Math 2552 Quiz 4 Review

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1 Method of Undetermined Coefficients

Find one particular real-valued solution of the following second-order linear ordinary differential equation:

$$y'' + 5y' + 4y = (t + 1)e^{2t}$$

Solution.

To find a particular real-valued solution of the given second-order linear ordinary differential equation, we can use the method of undetermined coefficients.

The characteristic equation of the associated homogeneous equation is:

$$r^2 + 5r + 4 = 0.$$

The roots of this characteristic equation are $r_1 = -1$ and $r_2 = -4$, which means the homogeneous solution is given by:

$$y_c(t) = c_1e^{-t} + c_2e^{-4t},$$

where c_1 and c_2 are constants.

Now, we can find the particular solution. We assume the particular solution has the form:

$$y_p(t) = (At + B)e^{2t}.$$

We know this from the following chart:

$d(x)$	Particular Solution:
$\alpha e^{\beta x}$	$Ae^{\beta x}$
$\alpha \cos(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$\alpha \sin(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$\alpha x^n + \cdots + \gamma x + \delta$	$Ax^n + \cdots + Yx + Z$
$(\alpha x^n + \cdots + \gamma x + \delta)e^{\beta x}$	$(Ax^n + \cdots + Yx + Z)e^{\beta x}$

Figure 1: Our table of guesses for the particular solution.

Now, we'll find y_p' and y_p'':

$$y_p'(t) = Ae^{2t} + 2Ate^{2t} + 2Be^{2t},$$

$$y_p''(t) = 4Ae^{2t} + 4Ate^{2t} + 4Be^{2t}.$$

Now, substitute these into the original ODE:

$$(4Ae^{2t} + 4Ate^{2t} + 4Be^{2t}) + 5(Ae^{2t} + 2Ate^{2t} + 2Be^{2t}) + 4(Ate^{2t} + Be^{2t}) = (t+1)e^{2t}.$$

Expand:

$$4Ae^{2t} + 4Ate^{2t} + 4Be^{2t} + 5Ae^{2t} + 10Ate^{2t} + 10Be^{2t} + 4Ate^{2t} + 4Be^{2t} = (t+1)e^{2t}.$$

Simplify:

$$9Ae^{2t} + 18Ate^{2t} + 18Be^{2t} = (t+1)e^{2t}.$$

Group like terms:

$$(9A + 18B)e^{2t} + (18A)te^{2t} = (t+1)e^{2t}.$$

We equate the coefficients of like terms on both sides:

$$9A + 18B = 1 \quad (\text{coefficient of } e^{2t}),$$
$$18A = 1 \quad (\text{coefficient of } te^{2t}),$$

Solving these equations, we find $A = \frac{1}{18}$ and $B = \frac{1}{36}$. Therefore, the particular solution is:

$$y_p(t) = (\frac{1}{18}t + \frac{1}{36})e^{2t}.$$

If we were asked to find the general solution, we compute the sum of the complementary and particular solutions:

$$y(t) = y_c(t) + y_p(t) = c_1e^{-t} + c_2e^{-4t} + (\frac{1}{18}t + \frac{1}{36})e^{2t}.$$

2 Method of Undetermined Coefficients (again!)

Find one particular real-valued solution of the following second-order linear ordinary differential equation:

$$y'' + 9y' = \cos(3t) + 9$$

Solution.

First, we solve for the complementary solution:

$$r^2 + 9 = 0,$$
$$r^2 = -9,$$
$$r = \pm 3i.$$

We have complex roots. The complementary solution for a given complex root $r = \alpha + \beta i$ is of the form

$$y_c = e^{\alpha t}(C_1\cos(\beta t) + C_2\sin(\beta t)).$$

Therefore, for the root $0 + 3i$ we have

$$y_c = e^{0t}(C_1\cos(3t) + C_2\sin(3t))$$

or

$$y_c = C_1\cos(3t) + C_2\sin(3t).$$

Now, we need an appropriate guess for our RHS. The RHS of the differential equation is $\cos(3t) + 9$. Let's consider y_p as the sum of two parts, y_{p1} and y_{p2}. For the first term (y_{p1}), if our RHS is $\alpha\cos(\beta t)$, then a reasonable guess is $A\cos(\beta t) + B\sin(\beta t)$, where A and B are some unknown coefficients. For the second term (y_{p2}), we know that a constant is a zeroth-degree polynomial, so our guess is just some constant C[1]. We can identify these guesses from Figure 1.

Therefore, our guess could be $y_p = y_{p1} + y_{p2} = (A\cos(2t) + B\sin(2t)) + C$. However, there is an issue with y_{p1}. Since we already have that $\cos(2t)$ is a solution (in our complementary solution), we need to multiply our guess y_{p1} by t. Our final guess for the particular solution becomes:

$$t(A\cos(3t) + B\sin(3t)) + C,$$

[1] This constant C is independent from C_1 and C_2 in our complementary solution.

or

$$At\cos(3t) + Bt\sin(3t) + C.$$

Now, we can solve for A and B. First, let's find y_p' and y_p''[2]:

$$y_p' = A(\cos(3x) - 3t(\sin(3t)) + B(\sin(3t) + 3t\cos(3t))$$

$$y_p'' = A(-9t\cos(3t) - 6\sin(3t)) + B(6\cos(3t) - 9t\sin(3t))$$

Let's substitute this into our original differential equation $y'' + 9y = \cos(3t) + 9$:

$$A(-9t\cos(3t) - 6\sin(3t)) + B(6\cos(3t) - 9t\sin(3t)) + 9(At\cos(3t) + Bt\sin(3t) + C) = 9 + \cos(3t)$$

We expand terms:

$$-9At\cos(3t) - 6A\sin(3t) + 6B\cos(3t) - 9Bt\sin(3t) + 9At\cos(3t) + 9Bt\sin(3t) + 9C = 9 + \cos(3t)$$

We simplify:

$$-6A\sin(3t) + 6B\cos(3t) + 9C = 9 + \cos(3t)$$

Let's equate the coefficients on both sides.

$$-6A = 0 \quad (\text{coefficient of } \sin(3t),$$
$$6B = 1 \quad (\text{coefficient of } \cos(3t),$$
$$9C = 9 \quad (\text{constant term on the right side}).$$

Therefore, $A = 0$, $B = \frac{1}{6}$, and $C = 1$.
$y_p = At\cos(3t) + Bt\sin(3t) + C$, so

$$y_p = \frac{1}{6}t\sin(3t) + 1$$

.

If we wanted the general solution, we compute $y_g = y_c + y_p$, and we have

$$y_g = C_1\cos(3t) + C_2\sin(3t) + \frac{1}{6}t\sin(3t) + 1$$

,

and we are done!

3 LRC-series circuit

Find the charge on q on the capacitor on an LRC-series circuit when $L = .5$ Henry, $R = 1$ ohm, $C = .5$ Faraday, $E(t) = 0$ Volts, $q(0) = 2$, $q'(0) = 0$.

Solution.

Recall the differential equation:

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

Let's substitute in our values for L, R, and C:

$$.5q'' + 1q' + \frac{q}{.5} = 0$$

Putting this in standard form,

$$q'' + 2q' + 4q = 0$$

Let's use quadratic formula to find our roots:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

[2] We can use product rule to solve these!

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{-3}}{2}$$
$$\lambda_{1,2} = -1 \pm \sqrt{-3} = -1 \pm \sqrt{3}i$$

For a complex root $\alpha \pm \beta i$, our solution can be written:

$$q(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t),$$

so our general solution is

$$q(t) = e^{-t}(C_1 \cos(3t) + C_2 \sin(3t).$$

Now, we will solve the initial value problem.
We know that $q(0) = 2$, so we have

$$2 = e^{(0)}(C_1 \cos(3(0)) + C_2 \sin(3(0)) = C_1,$$

so

$$C_1 = 2$$

Since we are given that $q'(0) = 0$, we can differentiate[3] q:

$$q(t) = C_1 e^{-t} \cos(3t) + C_2 e^{-t} \sin(3t)$$
$$q'(t) = C_1(-3e^{-t} \sin(3t) - e^{-t} \cos(3t)) + C_2(3e^{-t} \cos(3t) - e^{-t} \sin(3t)$$

Using $q'(0) = 0$, we have

$$0 = C_1(-3 \sin(3(0)) - \cos(3(0))) + C_2(3 \cos(3(0)) - \sin(3(0))$$
$$0 = C_1(-1) + C_2(3)$$

Using $C_2 = 2$, we have

$$0 = -C_1 + 6,$$

so

$$C_1 = 6$$

.

The solution to the initial value problem is therefore

$$q(t) = e^{-t}(6 \cos(3t) + 2 \sin(3t).$$

4 Method of Undetermined Coefficients (theory)

Theory question: Consider a non-homogeneous second-order differential equation for which we would like to use the method of undetermined coefficients. We have some non-homogeneous term $d(x)$, and so we use the table in Figure 1 to inform us about the form of our particular solution. What if our guess for the particular solution is equal to a solution in the complementary solution? What if the complementary solution has repeated roots? Is the method of undetermined coefficients guaranteed to provide us with a correct particular solution for any second-order differential equation?

Solution.

If our guess is equal to to a solution in the complementary solution, we multiply our guess by t.

In the case of repeated roots in the complementary solution, we cannot multiply our particular solution by t, because this is also a solution (recall the general solution to a characteristic solution with repeated roots). Therefore, we must multiply guess by t again (introducing a t^2 variable). Any time our guess for the particular solution is linearly dependent on the fundamental set of solutions, we can multiply by t again.

No, the method of undetermined coefficients is not guaranteed to provide us with a correct particular solution. Undetermined coefficients will usually fail on equations with variable coefficients.

[3] product rule again!