Math 151B Homework 1

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Question 5.1.1(a)

Consider the initial value problem given by

$$y' = y\cos t,\ 0 \le t \le 1,\ y(0) = 1.$$

Note that $f(t, y) = y\cos t$, which is continuous on the set $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$. We have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = |\cos t| \le 1$$

for $t \in [0,1]$, so by Theorem 5.3 $f(t, y)$ satisfies a Lipschitz condition on D in y with Lipschitz constant $L = 1$. Hence, by Theorem 5.4, it follows that $y(t)$ has a unique solution for $0 \le t \le 1$. This unique solution is given by

$$\begin{aligned}
y' &= y\cos t \\
\frac{dy}{dt} &= y\cos t \\
\frac{dy}{y} &= \cos t\, dt \\
\int \frac{1}{y}\, dy &= \int \cos t\, dt \\
\ln y &= \sin t + c \\
e^{\ln y} &= e^{\sin t + c} \\
y &= e^c e^{\sin t} \\
y &= c_1 e^{\sin t}
\end{aligned}$$

where we let $c_1 = e^c$. Thus, $y(t) = c_1 e^{\sin t}$ and we use the initial condition to find that $y(0) = 1 = c_1 e^{\sin 0} = c_1$. Therefore, the unique solution to the initial value problem is $y(t) = e^{\sin t}$.

Question 5.1.2(a)

Consider the set $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$ and the IVP given by

$$y' = e^{t-y},\ 0 \le t \le 1,\ y(0) = 1.$$

Theorem 5.4 guarantees that the IVP has a unique solution for $0 \le t \le 1$ if f is continuous on D and satisfies a Lipschitz condition in y on D. We see that $f(t, y)$ is continuous on D, so this satisfies the first hypothesis of the theorem. Note that $y' = f(t, y) = e^{t-y} > 0$ for any arbitrary $(t, y) \in D$. Hence we can rewrite the IVP equivalently as the continuous piecewise-defined function

$$y' = \begin{cases} e^{t-y} & \text{if } y > 0, \\ e^t & \text{otherwise.} \end{cases}$$

We then find that

$$\left|\frac{\partial}{\partial y} f(t,y)\right| = \left|\frac{\partial}{\partial y} e^{t-y}\right| = |-e^{t-y}| = e^{t-y} \le e$$

for $y > 0$. Therefore, f satisfies a Lipschitz condition in y on D with Lipschitz constant $L = e$. Hence, we can apply Theorem 5.4 to conclude that the IVP has a unique solution $y(t)$ for $0 \le t \le 1$. We find this solution to be

$$\begin{aligned} y' &= e^{t-y} \\ \frac{dy}{dt} &= e^{t-y} \\ \frac{dy}{e^{-y}} &= e^t\,dt \\ e^y\,dy &= e^t\,dt \\ \int e^y\,dy &= \int e^t\,dt \\ e^y &= e^t + c. \end{aligned}$$

We use the initial condition to find that $y(0) = 1 = e^1 = e^0 + c$, so $c = e - 1$. It further follows that

$$\begin{aligned} y &= \ln(e^t + c) \\ y &= \ln(e^t + e - 1), \end{aligned}$$

so $y(t) = \ln(e^t + e - 1)$ is the unique solution to this initial value problem.

Question 5.1.3(a)

(i) Consider $f(t, y) = t^2 y + 1$ and $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$. By Theorem 5.3, we have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|t^2\right| \le 1$$

for $t \in [0,1]$. Therefore, f satisfies a Lipschitz condition in y on D with Lipschitz constant $L = 1$.

(ii) By Theorem 5.6, the IVP

$$y' = f(t, y),\ 0 \le t \le 1,\ y(0) = 1$$

has a unique solution $y(t)$ for $0 \le t \le 1$ if $f(t, y)$ is continuous on D satisfies a Lipschitz condition in y on D. Clearly, $f(t, y) = t^2 y + 1$ is continuous on D and in part (a) we showed that it satisfies a Lipschitz condition in y on D for $L = 1$. Therefore we can apply Theorem 5.6 to conclude that this IVP is well-posed.

Question 5.1.3(b)

(i) Consider $f(t, y) = ty$ and $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$. By Theorem 5.3, we have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = |t| \le 1$$

for $t \in [0,1]$. Therefore, f satisfies a Lipschitz condition in y on D for Lipschitz constant $L = 1$.

(ii) By Theorem 5.6, the IVP

$$y' = f(t, y),\ 0 \le t \le 1,\ y(0) = 1$$

has a unique solution $y(t)$ for $0 \le t \le 1$ if $f(t, y)$ is continuous on D satisfies a Lipschitz condition in y on D. Clearly, $f(t, y) = ty$ is continuous on D and in part (a) we showed that it satisfies a Lipschitz condition in y on D for $L = 1$. Therefore we can apply Theorem 5.6 to conclude that this IVP is well-posed.

Question 5.1.3(c)

(i) Consider $f(t, y) = 1 - y$ and $D = \{(t, y) : 0 \le t \le 1, -\infty < y < \infty\}$. By Theorem 5.3, we have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = |-1| \le 1$$

for $t \in [0,1]$. Therefore, f satisfies a Lipschitz condition in y on D for Lipschitz constant $L = 1$.

(ii) By Theorem 5.6, the IVP

$$y' = f(t,y),\ 0 \le t \le 1,\ y(0) = 1$$

has a unique solution $y(t)$ for $0 \le t \le 1$ if $f(t,y)$ is continuous on D satisfies a Lipschitz condition in y on D. Clearly, $f(t,y) = 1 - y$ is continuous on D and in part (a) we showed that it satisfies a Lipschitz condition in y on D for $L = 1$. Therefore we can apply Theorem 5.6 to conclude that this IVP is well-posed.

Question 5.1.3(d)

(i) Consider $f(t,y) = -ty + \frac{4t}{y}$ and $D = \{(t,y) : 0 \le t \le 1, -\infty < y < \infty\}$. We find that for $(t,y_1),(t,y_2) \in D$ that

$$\begin{aligned}
|f(t,y_2) - f(t,y_1)| &= \left| -ty_2 + \frac{4t}{y_2} + ty_1 - \frac{4t}{y_1} \right| \\
&= \left| t(y_1 - y_2) + 4t(1/y_2 - 1/y_1) \right| \\
&= \left| t(y_1 - y_2) + 4t\left(\frac{y_1 - y_2}{y_1 y_2}\right) \right| \\
&= |t| \left| (y_1 - y_2) + 4\left(\frac{y_1 - y_2}{y_1 y_2}\right) \right| \\
&= |t| \left| 1 + \frac{4}{y_1 y_2} \right| |y_2 - y_1|.
\end{aligned}$$

Hence, the coefficient of $|y_1 - y_2|$ is dependent on y_1 and y_2, so f cannot possibly satisfy a Lipschitz condition in y on D.

(ii) By Theorem 5.6, the IVP

$$y' = f(t,y),\ 0 \le t \le 1,\ y(0) = 1$$

has a unique solution $y(t)$ for $0 \le t \le 1$ if $f(t,y)$ is continuous on D satisfies a Lipschitz condition in y on D. However, we found that $f(t,y) = -ty + \frac{4t}{y}$ does not satisfy a Lipschitz condition in y on D. Therefore, we cannot apply Theorem 5.6 to guaranteed well-posedness.

Question 5.1.9

Proof. A set $D \subset \mathbb{R}^2 = \{(t,y) : a \le t \le b, -\infty < y < \infty\}$ is said to be convex if whenever $(t_1,y_1),(t_2,y_2) \in D$ and $\lambda \in [0,1]$ then

$$\big((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2\big) \in D.$$

Let $a,b \in \mathbb{R}$ be arbitrary constants and let $(t_1,y_1),(t_2,y_2) \in D$, meaning that $a \le t_1, t_2 \le b$ and $-\infty < y_1, y_2 < \infty$. We want to show that $a \le (1-\lambda)t_1 + \lambda t_2 \le b$. By hypothesis, it follows that for $\lambda \in [0,1]$,

$$(1-\lambda)a \le (1-\lambda)t_1 \le (1-\lambda)b \quad \text{and} \quad \lambda a \le \lambda t_2 \le \lambda b$$

as $\lambda, 1-\lambda \ge 0$. Adding these two inequalities yields

$$(1-\lambda)a + \lambda a \le (1-\lambda)t_1 + \lambda t_2 \le (1-\lambda)b + \lambda b$$

which we can simplify to

$$a \le (1-\lambda)t_1 + \lambda t_2 \le b.$$

This completes the first part of the proof. We must now show that $-\infty < (1-\lambda)y_1 + \lambda y_2 < \infty$. Since we let $(t_1,y_1),(t_2,y_2) \in D$, we know from before that $-\infty < y_1, y_2 < \infty$. Thus we clearly have that for $\lambda \in [0,1]$,

$$-\infty < (1-\lambda)y_1 + \lambda y_2 < \infty.$$

This completes the second part of the proof. As such, we have that $\big((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2\big) \in D$, so D is convex. And we're done. $\square$

Question 5.2.1(a)

Consider the given IVP

$$y' = te^{3t} - 2y,\ 0 \le t \le 1,\ y(0) = 0.$$

To apply Euler's Method with $h = 0.5$, we need to perform $N = (b-a)/h = (1-0)/0.5 = 2$ iterations. Given $y(0) = 0$, we have that $t_0 = 0$ and $w_0 = 0$. Performing our first iteration,

$$w_1 = w_0 + hf(t_0, w_0) = 0 + 0.5(0e^{3\times 0} - 2\times 0) = 0 \approx y(0.5).$$

For our next iteration, $t_1 = 0.5$ and $w_1 = 0$. It follows that

$$w_2 = w_1 + hf(t_1, w_1) = 0 + 0.5(0.5e^{3\times 0.5} - 2\times 0) = 0.25e^{1.5} = 1.120422 \approx y(1).$$

So Euler's Method gives us $w_1 = 0 \approx y(0.5)$ and $w_2 = 1.12042 \approx y(1)$. We can also summarize this in the table below.

	t_i	w_i
0	0.0	0.000000
1	0.5	0.000000
2	1.0	1.120422

Question 5.2.1(b)

Consider the given IVP

$$y' = 1 + (t-y)^2,\ 2 \le t \le 3,\ y(2) = 1.$$

To apply Euler's Method for $h = 0.5$, we perform $N = (b-a)/h = (3-2)/0.5 = 2$ iterations. Given $y(2) = 1$, we have that $t_0 = 2$ and $w_0 = 1$. Performing our first iteration,

$$w_1 = w_0 + hf(t_0, w_0) = 1 + 0.5\big(1 + (2-1)^2\big) = 2 \approx y(2.5).$$

For the next iteration, $t_1 = 2.5$ and $w_1 = 2$. We have that

$$w_2 = w_1 + hf(t_1, w_1) = 2 + 0.5\big(1 + (2.5-2)^2\big) = 2.625 \approx y(3).$$

Thus, Euler's Method gives us $w_1 = 2 \approx y(2.5)$ and $w_2 = 2.625 \approx y(3)$. We can summarize this in the table below too.

	t_i	w_i
0	2.0	1.000000
1	2.5	2.000000
2	3.0	2.625000

Question 5.2.3(a)

We are given that the actual solution to the IVP in 5.2.1(a) is

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

So the actual solution for $y(0.5)$ is then

$$y(0.5) = \frac{0.5}{5}e^{3(0.5)} - \frac{1}{25}e^{3(0.5)} + \frac{1}{25}e^{-2(0.5)} = \frac{1}{10}e^{1.5} - \frac{1}{25}e^{1.5} + \frac{1}{25}e^{-1} = 0.283616.$$

As such, the actual error between w_1 and $y(0.5)$ is $|w_1 - y(0.5)| = |0 - 0.283616| = 0.283616$. Moreover, the actual solution for $y(1)$ is

$$y(1) = \frac{1}{5}e^{3\times 1} - \frac{1}{25}e^{3\times 1} + \frac{1}{25}e^{-2\times 1} = \frac{1}{5}e^{3} - \frac{1}{25}e^{3} + \frac{1}{25}e^{-2} = 3.219099.$$

So the actual error between w_2 and $y(1)$ is $|w_2 - y(1)| = |1.120422 - 3.219099| = 2.098677$. Next, we will compute the error bound. Recall that the error bound is given by

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left(e^{L(t_i-a)} - 1\right)$$

for each $i = 1, 2$. Using Theorem 5.3, we have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|\frac{\partial}{\partial y}(te^{3t} - 2y)\right| = |-2| \leq 2$$

for $t \in [0,1]$. Hence f satisfies a Lipschitz condition in y on D with Lipschitz constant $L = 2$. We now want to compute M. First observe that

$$|y''(t)| = \left|\frac{1}{25}e^{3t}(45t+21) + \frac{4}{25}e^{-2t}\right| \leq \frac{1}{25}\max_{t\in[0,1]}\left|\underbrace{e^{3t}(45t+21) + 4e^{-2t}}_{=g(t)}\right|.$$

We want to identify any critical points of $g(t)$ on $[0,1]$ using the Extreme Value Theorem. We compute that

$$g'(t) = (135t + 108)e^{3t} - 8e^{-2t}$$

Using a graphing calculator, we have that $g'(t) = 0$ for $t \approx -0.388$, but this value of t is not contained in the interval $[0,1]$. Hence, our only potential extremizers (critical points) are the endpoints $t = 0$ and $t = 1$. We find that

$$\begin{aligned} g(0) &= e^{3(0)}(45(0)+21) + 4e^{-2(0)} = 25, \\ g(1) &= e^{3(1)}(45(1)+21) + 4e^{-2(1)} = 1326.186778. \end{aligned}$$

Clearly $|g(1)|$ is the maximum value of $|g(t)|$ on $[0,1]$. It follows that

$$|y''(t)| \leq \frac{1}{25}\max_{t\in[0,1]}\left|e^{3t}(45t+21) + 4e^{-2t}\right| = \frac{1}{25}|g(1)| = \frac{1326.186778}{25} = 53.047471$$

As such, $M = 53.047471$. We can now find the error bound for our first approximation. We have that $t_1 = 0.5$, $a = 0$, $h = 0.5$, $L = 2$, and $M = 53.047471$. So the error bound for this approximation is

$$|y(t_1) - w_1| \leq \frac{0.5 \times 53.047471}{2 \times 2}\left(e^{2(0.5-0)} - 1\right) = 11.393813.$$

For the second approximation, we now have that $t_2 = 1$. We similarly have that

$$|y(t_2) - w_2| \leq \frac{0.5 \times 53.047471}{2 \times 2}\left(e^{2(1-0)} - 1\right) = 42.365408.$$

We can summarize the results in the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i\rvert$	Error bound
1	0.5	0.000000	0.283616	0.283616	11.393813
2	1.0	1.120422	3.219099	2.098677	42.365408

Question 5.2.3(b)

We are given that the actual solution to the IVP in 5.2.1(b) is

$$y(t) = t + \frac{1}{1-t}.$$

So the actual solution for $y(2.5)$ is then

$$y(2.5) = 2.5 + \frac{1}{1-2.5} = 1.833333.$$

Hence the actual error between w_1 and $y(2.5)$ is $|w_1 - y(0.5)| = |2-1.833333| = 0.166667$. Additionally, the actual solution for $y(3)$ is

$$y(3) = 3 + \frac{1}{1-3} = 2.5.$$

This means that the actual error between w_2 and $y(3)$ is $|w_2 - y(3)| = |2.625-2.5| = 0.125$. We will now compute the error bound. Recall that the error bound is given by

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left(e^{L(t_i-a)} - 1\right)$$

for each $i = 1, 2$. Using Theorem 5.3 and the known solution $y(t)$, we have that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|\frac{\partial}{\partial y}\left(1+(t-y)^2\right)\right| = |-2(t-y)| = \left|-2\left(t-\left(t+\frac{1}{1-t}\right)\right)\right| = \left|\frac{-2}{1-t}\right| \leq 2$$

for $t \in [2,3]$. Hence f satisfies a Lipschitz condition in y on D with Lipschitz constant $L = 2$. We now want to compute M. We note that

$$|y''(t)| = \left|\frac{2}{(1-t)^3}\right| \leq 2 \max_{t\in[2,3]} \underbrace{\left|\frac{1}{(1-t)^3}\right|}_{=g(t)}$$

We want to identify any critical points of $g(t)$ on $[2,3]$. We find that

$$g'(t) = \frac{-3}{(1-t)^4} < 0$$

for all $t \in [2,3]$, so $g(t)$ is strictly decreasing on this interval. Hence, the only critical points are the endpoints $t = 2$ and $t = 3$. We find that

$$\begin{aligned} g(2) &= 1/(1-2)^3 = -1, \\ g(3) &= 1/(1-3)^3 = -1/8. \end{aligned}$$

Clearly, $|g(2)|$ is the maximum value of $|g(t)|$ for $t \in [2,3]$. It follows that

$$|y''(t)| = \left|\frac{2}{(1-t)^3}\right| \leq 2 \max_{t\in[2,3]} \left|\frac{1}{(1-t)^3}\right| = 2|g(2)| = 2|-1| = 2.$$

Hence, $M = 2$. We can proceed to find the error bound for our first approximation. We have that $t_1 = 2.5$, $a = 2$, $h = 0.5$, $L = 2$, and $M = 2$. We obtain

$$|y(t_1) - w_1| \leq \frac{0.5\times 2}{2\times 2}\left(e^{2(2.5-2)} - 1\right) = 0.429570.$$

And for our second approximation, $t_2 = 3$ and

$$|y(t_2) - w_2| \leq \frac{0.5\times 2}{2\times 2}\left(e^{2(3-2)} - 1\right) = 1.597264.$$

We can summarize our results in the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i\rvert$	Error bound
1	2.5	2.000000	1.833333	0.166667	0.429570
2	3.0	2.625000	2.500000	0.125000	1.597264

Question 5.2.5(a)

Consider the initial value problem given by

$$y' = y/t - (y/t)^2,\ 1 \leq t \leq 2,\ y(1) = 1.$$

For Euler's Method with $h = 0.1$, we will perform $N = (b-a)/h = (2-1)/0.1 = 10$ iterations. Given $y(1) = 1$, we have that $t_0 = 1$ and $w_0 = 1$. Using the Python code (available in euler.py), the output for the Euler's Method approximations from all 10 iterations is given below.

i	t_i	w_i
0	1.0	1.0000000
1	1.1	1.0000000
2	1.2	1.0082645
3	1.3	1.0216895
4	1.4	1.0385147
5	1.5	1.0576682
6	1.6	1.0784611
7	1.7	1.1004322
8	1.8	1.1232621
9	1.9	1.1467236
10	2.0	1.1706516

Question 5.2.5(b)

Consider the initial value problem given by

$$y' = 1 + y/t + (y/t)^2,\ 1 \le t \le 3,\ y(1) = 0.$$

Using Euler's Method with $h = 0.2$, we will perform $N = (b-a)/h = (3-1)/0.2 = 10$ iterations. Given $y(1) = 0$, we have that $t_0 = 1$ and $w_0 = 0$. Using the aforementioned Python code (included in euler.py), the outputted Euler's Method approximations for all 10 iterations are below.

i	t_i	w_i
0	1.0	0.0000000
1	1.2	0.2000000
2	1.4	0.4388889
3	1.6	0.7212428
4	1.8	1.0520380
5	2.0	1.4372511
6	2.2	1.8842608
7	2.4	2.4022696
8	2.6	3.0028372
9	2.8	3.7006007
10	3.0	4.5142774

Question 5.2.7(a)

The actual solution to the IVP in 5.2.5(a) is

$$y(t) = \frac{t}{1 + \ln t}.$$

Using the Euler's Method code (euler.py), we find the actual errors of the approximations calculated in 5.2.5(a). These results are outputted in the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i \rvert$
0	1.0	1.0000000	1.0000000	0.0000000
1	1.1	1.0000000	1.0042817	0.0042817
2	1.2	1.0082645	1.0149523	0.0066879
3	1.3	1.0216895	1.0298137	0.0081242
4	1.4	1.0385147	1.0475339	0.0090192
5	1.5	1.0576682	1.0672624	0.0095942
6	1.6	1.0784611	1.0884327	0.0099716
7	1.7	1.1004322	1.1106551	0.0102229

8	1.8	1.1232621	1.1336536	0.0103915
9	1.9	1.1467236	1.1572284	0.0105048
10	2.0	1.1706516	1.1812322	0.0105806

Question 5.2.7(b)

The actual solution to the IVP in 5.2.5(b) is

$$y(t) = t\tan(\ln t).$$

Using the Euler's Method code (euler.py), we can find the actual errors of the approximations computed in 5.2.5(b). The output is given by the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i\rvert$
0	1.0	0.0000000	0.0000000	0.0000000
1	1.2	0.2000000	0.2212428	0.0212428
2	1.4	0.4388889	0.4896817	0.0507928
3	1.6	0.7212428	0.8127527	0.0915100
4	1.8	1.0520380	1.1994386	0.1474006
5	2.0	1.4372511	1.6612818	0.2240306
6	2.2	1.8842608	2.2135018	0.3292410
7	2.4	2.4022696	2.8765514	0.4742818
8	2.6	3.0028372	3.6784753	0.6756382
9	2.8	3.7006007	4.6586651	0.9580644
10	3.0	4.5142774	5.8741000	1.3598226

Question 5.2.9(a)

We are given the initial value problem

$$y' = \frac{2}{t}y + t^2 e^t,\ 1 \le t \le 2,\ y(1) = 0$$

and the exact solution $y(t) = t^2(e^t - e)$. Using the Python code (euler.py) for Euler's Method with $h = 0.1$, we find the approximations, exact values, and corresponding actual errors at each step. The output is given below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i\rvert$
0	1.0	0.0000000	0.0000000	0.0000000
1	1.1	0.2718282	0.3459199	0.0740917
2	1.2	0.6847556	0.8666425	0.1818870
3	1.3	1.2769783	1.6072151	0.3302367
4	1.4	2.0935477	2.6203596	0.5268119
5	1.5	3.1874451	3.9676663	0.7802212
6	1.6	4.6208178	5.7209615	1.1001437
7	1.7	6.4663964	7.9638735	1.4974771
8	1.8	8.8091197	10.7936247	1.9845050
9	1.9	11.7479965	14.3230815	2.5750850
10	2.0	15.3982357	18.6830971	3.2848614

Question 5.2.9(c)

To begin, note that $D = \{(t, y) : 1 \le t \le 2, -\infty < y < \infty\}$. By Equation 5.10, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right)$$

where $f(t, y)$ is continuous and satisfies a Lipschitz condition on y with constant L on D and $|y''(t)| \le M$ for all $t \in [a, b]$. We want to find the value of h such that $|y(t_i) - w_i| \le 0.1$ for each i. Hence we need to find h such that

$$\frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right) \le 0.1.$$

Note that for each $i = 0, 1, 2, \ldots, N$,

$$\frac{hM}{2L}\left(e^{L(t_i-a)}-1\right) \le \frac{hM}{2L}\left(e^{L(b-a)}-1\right)$$

as the exponential function is increasing. From the IVP, we know that $a = 1$ and $b = 2$, so

$$\frac{hM}{2L}\left(e^{L(2-1)}-1\right) = \frac{hM}{2L}\left(e^{L}-1\right) \le 0.1.$$

We can now apply Theorem 5.3 to see that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|\frac{2}{t}\right| \le 2$$

for $t \in [1,2]$. Hence f satisfies a Lipschitz condition on y in D with Lipschitz constant $L = 2$. Now, to compute M, we note that

$$\left|y''(t)\right| = \left|(t^2+4t+2)e^t - 2e\right| \le \max_{t\in[1,2]} \big|\underbrace{(t^2+4t+2)e^t - 2e}_{=g(t)}\big|$$

for $t \in [1,2]$. We want to use the Extreme Value Theorem to find critical points on this interval. Hence, we have that

$$g'(t) = (t^2+6t+6)e^t.$$

Letting $g'(t) = 0$, we can use the quadratic formula to find that in addition to the endpoints $t = 1$ and $t = 2$, $t = -3-\sqrt{3}$ and $t = -3+\sqrt{3}$ are also critical points for $g(t)$ on the interval $[1,2]$. We compute

$$\begin{aligned}
g(1) &= (1^2+4(1)+2)e^1 - 2e = 13.591409,\\
g(2) &= (2^2+4(2)+2)e^2 - 2e = 98.010221,\\
g(-3-\sqrt{3}) &= \left((-3-\sqrt{3})^2 + 4(-3-\sqrt{3}+2\right)e^{-3-\sqrt{3}} - 2e = -5.388434,\\
g(-3+\sqrt{3}) &= \left((-3+\sqrt{3})^2 + 4(-3+\sqrt{3}+2\right)e^{-3+\sqrt{3}} - 2e = -5.848574.
\end{aligned}$$

As we can see, $|g(2)|$ is the maximum value of $|g(t)|$ on $[2,3]$. We thus have that

$$\left|y''(t)\right| = \left|(t^2+4t+2)e^t - 2e\right| \le \max_{t\in[1,2]}\left|(t^2+4t+2)e^t - 2e\right| = \left|(2^2+4(2)+2)e^2 - 2e\right| = 98.010222.$$

Therefore, $M = 98.010222$. Rearranging the above equation, we have that

$$h \le \frac{0.2L}{M(e^L-1)}.$$

We substitute in our known values of M and L to find that

$$h \le \frac{0.2\times 2}{98.010222(e^2-1)} = 0.00063878.$$

Hence, we require a step size of $h \le 0.00063878$ to guarantee that $|y(t_i) - w_i| \le 0.1$ for each $i = 0, 1, 2, \ldots, N$.

Question 5.3.5(a)

Consider the initial value problem given by

$$y' = y/t - (y/t)^2,\ 1 \le t \le 1.2,\ y(1) = 1.$$

We want to approximate the solution using Taylor's Method of order 2 for $h = 0.1$. As such, we need to perform $N = (b-a)/h = (1.2-1)/0.1 = 2$ iterations. We find that

$$\frac{d}{dt}f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f = -\frac{y}{t^2} + \frac{2y^2}{t^3} + \left(\frac{1}{t} - \frac{2y}{t^2}\right)\left(y/t - (y/t)^2\right).$$

So the setup for Taylor's Method of order 2 is

$$\begin{aligned}
w_0 &= 1\\
w_{i+1} &= w_i + \frac{h}{2}\left(-\frac{w_i}{t_i^2} + \frac{2w_i^2}{t_i^3} + \left(\frac{1}{t_i} - \frac{2w_i}{t_i^2}\right)\left(w_i/t_i - (w_i/t_i)^2\right)\right)
\end{aligned}$$

for $i = 0, 1$ and $h = 0.1$. Using the Python code for order 2 Taylor's Method (taylor_order_2.py), we obtain the approximations in the table below.

i	t_i	w_i
0	1.0	1.0000000
1	1.1	1.0050000
2	1.2	1.0160294

Question 5.3.5(b)

Consider the initial value problem given by

$$y' = \sin t + e^{-t},\ 0 \le t \le 1,\ y(0) = 0.$$

We want to approximate the solution using Taylor's Method (order 2) with $h = 0.5$. Hence, we require $N = (b-a)/h = (1-0)/0.5 = 2$ iterations. First, we find that

$$\frac{d}{dt} f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f = \cos t - e^{-t}.$$

The setup for Taylor's Method of order 2 is

$$\begin{aligned} w_0 &= 0 \\ w_{i+1} &= w_i + \frac{h}{2}\left(\cos t_i - e^{-t_i}\right) \end{aligned}$$

for $i = 0, 1$, and $h = 0.5$. Using the Python code for order 2 Taylor's Method (taylor_order_2.py), the approximations are outputted in the table below.

i	t_i	w_i
0	0.0	0.0000000
1	0.5	0.5000000
2	1.0	1.0768596

Question 5.3.9(a)

We are given the initial value problem

$$y' = \frac{2}{t} y + t^2 e^t,\ 1 \le t \le 2,\ y(1) = 0$$

and the exact solution $y(t) = t^2(e^t - e)$. We want to use Taylor's Method of order 2 with $h = 0.1$ to approximate the solution. To apply this method, since the exact solution $y(t)$ is given, we have that

$$f'(t,y) = y''(t) = (t^2 + 4t + 2)e^t - 2e.$$

The setup for this method is therefore

$$\begin{aligned} w_0 &= 0 \\ w_{i+1} &= w_i + h\left(\frac{2}{t_i} w_i + t_i^2 e^{t_i}\right) + \frac{h^2}{2}\left((t_i^2 + 4t_i + 2)e^{t_i} - 2e\right) \end{aligned}$$

for $i = 0, 1, \ldots, 9$ and $h = 0.1$. Using the Python code for Taylor's Method of order 2 (taylor_order_2.py), our code outputs the results in the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i \rvert$
0	1.0	0.0000000	0.0000000	0.0000000

1	1.1	0.3397852	0.3459199	0.0061346
2	1.2	0.8521941	0.8666425	0.0144484
3	1.3	1.5819293	1.6072151	0.0252857
4	1.4	2.5813316	2.6203596	0.0390279
5	1.5	3.9115682	3.9676663	0.0560981
6	1.6	5.6439945	5.7209615	0.0769671
7	1.7	7.8617134	7.9638735	0.1021601
8	1.8	10.6613611	10.7936247	0.1322636
9	1.9	14.1551478	14.3230815	0.1679337
10	2.0	18.4731918	18.6830971	0.2099053

Question 5.3.9(c)

Again, We are given the initial value problem

$$y' = \frac{2}{t}y + t^2 e^t,\ 1 \le t \le 2,\ y(1) = 0$$

and the exact solution $y(t) = t^2(e^t - e)$. We want to use Taylor's Method of order 4 with $h = 0.1$ to approximate the solution. From part (a), we know that

$$f'(t,y) = y''(t) = (t^2 + 4t + 2)e^t - 2e.$$

However, we also need $f''(t,y)$ and $f'''(t,y)$. So we have that

$$f''(t,y) = y'''(t) = (t^2 + 6t + 6)e^t$$

and lastly that

$$f'''(t,y) = y^{(4)}(t) = (t^2 + 8t + 12)e^t.$$

The setup for this method is therefore

$$\begin{aligned} w_0 &= 0 \\ w_{i+1} &= w_i + h\left(\frac{2}{t_i}w_i + t_i^2 e^{t_i}\right) + \frac{h^2}{2}\left((t_i^2 + 4t_i + 2)e^{t_i} - 2e\right) + \frac{h^3}{6}\left((t_i^2 + 6t_i + 6)e^{t_i}\right) + \frac{h^4}{24}\left((t_i^2 + 8t_i + 12)e^{t_i}\right) \end{aligned}$$

for $i = 0, 1, \ldots, 9$ and $h = 0.1$. Using the Python code for Taylor's Method of order 4 (taylor_order_4.py), we obtain the outputs in the table below.

i	t_i	w_i	$y(t_i)$	$\lvert y(t_i) - w_i\rvert$
0	1.0	0.0000000	0.0000000	0.0000000
1	1.1	0.3459127	0.3459199	0.0000072
2	1.2	0.8666258	0.8666425	0.0000167
3	1.3	1.6071861	1.6072151	0.0000290
4	1.4	2.6203152	2.6203596	0.0000443
5	1.5	3.9676032	3.9676663	0.0000631
6	1.6	5.7208758	5.7209615	0.0000857
7	1.7	7.9637608	7.9638735	0.0001127
8	1.8	10.7934801	10.7936247	0.0001446
9	1.9	14.3228996	14.3230815	0.0001819
10	2.0	18.6828717	18.6830971	0.0002253

euler.py

The following code provides my implementation of Euler's Method in Python. The example below solves 5.2.9(a).

```
import numpy as np
import pandas as pd
import math

# For more decimal places in output
pd.set_option("display.precision", 7)

## Argument specification

# Function
f = lambda t, y: (2/t)*y + t**2*math.exp(1)**t
# Left endpoint
t_0 = 1
# Right endpoint
t_1 = 2
# Step size
h = 0.1
# Initial condition
w0 = 0

N = int((t_1-t_0)/h)
t = np.arange(t_0, t_1+h, h)
w = np.zeros(len(t))
w[0] = w0

# Approximations
for i in range(0, len(t) - 1):
  w[i+1] = w[i] + h*f(t[i], w[i])

# Display
df = pd.DataFrame({('t_i'): t, 'w_i': w})
print(df)

# Error computation for when exact solution is known
y = lambda t: t**2*(math.exp(1)**t - math.exp(1))
yt = np.zeros(len(t))

# Exact values
for i in range(0, len(t)):
  yt[i] = y(t[i])

df2 = pd.DataFrame({('t_i'): t, 'w_i': w,
                    'y(t_i)': yt, '|y(t_i) - w_i|': abs(w-yt)})
print(df2)
```

taylor_order_2.py

The code chunk below is my implementation of Taylor's Method of order 2 in Python. For example below is the solution for 5.3.9(a).

```
import numpy as np
import pandas as pd
import math

# For more decimal places in output
pd.set_option("display.precision", 7)

## Argument specification

# Functions
f        = lambda t, y: (2/t)*y + t**2*math.exp(1)**t
f_prime  = lambda t, y: (t**2 + 4*t + 2)*math.exp(1)**t - 2*math.exp(1)
# Left endpoint
t_0 = 1
# Right endpoint
t_1 = 2
```

```
# Step size
h = 0.1
# Initial condition
w0 = 0

t = np.arange(t_0, t_1+h, h)
w = np.zeros(len(t))
w[0] = w0

# Approximations
for i in range(0, len(t) - 1):
  w[i+1] = w[i] + h*(f(t[i], w[i])) + h**2/2*f_prime(t[i], w[i])

# Display
df = pd.DataFrame({('t_i'): t, 'w_i': w})
print(df)

# Error computation for when exact solution is known
y = lambda t: t**2*(math.exp(1)**t - math.exp(1))
yt = np.zeros(len(t))

# Exact values
for i in range(0, len(t)):
  yt[i] = y(t[i])

# Display
df2 = pd.DataFrame({('t_i'): t, 'w_i': w,
                    'y(t_i)': yt, '|y(t_i) - w_i|': abs(w-yt)})
print(df2)
```

taylor_order_4.py

The code snippet below implements Taylor's Method of order 4 in Python. The specific instance below is for 5.3.9(c).

```
import numpy as np
import pandas as pd
import math

# For more decimal places in output
pd.set_option("display.precision", 7)

## Argument specification

# Functions
f        = lambda t, y: (2/t)*y + t**2*math.exp(1)**t
f_prime  = lambda t, y: (t**2 + 4*t + 2)*math.exp(1)**t - 2*math.exp(1)
f_dprime = lambda t, y: (t**2 + 6*t + 6)*math.exp(1)**t
f_tprime = lambda t, y: (t**2 + 8*t + 12)*math.exp(1)**t
# Left endpoint
t_0 = 1
# Right endpoint
t_1 = 2
# Step size
h = 0.1
# Initial condition
w0 = 0

t = np.arange(t_0, t_1+h, h)
w = np.zeros(len(t))
w[0] = w0

t = np.arange(t_0, t_1+h, h)
w = np.zeros(len(t))
w[0] = w0

# Approximations
for i in range(0, len(t) - 1):
  w[i+1] = w[i] + h*(f(t[i], w[i])) + h**2/2*f_prime(t[i], w[i])

```

```python
# Display
df = pd.DataFrame({('t_i'): t, 'w_i': w})
print(df)

# Error computation for when exact solution is known
y = lambda t: t**2*(math.exp(1)**t - math.exp(1))
yt = np.zeros(len(t))

# Exact values
for i in range(0, len(t)):
  yt[i] = y(t[i])

# Display
df2 = pd.DataFrame({('t_i'): t, 'w_i': w,
                    'y(t_i)': yt, '|y(t_i) - w_i|': abs(w-yt)})
print(df2)
```