MATH 151A MIDTERM 1 REVIEW

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1. Proofs

(i) Suppose that $f \in C[a,b]$ with $f(a)f(b) < 0$. The Bisection Method generates a sequence $\{p_n\}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n} \quad \text{when } n \geq 1.$$

First observe that

$$\begin{aligned}
|b_2 - a_2| &= \frac{1}{2}\big|b_1 - a_1\big| \\
|b_3 - a_3| &= \frac{1}{2}\big|b_2 - a_2\big| = \frac{1}{2^2}\big|b_1 - a_1\big| \\
&\vdots \\
|b_n - a_n| &= \frac{1}{2^{n-1}}\big|b_1 - a_1\big|
\end{aligned}$$

because we are cutting in the interval in half after each iteration of the Bisection Method. We can also observe that

$$\begin{aligned}
|p_1 - p| &\leq \frac{1}{2}\big|b_1 - a_1\big| \text{ because } \quad p_1 = \frac{a_1 + b_1}{2} \\
|p_2 - p| &\leq \frac{1}{2}\big|b_2 - a_2\big| \text{ because } \quad p_2 = \frac{a_2 + b_2}{2} \\
&\vdots \\
|p_n - p| &\leq \frac{1}{2}\big|b_n - a_n\big| \text{ because } \quad p_n = \frac{a_n + b_n}{2}
\end{aligned}$$

Proof. For $n \geq 1$, we observe that $|b_n - a_n| \leq \frac{1}{2^{n-1}}|b_1 - a_1|$. Also observe that $p \in (a_n, b_n)$ because $f(a_n)f(b_n) < 0$ (i.e., there is a root in this interval). Additionally, since $p_n = \frac{a_n+b_n}{2}$, we have that $|p_n - p| \leq \frac{1}{2}|b_n - a_n|$. Putting this together, since $b > a$, we have that

$$|p_n - p| \leq \frac{1}{2}\big|b_n - a_n\big| \leq \frac{1}{2} \cdot \frac{1}{2^{n-1}}\big|b_1 - a_1\big| = \frac{1}{2^n}\big|b_1 - a_1\big| = \frac{|b-a|}{2^n} = \frac{b-a}{2^n}.$$

□

(ii) (a) *Existence.* If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

Proof (a). Define $f(x) = x - g(x)$. Since x and $g(x)$ are continuous, $f \in C[a,b]$. We have that

$$\begin{aligned}
f(a) = a - g(a) \leq 0 \text{ but } g(a) \geq a \text{ because } g(x) \in C[a,b] \\
f(b) = b - g(b) \geq 0 \text{ but } g(b) \leq b \text{ because } g(x) \in C[a,b]
\end{aligned}$$

Therefore, this tells us that $f(a) \leq 0 \leq f(b)$. If $f(a) = 0$, then

$$a - g(a) = 0 \implies a = g(a) \implies a \text{ is a fixed point of } g.$$

Similarly, if $f(b) = 0$, then

$$b - g(b) = 0 \implies b = g(b) \implies b \text{ is a fixed point of } g.$$

Otherwise, $f(a) < 0 < f(b)$. By IVT, there is a $p \in (a, b) : f(p) = 0$, which implies that $p - g(p) = 0$ and that $p = g(p)$, meaning that p is a fixed point of g. □

(b) *Uniqueness.* If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$.

Proof (b). Suppose (by contradiction) that the fixed point is not unique. Let p and p' be fixed points. Then $g(p) = p$ and $g(p') = p'$. We therefore have that

$$|g(p) - g(p')| = |p - p'|.$$

We know that $g \in C[a, b]$, $g'(x)$ exists on (a, b), and that $|g'(x)| \leq k$ for all $x \in (a, b)$. By the Mean Value Theorem, there is a c between p and p' such that

$$\begin{aligned} |p - p'| = |g(p) - g(p')| &= |g'(c)(p - p')| \\ &= |g'(c)||p - p'| \\ &\leq k|p - p'| \\ &< |p - p'| \end{aligned}$$

This implies that $|p - p'| < |p - p'|$, which is a contradiction. Thus, the fixed point is unique. □

(iii) *Fixed-Point Theorem.* Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose in addition that g' exists on (a, b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then for any number $p_0 \in [a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \ n \geq 1,$$

converges to the unique fixed point p in $[a, b]$.

Proof. We know from Theorem 2.3 that g has a unique fixed point $p \in [a, b]$ such that $g(p) = p$ since $g(p) \in [a, b]$, then $p_n = g(p_{n-1}) \in [a, b]$ for $n \geq 1$. Since $g \in C[a, b]$ and g is differentiable, by MVT, there is a ξ_{n-1} between p_{n-1} and p such that

$$\begin{aligned} |p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_{n-1})||p_{n-1} - p| &\leq k|p_{n-1} - p| \\ &= k|g(p_{n-2}) - g(p)| \end{aligned}$$

Applying MVT again, there is a ξ_{n-2} between p_{n-2} and p such that

$$\begin{aligned} |p_n - p| \leq k|g(p_{n-2}) - g(p)| &= k|g'(\xi_{n-2})||p_{n-2} - p| \\ &\leq k^2|p_{n-2} - p| \end{aligned}$$

Applying MVT continuously yields

$$|p_n - p| \leq k^n|p_0 - p|.$$

It follows that $0 \leq |p_n - p| \leq k^n|p_0 - p|$. Applying Squeeze Theorem,

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p|.$$

Since $0 < k < 1$, we find that

$$0 \leq \lim_{n\to\infty} |p_n - p| \leq 0,$$

which implies that $p_n \to p$. □

2. Calculus Theorems

(i) *Rolle's Theorem.* Suppose $f \in [a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number $c \in (a,b)$ exists with $f'(c) = 0$.

(ii) *Mean Value Theorem.* If $f \in [a,b]$ and f is differentiable on (a,b), then a number $c \in (a,b)$ exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(iii) *Extreme Value Theorem.* If $f \in [a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. In addition, if f is differentiable on (a,b) then c_1, c_2 occur either at the endpoints or when f' is zero.

(iv) *Generalized Rolle's Theorem.* Suppose $f \in C(a,b)$ is n times differentiable on (a,b). If $f(x) = 0$ at $n+1$ distinct numbers $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then a number $c \in (a,b)$ exists such that $f^{(n)}(c) = 0$.

(v) *Intermediate Value Theorem.* If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = K$.

3. Taylor's Theorem

Theorem. Suppose $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on $[a,b]$ where $x_0 \in [a,b]$. For every $x \in [a,b]$, there is a number $\xi(x)$ between x_0 and x with

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \cdots + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}.$$

4. Numerical Methods

(1) *Bisection Method.*
- This method converges *at least* sub-linearly as long as $f \in C[a,b]$ and $f(a)f(b) < 0$.
- Typically not a first choice method; it's guaranteed to converge (albeit slowly), so it is good to check against other codes.

(2) *Fixed-Point Iteration.*
- Converges if $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$ where in addition $|g'(x)| \leq k < 1$ for all $x \in (a,b)$.
- For any $p_0 \in [a,b]$, $p_n = g(p_{n-1})$ for $n \geq 1$ converges to $p \in [a,b]$.
- If $g(p) = p$ and $g'(p) \neq 0$, then we have linear convergence. If $g(p) = p$ and $g'(p) = 0$, then we have quadratic convergence.
- Useful when problem is a clear fixed-point problem.

(3) *Newton's Method.*
- Converges when $f \in C^2[a,b]$ where $f(p) = 0$ and $p_0 \in [a,b]$ is an approximation to p where $f'(p_0) \neq 0$ and $|p - p_0|$ is "small."
- We make an initial guess p_0 and calculate $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$.
- When $f(p) = 0$ and $f'(p) \neq 0$, then Newton's Method converges *at least quadratically.* p is a simple root.
- When $f(p) = 0$ and $f'(p) = 0$, p is not a simple root and we are not guaranteed quadratic convergence.

(4) *Modified Newton's Method.*
- We can adjust Newton's Method to get quadratic convergence in cases of zeros with multiplicity ≥ 1.
- We define $\mu(x) = \frac{f(x)}{f'(x)}$ where $f(x) = (x - p)^m q(x)$.
- p is a simple root of $\mu(x)$ (i.e., $\mu(p) = 0$ but $\mu'(p) \neq 0$).

- We calculate p_n for $n \geq 1$ with
$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}.$$

(5) *Secant Method.*
- Useful when derivative is complex to compute.
- We need two initial guesses p_0 and p_1.
- We calculate p_n for $n \geq 2$ with
$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$
- Usually slower than Newton's Method.

(6) *Method of False Position.*
- This is a modification of Secant Method which ensures root bracketing.
- We choose p_0 and p_1 such that $f(p_0)f(p_1) < 0$. We then calculate p_2 using the Secant Method. If $f(p_0)f(p_2) < 0$, we choose p_0 and p_2 in the next step. If $f(p_1)f(p_2) < 0$, we choose p_1 and p_2 in the next step.

5. Error Analysis for Iterative Methods

Definition. If $(p_n) \to p$ with $p_n \neq p$ for all n and $\lambda > 0$ and $\alpha > 0$ exist with
$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$
then (p_n) converges to p with order α and asymptotic error constant λ.
- $\alpha = 1, \lambda < 1 \implies$ linear convergence
- $\alpha = 2 \implies$ quadratic convergence

6. Practice Problems

(1) Use the Bisection Method to find p_3 for $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.

Solution. We first evaluate f at 0 and 1. We find that
$$f(0) = 0 - \cos 0 = -1 < 0$$
$$f(1) = 1 - \cos 1 \approx 0.459 > 0$$
So $f(p) = 0$ for some $p \in [0, 1]$ by IVT. We choose $a_1 = 0$ and $b_1 = 1$. Therefore
$$p_1 = \frac{a_1 + a_2}{2} = \frac{0 + 1}{2} = 0.5.$$
Evaluating f at 0.5,
$$f(0.5) = \sqrt{0.5} - \cos 0.5 \approx -0.1704 < 0$$
So we choose $a_2 = 0.5$ and $b_2 = 1$. So we have
$$p_2 = \frac{a_2 + b_2}{2} = \frac{0.5 + 1}{2} = 0.75$$
Evaluating f at 0.75,
$$f(0.75) = \sqrt{0.75} - \cos 0.75 \approx 0.1343 > 0$$
So we choose $a_3 = 0.5$ and $b_3 = 0.75$. As a result,
$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.5 + 0.75}{2} = 0.625.$$

(2) Use Theorem 2.3 to show that $g(x) = \pi + 0.5 \sin\left(\frac{x}{2}\right)$ has a unique fixed point on $[0, 2\pi]$.

Solution. We will first show existence (a). We want to show that $g(x) \in [0, 2\pi]$ for all $x \in [0, 2\pi]$. Consider a $2\pi \times 2\pi$ "window." We can easily verify that $g(x)$ is continuous on $\mathbb{R}$. By Extreme Value Theorem,

$$g'(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right) = 0 \implies \frac{x}{2} = \cos^{-1}(0) \implies x = \frac{\pi}{2} \times 2 = \pi \text{ is critical.}$$

So π, 0, and 2π (the endpoints) are critical points. We find that

$$g(0) = \pi + 0.5\sin\left(\frac{0}{2}\right) = \pi$$
$$g(\pi) = \pi + 0.5\sin\left(\frac{\pi}{2}\right) = \pi + \frac{1}{2}$$
$$g(2\pi) = \pi + 0.5\sin\left(\frac{2\pi}{2}\right) = \pi$$

Hence, $0 \le \pi \le g(x) \le \pi + 0.5 \le 2\pi$, so Theorem 2.3(a) guarantees us a fixed point on $[0, 2\pi]$. But we don't yet know if it's unique. So we will now show uniqueness (b). From the previous part of this, we know that $g'(x)$ exists on (a,b). We want to show that there is a positive constant $k < 1$ such that $|g'(x)| \le k$ for all $x \in (a, b)$. We find that

$$|g'(x)| = \left|\frac{1}{4}\cos\left(\frac{x}{2}\right)\right| \le \left|\frac{1}{4}\right| \times |1| = \frac{1}{4} \text{ because } \left|\cos\left(\frac{x}{2}\right)\right| \le 1.$$

Hence $k = \frac{1}{4}$, and the fixed point of g on $[0, 2\pi]$ is unique.

(3) Estimate the number of iterations required to achieve 10^{-2} accuracy for fixed point iteration of $g(x)$ in (2).

Solution. In (2), we showed that the hypotheses of Theorem 2.3 are satisfied, so Theorem 2.5 says that for $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges to a unique fixed point in $[a, b]$ for $n \ge 1$. We can use Corollary 2.5, which says that the bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|$$

in addition to

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\}.$$

Use the initial guess $p_0 = \pi$. Then

$$p_1 = g(p_0) = g(\pi) = \pi + 0.5\sin\left(\frac{\pi}{2}\right) = \pi + 0.5.$$

Using Corollary 2.5,

$$10^{-2} \le \frac{0.25^n}{0.75}|\pi + 0.5 - \pi| = \frac{4}{3}(0.25^n)|0.5| = \frac{2}{3}(0.25)^n$$

Which implies that

$$\ln\left(\frac{3}{2} \times 10^{-2}\right) \le n\ln(0.25) \implies n \ge \frac{\ln\left(\frac{3}{2} \times 10^{-2}\right)}{\ln(0.25)} \approx 3.0295.$$

So we require $\lceil 3.0295 \rceil = 4$ iterations.

(4) Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's Method to find p_2.

Solution. We have that $f(x) = x^2 - 6$, so $f'(x) = 2x$. The setup for Newton's Method is

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \text{ for } n \geq 1.$$

Using $p_0 = 1$, we find that

$$p_1 = 1 - \frac{1^2 - 6}{2} = 1 + \frac{5}{2} = \frac{7}{2}.$$

Using $p_1 = \frac{7}{2}$, we find that

$$p_2 = \frac{7}{2} - \frac{\frac{49}{4} - \frac{24}{4}}{\frac{28}{4}} = \frac{7}{2} - \frac{25}{28} = \frac{98 - 25}{28} = \frac{73}{28}.$$

So $p_3 = \frac{73}{28}$ by Newton's Method.

(5) Let $f(x) = x^2 - 6$ and $p_0 = 3$ and $p_1 = 2$.

(a) Find p_3 using the Secant Method.

Solution. We find that

$$f(p_0) = f(3) = 3^2 - 6 = 3$$
$$f(p_1) = f(2) = 2^2 - 6 = -2$$

p_2 can be found using the Secant Method by

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2(2-3)}{2-3} = 2 + \frac{2}{5} = \frac{12}{5}.$$

It follows that

$$f(p_2) = \left(\frac{12}{5}\right)^2 - 6 = \frac{144}{25} - \frac{150}{5} = -\frac{6}{25}.$$

We find p_3 to be

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = \frac{12}{5} - \frac{-\frac{6}{25}\left(\frac{12}{5} - 2\right)}{-\frac{6}{25} + 2} = \cdots = \frac{27}{11}.$$

(6) Find a bound for the number of iterations of the Bisection Method needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ on the interval $[1, 4]$.

Solution. We know that the error bound for the Bisection Method is given by

$$|p_n - p| \leq \frac{b - a}{2^n} \text{ for } n \geq 1.$$

So we want to find n such that

$$\frac{1}{2^n}(4 - 1) \leq 10^{-3}.$$

Rearranging, we find that

$$\frac{\ln(3 \times 10^3)}{\ln 2} \leq n.$$

Taking the ceiling, we find that we need at most $n = 12$ iterations of the Bisection Method. The `BisectionMethod.m` code takes 12 iterations.