# SYSEN 5200: Systems Analysis Behavior and Optimization

## Probability & Statistics: Random Variables

Nick Kunz [NetID: nhk37] nhk37@cornell.edu

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1. Suppose $X$ is a continuous r.v. with the following p.d.f.

$$f(x) = \begin{cases} cx^{-5} & \text{for } 2 \leq x < \infty \\ 0 & \text{for } x < 2 \end{cases}$$

   (a) The constant $c$ can be computed as:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{\infty} cx^{-5}\,dx = 1 \\ = cx^{-5}\Big|_{2}^{\infty} \\ = \frac{c}{64} \Rightarrow c = 64 \end{aligned} \tag{1}$$

   (b) The c.d.f. can be computed as:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(y)\,dy \\ &= \int_{2}^{x} 64y^{-5}\,dy \\ &= 64y^{-5}\Big|_{2}^{x} \\ &= 64[-1/4y^{4}]\Big|_{2}^{x} \\ &= \begin{cases} 1 - (16/x^{4}) & \text{for } x \geq 2 \\ 0 & \text{for } x < 2 \end{cases} \end{aligned} \tag{2}$$

(c) The $\mathbb{E}[X]$ and $Var(X)$ can be computed as:

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\ &= \int_{2}^{\infty} x\,(64x^{-5})\, dx \\ &= \frac{8}{3}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2\, f(x)\, dx \\ &= \int_{2}^{\infty} x^2\,(64x^{-5})\, dx \\ &= 8\end{aligned} \tag{3}$$

$$\begin{aligned}Var(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= 8 - (8/3)^2 \\ &= \frac{8}{9}\end{aligned}$$

(d) The $\mathbb{E}[X^4]$ can be computed as:

$$\begin{aligned}\mathbb{E}[X^4] &= \int_{-\infty}^{\infty} x^4\, f(x)\, dx \\ &= \int_{2}^{\infty} x^4\,(64x^{-5})\, dx \\ &= \infty\end{aligned} \tag{4}$$

2. Suppose that $X$ is a continuous uniform r.v. with p.d.f. $U[10, 17]$. $P(X \geq 15)$ can be computed by evaluating all integrals with the c.d.f.:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases} \tag{5}$$

where:

$$U[a, b] \tag{6}$$

Hence:

$$\begin{aligned}P(X \geq 15) &= 1 - F(x) \\ &= 1 - (5/7) \\ &= \frac{2}{7}\end{aligned} \tag{7}$$

where:

$$F(15) = \frac{15 - 10}{17 - 10} = \frac{5}{7} \tag{8}$$

3. Suppose that $X$ is a continuous r.v. with p.d.f. $Expo(\lambda)$, and $a$ is some positive number such that $a < \frac{\lambda}{2}$. $Var(e^{aX})$ can be computed by evaluating all integrals.

   Given the p.d.f. of the exponential r.v. $X$:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

   The $Var(e^{aX})$ can be computed as:

$$\begin{aligned} \mathbb{E}[e^{aX}] &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\ &= \int_{0}^{\infty} e^{aX}\, \lambda e^{-\lambda x}\, dx \\ &= \lambda \left[ \frac{e^{-(\lambda-a)x}}{-(\lambda-a)} \right]_0^\infty \\ &= \frac{\lambda}{(\lambda-a)} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[e^{2aX}] &= \int_{-\infty}^{\infty} x^2\, f(x)\, dx \\ &= \int_{0}^{\infty} e^{2aX}\, \lambda e^{-\lambda x}\, dx \\ &= \lambda \left[ \frac{e^{-(\lambda-2a)x}}{-(\lambda-2a)} \right]_0^\infty \\ &= \frac{\lambda}{(\lambda-2a)} \end{aligned} \tag{10}$$

$$\begin{aligned} Var(e^{aX}) &= \mathbb{E}[e^{2aX}] - (\mathbb{E}[e^{aX}])^2 \\ &= (\lambda/\lambda - 2a) - (\lambda/(\lambda-a))^2 \\ &= \frac{\lambda}{(\lambda-2a)} - \frac{\lambda^2}{(\lambda-a)^2} \end{aligned}$$

4. Let $X$ and $Y$ be Bernoulli r.v.'s with $P(X = 1 \text{ and } Y = 1) = 1/3$ with the probability of success $p$ and $q$ respectively, such that:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}, \quad Y = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases}$$

(a) The $Cov(X, Y)$ can be computed as:

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}\big[(X - \mathbb{E}[X]) - (Y - \mathbb{E}[Y])\big] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \frac{1}{3} - p \cdot q \end{aligned} \tag{11}$$

where:

$$\begin{aligned} \mathbb{E}[X] &= p \\ \mathbb{E}[Y] &= q \end{aligned}$$

$$\begin{aligned} \mathbb{E}[XY] &= P(X = 1 \text{ and } Y = 1) \\ &= P(X = 1)P(Y = 1|X = 1) \\ &= \frac{1}{3} \end{aligned} \tag{12}$$

(b) The values of $X$ and $Y$ uncorrelated when $p$ and $q$ are:

$$\begin{aligned} Cov(X,Y) &= \frac{1}{3} - p \cdot q \\ &= \frac{1}{3} - \frac{1}{3} \\ &= 0 \end{aligned} \tag{13}$$

where:

$$p \cdot q = \frac{1}{3} \tag{14}$$

(c) $X$ and $Y$ *are* independent when $p \cdot q = 1/3$, as we have $P(X = 1 \text{ and } Y = 1) = P(X = 1)P(Y = 1)$, such that:

$$\begin{aligned} P(X = 1 \text{ and } Y = 1) &= \frac{1}{3} \\ &= p \cdot q \end{aligned} \tag{15}$$