MATH 151A FINAL REVIEW

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1. Proofs

(i) Let $f \in C^2[a,b]$, $h = (b-a)/n$, and $x_j = a + jh$ for each $j = 0, \dots, n$. There exists a $\mu \in (a,b)$ for which Composite Trapezoidal Rule for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Proof. Define $h = (b-a)/n$, $x_j = a + jh$ for each $j = 0, 1, \dots, n$, $x_0 = a$, and $x_n = b$. We have that

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx = \sum_{j=1}^{n} \left[\frac{h}{2}\big(f(x_{j-1}) + f(x_j)\big) - \frac{h^3}{12} f''(\xi_j)\right].$$

for $\xi_j \in (x_{j-1}, x_j)$. We expand the above expression to get

$$= \frac{h}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)\right] - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right] - \underbrace{\frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)}_{=\alpha}.$$

We want to rewrite the quantity α. If $f'' \in C[a,b]$, then by Extreme Value Theorem, f'' has a minimum and maximum on $[a,b]$. That is, for all $x \in [a,b]$,

$$\begin{aligned}
&\min_{x\in[a,b]} f''(x) \le f''(x) \le \max_{x\in[a,b]} f''(x) \\
\implies& \min_{x\in[a,b]} f''(x) \le f''(\xi_j) \le \max_{x\in[a,b]} f''(x) \qquad \forall j \in \{0,1,\dots,n\} \\
\implies& \sum_{j=1}^{n} \min_{x\in[a,b]} f''(x) \le \sum_{j=1}^{n} f''(\xi_j) \le \sum_{j=1}^{n} \max_{x\in[a,b]} f''(x) \\
\implies& n \times \min_{x\in[a,b]} f''(x) \le \sum_{j=1}^{n} f''(\xi_j) \le n \times \max_{x\in[a,b]} f''(x) \\
\implies& \min_{x\in[a,b]} f''(x) \le \frac{1}{n}\sum_{j=1}^{n} f''(\xi_j) \le \max_{x\in[a,b]} f''(x).\}
\end{aligned}$$

By Intermediate Value Theorem, there exists $\mu \in (a,b)$ such that $f''(\mu) = \frac{1}{n}\sum_{j=1}^{n} f''(\xi_j)$ where $j \in \{0,1,\dots,n\}$. This means that

$$f''(\mu) = \frac{1}{n}\sum_{j=1}^{n} f''(\xi_j) \implies nf''(\mu) = \sum_{j=1}^{n} f''(\xi_j).$$

This further implies that

$$-\frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j) = -\frac{h^3}{12} nf''(\mu) = -\frac{h^2(b-a)}{12n} nf''(\mu) = -\frac{b-a}{12}h^2 f''(\mu).$$

Using this result, since we defined $x_0 = a$ and $x_n = b$, it follows that

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

This concludes the proof. □

(ii) Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$

Proof. We will split this into two cases: (1) $P(x)$ has degree $< n$ and (2) $P(x)$ has degree $\geq n$ but $< 2n$.

(1) Suppose that $P(x)$ has degree $< n$. The Lagrange polynomial $P(x)$ with n nodes $x_1, \ldots, x_n$ is

$$P(x) = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) + \underbrace{\frac{P^{(n)}(\xi)}{n!}(x - x_1)\cdots(x - x_n)}_{=0}$$

where $P^{(n)}(\xi) = 0$ because $P(x)$ has degree $< n$. We integrate to get that

$$\begin{aligned}
\int_{-1}^{1} P(x)\, dx &= \int_{-1}^{1} \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} P(x_i)\, dx \\
&= \sum_{i=1}^{n} \left(\int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx \right) P(x_i) \\
&= \sum_{i=1}^{n} c_i P(x_i).
\end{aligned}$$

This takes care of the first case. We now want to show the second case.

(2) Suppose that $P(x)$ has degree $\geq n$ but $< 2n$. If we divide $P(x)$ by the nth Legendre polynomial $P_n(x)$, we get

$$P(x) = Q(x)P_n(x) + R(x)$$

where $Q(x), R(x)$ are polynomials of degree $< n$. We have that x_i are the roots of $P_n(x)$, so $P_n(x_i) = 0$ for each $i = 1, \ldots, n$. Evaluating $P(x)$ at each $x = x_i$, it follows that

$$P(x_i) = \underbrace{Q(x_i)P_n(x_i)}_{=0} + R(x_i).$$

This implies that $P(x_i) = R(x_i)$. We furthermore have that

$$\begin{aligned}
\int_{-1}^{1} P(x)\, dx &= \int_{-1}^{1} \underbrace{Q(x)P_n(x)}_{=0} + \int_{-1}^{1} R(x)\, dx \\
&= \int_{-1}^{1} R(x)\, dx
\end{aligned}$$

where $Q(x)P_n(x) = 0$ by the properties of Legendre polynomials. From case (1), we have that

$$\int_{-1}^{1} P(x) = \int_{-1}^{1} R(x) = \sum_{i=1}^{n} c_i R(x_i) = \sum_{i=1}^{n} c_i P(x_i).$$

This verifies case (2) and concludes the proof. $\square$

(iii) Show that the formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have degree of precision greater than $2n-1$, regardless of the choice of $c_1, \ldots, c_n$ and $x_1, \cdots, x_n$.

Proof. We will proceed by contradiction. Suppose that $Q(p)$ has degree of precision $2n$. We construct the polynomial
$$P(x) = (x - x_1)^2 (x - x_2)^2 \cdots (x - x_n)^2,$$
which has degree $2n$. Since the degree of precision is $2n$, we have that
$$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i),$$
but $\sum_{i=1}^{n} c_i P(x_i) = 0$. However, we have that
$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} (x - x_1)^2 (x - x_2)^2 \cdots (x - x_n)^2\,dx > 0.$$
Hence, this implies that $0 > 0$, so a contradiction has arisen. Therefore, the degree of precision of $Q(p)$ is at most $2n - 1$. □

2. Formulas

(a) **Trapezoidal rule**. Let $x_0 = a$ and $x_1 = b$ where $h = x_1 - x_0$. Then for $\xi \in (x_0, x_1)$ we have that
$$\int_a^b f(x)\,dx = \frac{h}{2}\Big[f(x_0) + f(x_1)\Big] - \frac{h^3}{12} f''(\xi).$$

(b) **Simpson's rule**. Let $x_0 = a$, $x_2 = b$, $h = (b - a)/2$, and $x_1 = x_0 + h$. Then for $\xi \in (x_0, x_2)$ we have that
$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big] - \frac{h^5}{90} f^{(4)}(\xi).$$

(c) **Difference formula.** To estimate $f'(x_0)$ where the distance between points is h, we have that
$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi)$$
where ξ is between x_0 and $x_0 + h$. If $h > 0$, this is the *forward difference formula*. If $h < 0$, this is the *backward difference formula*.

(d) **Centered difference formula**. For this formula, we require 3 equally-spaced points: $x_0, x_0 - h$, and $x_0 + h$. Then the formula is
$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi)$$
where $\xi \in (x_0 - h, x_0 + h)$.