MATH 151A MIDTERM 2 REVIEW

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1. Proofs

(i) Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$. Then for each x in $[a,b]$, a number $\xi(x)$ (generally unknown) between $\min\{x_0, x_1, \ldots, x_n\}$ and $\max\{x_0, x_1, \ldots, x_n\}$ and hence in (a,b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

where $P(x)$ is the nth Lagrange interpolating polynomial.

Proof. At $x = x_k$ for $k = 0, 1, \ldots, n$, by definition we have that $f(x_k) = P(x_k)$ as

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n) = 0$$

where $(x_k - x_k) = 0$ for some $k = 0, 1, \ldots, n$. So the expression holds for $x = x_k$ and we can choose $\xi(x)$ arbitrarily. We now want to show that the expression holds for $x \neq x_k$. Define g for $t \in [a,b]$ by

$$g(t) = f(t) - P(t) - \big(f(x) - P(x)\big)\frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}.$$

At $t = x_k$, we have that

$$g(x_k) = \underbrace{f(x_k) - P(x_k)}_{=0,\text{ by definition}} - \big(f(x) - P(x)\big)\underbrace{\frac{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}}_{=0,\text{ since } (x_k - x_k) = 0 \text{ where } 0 \le k \le n} = 0.$$

Now, at $t = x$, we have that

$$g(x) = f(x) - P(x) - \big(f(x) - P(x)\big)\underbrace{\frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}}_{=1} = 0.$$

Since $g \in C^{n+1}$, g is zero at $n+2$ distinct values $x_0, \ldots, x_n, x$. By Generalized Rolle's Theorem, there exists some $\xi(x) \in (a,b)$ such that $g^{(n+1)}(\xi(x)) = 0$. It follows that

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \underbrace{P^{(n+1)}(t)}_{=0} - \big(f(x) - P(x)\big)\frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

where $P^{(n+1)}(t) = 0$ since $P(x)$ is at most degree n. Plugging in $\xi(x)$,

$$g^{(n+1)}(\xi(x)) = 0 = f^{(n+1)}(\xi(x)) - \big(f(x) - P(x)\big)\frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}.$$

Rearranging to solve for $f(x)$, we have that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

This concludes the proof. □

(ii) Let f be defined at $x_0, x_1, \ldots, x_k$ and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i - x_j)}$$

is the kth Lagrange interpolating polynomial that interpolates f at the $k+1$ points $x_0, x_1, \ldots, x_k$.

Proof. Let $Q = P_{0,1,\ldots,i-1,i+1,\ldots,k}$ and $\hat{Q} = P_{0,1,\ldots,j-1,j+1,\ldots,k}$ where Q and $\hat{Q}$ both have degree less than or equal to $k-1$. Define

$$P(x) = \frac{(x - x_j)\hat{Q}(x) - (x - x_i)Q(x)}{x_i - x_j}$$

where $P(x)$ has degree at most k. We know that $Q(x_j) = f(x_j)$ and $\hat{Q}(x_i) = f(x_i)$. So it follows that

$$P(x_i) = \frac{(x_i - x_j)\hat{Q}(x_i) - (x_i - x_i)Q(x_i)}{x_i - x_j} = \hat{Q}(x_i) = f(x_i)$$

and

$$P(x_j) = \frac{(x_j - x_j)\hat{Q}(x_j) - (x_j - x_i)Q(x_j)}{x_i - x_j} = Q(x_j) = f(x_j).$$

This verifies the case of x_i and x_j. We want to show that the kth Lagrange polynomial also interpolates x_r where $r \neq i, j$. By definition, we know that $\hat{Q}(x_r) = f(x_r)$ and $Q(x_r) = f(x_r)$. We have that

$$\begin{aligned}
P(x_r) &= \frac{(x_r - x_j)\hat{Q}(x_r) - (x_r - x_i)Q(x_r)}{x_i - x_j} \\
&= \frac{(x_r - x_j)f(x_r) - (x_r - x_i)f(x_r)}{x_i - x_j} \\
&= \frac{(x_i - x_j)f(x_r)}{x_i - x_j} \\
&= f(x_r).
\end{aligned}$$

So $P(x) = P_{0,1,\ldots,k}(x)$ is the kth Lagrange polynomial that agrees with f at $k+1$ points $x_0, \ldots, x_k$. □

(iii) Suppose that $f \in C^n[a,b]$ where $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a,b]$. Then a number ξ exists in (a,b) with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof. Let $g(x) = f(x) - P_n(x)$ where $P_n(x)$ is the Newton's form interpolating polynomial, i.e.,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\cdots(x - x_{k-1}).$$

Since $f(x_i) = P_n(x_i)$ for $i = 0, 1, \ldots, n$, this implies that $g(x_i) = 0$, so g has $n+1$ distinct roots on $[a,b]$ and thus $g \in C^n[a,b]$ since $f \in C^n[a,b]$ and $P_n \in C^\infty[a,b]$ by construction. By Generalized Rolle's Theorem, there exists some $\xi \in (a,b)$ such that $g^{(n)}(\xi) = 0$. It follows that

$$\begin{aligned}
0 = g^{(n)}(\xi) &= f^{(n)}(\xi) - P_n^{(n)}(\xi) \\
&= f^{(n)}(\xi) - n! f[x_0, x_1, \ldots, x_n].
\end{aligned}$$

We can rearrange the above expression to get that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

This concludes the proof. □

2. Selected Exercises

(1) *Homework 4, Exercise 3.* Let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolating polynomials of degree at most one and at most two to approximate $f(0.45)$ for $f(x) = \cos x$. Find the absolute error.

Solution. We will first construct the Lagrange interpolating polynomial of degree at most one. Let $x_0 = 0$ and $x_1 = 0.6$. We have that

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

So we have

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.6}{0 - 0.6} = \frac{x - 0.6}{-0.6}$$

and also that

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{0.6 - 0} = \frac{x}{0.6}.$$

Using a calculator, we find that $f(0) = 1$ and $f(0.6) = 0.82533$. Hence, the degree one Lagrange interpolating polynomial is

$$P_1(x) = \frac{x - 0.6}{-0.6} + \frac{0.82533x}{0.6}.$$

Evaluating at $x = 0.45$,

$$P_1(0.45) = \frac{0.45 - 0.6}{0.6} + \frac{0.82533 \times 0.45}{0.6} = 0.86998.$$

We find that $f(0.45) = \cos 0.45 = 0.90045$, so the absolute error is

$$|P_1(0.45) - f(0.45)| = |0.868998 - 0.90045| = 0.03145.$$

Now we will construct the Lagrange interpolating polynomial of degree at most two. Let $x_0 = 0, x_1 = 0.6$, and $x_2 = 0.9$. This is given by

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2).$$

We calculate that $f(0) = 1$, $f(0.6) = 0.82533$, and $f(0.9) = 0.62160$. So we have that

$$P_2(x) = \frac{(x - 0.6)(x - 0.9)}{(0 - 0.6)(0 - 0.9)} + \frac{0.82533x(x - 0.9)}{(0.6 - 0)(0.6 - 0.9)} + \frac{0.62160x(x - 0.6)}{(0.9 - 0)(0.9 - 0.6)}.$$

Evaluating at $x = 0.45$,

$$P_1(0.45) = 0.8980963.$$

So the absolute error is

$$|P_2(0.45) - f(0.45)| = 0.00235.$$

(2) *Homework 3, Exercise 4.* Use Theorem 3.3 to find an error bound for the previous approximation.

Solution. By Theorem 3.3, the error term for the approximation $P_2(x)$ to $f(x)$ is

$$g(x) = \frac{f^{(3)}(\xi(x)}{3!}(x - 0)(x - 0.6)(x - 0.9).$$

We can easily calculate that $f^{(3)}(x) = \sin x$, so this means that the error term is

$$g(x) = \frac{1}{6} \sin\big(\xi(x)\big)(x - 0)(x - 0.6)(x - 0.9)$$

for $\xi(x) \in [0, 0.9]$. To simplify notation, let $\xi(x) = \xi$. It follows that

$$g(x) \leq \max_{\xi \in [0,0.9]} \left| \frac{1}{6} \sin(\xi) \right| \max_{x \in [0,0.9]} \big| x(x - 0.6)(x - 0.9) \big|.$$

On the interval $[0, 0.9]$, $\sin(\xi)$ realizes its maximum at $\xi = 0.9$. So this means that

$$g(x) \leq \frac{1}{6} \sin(0.9) \times \max_{x \in [0,0.9]} \big| x(x - 0.6)(x - 0.9) \big|.$$

Let $h(x) = x(x - 0.6)(x - 0.9)$. We find that the critical values are

$$x = 0, x = \frac{5 + \sqrt{7}}{10}, x = \frac{5 - \sqrt{7}}{10}, x = 0.9.$$

We can evaluate h at each x to find the maximum value, which occurs at $x = \frac{5-\sqrt{7}}{10}$. Hence,

$$\left| h\left(\frac{5 - \sqrt{7}}{10} \right) \right| = 0.057040.$$

This means that

$$|g(x)| \leq \frac{1}{6} \sin(0.9) \times = 0.057040 = 0.007447.$$

So 0.007447 is the maximum error bound for approximations to $f(x) = \cos x$ by $P_2(x)$ on $[0, 0.9]$.

(3) *Homework 5, Exercise 1.* Given the data $(8.1, 16.94410), (8.3, 17.56492), (8.6, 18.50515), (8.7, 18.82091)$, use Divided Differences with $x_0 = 8.1$, $x_1 = 8.3$, and $x_2 = 8.6$ to construct an interpolating polynomial of degree two. Use this to approximate $f(8.4)$.

Solution. Using the table, we fill in

$$\begin{aligned} f[x_0] &= 16.94410 \\ f[x_1] &= 17.56492 \\ f[x_2] &= 18.50515 \end{aligned}$$

From this, we calculate

$$\begin{aligned} f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{17.56492 - 16.94410}{8.3 - 8.1} = 3.1041 \\ f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{18.50515 - 17.56492}{8.6 - 8.3} = 3.1341 \end{aligned}$$

Using these, we find the last divided difference to be

$$f[x_1, x_2, x_3] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3.1341 - 3.1041}{8.6 - 8.1} - 0.06$$

Hence, the interpolating Divided Differences polynomial is

$$\begin{aligned} P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 16.94410 + 3.1041(x - 8.1) + 0.06(x - 8.1)(x - 8.3) \end{aligned}$$

Substituting $x = 8.4$, we find that

$$P_2(8.4) = 17.87713$$

(4) *Homework 5, Exercise 3.* Construct an approximating Hermite polynomial for the following data:

x	$f(x)$	$f'(x)$
0.1	-0.29004996	-2.8019975
0.2	-0.56079734	-2.6159201
0.3	-0.81401972	-2.9734038

Solution. Let $z_0, z_1 = x_0 = 0.1$, $z_2, z_3 = x_1 = 0.2$, and $z_4, z_5 = x_3 = 0.3$. We use the table to fill in

$$\begin{aligned} f[z_0] = f[z_1] &= f(x_0) = -0.29004996 \\ f[z_2] = f[z_3] &= f(x_1) = -0.56079734 \\ f[z_4] = f[z_5] &= f(x_2) = -0.81401972 \end{aligned}$$

Using this, we calculate

$$
\begin{aligned}
f[z_0, z_1] &= -2.8019975 \\
f[z_1, z_2] &= \frac{f[z_2] - f[z_1]}{z_2 - z_1} = \frac{-0.56079734 + 0.29004996}{0.2 - 0.1} = -2.7074738 \\
f[z_2, z_3] &= -2.6159201 \\
f[z_3, z_4] &= \frac{f[z_4] - f[z_3]}{z_4 - z_3} = \frac{-0.81401972 + 0.56079734}{0.3 - 0.2} = -2.5322238 \\
f[z_4, z_5] &= -2.9734038
\end{aligned}
$$

We continue with the Divided Differences pattern, which eventually gives us a Hermite polynomial of the form

$$
\begin{aligned}
H_5(x) = {} & f[z_0] + f[z_0, z_1](x - x_1) + f[z_0, z_1, z_2](x - x_1)^2 + f[z_0, z_1, z_2, z_3](x - x_1)^2(x - x_2) \\
& + f[z_0, z_1, z_2, z_3, z_4](x - x_1)^2(x - x_2)^2 + f[z_0, z_1, z_2, z_3, z_4, z_5](x - x_1)^2(x - x_2)^2(x - x_3).
\end{aligned}
$$

Substituting in our known values, we have that

$$
\begin{aligned}
H_5(x) = {} & -0.29004996 - 2.8019975(x - 0.1) + 0.945237(x - 0.1)^2 - 0.297(x - 0.1)^2(x - 0.2) \\
& - 0.47935(x - 0.1)^2(x - 0.2)^2 - 1299.97225(x - 0.1)^2(x - 0.2)^2(x - 0.3).
\end{aligned}
$$