Math 151B Midterm

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Question 2

We are given the initial value problem

$$y' = \frac{2-2ty}{t^2+1}, \ \ 0 \le t \le 1, \ \ y(0) = 1.$$

To approximate solutions to this IVP using second-order Taylor's method, we will perform $N = (b-a)/h = (1-0)/0.1 = 10$ iterations evaluated at equally-spaced mesh points between 0 and 1. Given $f(t,y) = (2-2ty)/(t^2+1)$, we find that

$$\begin{aligned}
\frac{d}{dt} f(t,y) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t,y) \\
&= \frac{2t^2y - 4t - 2y}{(t^2+1)^2} - \frac{2t}{t^2+1}\left(\frac{2-2ty}{t^2+1}\right) \\
&= \frac{2t^2y - 4t - 2y - 4t + 4t^2y}{(t^2+1)^2} \\
&= \frac{6t^2y - 8t - 2y}{(t^2+1)^2}
\end{aligned}$$

so second-order Taylor's method for this initial value problem is given by

$$\begin{aligned}
w_0 &= 1 \\
w_{i+1} &= w_i + h\left(\frac{2-2t_iw_i}{t_i^2+1}\right) + \frac{h^2}{2}\left(\frac{6t_i^2w_i - 8t_i - 2w_i}{(t_i^2+1)^2}\right)
\end{aligned}$$

for $h = 0.1$ and $i = 0, 1, \ldots, 9$. Hence, using taylor_order_2.py, we obtain as output the following table of approximations. The code is attached on the following page.

i	t_i	w_i
0	0.0	1.0000000
1	0.1	1.1900000
2	0.2	1.3492187
3	0.3	1.4712595
4	0.4	1.5546191
5	0.5	1.6019194
6	0.6	1.6184028
7	0.7	1.6103856
8	0.8	1.5840995
9	0.9	1.5450253
10	1.0	1.4976289

taylor_order_2.py

```
# Imports
import numpy as np
import pandas as pd
import math

# For more decimal places in output
pd.set_option("display.precision", 7)

# Functions
f        = lambda t, y: (2-2*t*y)/(t**2 + 1)
f_prime  = lambda t, y: (6*t**2*y-8*t-2*y)/(t**2 + 1)**2

# Left endpoint
t_0 = 0
# Right endpoint
t_1 = 1
# Step size
h = 0.1
# Initial condition
w0 = 1

# Initializing arrays
t = np.arange(t_0, t_1+h, h)
w = np.zeros(len(t))
w[0] = w0

# Looping to determine approximations
for i in range(0, len(t) - 1):
  w[i+1] = w[i] + h*(f(t[i], w[i])) + h**2/2*f_prime(t[i], w[i])

# Output
df = pd.DataFrame({('t_i'): t, 'w_i': w})
print(df)
```

Question 3

Proof. Consider the second order initial value problem given by

$$y'' = p(t)y' + q(t)y + r(t), \ \ a \le t \le b, \ \ y(a) = \alpha_1, y'(a) = \alpha_2.$$

We want to convert this into a system of first-order equations. Let $u_1 = y$ and $u_2 = y'$. From this, we clearly have that $u_1' = u_2$. Next, we find that

$$u_2' = y'' = p(t)y' + q(t)y + r(t) = p(t)u_2 + q(t)u_1 + r(t).$$

As such, the system is given by

$$\begin{cases} u_1' = u_2 \\ u_2' = p(t)u_2 + q(t)u_1 + r(t) \end{cases}$$

with initial conditions $u_1(a) = \alpha_1$ and $u_2(a) = \alpha_2$. Let $f_1(t, u_1, u_2) = u_1'$ and $f_2(t, u_1, u_2) = u_2'$. We know by Theorem 5.17 that this second-order system of first-order IVPs has a unique solution $u_1(t), u_2(t)$ for $a \le t \le b$ if $f_1(t, u_1, u_2)$ and $f_2(t, u_1, u_2)$ are continuous and satisfy Lipschitz conditions on the set

$$D = \{(t, u_1, u_2) : a \le t \le b, -\infty < u_1, u_2 < \infty\}.$$

Hence, we want to first show that $f_1(t, u_1, u_2) = u_2$ is continuous on D and also satisfies a Lipschitz condition in D. Clearly, $f_1(t, u_1, u_2) = u_2$ is continuous on $\mathbb{R}$, so it is continuous on D. We find that the partial derivative of f_1 with respect to u_1 satisfies the inequality

$$\left|\frac{\partial f_1(t, u_1, u_2)}{\partial u_1}\right| = \left|\frac{\partial}{\partial u_1}(u_2)\right| = 0 \le 0$$

and that the partial derivative of f_1 with respect to u_2 satisfies

$$\left|\frac{f_1(t, u_1, u_2)}{\partial u_2}\right| = \left|\frac{\partial}{\partial u_2}(u_2)\right| = 1 \le 1.$$

Therefore, if we set $L = \max\{0, 1\} = 1$, we have that

$$\left|\frac{f_1(t, u_1, u_2)}{\partial u_i}\right| \le L$$

for each $i = 1, 2$. Hence by Definition 5.16, $f_1(t, u_1, u_2)$ satisfies a Lipschitz condition on D with Lipschitz constant $L = 1$. Next, we want to show that $f_2(t, u_1, u_2) = p(t)u_2(t) + q(t)u_1(t) + r(t)$ is continuous on D and also satisfies a Lipschitz condition on D. We are given that $p(t)$, $q(t)$, and $r(t)$ on $[a, b]$, so it follows that these functions are continuous on D. In addition, we have have that u_1, u_2 are also continuous on $\mathbb{R}$ and by extension D. The product of two continuous functions is continuous, so we have that $p(t)u_2$ and $q(t)u_1$ are continuous on D as well. Furthermore, the sum of a finite number of continuous functions is continuous, so $f_2(t, u_1, u_2) = p(t)u_2 + q(t)u_1 + r(t)$ is continuous on D. Next, we show that the first partial derivative of f_1 satisfies

$$\left|\frac{\partial f_2(t, u_1, u_2)}{\partial u_1}\right| = \left|\frac{\partial}{\partial u_1}\big(p(t)u_2 + q(t)u_1 + r(t)\big)\right| = \big|q(t)\big| \le \max_{t\in[a,b]} |q(t)| = L_1$$

since $q(t)$ is continuous on $[a, b]$ and hence achieves its maximum L_1 on the closed interval by Extreme Value Theorem. Additionally,

$$\left|\frac{\partial f_2(t, u_1, u_2)}{\partial u_2}\right| = \left|\frac{\partial}{\partial u_2}\big(p(t)u_2 + q(t)u_1 + r(t)\big)\right| = \big|p(t)\big| \le \max_{t\in[a,b]} |p(t)| = L_2$$

since $p(t)$ similarly achieves its maximum L_2 on $[a, b]$ by Extreme Value Theorem. Letting $L' = \max\{L_1, L_2\}$, we have shown that

$$\left|\frac{\partial f_2(t, u_1, u_2)}{\partial u_i}\right| \le L'$$

for each $i = 1, 2$. By Definition 5.16, $f_2(t, u_1, u_2)$ satisfies a Lipschitz condition on D with Lipschitz constant $L' = \max\{L_1, L_2\}$. Therefore, since we also showed above that $f_1(t, u_1, u_2)$ satisfies a Lipschitz condition on D with Lipschitz constant $L = 1$, by Theorem 5.17, this second-order IVP has a unique solution $u_1(t), u_2(t)$ for $a \le t \le b$. And we're done. $\square$

Question 4

Consider the general initial value problem given by

$$y' = f(t,y), \;\; a \le t \le b, \;\; y(a) = \alpha.$$

Suppose that we have initial conditions $w_0 = \alpha$ and $w_1 = \alpha_1$. We want to determine if the multistep method

$$w_{i+1} = -w_i + 2w_{i-1} + \frac{h}{2}\left[5f(t_i, w_i) + f(t_{i-1}, w_{i-1}\right]$$

is stable for $i = 1, 2, \ldots, N-1$ and step size h when approximating initial value problems of the above form. By Definition 5.22, the characteristic polynomial associated with a general multistep method

$$\begin{aligned} w_0 &= \alpha, w_1 = \alpha_1, \ldots, w_{m-1} = \alpha_{m-1} \\ w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}) \end{aligned}$$

is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0.$$

For the multistep method given above, we can see that $a_1 = -1$, $a_0 = 2$, and $m = 2$, so then $P(\lambda) = \lambda^2 - a_1\lambda - a_0$. Hence, the characteristic polynomial is

$$P(\lambda) = \lambda^2 + \lambda - 2.$$

We now want to solve for the roots of $P(\lambda)$ and determine if they satisfy the root condition. We find that

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} \;\; \Rightarrow \;\; \lambda_1 = -2, \; \lambda_2 = 1.$$

For this multistep method to satisfy the root condition, we require that $|\lambda_i| \le 1$ for $i = 1, 2$. However, we have that $|\lambda_1| = 2 > 1$, so this method fails the root test. Therefore, by Definition 5.23(iii), the multistep method is unstable.