Monte Carlo simulation with scattering medium and specular reflection

ME 964-002 Final Project Report
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2025/05/06

Introduction:

For the final project, I developed a python-based Monte Carlo simulation, source code on GitHub (https://github.com/nickmedwards/me964_scattering_monte_carlo), that models a trough concentrating solar power (CSP) receiver and the air as a participating medium. This is a numerical approximation of the radiative transfer equation. The Solana Generation Station is a trough CSP plant that exists in Gila Bend, Arizona and will be the basis of the physical constants for the simulation. The receiver has two gray surfaces with specular reflection: a perfect cylinder and parabola, representing a heat transfer fluid (HTF) pipe and the concentrating mirror. The tube is centered at the origin of the simulations xz plane, and therefore the parabolic mirror must have its focus at the origin of the xz plane and is of the form below.

$$z = m \cdot x^2 - \frac{1}{4 \cdot m}$$

Where m will be known as the parabolic parameter for this report. The light incident on the receiver is emitted from a rectangle placed at the modeled length of air that is dense enough for Rayleigh scattering, given in the equation below.[1]

$$L_{actual} = \frac{L_{air}}{\sin\left(h + \frac{244}{165 + 47 \cdot h^{1.1}}\right)}$$

Where h is the apparent altitude in degrees, and L_{air} is 50 [km].[2] The altitude angle is measured from the xy plane rather than the z axis. The atmospheric extinction coefficient is a function of wavelength. In the right figure, a measurement of the extinction coefficient at Zilnez Mesa, Arizona is given

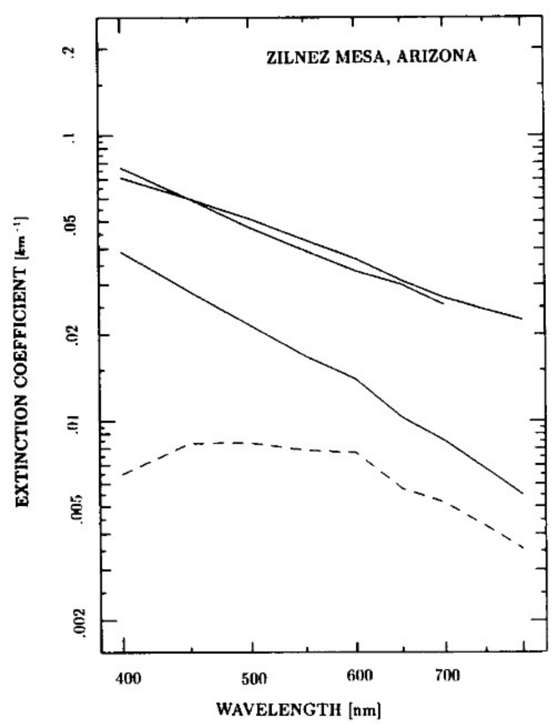


Fig. 5. Examples for extinction coefficients measured in the arid regions of Arizona. The telephotometer was located at an elevation of approx. 2000 m, surrounding mountains were used as targets. The lower curve is representative for extinction coefficients under normal conditions, disturbances by a severe smog episode in Los Angeles or forest fires caused an increase in extinction coefficients by a factor of two (upper curves). The dotted curve is the extinction coefficient of the particles only.

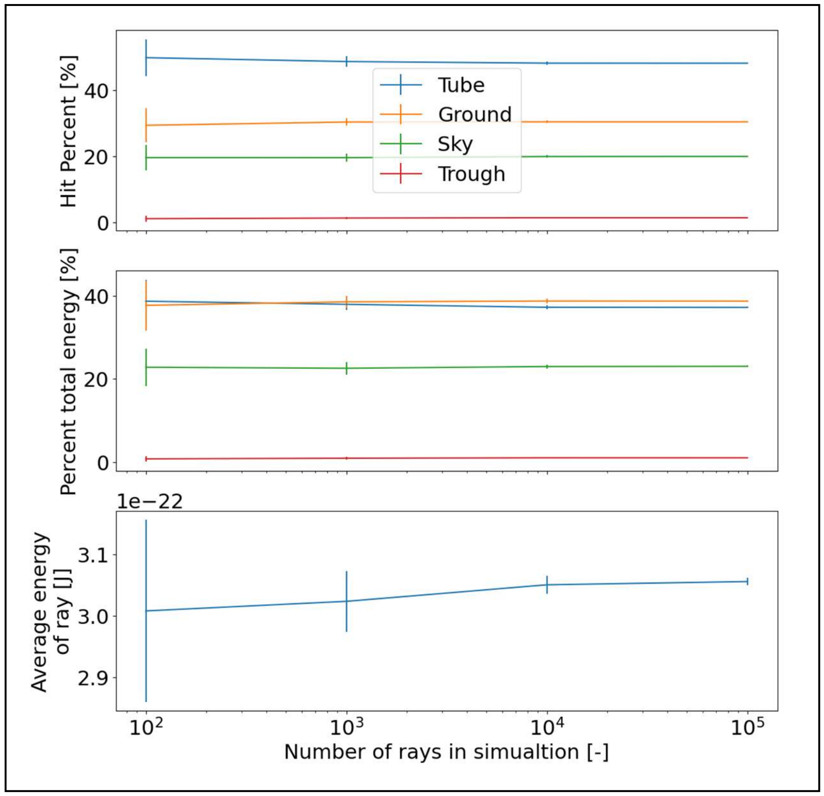
with the captioned with Fig. 5. from the paper. [3] The direction of rays emitted with azimuth from the south in the simulations coordinates is the positive y direction and polar angle equal to the solar zenith angle

This simulation models atmospheric absorption and scattering as separate phenomena, differing slightly compared to the procedure laid out in Modest Chapter 20. The air is assumed to be in radiative equilibrium so all absorbed is reemitted with polar and azimuthal symmetry. All scattering events are assumed to follow the Rayleigh scattering phase function. A decade of aerosol characterization at Mt. Lemmon, Arizona concluded that the single scattering albedo remained effectively constant for over a decade at .937[4] and is assumed to not be a function of wavelength for simplicity.

Ray tracing algorithm:

The core of radiation exchange Monte Carlo simulations are the ray tracing algorithms. While in class, the radiative transfer equation has been in terms of the optical length however, Modest Chapter 20 frames it in normal lengths. Compared to Modest where l_β is randomly chosen and then absorption and scattering is determined by the single scattering albedo. This simulation randomly choses l_σ and l_κ based on the calculated κ_λ and σ_λ from the measured β_λ. To keep the simulation manageable a ground length, l_g, or sky length, l_s, are calculated to stop rays going too low or too high in the simulations coordinate system. The ground length is calculated if the ray has a negative z component of its direction vector and by solving the length along the ray to a ground level of -15 [m] to accommodate all troughs. The sky length is calculated by approximating an azimuthally symmetric surface around the simulations coordinate system at L_{actual}, depending on the polar angle. The hittable geometry, the tube and trough, have the same length, L_t, of 200 [m] in y direction, and the tube has a radius of 1 [m], and the trough has a width, W_t, of 30 [m], and are both centered in the simulations coordinate system. The lengths to the tube and the trough are calculated by solving the system of equations for the ray and the hittable geometry in the xz plane because the tube and trough don't change in the y direction.

With all lengths calculated, the minimum is determined. If the shortest length is to the sky or ground, record which and end the algorithm. If the shortest length is the scattering length, record a scattering event and create a new ray at the scattering location in the random scattering direction. If the shortest length is the absorption length, record an absorption event and create a new ray at the absorption location in the random emission direction. If the shortest length is to the tube, check if the ray's absorbed. If it is then record the absorption and end the algorithm, if not then specularly reflect off the tube. If the shortest length is to the trough, check if the ray's reflected. If it is then specularly reflect off the trough, if not then record the absorption and end the algorithm. In total the algorithm returns the wavelength of the ray because all interactions are gray and don't change the wavelength, the final absorption location, the number of tube and trough reflection events, and the number of scattering and absorption events in the air.



Convergence Study:

To determine the number of rays to use in the parametric study, a convergence study was conducted with solar zenith (polar) angle of π/8, parabolic parameter, m of .033, and both the trough reflectance and tube absorptance of .98. The numbers of rays considered are 10^2, 10^3, 10^4, 10^5. As seen in the figure, 10^4 are enough rays to converge.

Parametric Study:

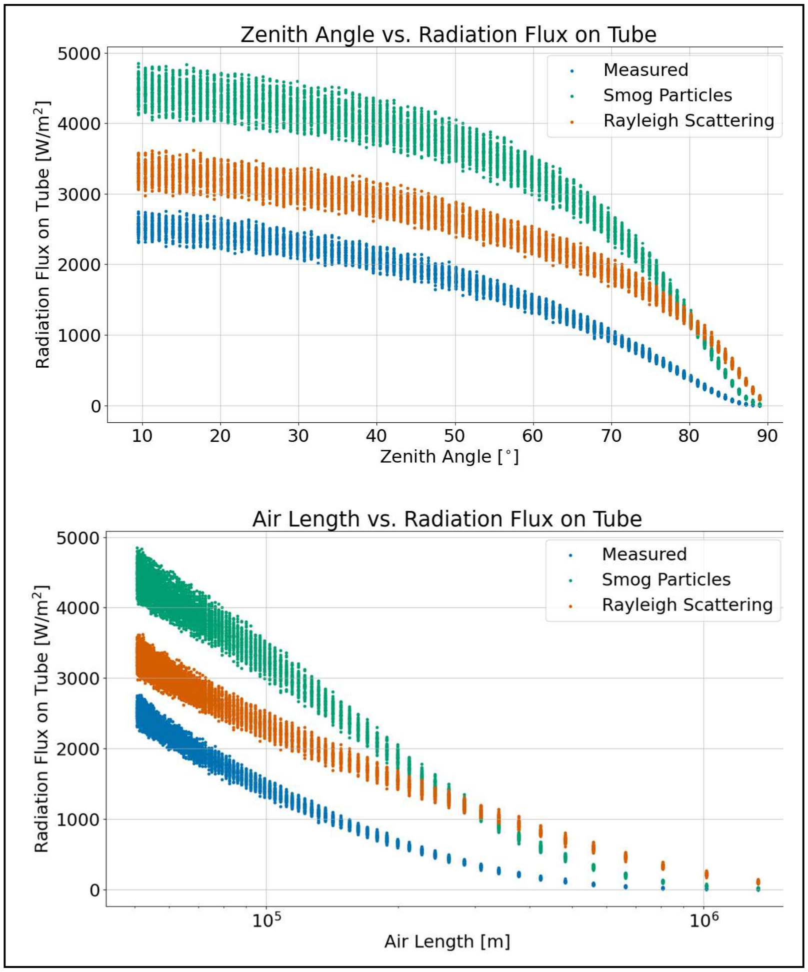
In this project, a parametric study of this system is conducted according to the table below with a total of 17472 Monte Carlo simulations:

Parameter	Values
Extinction Coefficient, β_λ $[km^{-1}]$	The lower and dotted curves from Fig. 5. and the difference of those curves for Rayleigh scattering.
Zenith Angle [°]	[9.48 to 89] with 91 steps. 9.48 is the highest the sun gets in the sky at Solana.
Parabolic Parameter, m	[.06, .033, .025, .0175]
Trough Reflectance [-]	[1.0, .98, .96, .94]
Tube Absorptance [-]	[1.0, .98, .96, .94]

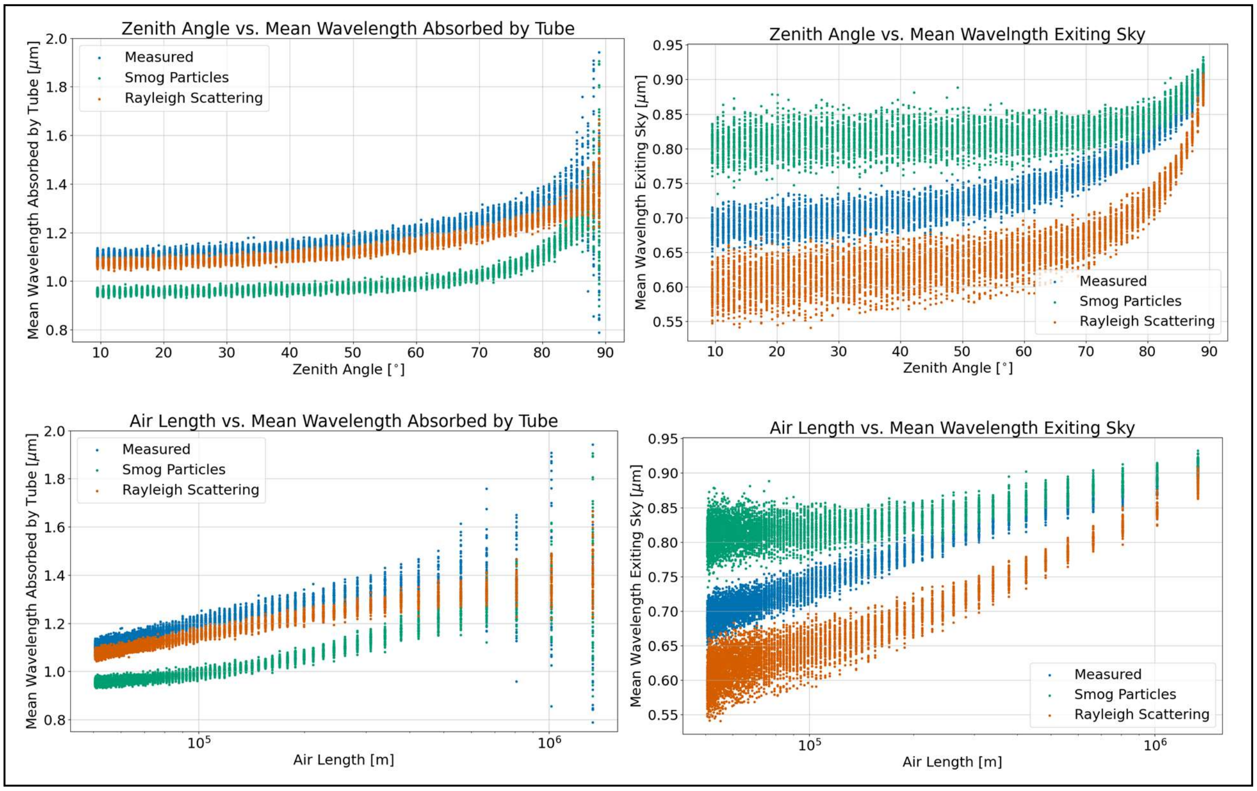
Results of parametric study:

One of the main physical properties of concern for CSP plants is the heat flux on the HTF tube. This simulation calculated the radiative heat gain on the tube rather than the net flux as the tube was not emitting. This simulation is also ignoring convection on the tube and conduction within the tube. While for zenith angles below about 54°, the incident radiation wouldn't have azimuthal angles directly south because the sun wouldn't be at noon, but for simplicity that azimuth was maintained because the amount of air could be mapped to other solar locations in future work. I've found a comparison of zenith angle and air length to be an interesting view at this simulation. The absorptances and reflectances did not span a large enough range to see significant differences in flux and could be explored in future work. The parabolic parameter didn't affect the radiation flux significantly either however this would show up if convection was added as the geometry would change the air flow around the tube or if a technoeconomic analysis was done and the amount of material was considered.

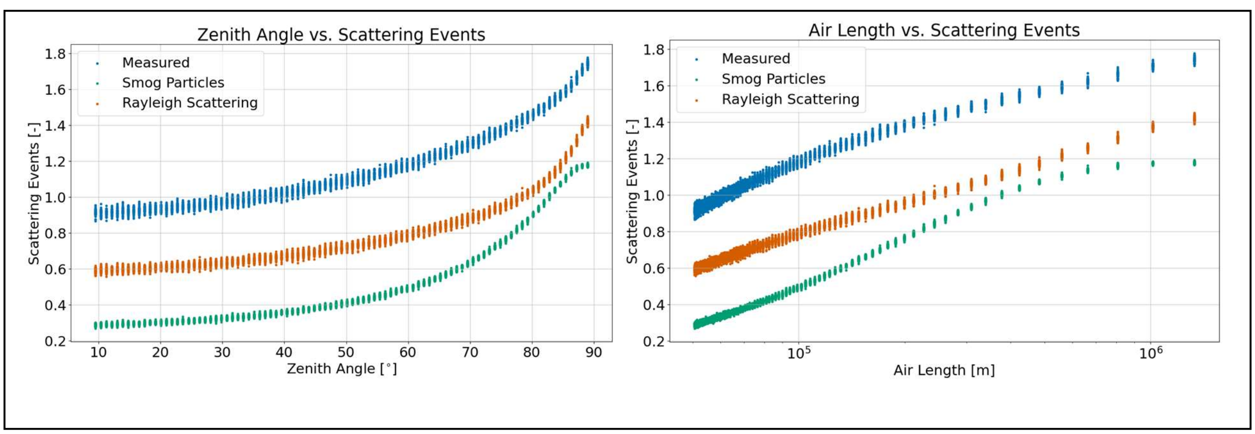
In the figure to the right, the radiation flux on the tube vs the sun's zenith angle and the length of air simulated between the sun and the trough. As expected, the flux goes down as more air affects the light traveling from the sun. The air length plot shows a much more linearizable function of the flux. The effects on the simulated fluxes of smog particles and Rayleigh scattering extinction coefficients combine to the dampened simulated flux for the measured extinction coefficients. A radiation flux measurement for a trough plant could not be found in time to validate this result.



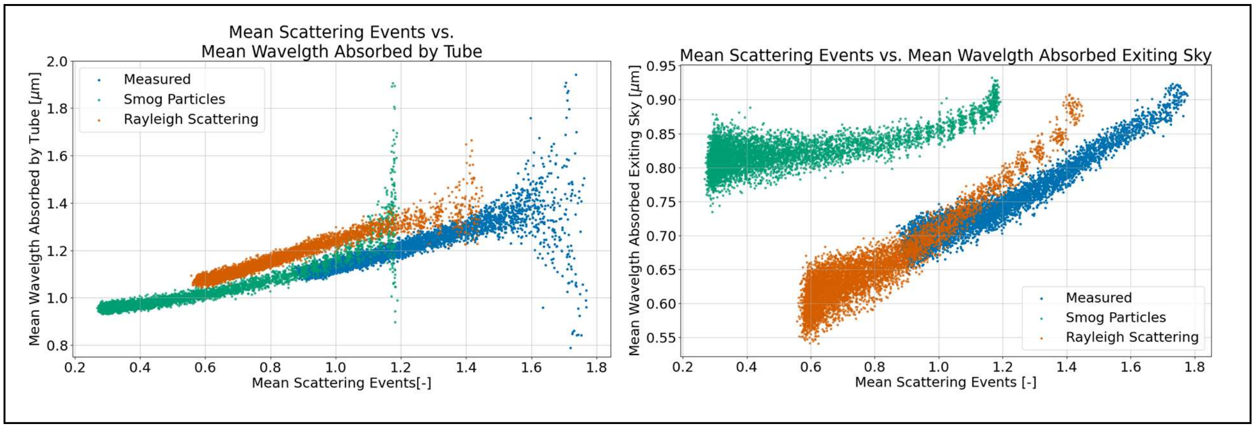
The zenith angle and air length should increase the mean wavelengths that are absorbed by the sky and tubes as more air can scatter the rays. As seen in the figure below, these relationships are reproduced, while not validated by measurements. The air length plots again show more linearizable functions. Because of the differences in how the smog particles and Rayliegh particles scatter, the wavelengths are affected differently.



The zenith angle and air length should also increase the mean scattering events per ray in the simulation for the same reason the mean wavelength should increase. This also could not be validated by measurement but is demonstrated in the figure below.



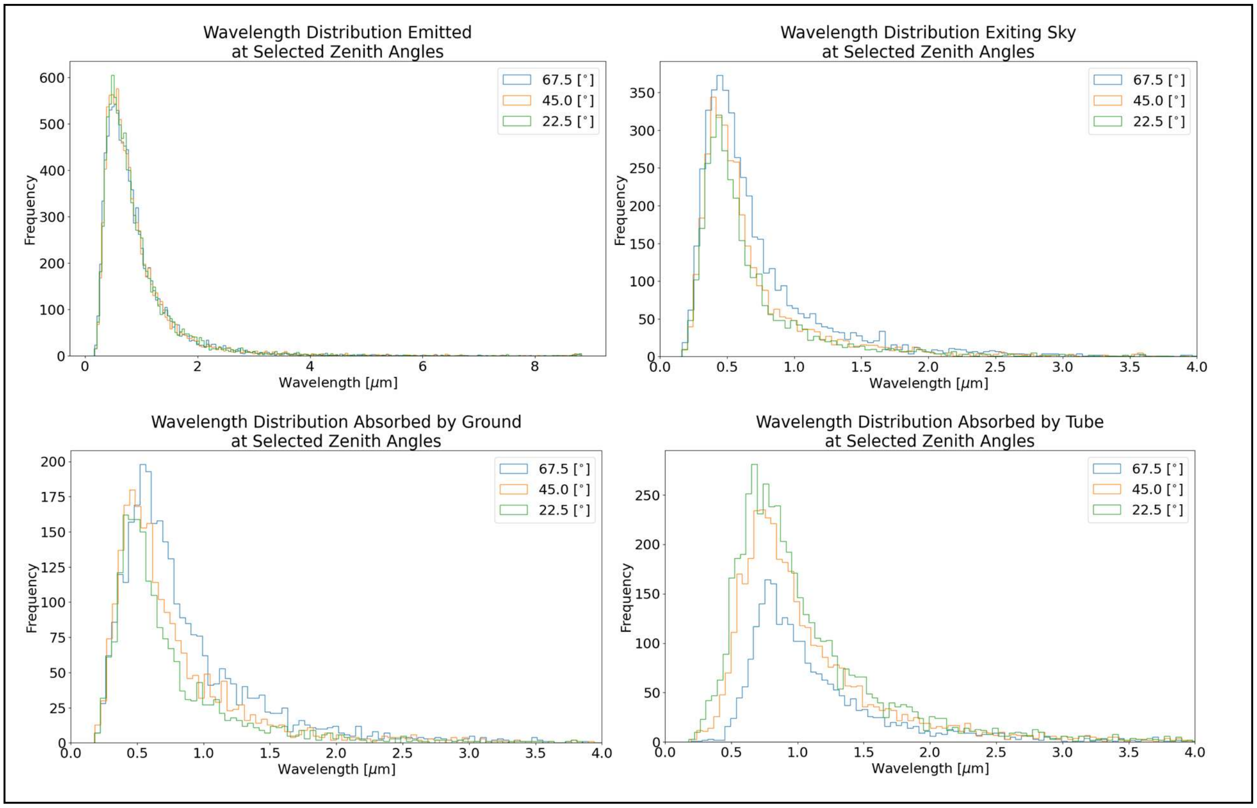
Another look at the parametric study is the mean wavelengths of the tube and the sky compared to the mean scattering events. Again, the pattern of increasing mean wavelength with the mean number of scattering events is demonstrated



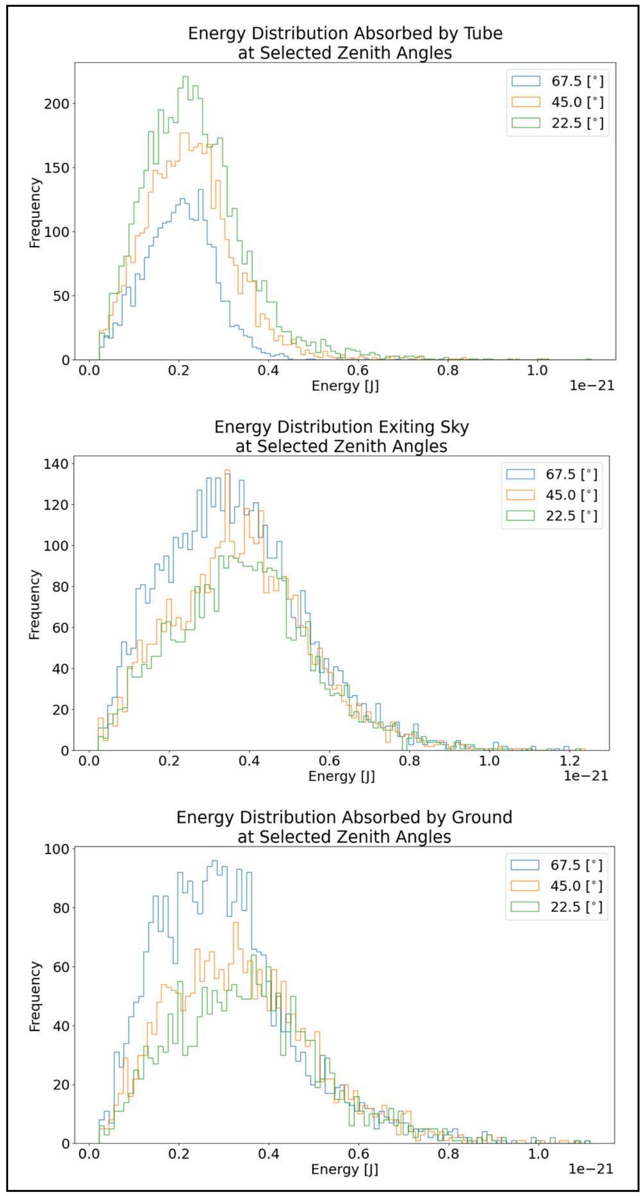
This simulation could be improved by treating the different extinction coefficients differently because the particles occupy different sections of the atmosphere.

Distributions from select zenith angles:

For zenith angles 22.5, 45.0, 67.5 [˚], distributions for the wavelengths emitted by the sun, and for the wavelengths and energies absorbed by the tube, sky, and ground. The trough was ignored because the reflectance is very high.

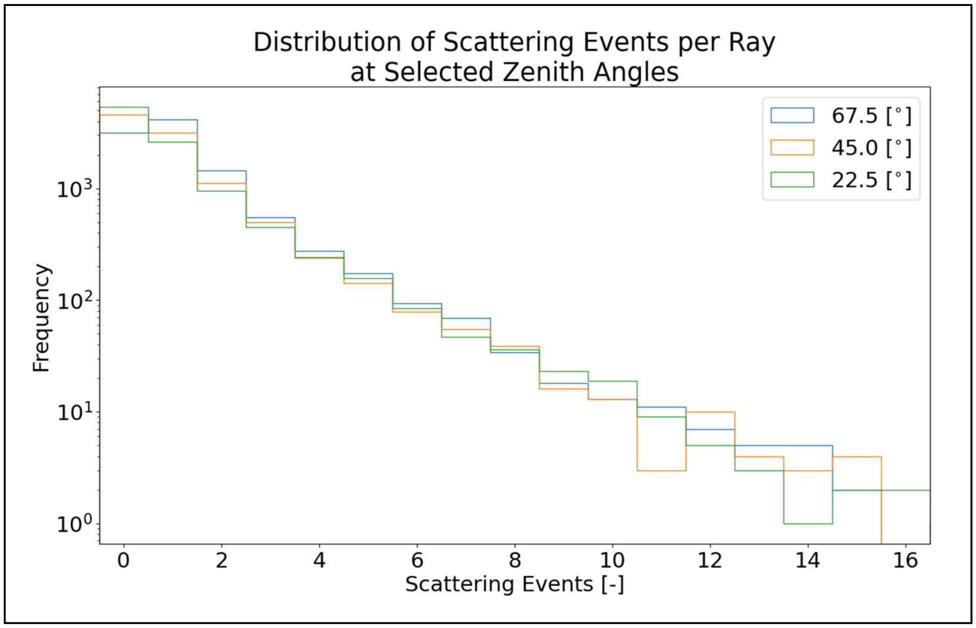


In the figure above, in the top left, the invariance in the input wavelength distribution is a blackbody and that the number of rays is sufficient to reproduce the distribution. The other plots are limited to 4 [μm] to show the detail. In the top right and bottom left, the peak of the sky and ground distributions increase with the zenith angle. In the bottom right plot, the minimum wavelength absorbed effective increases from .2 [μm] to .5 [μm] showing how dramatic the effect of scattering can be on solar energy systems.



In the figure to the right, the energy distributions of the tube, sky, and ground are shown. These plots more simply show how scattering effect solar energy systems. The frequency of rays with energy greater than 0.4e-21 [J] absorbed by the tube is effectively zero at the higher zenith angle. While as seen in the sky and ground plots more low energy rays are absorbed because they have a higher chance of being scattered away to no longer intersect the trough or the tube.

A final figure on how scattering effects rays emitted by the sun is below, showing the distribution of scattering events of the rays at these zenith angles. The key part to note is that at 67.5 [˚] rays are scattered once more than never. Another interesting note is that the simulation produces a line in log space. While this is speculation, it seems that at least in small regimes like the solar spectrum the count distribution would be linear. In a literature review for previous class in two-phase flow, bubble size distributions were linear within certain diameter size regimes.



Conclusion:

I developed a Monte Carlo simulation code that models a trough CSP receiver in Arizona with air participating. The radiation flux gain from the sun was calculated for several parameters. The main effects discussed are the zenith (polar) angle of the sun, and tandem the amount of air that the rays travel through, and the wavelength extinction coefficient used. The extinction coefficient functions include a measurement from Arizona, and calculated contributions from smog particles and Rayleigh scattering. While simulated radiative flux was not able to be validated, the qualitative effects of scattering are demonstrated.

Acknowledgements:

Thank you, Prof. Tervo, for putting this class together. I have been interested in how light works for a decade, and this class brought it into the world of engineering for me in a very fun way.

References:

[0] Modest, “Radiative Heat Transfer”. By 0, I mean that I needed this to even get the idea off the ground. I tried to get all the equation references in my hand-written notes attached.

[1] K. Pickering, “Reidentifcation of some entries in the Ancient Star Catalog” The International Journal of Scientific History. vol. 12, Sep 2002. Fn. 39

[2] National Oceanic and Atmospheric Administration. (n.d.). *Layers of the atmosphere*. Layers of the Atmosphere. https://www.noaa.gov/jetstream/atmosphere/layers-of-atmosphere

[3] H. Horvath, “Spectral extinction coefficients of background aerosols in Europe, North and South America: A comparison,” Atmospheric Environment. Part A. General Topics, vol. 25, no. 3–4, pp. 725–732, Jan. 1991, doi: 10.1016/0960-1686(91)90071-E. Fig. 5.

[4] R. Matichuk et al., “A Decade of Aerosol and Gas Precursor Chemical Characterization at Mt. Lemmon, Arizona (1992 to 2002),” Journal of the Meteorological Society of Japan, vol. 84, no. 4, pp. 653–670, 2006, doi: 10.2151/jmsj.84.653.

Hand written notes:

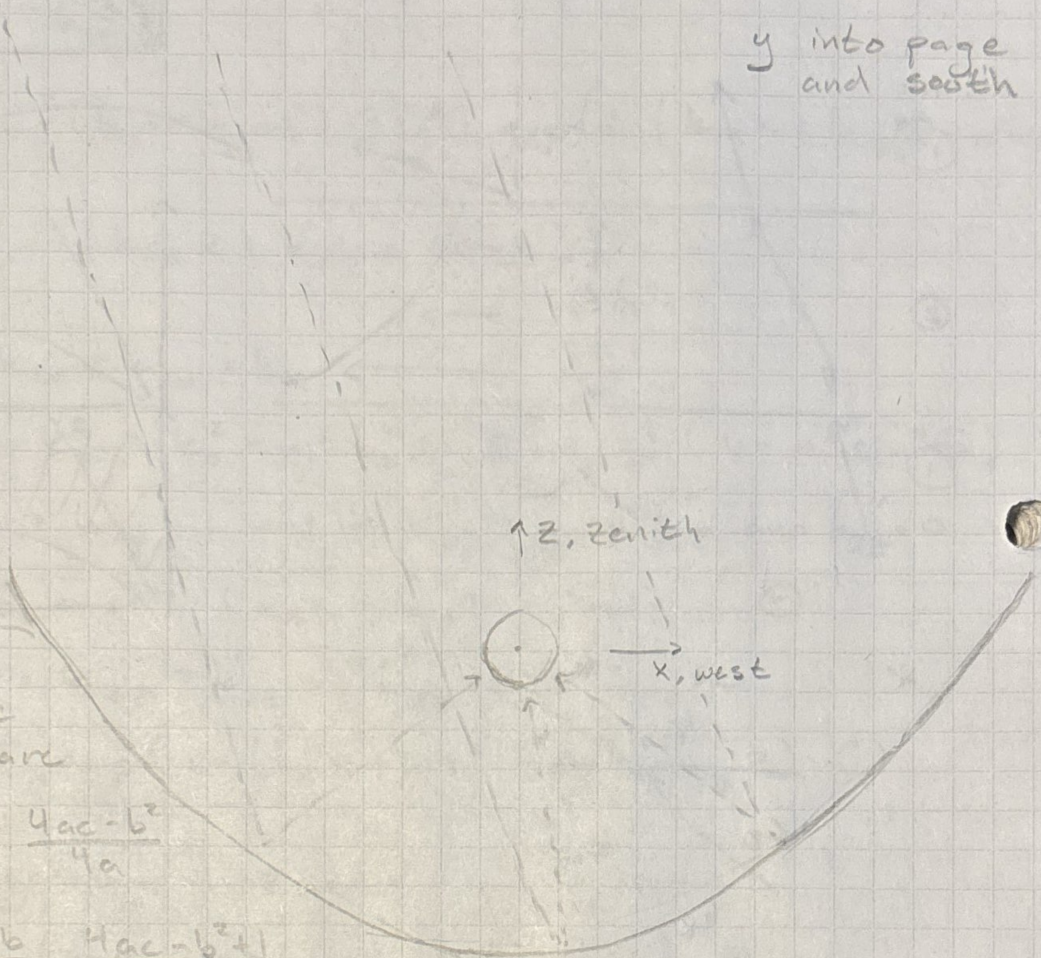
My notes are attached for completeness and if you are curious but it’s mostly fixing geometry mistakes.

ME 964 Thermal EM Radiation 2025/04/24

Final Project

Develop Monte Carlo simulation with scattering medium and specularly reflective surface.

Simulation of CSP trough, so geometry includes parabolic trough and cylindrical collector.
Using Solana Plant in Gila Bend, Az for sun positions.



$f(x) = ax^2 + bx + c$

completing square

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

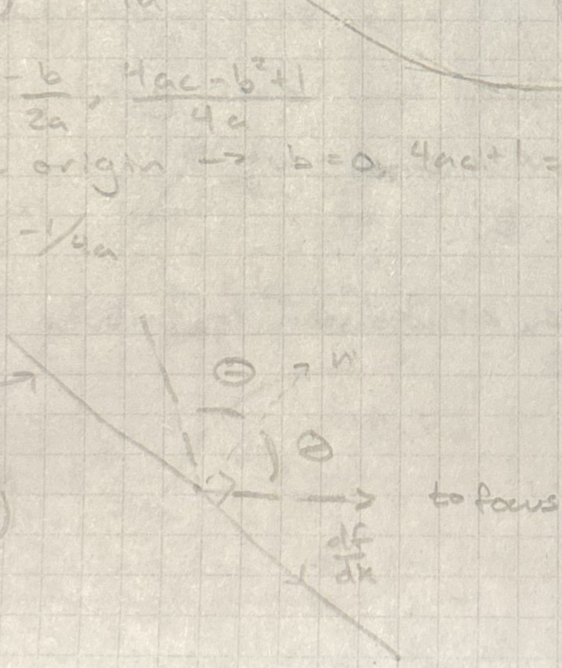
$\therefore$ focus @ $\left(-\frac{b}{2a}, \frac{4ac - b^2 + 1}{4a}\right)$

for focus @ origin $\rightarrow$ $b = 0$, $4ac + 1 = 0 \rightarrow c = -1/4a$

$$f(x) = ax^2 - 1/4a$$

$$\frac{df}{dx} = 2ax$$

$$n = (x, -1/2ax)$$

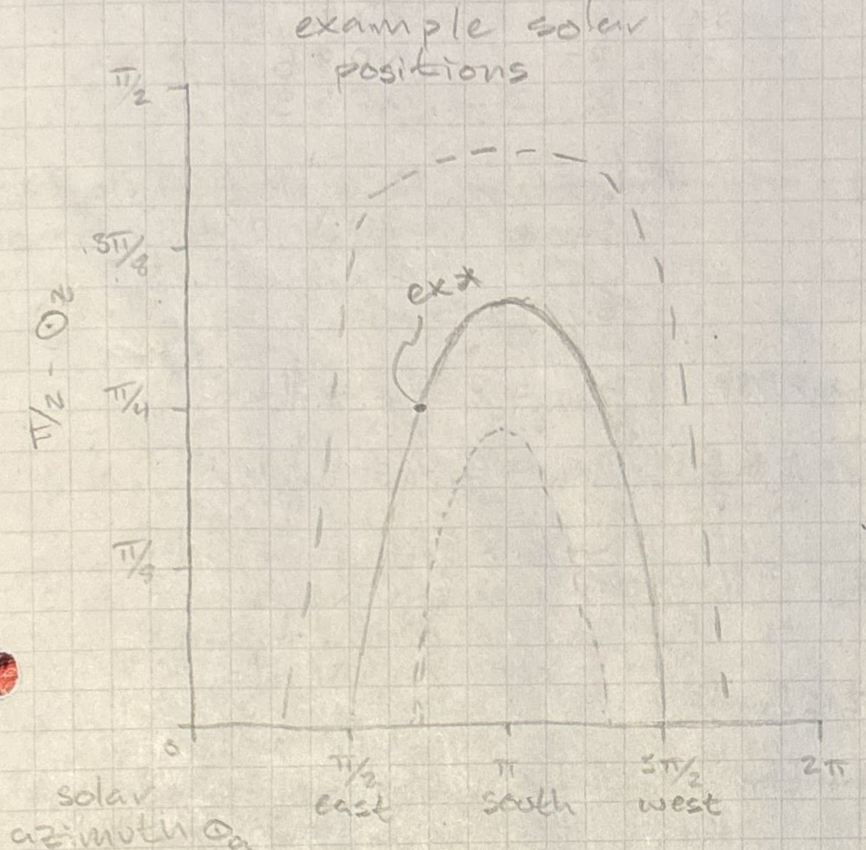


emission from sun thru atmosphere

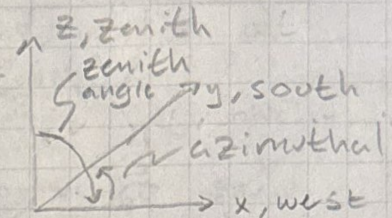
considering thermo, meso, strato, and troposhere layers

effective height at zenith 600 km = L_{air} (noaa.gov)

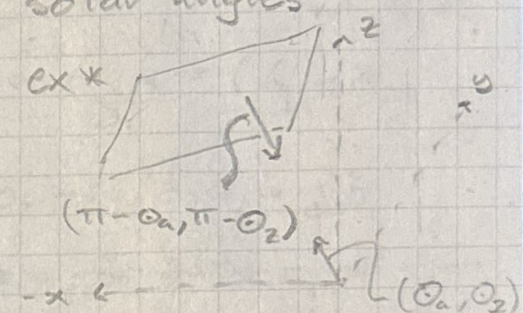
air length based on sun zenith Θ_z $L_{air}/\cos\Theta_z = L'_{air}$



sun position is normal to surface emitting photons



angle names for solar angles



emission location → random location on rectangle

direction → assuming perfectly colomated light hitting atmosphere
∴ $\pi - \Theta_a$, $\pi - \Theta_z$

wavelength → sample $E_{b\lambda}$ (5777 K)
but only take $400\,nm \leq \lambda \leq 2\,\mu m$
→ noaa atmospheric window

Rayleigh scattering and absorption efficiency → $Q_{sca} = \frac{8}{3}\left|\frac{m^2-1}{m^2+2}\right|^2 x^4$ (11.45)

$\propto 1/\lambda^4$ (11.47)

$\sigma_{s\lambda} = \pi a^2 N_T Q_{sca} \rightarrow \sigma_{sca} \propto 1/\lambda^4$

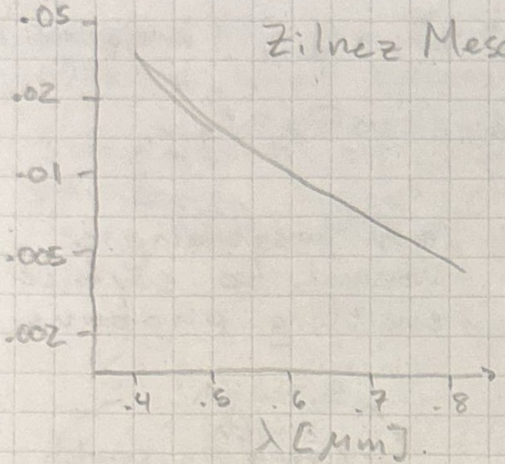
$Q_{abs} = -4\,\mathfrak{I}\left\{\frac{m^2-1}{m^2+2}\right\} x$ (11.46)

$\kappa_\lambda = \pi a^2 N_T Q_{abs} \rightarrow \sigma \propto 1/\lambda$

$\omega_\lambda = \frac{\sigma_{s\lambda}}{\kappa_\lambda + \sigma_{s\lambda}}$ ranges .9 - .96 (Ohta, S 1996, Mt. Lemmon Az)
↑ monthly average*

extinction coefficient, $\beta_\lambda = \sigma_{s\lambda} + \kappa_\lambda$ measured w/ U-Vienna telephotometer

β_λ [km^{-1}] (Horvath, H 1990) ←



Zilnez Mesa, Az

λ [μm]	β_λ [km^{-1}]
.4	.04
.5	.023
.6	.014
.7	.0085
.8	.0055

* better from (Matichok, R 2006) same location Mt. Lemmon

@ $\lambda = .5$ μm

base $\sigma_{s\lambda}, \kappa_\lambda$ on $\omega_{.5} = .937$

$\sigma_{s\lambda} = 8.57E{-3}$ [km^{-1}]

$\sigma_{s\lambda}(\lambda^4) = \dfrac{\hat{\sigma}_{s\lambda}\lambda^4}{.5^4} \to l_\sigma = \dfrac{1}{\sigma_\lambda} \ln \dfrac{1}{R_\sigma}$ (20.23)

$\kappa_\lambda = 5.66E{-4}$ [km^{-1}]

calculated from $PM_{2.0}$ measurements didn't find long term change in $\omega = .937$

$\kappa_\lambda(\lambda) = \dfrac{\hat{\kappa}_\lambda \lambda}{.5} \to l_\kappa = \dfrac{1}{\kappa_\lambda} \ln \dfrac{1}{R_\kappa}$ (20.18)

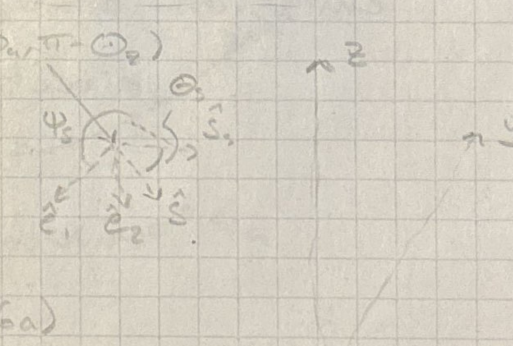
$\hat{\sigma}_{s.5} = .023\,\omega_{.5} = .0216$ [km^{-1}]

$\hat{\kappa}_{.5} = .023 - \hat{\sigma}_{s.5} = .0014$ [km^{-1}] ← b/c of measurement technique only model strato, troposphere $L_{air} = 50$ km

Rayleigh scattering

$\frac{1}{4\pi}\int_{4\pi} \Phi(\hat{s}_s, \hat{s})\, d\Omega_s = 1$

$\Phi(\hat{s}_s, \hat{s}) = 3/4(1 + \cos^2\Theta_s) = \Phi(\Theta_s)$



$\varphi = \int_0^\pi \Phi(\Theta_s) \sin\Theta_s\, d\Theta_s = 2$

$R_{\psi_s} = \int_0^{\psi_s} \varphi\, d\psi_s \Big/ \int_0^{2\pi} \varphi\, d\psi_s = \psi_s/2\pi$ (20.26a)

$R_{\Theta_s} = \int_0^{\Theta_s} \Phi(\Theta_s) \sin\Theta_s\, d\Theta_s / \varphi$ (20.26b)

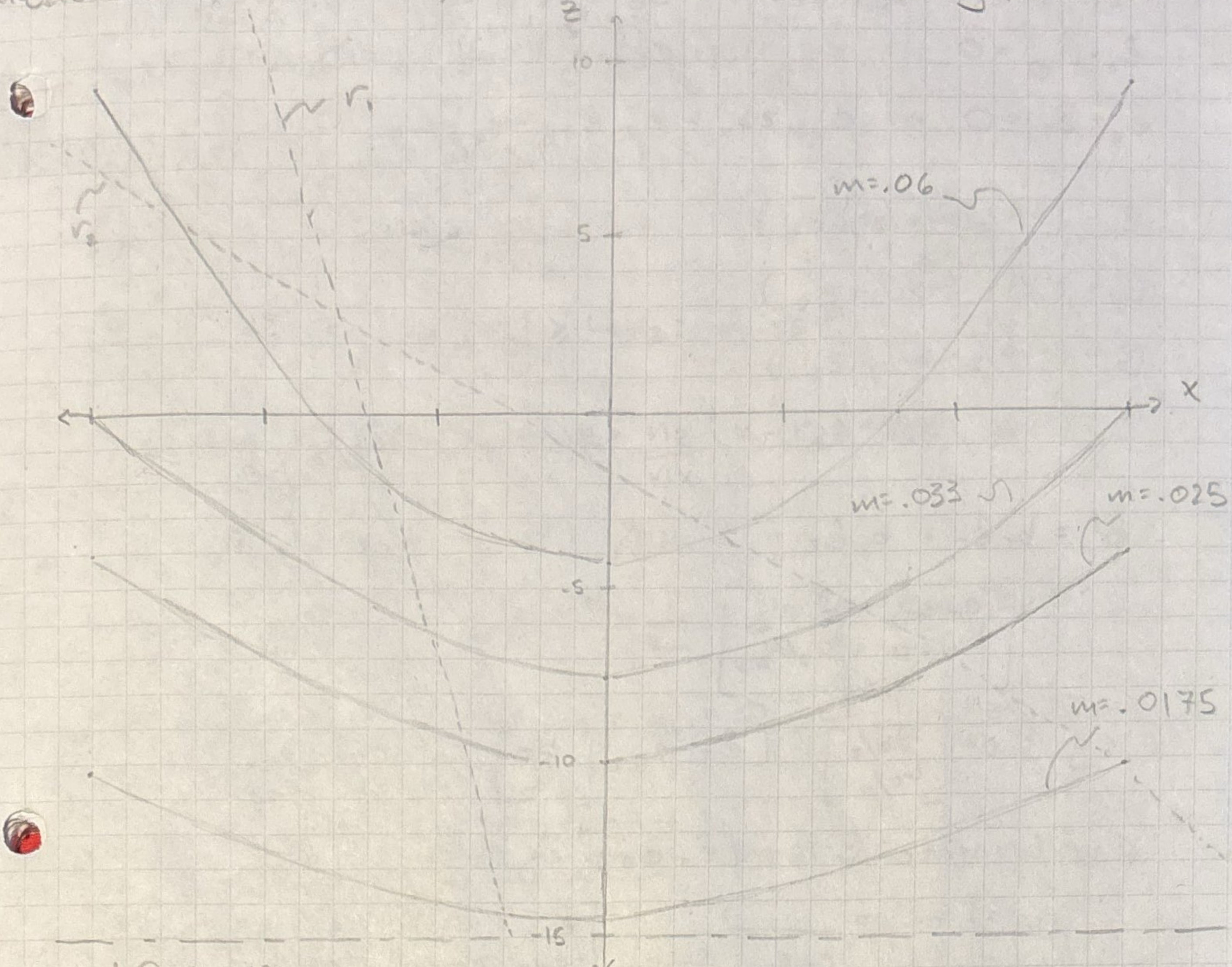
emission after absorption

$\int_0^{\Theta_s} \Phi(\Theta_s) \sin\Theta_s\, d\Theta_s = -3/4\left(1/3\cos^3\Theta_s + \cos\Theta_s\right)\Big|_0^{\Theta_s}$

assumining polar/azimuthal symmetry

is invertible ∴ for $0 \le \Theta_s \le \pi$ $\quad R_{\Theta_s} = \dfrac{1 - 3/4(1/3\cos^3\Theta_s + \cos\Theta_s)}{2}$

$R_{\psi_e} = \psi_e/2\pi$; $R_{\Theta_e} = \dfrac{(1 + \cos\Theta_e)}{2}$ (20.14a,c) hitdeck for $z = mx^2 - 1/4m$; for $-15 \leq x \leq 15$ ← renaming parameter a→m



ground @ z = -15

trough has length L_t and parabola w/ parameter m, width W_t

$W_t = 30$

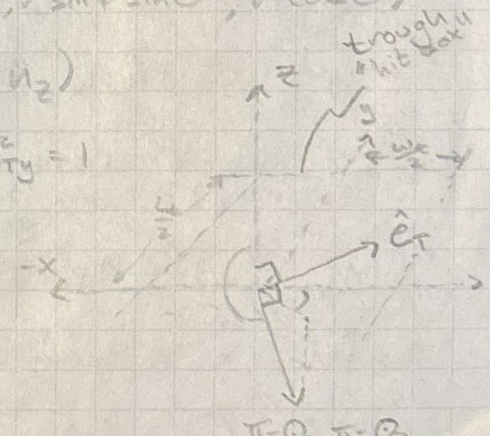
$T \subseteq \mathbb{R}^3$, $-\frac{W_t}{2} \leq x \leq \frac{W_t}{2}$, $-L_t/2 \leq y \leq L_t/2$, $z = mx^2 - 1/4m$

rays emitted from plane w/ normal = -(solar position).

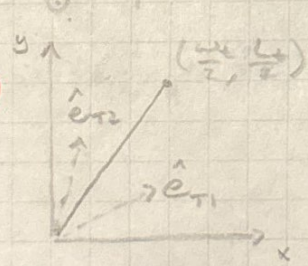
sph2rec: $\mathbb{R}^3 \to \mathbb{R}^3$, $(r, \phi, \Theta) = (r\cos\phi\sin\Theta, r\sin\phi\sin\Theta, r\cos\Theta)$

$\hat{n}_\odot = \text{sph2rec}(1, \pi - \Theta_a, \pi - \Theta_z) = (n_x, n_y, n_z)$

there is some $\hat{e}_T = (e_{Tx}, e_{Ty}, 0)$; $e_{Tx}^2 + e_{Ty}^2 = 1$
that lies in x-y plane



$\hat{n}_\odot \cdot \hat{e}_T = 0 = e_{Tx} \cdot n_x + e_{Ty} \cdot n_y$



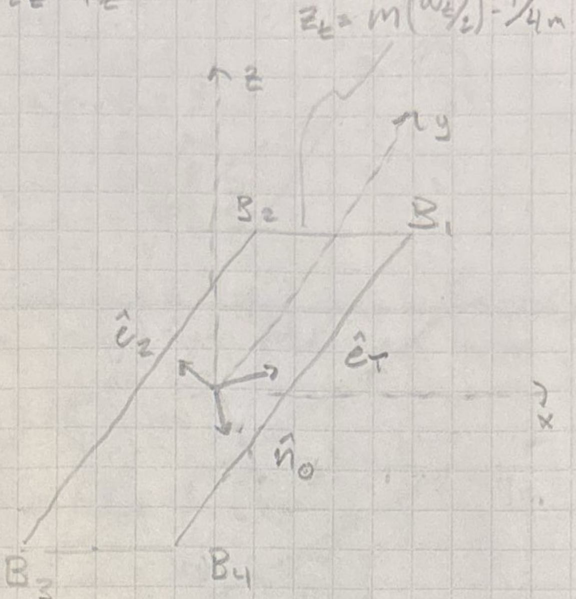
$\hat{e}_{T1} \to e_{Ty}/e_{Tx} < L_t/W_t$

$\hat{e}_{T2} \to e_{Ty}/e_{Tx} > L_t/W_t$

$\hat{e}_z = (e_{zx}, e_{zy}, e_{zz})$; $e_{zx}^2 + e_{zy}^2 + e_{zz}^2 = 1$

$\hat{e}_z \cdot \hat{n}_\odot = 0 \rightarrow e_{zx} n_x + e_{zy} n_y + e_{zz} n_z = 0$

$\hat{e}_z \cdot \hat{e}_T = 0 \rightarrow e_{zx} e_{Tx} + e_{zy} e_{Ty} + e_{zz} e_{Tz} = 0$



$B_1 = (w_t/2, L_t/2, z_t)$

$B_2 = (-w_t/2, L_t/2, z_t)$

$B_3 = (-w_t/2, -L_t/2, z_t)$

$B_4 = (w_t/2, -L_t/2, z_t)$

$$B' = b_T \hat{e}_T + b_z \hat{e}_z = \begin{bmatrix} b_T e_{Tx} + b_z e_{zx} \\ b_T e_{Ty} + b_z e_{zy} \\ b_z e_{zz} \end{bmatrix}$$

eg B_1': $\begin{bmatrix} w_t/2 \\ L_t/2 \end{bmatrix} = [\hat{e}_T \ \hat{e}_z] \begin{bmatrix} b_T \\ b_z \end{bmatrix}$; $\begin{bmatrix} e_{Tx} & e_{zx} \\ e_{Ty} & e_{zy} \end{bmatrix}$

truncate B so $[\hat{e}_T \hat{e}_z]^{-1}$ exists

$$b_1 = \begin{bmatrix} b_T \\ b_z \end{bmatrix} = \frac{1}{e_{Tx}e_{zy} - e_{Ty}e_{zx}} \begin{bmatrix} e_{zy} & -e_{zx} \\ -e_{Ty} & e_{Tx} \end{bmatrix} \begin{bmatrix} w_t/2 \\ L_t/2 \end{bmatrix}$$

* rather than tilt plane, tilt emission, $L_{air}' \gg L_t$

$\odot \subseteq \{B_1, B_2, B_3, B_4\} - \hat{n}_\odot L_{air}'$ ← surface emitting

$\hat{r} = \hat{n}_\odot$ ← direction of emission

$r(l) = \text{sample}(\odot) + l\,\hat{n}_\odot$; > 0

$= (\odot_x + l n_x, \odot_y + l n_y, \odot_z + l n_z)$

hit check assuming no scatter/absorption
↳ $r(l)$ = s/a location + $l \cdot$ (s/a direction)

$m_{xz}\, r(l) = n_z / n_x$

find x intercept → $r_z = \odot_z + l n_z = 0 \rightarrow l = -\odot_z / n_z$

$r_x = \odot_x + l n_x = \odot_x - \odot_z n_x / n_z$

find r_x, r_y @ $r_z = z_t$

$$r_z = \odot_z + l n_z = z_t \rightarrow l = \frac{z_t - \odot_z}{n_z}$$

$$r_x = \odot_x + l n_x = \odot_x + \frac{n_x(z_t - \odot_z)}{n_z}$$

$$r_y = \odot_y + l n_y = \odot_y + \frac{n_y(z_t - \odot_z)}{n_z}$$

also check if direct hit

if $|r_x| \geq w_t/2$ or $|r_y| \geq L_t/2$: miss

else:

if $n_x > 0$:

$$z - z_t = \frac{n_z}{n_x}(x - r_x) \rightarrow z = \frac{n_z}{n_x}(x - r_x) + z_t$$

parabola $\rightarrow z = mx^2 - 1/4m$

hit location $\rightarrow mx^2 - \frac{1}{4m} = \frac{n_z}{n_x}(x - r_x) + z_t$

$$\therefore mx^2 - \frac{n_z}{n_x}x - \frac{1}{4m} + \frac{n_z}{n_x}r_x - z_t = 0 \; ; \; c = \frac{1}{4m} + \frac{n_z r_x}{n_x} - z_t$$

$$x_h = \frac{(n_z/n_x) \pm \sqrt{(n_z/n_x)^2 - 4mc}}{2m} \; ; \text{recheck } r_y$$

else: $x_h = r_x$

$$z_h = m x_h^2 - 1/4m$$

derivative @hitpoint $\frac{dz}{dx} = 2mx \rightarrow \left.\frac{dz}{dx}\right|_{x_h} = 2mx_h$

$$d_h = (1, 2mx_h), \; |d_h| = \sqrt{1 + 4m^2 x_h^2}$$

$$\hat{d}_h = (1/|d_h|, 2mx_h/|d_h|)$$

normal @hitpoint

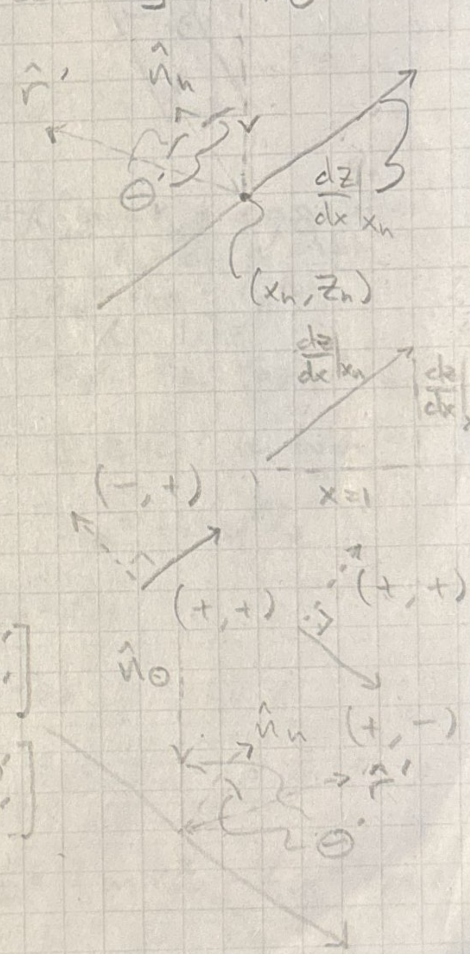
$$\hat{n}_h = \left(\frac{-2mx_h}{|d_h|}, \frac{1}{|d_h|}\right) = (n_{hx}, n_{hz})$$

$$(-\hat{n}_\odot) \cdot \hat{n}_h = -n_x n_{hx} - n_z n_{hz} = \cos\Theta'$$

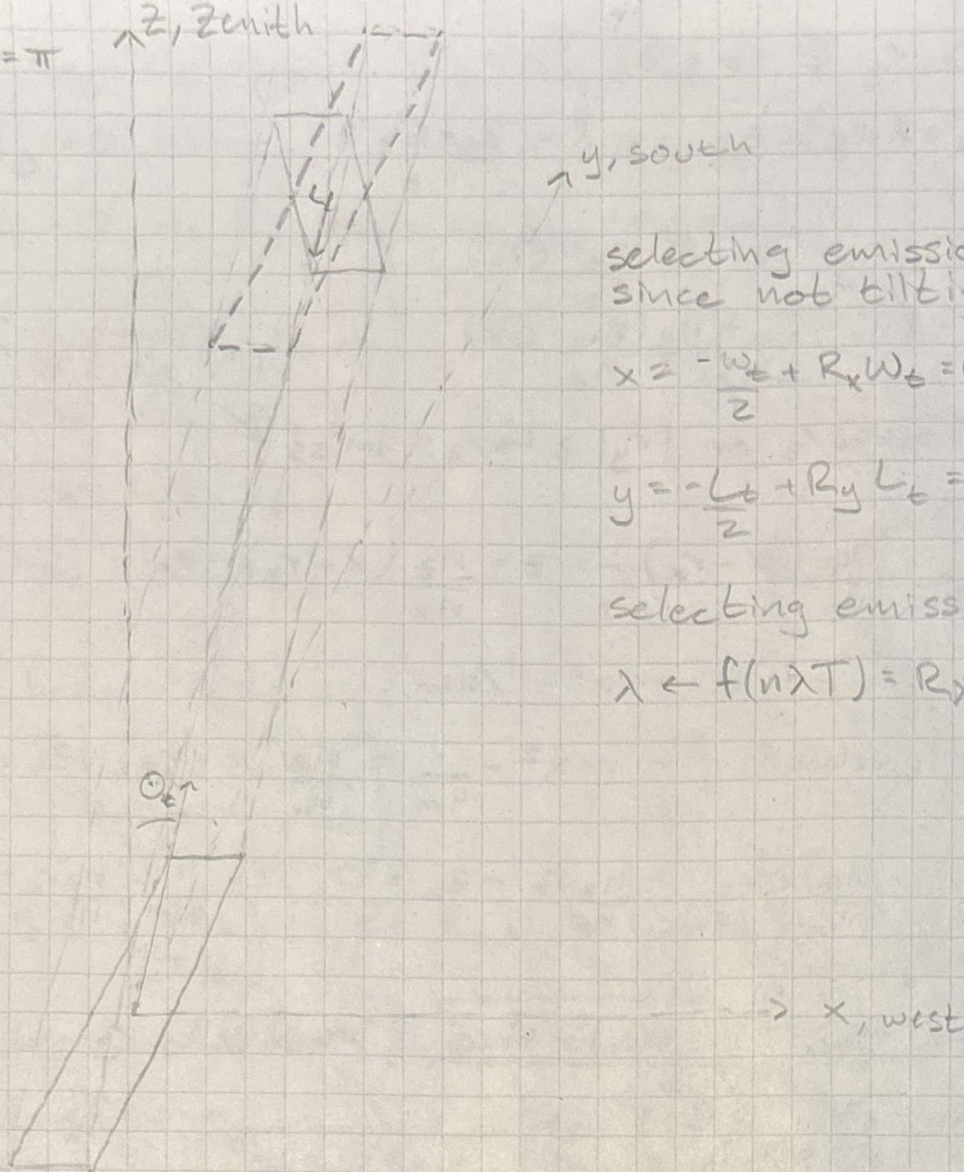
$-\hat{n}_\odot$ \ $\hat{n}_h$	+, +	−, +
$-n_x > n_{hx}$	cw	ccw
$-n_x < n_{hx}$	ccw	cw

$$ccw = \begin{bmatrix} \cos 2\Theta' & -\sin 2\Theta' \\ \sin 2\Theta' & \cos 2\Theta' \end{bmatrix}$$

$$cw = \begin{bmatrix} \cos 2\Theta' & \sin 2\Theta' \\ -\sin 2\Theta' & \cos 2\Theta' \end{bmatrix}$$



set $\Theta_a = \pi$



selecting emission location since not tilting

$$x = -\frac{W_t}{2} + R_x W_t = (R_x - 1/2)\, W_t$$

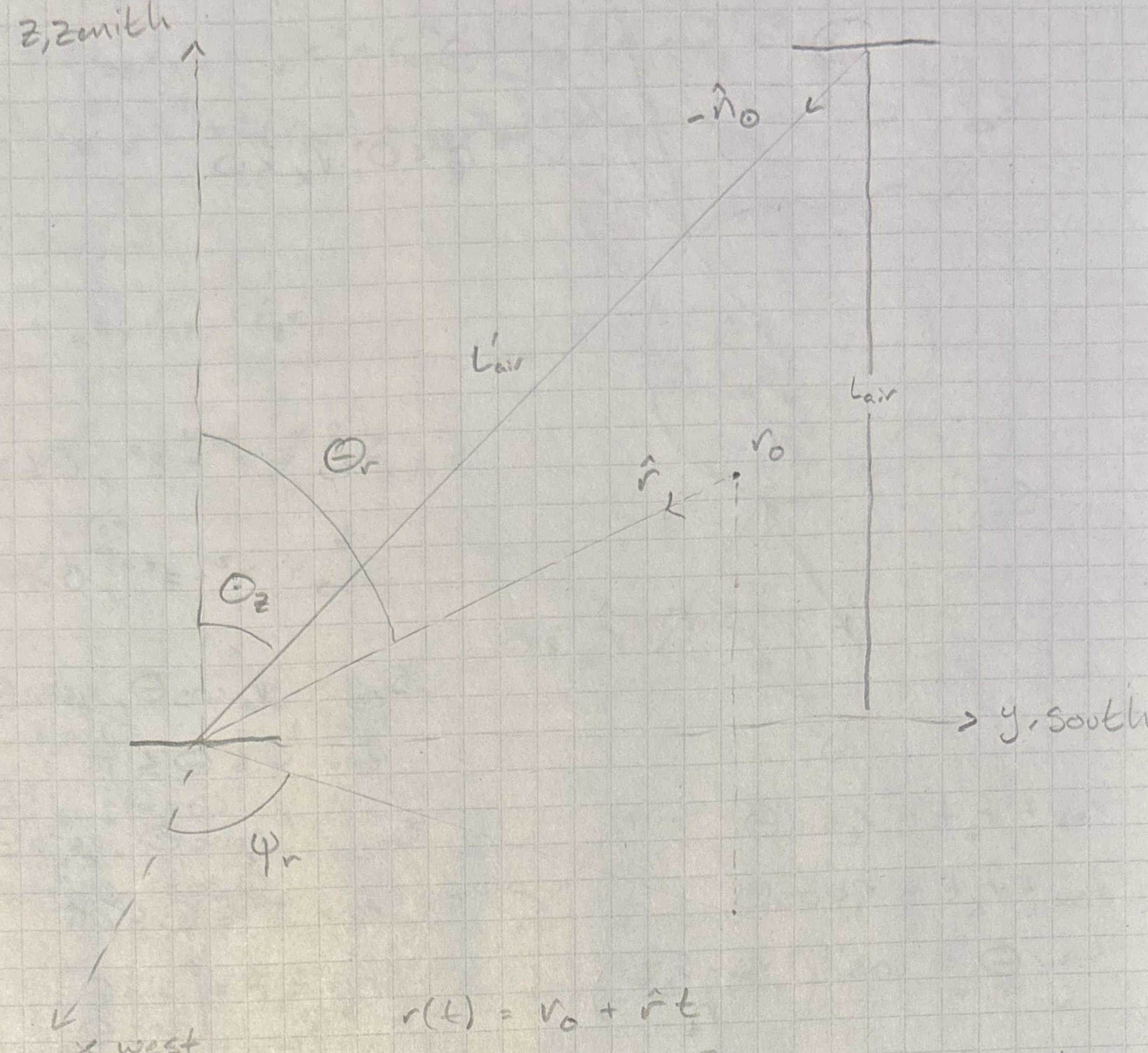
$$y = -\frac{L_t}{2} + R_y L_t = (R_y - 1/2)\, L_t$$

selecting emission wavelength

$$\lambda \leftarrow f(n\lambda T) = R_\lambda$$

$$\beta_\lambda = b_0 + b_1 \lambda + b_4 \lambda^4 \qquad \Delta \in R^{5\times 3},\ b \in R^{5},\ \beta_\lambda \in R^{5}$$

$$= \begin{bmatrix} 1 & \lambda & \lambda^4 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_4 \end{bmatrix} = \Delta b \rightarrow b = \Delta^{+} \beta_\lambda$$



$$r(t) = r_0 + \hat{r}t$$

$$= \begin{bmatrix} r_{0x} + \hat{r}_x t \\ r_{0y} + \hat{r}_y t \\ r_{0z} + \hat{r}_z t \end{bmatrix}$$ ← find x, y position → @ $z = z_t$ ie trough →

$$z_t = r_{0z} + \hat{r}_z t$$

$$t_t = (z_t - r_{0z}) / \hat{r}_z$$

@ $z = 0$ ie tube →

$$t_0 = -r_{0z}/\hat{r}_z$$

$(x, z) = (r_{0x} + \hat{r}_x t,\ r_{0z} + \hat{r}_{0z} t)$
↳ ray in x-z, $t > 0$

$(x, z) = ([-w/2, w/2],\ mx^2 - 1/4m)$

$$r_{0z} + \hat{r}_z t = m(r_{0x} + \hat{r}_x t)^2 - 1/4m$$

$$= m(r_{0x}^2 + 2r_{0x}\hat{r}_x t + \hat{r}_x^2 t^2) - 1/4m$$

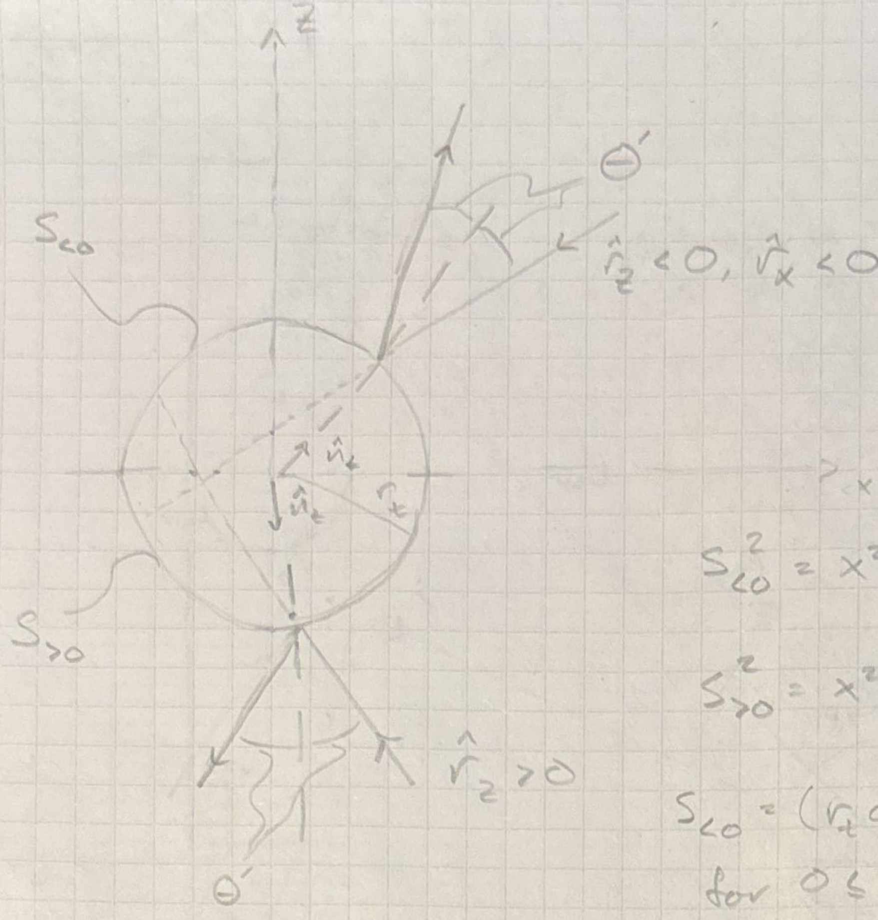
$$0 = m\hat{r}_x^2 t^2 + (2m r_{0x}\hat{r}_x - \hat{r}_z)t + m r_{0x}^2 - 1/4m - r_{0z}$$

$a = m\hat{r}_x^2$, $b = 2m r_{0x}\hat{r}_x - \hat{r}_z$, $c = m r_{0x}^2 - 1/4m - r_{0z}$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $|\hat{r}_x| < \varepsilon$: $r_{0z} + \hat{r}_z t = m r_{0x}^2 - 1/4m \rightarrow t = \dfrac{(m r_{0x}^2 - 1/4m - r_{0z})}{\hat{r}_z}$

$\hat{r}_x t > 0$



$S_{<0}^2 = x^2 + z^2 ; 0 \le x \le r$

$S_{>0}^2 = x^2 + z^2 ; 0 \ge x \ge -r$

$S_{<0} = (r_t \cos\Theta, r_t \sin\Theta)$

for $0 \le \Theta \le \pi$

$S_{>0} = (r_t \cos\Theta, r_t \sin\Theta)$

for $\pi \le \Theta \le 2\pi$

$\frac{r_0 + t\hat{r}}{r_t} = \begin{bmatrix} \sin\Theta \\ \cos\Theta \end{bmatrix}$

$r_{0z} + \hat{r}_z t = r_t \sin\Theta$

$r_{0x} + \hat{r}_x t = r_t \cos\Theta$

$\hookrightarrow \Theta = \cos^{-1}\left(\frac{r_{0x} + \hat{r}_x t}{r_t}\right)$

$t = \frac{r_t \sin\Theta - r_{0z}}{\hat{r}_z}$

for $\hat{r}_z, \hat{r}_x < 0$

$z = \sqrt{r_t^2(1-\cos^2\Theta)} = \sqrt{r_t^2 - r_t^2\cos^2\Theta} ; 0 \le \Theta \le \pi/2$

$n_t(t) = t \cdot \hat{n}_t = \begin{bmatrix} t\cos\Theta \\ 0 \\ t\sin\Theta \end{bmatrix}$

$(\hat{r}_z, -\hat{r}_x)$

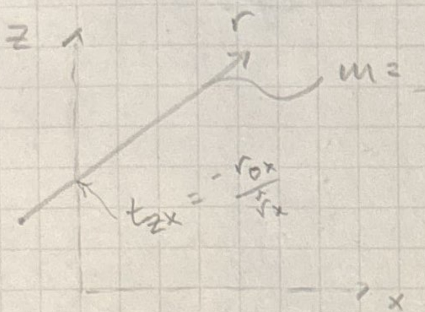
$(r_{ox}+\hat{r}_x t)^2 + (r_{oz}+\hat{r}_z t)^2 = r_t^2$

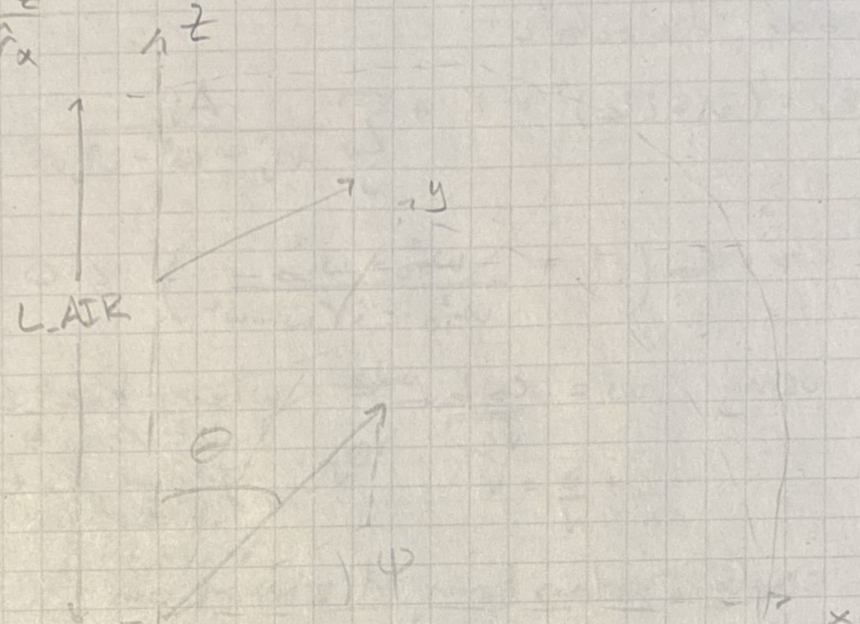
$r_{ox}^2 + 2 r_{ox}\hat{r}_x t + \hat{r}_x^2 t^2 + r_{oz}^2 + 2 r_{oz}\hat{r}_z t + \hat{r}_z^2 t^2 = r_t^2$

$(\hat{r}_x^2 + \hat{r}_z^2) t^2 + 2(r_{ox}\hat{r}_x + r_{oz}\hat{r}_z) t + r_{ox}^2 + r_{oz}^2 - r_t^2 = 0$

$ax^2 - 2ax + 1$

$(r_{ox}+\hat{r}_x t, r_{oz}+\hat{r}_z t)$





$(-\hat{r})\cdot\hat{n} = 0$

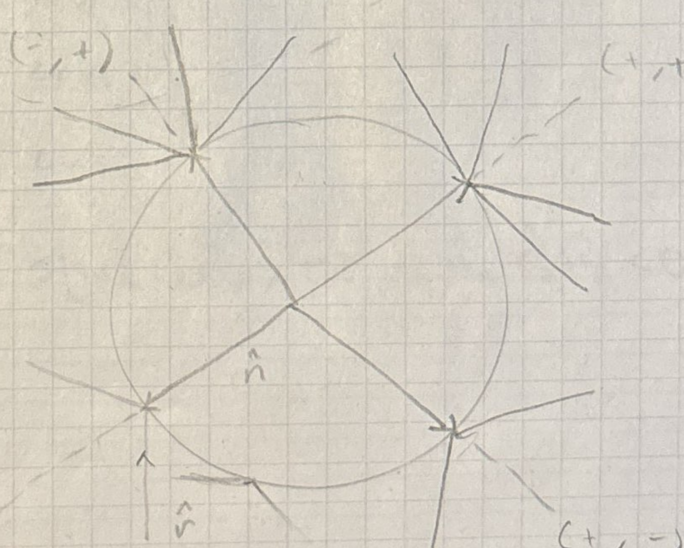
$\|\hat{r}\|\|\hat{n}\|\cos\Theta = 0$

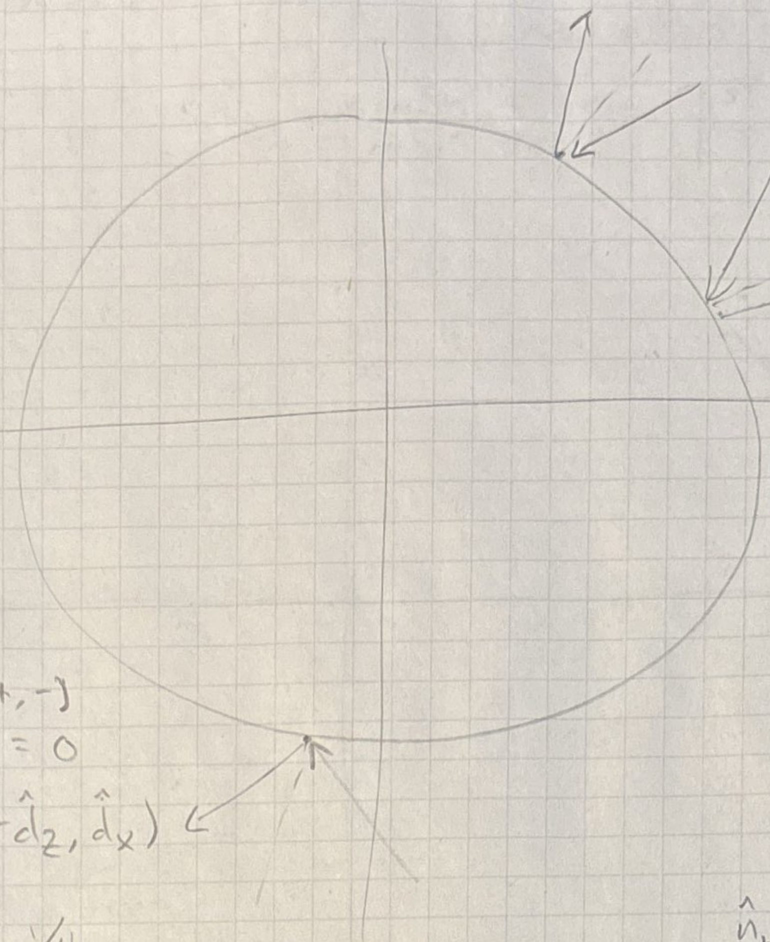
$-\hat{r}_x n_x - \hat{r}_z n_z = \cos\Theta$

Θ_z

$\hat{n}_\Theta$

$\tan(\pi/2 - \Theta_z) = H_t/L$





(+,+)

$\hat{d}_n \cdot \hat{n}_n = 0$

$\hat{d}_x \hat{n}_x + \hat{d}_z \hat{n}_z = 0$

$\hat{n}_n = (-\hat{d}_z, \hat{d}_x)$

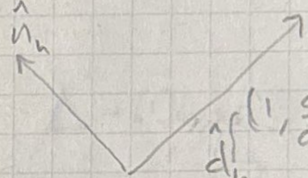
(+,-)

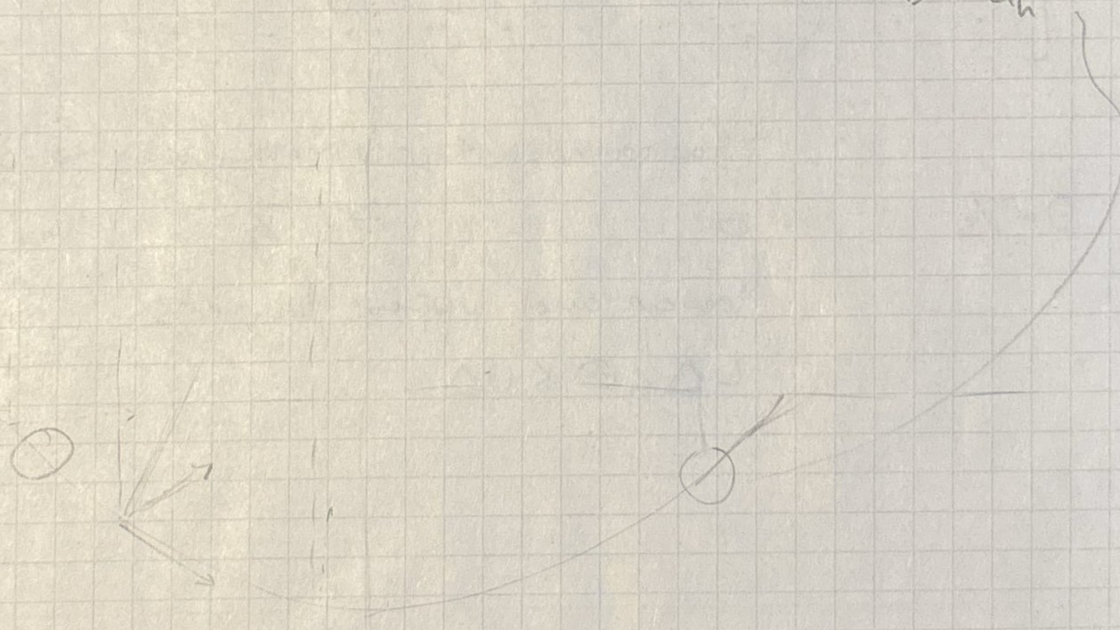
$\hat{n}_n \cdot \hat{d}_n = 0$

$\hat{n}_n = (-\hat{d}_z, \hat{d}_x)$

$z = mx^2 - 1/4m$

$\frac{dz}{dx} = 2mx$

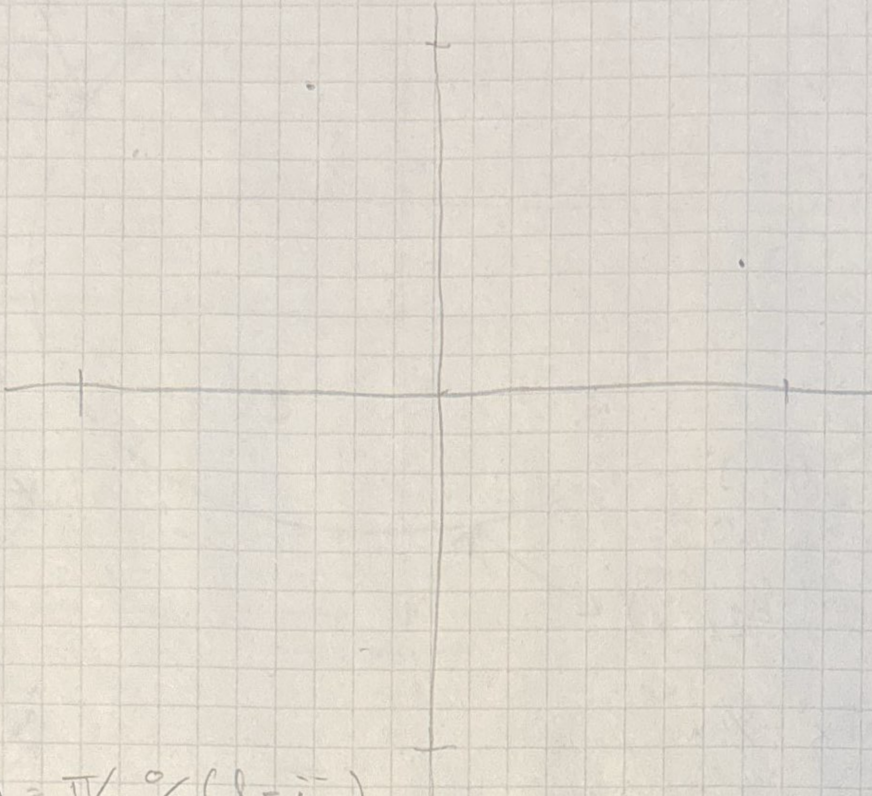




9.86364

radius of curvature

$$R = \left| \frac{(1+z'^2)^{3/2}}{z''} \right| \rightarrow \begin{array}{l} z' = 2mx \\ z'' = 2m \end{array}$$



new[1] = $\Delta = \pi/2 \,\%\, (l - i_a)$

[. . . . . L . U . .] ← length l

found index ← | airmass_index → i_a

$\theta = \pi/3$ $\quad \theta \quad \Theta \quad \pi/2$

lower and upper bounds

$L\Delta < \Theta < U\Delta$