, y + len):
            img[i, j] = 0

    return img

# ## sierpinski carpet
def sierp_carpet(lev = 4):

    len = 3 ** lev

    img_sq = np.full(
        shape = (len, len),
        fill_value = 1
    )

    len_i = lev + 1
    for i in range(1, len_i):
        len_sq = int(
            len / (3**i)
        )

        len_j = 3 ** i
        for j in range(0, len_j, 3):
            x_j = int(
                (j + 1) * len_sq
            )

            len_k = 3 ** i
            for k in range(0, len_k, 3):
                y_k = int(
                    (k + 1) * len_sq
                )

                img_sq = square(
                    x = x_j,
                    y = y_k,
                    len = len_sq,
                    img = img_sq
                )

    return img_sq

## make carpet
carpet = sierp_carpet()

## plot
plt.axis('off')
plt.imshow(carpet, cmap = 'Reds')
plt.show()
```

Python 3: Fractal Geometry for Sierpenski Carpet